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On cyclic branched coverings of prime knots

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Abstract

We prove that a prime knot K is not determined by its p-fold cyclic branched cover for at most two odd primes p. Moreover, we show that for a given odd prime p, the p-fold cyclic branched cover of a prime knot K is the p-fold cyclic branched cover of at most one more knot K' non-equivalent to K. To prove the main theorem, a result concerning symmetries of knots is also obtained. This latter result can be interpreted as a characterisation of the trivial knot.

1. *Introduction*

Two knots K and K' are *equivalent* if there is a homeomorphism of $\mathbb{S}^3$ sending K to K'. Given a knot $K \subset \mathbb{S}^3$ and an integer $p \geqslant 2$, one can construct the (total space of the) p-fold cyclic cover $M_p(K)$ of $\mathbb{S}^3$ branched along K: it is a fundamental object in knot theory.

Let K be an oriented knot and denote by K^* the same knot with the reversed orientation. If there is no homeomorphism of $\mathbb{S}^3$ sending K to K^* and preserving their orientations, that is, if K is *non-invertible*, then the composite knots $K\sharp K^*$ and $K\sharp K$ are non-equivalent, but all their cyclic branched covers are homeomorphic. Non-invertible knots exist, according to [**32**], and thus there are composite knots all of whose cyclic branched covers are homeomorphic. This is no longer true for prime knots: Kojima [**13**] proved that for each prime knot $K \subset \mathbb{S}^3$, there is an integer $n_K \geqslant 2$ such that two prime knots K and K' are equivalent if their p-fold cyclic branched covers are homeomorphic for some $p > \max(n_K, n_{K'})$.

There are many examples, due to Giller [**9**], Livingston [**17**], Nakanishi [**23**] and Sakuma [**27**], of prime knots in $\mathbb{S}^3$ which are not equivalent but share homeomorphic p-fold cyclic branched covers and which moreover show that there is no universal bound for n_K. For a survey of the subject, see, for instance, [**24**].

The main goal of this article is to study the relationship between prime knots and their cyclic branched covers when the number of sheets is an odd prime number.

Definition 1. Let $K \subset \mathbb{S}^3$ be a prime knot. A knot $K' \subset \mathbb{S}^3$ which is not equivalent to K and which has the same p-fold cyclic branched cover as K is called a *p-twin* of K.

The classification of Montesinos knots (see [**5**, Chapter 12]) shows that there are examples of prime knots, even hyperbolic knots, with an arbitrarily large number of non-equivalent 2-twins; see also [**12**, Example 1.3] for another construction. In contrast, for an odd prime number p, the number of p-twins is very restricted, according to our main result.

Theorem 1. *Let $K \subset \mathbb{S}^3$ be a prime knot. Then:*
(i) *there are at most two odd prime numbers p for which K admits a p-twin;*
(ii) *for a given odd prime number p, K admits at most one p-twin.*

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For hyperbolic knots, Theorem 1 is in fact a consequence of Zimmermann's result in [**34**] whose proof uses the orbifold theorem and the Sylow theory for finite groups. The result in Theorem 1 is sharp: for any pair of coprime integers $p > q > 2$, Zimmermann has constructed examples of prime hyperbolic knots with the same p-fold and q-fold branched coverings [**33**].

The second author has proved that a hyperbolic knot is determined by three cyclic branched covers of pairwise distinct orders [**25**]. The following straightforward corollary of Theorem 1 shows that a stronger conclusion holds for arbitrary prime knots when we focus on branched coverings with odd prime orders.

Corollary 1. *A prime knot is determined by three cyclic branched covers of pairwise distinct odd prime orders. More specifically, for every knot K there is at least one integer $p_K \in \{3, 5, 7\}$, such that K is determined by its p_K-cyclic branched cover.*

Another straightforward consequence of Theorem 1 is the following corollary.

Corollary 2. *Let $K = K_1\sharp \dots \sharp K_t$ and $K' = K'_1\sharp \dots \sharp K'_t$ be two composite knots with the same cyclic branched covers of orders p_j, $j = 1, 2, 3$, for three fixed, pairwise distinct, odd prime numbers. Then, after a reordering, the (non-oriented) knots K_i and K'_i are equivalent for all $i = 1, \dots, t$.*

Part (ii) of Theorem 1 states that for a given odd prime number p, a closed, orientable 3-manifold can be the p-fold cyclic branched cover of at most two non-equivalent prime knots in $\mathbb{S}^3$. In [**2**] it has been shown that an integer homology sphere that is an n-fold cyclic branched cover of $\mathbb{S}^3$ for four distinct odd prime numbers n is in fact $\mathbb{S}^3$. By putting together these two results, we get the following.

Corollary 3. *Let M be an irreducible integer homology 3-sphere. Then there are at most three distinct knots in $\mathbb{S}^3$ having M as cyclic branched cover of odd prime order.*

Our main task will be to prove Theorem 1 for a satellite knot: that is, a knot whose exterior $\mathbb{S}^3\setminus\mathcal{U}(K)$ has a non-trivial Jaco–Shalen–Johannson decomposition [**15**, **16**] (in what follows, we use 'JSJ-decomposition' for short). Otherwise the knot is called 'simple': in this case, due to Thurston's hyperbolization theorem [**31**], its exterior is either hyperbolic, and the proof follows from the work in [**25**] and [**34**], or it is a torus knot and a simple combinatorial argument applies.

For a hyperbolic knot, a key ingredient for the proof of Theorem 1 is that the existence of a p-twin for an odd prime number p implies that the knot has a *rotational symmetry of order p with trivial quotient*. A *rotational symmetry of order p* of a knot $K \subset \mathbb{S}^3$ is an orientation-preserving periodic diffeomorphism ψ of the pair $(\mathbb{S}^3, K)$ with period p and with non-empty fixed point set disjoint from K. We say that the rotational symmetry ψ has *trivial quotient* if K/ψ is the trivial knot in $\mathbb{S}^3$. This special symmetry for a hyperbolic knot admitting a p-twin is induced by the covering transformation associated to the p-twin and acting on the shared p-fold cyclic branched cover; see [**34**]. This fact is no longer true in general for a prime satellite knot K. In this case, the covering transformation associated to a p-twin induces a symmetry with non-empty fixed-point set only on some submanifold of the knot exterior $E(K)$, which we call *partial symmetry* of the knot K. The proof of Theorem 1 relies on the study of these partial symmetries induced by the covering transformations associated to the twins of K and on the localisation of their axes of fixed points in the components of the JSJ-decomposition of the exterior $E(K)$.

In particular, the proof uses the following result concerning rotational symmetries of prime knots which is of interest in its own right.

THEOREM 2. *Let K be a knot in $\mathbb{S}^3$ admitting three rotational symmetries with trivial quotients and with orders that are three pairwise distinct integers greater than* 2. *Then K is the trivial knot.*

Since the trivial knot admits a rotational symmetry with trivial quotient of order p for each integer $p \geqslant 2$, Theorem 2 can be interpreted as a characterisation of the trivial knot; that is, a knot is trivial if and only if it admits three rotational symmetries of pairwise distinct orders greater than 2 and with trivial quotients.

If a prime knot $K \subset \mathbb{S}^3$ has a p-twin and a q-twin for two distinct odd prime numbers p and q, it is natural to ask whether these two twins may be equivalent. The following result shows that in this case the knot K inherits two true symmetries, and it behaves just like a hyperbolic knot.

PROPOSITION 1. *Let $K \subset \mathbb{S}^3$ be a prime knot. Suppose that K admits the same knot K' as a p-twin and a q-twin for two distinct odd prime numbers p and q. Then K has two commuting rotational symmetries of order p and q with trivial quotients.*

The paper is organised as follows.

In Section 2, we study rotational symmetries of a prime knot K and prove Theorem 2. In Section 2, we also prove a result about the localisation of the axes of fixed points of two rotational symmetries with trivial quotient and of pairwise distinct odd orders in the exterior $E(K)$ (see Theorem 3): the two axes must sit in the JSJ-component of $E(K)$ containing $\partial E(K)$. This is one of the key ingredients for the proof of Theorem 1.

The proof of Theorem 1 is given in Section 3. The existence of a partial symmetry of order p for K, associated to a p-twin K' of K, is proved in Proposition 4: this is done by studying the actions of the respective deck transformations h and h', associated to K and K', on their common p-fold cyclic branched cover M. The idea is to show that, up to conjugacy, h and h' commute on a non-empty connected submanifold of M that contains the fixed-point sets of h and h' (see Proposition 3), and then to consider the quotient of this submanifold by h. In Proposition 5, we show that there is at most one odd prime number p such that a p-twin of K induces a partial symmetry of K that does not extend to a symmetry of K. Moreover, if such a p-twin exists, then any q-twin, for an odd prime number $q \neq p$, induces a rotational symmetry of K whose axis of fixed points in $E(K)$ cannot sit in the JSJ-component of $\partial E(K)$. These results, together with Theorem 3, show that there are at most two distinct odd prime numbers p and q for which K admits a p-twin and a q-twin. The uniqueness of the p-twin for a given odd prime number p follows from Lemma 10: this lemma states that if the actions of the deck transformation groups associated to the p-fold cyclic branched covers of two prime knots preserve a JSJ-piece or a JSJ-torus of the p-fold cyclic branched cover and coincide on it, then they are conjugate.

In Section 4, we show that all the possible types of symmetries (global and/or partial) induced by twins do occur for a prime satellite knot. In particular, for each given odd prime p, we construct examples of p-twins that induce symmetries that are not global. We also give examples of prime satellite knots with two distinct twins, one of which induces only a partial symmetry.

In Section 5, we prove Corollary 3, which improves the main result of [**2**] by showing that an irreducible integer homology sphere is homeomorphic to $\mathbb{S}^3$ if and only if it admits four pairwise non-conjugate rotational symmetries with trivial quotients and with orders that are odd prime integers.

We end the introduction by remarking that the hypothesis on the prime integer p being odd is crucial to the proof of Theorem 1. This hypothesis implies that the axes of fixed points of the deck transformations associated to K and its p-twin do not meet any torus of the JSJ-splitting of their common p-fold cyclic branched cover M, and also that they cannot meet each other. This last fact is a key point in proving that the restrictions of the two deck transformations commute, up to conjugacy, on any invariant hyperbolic piece of the JSJ-decomposition of M; see case (b) of the proof of statement (iii) in Proposition 3, or the remark at the end of [**34**, Section 2].

It is worth noticing that, completing previous works by Reni [**26**] and Mecchia and Zimmermann [**19**], Kawauchi [**12**] has shown that there exist nine mutually inequivalent knots in $\mathbb{S}^3$ whose 2-fold branched covers are homeomorphic to the same hyperbolic manifold. According to [**26**], this result is sharp and shows that the case of 2-twins is more subtle, even when the 2-fold branched cover is hyperbolic.

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2. *Rotational symmetries of knots*

A *rotational symmetry* of order p of a knot $K \subset \mathbb{S}^3$ is an orientation-preserving, periodic diffeomorphism ψ of the pair $(\mathbb{S}^3, K)$ of order p and non-empty fixed-point set disjoint from K. We say that the rotational symmetry ψ has *trivial quotient* if K/ψ is the trivial knot in $\mathbb{S}^3$.

REMARK 1. Let K be a knot and let ψ be a rotational symmetry of K of order p. The symmetry ψ lifts to a periodic diffeomorphism $\tilde{\psi}$ of the p-fold branched cover $M_p(K)$ with order p and non-empty fixed-point set, which commutes with the covering transformation h of K acting on $M_p(K)$. Then, as a consequence of the proof of the Smith conjecture, the symmetry ψ has trivial quotient if and only if $(M_p(K), \mathrm{Fix}(\tilde{\psi}))/\langle\tilde{\psi}\rangle \cong (\mathbb{S}^3, K')$. Moreover, in this case, K and K' have a common quotient link with two trivial components (see [**34**]).

In particular, a symmetry of a knot K induced by the covering transformation associated to a p-twin K' of K is a p-rotational symmetry with trivial quotient. This follows from the fact that the two commuting deck transformations associated to the two twins induce on $M_p(K)$ a $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$-cover of $\mathbb{S}^3$ branched over a link with two unknotted components.

The main result of this section is the following theorem; its assertion (i) is Theorem 2.

THEOREM 3. *Let K be a knot in $\mathbb{S}^3$.*
(i) *Assume that K admits three rotational symmetries with trivial quotients and with orders that are three pairwise distinct integers greater than 2. Then K is the trivial knot.*
(ii) *Assume that K admits two rotational symmetries ψ and φ with trivial quotients and of distinct orders greater than 2. Then the fixed-point sets* $\mathrm{Fix}(\psi)$ *and* $\mathrm{Fix}(\varphi)$ *sit in the JSJ-component of $E(K)$, which contains $\partial E(K)$.*

We prove first a weaker version of Theorem 2, which we shall use in the remainder of this section (see also [**25**, Scholium]).

PROPOSITION 2. *Let K be a knot in $\mathbb{S}^3$ admitting three commuting rotational symmetries of orders $p > q > r \geqslant 2$. If the symmetries of order q and r have trivial quotients, then K is the trivial knot.*

Proof. Denote by φ, ψ and ρ the three symmetries. If two of them — say φ, ψ — have the same axis, then, by hypothesis, the one with smaller order — say ψ — must have trivial quotient; that is, K/ψ is the trivial knot. Since the three symmetries commute, φ induces a rotational symmetry of K/ψ that is non-trivial, since the order of φ is larger than that of ψ. The axis $\mathcal{A}$ of this induced symmetry is the image of $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\psi)$ in the quotient by the action of ψ. In particular, K/ψ and $\mathcal{A}$ form a Hopf link and K is the trivial knot: this follows from the equivariant Dehn lemma; see [**11**]. We can thus assume that the axes are pairwise disjoint. Note that even if $r = 2$, since the symmetries commute, the symmetry of order 2 cannot act as a strong inversion on the axes of the other two symmetries. In this case we would see that the axis of ρ, which is a trivial knot, admits two commuting rotational symmetries, φ and ψ, with distinct axes, which is impossible: this follows, for instance, from the fact (see [**8**, Theorem 5.2]) that one can find a fibration of the complement of the trivial knot that is equivariant with respect to the two symmetries. □

The proof of Theorem 3 is based on a series of lemmata.

The first result concerns the structure of the JSJ-decomposition of the p-fold cyclic branched cover M of a prime knot $K \subset \mathbb{S}^3$. Let h be the covering transformation; then the quotient space $M/\langle h\rangle$ has a natural orbifold structure, denoted by $\mathcal{O}_p(K)$, with underlying space $\mathbb{S}^3$ and singular locus K with local group a cyclic group of order p (cf. [**1**, Chapter 2]). According to Bonahon and Siebenmann [**4**] and the orbifold theorem [**3**, **6**], such an orbifold admits a characteristic collection of toric 2-suborbifolds (formed by tori and 2-spheres with four points with branching index 2), which split $\mathcal{O}_p(K)$ into geometric suborbifolds. Moreover, this characteristic collection of toric 2-suborbifolds lifts to the JSJ-collection of tori for M. It follows that for $p > 2$, the Bonahon–Siebenmann characteristic collection of toric 2-suborbifolds contains only tori that do not meet K, and thus coincides with the JSJ-collection of tori for the exterior $E(K) = \mathbb{S}^3 \backslash \mathcal{U}(K)$ of K.

LEMMA 1. *Let $p > 2$ be an integer and let M be the p-fold cyclic branched cover of a prime knot K in the* 3*-sphere. Then:*

(i) *the dual graph associated to the JSJ-decomposition of M is a tree*;

(ii) *the fixed-point set of the group of deck transformations is entirely contained in one geometric piece of the decomposition.*

Proof. (i) By the equivariant sphere theorem [**20**], M is irreducible since K is prime. Hence the Bonahon–Siebenmann decomposition of the orbifold $\mathcal{O}_p(K)$ lifts to the JSJ-collection for M. Moreover, the graph dual to the Bonahon–Siebenmann decomposition of the orbifold $\mathcal{O}_p(K)$, which lifts to the JSJ-decomposition for M, is a tree. Cutting along a torus of the former decomposition and considering the component C which does not contain K, one gets the complement of a knot in $\mathbb{S}^3$. The lemma now follows from the fact that each connected component of a cyclic unbranched cover of C has a unique boundary component.

(ii) Note that the group of deck transformations preserves the JSJ-collection of tori. If $p > 2$, the fixed-point set of this group does not meet any torus of the JSJ-decomposition, because each JSJ-torus is separating and the fixed-point set is connected. Since the fixed-point set is connected, it is entirely contained in one geometric piece of the JSJ-decomposition. □

REMARK 2. Note that the conclusion of the first part of the lemma also holds for covers of order 2. For covers of prime order, this property follows also from the fact that $M_p(K)$ is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere (see [**10**]).

LEMMA 2. *If a knot $K \subset \mathbb{S}^3$ has a rotational symmetry with trivial quotient, then K is prime.*

Proof. Sakuma [**28**, Theorem 4] showed that the only possible rotational symmetries of a composite knot must either permute its prime summands cyclically, or act as a symmetry of one prime summand while permuting the remaining ones. In particular, the quotient knot cannot be trivial. □

The following is a key lemma for the proofs of Theorems 1 and 3.

LEMMA 3. *Let K be a knot admitting a rotational symmetry ψ of order $p > 2$ and consider the JSJ-decomposition of its exterior $E(K) = \mathbb{S}^3 \backslash \mathcal{U}(K)$.*

(i) *T is a torus of the decomposition which does not separate $\partial E(K)$ from* $\mathrm{Fix}(\psi)$ *if and only if the orbit of T under the cyclic group generated by ψ has p elements.*

(ii) *Under the assumption that ψ has trivial quotient, each torus that separates $\partial E(K)$ from* $\mathrm{Fix}(\psi)$ *corresponds to a prime companion of K on which ψ acts with trivial quotient.*

Proof. Let T be a torus of the *JSJ*-decomposition of $E(K)$ considered as a torus inside $\mathbb{S}^3$: T separates the 3-sphere into a solid torus containing K and the exterior of a non-trivial knot K_T, which is a companion of K. Note that, since the order of the symmetry ψ is greater than 2, its axis cannot meet T. Assume that the axis $\mathrm{Fix}(\psi)$ of the symmetry is contained in the solid torus.

If the orbit of T under ψ does not contain p elements, then a non-trivial power of ψ leaves T invariant, and thus it also leaves the solid torus and the knot exterior invariant. The restriction of this power of ψ to the solid torus acts as a rotation of order $m > 1$ around its core and leaves each meridian invariant. This non-trivial power of ψ would then be a rotational symmetry about the non-trivial knot K_T, which is absurd because of the proof of the Smith conjecture (see [**22**]).

For the reverse implication, it suffices to observe that the geometric pieces of the decomposition containing $\partial E(K)$ and $\mathrm{Fix}(\psi)$ must be invariant by ψ, and so must be the unique geodesic segment joining the corresponding vertices in the tree dual to the decomposition.

For the second part of the lemma, note that K_T/ψ is a companion of K/ψ, which is trivial by hypothesis. In particular, K_T/ψ is also trivial and thus, by Lemma 2, K_T must be prime. □

The following lemma gives a weaker version of assertion (ii) of Theorem 3 under a commutativity hypothesis.

LEMMA 4. *Let K be a prime knot admitting two commuting rotational symmetries ψ and φ of orders $p > 2$ and $q > 2$, respectively. Then:*

(i) *the fixed-point sets of ψ and φ are contained in the same geometric component of the JSJ-decomposition for $E(K)$;*

(ii) *if ψ has trivial quotient and $q > p$, the fixed-point sets of ψ and φ sit in the component that contains $\partial E(K)$.*

Proof. Part (i) Let v_ψ (respectively, v_φ) be the vertex of the graph Γ_K, dual to the JSJ-decomposition of $E(K)$, corresponding to the geometric component containing $\mathrm{Fix}(\psi)$ (respectively $\mathrm{Fix}(\varphi)$). Since the two rotational symmetries commute, ψ (respectively φ) must leave $\mathrm{Fix}(\varphi)$ (respectively $\mathrm{Fix}(\psi)$), invariant, and so the geodesic segment of Γ_K joining v_ψ to v_φ must be fixed by the induced actions of ψ and φ on Γ_K. If this segment contains an edge e, the corresponding JSJ-torus T in $E(K)$ cannot separate both $\mathrm{Fix}(\varphi)$ and $\mathrm{Fix}(\psi)$ from $\partial E(K)$. This would contradict Lemma 3(i).

Part (ii) If ψ and φ have the same axis, the argument given at the beginning of the proof of Proposition 2 shows that K is unknotted in $\mathbb{S}^3$, since ψ has trivial quotient and $q > p$. Then the proof follows from the fact that there is only one piece in the JSJ-decomposition for $E(K)$.

So we can assume that the axes of ψ and φ are disjoint. Let M be the p-fold cyclic branched cover of K and let h be the associated covering transformation. According to Remark 1, the lift $\tilde{\psi}$ of ψ to M is the deck transformation of a p-fold cyclic cover of $\mathbb{S}^3$ branched along a knot K'. Note that both $\tilde{\psi}$ and $\tilde{\varphi}$ (the lift of φ to M) normalise the cyclic covering group generated by h and hence commute on M with h, since h has a non-empty connected fixed-point set invariant under $\tilde{\psi}$ and $\tilde{\varphi}$. In particular, $\tilde{\varphi}$ and h induce commuting non-trivial rotational symmetries of K' with orders q and p, respectively. According to part (i), $\mathrm{Fix}(\tilde{\varphi})$ and $\mathrm{Fix}(h)$ belong to the same piece of the JSJ-decomposition of M. Since $\mathrm{Fix}(h)$ maps to K, $\mathrm{Fix}(\varphi)$ sits in the JSJ-piece of $E(K)$ that contains $\partial E(K)$. Then the conclusion follows, since $\mathrm{Fix}(\psi)$ belongs to the same JSJ-piece as $\mathrm{Fix}(\varphi)$. □

Let K be a knot in $\mathbb{S}^3$ and let T be an essential torus embedded in the exterior $E(K)$. Let V be the solid torus bounded by T in $\mathbb{S}^3$; then K sits in the interior of V. *The winding number of T with respect to K* is defined as the absolute value of the linking number of K with the boundary of a meridian disc of V, which is also the absolute value of the algebraic intersection number of K with a meridian disc of V. We need to take the absolute value, since the knot K is not oriented.

LEMMA 5. *Let K be a knot admitting a rotational symmetry ψ with trivial quotient and of order $p > 2$. Let M be the p-fold cyclic branched cover of K and denote by $\pi : M \longrightarrow (\mathbb{S}^3, K)$ the associated branched cover. Let T be a torus in the JSJ-collection of tori of $E(K)$.*

(i) *The torus T is left invariant by ψ if and only if $\pi^{-1}(T)$ is connected.*

(ii) *If $\pi^{-1}(T)$ is connected, then the companion K_T of K corresponding to T is prime and the winding number of T with respect to K is coprime with p, so in particular, it is not zero.*

(iii) *The torus T is not left invariant by ψ if and only if $\pi^{-1}(T)$ has p components.*

Proof. Part (i) According to Remark 1, the p-fold cyclic branched cover M of K admits two commuting diffeomorphisms of order p, h and $h' = \tilde{\psi}$, such that: $(M, \mathrm{Fix}(h))/\langle h\rangle \cong (\mathbb{S}^3, K)$ on which h' induces the p-rotational symmetry ψ with trivial quotient, and $(M, \mathrm{Fix}(h'))/\langle h'\rangle \cong (\mathbb{S}^3, K')$ on which h induces a p-rotational symmetry ψ' with trivial quotient. The preimage $\pi^{-1}(T) = \tilde{T}$ is connected if and only if it corresponds to a torus $\tilde{T}$ of the JSJ-decomposition of M which is left invariant by h. Since h and h' play symmetric roles, it suffices to show that such a torus is invariant by ψ. Looking for a contradiction, let us assume that ψ does not leave T invariant. Then the h'-orbit of $\tilde{T}$ consists of $m > 1$ elements. Cutting M along these m separating tori, one gets $m + 1$ connected components.

CLAIM 1. *Both* $\mathrm{Fix}(h)$ *and* $\mathrm{Fix}(h')$ *must be contained in the same connected component.*

Proof. Assume that $\mathrm{Fix}(h)$ and $\mathrm{Fix}(h')$ are not contained in the same connected component. The diffeomorphism h' cyclically permutes the m connected components that do not contain $\mathrm{Fix}(h')$. Since by hypothesis $\mathrm{Fix}(h)$ is contained in one of these m components and h and h' commute, h leaves invariant each of these components and it acts in the same way on each of them (that is, the restrictions of h to each component are conjugate). This contradicts the fact that the set $\mathrm{Fix}(h)$ is connected and the claim follows. □

The m components permuted by h' project to a connected submanifold of the exterior $E(K')$ of the knot K' with connected boundary, the image T' of $\tilde{T}$. This submanifold is invariant by the action of ψ', but does not contain $\mathrm{Fix}(\psi')$. This contradicts Lemma 3(i). This concludes the proof of Lemma 5(i).

Part (ii) The first part of assertion (ii) is a straightforward consequence of assertion (i) and of Lemma 3. The second part follows from the fact that for $\pi^{-1}(T)$ to be connected, the winding number of T and p must be coprime.

Part (iii) Reasoning as in the proof of Claim 1, it is not difficult to prove that a torus of the JSJ-decomposition for M is left invariant by h if and only if it is left invariant by h'. Using this remark and the fact that h and h' play symmetric roles, part (iii) is then a consequence of the proof of part (i) of Lemma 3. □

Proof of Theorem 3. The proof is achieved in three steps.

Step 1: Theorem 3 is true under the assumption that the rotational symmetries commute pairwise.

In this case, assertion (i) is the statement of Proposition 2. Assertion (ii) follows from Lemma 4.

Step 2: Theorem 3 is true under the assumption that every companion of K is prime (that is, K is *totally prime*) and has non-vanishing winding number (thai is, K is *pedigreed*).

Since K admits at least one rotational symmetry with trivial quotient, Lemma 2 ensures that K is a prime knot. If K is also totally prime and pedigreed, then Sakuma [**28**, Theorem 4 and Lemma 2.3] proved that, up to conjugacy, the rotational symmetries belong either to a finite cyclic subgroup or to an $\mathbb{S}^1$-action in $\mathrm{Diff}^{+,+}(\mathbb{S}^3, K)$. Thus, after conjugacy, Step 1 applies. For part (ii), note that the distances of the fixed-point set of the symmetries to the vertex containing $\partial E(K)$ in the JSJ-graph Γ_K do not change by conjugacy.

Step 3: Reduction of the proof to Step 2. If K is not totally prime or not pedigreed, then it is non-trivial. We shall construct a non-trivial, totally prime and pedigreed knot verifying the hypothesis of Theorem 3. Assertion (i) then follows by contradiction. For assertion (ii), we need to verify that the construction does not change the distance of the pieces containing the axes of rotations to the root containing $\partial E(K)$. Roughly speaking, we consider the JSJ-tori closest to $\partial E(K)$ and corresponding either to non-prime or to winding number zero companions. Then we cut $E(K)$ along these tori and keep the component W containing $\partial E(K)$ and suitably Dehn-fill W along these tori to get the exterior of a non-trivial knot $\hat{K}$ in $\mathbb{S}^3$, which verifies Sakuma's property. Note that, by [**15**, Lemma VI.3.4], a JSJ-torus corresponding to a non-prime companion of K belongs to the boundary of a composing space (that is, a space homeomorphic to a product $\mathbb{S}^1 \times B$, where B is an n-punctured disc with $n \geqslant 2$).

More precisely, let Γ_K be the tree dual to the JSJ-decomposition of $E(K)$ and let Γ_0 be its maximal (connected) subtree with the following properties.

(i) Γ_0 contains the vertex v_∂ corresponding to the geometric piece whose boundary contains $\partial E(K)$. Note that the geometric piece of the decomposition corresponding to v_∂ cannot be a composing space for K is prime.

(ii) No vertex of Γ_0 corresponds to a composing space.

(iii) No edge of Γ_0 corresponds to a torus with winding number 0 with respect to K.

Denote by $X(\Gamma_0)$ the submanifold of $E(K)$ corresponding to Γ_0.

The following claim describes certain properties of $X(\Gamma_0)$ with respect to a rotational symmetry ψ of $(\mathbb{S}^3, K)$.

Claim 2. *Let ψ be a rotational symmetry of $(\mathbb{S}^3, K)$ with order $p > 2$ and trivial quotient. Then:*

(i) *the fixed-point set of ψ is contained in $X(\Gamma_0)$;*

(ii) *the tree Γ_0 is invariant by the automorphism of Γ_K induced by ψ and the submanifold $X(\Gamma_0)$ is invariant by ψ.*

Proof. *Assertion* (i). Let γ be the unique geodesic segment in Γ_K that joins the vertex v_∂ to the vertex corresponding to the geometric piece containing $\mathrm{Fix}(\psi)$ (see Lemma 1; note that here we use $p > 2$). According to assertion (ii) of Lemma 3, no vertex along γ_i can be a composing space. Since the linking number of K and $\mathrm{Fix}(\psi)$ must be coprime with p, no torus corresponding to an edge of γ can have winding number 0 (see Lemma 5).

Assertion (ii). This is just a consequence of the maximality of Γ_0 and the fact that elements of the group $\langle\psi\rangle$ generated by ψ must preserve the JSJ-decomposition of $E(K)$ and the winding numbers of the JSJ-tori, as well as send composing spaces to composing spaces. □

Let $\pi : M_p(K) \longrightarrow (\mathbb{S}^3, K)$ be the p-fold cyclic branched cover of K. Let T be a torus of the JSJ-collection of tori for $E(K)$. Denote by E_T the manifold obtained as follows: cut $E(K)$ along T and choose the connected component whose boundary consists only of T. Note that E_T is the exterior of the companion K_T of K corresponding to T.

Claim 3. *Let T be a torus of $\partial X(\Gamma_0)\backslash\partial E(K)$. The preimage $\pi^{-1}(T)$ consists of p components, each bounding a copy of E_T in $M_p(K)$. In particular, there is a well-defined meridian-longitude system (μ_T, λ_T) on each boundary component of $X(\Gamma_0)$, different from $\partial E(K)$, which is preserved by taking the p-fold cyclic branched covers.*

Proof. According to Lemma 5, the preimage of T is either connected or consists of p components. If the preimage of T were connected, the tree Γ_0 would not be maximal according to Lemma 5(ii). The remaining part of the claim is then easy. □

We now wish to perform Dehn fillings on the boundary of $X(\Gamma_0)$ in order to obtain a totally prime and pedigreed knot admitting pairwise distinct rotational symmetries with trivial quotients. On each component T of $\partial X(\Gamma_0)\backslash\partial E(K)$, we fix the curve $\alpha_n = \lambda_T + n\mu_T$.

Claim 4. *For all but finitely many $n \in \mathbb{Z}$ the Dehn filling of each component T of $\partial X(\Gamma_0)\backslash\partial E(K)$ along the curve α_n produces the exterior of a non-trivial, prime and pedigreed knot $\hat{K}$ in $\mathbb{S}^3$.*

Proof. Note that by the choice of surgery curves, the resulting manifold $\hat{X}(\Gamma_0)$ is the exterior of a knot $\hat{K}$ in the 3-sphere, that is, $\hat{X}(\Gamma_0) \subset \mathbb{S}^3$, and thus it is irreducible. We distinguish two cases.

(i) The JSJ-component X_T of $X(\Gamma_0)$ adjacent to T is Seifert fibred. Then, by the choice of Γ_0, X_T is a cable space (that is, the exterior of a (a, b)-torus knot in the solid torus bounded by T in $\mathbb{S}^3$). Moreover, the fibre f of the Seifert fibration of $X(\Gamma_0)$ is homologous to $a\mu_T + b\lambda_T$ on T and the intersection number $|\Delta(f, \mu_T)| = b > 1$. The intersection number of the filling curve α_n with the fibre f is then $|\Delta(f, \alpha_n)| = |na - b|$ and is greater than 1 for all but finitely many $n \in \mathbb{Z}$. In this case, the resulting manifold $X_T(\alpha_n)$ is the exterior of a non-trivial torus knot, which is prime and pedigreed [**7**].

(ii) The JSJ-component X_T of $X(\Gamma_0)$ adjacent to T is hyperbolic. By Thurston's hyperbolic Dehn filling theorem [**30**, Chapter 5] (see also [**3**, Appendix B]) for all but finitely many $n \in \mathbb{Z}$, the Dehn filling of each component $T \subset \partial X_T \cap (\partial X(\Gamma_0)\backslash\partial E(K))$ along the curve α_n produces a hyperbolic manifold $X_T(\alpha_n)$ with finite volume.

Therefore for all but finitely many integers n, the Dehn filling of each component $T \subset \partial X(\Gamma_0)\backslash\partial E(K)$ along the curve α_n produces a ∂-irreducible 3-manifold $\hat{X}(\Gamma_0) \subset \mathbb{S}^3$, such that each Seifert piece of its JSJ-decomposition is either a Seifert piece of $X(\Gamma_0)$ or a non-trivial torus knot exterior. Hence, it corresponds to the exterior of a non-trivial knot $\hat{K} \subset \mathbb{S}^3$, which is totally prime. It is also pedigreed by the choice of Γ_0. □

Let ψ be a rotational symmetry of $(\mathbb{S}^3, K)$ with order $p > 2$. Then the restriction $\psi_{|_{X(\Gamma_0)}}$, given by Claim 2, extends to $\hat{X}(\Gamma_0)$, giving a p-rotational symmetry $\hat{\psi}$ of the non-trivial, totally prime and pedigreed knot $(\mathbb{S}^3, \hat{K})$. In order to be able to apply Step 2 to the knot $\hat{K}$ and the induced rotational symmetries, we still need to check that the rotational symmetry $\hat{\psi}$ has trivial quotient when ψ has trivial quotient. This is the aim of the following claim.

Claim 5. *If the knot K/ψ is trivial, then the knot $\hat{K}/\hat{\psi}$ is trivial.*

Proof. Let $\pi : M_p(K) \longrightarrow (\mathbb{S}^3, K)$ be the p-fold cyclic branched cover. Let h be the deck transformation of this cover and h' the lift of ψ. According to Remark 1, h' is the deck transformation for the p-fold cyclic cover of the 3-sphere branched along a knot K'. Note that, by Claim 3, $M_p(K)\backslash\pi^{-1}(X(\Gamma_0) \cup \mathcal{U}(K))$ is a disjoint union of p copies of $E(K)\backslash X(\Gamma_0)$. It follows that the p-fold cyclic branched cover $M_p(\hat{K})$ of $\hat{K}$ is the manifold obtained by a $(\lambda_T + n\mu_T)$-Dehn filling on all the boundary components of $\pi^{-1}(X(\Gamma_0) \cup \mathcal{U}(K))$. The choice of the surgery shows that both h and h' extend to diffeomorphisms $\hat{h}$ and $\hat{h}'$ of order p of $M_p(\hat{K})$. By construction, we see that $M_p(\hat{K})/\langle\hat{h}\rangle \cong \mathbb{S}^3$. In the same way, $M_p(\hat{K})/\langle\hat{h}'\rangle$ is obtained from $M_p(K)/\langle h'\rangle \cong \mathbb{S}^3$ by cutting off a copy of $E(K)\backslash X(\Gamma_0)$ and Dehn filling along $\partial X(\Gamma_0)$. The choice of the surgery curve assures that the resulting manifold is again $\mathbb{S}^3$ and the conclusion follows from Remark 1. □

From the non-trivial prime knot K, we have thus constructed a non-trivial, totally prime and pedigreed knot $\hat{K}$, which has the property that every rotational symmetry ψ of K with trivial quotient and order greater than 2 induces a rotational symmetry $\hat{\psi}$ of $\hat{K}$ with trivial quotient and the same order. Moreover, by the choice of the Dehn filling curve in the construction of $\hat{K}$, the vertex containing $\mathrm{Fix}(\hat{\psi})$ remains at the same distance from the vertex containing $\partial E(\hat{K})$ in the *JSJ*-tree $\Gamma_{\hat{K}}$ as the vertex containing $\mathrm{Fix}(\psi)$ from the vertex containing $\partial E(K)$ in the *JSJ*-tree Γ_K. Then the conclusion of Theorem 3 is a consequence of Step 2. □

3. *Twins of a prime knot*

In this section, we prove Theorem 1. If K is trivial, the theorem is a consequence of the proof of Smith's conjecture (see [**22**]). We shall thus assume in the remainder of this section that K is non-trivial and p is an odd prime number.

Let M be the common p-fold cyclic branched cover of two prime knots K and K' in $\mathbb{S}^3$. Let h and h' be the deck transformations for the coverings of K and K', respectively. By the orbifold theorem [**3**] (see also [**6**]), one can assume that h and h' act *geometrically* on the geometric pieces of the JSJ-decomposition of M, that is, by isometries on the hyperbolic pieces and respecting the fibration on the Seifert fibred ones.

The following lemma describes the Seifert fibred pieces of the JSJ-decomposition of the p-fold cyclic branched cover M (see also [**14**] and [**13**, Lemma 2]).

Lemma 6. *Let p be an odd prime integer and let M be the p-fold cyclic branched cover of $\mathbb{S}^3$ branched along a prime, satellite knot K. If V is a Seifert piece in the JSJ-decomposition for M, then the base B of V can be:*

(i) *a disc with 2, p or $p+1$ cone points corresponding to singular fibres;*
(ii) *a disc with one hole, that is, an annulus, with one or p cone points;*
(iii) *a disc with $p-1$ holes and one cone point;*
(iv) *a disc with p holes and one cone point;*
(v) *a disc with n holes, $n \geqslant 2$.*

Proof. It suffices to observe that V projects to a Seifert fibred piece V' of the Bonahon–Siebenmann decomposition for the orbifold $\mathcal{O}_p(K)$. There are four possible cases.

(i) V' contains K: V' is topologically a non-trivially fibred solid torus and K is a regular fibre of the fibration, that is, a torus knot $K(a,b)$, since it cannot be the core of the fibred solid torus. The knot K lifts to a singular fibre of order p if p does not divide ab and to a regular fibre otherwise. The core of the solid torus is a singular fibre of order, say, a. It lifts to a regular fibre if $a = p$, to a singular fibre of order $a/(a,p)$ if p does not divide b, or to p singular fibres of order a if p divides b. Thus, V has p boundary components if p divides a, and 1 otherwise. An Euler characteristic calculation shows that B is either a disc with two or p cone points, or a disc with $p-1$ holes and with at most one cone point.

(ii) V' is the complement of a torus knot $K(a,b)$ in $\mathbb{S}^3$. In this case, V is either a copy of V', and B is a disc with two cone points or V is a true p-fold cover of V'. In this case, V has exactly one boundary component. Reasoning as in case (i), we see that the two singular fibres of V' must lift to either two singular fibres, or one regular fibre and p singular fibres, or one singular fibre and p singular fibres. In particular, B is a disc with two, p or $p+1$ cone points.

(iii) V' is the complement of a torus knot $K(a,b)$ in a solid torus, that is, a cable space, and its base is an annulus with one cone point. Reasoning as in (ii), we find that B can be a disc with one hole and one or p cone points, or a disc with p holes and at most one cone point.

(iv) V' is a composing space with at least three boundary components, and thus so is V. More precisely, note that either V' lifts to p disjoint copies of itself, or V and V' are homeomorphic and V' is obtained by quotienting V via the p-translation along the $\mathbb{S}^1$ fibre. In this case, B is a disc with at least two holes.

This analysis ends the proof of Lemma 6. □

Proposition 3. *Let M be the common p-fold cyclic branched cover of two prime knots K and K' in $\mathbb{S}^3$, p an odd prime number, and let h be the deck transformation for the covering of K. Let Γ be the tree dual to the JSJ-decomposition of M. The deck transformation h' for the covering of K' can be chosen* (*up to conjugacy*) *in such a way that the following hold.*

(i) *There exists a subtree Γ_f of Γ on which the actions induced by h and h' are trivial.*

(ii) *The vertices of Γ corresponding to the geometric pieces of the decomposition which contain* $\mathrm{Fix}(h)$ *and* $\mathrm{Fix}(h')$ *belong to Γ_f.*

(iii) *Let M_f the submanifold of M corresponding to Γ_f. The restrictions of h and h' to M_f commute.*

Proof. The proof relies on the study of the actions of the two covering transformations h and h' on the JSJ-decomposition of the common p-fold cyclic branched covering M. Since Γ is finite, the group generated by the tree automorphisms induced by h and h' is finite as well. Standard theory of group actions on trees assures that a finite group acting on a tree without inversions must have a global fixed point and that its fixed-point set is connected. Thus, part (i) of the proposition follows, using the fact that h and h' have odd orders.

Choose now h', up to conjugacy in $\mathrm{Diff}^+(M)$, in such a way that Γ_f is maximal. We want to show that, in this case, M_f contains $\mathrm{Fix}(h)$ and $\mathrm{Fix}(h')$. Assume by contradiction that the vertex v_h of Γ corresponding to the geometric piece containing $\mathrm{Fix}(h)$, whose existence is ensured by Lemma 1, does not belong to Γ_f. Let γ_h be the unique geodesic path in Γ connecting v_h to Γ_f. Let e_h be the edge in γ_h adjacent to Γ_f and denote by T the corresponding torus of the JSJ-collection of tori for M. Let U be the connected component of $M \backslash T$ which contains $\mathrm{Fix}(h)$. Consider the $\langle h, h'\rangle$-orbit of U. This orbit is the disjoint union of h (and h') orbits of U. Note that the h-orbit of U is $\{U\}$.

Claim 6. *The orbit $\langle h, h'\rangle U$ must contain an h-orbit, different from $\{U\}$ and containing a unique element.*

Proof. Otherwise all the h-orbits in $\langle h, h' \rangle U$ different from $\{U\}$ would have p elements, since p is prime. In particular, the cardinality of $\langle h, h' \rangle U$ would be of the form $kp + 1$. This implies that at least one of the h'-orbits in $\langle h, h' \rangle U$ must contain one single element U'. Up to conjugacy with an element of $\langle h, h' \rangle$ (whose induced action on Γ_f is trivial), we can assume that $U = U'$, contradicting the hypothesis that h' was chosen up to conjugacy in such a way that Γ_f is maximal. □

Let $U' \neq U$ be the element of the orbit $\langle h, h' \rangle U$ such that $h(U') = U'$. Note that U and U' are homeomorphic, since they belong to the same $\langle h, h' \rangle$-orbit.

CLAIM 7. *U is homeomorphic to the exterior $E(K_0)$ of a knot $K_0 \subset \mathbb{S}^3$ admitting a free symmetry of order p.*

Proof. The first part of the claim follows from the fact that, by maximality of Γ_f, h' cannot leave U invariant, so it must freely permute p copies of U belonging to $\langle h, h' \rangle U$. Thus, U must appear as a union of geometric pieces of the JSJ-splitting of $E(K')$. The second part follows from the fact that h must act freely on U', which is homeomorphic to U. □

REMARK 3. Note that the quotient of U by the action of its free symmetry of order p is also a knot exterior because h acts freely on U', and U' must project to a union of geometric pieces of the JSJ-splitting of $E(K)$.

CLAIM 8. *U admits a rotational symmetry of order p whose quotient $U/\langle h \rangle$ is topologically a solid torus.*

Proof. The quotient $U/\langle h \rangle$ is obtained by cutting $\mathbb{S}^3$ along an essential torus in $E(K)$. Since $K \subset U/\langle h \rangle$, it must be a solid torus. □

Claim 8 shows that K_0 admits a rotational symmetry of order p and trivial quotient. It follows from Lemma 2 that the knot K_0 is prime. Moreover, according to Claim 7, K_0 also admits a free symmetry, and both the free symmetry and the rotational one have order p. This is however impossible, because Sakuma [**28**, Theorem 3] showed that a prime knot can only have one symmetry of odd order up to conjugacy. This contradiction proves part (ii) of Proposition 3.

To prove part (iii) we shall consider two cases, according to the structure of Γ_f.

Case (a): Γ_f *contains an edge.* Choose an edge in Γ_f and let T be the corresponding torus in the JSJ-collection of tori for M. Let V be a geometric piece of the JSJ-decomposition of M adjacent to T. Then Lemma 7 below, together with a simple induction argument, shows that h' can be chosen (up to conjugacy) in such a way that its restriction to M_f commutes with the restriction of h.

LEMMA 7. *If the covering transformations h and h' preserve a JSJ-torus T of M then, up to conjugacy in* $\mathrm{Diff}^+(M)$, *h and h' commute on the union of the geometric components of the JSJ-decomposition adjacent to T.*

Proof. First we show that h and h' commute on each geometric component adjacent to T. Since h and h' preserve the orientation of M, we deduce that $h(V) = V$ and $h'(V) = V$, and that h and h' act geometrically on the geometric piece V. A product structure on T can always be induced by the geometric structure on V: either by considering the induced Seifert fibration on T, if V is Seifert fibred, or by identifying T with a section of a cusp in the complete hyperbolic manifold V. Since h and h' are isometries of order p for such a product structure on T, they act as (rational) translations; that is, their action on $T = \mathbb{S}^1 \times \mathbb{S}^1$ is of the form $(\zeta_1, \zeta_2) \mapsto (e^{2i\pi r_1/p}\zeta_1, e^{2i\pi r_2/p}\zeta_2)$, where p and at least one of r_1 and r_2 are coprime. Thus, h and h' commute on T.

If V is hyperbolic, we have just seen that h and h' are two isometries of V that commute on the cusp corresponding to T. Thus, they must commute on V.

If V is Seifert fibred, then the Seifert fibration is unique up to isotopy, since ∂V is not empty and incompressible, and h and h' preserve this fibration.

Remark 4. Note that, if V is Seifert fibred, the quotient of V by a fibre-preserving diffeomorphism of finite order h depends only on the combinatorial behaviour of h, that is, its translation action along the fibre and the induced permutation on cone points and boundary components of the base. In particular, the conjugacy class of h depends only on these combinatorial data. Note, moreover, that two geometric symmetries having the same combinatorial data are conjugate via a diffeomorphism isotopic to the identity.

Since h and h' are fibre preserving and have odd order, they both commute with the translation along the fibres. Hence it suffices to see whether h and h' commute, up to a conjugation of h', on the base B of V. It is enough then to consider the possible actions of order p on the possible bases. According to Lemma 6, the possible actions of h and h' are described below.

(1) If B is a disc with two cone points, or an annulus with one cone point, or a disc with n holes, $n \neq p$, or a disc with $p-1$ holes and one cone point, then the action on B is necessarily trivial and there is nothing to prove. Note that, according to the proof of Lemma 6, if B is a disc with $p-1$ holes with one cone point, no boundary torus is left invariant, so this possibility in fact does not occur here.

(2) If B is a disc with p holes and one cone point or a disc with $p+1$ cone points, then the only possible action is a rotation about a cone point cyclically permuting the holes or the remaining cone points.

(3) If B is a disc with p cone points, then the action must be a rotation about a point corresponding to a regular fibre that cyclically exchanges the p cone points.

(4) If B is an annulus with p cone points, the action must be a free rotation, cyclically exchanging the cone points. Note that in the latter three cases, the action can never be trivial on the base.

(5) If B is a disc with n holes, then two situations can arise: either the action is trivial on the base (case (iv) in the proof of Lemma 6; note that in case (i), when $n = p-1$, all boundary components must be cyclically permuted), or $n = p$ and the action is a rotation about a point corresponding to a regular fibre, which cyclically permutes the p holes (see part (iii) of Lemma 6).

We shall now show that, if both h and h' induce non-trivial actions on the base of V, then, up to conjugacy in $\mathrm{Diff}^+(M)$, h and h' can be chosen so that their actions on B coincide. Note that for h and h' to commute, it suffices that the action of h' on B coincides with the action of some power of h; however, this stronger version will be needed in the proof of Corollary 10.

First of all remark that, if B is a disc with $p+1$ cone points (case 2) and h and h' leave invariant distinct singular fibres, then all the singular fibres must have the same order (in fact, they must have the same invariants). This means that, after conjugating h' by a homeomorphism of V, which is either an isotopy exchanging two regular fibres or a Dehn twist along an incompressible torus exchanging two singular fibres, one can assume that, in cases 2 and 3, h and h' leave set-wise invariant the same fibre. Note that this homeomorphism is isotopic to the identity on ∂V and thus extends to M. In fact, using Lemma 6 one can show that the fibres cannot all have the same order.

Since the actions of h and h' consist in permuting exactly p holes or singular fibres, it suffices to conjugate h' via a homeomorphism of V (which is a composition of Dehn twists along incompressible tori) in such a way as to exchange the order of the holes or singular fibres, so that h' and h cyclically permute them in the same order. Note that in the case of singular fibres, this product of Dehn twists is isotopic to the identity on ∂V and thus extends to M. In the case of holes, the product of Dehn twists extends to M, since it induces the identity on the fundamental groups of the tori of ∂V and the connected components of $M \backslash V$ adjacent to boundary tori different from T are necessarily homeomorphic.

Once the two diffeomorphisms h and h' commute on the two geometric pieces adjacent to T, the commutation can be extended on a product neighbourhood of T, since the two finite abelian groups generated by the restrictions of h and h' on each side of T have the same action on T. Indeed, the slope of the translation induced by h' on T has been left unchanged by the conjugation. $\square$

REMARK 5. Note that in case 1 of the proof of the above lemma, the actions of h and h' must coincide after taking a power, that is, h and h' generate the same cyclic group. This is not necessarily true in the remaining cases, even if h and h' induce the same action on B. Indeed, they can induce different translations along the fibres. Nevertheless, in both cases, to ensure that the actions of h and h' coincide on V, it suffices to check that they coincide on T.

Case (b): Γ_f *is a single vertex.* Let $V = M_f$ be the geometric piece corresponding to the unique vertex of Γ_f. According to part (ii) of Proposition 3, we can assume that the fixed-point sets of h and h' are contained in V.

Assume that V is Seifert fibred. Since h and h' are fibre preserving and have odd order, the fixed-point sets $\mathrm{Fix}(h)$ and $\mathrm{Fix}(h')$ are fibres of V. If $V = M$, then the exteriors of the knots K and K' are Seifert fibred; hence they are torus knots in $\mathbb{S}^3$. Two torus knots K and K' with the same p-fold cyclic branched covering are equivalent; therefore, up to conjugacy, h and h' coincide. We can thus assume that $V \neq M$, then, case (i) of the proof of Lemma 6 shows that the base B of V is either a disc with two or $p+1$ cone points, or a disc with $p-1$ holes and with one or two cone points. In the first case, the boundary torus of V is preserved by h and h' and the assertion follows from Lemma 7. In the second case, the action on the base is necessarily a rotation, fixing two points (either the unique point corresponding to a singular fibre and a point corresponding a regular one, or the two cone points) and cyclically permuting the p boundary components. Then conjugating h' by a product of Dehn twists along incompressible tori, which extends to M as in the proof of Lemma 7, leads to the desired conclusion.

The case where V is hyperbolic is due to Zimmermann [**34**]. We give the argument for completeness. Since V is hyperbolic, we consider the group $\mathcal{I}_V$ of isometries of V induced by diffeomorphisms of M which leave V invariant. Since V has finite volume (being either closed or with toric boundary components), $\mathcal{I}_V$ is finite. Let $\mathcal{S}$ be the p-Sylow subgroup of $\mathcal{I}_V$. Up to conjugacy, we can assume that both $h = h_{|_V}$ and $h' = h'_{|_V}$ belong to $\mathcal{S}$. If the groups $\langle h \rangle$ and $\langle h' \rangle$ generated by h and h' are conjugate, we can assume that $h = h'$ and this completes the argument. So we assume that $\langle h \rangle$ and $\langle h' \rangle$ are not conjugate. Then it suffices to prove that h' normalises $\langle h \rangle$ because each element normalising $\langle h \rangle$ must leave invariant $\mathrm{Fix}(h) \subset V$, and the subgroup of $\mathcal{I}_V$ which leaves invariant a simple closed geodesic, like $\mathrm{Fix}(h)$, must be a finite subgroup of $\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$. In particular, elements of odd order must commute. Assuming that $\langle h \rangle$ and $\langle h' \rangle$ are not conjugate, we see that $\langle h \rangle \subsetneq \mathcal{S}$ and, by [**29**, Chapter 2, 1.5], either $\langle h \rangle$ is normal in $\mathcal{S}$ and we have reached the desired conclusion, or there exists an element $\hat{h} = ghg^{-1}$, conjugate to h in $\mathcal{S}$, which normalises $\langle h \rangle$ and such that $\langle h \rangle \cap \langle \hat{h} \rangle = \{1\}$.

We want to show that h' normalises $\langle h \rangle$. Assume, by contradiction that h' is not contained in $\langle h, \hat{h} \rangle = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Then this group is smaller than $\mathcal{S}$ and again we are able to find a new cyclic group H of order p whose intersection with $\langle h, \hat{h}\rangle$ is reduced to the identity and which normalises $\langle h, \hat{h}\rangle$. Since the order of H is an odd prime number and since $\langle h\rangle$ and $\langle \hat{h}\rangle$ are the only subgroups of $\langle h, \hat{h}\rangle$ which fix point-wise a geodesic by [**18**, Proposition 4], H would commute with $\langle h, \hat{h}\rangle$, which is a contradiction to the structure of a group leaving a geodesic invariant. This final contradiction shows that, up to conjugacy, the subgroups $\langle h\rangle$ and $\langle h'\rangle$ either commute or coincide on V. This finishes the proof of Proposition 3. □

The idea is now to understand which is the largest submanifold of M, containing M_f (see Proposition 3(iii)) and made up of geometric pieces, on which h and h' commute. Note that, if h and h' commute on M, the $\langle h, h'\rangle$-orbit of a JSJ-torus for M contains either one or p^2 elements (see proof of Lemma 5). We shall see (in the proof of Proposition 4) that this property is also sufficient for h and h' to commute.

The following proposition shows that a prime knot K having a p-twin either admits a rotational symmetry of order p, or a well-specified submanifold $E_p(K)$ built up of geometric pieces of the JSJ-decomposition of $E(K)$ admits a *symmetry*, that is, a periodic diffeomorphism, of order p with non-empty fixed-point set.

Definition 2. Let K be a prime knot in $\mathbb{S}^3$. For each odd prime number p, we define $E_p(K)$ to be the connected submanifold of $E(K)$ containing $\partial E(K)$ and such that $\partial E_p(K)\backslash\partial E(K)$ is the union of the JSJ-tori of $E(K)$ with winding number p which are closest to $\partial E(K)$.

Proposition 4. *Let K be a prime knot and let p be an odd prime number. Then for any p-twin K', the deck transformation of the branched cover $M \longrightarrow (\mathbb{S}^3, K')$ induces on $E_p(K)$ a symmetry of order p, with non-empty fixed-point set and which extends to $\mathcal{U}(K)$.*

Proof. First we show that the deck transformation of the branched cover $M \longrightarrow (\mathbb{S}^3, K')$ associated to a p-twin of K induces on $E_p(K)$ a symmetry of order p.

Let K' be a p-twin of K. Let h and h' be the deck transformations on M for the p-fold cyclic branched covers of K and K'. We shall start by understanding the behaviour of h and h' on M. We have seen in Proposition 3 that h and h' can be chosen to commute on the submanifold M_f of M corresponding to the maximal subtree of Γ on which both h and h' induce a trivial action. Let Γ_c be the maximal $\langle h, h'\rangle$-invariant subtree of Γ containing Γ_f such that, up to conjugacy in $\mathrm{Diff}^+(M)$, h and h' can be chosen to commute on the corresponding submanifold M_c of M. It is sufficient to show that $E_p(K) \subset M_c/\langle h\rangle$: then the symmetry of order p induced by h' on $M_c/\langle h\rangle$ must preserve $E_p(K)$, since each JSJ-torus of $E(K) \cap M_c/\langle h\rangle$ can only be mapped to another torus of the family with the same winding number and the same distance from $\partial E(K)$.

If $M_c = M$, then, after conjugation, h' commutes with h on M, but is distinct from h because the knots K and K' are not equivalent. Hence it induces a rotational symmetry of order p of the pair $(\mathbb{S}^3, K)$ and this completes the argument.

So we consider now the case where ∂M_c is not empty. First we show the following claim.

Claim 9. *Let T be a connected component of ∂M_c. The h-orbit of T consists of p elements that are permuted in the same way by h and h'.*

Proof. Let T be a torus in ∂M_c and let U be the connected component of $M\backslash M_c$ adjacent to T. Because of Lemma 7, T cannot be preserved by both h and h' or else M_c would not be maximal. Without loss of generality, we can assume that either:

(i) $h(T) \neq T$ and $h'(T) \neq T$;

or

(ii) $h(T) = T$ but $h'(T) \neq T$; in this case, since h and h' commute on M_c, we see that $h(h'^{\alpha}(U)) = h'^{\alpha}(U)$. Then part (ii) of Proposition 3 implies that h acts freely on $h'^{\alpha}(U)$ for each $\alpha = 0, \dots, p-1$.

In case (i), the orbit of T by the action of the group $\langle h, h' \rangle$ consists of p or p^2 elements which bound on one side M_c and on the other side a manifold homeomorphic to U. If the orbit consist of p elements, since h and h' commute on M_c, up to choosing a different generator in $\langle h' \rangle$ we can assume that h and h' permute the elements of the orbit in the same way. Indeed, we have $h'h(T) = hh'(T) = h(h^{\alpha}(T)) = h^{\alpha}(h(T))$.

If the orbit consist of p^2 elements, U is a knot exterior and there is a well-defined longitude-meridian system on each component of the $\langle h, h' \rangle$-orbit of T. In particular, there is a unique way to glue a copy of U along the projection of T in $M_c/\langle h, h' \rangle$. Denote by N the manifold obtained by gluing, in the prescribed way, a copy of U along the boundary component $\bar{T}$ of $M_c/\langle h, h' \rangle$, corresponding to the projection of T. The $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$-regular cover $M_c \longrightarrow M_c/\langle h, h' \rangle$ induces a unique $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$-regular cover $M_c \cup \langle h, h' \rangle U \longrightarrow N$, thanks to the fact that $\pi_1(\bar{T})$ maps to 0 in $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. In particular, $(M_c \cup \langle h, h' \rangle U)/\langle h \rangle$ and $(M_c \cup \langle h, h' \rangle U)/\langle h' \rangle$, respectively, can be obtained as p-fold cyclic covers of N. This implies that h and h' commute on $M_c \cup \langle h, h' \rangle U$, contradicting the maximality of M_c. Note also that, in this latter case, the stabiliser of each component of $\langle h, h' \rangle U$ is reduced to the identity which clearly extends to $\langle h, h' \rangle U$.

Assume that we are in case (b). Consider the restriction of h and $h_{\alpha} = h'^{-\alpha} h h'^{\alpha}$ to U. Since h and h' commute on M_c, h and h_{α} coincide on T. Let V be the geometric piece of the JSJ-decomposition for M adjacent to T and contained in U. Using Lemma 7, we see that h and h_{α} commute on V and thus coincide on it, because they coincide on T. Thus h and h' commute on $M_c \bigcup_{\alpha=0}^{p-1} h'^{\alpha}(V)$, and again we reach a contradiction to the maximality of M_c. □

We can thus assume that, if $T \in \partial M_c$, then $h(T) \neq T$, $h'(T) \neq T$, and the $\langle h, h' \rangle$-orbit of T has p elements.

Claim 10. *Each torus in the boundary of $M_c/\langle h \rangle$ has winding number p with respect to K.*

Proof. Since a boundary component T of $M_c/\langle h \rangle$ lifts to p boundary components of M_c, the winding number of T with respect to K must be a multiple of p. We shall now reason by induction on the number n of boundary components of $M_c/\langle h \rangle$. If $n = 0$, there is nothing to prove.

If $n = 1$ the quotient spaces $M_c/\langle h \rangle$ and $M_c/\langle h' \rangle$ are solid tori, that is, the exterior of a trivial knot that can be identified with the boundary of a meridian disc of each solid torus. By definition, the winding number of T with respect to K is the (absolute value of the) linking number of K with such a meridian. Note, moreover, that the spaces $M_c/\langle h \rangle$ and $M_c/\langle h' \rangle$ have a common quotient $\mathcal{O}$ which is obtained by quotienting $M_c/\langle h \rangle$ via the symmetry ψ of order p and with non-empty fixed-point set, induced by h', or by quotienting $M_c/\langle h' \rangle$ via the symmetry ψ' of order p and with non-empty fixed-point set, induced by h. Since ψ' preserves $\partial(M_c/\langle h' \rangle)$ and has non-empty fixed-point set, $\mathrm{Fix}(\psi')$ and the meridian of $\partial(M_c/\langle h' \rangle)$ must form a Hopf link; in particular, their linking number has modulus 1. The image of $\mathrm{Fix}(\psi')$ and of the meridian of $\partial(M_c/\langle h' \rangle)$ form again a Hopf link in $\mathcal{O} = (M_c/\langle h' \rangle)/\psi$. By lifting them up to $M_c/\langle h \rangle$, we see that the meridian lifts to a meridian and the image of $\mathrm{Fix}(\psi')$ lifts to K, which thus have linking number p. Hence, the property is proved in this case.

If $n > 1$, we shall perform trivial Dehn surgery on $n-1$ boundary components of $M_c/\langle h \rangle$. Two distinct boundary components of $M_c/\langle h \rangle$ correspond to JSJ-tori T, T' of $E(K)$ which are not nested: this means that the corresponding edges do not lie in Γ_K on the same geodesic emanating from the vertex corresponding to the geometric piece containing $\partial E(K)$. It follows that the solid torus containing K and bounded by, say, T' must contain $E(K_T)$, that is, the knot exterior that does not contain K and is bounded by T; hence the two tori lie in disjoint balls in $\mathbb{S}^3$. Therefore, such a surgery does not change the winding number of the remaining boundary components. Moreover, the symmetry of order p of $M_c/\langle h\rangle$ extends to the resulting solid torus, and the surgery can be lifted on M_c in such a way that the quotient of the resulting manifold by the action of the diffeomorphism induced by h' is again a solid torus. This last property follows from the fact that each connected component of $(E(K)\backslash(M_c/\langle h\rangle)$ is the exterior of a knot that lifts in M to p diffeomorphic copies. These p copies of the knot exterior are permuted by h' and a copy appears in the JSJ-decomposition of $E(K')$. This means that on each boundary component, there is a well-defined meridian-longitude system that is preserved by h and h', and by passing to the quotient. The claim follows now from case $n = 1$. □

Now Claims 9 and 10 imply that $E_p(K)$ is a submanifold of $M_c/\langle h\rangle \cap E(K)$. Note, moreover, that by part (ii) of Proposition 3, the fixed-point set of the symmetry induced by h' is contained in $M_f/\langle h\rangle \subset M_c/\langle h\rangle$. In particular, each torus of the JSJ-family separating such a fixed-point set from K lifts to a single torus of the JSJ-family for M and its winding number cannot be a multiple of p. We can thus conclude that the fixed-point set of the symmetry induced by h' is contained in $E_p(K)$. This finishes the proof of Proposition 4. □

Remark 6. Note that $M_c/\langle h\rangle \cap E(K)$ can be larger than $E_p(K)$, for there might be tori of the JSJ-collection for M that have an $\langle h, h'\rangle$-orbit containing p^2 elements and that project to tori with winding number p. Note also that $E_p(K)$ coincides with $E(K)$ if there are no JSJ-tori in $E(K)$ with winding number p.

Remark 7. The deck transformations h and h' cannot commute on the submanifolds U of M corresponding to branches of Γ whose h- and h'-orbits coincide and consist of p elements, unless h and h' are the same; that is, the stabiliser $h'h^{-1}$ is a finite-order diffeomorphism of U if and only if it is trivial. To see this, assume that there is a unique orbit of this type and assume by contradiction that h and h' commute on M and are distinct. The diffeomorphism h' would induce a non-trivial symmetry of $E(K)$ of order p and non-empty fixed-point set which fixes set-wise the projection of U and acts freely on it. This contradicts the first part of Lemma 3. If there are $n > 1$ such orbits, an equivariant Dehn surgery argument on $n - 1$ components again leads to a contradiction.

Here is a straightforward corollary of Proposition 4 which generalises a result proved by Zimmermann [**34**] for hyperbolic knots.

Corollary 4. *Let K be a prime knot and let p be an odd prime number. If K has no companion of winding number p and has a p-twin, then K admits a rotational symmetry of order p with trivial quotient.*

So far we have proved that if a prime knot K has a p-twin, either $E(K)$ admits a p-rotational symmetry, or a well-specified submanifold $E_p(K)$ of $E(K)$ admits a symmetry of order p with non-empty fixed-point set. We shall say that the p-twin induces a *symmetry* or a *partial symmetry* of K, respectively.

Proposition 5. *Let K be a prime knot. Assume that K has a p-twin and a q-twin for two distinct odd prime numbers.*

(i) *At least one twin, say the q-twin, induces a q-rotational symmetry ψ_q of K.*

(ii) *Moreover, if the p-twin induces a partial p-symmetry of K, then $\partial E_p(K)\backslash\partial E(K)$ is a JSJ-torus which separates the fixed-point set* $\mathrm{Fix}(\psi_q)$ *from* $\partial E(K)$.

First we study some properties of partial symmetries induced by p-twins for an odd prime number p.

LEMMA 8. *Let K be a prime knot and let ψ_p be the partial symmetry of order p induced on $E_p(K)$ by a p-twin. Let T be a torus of the JSJ-collection of $E_p(K)$ which is not in the boundary. Then T does not separate $\partial E(K)$ from* $\mathrm{Fix}(\psi_p)$ *if and only if its ψ_p-orbit has p elements. Moreover, this is the case if and only if the lift of T to the p-fold cyclic branched cover of K has p elements.*

Proof. It suffices to perform ψ_p-equivariant Dehn fillings on the boundary components $\partial E_p(K)\backslash\partial E(K)$ of $E_p(K)$ in such a way that the resulting manifold is a knot exterior $E(\hat{K})$ and that the graph dual to the JSJ-decomposition of $E(\hat{K})$ remains unchanged after filling (see the proof of Theorem 3). Part (i) of Lemma 3 then applies to the resulting knot $\hat{K}$ and the induced rotational symmetry.

To prove the second part, we need to apply Lemma 5(iii). For its hypotheses to be verified, we need to perform Dehn filling in such a way that the resulting symmetry has trivial quotient. This can be done as in the proof of Claim 10, where the fillings were chosen in such a way that the induced fillings on the quotient $E_p(K)/\langle\psi_p\rangle$ also give a solid torus (see Remark 1). □

REMARK 8. In particular, case (ii) of the proof of Claim 9 cannot happen for a torus T in the situation of Lemma 8.

LEMMA 9. *Let K be a prime knot and let ψ_p be the partial symmetry of order p induced on $E_p(K)$ by a p-twin. Let $T\subset\partial E_p(K)\backslash\partial E(K)$ be a torus which is ψ-invariant. Let e_T be the corresponding edge in the tree dual to the JSJ-decomposition of $E_p(K)$. Let v_K and v_{ψ_p} be the vertices corresponding to the geometric pieces containing $\partial E(K)$ and* $\mathrm{Fix}(\psi_p)$*, respectively. Then v_{ψ_p} belongs to the unique geodesic joining v_K to e_T in the JSJ-tree.*

Proof. If we cut $\mathbb{S}^3$ along a torus of the JSJ-collection of $E_p(K)$, the connected component which does not contain K is a knot exterior and is thus contained in a ball in $\mathbb{S}^3$. If the conclusion of the lemma were false, then we could find two tori of the JSJ-decomposition of $E_p(K)$, one torus separating $\mathrm{Fix}(\psi_p)$ from $\partial E(K)$ and the other coinciding with T or separating it from $\partial E(K)$, such that the corresponding edges do not lie, in the tree dual to the JSJ-decomposition of $E_p(K)$, on the same geodesic with origin v_K. These two JSJ-tori would not be nested and hence would be contained in two disjoint balls (cf. the proof of Claim 10). In particular, the torus T would sit in a ball disjoint from the axis $\mathrm{Fix}(\psi_p)$ and this is impossible, since ψ_p leaves T set-wise invariant. □

COROLLARY 5. *Let K be a prime knot and let ψ_p be the partial symmetry of order p induced on $E_p(K)$ by a p-twin. Let $T\subset\partial E_p(K)\backslash\partial E(K)$ be a torus which is ψ_p-invariant. Then* $\mathrm{Fix}(\psi_p)$ *and $\partial E(K)$ belong to the same geometric piece of the JSJ-decomposition of $E_p(K)$. Moreover, the boundary torus T is adjacent to the geometric component containing* $\mathrm{Fix}(\psi_p)$ *and $\partial E(K)$.*

Proof. Let K' be a p-twin inducing a partial symmetry ψ_p on $E_p(K)$. Let M be the common p-fold cyclic branched cover of K and K', with covering transformations h and h', respectively. The lift to M of the ψ_p-invariant torus $T\subset\partial E_p(K)\backslash\partial E(K)$ consists of p tori, since the winding number of T is p by construction. These p tori correspond to the orbit of a JSJ-torus, say $\widetilde{T}$, under the action of $\langle h, h'\rangle$. This follows from the fact that T is ψ_p-invariant. According to the action of h' on the orbit, two situations can occur: $h'(\widetilde{T}) = \widetilde{T}$ or $h'(\widetilde{T}) \neq \widetilde{T}$.

Assume that we are in the first case, and let V be the geometric piece of M adjacent to $\widetilde{T}$ and such that $V/\langle h\rangle$ is not contained in $E_p(K)$. Reasoning as in part (ii) of Claim 9, we see that $V \subset M_c$; that is, h and h' commute on $\langle h, h'\rangle V$. In particular, ψ_p extends to $E_p(K) \cup V/\langle h\rangle$. Repeating the argument used in the first part of the proof of Lemma 8, we see that T should separate $\partial E(K)$ from $\mathrm{Fix}(\psi_p)$, which is impossible since $\mathrm{Fix}(\psi_p)$ is contained in the interior of $E_p(K)$, while T belongs to its boundary (see also Proposition 4).

We can thus assume that we are in the second case. Projecting $\widetilde{T}$ to $E(K')$, we obtain a ψ'_p-invariant torus T', where ψ'_p denotes the partial symmetry induced by h. Let Γ be the tree dual to the JSJ-decomposition of M and let v_h, v'_h be the vertices of Γ corresponding to the geometric pieces containing $\mathrm{Fix}(h)$ and $\mathrm{Fix}(h')$, respectively. Let $e_{\widetilde{T}}$ be the edge of Γ corresponding to the torus $\widetilde{T}$. Note now that T separates $\mathrm{Fix}(\psi_p)$ from $\partial E(K)$ if and only if $e_{\widetilde{T}}$ belongs to the geodesic joining v_h and v'_h, that is, if and only if T' separates $\mathrm{Fix}(\psi'_p)$ from $\partial E(K')$. Using the fact that T' is ψ'_p-invariant and applying Lemma 8, we deduce that T' cannot be in the interior of $E_p(K')$.

We want to show that T' belongs to the boundary of $E_p(K')$. Let T'' be the boundary torus of $E_p(K')$ which, in $E(K')$, separates T' from $\partial E(K')$. Since T' is ψ'_p-invariant, so is T''. Applying to T'' the reasoning used for T, we see that the projection in $E(K)$ of the lift to M of T'' cannot be in the interior of $E_p(K)$. However, this torus corresponds to an edge, in the tree dual to the JSJ-decomposition for $E(K)$, contained in the geodesic segment joining the vertex corresponding to $\mathrm{Fix}(\psi_p)$ to the edge corresponding to T. The submanifold of $E(K)$ associated to this geodesic segment is entirely contained in $E_p(K)$, and we conclude that $T' = T''$.

The above discussion shows in particular that T and T' have the same properties.

Considering now M, we see that Lemma 9 implies that v'_h belongs to the unique geodesic joining v_h to $e_{\widetilde{T}}$. Since T and T' have the same properties, h and h' play symmetric roles, and thus v_h must belong to the unique geodesic joining v'_h to $e_{\widetilde{T}}$. It follows that $v_h = v'_h$, and thus $\mathrm{Fix}(h)$ and $\mathrm{Fix}(h')$ belong to the same geometric piece of the JSJ-decomposition of M. Therefore, $\mathrm{Fix}(\psi_p)$ and $\partial E(K)$ belong to the same geometric piece of the JSJ-decomposition of $E_p(K)$.

The ψ_p-invariant torus T must be adjacent to the geometric component containing $\mathrm{Fix}(\psi)$ and $\partial E(K)$, or else any torus corresponding to an edge of the geodesic joining the vertex representing $\partial E(K)$ and that representing T would be ψ_p-invariant, and we would get a contradiction to Lemma 8. $\square$

Proof of Proposition 5. *Part* (i). Assume that the p-twin of K induces a partial symmetry of $E(K)$. Then $\partial E_p(K)\backslash\partial E(K)$ is not empty. Moreover, we must have $E(K)\backslash E_p(K) \subset E_q(K)$, since the winding number along nested tori is multiplicative and thus the winding number of any JSJ-torus contained in $E(K)\backslash E_p(K)$ must be of the form kp and cannot be q. In particular, $\partial E_p(K)\backslash\partial E(K) \subset \mathrm{int}(E_q(K))$.

Let $T_p \in \partial E_p(K)\backslash\partial E(K)$ be a torus and let ψ_q be the q-symmetry with non-empty fixed-point set induced on $E_q(K)$ by the q-twin (ψ_q is perhaps the restriction of a global symmetry). Since the winding number of T_p is p, its lift to the q-fold cyclic branched cover of K is connected. According to Lemma 8, T_p must separate $\partial E(K)$ from $\mathrm{Fix}(\psi_q)$. Since $\mathrm{Fix}(\psi_p)$ is connected, we see that so must be $\partial E_p(K)\backslash\partial E(K) = T_p$.

We argue now by contradiction. If the q-twin of K induces only a partial symmetry, by Corollary 5, $\mathrm{Fix}(\psi_q)$ and $\partial E(K)$ belong to the same geometric piece of the JSJ-decomposition of $E_q(K)$ and cannot be separated by T_p. This gives the desired contradiction.

Part (ii). This is a consequence of the first part of the proof of part (i), but does not use the requirement that ψ_q is a partial symmetry. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. *Part* 1. We argue by contradiction, assuming that K admits twins for three distinct, odd prime numbers p, q, r. Under this assumption, it follows that K is a non-trivial knot.

If the three twins induce rotational symmetries of the knot K, then part (i) of Theorem 3 gives a contradiction. Therefore part (i) of Proposition 5 implies that twins of orders, say q and r, induce rotational symmetries ψ_q and ψ_r of K having order q and r, respectively, while a p-twin induces only a partial rotational symmetry of $E(K)$ of order p.

Then part (ii) of Proposition 5 shows that $\partial E_p(K) \backslash \partial E(K)$ is a JSJ-torus in $E(K)$ that separates $\partial E(K)$ from both $\mathrm{Fix}(\psi_q)$ and $\mathrm{Fix}(\psi_r)$. This contradicts part (ii) of Theorem 3, which states that $\mathrm{Fix}(\psi_q)$ and $\mathrm{Fix}(\psi_r)$ must sit in the JSJ-component containing $\partial E(K)$.

Part (ii). Let K be a prime knot and let p be an odd prime number. We assume that K has at least two non-equivalent p-twins K_1 and K_2 and look for a contradiction.

If both ψ_1 and ψ_2 are rotational symmetries of order p of K, then by [**28**, Theorem 3] they are conjugate, since K is prime. This would contradict the hypothesis that the knots K_1 and K_2 are not equivalent.

Assume now that at least one symmetry, say ψ_1, is partial. Then ψ_1 and ψ_2 are rotational symmetries of order p of the submanifold $E_p(K) \subset E(K)$. Let X_0 be the geometric piece of the JSJ-decomposition of $E(K)$ containing $\partial E(K)$. Then ψ_1 and ψ_2 generate finite cyclic subgroups G_1 and G_2, respectively, of the group $\mathrm{Diff}^{+,+}(X_0, \partial E(K))$ of diffeomorphisms of the pair $(X_0, \partial E(K))$ which preserve the orientations of X_0 and of $\partial E(K)$. Moreover, one can assume that G_1 and G_2 act geometrically on X_0.

If X_0 admits a hyperbolic structure, it is a consequence of the proof of the Smith conjecture (see, for example, [**28**, Lemma 2.2]) that the subgroup of $\mathrm{Diff}^{+,+}(X_0, \partial E(K))$ consisting of restrictions of isometries of X_0 is finite cyclic. Hence $G_1 = G_2$ and, up to taking a power, $\psi_1 = \psi_2$ on X_0.

If X_0 is Seifert fibred, then it must be a cable space, since K is prime. The uniqueness of the Seifert fibration and the fact that the basis of the Seifert fibration has no symmetry of finite odd order imply that the cyclic groups G_1 and G_2 belong to the circle action $\mathbb{S}^1 \subset \mathrm{Diff}^{+,+}(X_0, \partial E(K))$ inducing the Seifert fibration of X_0; see [**28**, Lemma 2.3]. Since G_1 and G_2 have the same prime order, up to taking a power $\psi_1 = \psi_2$ on X_0.

Let h_1 and h_2 be the deck transformations on M associated to the p-fold cyclic coverings branched along K_1 and K_2, and inducing ψ_1 and ψ_2. Then by taking a suitable power, h_1 and h_2 coincide up to conjugacy on the geometric piece $\widetilde{X}_0$ of the JSJ-decomposition of M containing the preimage of K. The following lemma shows that they will coincide on M, contradicting our hypothesis. □

Lemma 10. *If the covering transformations h_1 and h_2 preserve a JSJ-piece or a JSJ-torus of M and coincide on it, then they can be chosen, up to conjugacy, to coincide everywhere.*

Proof. This is a consequence of the proofs of Propositions 3 and 4. We shall start by showing that we can always assume that there is a piece V of the JSJ-decomposition on which h_1 and h_2 coincide. For this purpose, assume that h_1 and h_2 coincide only on a JSJ-torus T. According to Lemma 7 and Remark 5, h_1 and h_2 coincide on the geometric pieces of the decomposition adjacent to T, which are also invariant. Consider now the maximal subtree Γ_1 of Γ such that the restrictions of h_1 and h_2 to the corresponding submanifold M_1 of M coincide, up to conjugacy, and such that $V \subset M_1$. Let S be a JSJ-torus for M in the boundary of M_1.

Since h_1 and h_2 coincide on M_1, the h_1-orbit and the h_2-orbit of S coincide as well and consist of either one single element $\{S\}$ or p elements $\{S, h(S) = h'(S), \dots, h^{p-1}(S) = h'^{p-1}(S)\}$. In the former case, according to Lemma 7, Γ_1 would not be maximal. In the latter case, we are precisely in the situation described in part (i) of Claim 9. Once more, Γ_1 is not maximal because one can impose the condition that h_1 and h_2 act in the same way on the p connected components $\{U, h(U) = h'(U), \dots, h^{p-1}(U) = h'^{p-1}(U)\}$ with connected boundary obtained by cutting M along the $\langle h_1, h_2\rangle$-orbit of S (see Remark 7). To see this, note that these p connected components are all homeomorphic to the exterior of a knot in $\mathbb{S}^3$, because they are freely permuted by both h and h', and thus U appears in the JSJ-decomposition of both $E(K)$ and $E(K')$. As a consequence, there is a well-defined longitude-meridian system on each boundary component, which is preserved by the action of both h and h'. By considering the quotient $M_1/\langle h\rangle = M_1/\langle h'\rangle$ as in the proof of Claim 9 (the case of an orbit with p^2 elements), one sees that there is a unique way to extend the p-fold cyclic (branched) cover defined by $h = h'$ on the total space M_1 to the manifold $N = M_1 \bigcup_{i=0}^{p-1} h^i(U)$, so the actions of h and h' must coincide on N. This contradiction shows that $M = M_1$ and the lemma is proved. □

Theorem 1 readily implies Corollary 1, which states that a prime knot is determined by three cyclic branched covers of pairwise distinct odd prime orders. For a composite knot, these data do not determine the knot but only the prime summands, up to permutation. This is the content of Corollary 2, which we prove now.

Proof of Corollary 2. First of all note that, because of the uniqueness of the Milnor–Kneser decomposition of the covers of K and K', the number of prime summands of K and K' is the same. After ditching components of K and K' that appear in both decompositions in equal numbers, we can assume that K_i is not equivalent to K'_ℓ, for all $i, \ell = 1, \dots, t$. If K and K' have three common cyclic branched covers of odd prime orders, we deduce that for each $i = 1, \dots, t$, K_i is not determined by its p_j-fold cyclic branched cover, $j = 1, 2, 3$, for it is also the p_j-fold cyclic branched cover of some K'_{i_j} not equivalent to K_i. Hence, K_i would have twins for three distinct odd prime orders, which is impossible by Theorem 1. □

We end this section with the proof of Proposition 1.

Proof of Proposition 1. First we analyse the case of a knot admitting two twins, one of which induces a partial symmetry.

PROPOSITION 6. *Let K be a prime knot admitting a p-twin K' and a q-twin K'' for two distinct odd prime numbers p and q. If K' induces a partial symmetry of K, then K' and K'' are not equivalent.*

Proof. By part (ii) of Proposition 5, $E_p(K)$ has a unique boundary component that separates $\partial E(K)$ from the fixed-point set of the q-rotational symmetry ψ induced by K''. By cutting $\mathbb{S}^3$ along $T = \partial E_p(K)$, we obtain a solid torus $V = E_p(K) \cup \mathcal{U}(K)$ containing K, and a knot exterior E_T. K admits a q-rotational symmetry ψ induced by K'' which preserves this decomposition and induces a q-rotational symmetry with trivial quotient (see Lemma 3) on E_T and a free q-symmetry $\tilde{\psi}$ on V. The covering transformation for the knot K' induces a p-symmetry φ of V with non-empty fixed-point set.

Assume now by contradiction that $K' = K''$. Since K' induces a partial symmetry of K and vice versa, $\mathbb{S}^3$ admits a decomposition into two pieces: $V' = E_p(K') \cup \mathcal{U}(K')$ and E_T. On the other hand, since K'' induces a genuine q-rotational symmetry of K, K'' admits a q-rotational symmetry ψ'' induced by K which preserves the aforementioned decomposition and induces a q-rotational symmetry with trivial quotient on E_T. Using the fact that E_T is the exterior of a prime knot (see Lemma 2) and Sakuma's result [**28**, Theorem 3], we see that the two

q-rotational symmetries with trivial quotient induced by ψ and ψ'' on E_T act in the same way. Now let E_0 be the smallest knot exterior in the JSJ-decomposition of E_T on which $\psi = \psi''$ induces a q-rotational symmetry with trivial quotient. (This is obtained by cutting E_T along the torus of the JSJ-decomposition closest to $\mathrm{Fix}(\psi)$ or, respectively, $\mathrm{Fix}(\psi'')$—and separating it from T.) Consider now the lift, denoted by $(X, \mathcal{K})$, to $(\mathbb{S}^3, K'')$ of $(E_0, \mathrm{Fix}(\psi))/\psi$. We claim that $(X, \mathcal{K}) = (V', K')$. Indeed, X contains $K'' = K'$ by construction, and its boundary is the unique torus of the JSJ-decomposition that is left invariant by the q-rotational symmetry of K''—by construction again—and that is closest to K'' (compare Corollary 5). Since $E_0/\psi = E_0/\psi''$, and a solid torus has a unique q-fold cyclic cover, we deduce that $(V', K') = (X, \mathcal{K}) = (V, K)$. In particular, the deck transformations for K and K' on their common p-fold cyclic branched cover can be chosen to coincide on the lift of $V = V'$. Lemma 10 implies that $K = K'$, contradicting the fact that K' is a p-twin. □

Let K' be a p-twin and a q-twin of K for two distinct odd prime numbers p and q. Proposition 6 implies that K' induces two rotational symmetries ψ_p and ψ_q of K with trivial quotients and orders p and q. Part (ii) of Theorem 3 shows that the fixed-point sets $\mathrm{Fix}(\psi_p)$ and $\mathrm{Fix}(\psi_q)$ lie in the JSJ-component of $E(K)$ that contains $\partial E(K)$. Then the proof of the proposition follows from the following lemma.

LEMMA 11. *Let K be a prime knot admitting two rotational symmetries ψ and φ of odd prime orders $p > q$. If the fixed-point sets of ψ and φ lie in the component that contains $\partial E(K)$, then the two symmetries commute up to conjugacy.*

Proof. Reasoning as in the proof of part (ii) of Theorem 1, one can show that ψ and φ commute on the component that contains $\partial E(K)$. Since all other components are freely permuted according to part (i) of Lemma 3, the conclusion follows as in the proof of part (i) of Claim 9. □

4. *Examples*

Examples of prime knots admitting a p-twin that induces a global rotational symmetry of order p were first constructed by Nakanishi [**23**] and Sakuma [**27**]. They considered a prime link with two trivial components whose linking number is 1. By taking the p-fold cyclic cover of $\mathbb{S}^3$ branched along the first component of the link, one again gets $\mathbb{S}^3$, and the second component lifts to a prime knot. In the same way, by taking the p-fold cyclic cover of $\mathbb{S}^3$ branched along the second component of the link, one again gets $\mathbb{S}^3$, and the first component lifts to a prime knot. The two knots thus constructed have the same p-fold cyclic branched cover by construction (see also Remark 1); moreover, by computing their Alexander polynomial they are shown to be distinct.

In [**34**, Theorem 3 and Corollary 1], Zimmerman showed that if a hyperbolic knot has a p-twin, for $p \geqslant 3$, then the p-twin induces a global symmetry and the two knots are thus obtained by the Nakanishi–Sakuma construction where the quotient link is hyperbolic and admits no symmetry that exchanges its two components.

As a matter of fact, the links considered by Nakanishi and Sakuma are hyperbolic and so are the resulting twins if p is at least 3, according to the orbifold theorem [**3**]; see also [**6**]. Note that, when $p = 2$, the situation, even in the case of hyperbolic knots, is much more complex and there are several ways to construct 2-twins of a given knot. A standard method to construct a 2-twin is via Conway mutation. Montesinos knots provide the simplest examples of hyperbolic knots admitting 2-twins that are Conway mutants. Given a Montesinos knot with at least four rational tangles, one may obtain a 2-twin by changing the order of its tangles [**21**].

Exchanging two adjacent rational tangles corresponds precisely to performing a Conway mutation along a sphere containing the two tangles. Conway mutants are 2-twins which can only appear when the corresponding 2-fold branched cover is toroidal and the fixed-point set of the covering transformation meets some essential torus. However, other types of 2-twins, which are not obtained by the Nakanishi–Sakuma construction, can arise also in the situation where the 2-fold branched cover is hyperbolic, and hence atoroidal. The simplest construction in this setting is a modification of Nakanishi–Sakuma, in which three different knots having the same 2-fold branched cover are obtained by lifting the different components of a theta-curve; see [**34**, Chapter 5] for details. What happens in this case is that the fixed-point sets of the covering transformations are not disjoint. For other basic methods of constructing 2-twins of hyperbolic, and more generally prime knots, the reader can consult [**24**] and the references cited therein.

In this section we shall see how one can construct, for each given odd prime p, two prime, non-simple knots that are p-twins, and such that the symmetries they induce are not global.

The first construction shows that the number ν of components of $\partial E_p(K)\backslash\partial E(K)$ can be arbitrarily large. This means that the situation encountered in Proposition 5(ii) is extremely special. The second construction shows that our result is indeed best possible even for prime knots with p-twins inducing partial symmetries: we shall construct prime knots admitting a p-twin inducing a partial symmetry and a q-twin inducing a global rotational symmetry.

4.1. *Knots admitting a p-twin inducing only a partial symmetry*

Assume that we are given a hyperbolic link $L = L_1 \cup \ldots \cup L_{\nu+2}$, with $\nu + 2 \geqslant 3$ components, satisfying the following requirements.

Property $*$

(1) the sublink $L_3 \cup \ldots \cup L_{\nu+2}$ is the trivial link;
(2) for each $i = 1, 2$ and $j = 3, \ldots, \nu + 2$, the sublink $L_i \cup L_j$ is a Hopf link;
(3) $\mathrm{lk}(L_1, L_2)$ is prime with p;
(4) no symmetry of L exchanges L_1 and L_2.

We shall consider the orbifold $\mathcal{O} = (\mathbb{S}^3, (L_1 \cup L_2)_p)\backslash\mathcal{U}(L_3 \cup \ldots \cup L_{\nu+2})$ which is the 3-sphere with singular set of order p the (sub)link $L_1 \cup L_2$, and with an open tubular neighbourhood of the (sub)link $L_3 \cup \ldots \cup L_{\nu+2}$ removed. $\mathcal{O}$ is hyperbolic if $p \geqslant 3$, and will represent the quotient of $\mathcal{O}_p = E_p(K) \cup \mathcal{U}(K)$ and $\mathcal{O}'_p = E_p(K') \cup \mathcal{U}(K')$ via the action of the partial p-symmetries. Indeed, to obtain $\mathcal{O}_p$ and $\mathcal{O}'_p$, respectively, take the p-fold cyclic orbifold cover of $(\mathbb{S}^3, (L_1 \cup L_2)_p)\backslash\mathcal{U}(L_3 \cup \ldots \cup L_{\nu+2})$ which desingularises L_2 and L_1, respectively. Observe that one can fix a longitude-meridian system on each boundary component of $\mathcal{O}$, induced by those of L_i, $i = 3, \ldots, \nu + 2$. Note that, because of condition 4 of Property $*$, the two orbifolds $\mathcal{O}_p$ and $\mathcal{O}'_p$ with the fixed peripheral systems are distinct.

Remark that $\mathcal{O}_p$ and $\mathcal{O}'_p$ can be obtained by the orbifold covers, analogous to those described above, of $(\mathbb{S}^3, (L_1 \cup L_2)_p)$ (which are topologically $\mathbb{S}^3$) by removing open regular neighbourhoods of the lifts of the components $L_3 \cup \ldots \cup L_{\nu+2}$. Note that these components lift to trivial components whose linking number with the lift of L_i, $i = 1, 2$, is precisely p, because of condition 2, and which form again a trivial link.

For each $j = 3, \ldots, \nu + 2$, choose a non-trivial knot exterior $E(\mathcal{K}_j)$ to be glued along the jth boundary component of $\mathcal{O}_p$ and $\mathcal{O}'_p$ in such a way that a fixed longitude-meridian system on $E(\mathcal{K}_j)$ is identified with the lift of the longitude-meridian system on the jth boundary component of $\mathcal{O}$. The underlying spaces of the orbifolds $\mathcal{O}_p \bigcup_{j=3}^{\nu+2} E(\mathcal{K}_j)$ and $\mathcal{O}'_p \bigcup_{j=3}^{\nu+2} E(\mathcal{K}_j)$ are topologically $\mathbb{S}^3$ and it is easy to see that their singular sets are connected (see condition 3). The resulting knots have the same p-fold cyclic branched cover; however, since $\mathcal{O}_p$ and $\mathcal{O}'_p$ are distinct, they are not equivalent.

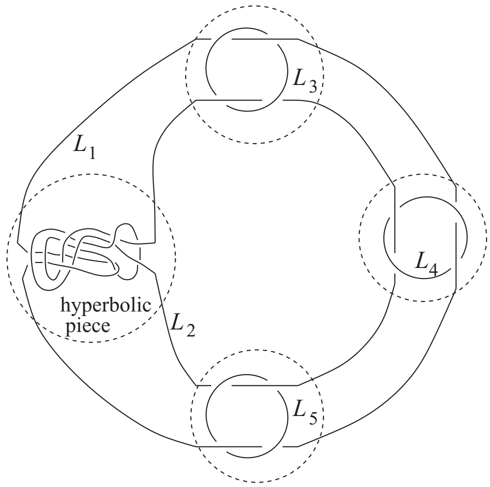

Figure 1. *The link L and its Bonahon–Siebenmann decomposition.*

Remark 9. Observe that we have just shown that the number of connected components of $\partial E_p(K)\backslash\partial E(K)$, which is precisely ν, can be arbitrarily large. Note also that if $\nu \geqslant 2$, according to Proposition 5, the knot K has no q-twins for any odd prime $q \neq p$.

We shall now prove that links with Property $*$ exist. Notice that for $\nu = 1$, links satisfying all the requirements were constructed by Zimmermann in [**33**] (see also [**24**]).

Consider the link given in Figure 1 for $\nu = 3$ (the generalisation for arbitrary $\nu \geqslant 1$ is obvious). Most conditions are readily checked just by looking at the figure, and we need only to show that L is hyperbolic and has no symmetries that exchange L_1 and L_2. To this purpose, we shall describe the Bonahon–Siebenmann decomposition of the orbifold $(\mathbb{S}^3, (L)_2)$, where all components have $\mathbb{Z}/2\mathbb{Z}$ as local group. The decomposition consists of one single hyperbolic piece (see Figure 1) and either $\nu+1$ Seifert fibred pieces if $\nu \geqslant 2$ or one Seifert fibred piece if $\nu = 1$. Since the Seifert fibred pieces contain no incompressible torus, the hyperbolicity of L follows.

Note now that every symmetry of L must leave invariant the unique hyperbolic piece of the decomposition. This piece is obtained by quotienting the hyperbolic knot 10_{155} via its full symmetry group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and thus has no symmetries (for more details, see [**24**]), so we conclude that the components L_1 and L_2 are non-exchangeable.

4.2. *Knots admitting a p-twin inducing a partial symmetry and a q-twin inducing a global symmetry*

Let $\mathcal{K}$ be a hyperbolic knot admitting a p-twin and a q-twin; the twins of $\mathcal{K}$ induce global symmetries, so that $\mathcal{K}$ admits a p- and a q-rotational symmetry with trivial quotient (see [**33**], where a method of constructing hyperbolic knots with two twins is described). Remove a tubular neighbourhood of the axis of the symmetry of order q (note that the two symmetries have disjoint axes), and use the resulting solid torus V to perform Dehn surgery on the exterior E of the $(2, q)$-torus knot. Denote by K the image of $\mathcal{K}$ after surgery. We require that:

(1) the resulting manifold is $\mathbb{S}^3$;
(2) the q-rotational symmetry of E and the restriction of the q-rotational symmetry of $\mathcal{K}$ to V give a global q-rotational symmetry of K;
(3) the q-rotational symmetry of K has trivial quotient.

Note that the last requirement can be met by choosing the longitude appropriately when satellising, as illustrated in Figure 2. We claim that K admits a q-twin, K'', and a p-twin, K'.

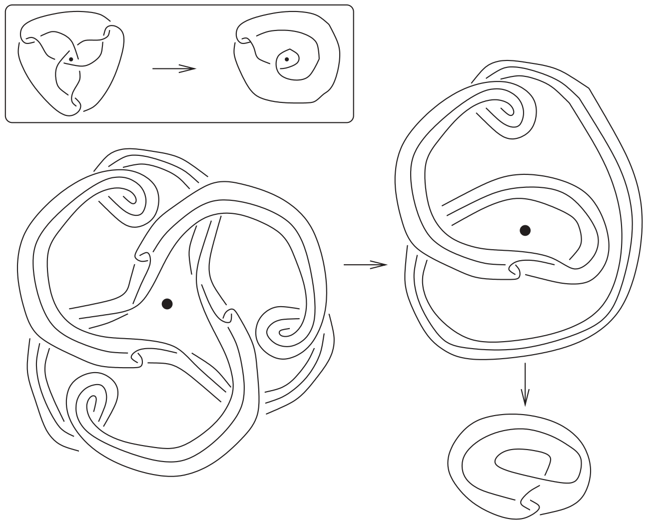

FIGURE 2. *Satellising so that the induced rotation has trivial quotient.*

K'' is obtained by the standard method described in Remark 1. Note that $K \neq K''$, for the roots of the JSJ-decompositions of the exteriors of K and K'' are hyperbolic and Seifert fibred, respectively. To construct K', consider the p-twin $\mathcal{K}'$ of $\mathcal{K}$ and let V' be the solid torus obtained by removing the axis of the q-rotational symmetry of $\mathcal{K}'$. Note that V and V' have a common quotient obtained by taking the space of orbits of the p-rotational symmetries; however V and V' are different orbifolds by construction. Fix a longitude-meridian system on V (the one used for the surgery): by first quotienting and then lifting it, we get a longitude-meridian system on V' that must be used to perform surgery along a copy of E. The image of $\mathcal{K}'$ after the surgery will be K'. Note that, when taking the p-fold cyclic branched covers of K and K', the hyperbolic orbifolds V and V' lift to the same manifold by construction, while the Seifert fibred part lifts, in both cases, to p copies of E. Again by construction, the gluings are compatible and the two covers coincide. It is also evident that K' can only induce a partial symmetry of K, and the claim is proved.

REMARK 10. Note that according to Proposition 6, the p-twins and q-twins obtained in this construction cannot be equivalent.

5. *Homology spheres as cyclic branched covers*

By the proof of the Smith conjecture, Corollary 3 is true for the 3-sphere $\mathbb{S}^3$. So from now on, we assume that the integral homology sphere M is not homeomorphic to $\mathbb{S}^3$. Then by [**2**, Theorem 1], M can be a p_i-fold cyclic branched cover of $\mathbb{S}^3$ for at most three pairwise distinct odd prime numbers p_i. Moreover, if M is irreducible and is the p_i-fold cyclic branched cover of $\mathbb{S}^3$ for three pairwise distinct odd prime numbers p_i, then the proof of [**2**, Corollary 1(i)] shows that for each prime p_i, M is the p_i-fold cyclic branched cover of precisely one knot. Since a knot admits at most one p-twin for an odd prime integer p, we need only to consider the case when the irreducible integral homology sphere M is the branched cover of $\mathbb{S}^3$ for precisely two distinct odd primes, say p and q. Moreover, [**2**, Corollary 1(ii)] shows that M has a non-trivial JSJ-decomposition.

Looking for a contradiction, we can assume that, for each prime, M is the branched covering of two distinct knots with covering transformations ψ, ψ' of order p and φ, φ' of order q.

We shall say that an orientation-preserving diffeomorphism of M is a *rotation* if it has finite order and non-empty and connected fixed-point set; in particular, the deck transformation of a cyclic branched cover of a knot is a rotation.

If each rotation of order p commutes with each rotation of order q up to conjugacy, then the contradiction follows from the following claim, which is an easy consequence of Sakuma's result [**28**, Theorem 3] (see [**2**, Claim 8]).

Claim 11. *Let $n \geqslant 3$ be a fixed odd integer. Let ρ be a rotation of an irreducible manifold M such that $M/\langle\rho\rangle = \mathbb{S}^3$. All the rotations of M of order n which commute with ρ are conjugate in* Diff(M) *into the same cyclic group of order n.*

Otherwise, consider the subgroup $G = \langle\psi, \psi', \varphi, \varphi'\rangle$ of diffeomorphisms of M. According to the proof of [**2**, Proposition 4], each rotation of order p commutes with each rotation of order q up to conjugacy, unless the induced action of G on the dual tree of the JSJ-decomposition for M fixes precisely one vertex corresponding to a hyperbolic piece V of the decomposition and $\{p, q\} = \{3, 5\}$. In this case, one deduces as in the proof of [**2**, Corollary 1(ii)] that the restrictions of ψ and ψ' (and, respectively, φ and φ') coincide up to conjugacy on V. Then the desired contradiction follows from Lemma 10, which implies that ψ and ψ' (and, respectively, φ and φ') coincide up to conjugacy on M.

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