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ON THE CLASSIFICATION OF KIM AND KOSTRIKIN MANIFOLDS

ALBERTO CAVICCHIOLI

Dipartimento di Matematica,
Università di Modena e Reggio Emilia,
Via Campi 213/B, 41100 Modena, Italy
cavicchioli.alberto@unimo.it

LUISA PAOLUZZI

Institut de Mathématiques de Bourgogne,
UMR 5584 du CNRS, Université de Bourgogne,
BP 47870, 9 Avenue Alain Savary,
21078 Dijon Cedex, France
paoluzzi@u-bourgogne.fr

FULVIA SPAGGIARI

Dipartimento di Matematica,
Università di Modena e Reggio Emilia,
Via Campi 213/B, 41100 Modena, Italy
spaggiari.fulvia@unimo.it

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ABSTRACT

We completely classify the topological and geometric structures of some series of closed connected orientable 3-manifolds introduced by Kim and Kostrikin in [20, 21] as quotient spaces of certain polyhedral 3-cells by pairwise identifications of their boundary faces. Then we study further classes of closed orientable 3-manifolds arising from similar polyhedral schemata, and describe their topological properties.

Keywords: 3-manifolds; group presentations; spines; orbifolds; polyhedral schemata; branched coverings.

Mathematics Subject Classification 2000: 57M12, 57M50, 57M60

1. Introduction

In [20, 21], Kim and Kostrikin constructed and studied five series of groups with finite presentations $G_i(n)$, $n \geq 1$, $i = 1, 2, 3, 4, 5$, for which they described, for instance, their derived quotients and proved that almost all of them are infinite. Moreover, they proved that some of these groups are 3-manifold groups by constructing five series of (not necessarily closed) orientable 3-manifolds $M_i(n)$, $n \geq 1$, $i = 1, 2, 3, 4, 5$, arising from polyhedral schemata, with the property that

$\pi_1(M_i(n)) \cong G_i(n)$ for all $n \geq 1$ if $i = 1, 4$, for all $n \geq 3$ if $i = 5$, and for small values of n (i.e. $n = 1, 2, 4$) if $i = 2$; (for $i = 3$ see our comment at the end of Sec. 3). Then they asked if the corresponding 3-manifolds are branched coverings of some knots or links and which of them admit a hyperbolic structure. The manifolds $M_1(n)$ can be constructed as special honey-comb spaces and their topological and geometric structures have been determined in [2]. In particular, the manifolds $M_1(n)$ are hyperbolic and can be represented as $(3n)$-fold cyclic coverings of $\mathbb{S}^3$ branched over the Whitehead link, where the branching indices of its components are 3 and $3n$, respectively. The study of the whole family of (not necessarily strongly) cyclic branched coverings of the Whitehead link can be found in [5] (see also [22, 32] for some related results). The manifolds $M_5(n)$ were completely classified in [1], and they are Seifert fibered spaces. In this paper, we will provide the topological classification of the manifolds $M_i(n)$, for $2 \leq i \leq 4$. More precisely, for every $n \geq 1$, we construct a polyhedral scheme $P_{n,p,q}$, depending on non negative integers p and q, which extends the combinatorial construction of the closed 3-manifolds $M_2(n)$ given in [20, 21] (see Sec. 2). Then we represent the corresponding manifolds as n-fold coverings of a connected sum of lens spaces branched over a $(2, 1)$-knot, and discuss their geometric structure (see Sec. 4). In Sec. 3, we consider the manifolds $M_4(n)$ and prove that $M_4(n)$ are homeomorphic to the manifolds $M_2(3n+1)$. Furthermore, we observe that the balanced groups $G_3(n)$ of [21] cannot be the fundamental groups of the manifolds $M_3(n)$, as claimed there. In fact, the manifolds $M_3(n)$ defined in [21] are simply lens spaces. Finally, in Sec. 5 we consider the quotient spaces obtained from the polyhedron $P_{n,p,q}$ by using all possible pairwise identifications of its boundary faces, and study the topology and geometry of the corresponding 3-manifolds. Our geometric constructions provide at least two infinite classes of hyperbolic closed orientable 3-manifolds.

2. Kim–Kostrikin Groups $G_2(n)$ and Manifolds $M_2(n)$

The following family $G_2(n)$, $n \geq 1$, of balanced group presentations was defined in [20, 21]:

$$G_2(n) = \langle \alpha_1, \alpha_2, \ldots, \alpha_{2n} : \alpha_{i+3}\alpha_i\alpha_{i+4}\alpha_{i+2} = \alpha_{i+1} \quad (i = 1, \ldots, 2n)\rangle,$$

where the subscripts are taken mod $2n$. As shown in the quoted papers, the groups $G_2(1)$, $G_2(2)$, and $G_2(4)$ are of geometric origin, that is, they correspond to spines of certain closed connected orientable 3-manifolds $M_2(n)$, $n = 1, 2, 4$. In fact, the authors constructed a tessellation of a 2-sphere (see Fig. 1 for $n = 2$ and Fig. 2 for $n = 4$) consisting of $4n$ pentagons. There are n pentagons with a common vertex at the north pole N, n pentagons with a common vertex at the south pole S, and $2n$ pentagons in the equatorial zone. Identifying the pairs of faces with the same labels, as well as the corresponding edges and vertices, they obtained a 3-dimensional complex $M_2(n)$, which is a closed 3-manifold since its Euler characteristic vanishes, and $\pi_1(M_2(n)) = G_2(n)$, $n = 1, 2, 4$. The following question was stated in [21] (see

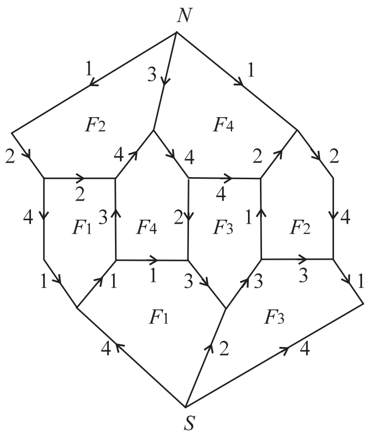

Fig. 1. A polyhedral representation for the manifold $M_2(2)$.

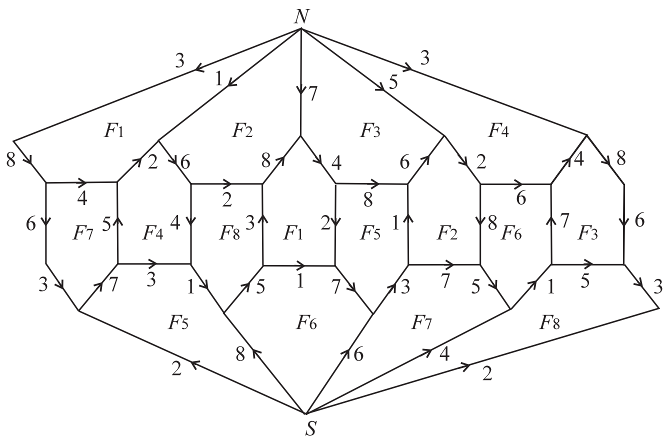

Fig. 2. A polyhedral representation for the manifold $M_2(4)$.

also [20]): *For which* $n \neq 1, 2, 4$, *the quotient complex* $M_2(n)$ *is a closed hyperbolic 3-manifold*? In this section we shall study a family of 3-dimensional complexes, generalizing those introduced in [21], and determine which of them are in fact closed 3-manifolds. It turns out that this is the case if and only if $n \not\equiv 0 \pmod 3$, showing that $M_2(n)$ is a closed 3-manifold for all $n \not\equiv 0 \pmod 3$. We also provide new balanced presentations for the fundamental groups of the obtained 3-manifolds.

For every $n \geq 1$, $0 \leq p \leq n-1$ and $1 \leq q \leq n$, let us consider the combinatorial 3-cell $P_{n,p,q}$ whose 2-sphere boundary consists of n pentagons labelled by Y_i' in the northern hemisphere, n pentagons labelled by X_i in the southern hemisphere, and $2n$ pentagons labelled by X_i' and Y_i, $i = 1, 2, \ldots, n$, in the equatorial zone (see Fig. 3). The side pairing is determined by identifying the pairs of faces (X_i, X_i') and

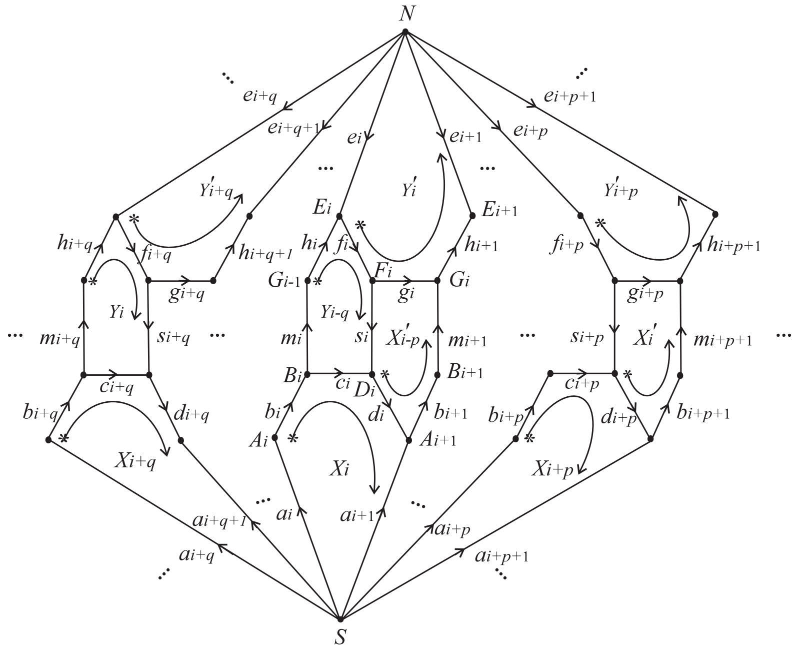

Fig. 3. The polyhedral scheme $P_{n,p,q}$.

(Y_i, Y_i') so that the corresponding oriented boundary edges with the same labels are glued together. The integer p (respectively, q) is the number of pentagons we have to shift before gluing the face X_i (respectively, Y_i') to X_i' (respectively, Y_i).

Let x_i and y_i be the identifications between the pairs of faces (X_i, X_i') and (Y_i, Y_i'), respectively, as follows:

$$
\begin{aligned}
x_i : X_i &\equiv X_i' \\
b_i &\to d_{i+p} \\
c_i &\to b_{i+p+1} \\
d_i &\to m_{i+p+1} \\
a_{i+1} &\to g_{i+p} \\
a_i &\to s_{i+p}
\end{aligned}
$$

and

$$
\begin{aligned}
y_i^{-1} : Y_i' &\equiv Y_i \\
f_i &\to h_{i+q} \\
g_i &\to f_{i+q} \\
h_{i+1} &\to s_{i+q} \\
e_{i+1} &\to c_{i+q} \\
e_i &\to m_{i+q}.
\end{aligned}
$$

Then we get the cycles of equivalent edges

$$a_i \xrightarrow{x_i} s_{i+p} \xrightarrow{y_{i+p-q}} h_{i+p-q+1} \xrightarrow{y_{i+p-2q+1}} f_{i+p-2q+1} \xrightarrow{y_{i+p-3q+1}} g_{i+p-3q+1} \xrightarrow{x_{i-3q+1}^{-1}} a_{i-3q+2}$$

with the arithmetic condition $2 - 3q \equiv 0 \pmod{n}$. This gives the relations

$$x_i y_{i+p-q} y_{i+p-2q+1} y_{i+p-3q+1} x_{i-3q+1}^{-1} = 1$$

for every $i = 1, \ldots, (n, 3q-2) = n$. Furthermore, we have also the cycles of equivalent edges

$$b_i \xrightarrow{x_i} d_{i+p} \xrightarrow{x_{i+p}} m_{i+2p+1} \xrightarrow{y_{i+2p-q+1}} e_{i+2p-q+1} \xrightarrow{y_{i+2p-q}^{-1}} c_{i+2p} \xrightarrow{x_{i+2p}} b_{i+3p+1}$$

with the arithmetic condition $3p + 1 \equiv 0 \pmod{n}$. This gives the relations

$$x_i x_{i+p} y_{i+2p-q+1} y_{i+2p-q}^{-1} x_{i+2p} = 1$$

for every $i = 1, \ldots, (n, 3p+1) = n$. So we have exactly $(n, 2-3q) = n$ classes of equivalent edges a_i and $(n, 3p+1) = n$ classes of equivalent edges b_i. The resulting identification space $M_{n,p,q}$ of $P_{n,p,q}$ has a cellular decomposition with one vertex, $(n, 3p+1) + (n, 2-3q) = 2n$ edges, $2n$ 2-cells, and one 3-cell, hence its Euler characteristic vanishes (compare with [27]). Thus we have the following characterization result:

Theorem 2.1. *With the above notation, the quotient space* $M_{n,p,q}$, $n \geq 1$, $0 \leq p \leq n-1$, *and* $1 \leq q \leq n$, *is a closed connected orientable* 3*-manifold if and only if* $3p + 1 \equiv 0 \pmod{n}$, $2 - 3q \equiv 0 \pmod{n}$, *and* $n \equiv 1, 2 \pmod{3}$.

Let us denote by $G_{n,p,q}$ the fundamental group of the manifold $M_{n,p,q}$. We can obtain a finite presentation of $G_{n,p,q}$ by considering the maps x_i and y_i $(i = 1, 2, \ldots, n)$ which identify the pairs of faces (X_i, X_i') and (Y_i, Y_i'), respectively. Therefore we have

Theorem 2.2. *Under the arithmetic conditions of Theorem* 2.1, *the polyhedral* 3*-cell* $P_{n,p,q}$ *with the identifications described above defines a closed connected orientable* 3*-manifold* $M_{n,p,q}$ *which has a spine modeled on the finite geometric presentation*

$$\begin{aligned} G_{n,p,q} \cong \langle x_1, \ldots, x_n, y_1, \ldots, y_n : \; & x_i y_{i+p-q} y_{i+p-2q+1} y_{i+p-3q+1} x_{i-3q+1}^{-1} = 1 \\ & (i = 1, \ldots, (n, 3q-2) = n) \\ & y_i^{-1} x_{i+q} x_{i+q-2p} x_{i+q-p} y_{i+1} = 1 \quad (i = 1, \ldots, (n, 3p+1) = n) \rangle, \end{aligned}$$

where the indices are taken mod n.

Since the quotient cellular complex $M_{n,p,q}$ has exactly one vertex, we can obtain a further geometric presentation for the fundamental group $G_{n,p,q} = \pi_1(M_{n,p,q})$ with generators a_i, $i = 1, \ldots, (n, 2-3q) = n$, and b_i, $i = 1, \ldots, (n, 3p+1) = n$, and relations arising from the boundaries of the 2-cells $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ of the polyhedral scheme $P_{n,p,q}$. Then we have the following result.

Theorem 2.3. *The fundamental group $G_{n,p,q}$ of the manifold $M_{n,p,q}$ admits the finite geometric presentation*

$$G_{n,p,q} \cong \big\langle a_i, b_i : a_i b_i b_{i-2p} b_{i-p} a_{i+1}^{-1} = 1 \quad (i = 1, \dots, n) \\ b_i a_{i+p+q} a_{i+p+2q} a_{i+p+1} b_{i+1}^{-1} = 1 \quad (i = 1, \dots, n) \big\rangle,$$

$n \equiv 1, 2 \pmod 3$, $3p + 1 \equiv 0 \pmod n$, $2 - 3q \equiv 0 \pmod n$, *which corresponds to a spine of the manifold.*

The algebraic conditions of Theorem 2.1 determine the following two cases:

First case: $\boldsymbol{n = 3m + 1}$, $\boldsymbol{m \geq 0}$. The conditions $3p + 1 \equiv 0 \pmod n$ and $0 \leq p \leq n - 1$ imply that $3p + 1 = an \geq 1$, $n \geq 1$, $a \geq 1$. Since $3p + 1 \leq 3n - 2$, we get $an \leq 3n - 2$ hence $3 - a \geq 2$ as $n \geq 1$. This gives $a \leq 1$, hence $a = 1$ and $p = m$. In a similar way, the conditions $3q - 2 \equiv 0 \pmod n$ and $1 \leq q \leq n$ imply that $q = m + 1$. Then we obtain the first class of Kim and Kostrikin manifolds $M_2(n) = M_{n,m,m+1}$, $n = 3m + 1$, whose fundamental groups $G_{n,m,m+1}$ admit the following geometric presentations

$$G_{n,m,m+1} \cong \langle x_1, \dots, x_n, y_1, \dots, y_n : \\ \text{(I)} \quad x_i y_{i-1} y_{i-m-1} y_{i+m-1} x_{i-1}^{-1} = 1 \quad (i = 1, \dots, (n, 3q - 2) = n) \\ \text{(II)} \quad y_i^{-1} x_{i+m+1} x_{i-m+1} x_{i+1} y_{i+1} = 1 \quad (i = 1, \dots, (n, 3p + 1) = n) \big\rangle$$

and

$$G_{n,m,m+1} \cong \big\langle a_1, \dots, a_n, b_1, \dots, b_n : \\ \text{(I}'\text{)} \quad a_i b_i b_{i+m+1} b_{i+2m+1} a_{i+1}^{-1} = 1 \quad (i = 1, \dots, n) \\ \text{(II}'\text{)} \quad b_i a_{i+2m+1} a_{i+1} a_{i+m+1} b_{i+1}^{-1} = 1 \quad (i = 1, \dots, n) \big\rangle$$

where the indices are taken mod n, and $n = 3m+1$, $m \geq 0$. To show the equivalence between the obtained presentations for $G_{n,m,m+1}$, $n = 3m + 1$, we set $a_i := y_i^{-1}$ and $b_i := x_{i+m+1}$, for every $i = 1, \dots, n$. Then relation (I$'$) becomes

$$y_i^{-1} x_{i+m+1} x_{i+2m+2} x_{i+3m+2} y_{i+1} = 1$$

or, equivalently,

$$y_i^{-1} x_{i+m+1} x_{i-m+1} x_{i+1} y_{i+1} = 1$$

which is relation (II). Relation (II$'$) becomes

$$x_{i+m+1} y_{i+2m+1}^{-1} y_{i+1}^{-1} y_{i+m+1}^{-1} x_{i+m+2}^{-1} = 1.$$

Taking its inverse and setting $j = i + m + 2$ we obtain relation (I).

Second case: $\boldsymbol{n = 3m + 2}$, $\boldsymbol{m \geq 0}$. Reasoning as in the first case, one gets $p = 2m + 1$ and $q = 2m + 2$. Then we obtain the second class of Kim and

Kostrikin manifolds $M_2(n) = M_{n,2m+1,2m+2}$, $n = 3m+2$, whose fundamental groups $G_{n,2m+1,2m+2}$ admit the following geometric presentations

$$\begin{aligned} &G_{n,2m+1,2m+2} \\ &\quad \cong \Big\langle x_1,\dots,x_n,y_1,\dots,y_n : \\ &\qquad \text{(I)} \quad x_i y_{i-1} y_{i+m} y_{i-m-2} x_{i-1}^{-1} = 1 \quad (i = 1,\dots,(n,3q-2) = n) \\ &\qquad \text{(II)} \quad y_i^{-1} x_{i+2m+2} x_{i+m+2} x_{i+1} y_{i+1} = 1 \quad (i = 1,\dots,(n,3p+1) = n)\Big\rangle \end{aligned}$$

and

$$\begin{aligned} G_{n,2m+1,2m+2} \cong \langle a_1,\dots,a_n,b_1,\dots,b_n :& \\ &\text{(I}'\text{)} \quad a_i b_i b_{i-m} b_{i+m+1} a_{i+1}^{-1} = 1 \quad (i = 1,\dots,n) \\ &\text{(II}'\text{)} \quad b_i a_{i+m+1} a_{i+1} a_{i+2m+2} b_{i+1}^{-1} = 1 \quad (i = 1,\dots,n)\Big\rangle \end{aligned}$$

where the indices are taken mod n, and $n = 3m+2$, $m \geq 0$. To show the equivalence between the obtained presentations for $G_{n,2m+1,2m+2}$, $n = 3m+2$, we set $a_i := y_i^{-1}$ and $b_i = x_{i+2m+2}$, for every $i = 1,\dots,n$. Then relation (I$'$) becomes

$$y_i^{-1} x_{i+2m+2} x_{i+m+2} x_{i+3m+3} y_{i+1} = 1$$

which is relation (II). Relation (II$'$) becomes

$$x_{i+2m+2} y_{i+m+1}^{-1} y_{i+1}^{-1} y_{i+2m+2}^{-1} x_{i+2m+3}^{-1} = 1.$$

Taking its inverse and setting $j = i+2m+3$ we obtain relation (I).

3. Kim–Kostrikin Groups $G_4(n)$ and Manifolds $M_4(n)$

A perfectly similar construction in [21] gives a series of closed connected orientable 3-manifolds $M_4(n)$, $n \geq 1$, whose polyhedral schemata are depicted in Figs. 4 and 5 for $n = 1$ and $n = 2$, respectively.

As shown in [21], the corresponding fundamental groups $\pi_1(M_4(n)) \cong G_4(n)$ have the following balanced presentations:

$$G_4(n) = \langle \alpha_1, \alpha_2, \dots, \alpha_{6n+2} : R_4^1(i), R_4^2(i), \quad 1 \leq i \leq 3n+1\rangle$$

where

$$\begin{aligned} R_4^1(i) &: \alpha_{6n+5-2i}\alpha_{4n+4-2i}\alpha_{2n+2-2i}\alpha_{6n+4-2i} = \alpha_{6n+3-2i} \quad (1 \leq i \leq 3n+1) \\ R_4^2(i) &: \alpha_{6n+6-2i}\alpha_{4n+5-2i}\alpha_{2n+3-2i}\alpha_{6n+5-2i} = \alpha_{6n+4-2i} \quad (1 \leq i \leq 3n+1). \end{aligned}$$

By construction the above polyhedral schemata coincide with those defining the Kim and Kostrikin manifolds $M_2(3n+1) = M_{3n+1,n,n+1}$. So we have the following

Theorem 3.1. *For every $n \geq 1$, the closed 3-manifolds $M_4(n)$ and $M_2(3n+1) = M_{3n+1,n,n+1}$ are homeomorphic. In particular, the groups $G_4(n)$ are isomorphic to the groups $G_{3n+1,n,n+1}$.*

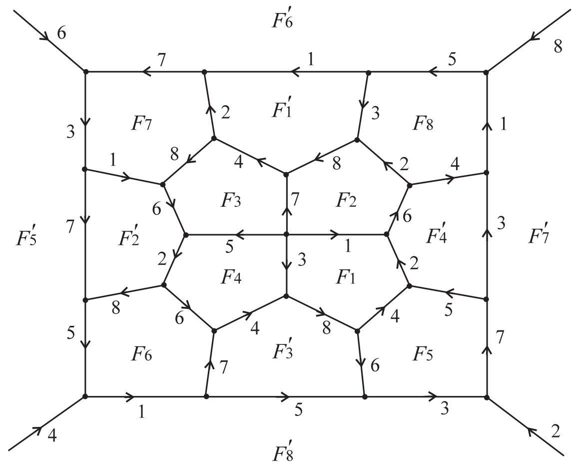

Fig. 4(a). A polyhedral representation for the manifold $M_4(1)$.

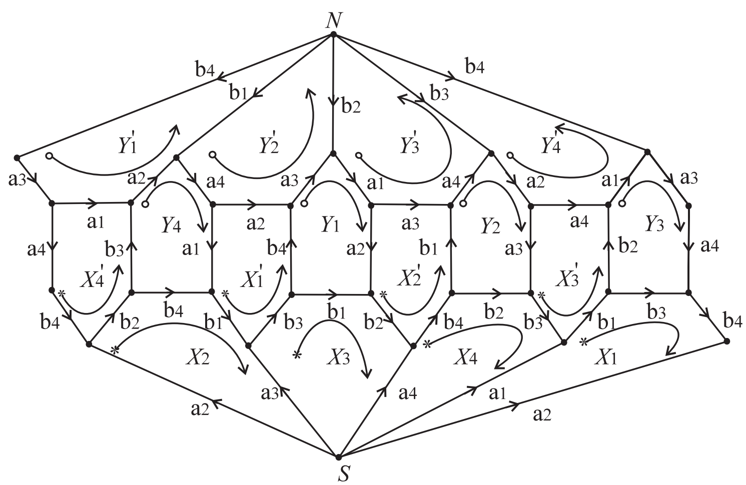

Fig. 4(b). The symmetric representation of $M_4(1)$.

To show explicitly the equivalence between the obtained presentations for the groups $G_4(n)$ and $G_{3n+1,n,n+1}$, $n \geq 1$, we set

$$b_i := \alpha_{4n+4-2i}$$
$$a_i := \alpha_{6n+5-2i} \quad 1 \leq i \leq 3n+1.$$

Then we have

$$\alpha_{2n+2-2i} = \alpha_{4n+4-2i-2n-2} = \alpha_{4n+4-2(i+n+1)} = b_{i+n+1}$$
$$\alpha_{6n+4-2i} = \alpha_{4n+4-2i+2n} = \alpha_{4n+4-2(i-n)} = b_{i-n} = b_{i+2n+1}$$
$$\alpha_{6n+3-2i} = \alpha_{6n+5-2i-2} = \alpha_{6n+5-2(i+1)} = a_{i+1},$$

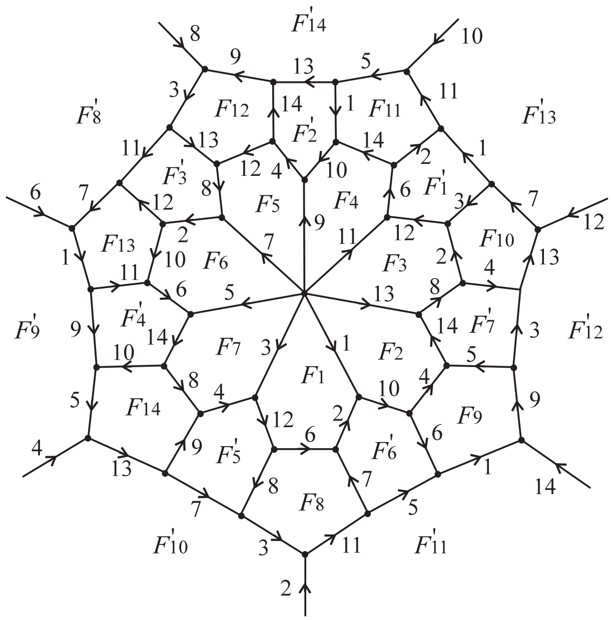

Fig. 5. A polyhedral representation for the manifold $M_4(2)$.

hence the relation $R_4^1(i)$ of $G_4(n)$ becomes

$$a_i b_i b_{i+n+1} b_{i+2n+1} a_{i+1}^{-1} = 1,$$

which is relation (I$'$) of $G_{3n+1,n,n+1}$. Furthermore, we get

$$\begin{aligned}
\alpha_{6n+6-2i} &= \alpha_{4n+4-2i+2n+2} = \alpha_{4n+4-2(i-n-1)} = b_{i-n-1}\\
\alpha_{4n+5-2i} &= \alpha_{6n+5-2i-2n} = \alpha_{6n+5-2(i+n)} = a_{i+n}\\
\alpha_{2n+3-2i} &= \alpha_{6n+5-2i-4n-2} = \alpha_{6n+5-2(i+2n+1)} = a_{i+2n+1}\\
\alpha_{6n+4-2i} &= b_{i-n},
\end{aligned}$$

hence the relation $R_4^2(i)$ of $G_4(n)$ becomes

$$b_{i-n-1} a_{i+n} a_{i+2n+1} a_i b_{i-n}^{-1} = 1$$

or, equivalently,

$$b_i a_{i+2n+1} a_{i+1} a_{i+n+1} b_{i+1}^{-1} = 1$$

which is relation (II$'$) of $G_{3n+1,n,n+1}$.

Remark. In [20] and [21] the authors introduced also the balanced groups $G_3(n)$, $n \geq 1$, with the following presentation

$$G_3(n) = \left\langle \alpha_1, \alpha_2, \ldots, \alpha_{4n+2} : R_3^1(i), R_3^2(i), R_3^3(i) \right\rangle,$$

where

$$\begin{array}{ll} R_3^1(i) : \alpha_{4i+1}\alpha_{4i+2}\alpha_{4i+3} = \alpha_{4i+5}\alpha_{4i+4} & (0 \leq i \leq n) \\ R_3^2(i) : \alpha_{4i-1}\alpha_{4i}\alpha_{4i+1} = \alpha_{4i+3}\alpha_{4i+2} & (1 \leq i \leq n) \\ R_3^3(i) : \alpha_{2i}\alpha_{4n+1-2i}\alpha_{2i+1}\alpha_{4n-1-2i} = \alpha_{2i+2} & (1 \leq i \leq 2n+1) \end{array}$$

where the indices are taken mod $4n+2$. Then they constructed a simply connected topological polyhedron, whose boundary consists of $m = 2n+1$ pentagons with a common vertex in the northern hemisphere, m pentagons with a common vertex in the southern hemisphere, and $2m$ pentagons in the equatorial zone. The numbering of edges and their orientation are as depicted in [21, Figs. 3.1 and 3.2]. The authors stated that after the identification of similarly oriented faces the resulting 3-dimensional complex is a closed connected 3-manifold $M_3(n)$, with $\pi_1(M_3(n)) \cong G_3(n)$. But the identification space $M_3(n)$ is a 3-dimensional cell complex with more than one vertex (and one can directly verify that it is a lens space), so the obtained presentation for $\pi_1(M_3(n))$ does not work. In a forthcoming paper we shall consider all possible side pairings of the boundary faces of the polyhedra related with $M_3(n)$. Then we shall study the topology and geometry of the obtained closed 3-manifolds whenever the pairings produce an infinite series of them.

4. Topological Properties of Manifolds $M_{n,p,q}$

To explain the statement of the next theorem, we recall the definition of *genus two* 1-*knots*, briefly called $(2,1)$-*knots*. They are knots in closed connected orientable 3-manifolds M of Heegaard genus two which admit a $(2,1)$-*decomposition* as follows. A knot $K \subset M$ is called a $(2,1)$-*knot* if there exists a Heegaard splitting of genus two $(M,K) = (V,A) \cup_\phi (V',A')$, where V and V' are orientable handlebodies of genus two, $A \subset V$ and $A' \subset V'$ are properly embedded trivial arcs, and $\phi : (\partial V, \partial A) \to (\partial V', \partial A')$ is an orientation-reversing attaching homeomorphism. The arc A is said to be *trivial* in V if there exists a disk $D \subset V$ such that $A \cap D = A \cap \partial D = A$ and $\partial D \backslash A \subset \partial V$. Note that $(2,1)$-knots are a particular case of the notion of (g,b)-*links* in closed connected orientable 3-manifolds of Heegaard genus g (see, for example [10, 12, 13, 18]). This concept generalizes in a natural way the classical one of bridge decomposition of links in the standard 3-sphere. As general references on the theory of knots and links, see for example [19, 26].

Theorem 4.1. *The closed connected orientable* 3*-manifolds* $M_{n,p,q}$, *with the arithmetic conditions* $3p+1 \equiv 0 \pmod n$, $2-3q \equiv 0 \pmod n$, *and* $n \equiv 1,2 \pmod 3$, *are* n*-fold cyclic coverings of the connected sum of lens spaces* $L(3,1)\#L(3,2)$ *branched over a* $(2,1)$*-knot which is independent of* n*. The singular set is the image of the north-south axis of the polyhedron* $P_{n,p,q}$ *under the* n*-rotational action.*

Proof. Let us consider the n-rotational symmetry of the polyhedron $P_{n,p,q}$ and denote by ρ the corresponding homeomorphism of the manifold $M_{n,p,q}$. The $1/n$–slice Π_n of $P_{n,p,q}$, pictured in Fig. 6(a), is the fundamental polyhedron for the quotient orbifold $M_{n,p,q}/\langle\rho\rangle$. The fundamental group of the quotient space obtained from Π_n is isomorphic to the group presented by $\langle a, b : ab^3a^{-1} = 1,\ ba^3b^{-1} = 1\rangle \cong \mathbb{Z}_3 * \mathbb{Z}_3$. A Heegaard diagram for the quotient space $M_{n,p,q}/\langle\rho\rangle$ obtained from Π_n by the induced side pairing of its boundary faces is pictured in Fig. 6(b). The axis of the rotation ρ is represented by a dotted curve in the figure. It lies below the diagram, inside the 3-ball whose boundary is being identified along the two disc pairs (X^+, X^-) and (Y^+, Y^-). To determine the singular set of the branched covering, we apply a method described in [16] for the figure-eight knot (and successively extended in [11] for link complements). Thus we can modify Fig. 6(b) to Fig. 6(d) as follows. Figure 6(c) is obtained from Fig. 6(b) by a simplification along the closed

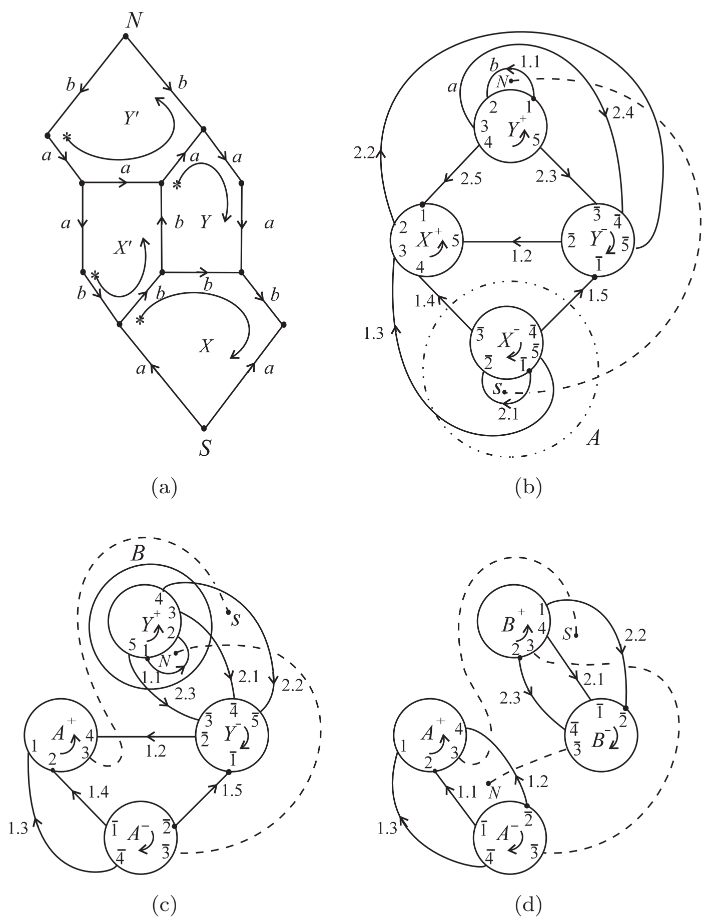

Fig. 6.

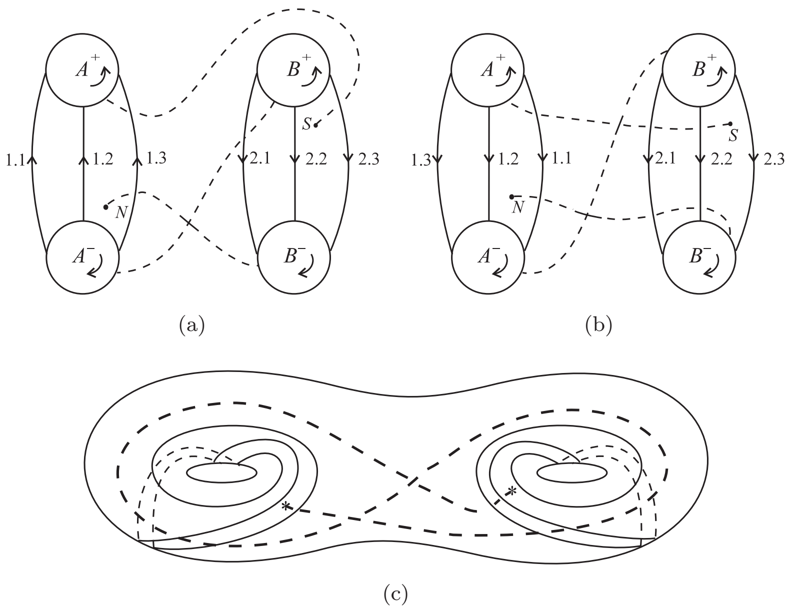

Fig. 7.

curve A which surrounds the "hole" X^-. Figure 6(d) is obtained from Fig. 6(c) by a simplification along the closed curve B which surrounds the "hole" Y^+.

The Heegaard diagram of Fig. 6(d) can be drawn as in Figs. 7(a) and 7(b). The Heegaard diagram in Fig. 7(b) shows clearly that the underlying space of the orbifold $M_{n,p,q}/\langle\rho\rangle$ is topologically homeomorphic to the connected sum $L(3,1)\#L(3,2)$. The singular set of the branched covering is the $(2,1)$-knot K represented in Fig. 7(b) by the dotted line whose endpoints N and S must be identified. Of course, the knot K is independent of n. Figure 7(c) shows a genus 2 handlebody of the Heegaard splitting of $L(3,1)\#L(3,2)$. One can see that the knot K intersects each handlebody of the splitting in a trivial arc. This completes the proof of the theorem. □

We want to show that the orbifold $\mathcal{O}_n(K) = M_{n,p,q}/\langle\rho\rangle$, whose underlying topological space is $L(3,1)\#L(3,2)$ and whose singular set of order n is the $(2,1)$-knot K, is hyperbolic for $n \geq 3$. We shall start by showing that the orbifold $\mathcal{O}_3(K)$ is hyperbolic. Let us consider the regular dodecahedron with dihedral angles of $2\pi/5$ used to construct the *Seifert–Weber dodecahedral space*, which is a classical example of hyperbolic closed 3-manifold (see, for instance, [24]). It can be seen as a metric version of the polyhedron P_3. Taking the quotient of P_3 (which is a compact hyperbolic 3-manifold with totally geodesic boundary) by the hyperbolic isometry which consists in a rotational symmetry of order 3 about the axis NS, we obtain a hyperbolic orbifold consisting in the polyhedron P_1 with singular axis NS of order 3, that is a 1/3–slice Π_3. Note that the boundary of P_1 (flattened out) is shown in Fig. 6(a), where the gluing needed to obtain $\mathcal{O}_n(K)$ is also described. To reach the conclusion it suffices to observe that the gluing can be performed via isometries and that the dihedral angles corresponding to identified edges add up to 2π. The first part follows from the fact that all the faces of the regular dodecahedron are pairwise isometric regular pentagons. The second part follows from the fact that the edges of P_1 are identified in two groups of five (compare Sec. 2 and Figs. 6(a) and 6(b)) and each dihedral angle is $2\pi/5$. Thurston's orbifold geometrization theorem [31] (see also [3, 8] for a proof) can now be applied to deduce that $\mathcal{O}_n(K)$ is hyperbolic for all $n \geq 3$. In particular, the manifolds $M_{n,p,q} = M_2(n)$ are hyperbolic for all $n > 3$, $n \not\equiv 0 \pmod 3$. Let us now prove that the manifolds $M_2(n)$, $n > 3$, $n \not\equiv 0$ (mod 3), are pairwise non homeomorphic. Assume by contradiction that the hyperbolic manifolds $M_2(n)$ and $M_2(n')$, where $n > n' > 3$, are homeomorphic. Mostow's rigidity theorem implies that $M_2(n)$ and $M_2(n')$ have the same volume v. The volumes of the quotient orbifolds $\mathcal{O}_n(K)$ and $\mathcal{O}_{n'}(K)$ are v/n and v/n', respectively, with $v/n < v/n'$. According to the Schäfli formula (see [17]; compare also with [15, 23]), the volume of the hyperbolic orbifold $\mathcal{O}_n(K)$ decreases with the cone angle of the singularity. This means that it increases with n, that is, $vol(\mathcal{O}_n(K)) \geq vol(\mathcal{O}_{n'}(K))$, which is a contradiction. Summarizing, we have the following result.

Theorem 4.2. *The Kim and Kostrikin manifolds $M_2(n)$, $n \equiv 1, 2 \pmod 3$, (and whence $M_4(m) = M_2(3m+1)$) are hyperbolic for every $n > 3$. In this case, the fundamental group of $M_2(n)$ is isomorphic to a discontinuous subgroup of the isometry group of the hyperbolic 3-space, hence it is infinite and torsion-free. Moreover, two manifolds $M_2(n)$ and $M_2(n')$ are homeomorphic if and only if $n = n'$.*

To make more clear the last sentence of Theorem 4.2, we add some final explanations to show that the manifolds $M_2(n)$ are pairwise non-homeomorphic also for the cases $n = 1, 2$. We note that the manifold $M_2(1) \cong L(3,1)\#L(3,2)$ is not prime so it cannot be homeomorphic to $M_2(n)$ which is prime for $n \geq 2$ (since it is a cyclic covering branched over a hyperbolic knot). Similarly, the manifolds $M_2(2)$ cannot be homeomorphic to $M_2(n)$, $n \geq 3$, for it is either hyperbolic and the same reasoning as above applies, or — because of Thurston's orbifold geometrization theorem — geometrizable but non hyperbolic.

5. Further Families of Manifolds Related to the Polyhedron $P_{n,p,q}$

We consider again the polyhedron $P_{n,p,q}$ in Fig. 3 and analyze all possible combinatorial identifications x_i and y_i between the pairs of faces (X_i, X_i') and (Y_i, Y_i'), respectively. For the identification x_i we have the following cases:

$$(0) \quad \begin{cases} a_i b_i c_i d_i a_{i+1} & \to s_{i+p} d_{i+p} b_{i+p+1} m_{i+p+1} g_{i+p} \\ S A_i B_i D_i A_{i+1} & \to F_{i+p} D_{i+p} A_{i+p+1} B_{i+p+1} G_{i+p} \end{cases}$$

$$(1) \quad \begin{cases} a_i b_i c_i d_i a_{i+1} & \to d_{i+p} b_{i+p+1} m_{i+p+1} g_{i+p} s_{i+p} \\ S A_i B_i D_i A_{i+1} & \to D_{i+p} A_{i+p+1} B_{i+p+1} G_{i+p} F_{i+p} \end{cases}$$

$$(2) \quad \begin{cases} a_i b_i c_i d_i a_{i+1} & \to b_{i+p+1} m_{i+p+1} g_{i+p} s_{i+p} d_{i+p} \\ S A_i B_i D_i A_{i+1} & \to A_{i+p+1} B_{i+p+1} G_{i+p} F_{i+p} D_{i+p} \end{cases}$$

$$(3) \quad \begin{cases} a_i b_i c_i d_i a_{i+1} & \to m_{i+p+1} g_{i+p} s_{i+p} d_{i+p} b_{i+p+1} \\ S A_i B_i D_i A_{i+1} & \to B_{i+p+1} G_{i+p} F_{i+p} D_{i+p} A_{i+p+1} \end{cases}$$

$$(4) \quad \begin{cases} a_i b_i c_i d_i a_{i+1} & \to g_{i+p} s_{i+p} d_{i+p} b_{i+p+1} m_{i+p+1} \\ S A_i B_i D_i A_{i+1} & \to G_{i+p} F_{i+p} D_{i+p} A_{i+p+1} B_{i+p+1}. \end{cases}$$

For the identification y_i^{-1} we have the following cases:

$$(0) \quad \begin{cases} e_i f_i g_i h_{i+1} e_{i+1} \to m_{i+q} h_{i+q} f_{i+q} s_{i+q} c_{i+q} \\ N E_i F_i G_i E_{i+1} \to B_{i+q} G_{i+q-1} E_{i+q} F_{i+q} D_{i+q} \end{cases}$$

$$(1) \quad \begin{cases} e_i f_i g_i h_{i+1} e_{i+1} \to h_{i+q} f_{i+q} s_{i+q} c_{i+q} m_{i+q} \\ N E_i F_i G_i E_{i+1} \to G_{i+q-1} E_{i+q} F_{i+q} D_{i+q} B_{i+q} \end{cases}$$

$$(2) \quad \begin{cases} e_i f_i g_i h_{i+1} e_{i+1} \to f_{i+q} s_{i+q} c_{i+q} m_{i+q} h_{i+q} \\ N E_i F_i G_i E_{i+1} \to E_{i+q} F_{i+q} D_{i+q} B_{i+q} G_{i+q-1} \end{cases}$$

$$(3) \quad \begin{cases} e_i f_i g_i h_{i+1} e_{i+1} \to s_{i+q} c_{i+q} m_{i+q} h_{i+q} f_{i+q} \\ N E_i F_i G_i E_{i+1} \to F_{i+q} D_{i+q} B_{i+q} G_{i+q-1} E_{i+q} \end{cases}$$

$$(4) \quad \begin{cases} e_i f_i g_i h_{i+1} e_{i+1} \to c_{i+q} m_{i+q} h_{i+q} f_{i+q} s_{i+q} \\ N E_i F_i G_i E_{i+1} \to D_{i+q} B_{i+q} G_{i+q-1} E_{i+q} F_{i+q}. \end{cases}$$

We denote by $(k\ell)$ the gluing determined by case (k) (respectively, (ℓ)) for the identification x_i (respectively, y_i^{-1}), $k, \ell = 0, 1, 2, 3, 4$. Note that the symmetries of the polyhedron $P_{n,p,q}$ imply that some of these gluings are indeed equivalent. Any orientation preserving symmetry, which exchanges the north and the south poles (N and S) and the edges of type a (respectively, b, c, and d) with those of type e (respectively, h, g, and f) while fixing those of type m and s, conjugates the gluing $(k\ell)$ to the gluing $(4-\ell 4-k)$. In a similar way, any orientation reversing symmetry which exchanges N and S and the edges of type a (respectively, b, c, d, and m) with those of type e (respectively, f, g, h, and s) conjugates the gluing $(k\ell)$ to the gluing (ℓk). Thus we are left to consider nine cases. We discuss completely the cases where the identification space is a closed connected orientable 3-manifold and the corresponding quotient 3-orbifold is topologically homeomorphic to $\mathbb{S}^3$. In these cases, we give nice representations for the knots or links which arise as singular sets of the corresponding branched coverings.

Case (00) (and (44)) corresponds to the Kim and Kostrikin manifolds $M_2(n)$ discussed in the previous sections.

Case (01) (and (10), (34), (43)). The identification space $M_{n,p,q}$ is a cell complex with two vertices, $(n, q-1) + (n, 3p+q+2) + (n, q)$ edges, $2n$ faces and one 3-cell. Hence it is a closed connected orientable 3-manifold if and only if $q = n$

and $n = 3p + 2$ or $q = 1$ and $n = 3p + 3$. For the "only if" part, note that if $1 < (n, q) < n$, then $1 < (n, q-1) < n$ (and similarly for $(n, q-1)$) and that in this case $(n, q) + (n, q-1) < n$. In these cases, the fundamental group of the underlying space of the quotient orbifold $\mathcal{O} = M_{n,p,q}/\langle\rho\rangle$ admits the following presentation

$$\pi_1(|\mathcal{O}|) = \big\langle x, y : xxyy^{-1}y^{-1}x = 1,\ y^{-1} = 1,\ xyx^{-1} = 1\big\rangle \cong \mathbb{Z}_3.$$

If $q = n$ and $n = 3p + 2$, then the fundamental group of the manifold $M_{n,p,q}$ has the cyclic presentation

$$\pi_1(M_{n,p,q}) \cong \langle x_1, \ldots, x_n : x_i x_{i+p} x_{i+2p+1} = 1 \quad (i = 1, \ldots, n)\rangle.$$

The asphericity problem for such presentations and some generalizations of them was studied in [7]. If $q = 1$ and $n = 3p + 3$, then we have

$$\begin{aligned}\pi_1(M_{n,p,q}) \cong \langle x_1, \ldots, x_n, y_1, \ldots, y_n : {} & y_1^{-1}y_2^{-1}\cdots y_n^{-1} = 1, \quad x_i y_{i+p-1} x_{i-1}^{-1} = 1,\\ & x_i x_{i+p} y_{i+2p} y_{i+2p+1}^{-2} x_{i+2p+2} = 1\\ & (i = 1, \ldots, n)\rangle.\end{aligned}$$

The second relation gives $y_{i+p-1} = x_i^{-1}x_{i-1}$. Substituting these formulae in the other relations yields the presentation

$$\begin{aligned}\pi_1(M_{n,p,q}) \cong \Big\langle x_1, \ldots, x_n : x_i x_{i+p} x_{i+p+1}^{-1} x_{i+p} \big(x_{i+p+1}^{-1} x_{i+p+2}\big)^2 x_{i+2p+2} = 1 \\ (i = 1, \ldots, n)\Big\rangle.\end{aligned}$$

In both cases, these manifolds are n-fold cyclic coverings of a lens space $L(3, \alpha)$, $\alpha = 1, 2$, branched over a $(1, 1)$-knot.

Case (02) (and (20), (24), (42)). The identification space $M_{n,p,q}$ is a cell complex with one vertex, $(n, 3p + q + 3)$ edges, $2n$ faces and one 3-cell, hence for every n it cannot be a closed 3-manifold.

Case (03) (and (14), (30), (41)). The identification space $M_{n,p,q}$ is a cell complex with one vertex, $(n, 3p + 2) + (n, q - 1)$ edges, $2n$ faces and one 3-cell. Hence it is a closed connected orientable 3-manifold if and only if $q = 1$ and $n = 3p + 2$. The fundamental group of the underlying space of the quotient orbifold $\mathcal{O} = M_{n,p,q}/\langle\rho\rangle$ admits the following presentation

$$\pi_1(|\mathcal{O}|) = \big\langle x, y : xyy^{-1}y^{-1}xxxyx^{-1} = 1,\ y^{-1} = 1\big\rangle \cong \mathbb{Z}_3.$$

If $q = 1$ and $n = 3p + 2$, then the fundamental group of the manifold $M_{n,p,q}$ has the cyclic presentation

$$\pi_1(M_{n,p,q}) \cong \big\langle x_1, \ldots, x_n : x_i x_{i+p} x_{i+2p+1} = 1 \quad (i = 1, \ldots, n)\big\rangle.$$

These manifolds are cyclic coverings of $L(3, 2)$ branched over a $(1, 1)$-knot.

Case (04) (and (40)). The identification space $M_{n,p,q}$ is a cell complex with one vertex, $(n, 3p - 3q + 5)$ edges, $2n$ faces and one 3-cell, hence for every n it cannot be a closed 3-manifold.

Case (11) (and (33)). The identification space $M_{n,p,q}$ is a cellular complex with $1 + 2(n, p + 1, q)$ vertices, $2(n, p + q + 1) + (n, p + 1) + (n, q)$ edges, $2n$ faces and one

3-cell. Note that if $n = p+q+1$, then $(n, p+1) = (n, q) = (n, p+1, q)$ and $M_{n,p,q}$ is a closed 3-manifold. On the other hand, assume that $M_{n,p,q}$ is a closed 3-manifold and consider the orbifold obtained by quotienting $M_{n,p,q}$ via the usual n-rotational symmetry ρ about the axis NS. The axis of the symmetry maps to a closed loop (for N and S are identified) with order of singularity n. A standard computation shows that the edges of type a and e map to singular edges in the quotient orbifold if and only if $(n, p+q+1) < n$. Since the edges a_i and e_i emanate from S and N, respectively, the orbifold would contain a non admissible singularity if the images of the edges of type a and e were in the singular set. So we obtain that the condition $p+q+1 = n$ is also necessary for $M_{n,p,q}$ to be a closed 3-manifold.

If $p + q + 1 = n$, then the fundamental group of the manifold $M_{n,p,q}$ has the geometric presentation

$$\begin{aligned}\pi_1(M_{n,p,q}) = \big\langle x_i, y_i, i = 1, \ldots, n : {} & x_i x_{i+p} y_{i+2p}^{-1} x_{i+p+q}^{-1} = 1,\\ & x_i y_{i+p-q+1} y_{i+p-q+2}^{-1} y_{i+p+1}^{-1} = 1,\\ & x_i x_{i+p+1} \cdots x_{i-p-1} = 1,\\ & y_i^{-1} y_{i+q}^{-1} \cdots y_{i-q}^{-1} = 1 \quad (i = 1, \ldots, n)\big\rangle.\end{aligned}$$

In this case, the fundamental group of the underlying space of the quotient orbifold $\mathcal{O} = M_{n,p,q}/\langle\rho\rangle$ admits the following presentation

$$\pi_1(|\mathcal{O}|) = \langle x, y : xxy^{-1}x^{-1} = 1,\ xyy^{-1}y^{-1} = 1,\ x = 1,\ y^{-1} = 1\rangle \cong 1,$$

and in fact $|\mathcal{O}|$ is the 3-sphere. The sequence of pictures in Fig. 8 shows that the manifolds are n-fold cyclic covers of the 3-sphere branched along a Montesinos link with two trivial components, in fact, a 2-bridge link. A Heegaard diagram for the quotient space is drawn in Fig. 8(a). Figure 8(b) is obtained from Fig. 8(a) by the cancellation of the handle 2.1 between the holes X^+ and X^- and the cancellation of the handle 4.1 between the holes Y^+ and Y^-. It represents the singular set of the n–fold cyclic covering $M_{n,p,q} \to \mathbb{S}^3$. The branching indices for the two components are n (this component is the image of the axis NS) and $k = n/(n, p+1) = n/(n, q)$ (this component is the image of the edges of type f and b). This link, shown in Fig. 8(c), is easily seen to be hyperbolic and even $(2\pi/3, \pi)$-hyperbolic. Indeed, one can obtain this link as the quotient of the mirror image of the hyperbolic 2-bridge knot 5_2 (which is $2\pi/3$-hyperbolic) via its 2-periodic symmetry, as illustrated in Fig. 8(d) (here we use Rolfsen's notation [26, p.391]). In particular, the manifolds $M_{n,p,q}$ are hyperbolic for all choices of $n > 2$.

Case (12) (and (21), (23), (32)). The identification space $M_{n,p,q}$ is a cell complex with one vertex, $(n, 2p+q+4) + (n, p+1)$ edges, $2n$ faces and one 3-cell. Hence it is a closed connected orientable 3-manifold if and only if $p = n-1$ and $q = n-2$. In this case, the fundamental group of the underlying space of the quotient orbifold $\mathcal{O} = M_{n,p,q}/\langle\rho\rangle$ admits the following presentation

$$\pi_1(|\mathcal{O}|) = \big\langle x, y : xxy^{-1}xyyy^{-1}y^{-1}x^{-1} = 1,\ x = 1\big\rangle \cong 1,$$

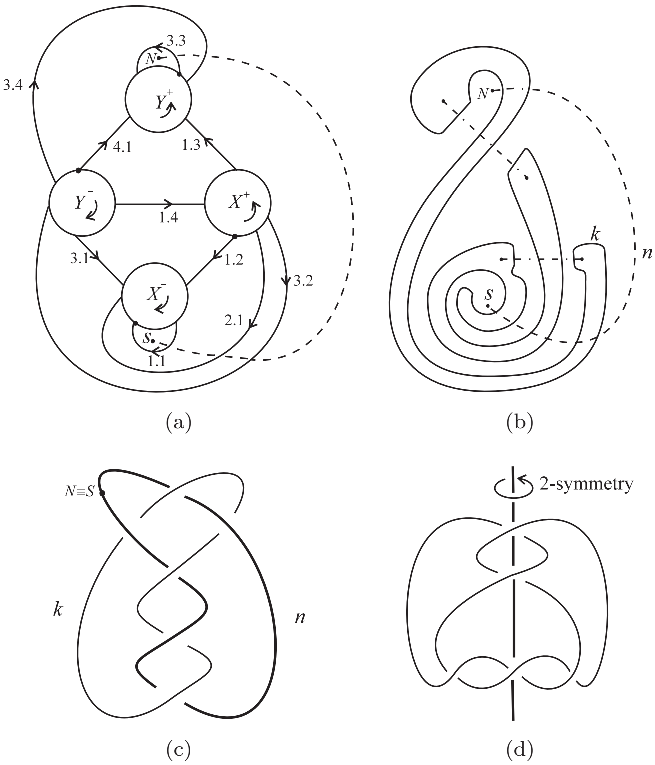

Fig. 8.

hence $|\mathcal{O}| \cong \mathbb{S}^3$ (recall that the Poincaré Conjecture is true for closed orientable 3-manifolds of Heegaard genus ≤ 2). If $p = n-1$ and $q = n-2$, then the fundamental group of the manifold $M_{n,p,q}$ has the cyclic presentation

$$\begin{aligned}
&\pi_1(M_{n,p,q}) \\
&\quad \cong \langle y_1, \ldots, y_n : y_{i+2p}^{-1} y_{i+3p+1} y_{i+3p-q+2} y_{i+3p-q+3}^{-1} y_{i+3p+3}^{-1} = 1 \quad (i = 1, \ldots, n)\rangle \\
&\quad \cong \langle y_1, \ldots, y_n : y_{i-2}^{-1} y_{i-2} y_{i+1} y_{i+2}^{-1} y_i^{-1} = 1 \quad (i = 1, \ldots, n)\rangle \\
&\quad \cong \langle y_1, \ldots, y_n : y_i y_{i+2} = y_{i+1} \quad (i = 1, \ldots, n)\rangle.
\end{aligned}$$

These presentations were first introduced in [28]; a geometric study of them can be found in [4] (we refer to [6] for more information on the topological properties of cyclically presented groups). The sequence of pictures in Fig. 9 shows that the manifolds $M_{n,p,q}$ are the n-fold cyclic coverings of the 3-sphere $\mathbb{S}^3$ branched over the trefoil knot, hence they are Seifert fibered manifolds.

A Heegaard diagram for the quotient space is drawn in Fig. 9(a). Figure 9(b) is obtained from Fig. 9(a) by the cancellation of the handle 2.1 between the holes X^+ and X^-. Figure 9(c) (respectively, 9(d)) is obtained from Fig. 9(b) (respectively, 9(c)) by a simplification along the closed curve A (respectively, B) which surrounds

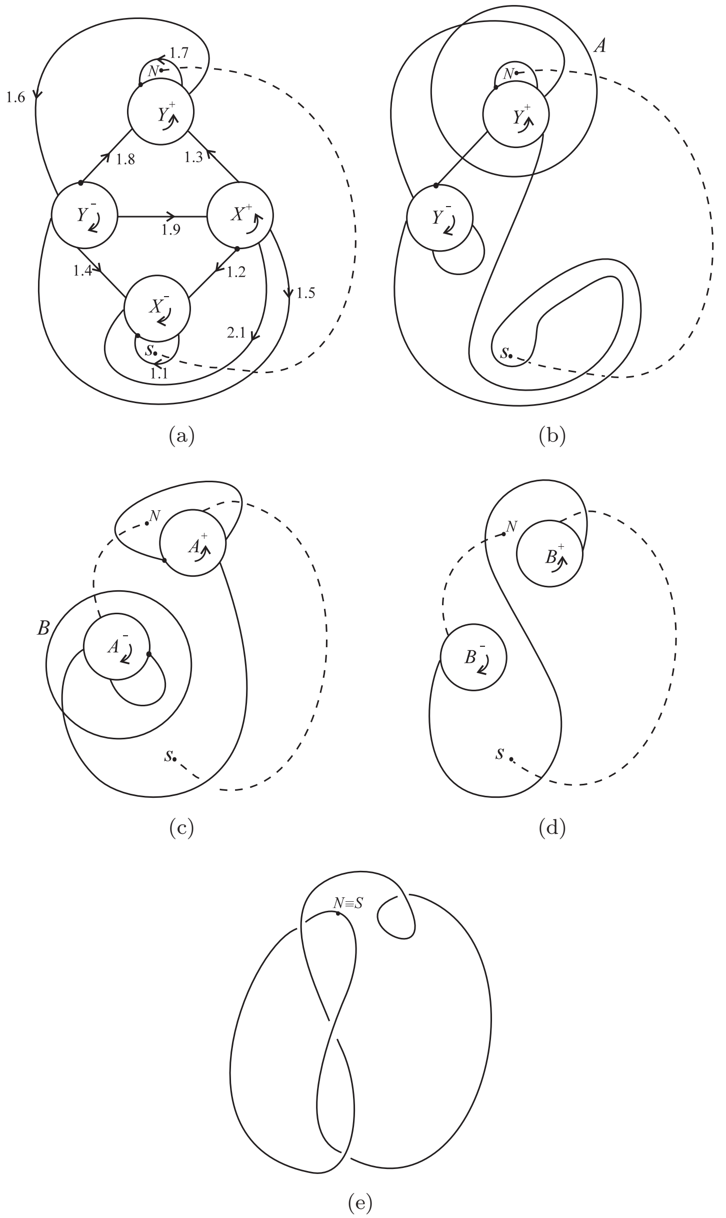

Fig. 9.

the hole Y^+ (respectively, A^-). Figure 9(e) is obtained from Fig. 9(d) by the cancellation of the remaining handle between B^+ and B^-. It represents the singular set of the n-fold cyclic covering $M_{n,p,q} \to \mathbb{S}^3$. Of course, this knot is equivalent to the trefoil knot.

Case (13) (and (31)). The identification space $M_{n,p,q}$ is a cell complex with two vertices, $1 + (n, p+1) + (n, q-1)$ edges, $2n$ faces and one 3-cell. Hence it is a closed connected orientable 3-manifold if and only if $p = n-1$ and $q = 1$. We can immediately see that in this case $M_{n,p,q} \cong \mathbb{S}^3$ for each n.

Case (22). The identification space $M_{n,p,q}$ is a cell complex with one vertex, $(n, 5) + (n, p+q)$ edges, $2n$ faces and one 3-cell. Hence it is a closed connected orientable 3-manifold if and only if $n = 5$ and $p+q = 5$. In this case, the fundamental group of the underlying space of the quotient orbifold $\mathcal{O} = M_{n,p,q}/\langle\rho\rangle$ admits the following presentation

$$\pi_1(|\mathcal{O}|) = \langle x, y : xxyyy^{-1}y^{-1}x^{-1}x^{-1} = 1,\ xy^{-1} = 1\rangle \cong \mathbb{Z},$$

and one can prove that $|\mathcal{O}| \cong \mathbb{S}^1 \times \mathbb{S}^2$. These manifolds are 5-fold coverings of $\mathbb{S}^1 \times \mathbb{S}^2$ branched over a $(1,1)$-knot. If $n = 5$ and $p+q = 5$, then the fundamental group of the manifold $M_{n,p,q}$ has the presentation

$$\begin{aligned}\pi_1(M_{n,p,q}) \cong \langle y_1, \ldots, y_5 : {} & y_{i+p}y_{i+2p+1}y_{i+3p+2}y_{i+4p+3} \\ & = y_{i+p+4}y_{i+2p+4}y_{i+3p+4}y_{i+4p+4} \quad (i = 1, \ldots, 5)\rangle.\end{aligned}$$

If $p = 0$ (and hence $q = 5$), then we get the presentation

$$\pi_1(M_{n,p,q}) \cong \left\langle y_1, \ldots, y_5 : y_i y_{i+1} y_{i+2} y_{i+3} = y_{i+4}^4 \quad (i = 1, \ldots, 5)\right\rangle$$

which defines the *generalized Neuwirth group* Γ_5^4. These groups and some generalizations of them were studied in [25, 30] (compare also with [29]).

Summarizing we have proved the following result

Theorem 5.1. *Considering all possible face-pairings* $x_i : X_i \to X_i'$ *with shift* p *and* $y_i^{-1} : Y_i' \to Y_i$ *with shift* q *on the boundary of the polyhedron* $P_{n,p,q}$, $n \geq 1$, $0 \leq p \leq n-1$, $1 \leq q \leq n$, *yields some infinite series of closed connected orientable* 3*-manifolds whenever the parameters satisfy certain arithmetic conditions listed above. These classes of manifolds contain cyclic coverings of* $L(3, \alpha)$, $\alpha = 1, 2$, *and* $L(3,1)\#L(3,2)$ *branched along* $(1,1)$*-knots and* $(2,1)$*-knots, respectively. The singular sets of the branched coverings are the images of the north-south axis of* $P_{n,p,q}$ *under the rotational actions. The above-constructed presentations of the fundamental groups are geometric, that is, they arise from Heegaard diagrams* (*or, equivalently, spines*) *of the considered manifolds. These constructions provide at least two infinite classes of hyperbolic closed orientable* 3*-manifolds.*

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