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Topology and its Applications 124 (2002) 85–101

TOPOLOGY AND ITS APPLICATIONS

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Non-equivalent hyperbolic knots

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Received 9 March 2001; received in revised form 3 July 2001

Abstract

We construct, for each integer $n \geqslant 3$, pairs of non-equivalent hyperbolic knots with the same 2-fold and n-fold cyclic branched covers. We also discuss necessary conditions for such pairs of knots to exist.

MSC: primary 57M25; secondary 57M12, 57M50

Keywords: Hyperbolic knots; Cyclic branched covers; Orbifolds; Bonahon–Siebenmann decomposition

1. Introduction

Let K be a knot in $\mathbb{S}^3$. Denote by $M(n, K)$, $n \geqslant 2$, the total space of the n-fold cyclic cover of $\mathbb{S}^3$ branched along K; by abuse of language we shall also refer to $M(n, K)$ as the n-fold (cyclic branched) cover of K. $M(n, K)$ is obviously an invariant for K. The problem of understanding whether and to what extent $M(n, K)$ is a "good" invariant for K has been widely studied. It is easy to see that n-fold covers are not "good" invariants for composite knots. On the other hand, for prime knots, a partial positive answer to this problem was given by Kojima. In [12], he proves that given two prime knots, K and K', there exists a constant C such that, if there exists an integer $n \geqslant C$ with the property that $M(n, K)$ is homeomorphic to $M(n, K')$, then K and K' are (*weakly*) *equivalent*, i.e., the pairs $(\mathbb{S}^3, K)$ and $(\mathbb{S}^3, K')$ are homeomorphic. However, for each fixed $n \geqslant 2$, there exist pairs of non-equivalent prime knots K_n and K'_n such that $M(n, K_n)$ is homeomorphic to $M(n, K'_n)$ (see [18,15]).

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[1] Partially supported by a grant of Consiglio Nazionale delle Ricerche.

PII: S0166-8641(01)00240-1

In [2] Boileau and Flapan asked whether there exists an $\bar{n} \geqslant 3$ with the property that, if $M(n, K)$ is homeomorphic to $M(n, K')$ for all $2 \leqslant n \leqslant \bar{n}$, where K and K' are prime knots, then K and K' are necessarily equivalent. The question is known to have positive answer in the case of hyperbolic knots (see [24]) and it is an easy exercise to prove that it has positive answer in the case of torus knots, while nothing is yet known in the case of arbitrary prime knots. Recall that a knot K is hyperbolic if the interior of its complement $\mathbb{S}^3 - \mathcal{U}(K)$ admits a complete hyperbolic structure of finite volume (here $\mathcal{U}(K)$ represents a tubular neighbourhood of K). In fact, the positive answer, in the case of hyperbolic knots, follows from a more general result due to Zimmermann who studied the determination up to equivalence of hyperbolic knots by means of couples of cyclic branched covers in [23, 24]. In the case when one of the two cyclic branched covers is the 2-fold one, however, Zimmermann gave only a partial answer to this problem; more precisely, he proved that the hyperbolic knots of a particular class (the π-hyperbolic ones) are determined, up to equivalence, by their 2-fold and n-fold cyclic branched covers if n is even, but it was not clear what happens when n is odd or if the knot is not π-hyperbolic. Recall that a knot is $2\pi/m$-hyperbolic, $m \geqslant 2$, if the orbifold, whose underlying topological space is $\mathbb{S}^3$ and whose singular set of order m is K, is hyperbolic (for basic definitions about orbifolds see [20]). Equivalently, K is $2\pi/m$-hyperbolic if its m-fold cyclic branched cover is a hyperbolic manifold and the group of deck transformations acts by isometries which fix a closed geodesic. In [16] and in the present paper we aim to completely resolve these questions. In [16] we proved that, for any given $n \geqslant 3$, a Conway irreducible hyperbolic knot is determined, up to equivalence, by its 2-fold and n-fold cyclic branched covers. We recall that a knot K is Conway reducible if it admits a Conway sphere, i.e., a sphere S^2 which meets K in four points and such that $S^2 - \mathcal{U}(K)$ is incompressible and boundary incompressible in the complement $\mathbb{S}^3 - \mathcal{U}(K)$ of K (for a general introduction to knot theory see [17,6]). The class of Conway irreducible hyperbolic knots contains in particular that of π-hyperbolic ones. Here we restrict our attention to the Conway reducible hyperbolic knots and we show that in this case the situation is totally different from the Conway irreducible case (and also from the case when $m > 2$). More precisely, in Section 2, we prove

Theorem 1. *Let $n \geqslant 3$. There exist pairs of non-equivalent hyperbolic knots with the same 2-fold and n-fold cyclic branched covers.*

This theorem gives in particular examples of hyperbolic knots with the same 2-fold and 3-fold cyclic branched covers, answering a question put by Boileau (see Kirby's list of open problems [11, Problem 1.75B]).

It is worth observing that the only known examples of knots which are not determined by their n-fold and m-fold cyclic branched covers for two given numbers $n > m \geqslant 2$ (see [23] for $m > 2$ and Section 2 for $m = 2$) are Conway reducible knots and this must be the case if $m = 2$ (see [16]). However, certain Conway reducible hyperbolic knots are determined by their 2-fold and n-fold cyclic branched covers for any given $n \geqslant 3$, as are the Conway irreducible hyperbolic ones. This is the case with the Montesinos knots, for instance (other examples will be given in Section 3).

To prove Theorem 1 we make an extensive use of the existence of a canonical decomposition for orbifolds into geometric pieces. This decomposition was studied by Bonahon and Siebenmann in [5]. We also use some results due to Zimmermann [24] whose proofs are based on certain algebraic considerations and on the Smith conjecture [14]. Similar methods will be applied in this paper as well.

In Sections 3 and 4 we shall state some results concerning the determination of hyperbolic knots via their cyclic branched covers. In particular, in Section 3 we find necessary conditions for a hyperbolic knot to fail to be determined among all knots (equivalently among hyperbolic knots as explained in Section 3) by its 2-fold and n-fold cyclic branched covers. We shall see, for instance, that the Jaco–Shalen–Johannson decomposition of its 2-fold cyclic branched cover must contain a hyperbolic piece. These results are summarized in Proposition 1 and show that the construction given in Section 2 is, in some sense, unique. The proofs of the results are rather technical and use methods similar to those of [16]. For this reason they will be only sketched or even omitted.

2. Proof of Theorem 1

In this section we shall construct non-equivalent hyperbolic knots with the same 2-fold and n-fold cyclic branched covers. We start by fixing some notation and terminology.

p_* denotes the projection of the covering induced by the deck transformation $*$ (i.e., $*$ generates the group of deck transformations);

$Fix(*)$ denotes the fixed-point set of the map $*$.

Let K be a non trivial knot. A *symmetry* h of K is a finite order diffeomorphism of the pair $(\mathbb{S}^3, K)$ preserving the orientation of $\mathbb{S}^3$. Let h be a symmetry of K with $Fix(h) \neq \emptyset$; by the Smith conjecture [14], $Fix(h)$ is the trivial knot. If $Fix(h) \cap K = \emptyset$ and the order of h is n we say that h is an *n-periodic symmetry*. If $Fix(h) \cap K \neq \emptyset$, then it consists of two points, the order of h is 2 and we say that h is a *strong inversion*. A knot is *strongly invertible* if it admits a strong inversion.

Let $L = L_1 \cup L_2$ be a two component link. We say that its two components are *exchangeable* if there exists a diffeomorphism of the pair $(\mathbb{S}^3, L)$ preserving the orientation of $\mathbb{S}^3$ and mapping L_1 (respectively L_2) to L_2 (respectively L_1).

To prove Theorem 1 we shall use the following:

Theorem 2. *Let $\overline{K} \cup \overline{K}'$ be a hyperbolic link whose two components are trivial and non exchangeable. Assume that for an $n \geqslant 3$ the following hold*:

(i) $\gcd(n, \mathrm{lk}(\overline{K}, \overline{K}')) = 1$;

(ii) *the Bonahon–Siebenmann decompositions of the orbifolds $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and $(\mathbb{S}^3, \overline{K}_n, \overline{K}'_2)$ are non trivial (and have the same geometric pieces and characteristic trees). Assume moreover that the Bonahon–Siebenmann decompositions of the orbifolds which are the 2-fold covers of $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and $(\mathbb{S}^3, \overline{K}_n, \overline{K}'_2)$ branched along the* *components of order* 2 *are the same, in the sense that they differ only by Dehn twists along the toric components of the characteristic family.*

Let K (*respectively* K') *be the preimages of* $\overline{K}$ (*respectively* $\overline{K}'$) *in the* n*-fold cyclic cover of* $\mathbb{S}^3$ *branched over* $\overline{K}'$ (*respectively* $\overline{K}$). *Then* K *and* K' *are non-equivalent hyperbolic knots—in* $\mathbb{S}^3$*—which have the same* 2*-fold and* n*-fold cyclic branched covers.*

Proof.

Remark 1: The orbifolds $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and $(\mathbb{S}^3, \overline{K}_n, \overline{K}'_2)$ are topologically $\mathbb{S}^3$ and their singular set is $\overline{K} \cup \overline{K}'$. Indices 2 and n stand for the orders of their singular sets. It is worth underlying that, although these two orbifolds have the same topological type, the orders of their singular sets are exchanged, so they are not the same orbifold.

K and K' are knots in $\mathbb{S}^3$: This follows from the fact that $\overline{K}$ and $\overline{K}'$ are trivial knots and that $\gcd(n, \mathrm{lk}(\overline{K}, \overline{K}')) = 1$.

K and K' are hyperbolic: All incompressible tori in the complement of K or K' should map onto incompressible tori inside the complement of $\overline{K} \cup \overline{K}'$ since $n \geqslant 3$ but this is impossible since $\overline{K} \cup \overline{K}'$ is hyperbolic. This also follows from Thurston's orbifold geometrization theorem [21,22] (see also [3] for a proof in the case of good orbifolds of cyclic type). In fact, since $\overline{K} \cup \overline{K}'$ is hyperbolic, it is $2\pi/n$-hyperbolic (being $n \geqslant 3$) and so are K and K'. The assertion follows.

K and K' are non-equivalent: If the knots were equivalent, the hyperbolic orbifolds $(\mathbb{S}^3, K_n)$ and $(\mathbb{S}^3, K'_n)$ would be isometric. According to Smith's conjecture, the n-periodic symmetries of the two orbifolds are unique and therefore would be conjugate by the isometry between them. In particular, such isometry would pass to the quotient $(\mathbb{S}^3, \overline{K}_n, \overline{K}'_n)$ and exchange the two components of the singular set which is absurd.

K and K' have the same n-fold cyclic branched cover: By construction this is the $\mathbb{Z}_n \oplus \mathbb{Z}_n$ branched cover of $\overline{K} \cup \overline{K}'$.

K and K' have the same 2*-fold cyclic branched cover*: Let L (respectively L') be the preimage of $\overline{K}'$ (respectively $\overline{K}$) in the 2-fold cyclic cover of $\mathbb{S}^3$ branched along $\overline{K}$ (respectively $\overline{K}'$). By construction the 2-fold cyclic branched cover of K (respectively K') is the n-fold cyclic branched cover of L (respectively L'). It then suffices to show that the n-fold cyclic branched covers of L and L' are the same. Requirement (ii) implies that these two manifolds have the the same Jaco–Shalen–Johannson decomposition and decomposition tree, moreover the gluing of the geometric pieces are isotopic and the claim follows.

Let us now construct explicit examples of links $\overline{K} \cup \overline{K}'$ satisfying the requirements of Theorem 2. We shall assume for the moment that n is odd and we shall see at the end how one can adapt the construction to the case when n is even. Consider the two component torus link $T(2, 32)$; the choice of $T(2, 32)$ is not the simplest possible but the advantage is that it works for all odd n and that the fibrations of the Seifert pieces are easier to visualize. Remove sixteen balls, as shown in Fig. 1, each containing a trivial tangle of the link. Replace the removed trivial tangles with π-hyperbolic tangles as follows. Let W be the complement of a hyperbolic knot $\mathcal{K}$ in the 3-sphere. Choose $\mathcal{K}$ in such a way that it

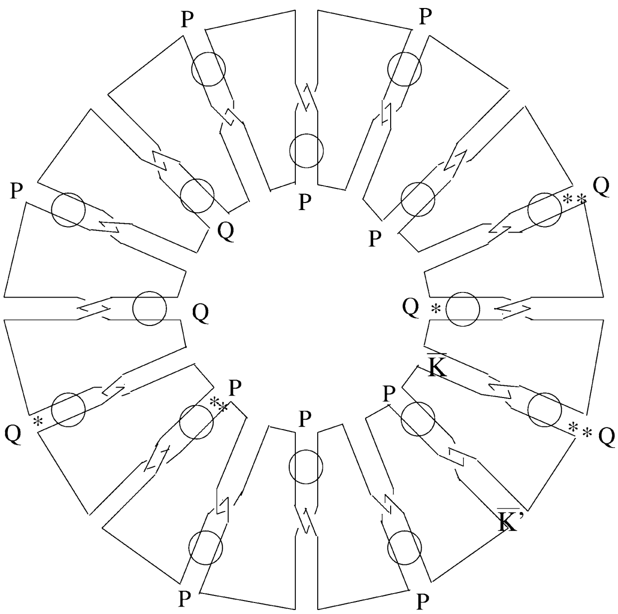

Fig. 1.

admits two non-equivalent strong inversions μ and ν. This choice ensures that the orbifolds $P := (p_\mu(W), p_\mu(Fix(\mu))_2)$ and $Q := (p_\nu(W), p_\nu(Fix(\nu))_2)$ have distinct topological types. Indeed, if P and Q were equivalent then, by the uniqueness of 2-fold cyclic branched covers (of a simply connected manifold) the strong inversions μ and ν would be conjugate. A possible choice for $\mathcal{K}$ is any 2-bridge chiral hyperbolic knot (see [19]). The choice of a chiral knot assures that μ and ν are not conjugate by an orientation-reversing symmetry of $\mathcal{K}$. In Fig. 2 we illustrate the case $\mathcal{K} = 5_2$. Let P and Q denote the tangles obtained from $\mathcal{K} = 5_2$ via μ and ν, respectively.

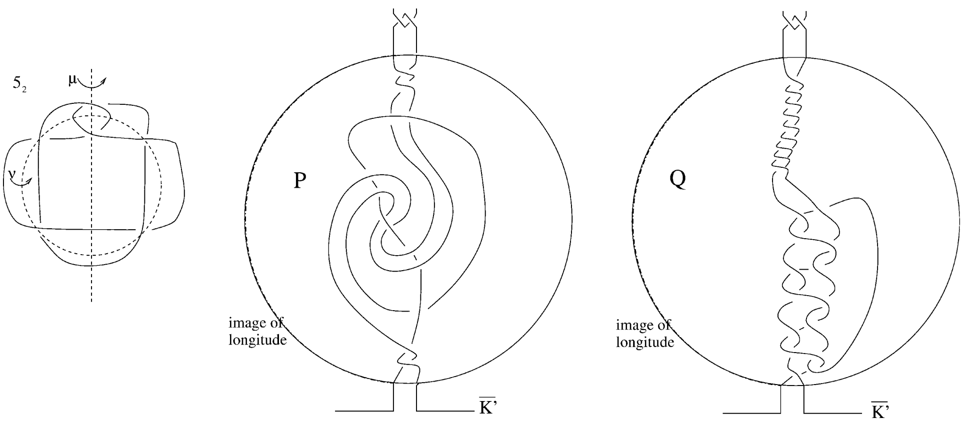

Fig. 2.

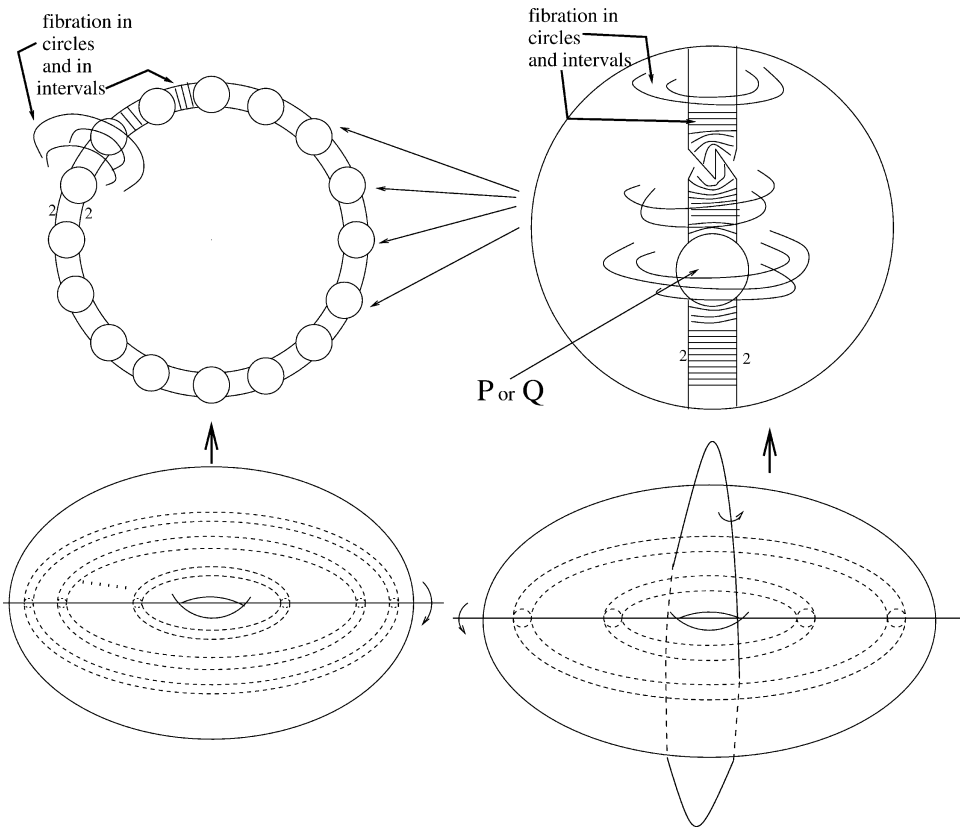

Fig. 3.

Replace the trivial tangles removed along the outer component $\overline{K}'$ of the link of Fig. 1 with the tangles P and Q in the manner indicated by Fig. 2. To decide how to replace the eight remaining trivial tangles, exploit the fact that one wants the outer component $\overline{K}'$ to be mapped to the inner one $\overline{K}$ by the homeomorphism φ shown in Fig. 4. The homeomorphism φ consists of a π-rotation about the dotted circle followed by an anticlockwise rotation by $7\pi/8$ about the centre C. To ensure that the gluings are well-behaved, we require that the image of a longitude of W in P and Q is glued in the same way on the boundary of all sixteen empty balls of the orbifold shown in Fig. 3, as suggested by Fig. 2 (for similar methods see [13]). Note that one thus obtains a link with two components which are trivial (for similar constructions and considerations see [23]). We shall denote such link $\overline{K} \cup \overline{K}'$. Note moreover that $\mathrm{lk}(\overline{K}, \overline{K}') = 16$.

We wish to stress that this is in fact the key point of the construction: one obtains the same result by removing all the geometric pieces along any of the two components but these are not exchangeable.

Claim 1. *The Bonahon–Siebenmann decomposition of* $(\mathbb{S}^3, \overline{K}_2, \overline{K}_2')$ *consists of six copies of Q, ten copies of P and seventeen Seifert fibred pieces.*

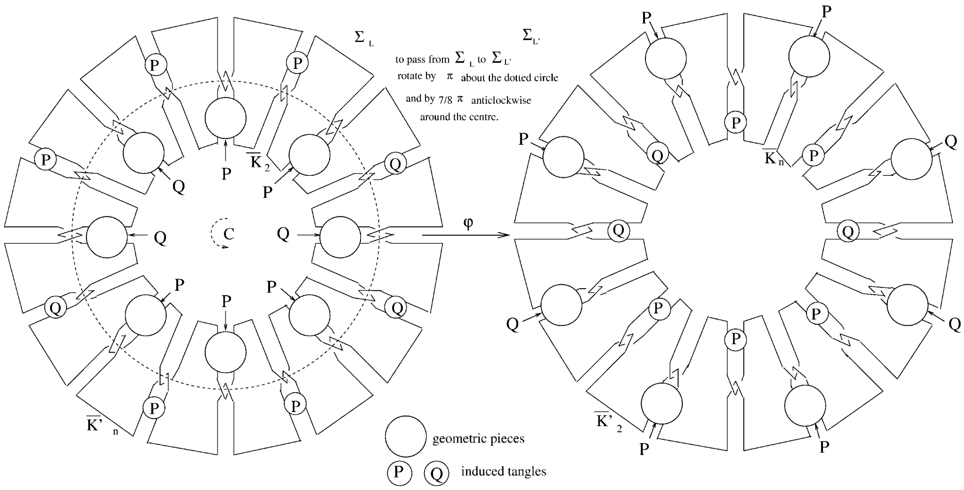

Fig. 4.

The pieces and the fibrations are shown in Fig. 3. Clearly the hyperbolic pieces must belong to the decomposition. The Seifert fibred pieces are obtained as quotients of trivially fibred solid tori with fifteen (respectively two) fibred tori drilled out from their interior. The axis of involutions giving the required quotients are shown in Fig. 3.

Claim 2. *The link $\overline{K} \cup \overline{K}'$ is hyperbolic.*

Since there are no incompressible toric suborbifolds which are tori in the decomposition of $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_2)$ and no incompressible tori in any of the Seifert pieces (they should be fibred), there are no incompressible tori in the decomposition of the complement of $\overline{K} \cup \overline{K}'$. The link is thus hyperbolic by the Thurston hyperbolization theorem.

Claim 3. *The Bonahon–Siebenmann decompositions of the orbifolds $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and $(\mathbb{S}^3, \overline{K}'_2, \overline{K}_n)$ have the same pieces and characteristic trees.*

Reasoning as in Claim 1 one sees that the Bonahon–Siebenmann decompositions of the orbifolds $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and $(\mathbb{S}^3, \overline{K}'_2, \overline{K}_n)$ are as given in Fig. 4: the geometric pieces are all hyperbolic, and precisely five copies of P, three copies of Q and an extra piece containing the component of order n, $\overline{\Sigma}_L$. We only need to prove hyperbolicity of $\overline{\Sigma}_L$. First of all remark that $\overline{\Sigma}_L$ is a geometric piece of the decomposition for it is atoroidal. Indeed, all incompressible toric suborbifolds would already appear in the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_2)$. Next observe that the link $\overline{K} \cup \overline{K}'$ contains Conway spheres which intersect $\overline{K}'$ (respectively $\overline{K}$) in four points. This follows from the fact that there are Conway spheres along both components of the singular set of $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_2)$. It is now sufficient to prove that, up to isotopy, the Conway spheres along $\overline{K}'$ (respectively $\overline{K}$) are contained in $\overline{\Sigma}_L$. If not, the ball determined by a Conway sphere along $\overline{K}'$ (respectively $\overline{K}$)

and containing only singular points of order n must intersect a ball determined by a Conway sphere along $\overline{K}$ (respectively $\overline{K}'$) and containing only singular points of order 2. Their intersection is however inessential, for it does not contain singular points. A standard argument of general position and minimal intersection proves that, up to isotopy, they must be disjoint. We have thus found a close incompressible hyperbolic surface contained in a geometric piece with non empty boundary and we deduce that the piece must be hyperbolic.

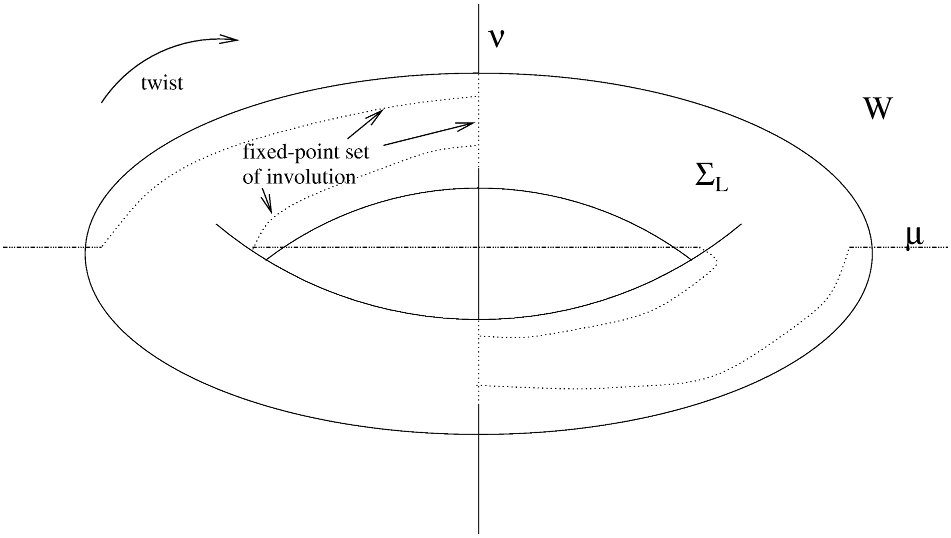

Fig. 5.

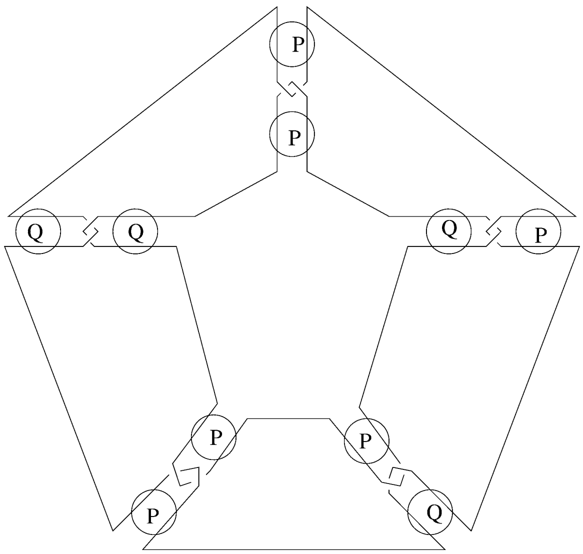

Fig. 6.

Claim 4. *The two components of $\overline{K} \cup \overline{K}'$ are not exchangeable.*

This follows from the fact that a homeomorphism exchanging them must preserve the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_2)$, sending hyperbolic pieces of type P (respectively Q) to hyperbolic pieces of the same type. Moreover, it can be chosen to have finite order since $\overline{K} \cup \overline{K}'$ is hyperbolic. Notice that any such homeomorphism must induce a fibre preserving homeomorphism of the Seifert fibred piece with sixteen boundary components: the group of such homeomorphisms is of the form $\mathbb{D}_{16} \oplus \mathbb{Z}_2$. In particular the two pieces of type Q marked with a $*$ in Fig. 4 should be exchanged but then one of the two pieces of type Q marked with $**$ should be mapped to the piece of type P marked with $**$, which is impossible.

To be able to apply Theorem 2 we still need to prove that the Bonahon–Siebenmann decompositions of the orbifolds $(\mathbb{S}^3, L_n)$ and $(\mathbb{S}^3, L'_n)$ which are the 2-fold covers of $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and $(\mathbb{S}^3, \overline{K}_n, \overline{K}'_2)$ branched along the components of order 2 are the same.

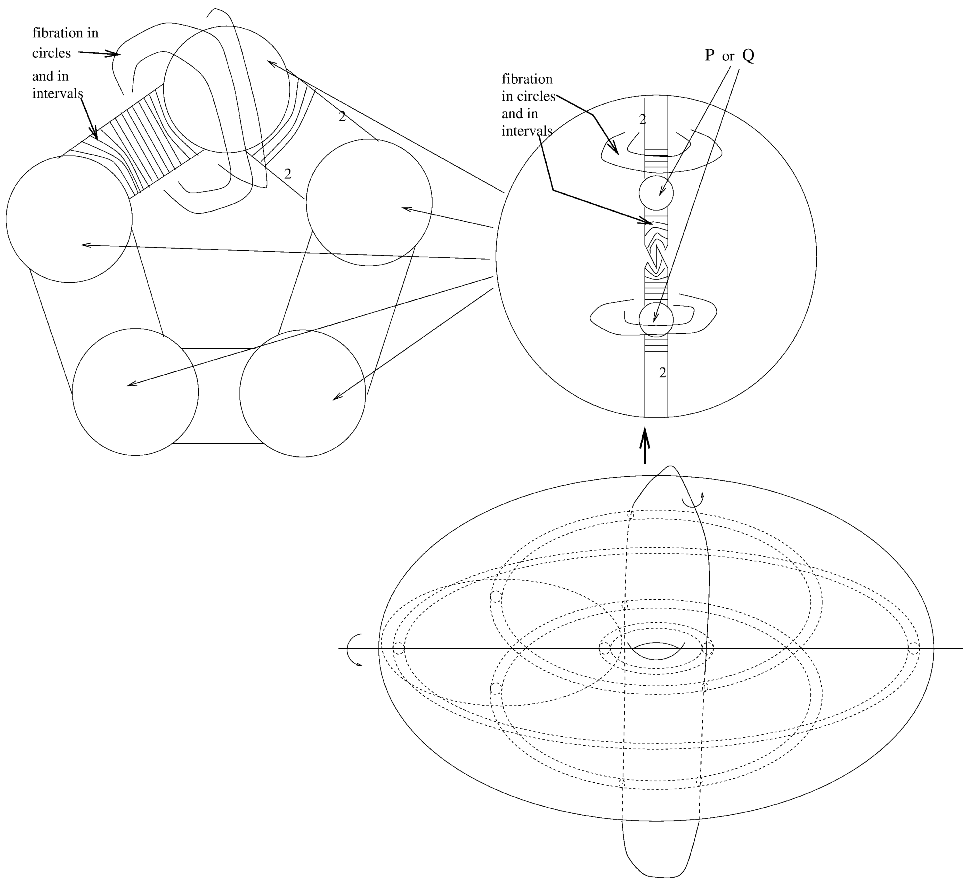

Fig. 7.

This is easily seen since these orbifolds differ only by $\pi/2$-Dehn twists along the boundary components of $\Sigma_L = \Sigma_{L'}$, the preimage of $\overline{\Sigma}_L = \overline{\Sigma}_{L'}$. To understand this, consider what happens locally near the common toric boundary of W and $\Sigma_L = \Sigma_{L'}$, as schematically explained in Fig. 5: the involution of $\Sigma_L = \Sigma_{L'}$ extends to ν or to μ up to $\pi/2$-Dehn twists. All the remaining pieces of the Bonahon–Siebenmann decomposition of the orbifolds $(\mathbb{S}^3, L_n)$ and $(\mathbb{S}^3, L'_n)$ are equal to W.

To conclude this section, we want to explain how one can proceed when n is even. Consider in this case the torus links $T(2, 2b)$ with b odd and replace trivial tangles with π-hyperbolic ones as suggested in Fig. 6 for the case $b = 5$. Here P and Q are as described at the beginning of the section. Note that b must be sufficiently large in order to obtain two trivial components which are not exchangeable, even if the tangles are disposed in the same way along both components (compare the construction made for n odd). The Bonahon–Siebenmann decomposition of the orbifold which is topologically $\mathbb{S}^3$ and has this link as singular set of order 2 is shown in Fig. 7 for the case $b = 5$ and consists of six hyperbolic pieces of type P, four of type Q and six Seifert fibred pieces. Clearly, in this way, one can construct pairs of knots with the same 2-fold and n-fold cyclic branched covers but only for those n (either even or odd) which are prime with b. One can also use a π-hyperbolic tangle and its Conway mutant instead of P and Q: in this case it is easier to prove that the 2-fold cyclic branched covers of the two knots are the same, although it is perhaps more difficult to ensure that the two components of the link are not exchangeable.

3. Non equivalent hyperbolic knots with two common covers

In this section we shall explain why the constructions of Section 2 are (in some sense) unique. We shell see that Proposition 1, at the end of this section, forces somehow the hypotheses of Theorem 2.

Let K be a hyperbolic knot and K' another knot, non-equivalent to K, but with the same 2-fold and n-fold cyclic branched covers as K, for a fixed $n \geqslant 3$. Let M be their 2-fold cyclic branched cover and τ, τ' the involutions of M which are deck transformations for $\mathbb{S}^3$ with branching set K and K', respectively. Since K is hyperbolic, it follows from Thurston's orbifold geometrization theorem and Dunbar's list of non hyperbolic orbifolds with underlying space $\mathbb{S}^3$ [7], that K is also $2\pi/n$-hyperbolic, unless $n = 3$ and K is the figure-eight knot 4_1. Since the figure-eight knot is determined, up to equivalence, by its 2-fold cyclic branched cover [8], from now on we shall always assume K (and K') not to be the figure-eight knot. We can thus apply the result of Zimmermann in [24], which says that if a $2\pi/n$-hyperbolic knot K is not determined by its n-fold cyclic branched cover, $n \geqslant 3$, it admits an n-periodic symmetry $\bar{h}$ such that $p_{\bar{h}}(K)$ is the trivial knot. Moreover, the preimage of $p_{\bar{h}}(Fix(\bar{h}))$ in the n-fold cyclic cover of $\mathbb{S}^3$ branched along $p_{\bar{h}}(K)$ (which is again $\mathbb{S}^3$) is the unique knot K' non-equivalent to K with the same n-fold cyclic branched cover as K. In particular, $p_{\bar{h}}(K \cup Fix(\bar{h}))$ is a link, that we shall denote by $\overline{K} \cup \overline{K}'$, whose two components are trivial and non exchangeable. This fact is proved in [24, pp. 668–669] for all n's which are not powers of 2, but in fact, following the same lines, one can extend the result to the case when $n > 2$ is a power of 2.

Let $\widehat{M}$ be the manifold which is the n-fold cyclic cover of $\mathbb{S}^3$ branched along K and K'. Because of Thurston's orbifold geometrization theorem and Mostow's rigidity theorem (see, for instance, [1] for basic results in hyperbolic geometry), $\widehat{M}$ is hyperbolic and the covering transformations for K and K' can be chosen to be isometries for the unique hyperbolic structure of $\widehat{M}$. It follows that K' is $2\pi/n$-hyperbolic and thus hyperbolic.

Let then $\bar{h}'$ be the n-periodic symmetry with trivial quotient for K' and let h, h' be lifts of $\bar{h}$ and $\bar{h}'$, respectively, to the 2-fold cyclic branched cover M. Note that p_h is the projection of M over $\mathbb{S}^3$ branched along a link L. Indeed $p_h(M)$ is the 2-fold cyclic cover of $\mathbb{S}^3$ branched along $\overline{K}$ and L is the preimage of $\overline{K}'$. Notice that L is either a knot or a two component link according to the parity of $\mathrm{lk}(\overline{K}, \overline{K}')$. Similar considerations hold for h' and we have the following two diagrams of orbifold covers:

$$\begin{tikzcd} M \arrow[r, "p_h"] \arrow[d, "p_\tau"] & (\mathbb{S}^3, L_n) \arrow[d, "p_{\bar{\tau}}"] \\ (\mathbb{S}^3, K_2) \arrow[r, "p_{\bar{h}}"] & (\mathbb{S}^3, \overline{K}_2, \overline{K}'_n) \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "p_{h'}"] \arrow[d, "p_{\tau'}"] & (\mathbb{S}^3, L'_n) \arrow[d, "p_{\bar{\tau}'}"] \\ (\mathbb{S}^3, K'_2) \arrow[r, "p_{\bar{h}'}"] & (\mathbb{S}^3, \overline{K}'_2, \overline{K}_n) \end{tikzcd}$$

here the maps $\bar{\tau}$ and $\bar{\tau}'$ are induced by the maps τ and τ' on the orbifolds $(\mathbb{S}^3, L_n)$ and $(\mathbb{S}^3, L'_n)$, respectively.

Notice that the two orbifolds $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and $(\mathbb{S}^3, \overline{K}'_2, \overline{K}_n)$ have the same topological type but the orders of their singular sets are exchanged so they are different in general.

Because of the result in [16], we can assume K to be Conway reducible. This means in particular that M contains an incompressible torus. Any such torus must project in $(\mathbb{S}^3, K'_2)$ to a Euclidean orbifold. Since K' is hyperbolic, it is atoroidal and thus the image of the torus must be a sphere with four points of order 2, i.e., a Conway sphere for K'.

We want now to describe in some detail certain properties of the Bonahon–Siebenmann decompositions of the orbifolds we are dealing with. Recall that, according to [5], we can cut an irreducible orbifold along a minimal family of toric 2-dimensional suborbifolds in such a way that the pieces we get are either Seifert fibred or atoroidal orbifolds. Because of Thurston's orbifold geometrization theorem, the atoroidal orbifolds which are not Seifert fibred must be hyperbolic. Since K is hyperbolic and thus atoroidal, the family of 2-dimensional suborbifolds in the Bonahon–Siebenmann decomposition for $(\mathbb{S}^3, K_2)$ consists of certain Conway spheres along K. The graph associated to this decomposition is a tree, since all spheres in $\mathbb{S}^3$ are separating. Moreover notice that the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, K_2)$ lifts to the Jaco–Shalen–Johannson decomposition for M [9,10]. In particular the graph associated to the Jaco–Shalen–Johannson decomposition for M is combinatorially the same as that associated to the

Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, K_2)$ (although edges and vertices do not have the same meaning); in particular it is a tree (this follows also from the fact that M is a rational homology sphere).

We will distinguish two cases, i.e., the family of 2-dimensional suborbifolds in the Bonahon–Siebenmann decomposition is empty or not.

In the first case, the 2-fold cyclic branched cover of K must be geometric and, since it contains incompressible tori, must be Seifert fibred. Since K is hyperbolic, it is a Montesinos knot with at least four tangles for it is Conway reducible. In particular K is a non elliptic Montesinos knot and its n-periodic symmetry $\bar{h}_M$ is necessarily of the type described in Fig. 8 (symmetries of non elliptic Montesinos links are studied in [4]). This means that the number of tangles of K is of the form $n\delta$. If $\delta \geqslant 4$, $p_{\bar{h}_M}(K)$ admits a Conway sphere and thus cannot be trivial, then $\delta \leqslant 3$ and $p_{\bar{h}_M}(K)$ is a trivial Montesinos knot with invariants, say, $(e; (\alpha_1, \beta_1), \dots, (\alpha_\delta, \beta_\delta))$. We have that the Seifert manifold M has invariants $(ne; (\alpha_1, \beta_1), \dots, (\alpha_\delta, \beta_\delta), \dots, (\alpha_1, \beta_1), \dots, (\alpha_\delta, \beta_\delta))$ with the pair (α_i, β_i) appearing n times for all $1 \leqslant i \leqslant \delta$. Since the fibration of M is unique, we can repeat the same reasoning for the knot K' and obtain that $p_{\bar{h}'_M}(K')$ is a trivial Montesinos knot with invariants $(e; (\alpha_{\sigma(1)}, \beta_{\sigma(1)}), \dots, (\alpha_{\sigma(\delta)}, \beta_{\sigma(\delta)}))$, where σ is a permutation of $1, \dots, \delta$. Clearly $p_{\bar{h}_M}(K)$ and $p_{\bar{h}'_M}(K')$ are the same Montesinos knot (see [6]); moreover the links $p_{\bar{h}_M}(K \cup Fix(\bar{h}_M))$ and $p_{\bar{h}'_M}(K' \cup Fix(\bar{h}'_M))$ are the same componentwise and so are the knots K and K'. Let us summarize the above considerations in the following:

Corollary 1. *Let $n \geqslant 3$. A hyperbolic Montesinos knot is determined up to equivalence by its 2-fold and n-fold cyclic branched covers.*

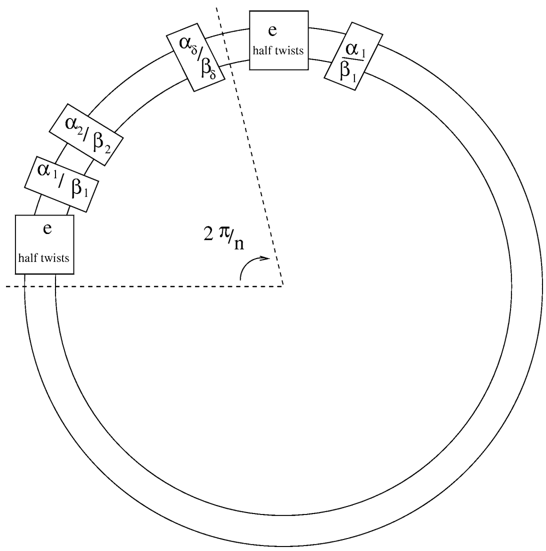

Fig. 8.

We can then assume that the family of toric suborbifolds of the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, K_2)$ is not empty. We can also assume, up to isotopy, that $\bar{h}$ preserves the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, K_2)$. Notice now that, being of order $n \geqslant 3$, $\bar{h}$ must act freely on the family of toric 2-dimensional suborbifolds, since they are Conway spheres. Thus the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, K_2)$ induces a Bonahon–Siebenmann decomposition for $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$: all toric 2-dimensional suborbifolds intersect the component of order 2 but miss the component of order n, coming from the fixed-point set of $\bar{h}$. Clearly every toric suborbifold of the family divides $\mathbb{S}^3$ in two balls, exactly one of which contains the component of order n. Moreover, such toric suborbifolds are Conway spheres for the link $\overline{K} \cup \overline{K}'$.

Claim 5. *The intersection of all the balls determined by the toric suborbifolds of the family and containing the component of order n is a hyperbolic orbifold.*

The orbifold we obtain is clearly a geometric piece of the decomposition. To show that it is hyperbolic repeat the same argument used in the proof of Claim 3.

We shall denote N the preimage in M of the hyperbolic piece we have just determined. N is connected since it intersects both $Fix(h)$ and $Fix(\tau)$. Repeating the same reasoning for the knot K' we obtain another hyperbolic piece N' of M. Up to isotopy, we can assume that N and N' belong to the same Jaco–Shalen–Johannson decomposition for M. Observe that N and N' contain the fixed-point sets of h and h', respectively. More precisely h and h' preserve N and N', respectively, while freely permuting all the tori of the Jaco–Shalen–Johannson decomposition for M. Indeed $\bar{h}$ and $\bar{h}'$ both permute freely the toric suborbifolds of the Bonahon–Siebenmann family. We affirm that N and N' are the same piece of the decomposition. This claim is a straightforward consequence of the standard fact that if two automorphisms of a tree have non empty fixed-point set then their fixed-point sets have non trivial intersection. This fact has the following consequence:

Corollary 2. *Let K be a hyperbolic knot whose 2-fold cyclic branched cover is a graph manifold of Waldhausen and let $n > 2$ be an integer. K is determined up to equivalence by its 2-fold and n-fold cyclic branched covers.*

Consider now the group G of all isometries g of N induced by diffeomorphisms g of M which preserve N. Note that, by Thurston's orbifold geometrization theorem and Mostow's rigidity theorem, the diffeomorphisms τ, τ', h and h' induce isometries of N belonging to G that we shall again denote by τ, τ', h and h', respectively. If we consider the group generated by h and h' in G, we remark that three possible situation can arise:

Case A. *The groups generated by h and h' are not conjugate in G.*

Case B. *The groups generated by h and h' are conjugate in G.*

In this second case, up to conjugation and perhaps a change of generator in $\langle h' \rangle$, we can assume $h = h'$. The subgroup of G generated by h, τ and τ' is isomorphic to $\mathbb{D}_t \oplus \mathbb{Z}_n$. We will distinguish two subcases:

Case B1. *t is even.*

Case B2. *t is odd, in particular τ and τ' are conjugate and we can assume $\tau = \tau'$.*

Notice that since the group $\mathbb{D}_t \oplus \mathbb{Z}_n$ preserves $Fix(h)$, either $Fix(h)$ is not connected or $t = 1$. Indeed the group of isometries leaving invariant a closed geodesic of a compact hyperbolic manifold is a finite subgroup of $\mathbb{Z}_2 \ltimes (\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}_2$ acts by inverting the orientation of the geodesic itself.

It is possible to prove that, under the hypothesis that K and K' are not equivalent, only case B2 can happen. This can be done by reducing the study to the closed π-hyperbolic case via equivariant hyperbolic Dehn surgery on the boundary components of N.

Assume now that we are in case B2. Let us introduce the following notation: $\Sigma_L := p_h(N)$, $\Sigma_{L'} := p_{h'}(N)$, $\overline{\Sigma}_L := p_{\bar{\tau}}(\Sigma_L)$, $\overline{\Sigma}_{L'} := p_{\bar{\tau}'}(\Sigma_{L'})$, where $\bar{\tau}$ and $\bar{\tau}'$ are the maps induced by τ and τ' on the orbifolds Σ_L and $\Sigma_{L'}$, respectively. Notice that these two orbifolds have, respectively, L and L' as singular sets, both of order n. Moreover they are naturally embedded in the orbifolds $(\mathbb{S}^3, L_n)$ and $(\mathbb{S}^3, L'_n)$, respectively, as a hyperbolic piece of their Bonahon–Siebenmann decomposition, which is induced by the Jaco–Shalen–Johannson decomposition of M. Since in this case $h = h'$ and $\tau = \tau'$, we have that $\Sigma_L = \Sigma_{L'}$ and $\overline{\Sigma}_L = \overline{\Sigma}_{L'}$, even if L and L' are not the same link in general. Notice that L is necessarily a two component link if $t > 1$ but it can be a knot if $t = 1$. Denote now by $\mathcal{E}_L$ and $\mathcal{E}_{L'}$ the two orbifolds which have the same topological type as $\overline{\Sigma}_L$ and $\overline{\Sigma}_{L'}$, respectively, but where the orders of singularity of the singular sets are exchanged. Let f, f' the natural embeddings of the orbifolds $\overline{\Sigma}_L$ and $\overline{\Sigma}_{L'}$ into the orbifolds $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and $(\mathbb{S}^3, \overline{K}'_2, \overline{K}_n)$, respectively, as hyperbolic pieces of the Bonahon–Siebenmann decomposition. The embeddings f and f' clearly induce embeddings $f_{\mathcal{E}}$ and $f'_{\mathcal{E}}$ of the orbifolds $\mathcal{E}_L$ and $\mathcal{E}_{L'}$ into the orbifolds $(\mathbb{S}^3, \overline{K}_n, \overline{K}'_2) = (\mathbb{S}^3, \overline{K}'_2, \overline{K}_n)$ and $(\mathbb{S}^3, \overline{K}'_n, \overline{K}_2) = (\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$, respectively.

Claim 6. *All the geometric pieces of the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ different from $\overline{\Sigma}_L$ appear in the decomposition of $\mathcal{E}_{L'}$.*

First of all, we show that $\mathcal{E}_{L'}$ does not contain toric 2-suborbifolds T which are topologically tori. Indeed a compressing disk for T in $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ would intersect the incompressible surfaces of $\partial\mathcal{E}_{L'}$ along inessential loops and could be isotoped inside $\mathcal{E}_{L'}$. Moreover, up to isotopy, $\mathcal{E}_{L'}$ contains all toric 2-suborbifolds of the Bonahon–Siebenmann family for $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ and they are clearly not boundary parallel since the boundary of $\mathcal{E}_{L'}$ consists of hyperbolic 2-dimensional orbifolds.

This means that the Bonahon–Siebenmann decomposition for $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ induces the Bonahon–Siebenmann decomposition for $\mathcal{E}_{L'}$ via $f'_{\mathcal{E}}$. The same obviously holds for $(\mathbb{S}^3, \overline{K}'_2, \overline{K}_n)$ and $\mathcal{E}_L$ via $f_{\mathcal{E}}$. We thus obtain that the map $f_{\mathcal{E}} f'^{-1}_{\mathcal{E}}$ sends the geometric pieces of the Bonahon–Siebenmann decomposition for $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ different from $\overline{\Sigma}_L$ to the geometric pieces of the Bonahon–Siebenmann decomposition for $(\mathbb{S}^3, \overline{K}'_2, \overline{K}_n)$ different from $\overline{\Sigma}_{L'}$. The considerations of this section give the following:

Proposition 1. *Let $n \geqslant 3$ and let K and K' be two non-equivalent hyperbolic knots with the same 2-fold and n-fold cyclic branched covers. Then K and K' are Conway reducible* *and the geometric pieces and the characteristic trees of the Bonahon–Siebenmann decompositions of the two orbifolds* $(\mathbb{S}^3, \overline{K}_2, \overline{K}'_n)$ *and* $(\mathbb{S}^3, \overline{K}'_2, \overline{K}_n)$ *are the same. Moreover the piece of the decomposition containing the component of order* n *is hyperbolic.*

Corollary 3. *Let* $n \geqslant 3$ *and let* K *be a hyperbolic knot such that the Bonahon–Siebenmann decomposition of the orbifold* $(\mathbb{S}^3, K_2)$ *contains at most three pieces. Then* K *is determined up to equivalence by its* 2*-fold and* n*-fold cyclic branched covers.*

Proof. This follows from the fact that n must divide the cardinality of the family of 2-dimensional toric suborbifolds of the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, K_2)$. Indeed, if K is not determined by its 2-fold and n-fold cyclic branched covers, the constructions of Section 2 suggest that the number of pieces in the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, K_2)$ must be much larger. Unfortunately, we are not able to give the maximal number of pieces in the Bonahon–Siebenmann decomposition of $(\mathbb{S}^3, K_2)$ ensuring K to be determined by its 2-fold and n-fold cyclic branched covers.

4. Ambiguous coverings

We start by giving some definitions. Let $\mathcal{N}$ be a finite set of integers such that $n \geqslant 2$ for all $n \in \mathcal{N}$. We say that a knot K is $\mathcal{N}$*-determined* if, whenever there exists a knot K' such that $M(n, K)$ and $M(n, K')$ are homeomorphic for all $n \in \mathcal{N}$, we have that K and K' are equivalent. If $\mathcal{N}$ consists of a unique element n we shall say that a knot is n*-determined* rather than $\mathcal{N}$-determined. Finally, given a knot K we shall say that $M(n, K)$ is ambiguous if K is not n-determined.

A straightforward consequence of Theorem 1 and [23] is that, in general, a hyperbolic knot admits ambiguous cyclic branched covers. The aim of this section is to give certain properties of the ambiguous cyclic branched covers of a hyperbolic knot. We collect them in the following:

Proposition 2. *A hyperbolic knot* K *has at most three ambiguous cyclic branched covers, and at most two of orders strictly larger than* 2*. If the* n*-fold cyclic branched cover of* K*,* $n \geqslant 3$*, is ambiguous, then* $(n-1)$ *divides* $2g$*, where* g *is the genus of* K*. Moreover if* K *has two ambiguous cyclic branched covers of orders* $n > m > 2$*, then* n *and* m *are coprime and we have* $2g = (n-1)(m-1)$*.*

Proof. Let $n \geqslant 3$. By Zimmermann's result in [24], if the n-fold cyclic branched cover of K is ambiguous, then K admits an n-periodic symmetry with trivial quotient. The proposition follows by an easy application of the Riemann–Hurwitz formula. The reader is referred to [16] (in particular the lemma) for more details. Notice that Proposition 2 holds also in the case of the figure-eight knot, for which Zimmermann's result applies only when $n \geqslant 4$. Indeed, exploiting the result in [8] and since the genus of 4_1 is 1, the only possible ambiguous cyclic branched cover is the 3-fold one. In fact, by considering Dunbar's list [7], we see that the figure-eight knot has no ambiguous cyclic branched covers.

Notice that, in accordance with Kojima's result [12], ambiguous cyclic branched covers, even of hyperbolic knots, can have arbitrarily large orders, linearly bounded by the genus of the knot. On the other hand, Proposition 2 says that there are only "few" ambiguous cyclic branched covers of a hyperbolic knot. We remark that it is not difficult to construct hyperbolic knots with three ambiguous cyclic branched covers. Indeed it is sufficient to consider the examples given by Zimmermann in [23] of pairs of non-equivalent hyperbolic knots with the same m-fold and n-fold cyclic branched covers, $n > m > 2$ coprime. All these knots are Conway reducible and for appropriate choices of the π-hyperbolic tangles used to construct them (i.e., asymmetric tangles) they admit non-equivalent Conway mutants. It would be interesting to understand whether there exists a hyperbolic knot K and two coprime integers $m, n > 2$ such that K is not $\{2, m, n\}$-determined.

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