1. Introduction

In the study of knots, both their cyclic branched coverings and the infinite abelian coverings of their complements in $\mathbf{S}^3$ have turned out to be very effective to distinguish them. It is well known, for instance, that the first algebraic invariant of a knot ever considered, the Alexander polynomial, is related to the homology of the infinite abelian covering of the complement [15]. On the other hand, it was proved by Kojima [8] that prime knots are actually determined by cyclic branched coverings of sufficiently large order. Nevertheless, there is no uniform bound on the order of the covering determining the knot [16], [12]. Zimmermann proved that $2\pi/p$-hyperbolic knots (p any odd prime number) are determined by their p-fold and p^2-fold branched coverings and that there are at most two non equivalent $2\pi/n$-hyperbolic knots (n not a power of 2) with the same n-fold branched covering [22]. Moreover these two non equivalents knots are related in a well specified way. It was also proved that there exist a) arbitrary many hyperbolic knots with the same 2-fold branched covering [22]; b) infinitely many sets of four different π-hyperbolic knots with the same 2-fold branched covering [14]. However, in certain cases, the 2-fold branched covering of a link determines the link. This is the case of the trivial knot [21], the trivial link [19], the 2-bridge links [6] and the non strongly invertible double knots [9]. Moreover in [1] Boileau and Flapan give sufficient conditions for π-hyperbolic links to be determined by their 2-fold branched covering.

Here we restrict our attention to the class $\mathcal{F}$ of links defined as follows:
Let M be the 2-fold cyclic covering of $\mathbf{S}^3$ branched along a prime link L. By Thurston's orbifold geometrization theorem [2], the 3-orbifold which is topologically the 3-sphere with singular set of order 2 the link L is geometric. Thus M is naturally endowed with a metric induced by the quotient orbifold. We say that L *belongs to the class* $\mathcal{F}$ if the group of isometries of M with respect to the defined metric is finite.

Indeed a weaker assumption can be made, i.e. we want the subgroup of $Iso(M)$ generated by all elements of order 2 to be of finite order.

Notice that the class $\mathcal{F}$ is not empty since it contains in particular all π-hyperbolic links.

Suppose now that L and L' are two different links with the same 2-fold cyclic branched covering M. Then the geometric structures induced on M by the two links coincide. In fact, M is either hyperbolic (and the statement follows from Mostow's rigidity theorem), Euclidean or Seifert fibred. This means in particular that any set containing all the links with the same 2-fold cyclic branched covering is either included in $\mathcal{F}$ or disjoint from it. In Section 6 we shall then give a partial answer (i.e. for the links of $\mathcal{F}$) to the question in [7] of how two non equivalent links with the same 2-fold cyclic branched covering are related. This will be accomplished using the results of Sections 3, 4 and 5.

Recall that two links which differ by a Conway mutation (see [20]) have the same 2-fold cyclic branched covering. Remark that restricting our attention to the class $\mathcal{F}$ we avoid this phenomenon. However it is still unknown whether two links with the same 2-fold branched covering are either Conway mutants or related as the links of $\mathcal{F}$.

Let us give some definitions. Let L be a link in $\mathbf{S}^3$, and consider the group of isometries of the (geometric) orbifold whose underlying topological space is $\mathbf{S}^3$ and whose singular set of order 2 is the link L. An isometry of the orbifold of order 2 with non empty fixed-point set (necessarily a trivial knot by Smith's conjecture) is called a 2-*periodic symmetry* of L if its fixed-point set does not intersect L; it is called a *strong inversion* of L if its fixed-point set intersect each component of L in exactly two points. In the latter case L is said to be *strongly invertible*. With this terminology the main result of the paper reads:

Theorem 1:

Let L be a link of $\mathcal{F}$ with r components. Assume that L is not strongly invertible. Then the following conditions are sufficient for L to be determined by its 2-branched covering:

i) $r \geq 3$;

ii) $r = 2$ and L does not admit a 2-periodic symmetry exchanging the two components or, if it does, the two components are not unknotted;

iii) $r = 1$ the quotient of L with respect to the action of any 2-periodic symmetry is not the trivial knot.

In case iii) one can easily derive a simple condition on the Alexander polynomial of L which ensures that the knot is determined by its 2-fold branched covering. The condition is given in

Theorem 2:

Let $\Delta_L(t) = \sum_{i=-n}^{n} a_i t^i \in \mathbb{Z}[t, t^{-1}]$ be the Alexander polynomial of a non strongly invertible knot L of $\mathcal{F}$. Recall that $a_i = a_{-i}$, $0 \leq i \leq n$, and that, since $\Delta_L(1) = \pm 1$, a_0 must be odd [15]. *Let $j = \max_{0 \leq i \leq n}\{a_i \equiv 1 (\mathrm{mod}\ 2)\}$.*
Assume that there exists i_0, $0 < i_0 < j$ such that a_{i_0} is even. Then L is determined by its 2-fold cyclic branched covering.

The proofs of Theorems 1 and 2 are given in Sections 2 and 3. One can compare Theorem 1 above with the sufficient conditions given in [1]. Indeed the techniques used here are basically the same as those used in [1]: finite group theory, theory of coverings, Smith's theory.

To conclude, in Section 7 we shall give an example of infinitely many distinct genus 2 fibred knots which are not determined by their 2-fold cyclic branched coverings. These knots were first considered by Edmonds and Livingston in [4] where they prove that these knots are distinct and that they all admit a 2-periodic symmetry. Thanks to this example, we shall see that two knots having the same 2-fold cyclic branched covering need not be both either fibred or non fibred. Moreover their genera need not be the same; in fact (the absolute value of) the difference of their genera can be arbitrarily large.

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2. Diagrams of coverings

Let us start by fixing some notation:
p_* denotes the covering projection induced by the covering transformation $*$ (i.e. $*$ generates the group of covering transformations);
$Fix(*)$ denotes the fixed-point set of the map $*$.

Let $L, L' \in \mathcal{F}$ be two distinct links with the same 2-fold cyclic branched covering M. From now on we shall call two such links *$\mathcal{F}$-mutants*. Let $\tau, \tau' \in Iso(M)$ be covering transformations such that $p_\tau(M)$ (resp. $p_{\tau'}(M)$) is the orbifold $(\mathbf{S}^3, L)$ (resp. $(\mathbf{S}^3, L')$) with singular set of order 2 the link L (resp. L'). By our assumptions the subgroup generated by τ and τ' in $Iso(M)$ is a dihedral group of finite order $2n$, $\mathbf{D}_n = \langle \tau\tau' \mid \tau^2, \tau'^2, (\tau\tau')^n \rangle$.

Remark:
One can assume that $n = 2^d$, $d \geq 1$. Indeed, if this is not the case we have $n = 2^d(2m+1)$. It is now possible to replace τ' by its conjugate $(\tau'\tau)^m\tau'(\tau\tau')^m$. Call this new involution again τ'. It is clear that the element $\tau\tau'$ has now order 2^d. Obviously $d > 0$ else $\tau = \tau'$ against our hypothesis that L and L' are distinct.

Define now

$M_1 := M$;

$\tau_1 := \tau$, $\tau'_1 := \tau'$;

$h_1 := (\tau_1\tau'_1)^{2^{d-1}}$;

$L_1 := L$, $L'_1 := L'$;

$A_1 = \bar{A}_1 := p_{\tau_1}(Fix(h_1))$, $A'_1 = \bar{A}'_1 := p_{\tau'_1}(Fix(h_1))$;

$\bar{h}_1$ (resp. $\bar{h'}_1$) the map induced by h_1 on $(\mathbf{S}^3, L_1)$ (resp. $(\mathbf{S}^3, L'_1)$);

and recursively

$M_{i+1} := p_{h_i}(M_i)$, $1 \leq i \leq d$;

τ_{i+1} (resp. τ'_{i+1}) the map induced by τ_i (resp. τ'_i) on M_{i+1}, $1 \leq i \leq d-1$;

$h_{i+1} := (\tau_{i+1}\tau'_{i+1})^{2^{d-i-1}}$, $1 \leq i \leq d-1$ which is also the map induced by $(\tau_1\tau'_1)^{2^{d-i-1}}$ on M_{i+1};

$L_{i+1} := p_{\bar{h}_i} p_{\tau_i}(Fix(\tau_i) \cup Fix(h_i\tau_i))$, $L'_{i+1} := p_{\bar{h'}_i} p_{\tau'_i}(Fix(\tau'_i) \cup Fix(h_i\tau'_i))$, $1 \leq i \leq d$;

$A_{i+1} := p_{\bar{h}_i} p_{\bar{h}_{i-1}} ... p_{\bar{h}_1} p_{\tau_1}(Fix(h_1))$, $A'_{i+1} := p_{\bar{h'}_i} p_{\bar{h'}_{i-1}} ... p_{\bar{h'}_1} p_{\tau'_1}(Fix(h_1))$, $1 \leq i \leq d$

$\bar{A}_{i+1} := p_{\tau_{i+1}}(Fix(h_{i+1}))$, $\bar{A'}_{i+1} := p_{\tau'_{i+1}}(Fix(h_{i+1}))$, $1 \leq i \leq d$.

$\bar{h}_{i+1}$ (resp. $\bar{h'}_{i+1}$) the map induced by h_{i+1} on $(\mathbf{S}^3, L_{i+1} \cup A_{i+1})$ (resp. $(\mathbf{S}^3, L'_{i+1} \cup A'_{i+1})$, $1 \leq i \leq d-1$;

We have the following diagram of coverings:

$$
(*) \qquad \begin{array}{ccccc}
(\mathbf{S}^3, L_1) & \xleftarrow{p_{\tau_1}} & M_1 & \xrightarrow{p_{\tau'_1}} & (\mathbf{S}^3, L'_1) \\
\downarrow p_{\bar{h}_1} & & \downarrow p_{h_1} & & \downarrow p_{\bar{h'}_1} \\
(\mathbf{S}^3, L_2 \cup A_2) & \xleftarrow{p_{\tau_2}} & M_2 & & \\
\vdots & & \vdots & & \vdots \\
(\mathbf{S}^3, L_d \cup A_d) & \xleftarrow{p_{\tau_d}} & M_d & \xrightarrow{p_{\tau'_d}} & (\mathbf{S}^3, L'_d \cup A'_d) \\
\downarrow p_{\bar{h}_d} & & & & \downarrow p_{\bar{h'}_d} \\
(\mathbf{S}^3, L_{d+1} \cup A_{d+1}) & & = & & (\mathbf{S}^3, L'_{d+1} \cup A'_{d+1})
\end{array}
$$

Remarks:

The map $\bar{h}_i$ (resp. $\bar{h'}_i$) is a non trivial involution with non empty fixed-point set $p_{\tau_i}(Fix(h_i) \cup Fix(h_i\tau_i))$ (resp. $p_{\tau'_i}(Fix(h_i) \cup Fix(h_i\tau'_i))$). Indeed $\bar{h}_i$ (resp. $\bar{h'}_i$) lifts on M_i to h_i and $h_i\tau_i$ (resp. h_i and $h_i\tau'_i$). Now $h_i\tau_i$ (resp. $h_i\tau'_i$) is conjugated to τ_i (resp. τ'_i) when $i < d$ or to τ'_i (resp. τ_i) when $i = d$, so $\bar{h}_i$ (resp. $\bar{h'}_i$) has a non empty fixed-point set which must be a trivial knot by Smith's conjecture.

In the singular set of the orbifold $(\mathbf{S}^3, L_i \cup A_i)$ (resp. $(\mathbf{S}^3, L'_i \cup A'_i)$) the components of L_i (resp. L'_i) have order 2, while the components of A_i (resp. A'_i) can have different orders, all of them powers of 2, perhaps $1 = 2^0$. In this latter case, $Fix(h_1)$ is empty. Notice, moreover that $L_i = p_{\tau_i}(Fix(\tau_i))$ (resp. $L'_i = p_{\tau'_i}(Fix(\tau'_i))$).

The maps h_i, τ_i and $h_i\tau_i$ (resp. h_i, τ_i' and $h_i\tau_i'$) commute.

Notice that, since τ_d and τ_d' commute, the singular sets L_{d+1} and L_{d+1}' coincide. For the same reason A_{d+1} and A_{d+1}' coincide as well.

Proposition 1:

The map h_i acts freely if and only if $Fix(\tau_i) \cap Fix(h_i\tau_i)$ is empty (equivalently if and only if $Fix(\tau_i) \cap Fix(h_i\tau_i)$ is empty). Moreover $Fix(\tau_i) \cap Fix(h_i\tau_i)$ and $Fix(\tau_i') \cap Fix(h_i\tau_i')$ are contained in $Fix(h_i)$.

Proof:

Let $x \in Fix(\tau_i) \cap Fix(h_i\tau_i)$. Then $h_i(x) = h_i\tau_i(x)$ since $x \in Fix(\tau_i)$ and $h_i\tau_i(x) = x$ because $x \in Fix(h_i\tau_i)$. So h_i fixes x and this implies that if h_i acts freely, $Fix(\tau_i) \cap Fix(h_i\tau_i)$ must be empty. Suppose now that $Fix(h_i)$ is not empty. Now $Fix(\bar{h}_i) = p_{\tau_i}(Fix(h_i) \cup Fix(h_i\tau_i))$ is the unknot and τ_i preserves the fixed-point sets of both h_i and $h_i\tau_i$. Since the image is connected the two fixed-point sets must intersect. It is now clear that the intersection is contained in the fixed-point set of τ_i since these map commute.

Remark:

Notice that if h_i is free so is h_{i+1}. Indeed let $x \in Fix(\tau_{i+1}) \cup Fix(h_{i+1}\tau_{i+1})$ and let $y \in M_i$ be such that $p_{h_i}(y) = x$. We must have:

$$(1) \qquad (\tau_i\tau_i')^{2^{d-i-1}}\tau_i(y) = \begin{cases} y \\ h_i(y) = (\tau_i\tau_i')^{2^{d-i}}(y) \end{cases}$$

and

$$(2) \qquad \tau_i(y) = \begin{cases} y \\ h_i(y) = (\tau_i\tau_i')^{2^{d-i}}(y) \end{cases}$$

Assume that $\tau_i(y) = y$, then from (1) one obtains $(\tau_i\tau_i')^{2^{d-i}}(y) = y$, contrary to our hypothesis. Then it must be $\tau_i(y) = h_i(y)$ but even in this case one reaches the same conclusion.

One can also prove the assertion remembering that h_i is the map induced by $(\tau_1\tau_1')^{2^{d-i}}$, then if $(\tau_1\tau_1')^{2^{d-i}}$ is free so must be $(\tau_1\tau_1')^{2^{d-i-1}}$. Arguing in similar way, one can check that if h_{i+1} is not free neither is h_i.

Note, moreover, that $\bar{A}_i \subset A_i$ (resp. $\bar{A'}_i \subset A_i'$) and that $p_{\bar{h}_i}(\bar{A}_i)$ (resp. $p_{\bar{h'}_i}(\bar{A'}_i)$) contains $\bar{A}_{i+1}$ (resp. $\bar{A'}_{i+1}$).

We are now able to prove part i) of Theorem 1.

Proof of Theorem 1 i):

Since by hypothesis $\bar{h}_1$ cannot be a strong inversion, $p_{\tau_1}^{-1}(Fix(\bar{h}_1)) = Fix(h_1\tau_1)$, according to Proposition 1, and it must consist of at most two components since it is a 2-fold covering of a trivial knot. On the other hand, the number of components of $Fix(h_1\tau_1)$ is equal to the number of components of $Fix(\tau_1)$ if $d > 1$ or of $Fix(\tau_1')$ if $d = 1$ since the maps are conjugate (see the Remarks above), but this is a contradiction. Indeed in the case when $d = 1$, one can exchange the roles of τ_1 and τ_1'.

Now we wish to understand more clearly diagram $(*)$. To do this, we consider three cases according to the behaviour of the maps h_i:

Case $\mathcal{P}$ – h_1 is free; equivalently h_i is free for all $1 \leq i \leq d$.

Case $\mathcal{I}$ – h_d is not free; equivalently h_i is not free for all $1 \leq i \leq d$.

Case $\mathcal{M}$ – there exists a j, $1 < j \leq d$, such that h_j is free but h_{j-1} is not.

We shall discuss these cases in the given order in the following three sections.

3. The free case $\mathcal{P}$

The main aim of this Section is to give the proof of last part of Theorem 1 and of Theorem 2.

We shall consider links which admit $\mathcal{F}$-mutants and, consequently a diagram as $(*)$. By our extra assumption h_i is free for all $1 \leq i \leq d$. By Proposition 1 we know that $\bar{h}_i$ can never be a strong inversion. By Theorem 1 i), we are left to consider only knots and links with two components.

Statement 1:

Let $L = L_1$ be a knot and let $d = 1$. Then $\bar{h}_1$ is a 2-periodic symmetry of L_1 and $p_{\bar{h}_1}(L_1)$ is a trivial knot. L_2 is a link with two trivial components which cannot be exchanged.

Proof:

Since $h_1\tau_1' = \tau_1$, we have $p_{\bar{h}_1}(L_1) = p_{\bar{h}_1}p_{\tau_1}(Fix(\tau_1)) = p_{\bar{h}_1}p_{\tau_1}(Fix(h_1\tau_1')) = p_{\bar{h'}_1}p_{\tau_1'}(Fix(h_1\tau_1')) = p_{\bar{h'}_1}(Fix(\bar{h'}_1))$. Since $Fix(\bar{h'}_1)$ is a trivial knot by Smith's conjecture, the first part of the statement is proved.

For the second part it is sufficient to observe that if the two components of L_2 were exchangeable, L_1 and L_1' would coincide, contrary to the hypothesis that they are $\mathcal{F}$-mutants.

Statement 2:

Let L_1 be a knot and $d > 1$. Then L_i, $1 < i \leq d+1$, is a link with two trivial components: for $2 \leq i \leq d$, $\bar{h}_i$ exchanges them, while the two components of L_{d+1} cannot be exchanged.

Proof:

Recall that L_2 is the disjoint union of $p_{\bar{h}_1}(L_1)$ and of $p_{\bar{h}_1}(Fix(\bar{h}_1))$. Since $Fix(\bar{h}_1)$ is a trivial knot, to prove the Statement it is enough to prove that $\bar{h}_2$ exchanges the components of L_2. Suppose, on the contrary, that $\bar{h}_2$ preserves each component: L_3 has three components. Now assume $d > 3$, then $L_3 = p_{\tau_3}(Fix(\tau_3))$ which must have the same number of components of $p_{\tau_3}^{-1}(Fix(\bar{h}_3))$ and this is absurd. If $d = 3$ instead, $p_{\tau_3'}(Fix(\tau_3)) = p_{\tau_3'}(Fix(h_3\tau_3'))$ has only one component and, by commutativity of the diagram, $p_{\bar{h}_3}(L_2)$ must have one component.

With similar arguments one can prove

Statement 3:

Let L_1 a two component link. Then L_i, $1 \leq i \leq d+1$, is a link with two trivial components which are exchanged by $\bar{h}_i$ if $1 \leq i \leq d$. The two components of L_{d+1} are not exchangeable.

Proof of Theorem 1 ii) and iii):
The proof is an easy consequence of the preceding Statements since cases $\mathcal{I}$ and $\mathcal{M}$ are possible only for strongly invertible links.

Corollary 1:

Let $L \in \mathcal{F}$ be a non strongly invertible two-component link. Let $D_L(x,t) \in \mathbb{Z}[x, x^{-1}, t, t^{-1}]$ be its two-variable Alexander polynomial (see [3] *for the definition). Assume that $D(x,t) \neq D(t,x)$. Then L is determined by its 2-fold cyclic branched covering.*

Proof:
It is sufficient to observe that the two components cannot be exchangeable in this case.

Proof of Theorem 2:
Let L be a non strongly invertible knot and $\Delta_L(t)$ its Alexander polynomial. If L admits a 2-periodic symmetry h, it must be [11]

$$\Delta_L(t) \equiv (\Delta_{p_h(L)}(t))^2 \sum_{i=0}^{\lambda-1} t^i \pmod 2, \quad \lambda := lk(L, Fix(h))$$

where $p_h(L)$ is the quotient of L with respect to the action of h and $\Delta_{p_h(L)}(t)$ is its Alexander polynomial. Assume that $p_h(L)$ is trivial so that $\Delta_{p_h(L)}(t) = 1$ and $\Delta_L(t) \equiv \sum_{i=0}^{\lambda-1} t^i \pmod 2$. If this equivalence does not hold, it follows from Theorem 1 iii) that the knot is determined.

Corollary 2:

Let $L \in \mathcal{F}$ be non strongly invertible fibred knot. If its Alexander polynomial has an even coefficient, then L is determined by its 2-fold cyclic branched covering.

Proof:
It is sufficient to observe that, in this case and with the notation of Theorem 2, $a_n = \pm 1$ and $j = n$.

4. The strongly invertible case $\mathcal{I}$

Throughout this section we shall consider links with $\mathcal{F}$-mutants, thus admitting a diagram of type $(*)$, satisfying the extra condition $Fix(h_i) \supset (Fix(\tau_i) \cap Fix(h_i\tau_i)) \neq \varnothing$. We start with the following

Proposition 2:
$Fix(\tau_i) \cap Fix(h_i\tau_i)$ consists of a finite number of points. Each connected component of $Fix(h_i)$, $Fix(\tau_i)$ and $Fix(h_i\tau_i)$ contains exactly two points of $Fix(\tau_i) \cap Fix(h_i\tau_i)$. In particular $Fix(\tau_i) \cap Fix(h_i\tau_i) = Fix(h_i) \cap Fix(h_i\tau_i) = Fix(\tau_i) \cap Fix(h_i) = Fix(h_i) \cap Fix(\tau_i) \cap Fix(h_i\tau_i)$ and the fixed-point sets of h_i, τ_i, $h_i\tau_i$ have the same number of connected components.

Proof:
Suppose that the three involutions have a common fixed-point component K. If L_i is a knot, it must be the trivial one: $\bar{h}_i$ would fix exactly L_i which would be trivial because of Smith's conjecture. This is absurd by Waldhausen result [21]. Then both τ_i and $h_i\tau_i$ must have at least two fixed-point components, which is again absurd.

Assume now that there exists a component of $Fix(h_i)$ or of $Fix(h_i\tau_i)$ which does not intersect $Fix(\tau_i)$. In this case again $Fix(\bar{h}_i)$ would have at least two components: a contradiction.

Since τ_i cannot act as the identity on any of the components of $Fix(h_i)$ and of $Fix(h_i\tau_i)$ and since it fixes at least one point on all of them, it turns out that it must act as a reflection and fix exactly two points on all of them.

Suppose that there exists a component K of $Fix(\tau_i)$ which does not intersect $Fix(h_i\tau_i)$. If $i = d$ this cannot happen because otherwise $Fix(\bar{h}'_i)$ would have at least two components. So $i < d$ and $h_i\tau_i = (\tau_i\tau_i')^{2^{d-i-1}}\tau_i(\tau_i'\tau_i)^{2^{d-i-1}}$. We have $Fix(\tau_i) = (\tau_i'\tau_i)^{2^{d-i-1}}(Fix(h_i\tau_i))$ so that $\bar{K} := (\tau_i\tau_i')^{2^{d-i-1}}(K) \subset Fix(h_i\tau_i)$. Now there exists, by the above discussion, $x \in Fix(\tau_i) \cap \bar{K} \subset Fix(h_i)$. We have that $(\tau_i'\tau_i)^{2^{d-i-1}}(x) \in K$ is fixed by h_i, for h_i and $(\tau_i'\tau_i)^{2^{d-i-1}}$ commute, thus it is fixed also by $h_i\tau_i$: a contradiction.

Remark:

Both $Fix(\tau_i) \cup Fix(h_i)$ and $Fix(h_i) \cup Fix(h_i\tau_i)$ are connected. In the second case this is clear since the image $p_{\tau_i}(Fix(h_i) \cup Fix(h_i\tau_i))$ must be the unknot and the map p_{τ_i} preserves all components of the fixed-point sets. In the first case it is enough to note that either $i = d$ so that $p_{\tau'_i}(Fix(\tau_i) \cup Fix(h_i)) = Fix(\bar{h'}_i)$, or $i < d$ and one can exchange the roles of τ_i and $h_i\tau_i$. Indeed $((h_i\tau_i)\tau'_i)^{2^{d-i}} = (\tau_i\tau'_i)^{2^{d-i}} = h_i$, because of the commutativity of h_i, and $h_i(h_i\tau_i) = \tau_i$.

Statement 4:

For $1 \leq i \leq d$, L_i is a strongly invertible link and $\bar{A}_i$ is not empty. Moreover $\bar{h}_i$ acts on such link as a strong inversion which preserves A_i. In particular $Fix(\bar{h}_i) \cap A_i = \bar{A}_i$ and the map $\bar{h}_i$ must preserve set-wise $p_{\bar{h}_{i-1}} p_{\bar{h}_{i-2}} ... p_{\bar{h}_j}(\bar{A}_j)$ for all $j < i$, since it must preserve the components of the singular set of the orbifold with the same order of singularity. The singular set of the orbifolds appearing in diagram $(*)$ is a trivalent graph. The two subgraphs $p_{\bar{h}_i} p_{\tau_i}(Fix(h_i) \cup Fix(\tau_i))$ and $p_{\bar{h}_i} p_{\tau_i}(Fix(h_i) \cup Fix(h_i\tau_i))$ are trivial knots. Again the link L_{d+1} does not admit a strong inversion preserving A_{d+1} in the way described.
If L_1 is a knot, then the trivalent graph which constitutes the singular set of the orbifolds is a θ-curve with (at least) two trivial constituent knots. In this case A_i and $\bar{A}_i$ coincide for all i.

Proof:
The Statement follows readily from Proposition 2 and the above Remark. We only wish to point out that since A_i is the image of $Fix(h_1)$, the map induced by h_i must preserve it. Notice that in fact the number of connected components of $Fix((\tau_1\tau'_1)^{2^{d-i}})$ is in general larger that the number of components of $Fix((\tau_1\tau'_1)^{2^{d-i-1}})$, unless L is a knot. This implies in particular that the number of connected components of $Fix(\tau_i)$ can decrease as i increases.

5. The mixed case $\mathcal{M}$

The description of this case is the same of that given for case $\mathcal{I}$ whenever $i \leq j - 1$.

Statement 5:

The link L_j has at most two components and $\bar{h}_j$ is a 2-periodic symmetry of the link which preserves A_j. Moreover, if L_j has exactly one component $\bar{h}_j$ acts as a strong inversion on $p_{\bar{h}_{j-1}}(\bar{A}_{j-1})$. In this case L_i, $j+1 \leq i \leq d$, is a two trivial component link whose components are exchanged by $\bar{h}_i$. As usual $\bar{h}_i$ preserves A_i. If L_j is a two component link, then its components are trivial and $\bar{h}_j$ exchanges them. Again $\bar{h}_j$ preserves A_j but nothing can be said about its action on $p_{\bar{h}_{j-1}}(\bar{A}_{j-1})$.

For $j+1 \leq i \leq d$, the behaviour is the same described in the one component case. In any case L_{d+1} is a link with two trivial components which cannot be exchanged by a 2-periodic symmetry preserving A_{d+1}.

Proof:
The only thing to prove is the fact that the number of components cannot exceed two, the remaining considerations being proved as in case $\mathcal{P}$. But this is again easily seen to be true since either τ_j is conjugated to $h_j\tau_j$, whose fixed-point set cannot have more than two components, or because $Fix(\tau_j)$ itself maps to $Fix(\bar{h'}_j)$ (case $j = d$).

6. $\mathcal{F}$-mutant links

We want now to show how the results of the previous Sections lead to the construction of all $\mathcal{F}$-mutants of a given link L in $\mathcal{F}$. We want to do so without knowing M and $Iso(M)$. If an $\mathcal{F}$-mutant L' of L exists, then L admits a 2-periodic symmetry or a strong inversion and a sequence of quotients satisfying the requirements given in Statements 1 to 5. All we need to know is:

- Q1 when do we reach L_{d+1}?
- Q2 in the case of strong inversions, which part of the fixed-point set comes from $Fix(h_i)$ and which from $Fix(h_i\tau_i)$?
- Q3 are there quotients which will not give $\mathcal{F}$-mutants (but only, say, again the link L or some link in a manifold different from $\mathbf{S}^3$)?

It is then clear that one can reconstruct L' by successive lifts of a suitable component (or sub-arcs) of L_{d+1}. Note that lifting a knot (or an arc) with respect to the unknotted (in the case of the arc, its endpoints belong to the unknotted) to the 2-fold covering corresponds to symmetrize the knot (or the arc) with respect to the unknotted itself.

Before answering to the above questions, we underline the fact that each L_i admits only a finite number of 2-periodic symmetries and strong inversions, for $Iso(M)$ is finite.

Let us now consider separately the cases when we quotient by means only of 2-periodic symmetries, only of strong inversions or by means of both. The $\mathcal{F}$-mutants obtained in these three cases will be called $(\mathcal{F},\mathcal{P})$-, $(\mathcal{F},\mathcal{I})$- and $(\mathcal{F},\mathcal{M})$-mutants respectively.

Case $\mathcal{P}$

This is the simplest case to consider. Recall that only knots and links with two trivial components can have $(\mathcal{F},\mathcal{P})$-mutants.

Let $L = L_1$ be a link with two trivial components. We must consider all 2-periodic symmetries exchanging them. If 2-periodic symmetries of this type do not exist, then L does not have $(\mathcal{F}, \mathcal{P})$-mutants. Let $\bar{h}_1$ be any such 2-periodic symmetry. Define $L_2 := p_{\bar{h}_1}(L_1 \cup Fix(\bar{h}_1))$ and repeat the reasoning. We reach L_{d+1} when the two components of the link cannot be any longer exchanged. Indeed, if the components can be exchanged, the two links obtained by lifting one with respect to the other coincide. L' can be recovered by lifting $p_{\bar{h}_d}(Fix(\bar{h}_d))$ with respect to $p_{\bar{h}_d}(L_d)$, thus obtaining L'_d, and successively by lifting any of the components of L'_i with respect to the other. Remark, once more, that the map $\bar{h}_i$ need not be unique (see [9], [17], [18]) and its necessary to consider all such maps.

Let $L = L_1$ be a knot admitting a 2-periodic symmetry $\bar{h}_1$ such that $p_{\bar{h}_1}(L_1)$ is the trivial knot. Let $L_2 := p_{\bar{h}_1}(L_1 \cup Fix(\bar{h}_1))$ and repeat the considerations above. Notice again that $\bar{h}_1$ is not necessarily uniquely determined even if there are classes of knots for which it is (e.g. hyperbolic knots, knots which are prime and pedigreed [17]).

Remark:

There exist knots and two components links which admit an infinite sequence of quotients $p_{\bar{h}_i}$. This is the case of the torus knots and links of type $(2, n)$.

Case $\mathcal{I}$

Let now $\bar{h}_1$ be a strong inversion of $L = L_1$. We have that $p_{\bar{h}_1}(L_1 \cup Fix(\bar{h}_1))$ is a trivalent graph. There exist exactly two bivalent closed subgraphs (i.e. links) containing $p_{\bar{h}_1}(L_1)$. These correspond to the two possible choices $p_{\bar{h}_1}p_{\tau_1}(Fix(h_1))$ and $p_{\bar{h}_1}p_{\tau_1}(Fix(h_1\tau_1))$ of their complement in the graph. If both of them are not the trivial knot, then we shall not obtain an $(\mathcal{F}, \mathcal{I})$-mutant. Indeed $p_{h_1\tau_1}(M) \cong \mathbf{S}^3$ which is the 2-fold covering of $\mathbf{S}^3$ branched along the complement of the arcs $p_{\bar{h}_1}p_{\tau_1}(Fix(h_1\tau_1))$ by commutativity.

Suppose now that both components are trivial. In this case $d = 1$ because $p_{h_1}(M) \cong \mathbf{S}^3$ and $p_{\bar{h'}_1}(M)$ is the lift of the quotient of $p_{h_1}(M)$ by the action induced by either τ_1 or τ'_1. So the (possible) $(\mathcal{F}, \mathcal{I})$-mutants of L are obtained by lifting the complements in the trivalent graph of the two trivial knots. In this case then we may obtain two $(\mathcal{F}, \mathcal{I})$-mutants.

Assume now that exactly one subgraph is a trivial knot. Define L_2 to be the other subgraph and A_2 to be its complement. Now repeat the considerations above keeping in mind that the whole trivalent graph must be preserved by the new strong inversion of L_2 according to the requirements of Statement 4. Note that $\bar{A}_i = A_i \cap Fix(\bar{h}_i)$. Moreover it is convenient to work with orbifolds: not all the components of A_i have the same order of singularity and $\bar{h}_i$ must respect the orders as well. Indeed $p_{\bar{h}_i}(\bar{A}_i)$ has order of singularity equal to 2^i. Remark that we reach L_{d+1} when either there does not exists a strong inversion satisfying these requirements or the two subgraphs described before are trivial knots. To recover L', consider the trivalent graph $p_{\bar{h}_d}(L_d \cup Fix(\bar{h}_d))$. L'_d is the lift of the components of $p_{\bar{h}_d}(Fix(\bar{h}_d))$ which are not in A_{d+1}, with respect to the union of the components which are in A_{d+1} and of $p_{\bar{h}_d}(L_d)$. The lift of A_{d+1} will be A'_d. L'_{i-1} will then be the lift of any of the halves of L'_i with respect to the other half together with all the components of A'_i of maximal singular order.

Proposition 3:

Let $L \in \mathcal{F}$ be a knot. Then the number of $(\mathcal{F},\mathcal{I})$-mutants of L is lesser or equal to $2s$, where s is the number of non equivalent strong inversions of L. Moreover if the 2-fold branched covering of $\mathbf{S}^3$ branched along L is not an integral homology 3-sphere, then the number of $(\mathcal{F},\mathcal{I})$-mutants of L is lesser or equal to s.

Proof:

Recall that in our case the trivalent graph is a θ-curve and all maps are isometries of orbifolds. If $d = 1$ and all constituent knots of the θ-curve are trivial, then we obtain at most two $(\mathcal{F},\mathcal{I})$-mutants of L. Otherwise it is enough to show that, fixed $\bar{h}_1$, there can exist at most one $\bar{h}_2$. Indeed $\bar{h}_2$ must act as a strong inversion on the non trivial constituent knot of the θ-curve and must contain in its axis the remaining arc. Suppose there were two such strong inversions. Their product fixes the θ-curve and must have finite order since $Iso(M)$ is finite. By Smith's theory, such product is thus the identity and the two strong inversions coincide. It is now clear that for each strong inversion of L we can obtain at most two $(\mathcal{F},\mathcal{I})$-mutants. In particular we can obtain at most one mutant if $d > 1$. Indeed if $M_{d+1} \cong \mathbf{S}^3$, $p_{h_d}(Fix(h_d))$ cannot be the trivial knot, otherwise M would be $\mathbf{S}^3$ contrary to Waldhausen result [21]. In particular M_i, $i \leq d$, cannot be $\mathbf{S}^3$.

To prove the second part of the Proposition, we exploit the fact that the first homology group of the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-fold branched covering of a θ-curve is isomorphic to the direct sum of the first homology groups of the 2-fold branched covering of the constituent knots (the proof of this fact can be found in [13]). In our situation we have $H_1(M;\mathbb{Z}) \cong H_1(M_i;\mathbb{Z})$ for all $1 \leq i \leq d+1$, independently of the choice of $\bar{h}_1$. If we have $d = 1$ and $M_2 \cong \mathbf{S}^3$ for some choice of $\bar{h}_1$, then M must be an integral homology 3-sphere. This proves the Proposition.

Case $\mathcal{M}$

Assume that in case $\mathcal{I}$ we find an L_j, $1 < j \leq d$ with at most two components. More precisely, if L_j has two components they must be trivial knots while if it has only one component it cannot be trivial (else $j = d+1$). In these cases we must consider also 2-periodic symmetries. For such 2-periodic symmetries one reasons as in case $\mathcal{P}$, remembering that in this case $\bar{h}_i$, $j \leq i \leq d$, must now satisfy the extra requirements of Statement 5. Moreover remember that we need to recover not only L'_j but also A'_j to be able to proceed as in case $\mathcal{I}$.

To conclude we make a few considerations for the situation when L is a knot. In this case $\bar{h}_j$ is uniquely determined (as in the proof of Proposition 3) since it must rotate L_j and act as a strong inversion on the remaining arc of the θ-curve corresponding to A_j. Now L_{j+1} is a link with two trivial components and A_{j+1} is an arc with an endpoint on each of them. Again $\bar{h}_{j+1}$ must exchange the two components acting as a strong inversion on A_{j+1}. It is easy to see that one can have at most two such $\bar{h}_{j+1}$ (according to the fact that $\bar{h}_{j+1}$ can preserve or reverse arbitrary orientations of the two components of the link). This means that for each strong inversion $\bar{h}_1$ of L there are at most $\sum_{j=2}^{l} 2^{d-j} = 2^d - 1$ $(\mathcal{F}, \mathcal{M})$-mutants of L. Unfortunately d cannot be estimated only in terms of the symmetries of L.

7. Fibred knots

The aim of this Section is to give an example of a family of knots which are not determined by their 2-fold branched coverings. The example shows that certain properties of a knot are not necessarily carried by its $\mathcal{P}$-mutants. These properties are in particular the genus and the fact of being fibred.

In [4] the infinite family of closed braids of Figure 1 is considered.

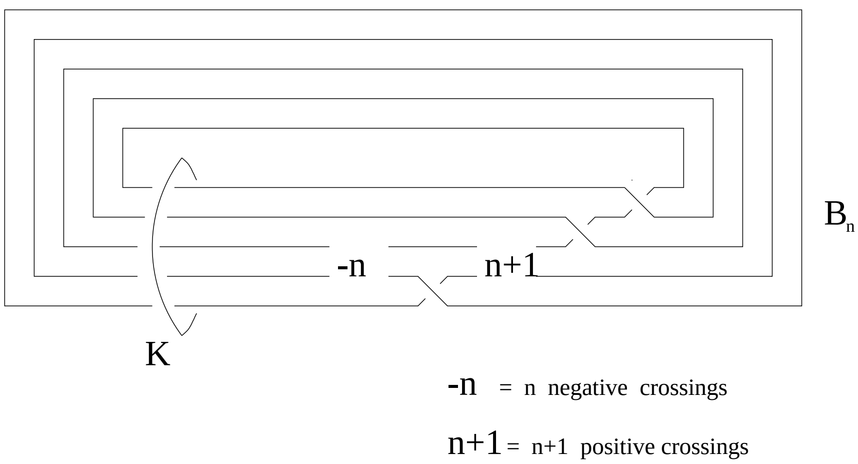

Figure 1

For all $n \geq 1$, B_n is a trivial knot. Let K_n denote the lift of the axis of the closed braid B_n to the 2-fold covering of $\mathbf{S}^3$ branched along B_n. For all n, K_n is a fibred knot by [7]. The genus of K_n is 2 and its Alexander polynomial is given by the expression

$$\Delta_{K_n}(x) = x^4 + (n^2 + n - 1)x^3 + (-2n^2 - 2n + 1)x^2 + (n^2 + n - 1)x + 1.$$

Let now $\widetilde{B_n}$ be the lift of $B_n$to the 2-fold cyclic covering of $\mathbf{S}^3$ branched along K. using the reduced Burau representation $R_n(t)$ of the braid B_n one can compute the Alexander polynomial $D_n(x,t)$ of the link $K \cup B_n$ (see [10]). One has $D_n(x,t) = \det(R_n(t) - xI)$. Using $D_n(x,t)$ one can compute both $\Delta_{K_n}(x) = D_n(x,-1)$ and $\Delta_{\widetilde{B_n}}(t) = D_n(-1,t)$ (see [11]). We obtain

$$\Delta_{\widetilde{B_n}}(t) = 2(1-t)f_n(t)f_{n+1}(t) - (-t)^{n-2}(t^4 - t^3 + t^2 - t + 1)$$

where $f_n(t) := \sum_{i=0}^{n-1}(-t)^i$. One has

$$\deg(\Delta_{\widetilde{B_n}}(t)) = \begin{cases} 4 & n = 1,2 \\ 2n & n \geq 3 \end{cases}$$

and

$$\Delta_{\widetilde{B_n}}(0) = \begin{cases} 1 & n = 1,2 \\ 2 & n \geq 3 \end{cases} .$$

It is now easy to see that the knots $\widetilde{B_n}$ and K_n are different exactly for $n \geq 2$ (for $n = 1$ one can check directly that they coincide). If $n \geq 3$ the knots $\widetilde{B_n}$ cannot be fibred since $\Delta_{\widetilde{B_n}}(0) \neq \pm 1$. Notice to conclude that the knots $\widetilde{B_n}$ have arbitrary large genera; indeed $g(\widetilde{B_n}) \geq n$ (see [15]).

Observe to conclude that the construction of this Sections work not only for 2-fold coverings but for arbitrary q-fold cyclic branched coverings. It is again true that the lifts of the axis of the braids are fibred knots (if $q \not\equiv 0 \pmod 5$) of genus $2q-2$ and with Alexander polynomials of degree $4(q-1)$. The degree of the Alexander polynomial of the lift of the braid, instead, increases with n so, for n large enough, we can construct infinitely many fibred knots which are not determined by their q-fold branched covering, whenever $q \not\equiv 0 \pmod 5$.

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