IN HOW MANY WAYS CAN A 3-MANIFOLD BE THE CYCLIC BRANCHED COVER OF A KNOT IN $\mathbf{S}^3$?

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1. Introduction

Given a knot K in the 3-sphere and an integer $n \geq 2$ it is well-known that one can construct a closed, connected, and orientable 3-manifold, $M(K,n)$, as follows: $M(K,n)$ is obtained by taking the total space of the n-fold cyclic cover of the exterior of K and by Dehn-filling its boundary so that the meridian of the added solid torus is mapped to the lift of the meridian of K. The manifold $M(K,n)$ is the total space of the n-fold cyclic cover of $\mathbf{S}^3$ branched along K. By abuse of language $M(K,n)$ is called *the cyclic branched cover of* K or *the cyclic cover of the* 3*-sphere branched over* K (cf. [17] for a different construction of $M(K,n)$).

It is not difficult to see that it is not possible to recover every closed connected orientable 3-manifold in this way. Possibly, the simpliest example of a manifold that is not of the form $M(K,n)$ for any K or n is $\mathbf{S}^2 \times \mathbf{S}^1$. Other examples are hyperbolic 3-manifolds whose orientation-preserving isometry groups are trivial, or any irreducible 3-manifold with non-trivial JSJ-decomposition such that the dual graph of the decomposition is not a tree. A key point to see that these manifolds are not cyclic branched covers of knots is the fact that in general the geometric properties of K are reflected in those of $M(K,n)$. Notably we have: K is a composite knot if and only if $M(K,n)$ is not prime; if K is a torus knot, $M(K,n)$ is Seifert fibred; if K is hyerbolic so is $M = M(K,n)$ provided $n \geq 3$ and M is not the 3-fold cyclic branched cover of the figure-eight knot 4_1, while K is always hyperbolic if so is $M(K,n)$; and if K is a prime satellite knot, then $M(K,n)$ is irreducible with a non trivial JSJ-decomposition dual to a tree.

Now, given any closed, connected, and orientable 3-manifold M, a natural question is to understand in how many manners M can be obtained as an $M(K,n)$ for some knot K and some integer $n \geq 2$. Of course, this question can be rephrased in more precise terms in different ways.

Possibly the most basic problem in this context is to determine the size of the set of knots K such that $M = M(K,n)$ for fixed M and n. This problem has been studied for a long time and is now well-understood. It is, for instance, known that for any $n \geq 2$ and any $N \in \mathbb{N}$ there is a non-prime manifold M which is the n-fold cyclic branched cover of at least N knots; the basic idea for constructing such manifolds can be found in [19]. For $n = 2$, it is also possible to find irreducible manifolds that are 2-fold branched covers of an arbitrary number of knots: this is the case, for example, of certain Seifert-fibred 3-manifolds that are double branched covers of Montesinos knots [12]. On the other hand, a hyperbolic manifold M is the n-fold cyclic branched cover of at most two knots if $n > 2$ [20], and of at most nine if $n = 2$ [14]. Similarly, any prime manifold is the n-fold cyclic branched cover of at most two knots, provided n is an odd prime [2].

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Conversely, given a manifold M and a quotient knot K one may want to bound the set of orders $n \geq 2$ for which $M = M(K, n)$. This question was analysed by A. Salgueiro in his Ph.D. thesis. It turns out that in many cases (e.g. if the JSJ-decomposition of the exterior of K contains a hyperbolic piece) there is at most one such n [18].

Here we shall be mainly concerned with the following two sets:

$$\mathcal{K}(M) = \{K \subset \mathbf{S}^3 \mid \exists n \geq 2,\ M = M(K, n)\}$$

$$\mathcal{O}(M) = \{n \in \mathbb{N},\ n \geq 2 \mid \exists K \subset \mathbf{S}^3,\ M = M(K, n)\}$$

that is, the set of knots that are cyclically branched covered by M and the set of orders of such cyclic branched covers.

The problem of understanding the size of these sets has received some attention in recent years [15, 3, 1]. The goal of this note is to provide a review of the different results known. We shall start by discussing several examples of manifolds M whose associated sets $\mathcal{K}(M)$ and $\mathcal{O}(M)$ exhibit different behaviours. These exemples will suggest in which context it makes sense to ask whether the cardinalities of the sets can be bounded independently of M. We have already seen that one cannot hope to find a universal bound for the cardinality of $\mathcal{K}(M)$, however we shall see that by imposing constraints on the manifolds, that is, by requiring that the manifolds considered are hyperbolic, a bound independent of the hyperbolic manifold M does exist [1].

As for the set $\mathcal{O}(M)$, again we shall see that its cardinality cannot be universally bounded. On the other hand it is possible to bound the cardinality of

$$\Pi(M) = \{p \in \mathcal{O}(M) \mid p \text{ prime}\}$$

the set of orders of covers that are prime, independently of M a closed, orientable and connected 3-manifold not homeomorphic to $\mathbf{S}^3$ [1].

The paper is organised as follows. In Section 2 we shall present several examples of manifolds M for which the sets $\mathcal{K}(M)$ and $\mathcal{O}(M)$ exhibit various behaviours. This will allow to restate and contextualise our initial question more appropriately. In Section 3 we will present bounds for the cardinality of $\mathcal{K}(M)$ and $\mathcal{O}(M)$ under the assumption that M is hyperbolic: here the proof relies on the study of finite group actions with certain elements acting in a prescribed way. In Section 4 we will see what happens if we impose homological restrictions on the manifold M. Finally, in Section 5 we will discuss the case of arbitrary manifolds, by distingushing the case where the manifolds are prime or not and, when they are prime, the case where they are Seifert fibred or admit a non trivial JSJ-decomposition.

2. Examples

In this section we shall analyse the structure of the sets $\mathcal{K}(M)$ and $\mathcal{O}(M)$ for some particular choices of M.

Let us start with $M = \mathbf{S}^3$. It follows from the positive solution to the Smith conjecture [13] that the 3-sphere is the n-fold cyclic branched cover of the trivial knot for all $n \geq 2$ and that it is the cyclic branched cover of no other knot, so that $\mathcal{K}(\mathbf{S}^3)$ contains a single element, the trivial knot, while $\mathcal{O}(\mathbf{S}^3)$ is infinite and contains all integers ≥ 2. The case of the 3-sphere is rather special. Indeed, it follows from the orbifold theorem that if M is not homeomorphic to $\mathbf{S}^3$ then $\mathcal{O}(M)$ is finite [7] but, of course, its maximum as well as its cardinality depend on M.

The maximum can obviously be arbitrarily large, nonetheless one might ask whether the cardinality of $\mathcal{O}(M)$ can be bounded above independently of $M \neq \mathbf{S}^3$. Unfortunately this is not true in general: for any $N > 0$, there are a Seifert fibred manifold M_N, pairwise distinct torus knots K_i, $i = 1, \dots N$, and pairwise distinct integers $n_i \geq 2$, $i = 1, \dots N$, such that $M_N = M(K_i, n_i)$ for all $i = 1, \dots N$. Moreover, one can choose M_N to be a circle bundle over a surface with Euler class ± 1 (see [1, Proposition 14]). This shows in particular the following fact:

Proposition 2.1. *There are irreducible manifolds M, different from $\mathbf{S}^3$, for which the cardinalities of the two sets $\mathcal{K}(M)$ and $\mathcal{O}(M)$ are both arbitrarily large.*

Of course, as it was already remarked in the Introduction, double branched covers of Montesinos knots already provide examples of irreducible manifolds M for which the set $\mathcal{K}(M)$ can be arbitrarily large. Note that they can be chosen so that, on the other hand, $\mathcal{O}(M) = \{2\}$: this is the case, for instance, for the Brieskorn $\mathbb{Z}$-homology spheres of type $\Sigma(2, p_1, p_2, \dots, p_k)$, where $k \geq 3$ and the p_i are pairwise distinct odd prime numbers. In this case M is the double cover of $k!/2$ distinct Montesinos knots.

Remark 2.2. It must be stressed that, for $N > 3$, the Seifert manifolds appearing in Proposition 2.1 cannot be rational-homology spheres (see Proposition 4.4). Since a manifold of the form $M(K, p)$, where p is a prime, is a $\mathbb{Z}/p$-homology sphere, this implies, in particular, that the set $\mathcal{O}(M)$ does not contain any prime integer in this case.

If we restrict our attention to cyclic branched coverings of odd prime order, an analysis of the Seifert invariants gives the following result which is [1, Proposition 14].

Proposition 2.3. *If $M \neq \mathbf{S}^3$ is a Seifert fibred manifold, then $\mathcal{O}(M)$ contains zero, one, or three odd prime numbers. In particular the cardinality of $\Pi(M)$ is always bounded by 4 in this case. Moreover, for each odd order in $\Pi(M)$, M is the cyclic branched cover of precisely one knot.*

Note that for all three odd primes $p < q < r$ the Brieskorn $\mathbb{Z}$-homology sphere $M = \Sigma(p, q, r)$ is such that $\mathcal{O}(M) = \{2, p, q, r\}$ and $\mathcal{K}(M)$ contains three torus knots ($T(p, q)$, $T(p, r)$, $T(q, r)$) and a hyperbolic bretzel knot (of type (p, q, r)). Observe also that if M is the Brieskorn sphere of type $\Sigma(2, p, q)$, M is the double branched cover of the torus knot $T(p, q)$ and of the bretzel knot of type $(2, p, q)$. These two knots may coincide (for instance, if $M = \Sigma(2, 3, 5)$ is the Poincaré sphere) or not (for instance, when p and q are large enough so that the bretzel knot is hyperbolic).

Remark 2.2 above implies that if $\Pi(M) \neq \emptyset$, then M is a $\mathbb{Q}$-homology sphere, so that Proposition 2.3 follows from Proposition 4.4.

3. Bounds for hyperbolic manifolds

Let M be a closed, connected, and orientable 3-manifold. There exist a knot K and an integer $n \geq 2$ such that $M = M(K, n)$ if and only if M admits a cyclic group H of order n of orientation-preserving diffeomorphisms such that each non-trivial element of H has the same non-empty and connected fixed-point set, and the space of orbits M/H is the 3-sphere. The group H is precisely the group of deck transformations of the cyclic branched covering $M \longrightarrow M/H$. Remark that the given conditions on H are equivalent to requiring that H is generated by an element φ of order n such that (i) $Fix(\varphi) = Fix(\varphi^k) = \mathbf{S}^1$, for all $0 < k < n$, and (ii) the space of orbits of φ is $\mathbf{S}^3$. We call an orientation-preserving diffeomorphism of M satisfying (i) a *rotation* of order n. If, in addition, φ satisfies (ii), then we shall say that the rotation is *hyperelliptic*.

Assume that $M = M(K_1, n_1) = M(K_2, n_2)$ with associated groups H_1 and H_2, respectively. We have that the pairs (K_1, n_1) and (K_2, n_2) are the same (that is, the knots are isotopic and the integers are equal) if and only if H_1 and H_2 are conjugate as subgroups of the group of orientation-preserving diffeomorphisms of M.

The initial problem of determining in how many ways a manifold can be presented as the cyclic branched cover of a knot can thus be answered by determining how many conjugacy classes of subgroups generated by hyperelliptic rotations are contained in the group of orientation-preserving diffeomorphisms of M.

If M is hyperbolic, then each finite group of orientation-preserving diffeomorphims is conjugate inside the (finite) group of orientation-preserving isometries of M. For the class of hyperbolic manifolds our geometric problem becomes a question concerning finite group actions: Given a finite group G acting on a manifold by orientation-preserving diffeomorphisms, is it possible to bound the number of conjugacy classes of cyclic subgroups of G generated by hyperelliptic rotations?

Since every finite group G is the group of orientation-preserving isometries of some hyperbolic 3-manifold [9], no restriction on G can be given a priori. On the other hand, the property that an element φ of G acts as a hyperelliptic rotation implies that the normaliser of $\langle\varphi\rangle$ must have a very specific structure (see, for instance, [1, Section 2]). Moreover, if p is an odd prime dividing the order of φ, then the structure of the Sylow p-subgroup of G and of its normaliser is also constrained; notably, such Sylow p-group must be either cyclic or the direct sum of two cyclic groups, and its centraliser has index 1, 2 or 4 in its normaliser. The different possibilities are related to the type of symmetries enjoyed by the associated quotient knot (see again [1, Section 2] for more details). These remarks allow to translate the initial question in purely group-theoretical terms.

A further constraint imposed by the geometry on the algebra is the fact that no solvable subgroup of a group G acting by orientation-preserving diffeomorphisms on a manifold different from $\mathbf{S}^3$ can contain more than three conjugacy classes of cyclic subgroups generated by hyperelliptic rotations and of orders that are not powers of 2 [1, Theorem 4].

In [1] the following group theoretical result was obtained:

Theorem 3.1. *Let G be a finite group of orientation-preserving diffeomorphisms of a closed, connected and orientable* 3*-manifold M. Then G can contain at most six conjugacy classes of cyclic subgroups generated by hyperelliptic rotations whose orders are not powers of* 2*.*

Concerning the cyclic subgroups generated by hyperelliptic rotations of order a power of 2, up to conugacy they are all contained in a Sylow 2-subgroup. The number of their conjugacy classes is bounded by nine according to work of Mecchia [10] and Reni [14].

It follows from the above results and discussion that, for hyperbolic manifolds M, there is a universal bound on the number of ways M can be presented as a branched cover of a knot:

Theorem 3.2. *Let M be a hyperbolic manifold, then we have that the cardinality of $\mathcal{K}(M)$ is bounded above by fifteeen and that of $\mathcal{O}(M)$ by nine.*

At this point it is not known whether these bounds are sharp. Examples of hyperbolic manifolds that are double branched covers of nine knots are known to exist, but the existence is not proved by an explicit construction [6]. One can also easily construct hyperbolic manifolds that are cyclic branched covers of three distinct knots with branching orders > 2 that are pairwise coprime [16]. If hyperbolic manifolds that are branched covers of at least four knots with orders > 2 exist, then they must admit finite group actions where the groups acting must have an imposed structure, see [1, Proposition 11] for more details.

4. Bounds under homological conditions

One of the difficulties in proving Thoerem 3.1 comes from the fact that every finite group acts as a group of orientation-preserving isometries of some hyperbolic 3-manifold. An essential ingredient of the proof is the classification of finite simple groups. This is needed to obtain an explicit upper bound. To conclude on the existence of a universal bound, though, it is sufficient to know that only a finite number of sporadic finite simple groups (that is, simple groups that are neither alternating nor of Lie type) exist.

Even if we were only interested in bounding the cardinality of $\Pi(M)$, we would not have any a priori restriction on the finite groups G acting on M. Indeed, every finite group G acts by orientation-preserving isometries on some $\mathbb{Q}$-homology sphere (see [4] for the case of free actions and [1, Section 10] for actions where certain elements of G fix some points). On the other hand, if M is required to be a $\mathbb{Z}$-homology sphere, then it was proved in [11] that there are restritions on the finite groups that can act by orientation-preserving diffeomorphisms on M:

Theorem 4.1. *Let G be a finite group of orientation-preserving diffeomorphisms of a $\mathbb{Z}$-homology sphere M. Then either G is solvable or it is isomorphic to one of the following groups:*

$$\mathbf{A}_5, \quad \mathbf{A}_5 \times \mathbb{Z}/2, \quad \mathbf{A}_5^* \times_{\mathbb{Z}/2} \mathbf{A}_5^*, \quad \mathbf{A}_5^* \times_{\mathbb{Z}/2} C,$$

where $\mathbf{A}_5$ is the dodecahedral group (alternating group on 5 elements), $\mathbf{A}_5^*$ is the binary dodecahedral group (isomorphic to $SL_2(5)$), C is a solvable group with a unique involution and $\times_{\mathbb{Z}/2}$ denotes a central product.

It follows at once that if M is a hyperbolic $\mathbb{Z}$-homology sphere then its group of isometries contains at most three conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of orders not a power of 2, so that the cardinalities of $\mathcal{K}(M)$ and $\mathcal{O}(M)$ are bounded above by twelve and six in this case. Under the same requirement on the homology, a weaker result that is nonetheless valid for all manifolds, not necessarily hyperbolic ones, is the following (see [3, Theorem 1]):

Theorem 4.2. *Assume that $M \neq \mathbf{S}^3$ is a $\mathbb{Z}$-homology sphere. Then the cardinality of $\Pi(M) \setminus \{2\}$ is at most three and there are manifolds for which the bound is attained. For instance, the Brieskorn spheres $M = \Sigma(p, q, r)$, where $2 < p < q < r$ are prime, are such that $\Pi(M) = \{2, p, q, r\}$.*

The proof of Theorem 4.2, and those of Theorem 5.1 and Corollary 5.2 follow the same strategy that will be briefly explained in the next section.

The situation for finite groups acting by orientation-preserving diffeomorphisms on $\mathbb{Z}/2$-homology spheres is somehow intermediate between that seen for $\mathbb{Z}$-homology spheres and that for $\mathbb{Q}$-homology spheres, in the sense that it is still possible to list all the non-solvable groups that are liable to act on them, but the list is larger [11]. In order to establish the list, Mecchia and Zimmermann rely on the Gorenstein-Harada classification of finite simple groups of sectional 2-rank bounded by 4 [5].

Theorem 4.3. *Let G be a finite group of orientation-preserving diffeomorphisms of a $\mathbb{Z}/2$-homology sphere. Then either G is solvable or G can be decomposed in the following way:*

$$1 \longrightarrow O \longrightarrow G \longrightarrow G/O \longrightarrow 1$$

and

$$1 \longrightarrow H \longrightarrow G/O \longrightarrow K \longrightarrow 1$$

where O is the maximal normal subgroup of G of odd order (which is characteristic in G), K is solvable (in fact either abelian or a 2-fold extension of an abelian group) and H belongs to the following list:

$$PSL_2(q), \quad PSL_2(q) \times \mathbb{Z}/2, \quad SL_2(q) \times_{\mathbb{Z}/2} C, \quad \hat{\mathbf{A}}_7, \quad SL_2(q) \times_{\mathbb{Z}/2} SL_2(q'),$$

where C is solvable with a unique element of order 2, $q, q' > 4$ are odd prime powers, $\hat{\mathbf{A}}_7$ denotes the unique perfect central extension of the alternating group on 7 elements, and $\times_{\mathbb{Z}/2}$ denotes a central product over $\mathbb{Z}/2$.

It turns out that the bounds obtained on the cardinality of $\mathcal{K}(M)$ and $\mathcal{O}(M)$ for hyperbolic $\mathbb{Z}/2$-homology spheres M, and on $\Pi(M)$ for arbitrary $\mathbb{Z}/2$-homology spheres are the same as those that one gets without any homological restriction.

We have just observed that the problem of bounding the number of ways a hyperbolic manifold can be presented as a cyclic branched cover of a knot is not simplified if one assumes that the manifold is a $\mathbb{Q}$-homology sphere. Moreover, the bounds one obtains are the same. In contrast with the hyperbolic case, Seifert fibred $\mathbb{Q}$-homology spheres M are much better behaved than arbitrary Seifert fibred manifolds, for which the cardinalities of both $\mathcal{K}(M)$ and $\mathcal{O}(M)$ cannot be bounded. Indeed, we have the following result which was stated without proof in [1].

Proposition 4.4. *If $M \neq \mathbf{S}^3$ is a Seifert fibred $\mathbb{Q}$-homology sphere, then there are at most three pairs (K, n), with $n > 2$, such that $M = M(K, n)$.*

Proof. Assume $M \neq \mathbf{S}^3$ is a Seifert manifold. According to the discussion in the Introduction, if $M = (K, n)$ and $n > 2$ then K is a torus knot unless $M = M(4_1, 3)$. In this latter case, one can check directly that M admits only this presentation as the cyclic branched cover of a knot.

Assume now that K is the torus knot $T(a, b)$, where $a \geq 2$ and $b \geq 2$ are two coprime integers, and let $n \geq 2$ be an integer. Let α be the GCD of a and n, and β that of b and n. In this case, a combinatorial analysis of the Seifert invariants (see, for instance, [8]) shows that $M(T(a, b), n)$ is a Seifert fibred space with base of genus $(\alpha-1)(\beta-1)/2$ admitting β fibres of order a/α, α fibres of order b/β, and one fibre of order $n/(\alpha\beta)$. Note that these fibres may not be exceptional. Note also that in all cases, the Seifert fibration of M is unique.

Now, if M is $\mathbb{Q}$-homology sphere, then necessarily $g = 0$, that is, at least one between α and β is equal to 1.

Assume that $\alpha = \beta = 1$. In this case the manifold has three exceptional fibres of orders a, b, and n and necessarily $M = M(T(a,b),n) = M(T(a,n),b) = M(T(n,b),a)$ are all the possibilities.

Else, without loss of generality, we can assume that $\alpha = 1$ and $\beta > 1$. So in this case we have that there must be $\beta > 1$ exceptional fibres of order a, and at most two other exceptional fibres. This means that M must be of the form $M = M(T(a, x\beta), y\beta)$, where if M has two more exceptional fibres, then x and y are their orders; if M has only one other exceptional fibre of order $c \geq 2$ then $\{x, y\} = \{1, c\}$; and if M does not have any other exceptional fibre, then $x = y = 1$. This shows that in this situation M is the cyclic branched cover of at most two torus knots, which concludes the proof. $\square$

Observe again that, even in the case of Seifert fibred $\mathbb{Q}$-homology spheres M, the number of pairs of the form $(K, 2)$ such that $M = M(K, 2)$ can be arbitrarily large, so that there is no universal bound on the cardinality of $\mathcal{K}(M)$ while that of $\mathcal{O}(M)$ is bounded by four, just like in the case of $\mathbb{Z}$-homology spheres, and clearly the bound is sharp.

5. Weaker bounds for non hyperbolic manifolds

As it was already remarked, Proposition 2.1 shows that one cannot hope to bound the cardinality of $\mathcal{O}(M)$, independently of $M \neq \mathbf{S}^3$. However, it is indeed possible to bound the cardinality of $\Pi(M)$ [1]:

Theorem 5.1. *If $M \neq \mathbf{S}^3$ is an irreducible manifold, then $\Pi(M) \setminus \{2\}$ contains at most six prime numbers. Moreover, the cardinality of*

$$\{K \in \mathcal{K}(M) \mid M = M(K, n),\ n \in \Pi(M) \setminus \{2\}\}$$

is at most six.

The proof of Theorem 5.1 follows directly from Theorem 3.2 for hyperbolic manifolds and from Proposition 4.4 for Seifert fibred ones. For manifolds admitting a non trivial JSJ-decomposition, the strategy is to understand the action of the group of orientation-preserving diffeomorphisms generated by hyperelliptic rotations, up to conjugacy. This group is not finite in general, but it preserves the JSJ-decomposition. Since the dual graph of the decomposition is a tree, there is at least one geometric piece that is left invariant. The idea is then to show that, up to taking different representatives in their conjugacy class, the hyperelliptic rotations commute on the fixed geometric piece. The next step is to show that one can adjust further the conjugation so that the hyperelliptic rotations commute on the entire manifold.

The proof in the case of manifolds with non-trivial JSJ decomposition is inspired from techniques developped in [2] and already exploited in [3], and relies heavily on the fact that the hyperelliptic rotations are required to have prime order > 2. At this point we do not know if the hypotheses of Theorem 5.1 on the type of orders can be relaxed and replaced by the condition that M is a $\mathbb{Q}$-homology sphere.

Consider a non prime manifold M admitting a hyperelliptic rotation φ. The equivariant sphere theorem assures that φ induces a hyperelliptic rotation of the same order as φ on each prime summand of M; in particular all summands of M are irreducible. This fact together with Theorem 5.1 has the following consequence.

Corollary 5.2. *The set $\Pi(M) \setminus \{2\}$ contains at most six prime numbers provided $M \neq \mathbf{S}^3$.*

Of course, it is not possible to give a bound on the set of knots that are covered by an arbitrary manifold M even if we only consider covers of odd prime orders.

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