SUPPLEMENTARY for
Exploring the depths of Autocorrelation Function: its departure from normality

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1. Results for simulated WN with length $n = 100$

In the main paper, in Section 4, we simulate $N_S = 5\,000$ white noises of length $n = 500$. Here, we simulate N_S WN with length $n = 100$.

1.1. Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk. Figure S1 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 100$, whereas Figure S2 gives the p-values provided by Shapiro-Wilk.

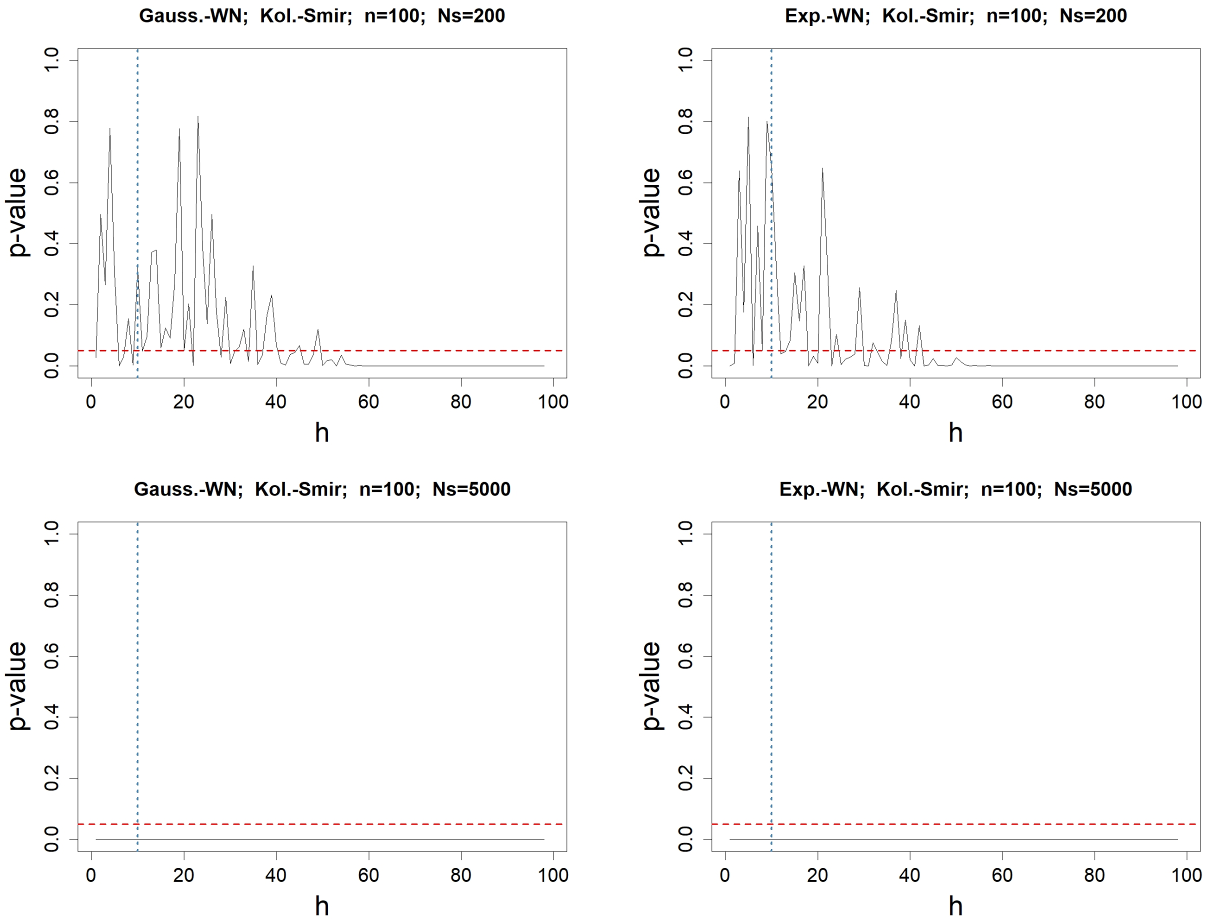

Figure S1.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns Gaussian WN whereas the right one deals with Exponential WN process. The length of the simulated WN process is $n = 100$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

The results in Figure S1, top line, show that $\hat{\Xi}(h)$ behaves roughly like a Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, as it was expected from Theorem 2.1, even for a lag h much greater than $\sqrt{n}$. But the graphics in the bottom line show that $\hat{\Xi}(h)$ deviates from this specific normal distribution

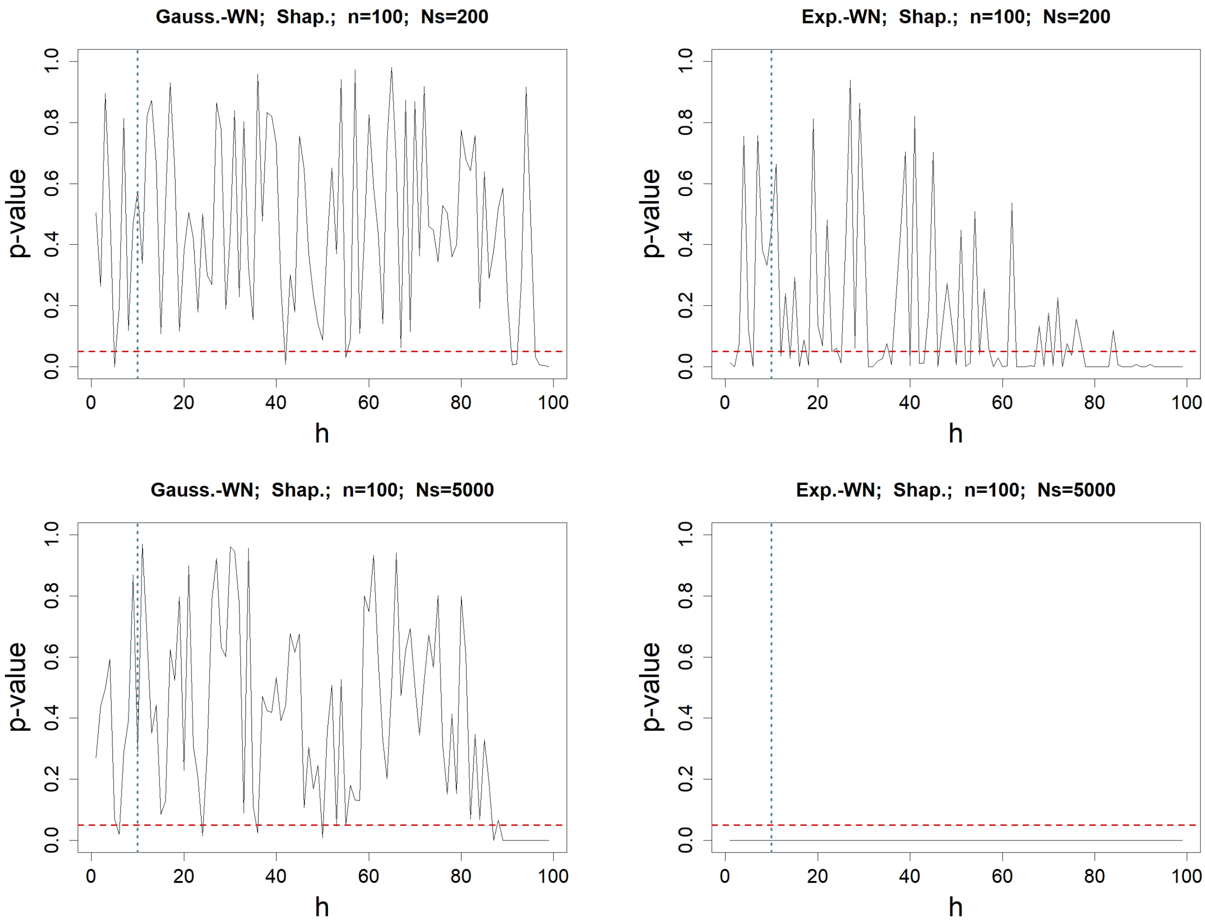

Figure S2.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian WN whereas the right one deals with Exponential WN process. The length of the simulated WN process is $n = 100$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

when we lead the normality tests on more simulations. Moreover, the same is observed for the Gaussian behavior in Figure S2.

1.2. *Check for the normality of $S_{ACF}(H)$ at a fixed lag H*

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S3 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 100$.

In Figure S3, we observe that the sum of sample ACF departs from normality for almost all the lags H, except maybe for the first lags H when the white noise is Gaussian, and/or when the number of simulations remains low. Of course, Kolmogorov-Smirnov test confirmed the departure of $S_{ACF}(H)$ from $\mathcal{N}\left(0, \frac{H}{n}\right)$ at any lag, whatever the nature of the WN (Gaussian or Exponential) and even for a low number of simulations (results not shown). Furthermore Figure S3 tells us that none of the variables $S_{ACF}(H)$ is Gaussian. Thus even the normality of the vector $(\hat{\Xi}(1), \cdots, \hat{\Xi}(H))$ is called into question. Indeed, if $\left(\boldsymbol{\mathcal{A}_H(\underline{\mu}, \Sigma)}\right)$ were true, $\left(\boldsymbol{\mathcal{S}_H(\underline{\mu}, \Sigma)}\right)$

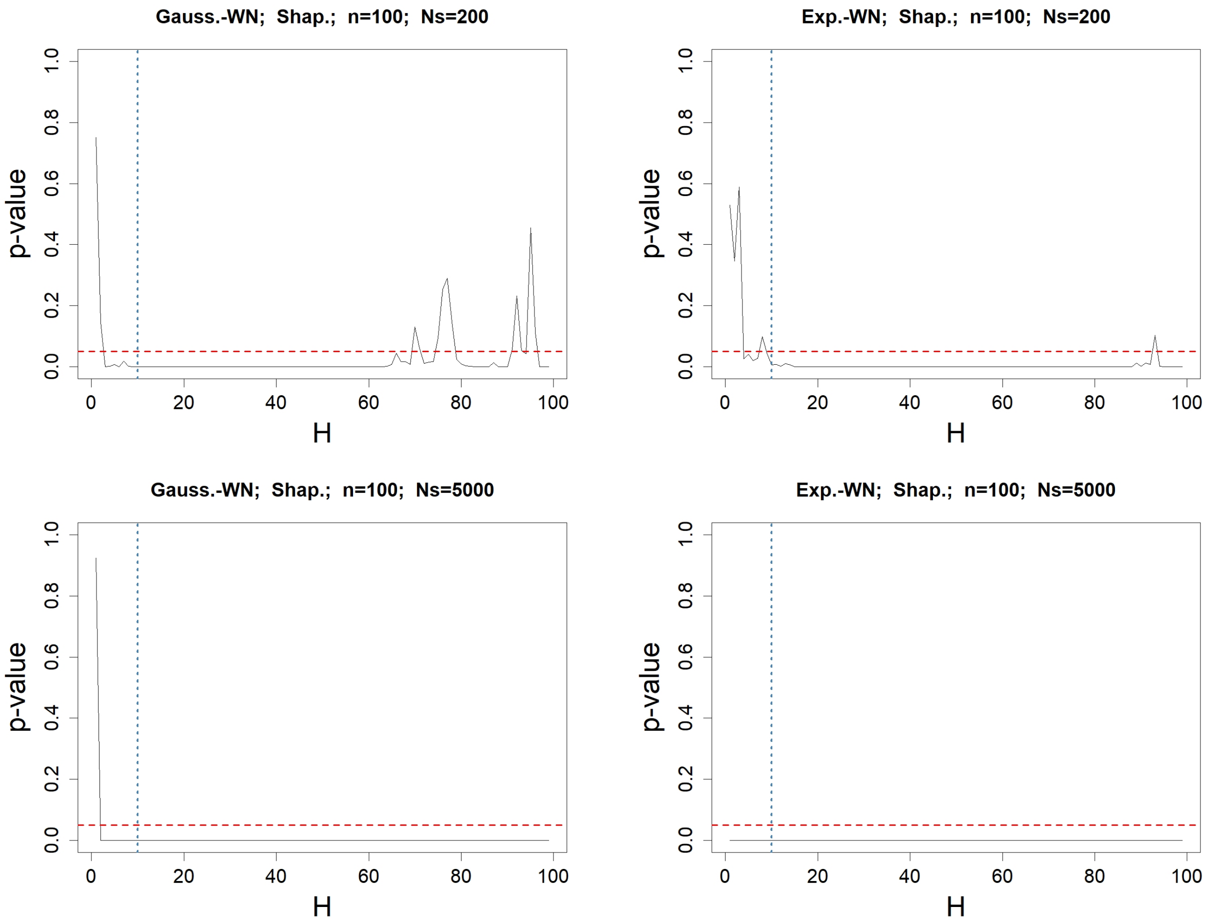

Figure S3.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian WN whereas the right one deals with Exponential WN process. The length of the simulated WN process is $n = 100$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

would also be.

Consequently, from Figures S2 and S3, we conclude that at a fixed lag h, $\hat{\Xi}(h)$ is roughly Gaussian, with $\mu \simeq 0$ or $\sigma^2 \simeq \frac{1}{n}$. But the vector $(\hat{\Xi}(1), \cdots, \hat{\Xi}(H))$ is not a Gaussian vector.

1.3. Check for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$

The previous finding raises questions about the methods used in practice to model a time series. Indeed, a model has to be validated by checking that the associated residuals are a WN, by using Box-Pierce or Ljung-Box test. We know that $(\mathcal{A}_H(\underline{0}, I_H))$ does not hold. But we wonder if it is a problem in practice. Then we adopt another point of view, that is more adequate with the practice, where we have to model a single time series, from the properties of its first $\hat{\rho}(h)$ values. For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words that they are the realizations of independent variables. Let us suppose that this hypothesis is satisfied. First we use Shapiro-Wilk to test for the normality behavior, and next Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$. Figure S4 gives the percentage of inadequate testing conclusions with Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 100$ and, whereas Figure S5 gives the same percentages with Kolmogorov-Smirnov.

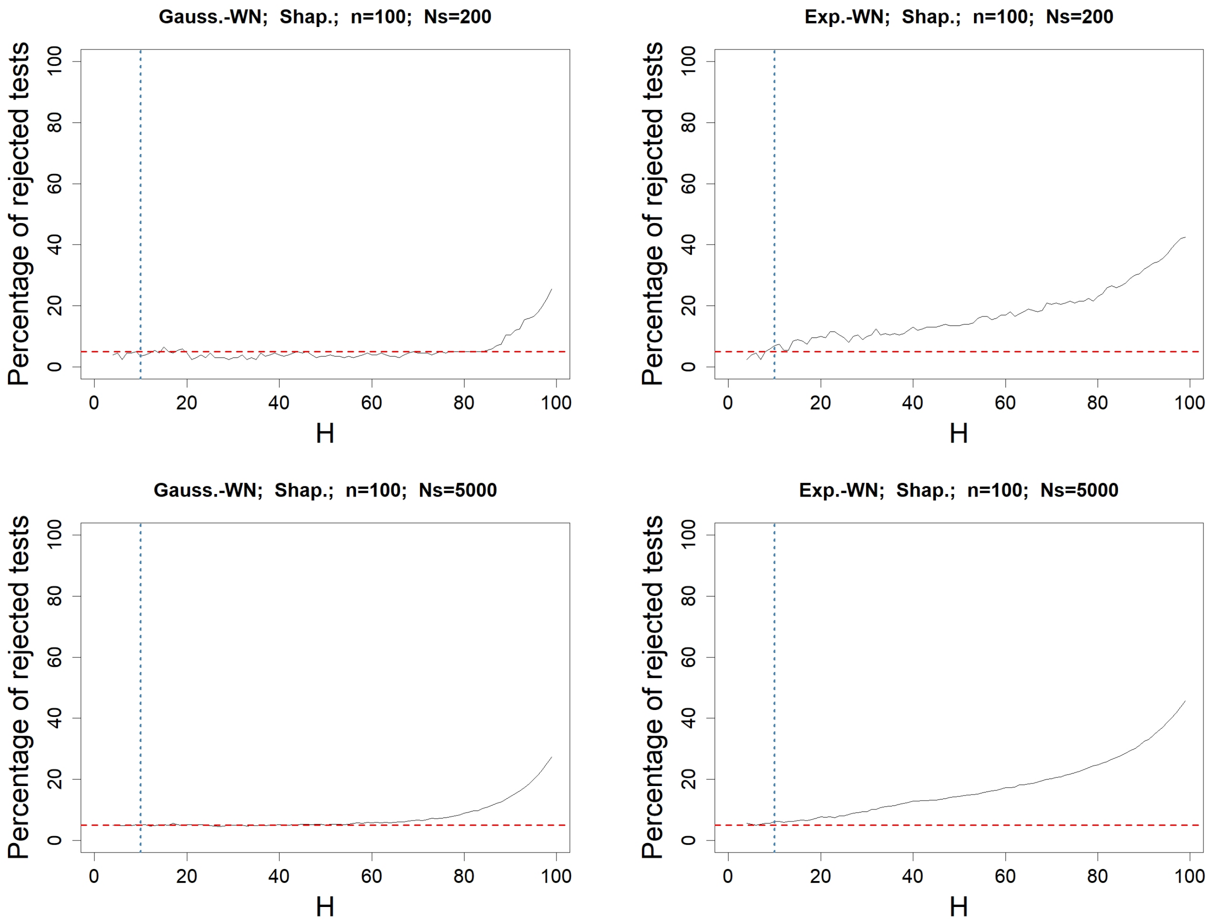

Figure S4.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with H varying from 1 to $n - 1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian WN whereas the right one deals with Exponential WN process. The length of the simulated WN process is $n = 100$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

In Figures S4 and S5, we observe that the percentage of p-values $< \alpha = 5\%$ is very close to 5%, which seems to comfort that, at least for Gaussian-WN, $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ as realizations of Gaussian variables until a lag H not too large, with the expectation and the covariance-matrix as stated in Theorem 2.1. So that these Figures seem to assess that $\left(\boldsymbol{\mathcal{A}_H}(\underline{\boldsymbol{\mu}}, \boldsymbol{\Sigma})\right)$ and even $\left(\boldsymbol{\mathcal{A}_H}(\underline{\mathbf{0}}, \boldsymbol{I_H})\right)$ are true. This last point seems contradictory with Subsection 1.1. But remember that the normality adequacy is sensitive to the number of observations used when applying the normality tests, and that a slight departure from normality is more likely to be detected when this number is large. Here, the normality of the successive $\hat{\rho}(1), \cdots, \hat{\rho}(H)$ values is assessed until a size of sample around $H = \frac{n}{2}$, which is low with respect to $N_S = 5\,000$. So that making a

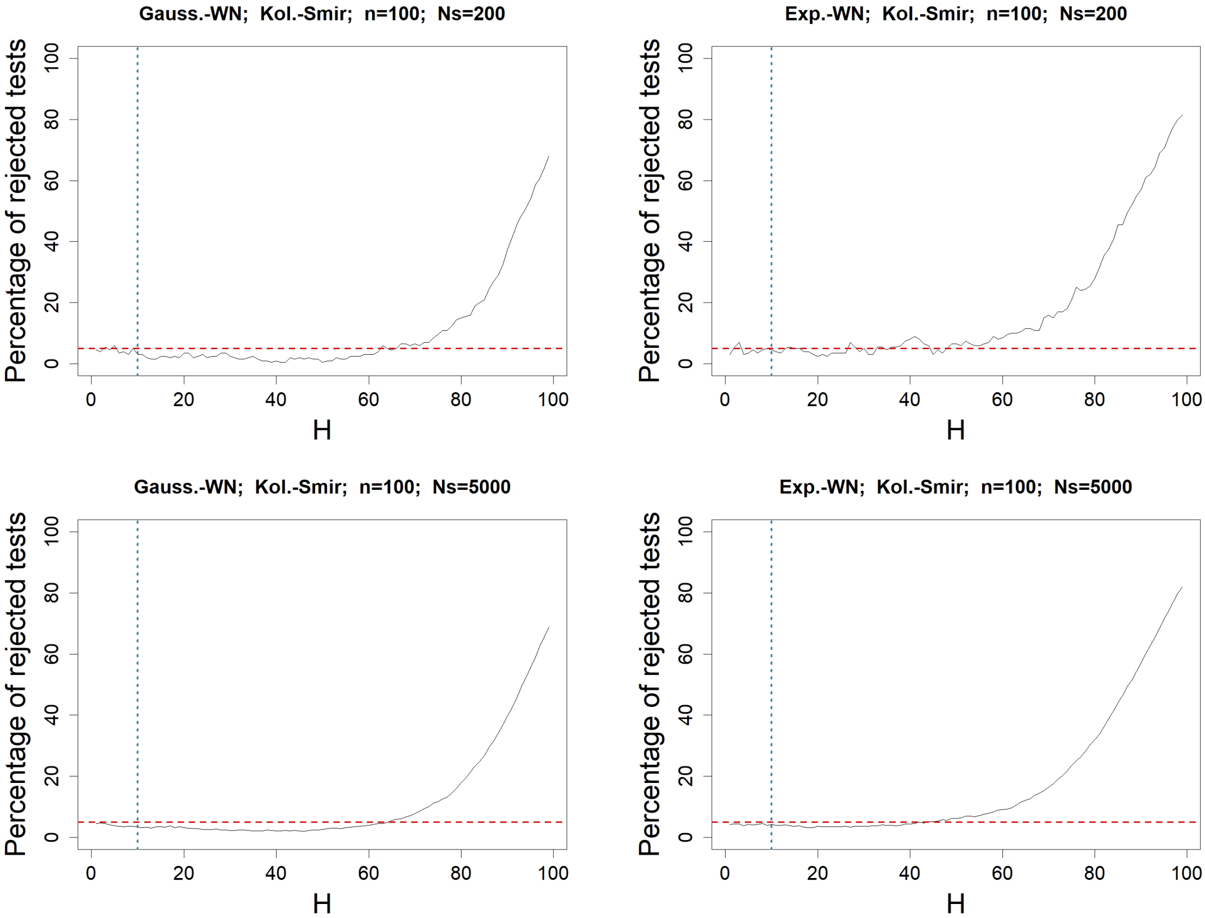

Figure S5.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns Gaussian WN whereas the right one deals with Exponential WN process. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

diagnosis from the first ACF might be acceptable. But keep in mind that the tests displayed in Figures S4 and S5 suppose that $\hat{\rho}(1), \cdots, \hat{\rho}(H)$ are realizations of independent variables, which is not guaranteed. Indeed, Hassani's $-\frac{1}{2}$-Theorem proves that $\hat{\rho}(1), \cdots, \hat{\rho}(n-1)$ can not be independent.

1.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S6 shows that Box-Pierce and Ljung-Box tests are not totally accurate. Indeed, Box-Pierce appears to be too conservative, whereas Ljung-Box is too liberal.

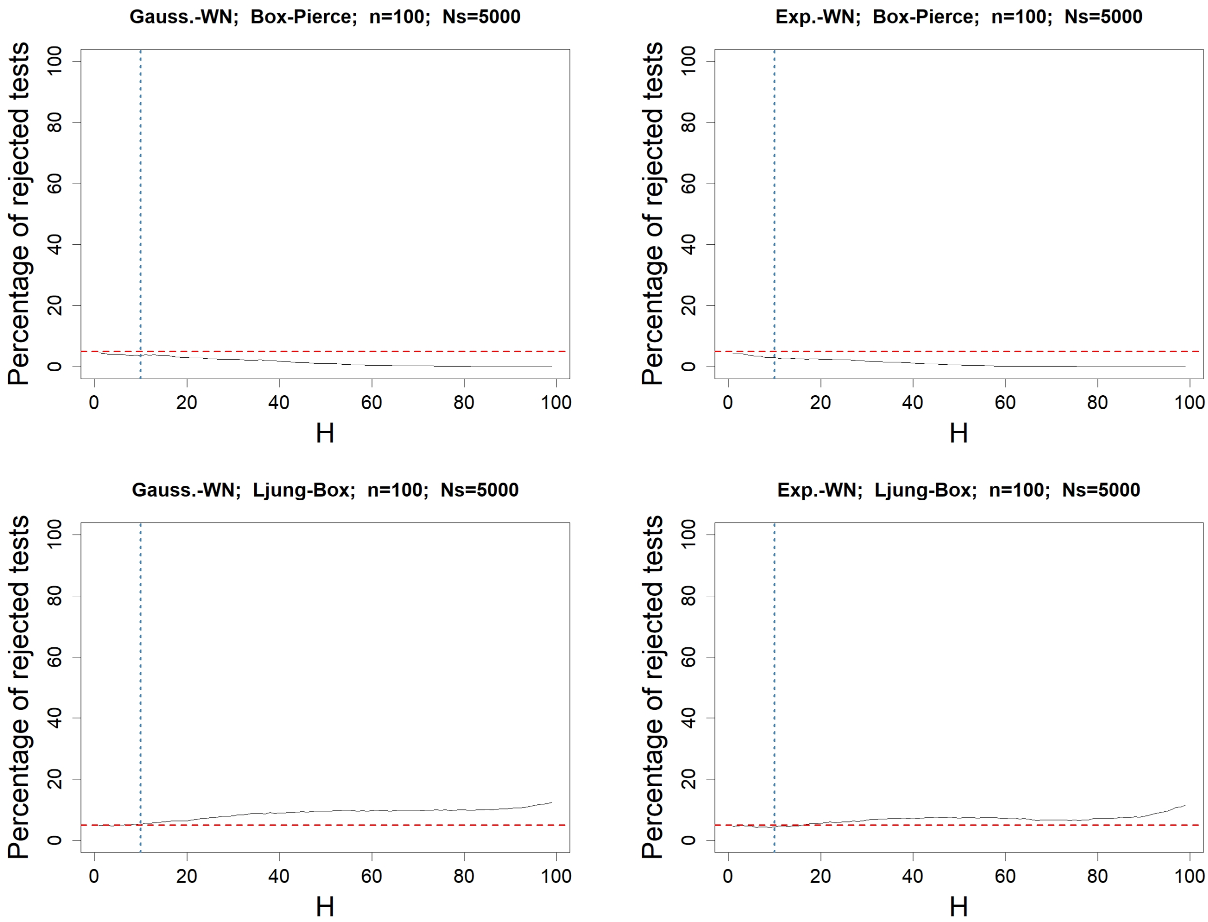

Figure S6.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns Gaussian WN whereas the right one deals with Exponential WN process. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

2. Simulation results for residuals of well-specified models

In the main paper, in Section 4, we simulate $N_S = 5\,000$ white noises of length $n = 500$. Here we also run simulations under several ARIMA(p,d,q) models, with either Gaussian or an Exponential underlying white noise. Let us denote by $(\mathcal{E}_t)_t$ a Gaussian- or an Exponential-WN, with length $n = 500$. The simulated models are

1. invertible MA(2) processes with equation

$$Z_t \;=\; \mathcal{E}_t \,-\, \frac{3}{4}\mathcal{E}_{t-1} \,+\, \frac{1}{8}\mathcal{E}_{t-2}\,,$$

2. stationary AR(2) processes with equation

$$Z_t \,+\, \frac{3}{4}Z_{t-1} \,-\, \frac{1}{8}Z_{t-2} \;=\; \mathcal{E}_t\,,$$

3. ARIMA(0,2,2) processes with equation

$$\Delta^2(Z_t) \;=\; \mathcal{E}_t \,-\, \frac{3}{4}\mathcal{E}_{t-1} \,+\, \frac{1}{8}\mathcal{E}_{t-2}\,,$$

4. ARIMA(1,1,1) processes with equation

$$\Delta(Z_t) \,-\, \frac{1}{4}\Delta(Z_{t-1}) \;=\; \mathcal{E}_t \,+\, \frac{1}{2}\mathcal{E}_{t-1}\,,$$

   which is a non-stationary ARMA(2,1) processes with equation

$$Z_t \,-\, \frac{5}{4}Z_{t-1} \,-\, \frac{1}{4}Z_{t-2} \;=\; \mathcal{E}_t \,+\, \frac{1}{2}\mathcal{E}_{t-1}\,.$$

We estimate every simulated series with the convenient ARIMA(p,d,q) model, and compute its residuals. All the testing-procedures introduced in the main paper in Section 4are applied to these residuals. We obtain very similar results for all the simulated models.

2.1. Results for simulated MA(2) with length $n = 500$

In the main paper, in Section 4, we simulate $N_S = 5\,000$ white noises of length $n = 500$. Here, we simulate N_S MA(2) processes with equation

$$Z_t \;=\; \mathcal{E}_t \,-\, \frac{3}{4}\mathcal{E}_{t-1} \,+\, \frac{1}{8}\mathcal{E}_{t-2}\,,$$

where $(\mathcal{E}_t)_t$ is either a Gaussian- or an Exponential-WN, with length $n = 500$. We estimate every simulated series as a MA(2) process, and compute its residuals.

2.1.1. Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0,\, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk. Figure S7 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to the residuals of either $N_S = 200$ or $N_S = 5\,000$ simulations of a MA(2) process, whereas Figure S8 gives the p-values provided by Shapiro-Wilk. Note that the same tests have also been led on WN with length $n = 100$, and the results are very similar.

The results in Figure S7, top line, show that $\hat{\Xi}(h)$ behaves roughly like a Gaussian distribution $\mathcal{N}\left(0,\, \frac{1}{n}\right)$, as it was expected from Theorem 2.1, even for a lag h much greater than $\sqrt{n}$. But the graphics in the bottom line show that $\hat{\Xi}(h)$ deviates from this specific normal distribution when we lead the normality tests on more simulations. Moreover, the same is observed for the Gaussian behavior in Figure S8.

2.1.2. Check for the normality of $S_{ACF}(H)$ at a fixed lag H

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S9 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$.

In Figure S9, we observe that the sum of sample ACF departs from normality for almost all the lags H, except maybe for the first lags H when the white noise is Gaussian, and/or when the number of simulations remains low. Of course, Kolmogorov-Smirnov test confirmed

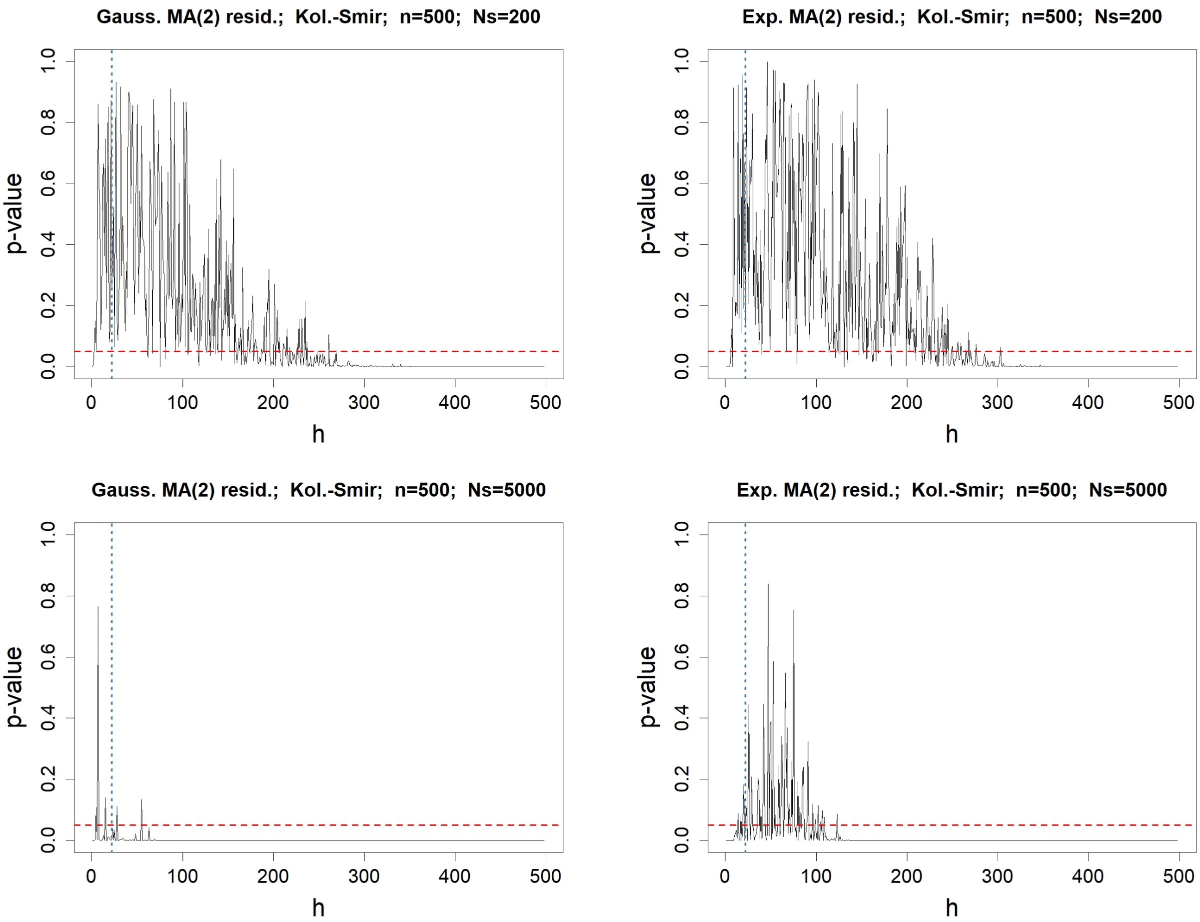

Figure S7.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns MA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated MA(2) process is $n = 500$. In the upper figures, the number of simulated MA(2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

the departure of $S_{ACF}(H)$ from $\mathcal{N}\left(0, \frac{H}{n}\right)$ at any lag, whatever the nature of the underlying WN (Gaussian or Exponential) and even for a low number of simulations (results not shown). Furthermore Figure S9 tells us that none of the variables $S_{ACF}(H)$ is Gaussian. Thus even the normality of the vector $(\hat{\Xi}(1), \cdots, \hat{\Xi}(H))$ is called into question. Indeed, if $\left(\boldsymbol{\mathcal{A}_H(\underline{\mu}, \Sigma)}\right)$ were true, $\left(\boldsymbol{\mathcal{S}_H(\underline{\mu}, \Sigma)}\right)$ would also be.

Consequently, from Figures S8 and S9, we conclude that at a fixed lag h, $\hat{\Xi}(h)$ is roughly Gaussian, with $\mu \simeq 0$ or $\sigma^2 \simeq \frac{1}{n}$. But the vector $(\hat{\Xi}(1), \cdots, \hat{\Xi}(H))$ is not a Gaussian vector.

2.1.3. Check for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$

The previous finding raises questions about the methods used in practice to model a time series. Indeed, a model has to be validated by checking that the associated residuals are a WN, by using Box-Pierce or Ljung-Box test. We know that $\boldsymbol{(\mathcal{A}_H(\underline{0}, I_H))}$ does not hold. But we wonder if it is a problem in practice. Then we adopt another point of view, that is more adequate with the

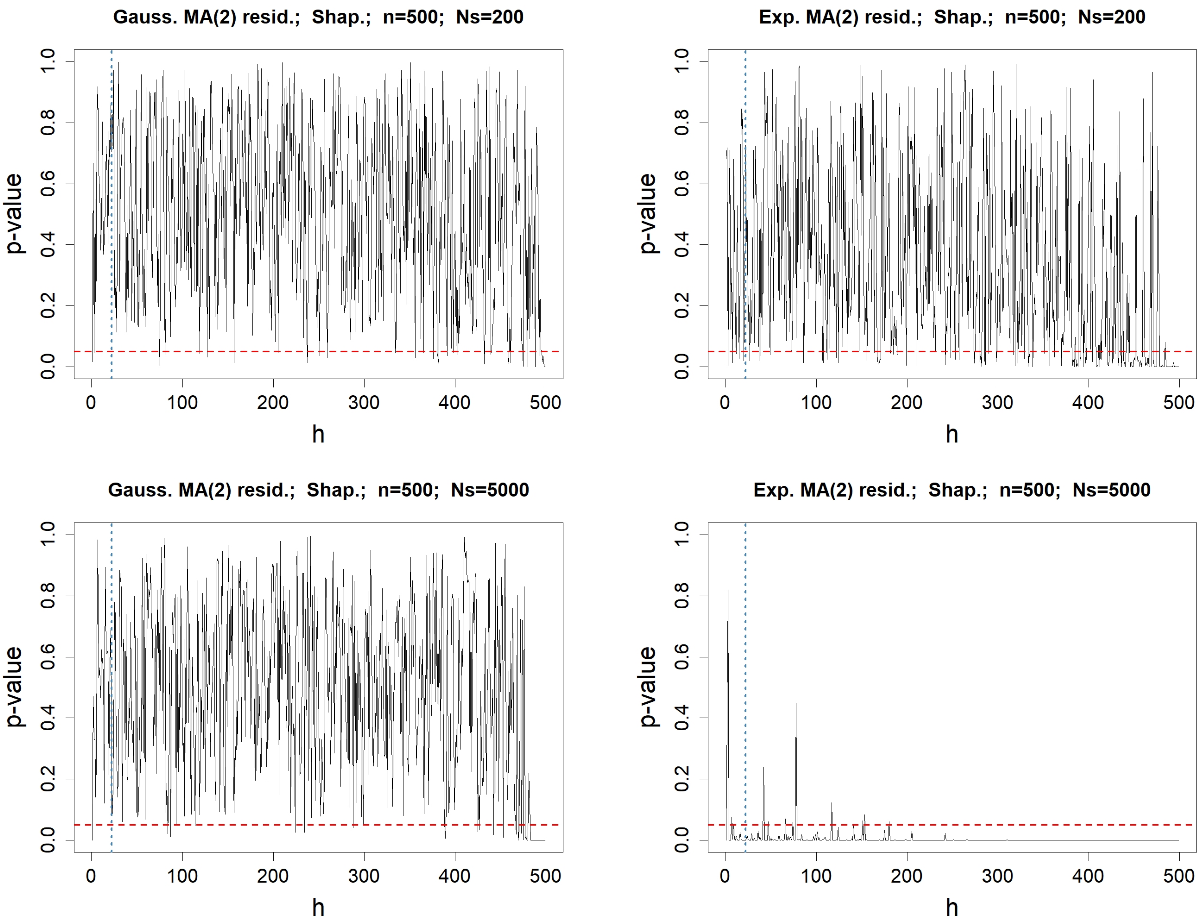

Figure S8.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian MA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated MA(2) processes is $n = 500$. In the upper figures, the number of simulated MA(2) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

practice, where we have to model a single time series, from the properties of its first $\hat{\rho}(h)$ values. For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words that they are the realizations of independent variables. Let us suppose that this hypothesis is satisfied. First we use Shapiro-Wilk to test for the normality behavior, and next Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$. Figure S10 gives the percentage of inadequate testing conclusions with Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of MA(2) with length $n = 500$ and, whereas Figure S11 gives the same percentages with Kolmogorov-Smirnov.

In Figures S10 and S11, we observe that the percentage of p-values $< \alpha = 5\%$ is very close to 5%, which seems to comfort that, at least for MA(2) processes associated to a Gaussian-WN, $\hat{\rho}(1), \cdots, \hat{\rho}(H)$ as realizations of Gaussian variables until a lag H not too large, with the expectation and the covariance-matrix as stated in Theorem 2.1. So that these Figures seem to

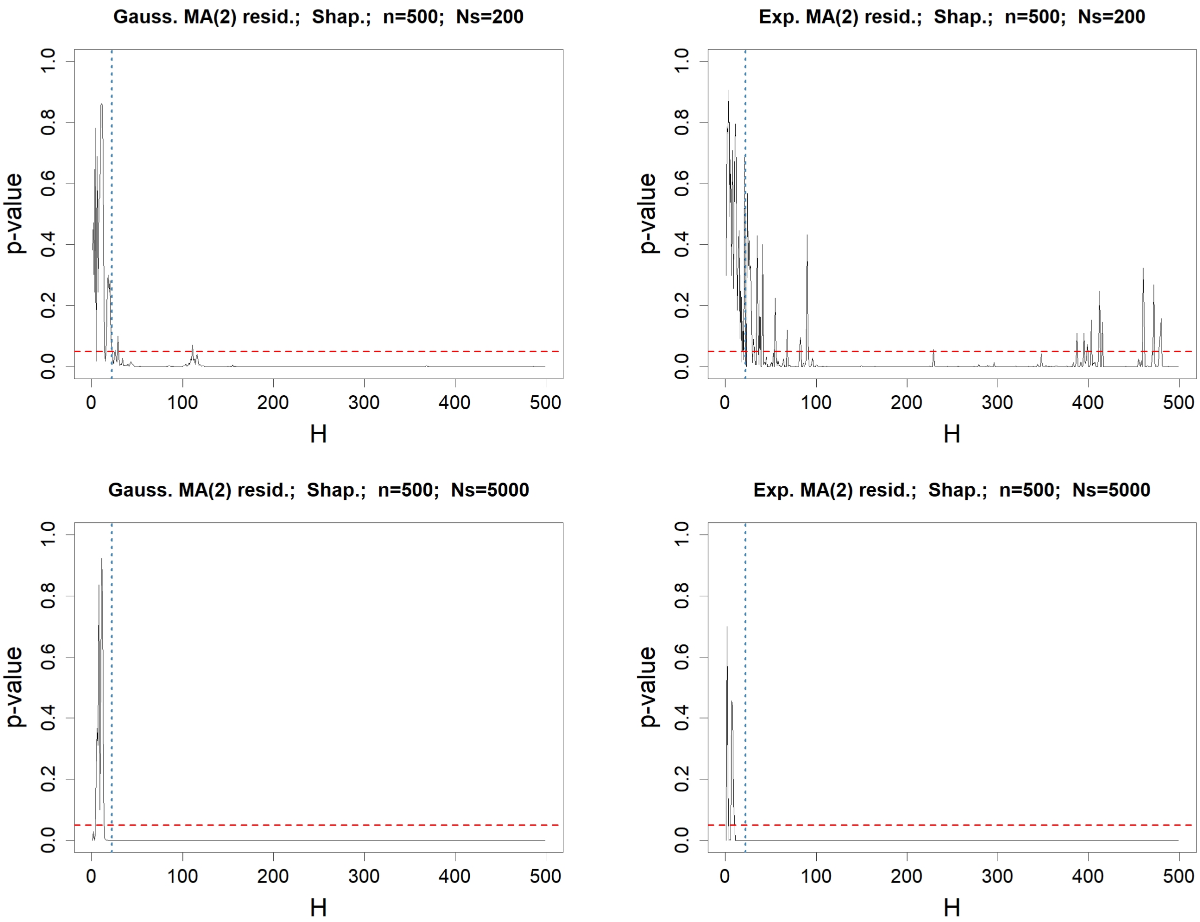

Figure S9.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns concerns MA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated MA(2) processes is $n = 500$. In the upper figures, the number of simulated MA(2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

assess that $\left(\boldsymbol{\mathcal{A}_H}(\underline{\boldsymbol{\mu}}, \boldsymbol{\Sigma})\right)$ and even $(\boldsymbol{\mathcal{A}_H}(\underline{\mathbf{0}}, \boldsymbol{I_H}))$ are true. This last point seems contradictory with Subsection 2.1.1. But remember that the normality adequacy is sensitive to the number of observations used when applying the normality tests, and that a slight departure from normality is more likely to be detected when this number is large. Here, the normality of the successive $\hat{\rho}(1), \cdots, \hat{\rho}(H)$ values is assessed until a size of sample around $H = \frac{n}{2}$, which is low with respect to $N_S = 5\,000$. So that making a diagnosis from the first ACF might be acceptable. But keep in mind that the tests displayed in Figures S10 and S11 suppose that $\hat{\rho}(1), \cdots, \hat{\rho}(H)$ are realizations of independent variables, which is not guaranteed. Indeed, Hassani's $-\frac{1}{2}$-Theorem proves that $\hat{\rho}(1), \cdots, \hat{\rho}(n-1)$ can not be independent.

2.1.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S12 shows that Box-Pierce

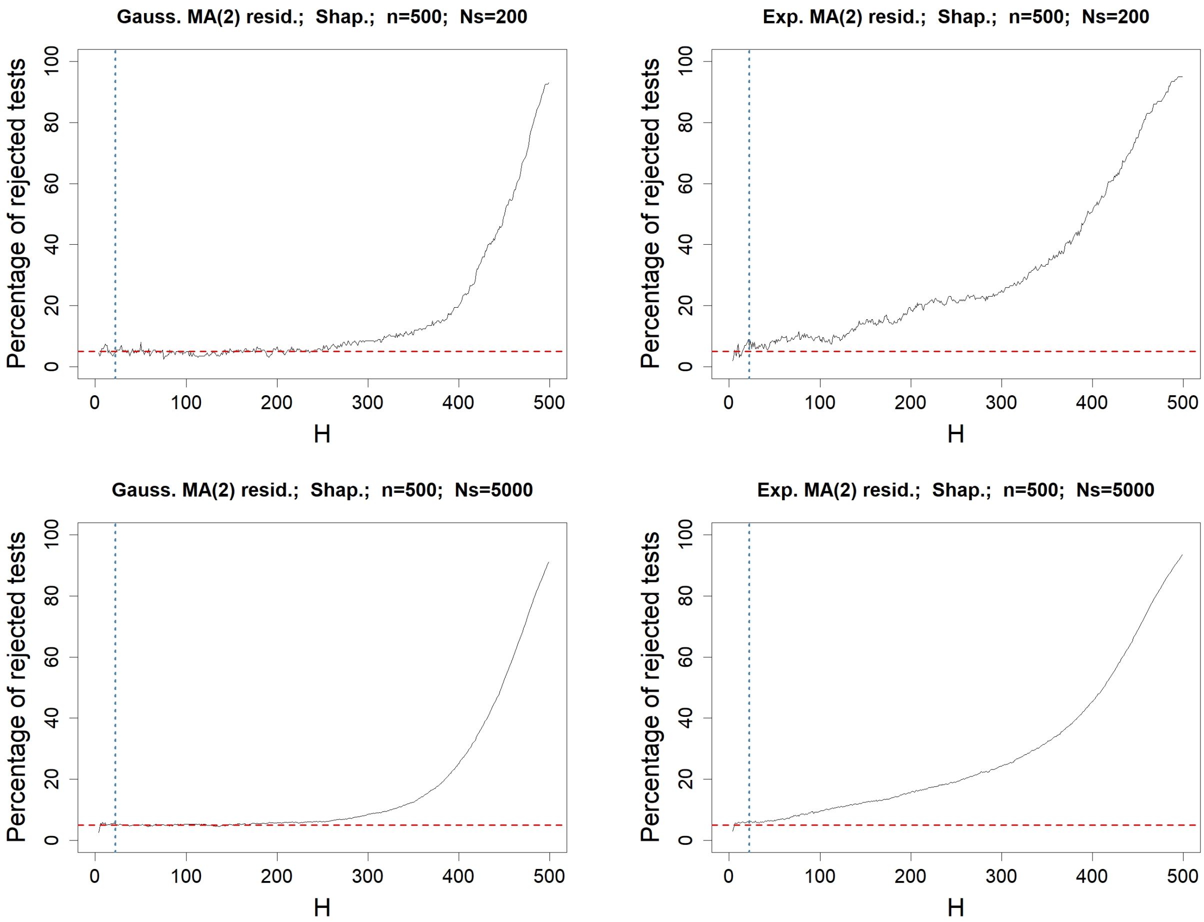

Figure S10.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns MA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential WN processes. The length of the simulated MA(2) processes is $n = 500$. In the upper figures, the number of simulated MA(2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

and Ljung-Box tests are not totally accurate. Indeed, Box-Pierce appears to be too conservative, whereas Ljung-Box is too liberal.

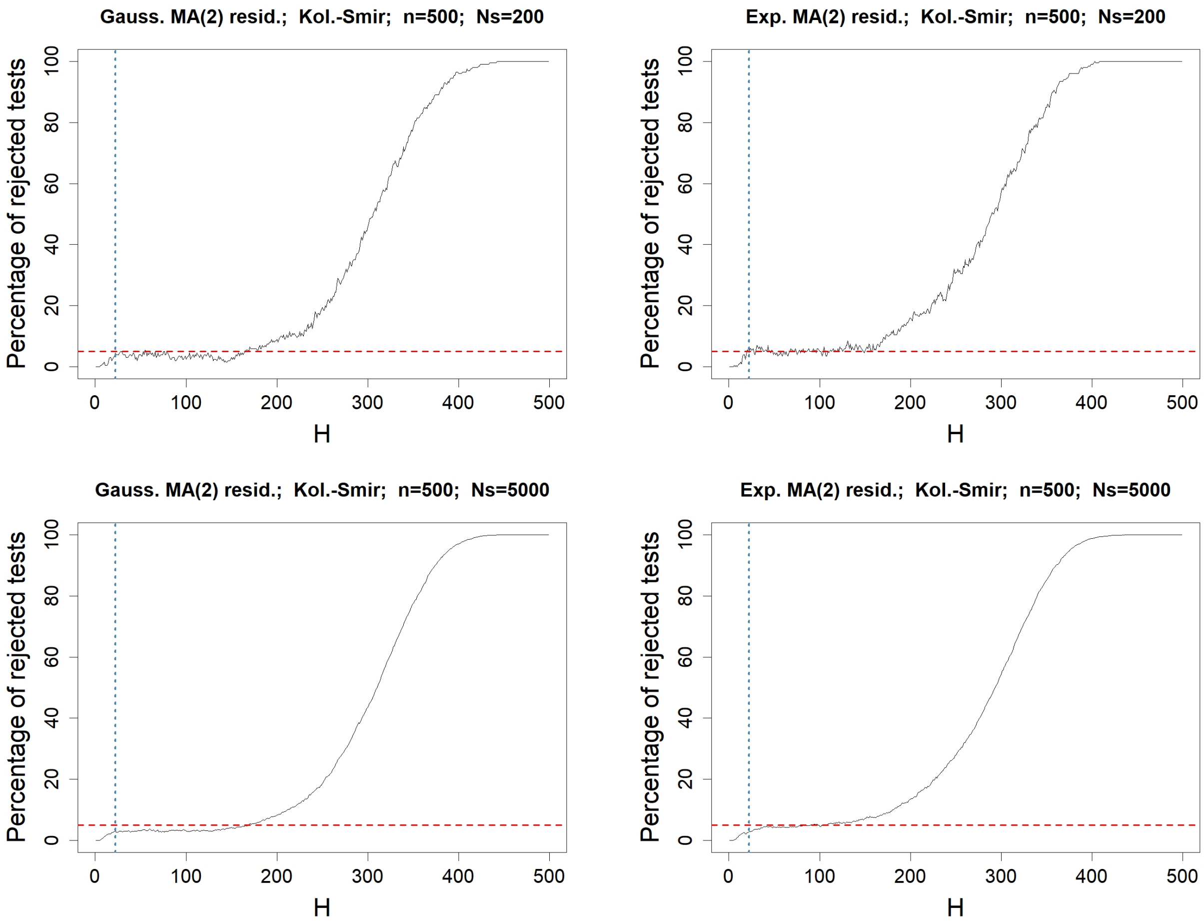

Figure S11.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns MA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. In the upper figures, the number of simulated MA(2) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

2.2. Results for simulated AR(2) with length $n = 500$

In the main paper, in Section 4, we simulate $N_S = 5\,000$ white noises of length $n = 500$. Here, we simulate N_S AR(2) processes with equation

$$Z_t + \frac{3}{4} Z_{t-1} - \frac{1}{8} Z_{t-2} = \mathcal{E}_t \,,$$

where $(\mathcal{E}_t)_t$ is either a Gaussian- or an Exponential-WN, with length $n = 500$. We estimate every simulated series as a AR(2) process, and compute its residuals. All the testing-procedures provide the same results as for WN- or MA(2)-simulations.

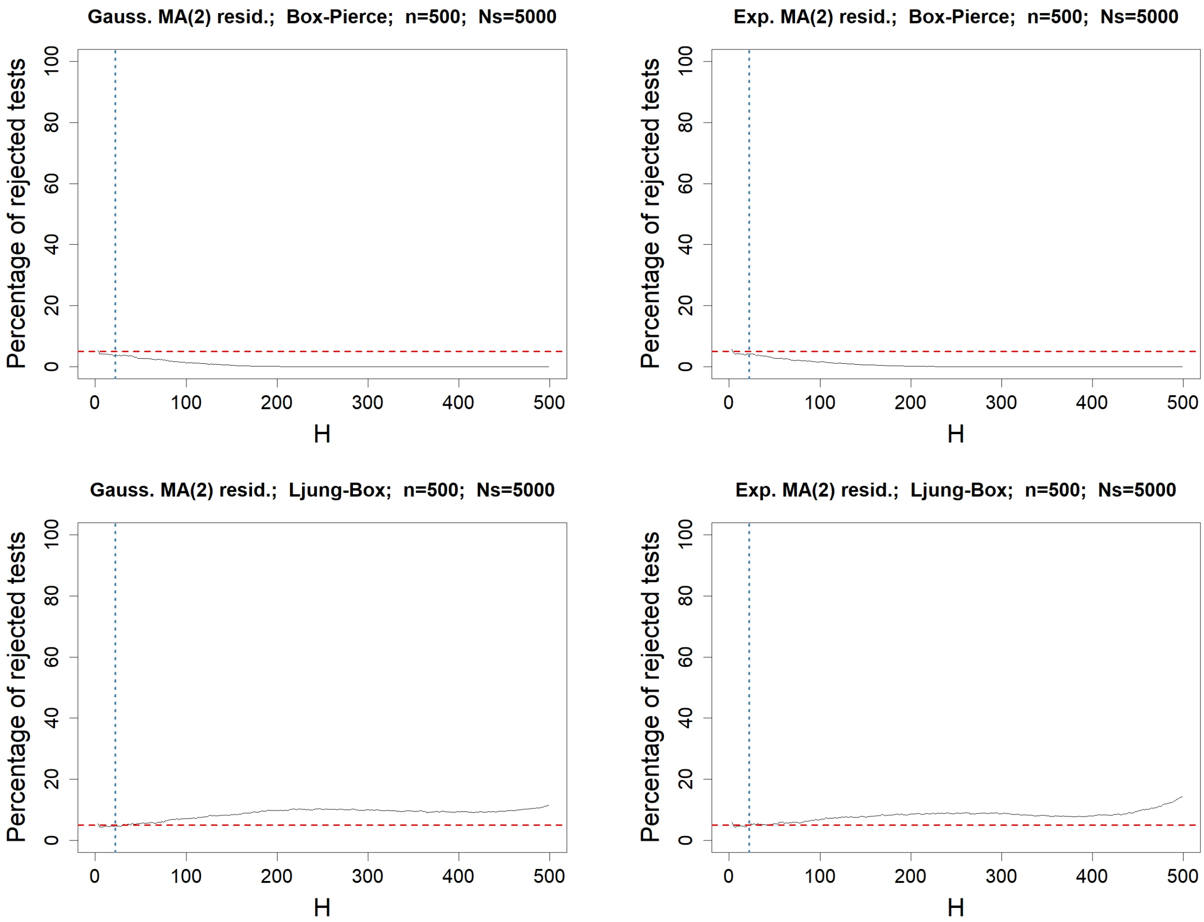

Figure S12.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns MA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

2.2.1. *Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h*

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk. Figure S13 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to the residuals of either $N_S = 200$ or $N_S = 5\,000$ simulations of a AR(2) process, whereas Figure S14 gives the p-values provided by Shapiro-Wilk.

2.2.2. *Check for the normality of $S_{ACF}(H)$ at a fixed lag H*

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S15 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$.

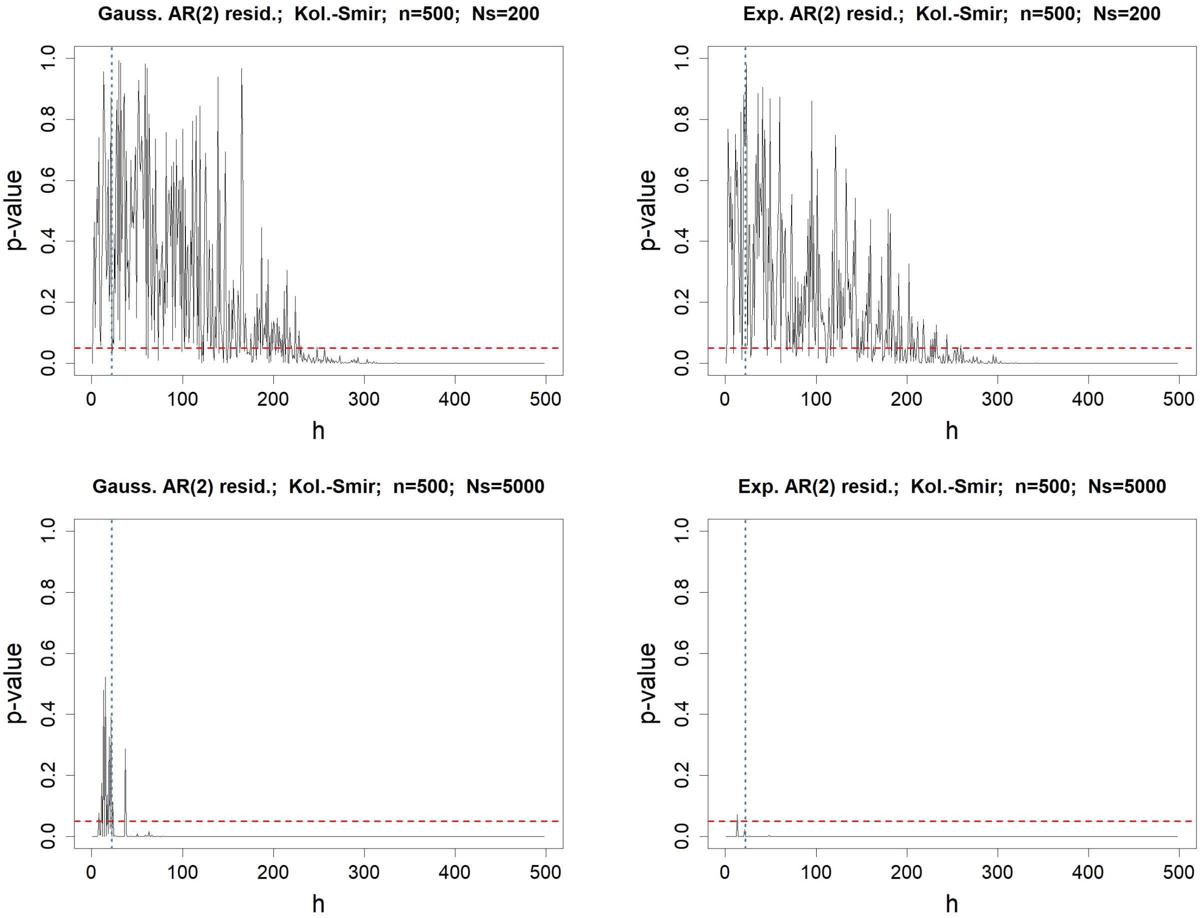

Figure S13.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns AR(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated AR(2) process is $n = 500$. In the upper figures, the number of simulated AR(2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

2.2.3. Check for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$

For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words that they are the realizations of independent variables. Let us suppose that this hypothesis is satisfied. First we use Shapiro-Wilk to test for the normality behavior, and next Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$. Figure S16 gives the percentage of inadequate testing conclusions with Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of AR(2) with length $n = 500$ and, whereas Figure S17 gives the same percentages with Kolmogorov-Smirnov.

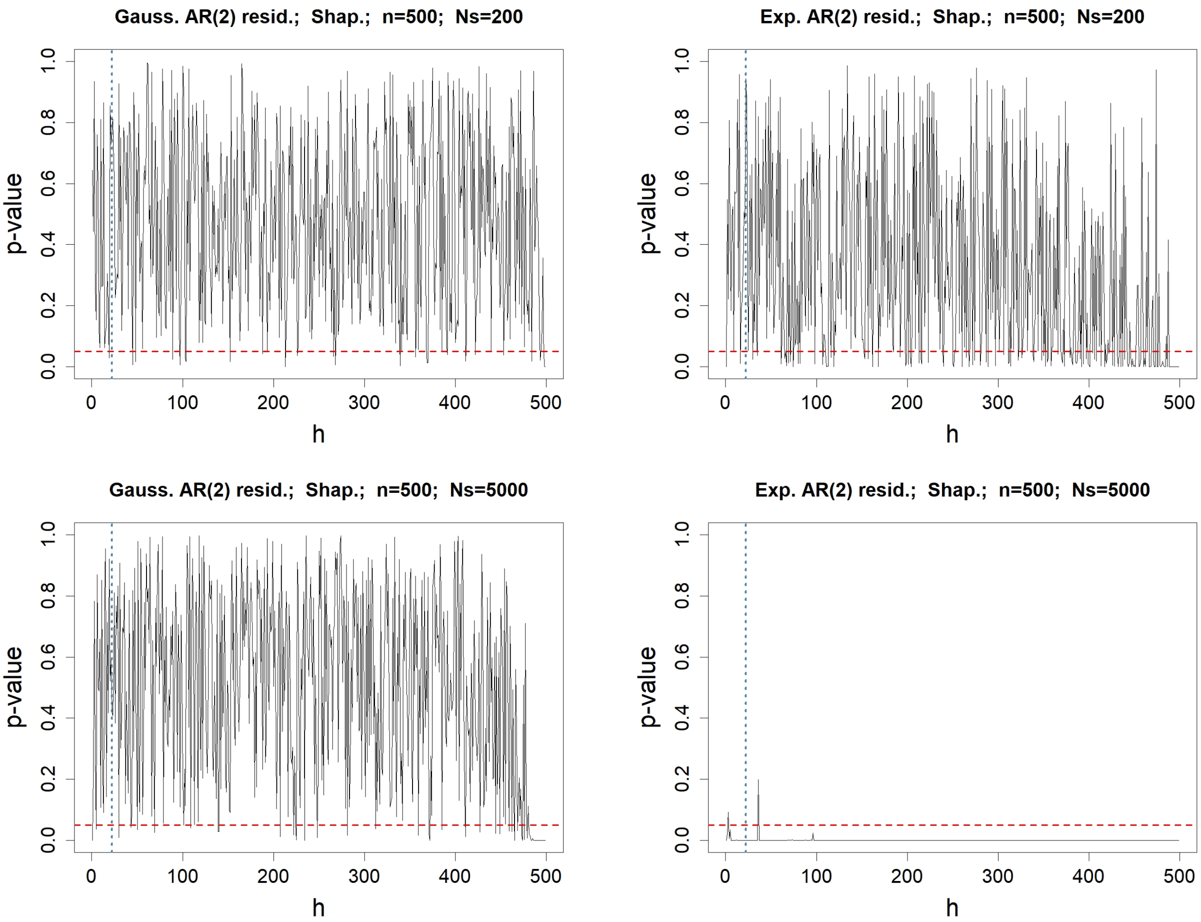

Figure S14.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian AR(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated AR(2) processes is $n = 500$. In the upper figures, the number of simulated AR(2) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

2.2.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S18 shows that Box-Pierce and Ljung-Box tests are not totally accurate. Indeed, Box-Pierce appears to be too conservative, whereas Ljung-Box is too liberal.

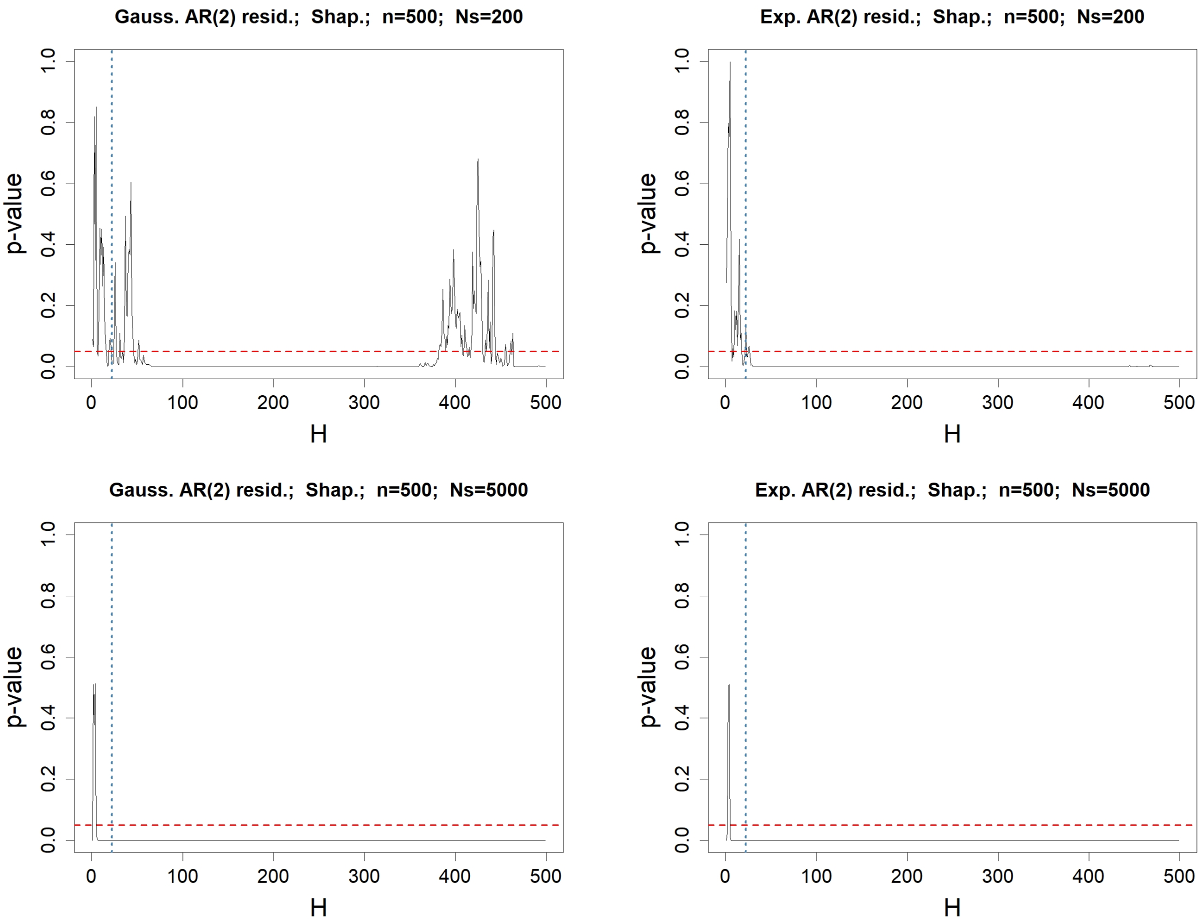

Figure S15.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns concerns AR(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated AR(2) processes is $n = 500$. In the upper figures, the number of simulated AR(2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

2.3. Results for simulated ARIMA(0,2,2) with length $n = 500$

Here, we simulate N_S ARIMA(0,2,2) processes with equation

$$\Delta^2(Z_t) \;=\; \mathcal{E}_t \;-\; \frac{3}{4}\mathcal{E}_{t-1} \;+\; \frac{1}{8}\mathcal{E}_{t-2}\,,$$

where $(\mathcal{E}_t)_t$ is either a Gaussian- or an Exponential-WN, with length $n = 500$. We estimate every simulated series as an ARIMA(0,2,2) process, and compute its residuals. All the testing-procedures provide the same results as for WN- or MA(2)- and AR(2)-simulations.

2.3.1. Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0,\, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk.

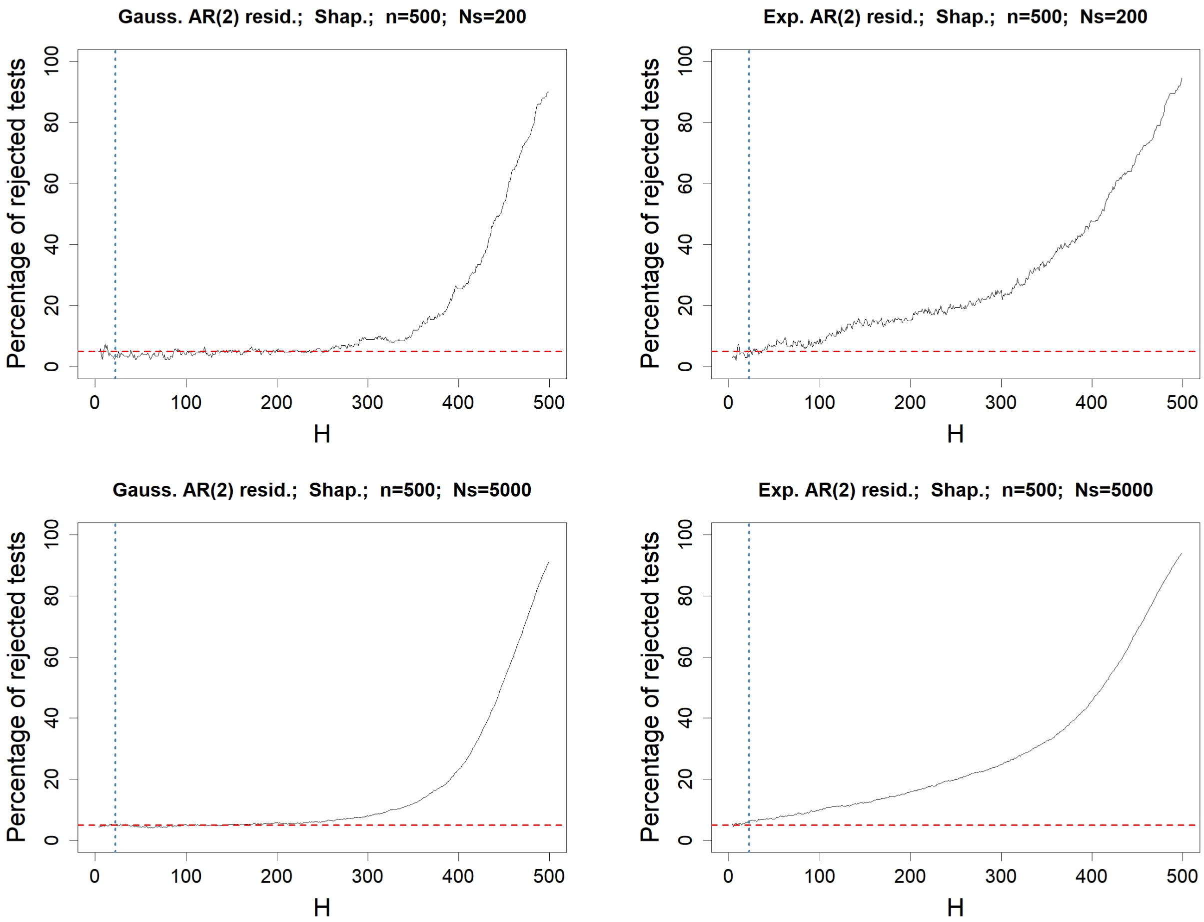

Figure S16.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns AR(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential WN processes. The length of the simulated AR(2) processes is $n = 500$. In the upper figures, the number of simulated AR(2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

Figure S19 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to the residuals of either $N_S = 200$ or $N_S = 5\,000$ simulations of an ARIMA(0,2,2) process, whereas Figure S20 gives the p-values provided by Shapiro-Wilk.

2.3.2. Check for the normality of $S_{ACF}(H)$ at a fixed lag H

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S21 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$.

2.3.3. Check for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$

For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words

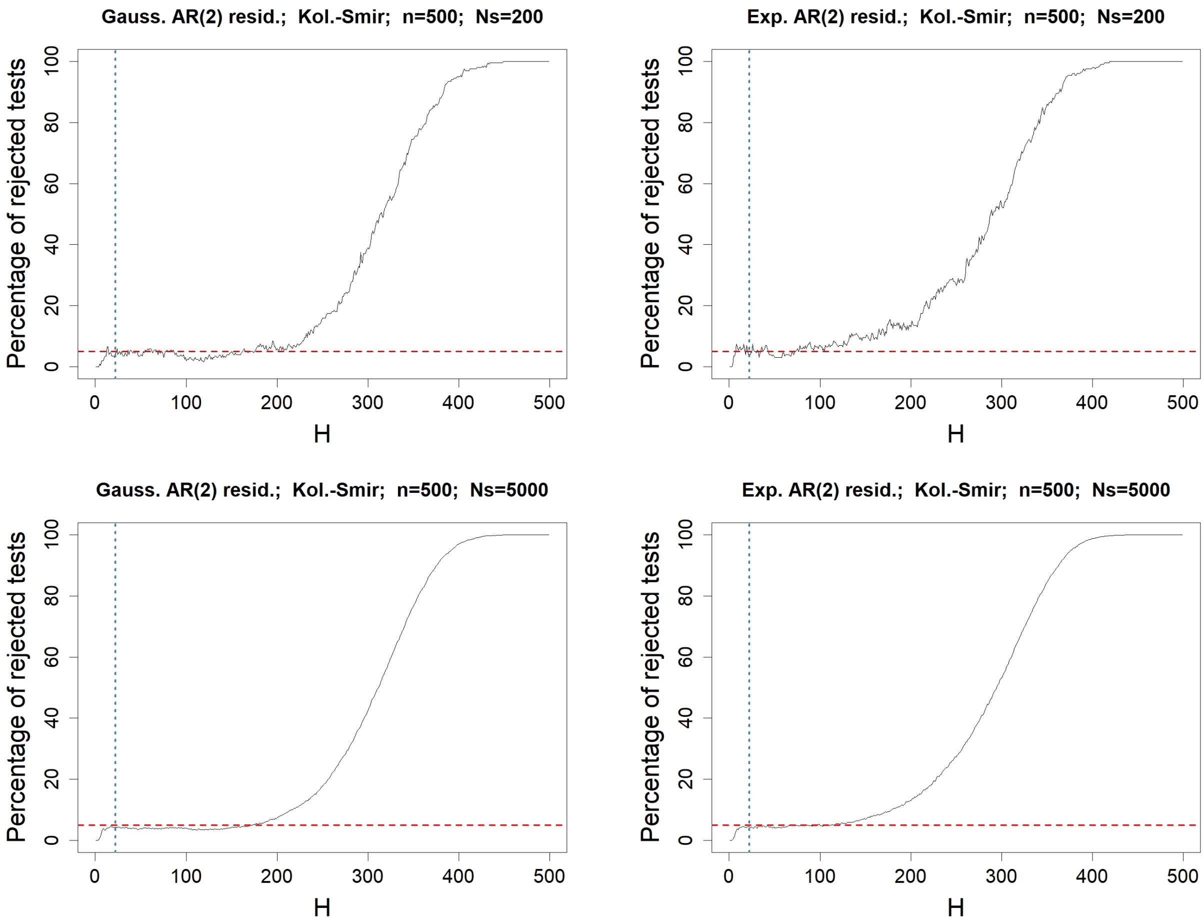

Figure S17.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns MAR(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. In the upper figures, the number of simulated AR(2) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

that they are the realizations of independent variables. Let us suppose that this hypothesis is satisfied. First we use Shapiro-Wilk to test for the normality behavior, and next Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$. Figure S22 gives the percentage of inadequate testing conclusions with Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of ARIMA(0,2,2) with length $n = 500$ and, whereas Figure S23 gives the same percentages with Kolmogorov-Smirnov.

2.3.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S24 shows that Box-Pierce and Ljung-Box tests are not totally accurate. Indeed, Box-Pierce appears to be too conservative, whereas Ljung-Box is too liberal.

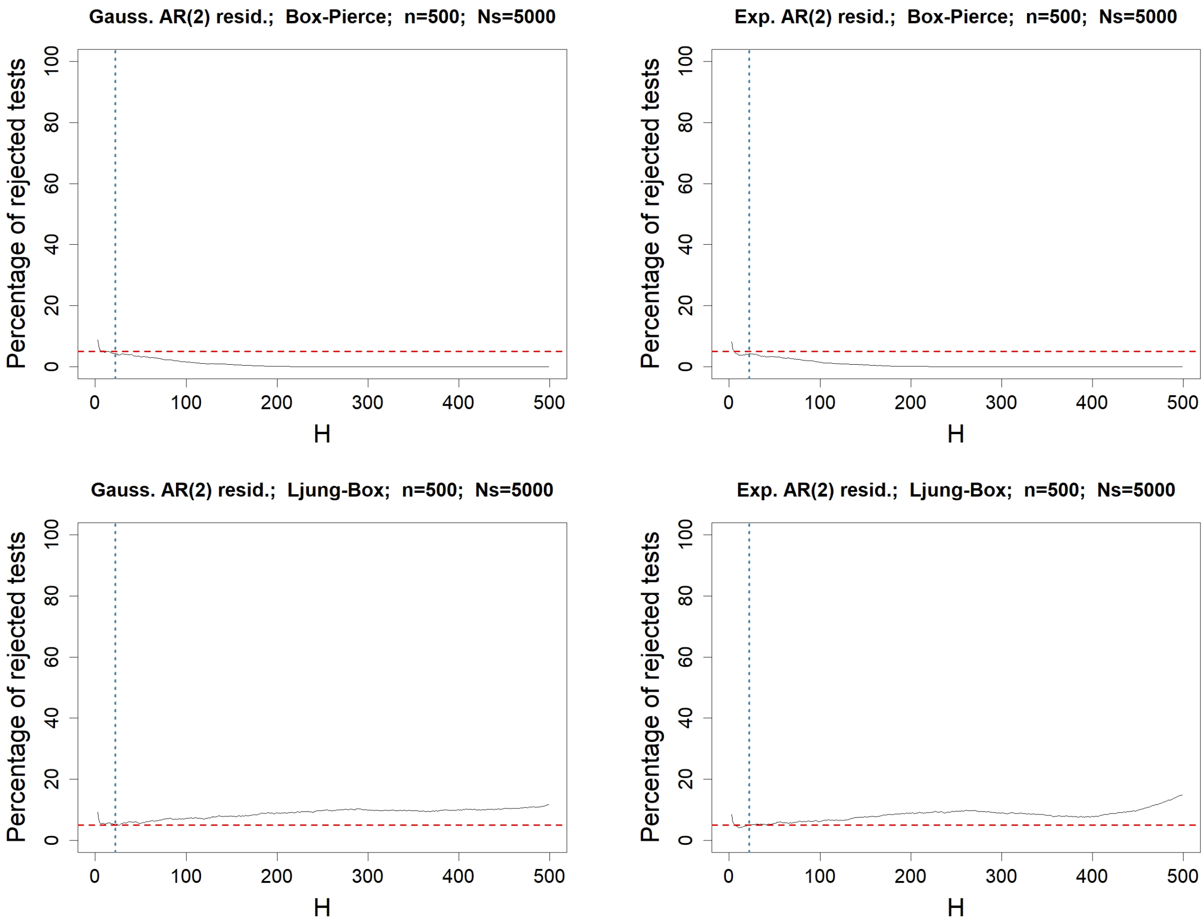

Figure S18.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns AR(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

2.4. Results for simulated ARIMA(1,1,1) with length $n = 500$

Here, we simulate N_S ARIMA(1,1,1) processes with equation

$$\Delta^2(Z_t) \;=\; \mathcal{E}_t \,-\, \frac{3}{4}\mathcal{E}_{t-1} + \frac{1}{8}\mathcal{E}_{t-2}\,,$$

where $(\mathcal{E}_t)_t$ is either a Gaussian- or an Exponential-WN, with length $n = 500$. We estimate every simulated series as an ARIMA(1,1,1) process, and compute its residuals. All the testing-procedures provide the same results as for WN-, MA(2)-, AR(2)- and ARMA(1,1)-simulations.

2.4.1. Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk. Figure S49 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to the

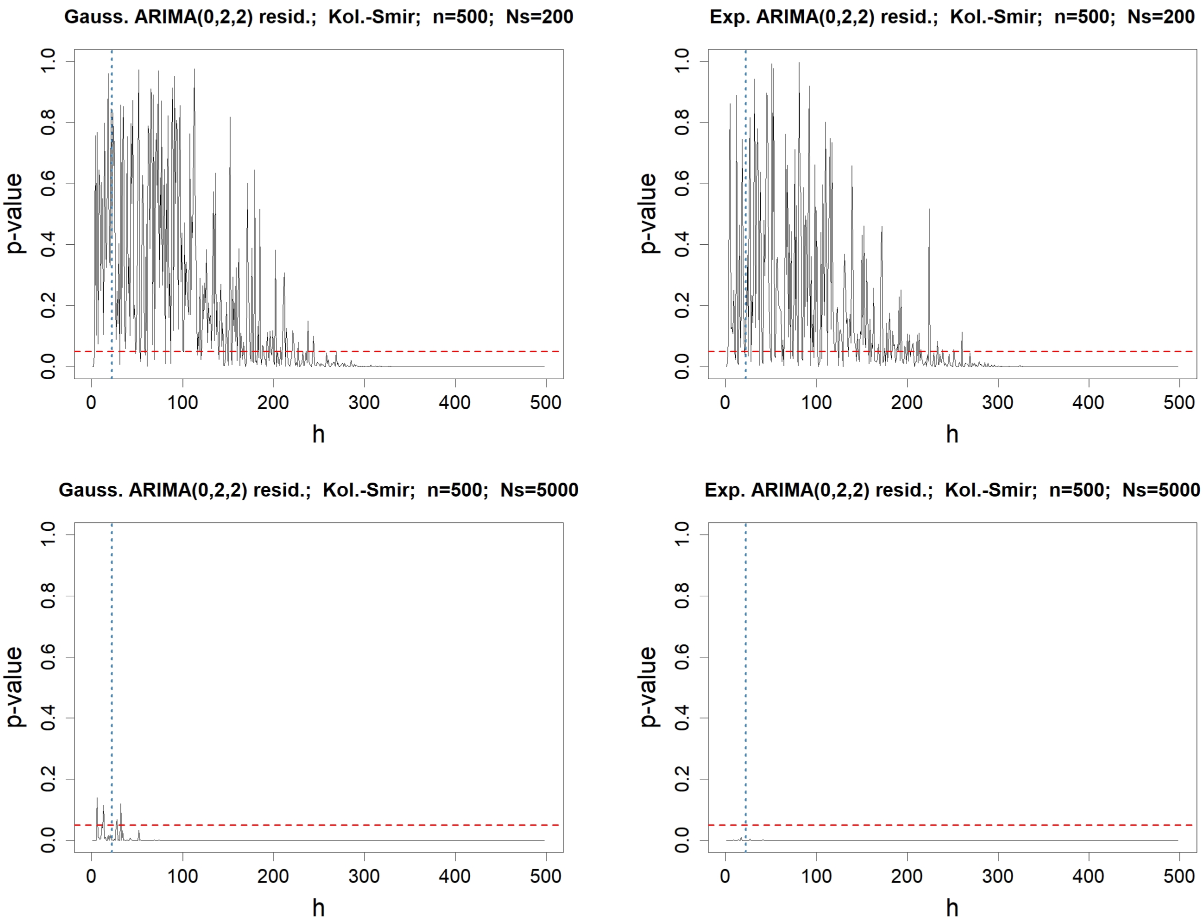

Figure S19.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns ARIMA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(0,2,2) process is $n = 500$. In the upper figures, the number of simulated ARIMA(0,2,2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

residuals of either $N_S = 200$ or $N_S = 5\,000$ simulations of an ARIMA(1,1,1) process, whereas Figure S50 gives the p-values provided by Shapiro-Wilk.

2.4.2. Check for the normality of $S_{ACF}(H)$ at a fixed lag H

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S51 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$.

2.4.3. Check for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$

For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words that they are the realizations of independent variables. Let us suppose that this hypothesis is

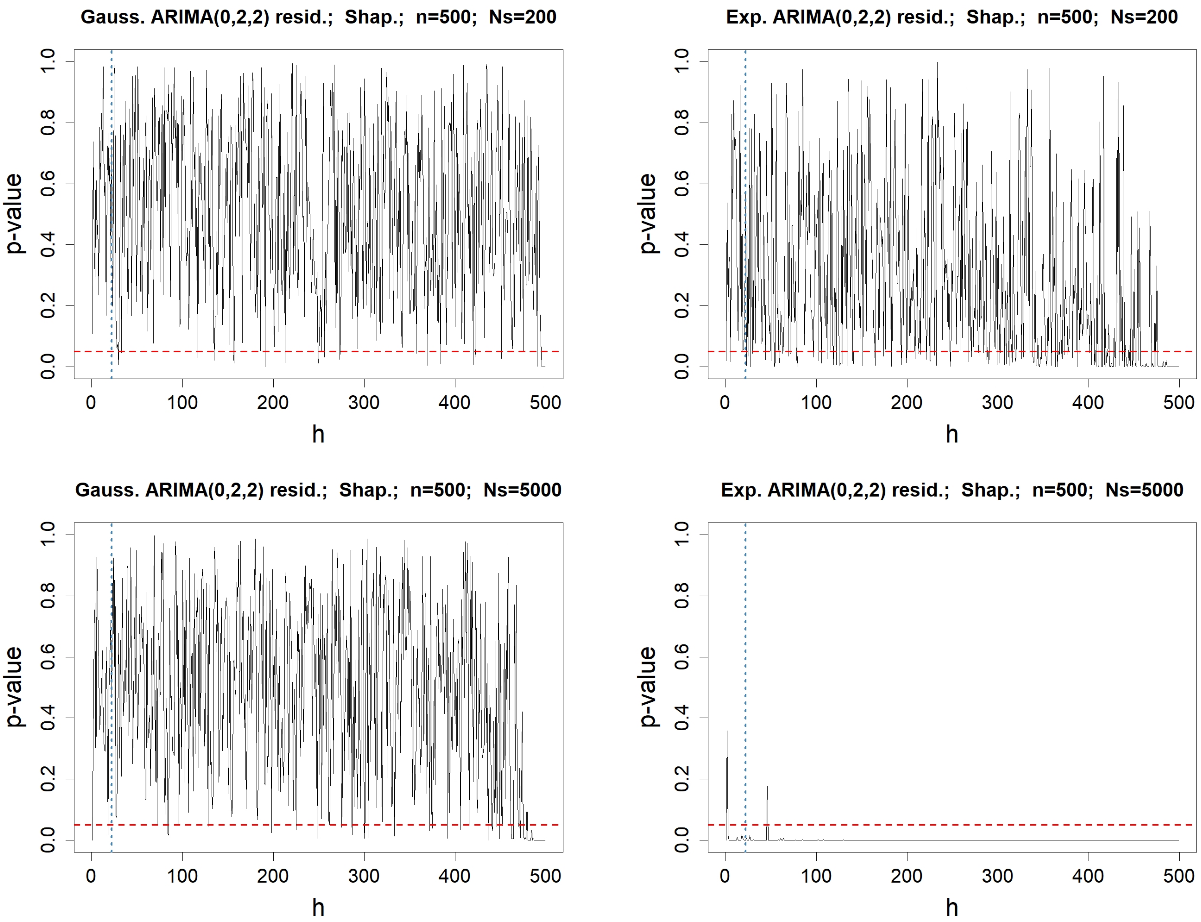

Figure S20.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian ARIMA(0,2,2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(0,2,2) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(0,2,2) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

satisfied. First we use Shapiro-Wilk to test for the normality behavior, and next Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$. Figure S52 gives the percentage of inadequate testing conclusions with Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of MA(2) with length $n = 500$ and, whereas Figure S53 gives the same percentages with Kolmogorov-Smirnov.

2.4.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S54 shows that Box-Pierce and Ljung-Box tests are not totally accurate. Indeed, Box-Pierce appears to be too conservative, whereas Ljung-Box is too liberal.

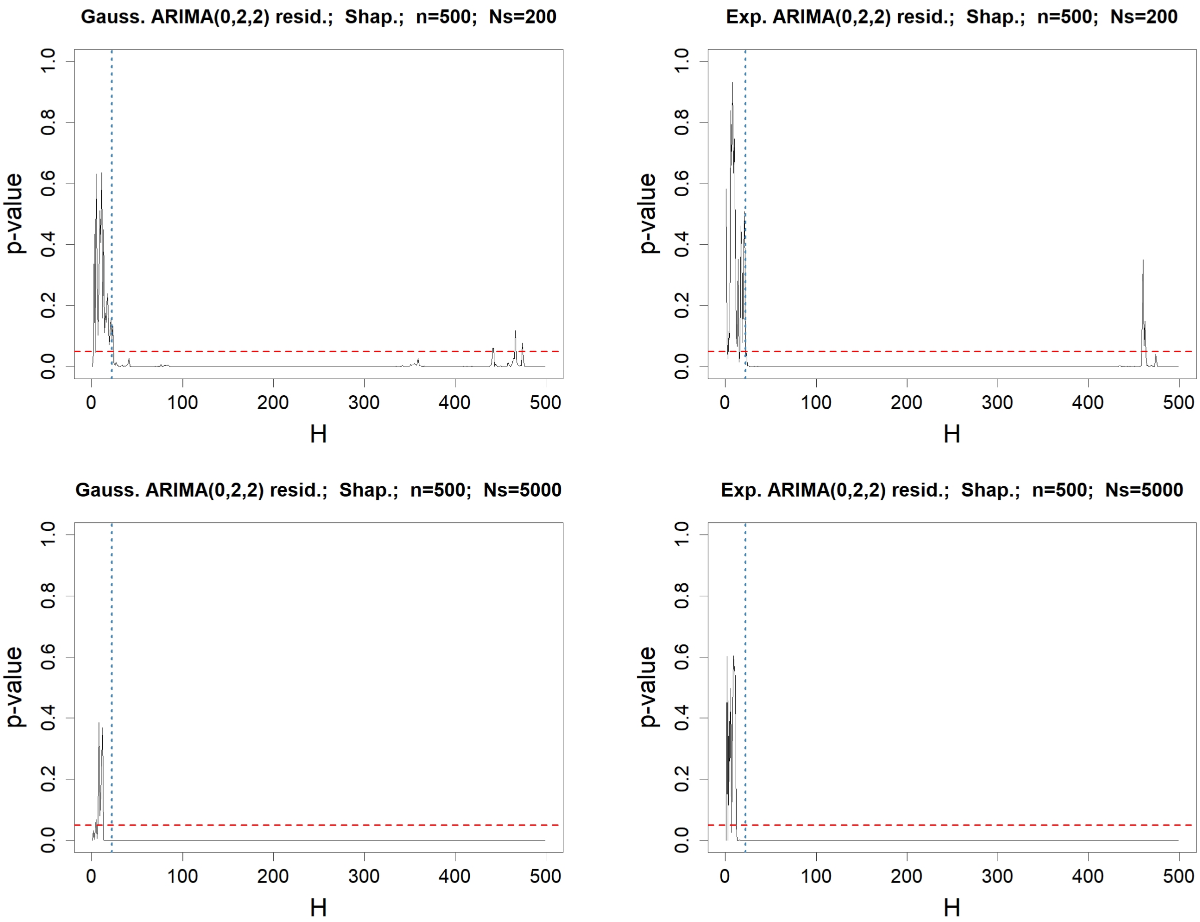

Figure S21.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns concerns ARIMA(0,2,2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(0,2,2) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(0,2,2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

3. Simulation results for residuals of misspecified models

In the previous case, the orders of the underlying were known, which is rarely the case, except in the case of simulations. In this section, we also explore the case where the estimated model is not the convenient one. We simulate ARIMA(p,d,q) processes, we estimate every simulated series with a misspecified ARIMA(p',d',q') model, and compute its residuals. In the main paper, we detailed the case of simulated ARIMA(0,2,2) processes, estimated by an ARIMA(0,2,0) model. Let us denote by $(\mathcal{E}_t)_t$ a Gaussian- or an Exponential-WN, with length $n = 500$. The simulated models are

1. invertible MA(2) processes with equation

$$Z_t \;=\; \mathcal{E}_t \,-\, \frac{3}{4}\mathcal{E}_{t-1} \,+\, \frac{1}{8}\mathcal{E}_{t-2}\,,$$

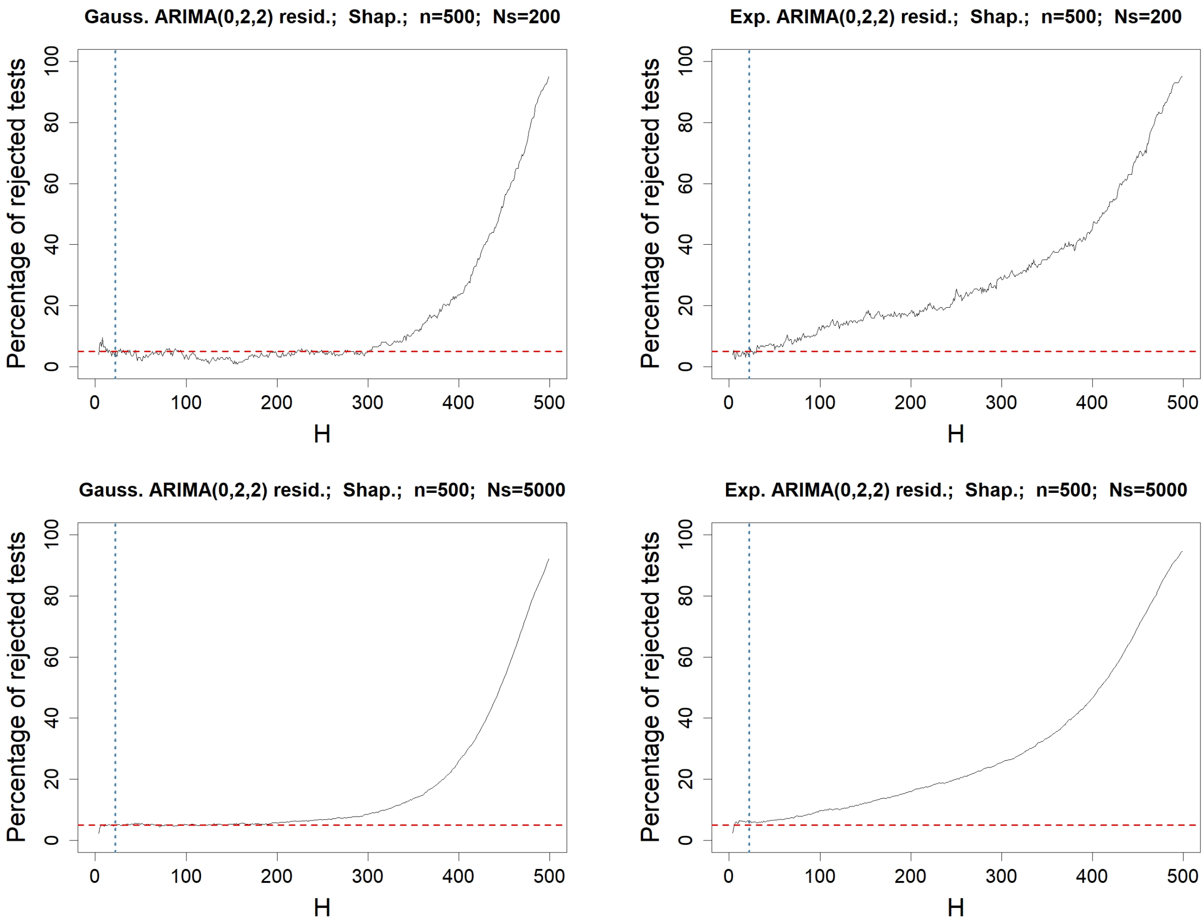

Figure S22.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns ARIMA(0,2,2) simulations associated to Gaussian-WN whereas the right one deals with Exponential WN processes. The length of the simulated ARIMA(0,2,2) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(0,2,2) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

which is estimated by a WN,

2. stationary AR(2) processes with equation

$$Z_t + \frac{3}{4} Z_{t-1} - \frac{1}{8} Z_{t-2} = \mathcal{E}_t \,,$$

which is estimated by a WN,

3. ARIMA(0,2,2) processes with equation

$$\Delta^2(Z_t) = \mathcal{E}_t - \frac{3}{4}\mathcal{E}_{t-1} + \frac{1}{8}\mathcal{E}_{t-2} \,,$$

which is estimated by an AR(1),

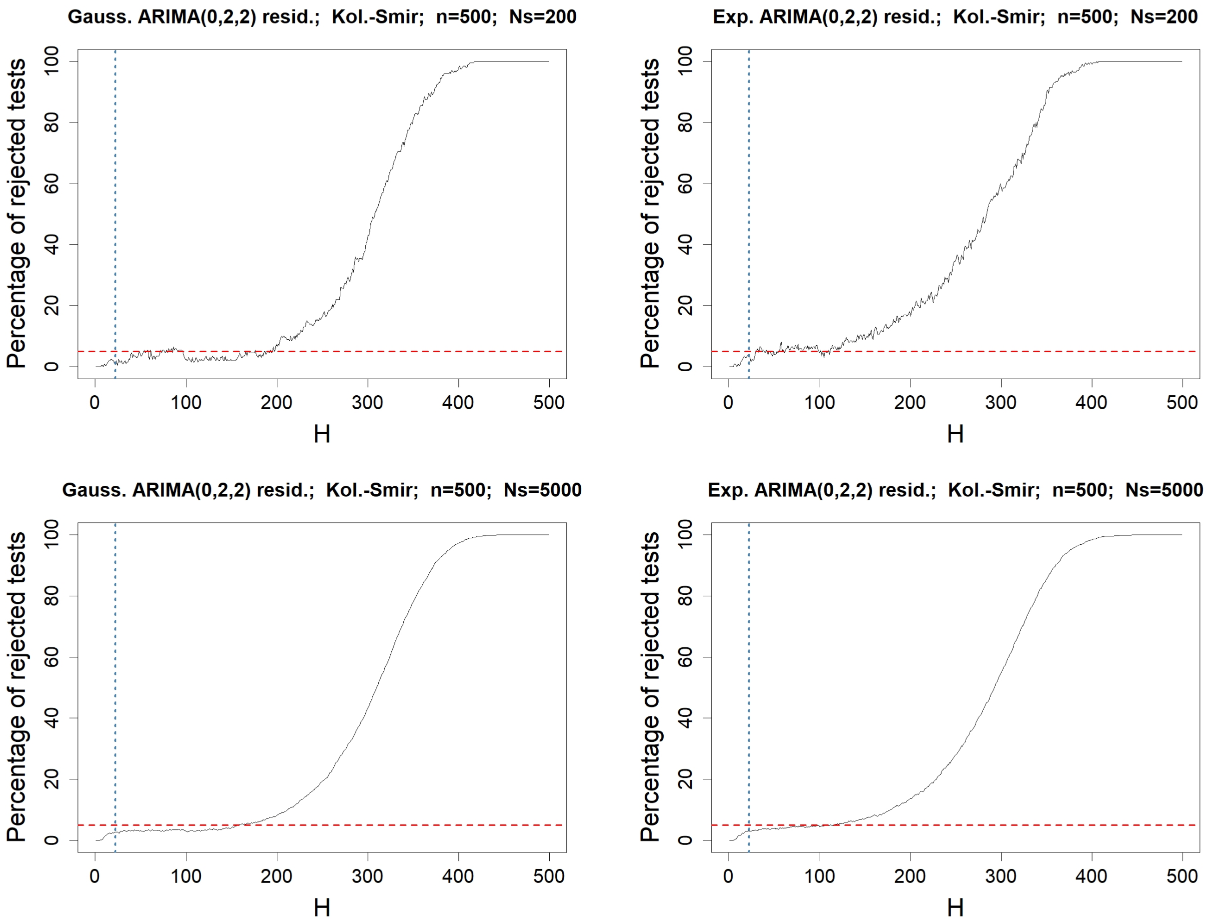

Figure S23.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns ARIMA(0,2,2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. In the upper figures, the number of simulated ARIMA(0,2,2) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

4. ARIMA(1,1,1) processes with equation

$$\Delta(Z_t) - \frac{1}{4}\Delta(Z_{t-1}) = \mathcal{E}_t + \frac{1}{2}\mathcal{E}_{t-1},$$

which is a non-stationary ARMA(2,1) processes with equation

$$Z_t - \frac{5}{4}Z_{t-1} - \frac{1}{4}Z_{t-2} = \mathcal{E}_t + \frac{1}{2}\mathcal{E}_{t-1}.$$

It is estimated by an ARIMA(2,1,2) process.

All the testing-procedures introduced in the main paper in Section 4are applied to these residuals. In the first case of simulations (MA(2)-simulations estimated by a WN), the results are very similar to those observed in the main paper for simulations of an ARIMA(0,2,2), estimated by an ARIMA(0,2,0). Namely, only Shapiro's test on successive ACFs, and the portmanteau

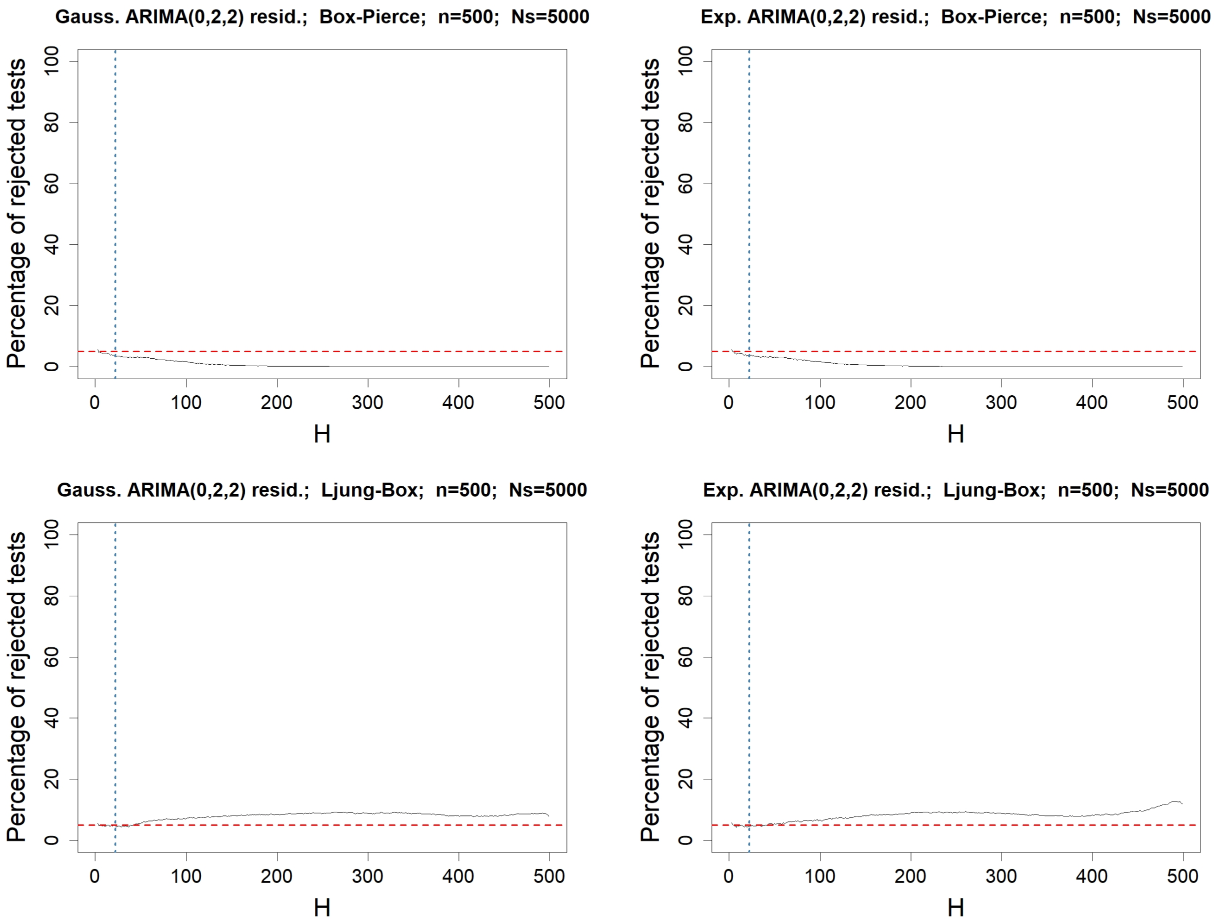

Figure S24.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns ARIMA(0,2,2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

tests detect model misspecification. Note that these 2 cases only imply a misspecification on the q-parameter. The following case (AR(2)-simulations estimated by a WN), involving a misspecification on the p-parameter, show specific behaviors, as if the misspecification were more pronounced. Indeed, Shapiro's test on successive ACFs, and the portmanteau tests still detect model misspecification, but additionnaly Kolmogorov-Smirnov's test applied either to the successive ACF or to the ACF at a fixed lag h, detects a departure to normality. The third case (ARIMA(0,2,2)-simulations estimated by an AR(1)) involves a misspecification on all the parameters p, d and q. The testing-procedures react as if the misspecification were even more marked, and so more easily detectable. Thus, all the procedures systematically reject the null hypothesis of normality. Finally the last simulations concern a misspecification, but with a more general model than the one used to generate the simulations (ARIMA(1,1,1)-simulations estimated by an ARIMA(2,1,2)). In this case the testing-procedures provide results very similar to WN or residuals of well-specified models.

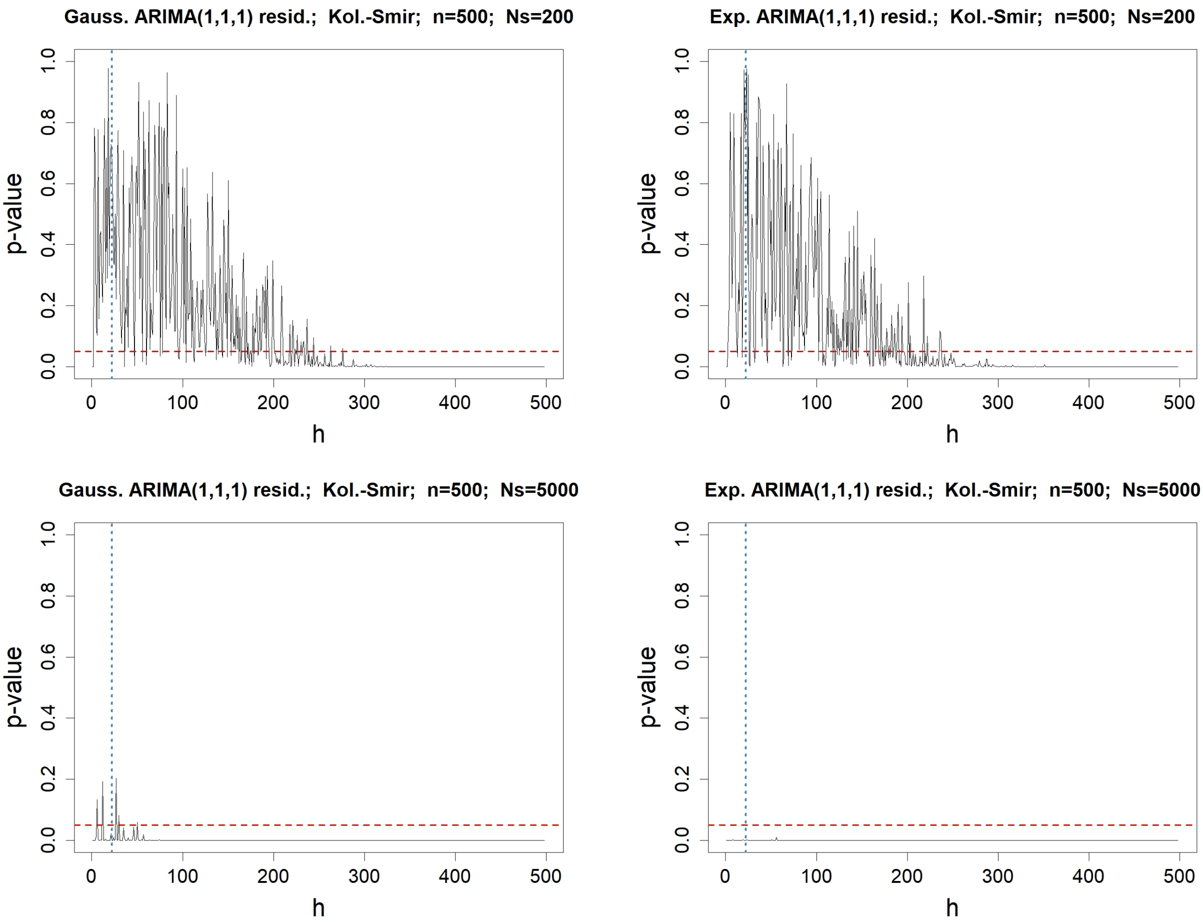

Figure S25.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns ARIMA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(1,1,1) process is $n = 500$. In the upper figures, the number of simulated ARIMA(1,1,1) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

3.1. Results for the residuals of simulated MA(2), estimated as WN, with length $n = 500$

We simulate a MA(2) with equation

$$Z_t \;=\; \mathcal{E}_t \,-\, \frac{3}{4}\mathcal{E}_{t-1} \,+\, \frac{1}{8}\mathcal{E}_{t-2}\,,$$

where $(\mathcal{E}_t)_t$ is either a Gaussian- or an Exponential-WN, with length $n = 500$. But instead of considering the convenient MA(2) model, we estimate the simulated process with a WN model.

3.1.1. Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk.

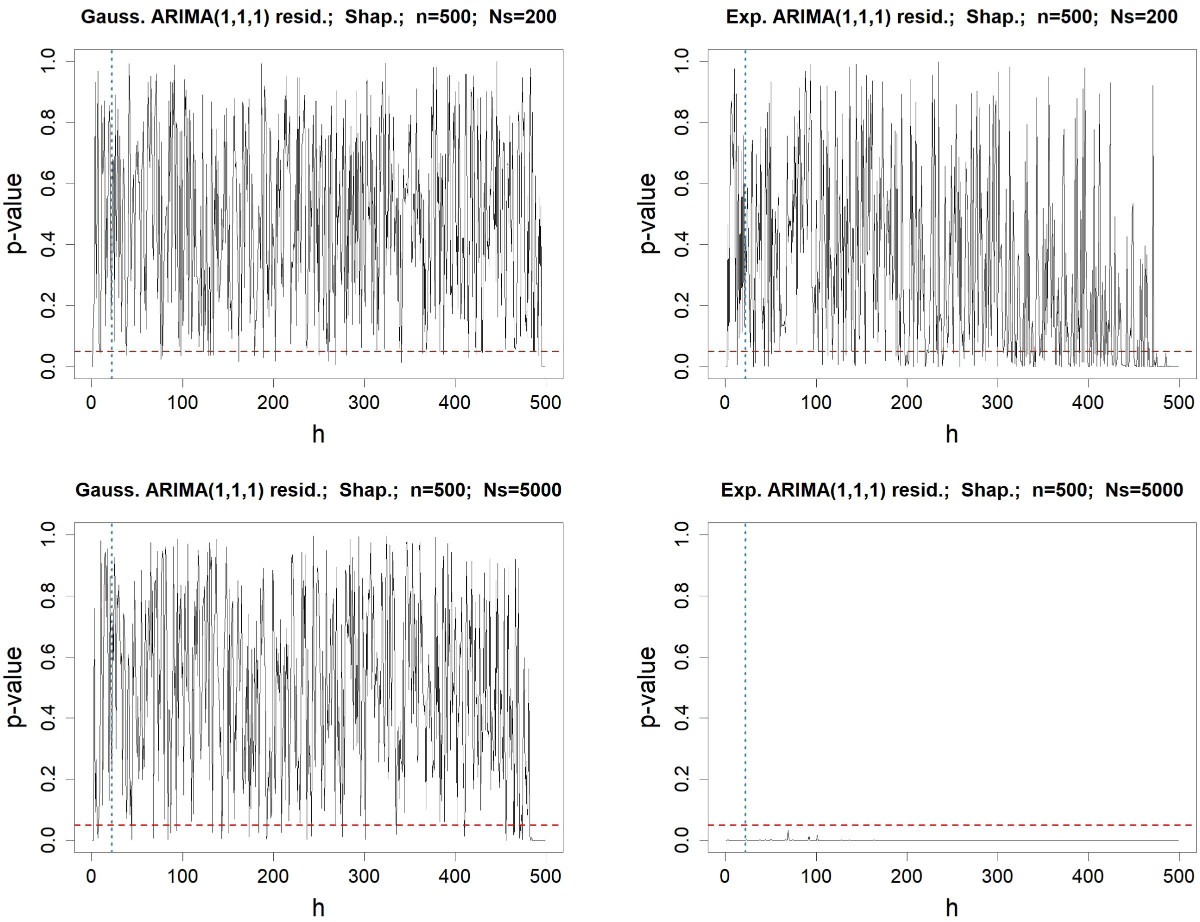

Figure S26.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(1,1,1) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(1,1,1) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

Figure S31 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$, whereas Figure S32 gives the p-values provided by Shapiro-Wilk.

Here, we observe the same behavior as for simulated WN. Indeed, in Figure S32, we see that Shapiro's test does not reject the normality of the ACFs at a fixed lag h, but Kolmogorov-Smirnov's test detects a lack of fit to the expected normal distribution $\mathcal{N}\left(0,\ \frac{1}{n}\right)$, as the number of simulations increases, see Figure S31. This means that the ACF follow a distribution close to a normal distribution, but with either an expectation different from 0 and/or variance different from $\frac{1}{n}$.

3.1.2. Check for the normality of $SACF(H)$ at a fixed lag H

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S33 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$.

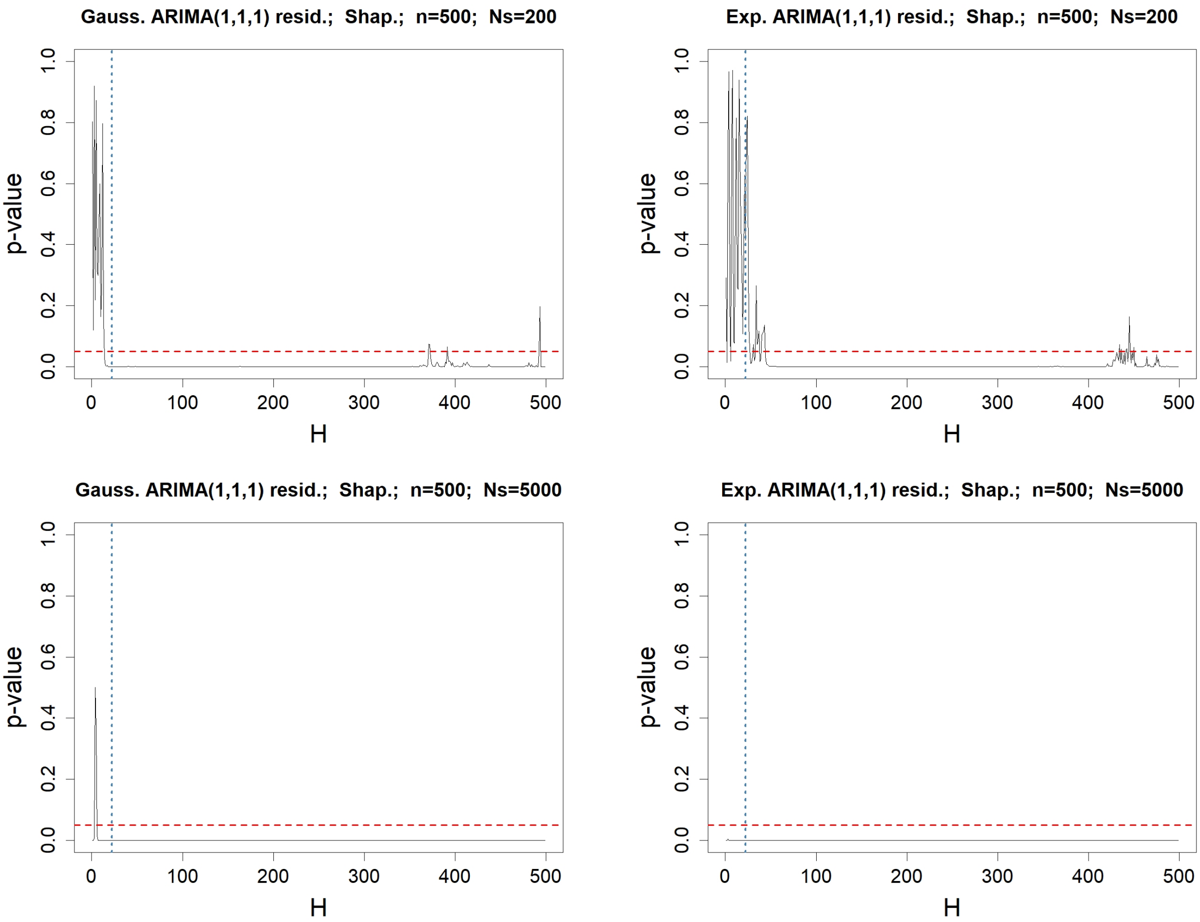

Figure S27.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns concerns ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(1,1,1) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(1,1,1) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

SACF associated with WN simulations or with residuals from well-specified models did not behave at all like normal distributions, contrary to what might be expected, even with a small number of simulations. Here, in Figure S33, the departure from normality is less obvious, especially when the underlying-WN is Gaussian.

3.1.3. Check for $\left(\boldsymbol{\mathcal{A}_H}(\underline{\boldsymbol{\mu}}, \boldsymbol{\Sigma})\right)$

For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words that they are the realizations of independent variables. Let us suppose that this hypothesis is satisfied. First we use Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and next Shapiro-Wilk to test for the normality behavior. Figure S35 gives the percentage of inadequate testing conclusions with Kolmogorov-Smirnov test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of MA(2) processes estimated

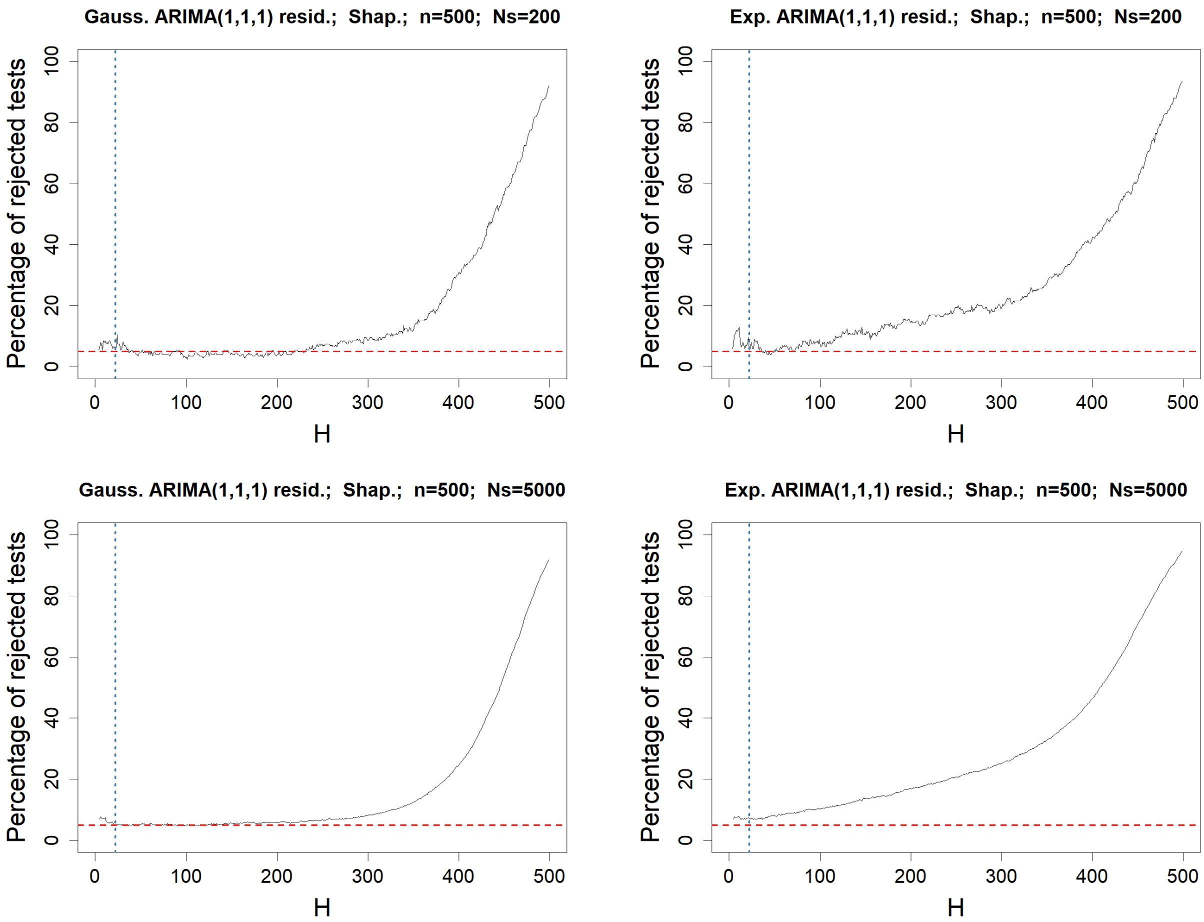

Figure S28.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential WN processes. The length of the simulated ARIMA(1,1,1) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(1,1,1) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

by a WN, whereas Figure S34 gives the same percentages with Shapiro-Wilk. In Figure S34, we observe that the percentage of p-values $< \alpha = 5\%$ for Kolmogorov-Smirnov test is very close to 5%, which seems to comfort that $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ can be considered as realizations of Gaussian variables $\mathcal{N}\left(0, \frac{1}{n}\right)$, until a lag H rather large, as if the residuals were a WN. But as seen in Figure S35, Shapiro's test largely rejects the normality condition for the successive ACFs. Shapiro's test on successive ACFs is the only normality test that is sensitive to the fact that the model is misspecified, alerting us to the fact that the residuals probably do not form a white noise.

3.1.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S36 shows that Box-Pierce and Ljung-Box tests have detected that the model was misspecified.

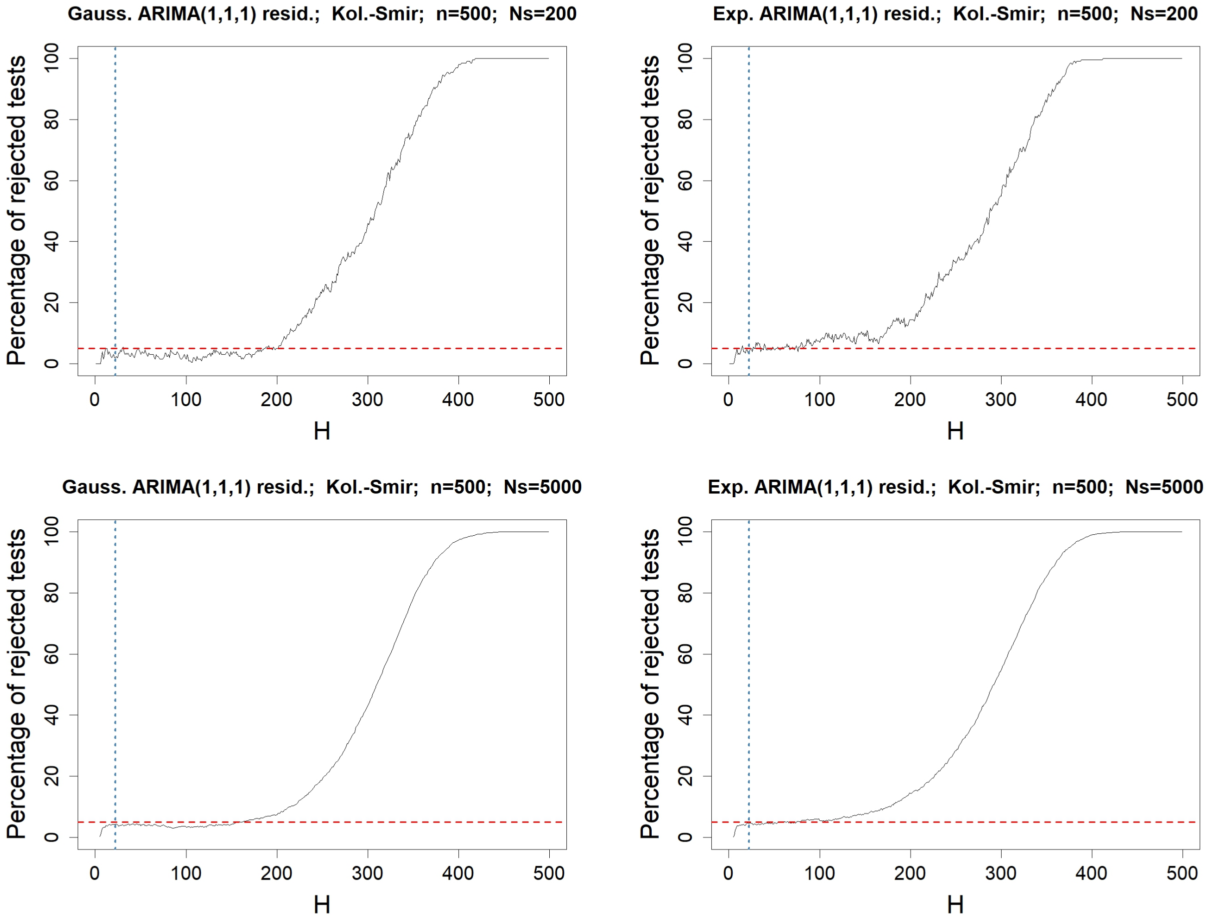

Figure S29.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. In the upper figures, the number of simulated ARIMA(1,1,1) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

3.2. Results for the residuals of simulated AR(2), estimated as WN, with length $n = 500$

We simulate a AR(2) with equation

$$Z_t + \frac{3}{4} Z_{t-1} - \frac{1}{8} Z_{t-2} = \mathcal{E}_t \,,$$

where $(\mathcal{E}_t)_t$ is either a Gaussian- or an Exponential-WN, with length $n = 500$. But instead of considering the convenient AR(2) model, we estimate the simulated process with a WN model.

3.2.1. Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk.

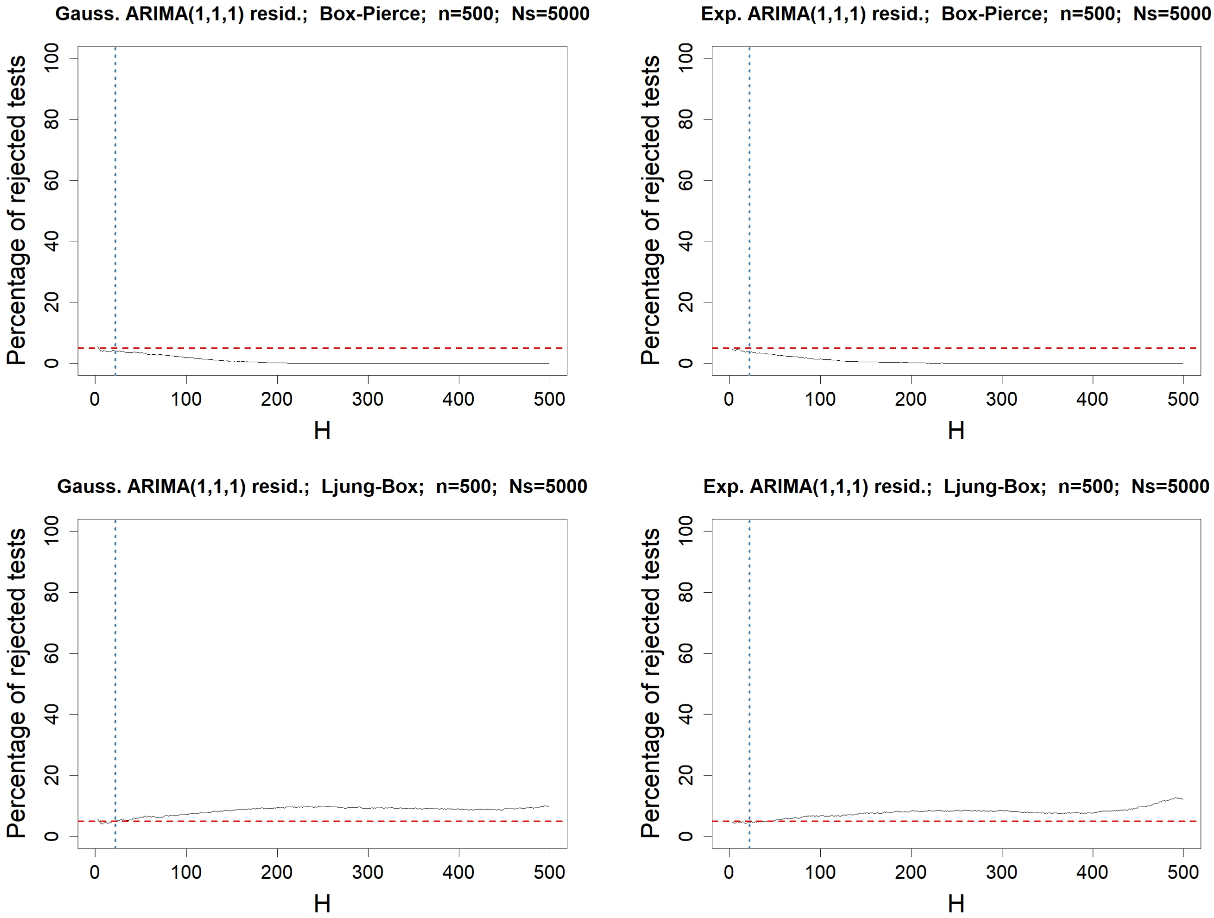

Figure S30.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

Figure S37 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$, whereas Figure S38 gives the p-values provided by Shapiro-Wilk.

Here, we observe a different behavior from the one observed for simulated-WN. There is an increased tendency to reject the null hypothesis of normality, especially for the Kolmogorov-Smirnov test. This means that the ACF follow a distribution close to a normal distribution, but with either an expectation different from 0 and/or variance different from $\frac{1}{n}$.

3.2.2. Check for the normality of $SACF(H)$ at a fixed lag H

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S39 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$.

SACF associated with WN simulations or with residuals from well-specified models did not behave at all like normal distributions, contrary to what might be expected, even with a small number of simulations. Here, in Figure S39, the departure from normality is less obvious, espe-

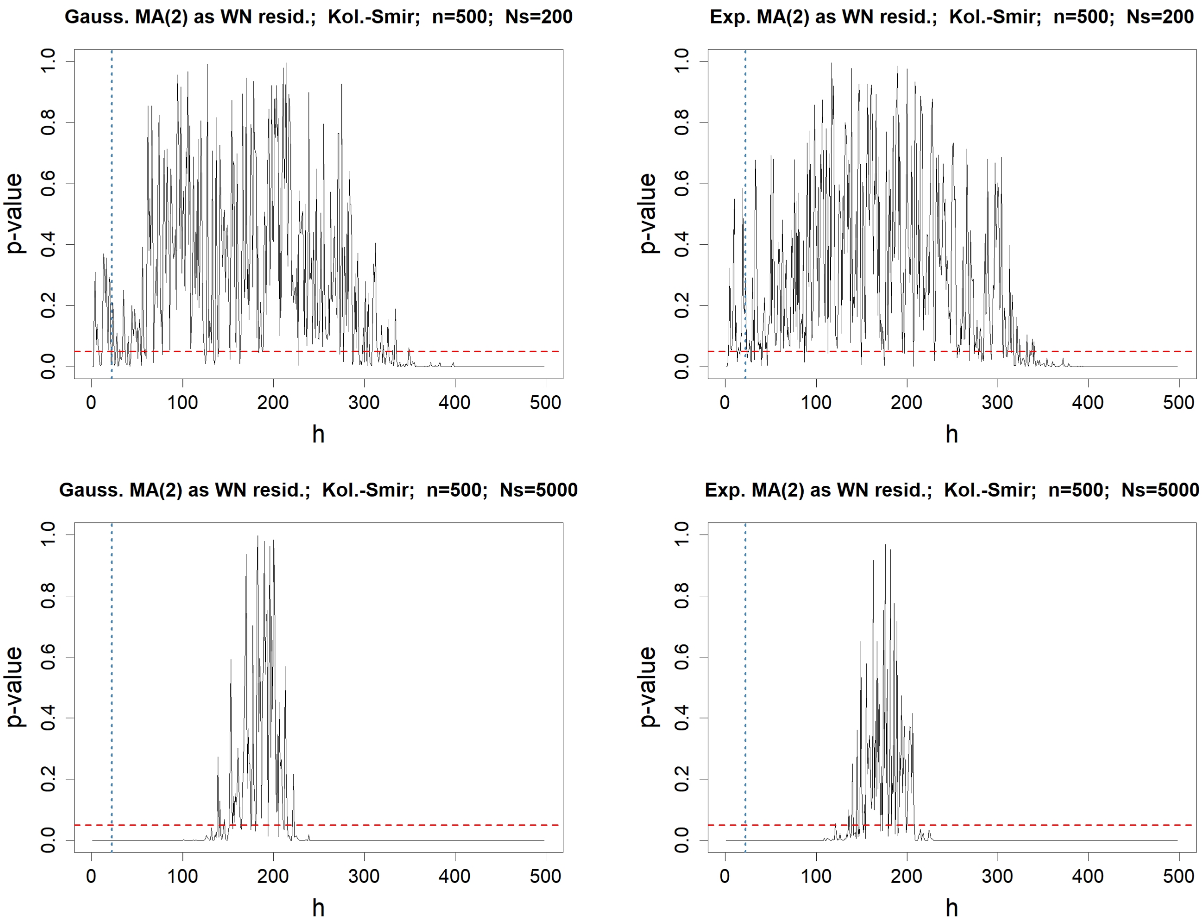

Figure S31.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential underlying-WN process. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

cially when the underlying-WN is Gaussian.

3.2.3. Check for $\left(\boldsymbol{\mathcal{A}_H}(\underline{\boldsymbol{\mu}}, \boldsymbol{\Sigma})\right)$

For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words that they are the realizations of independent variables. Let us suppose that this hypothesis is satisfied. First we use Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and next Shapiro-Wilk to test for the normality behavior. Figure S41 gives the percentage of inadequate testing conclusions with Kolmogorov-Smirnov test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of AR(2) processes estimated by a WN, whereas Figure S40 gives the same percentages with Shapiro-Wilk. In Figure S40 and S41, we observe that both Kolmogorov-Smirnov and Shapiro tests largely reject the normality condition for the successive ACFs, as if they had detected that the residuals were not a WN.

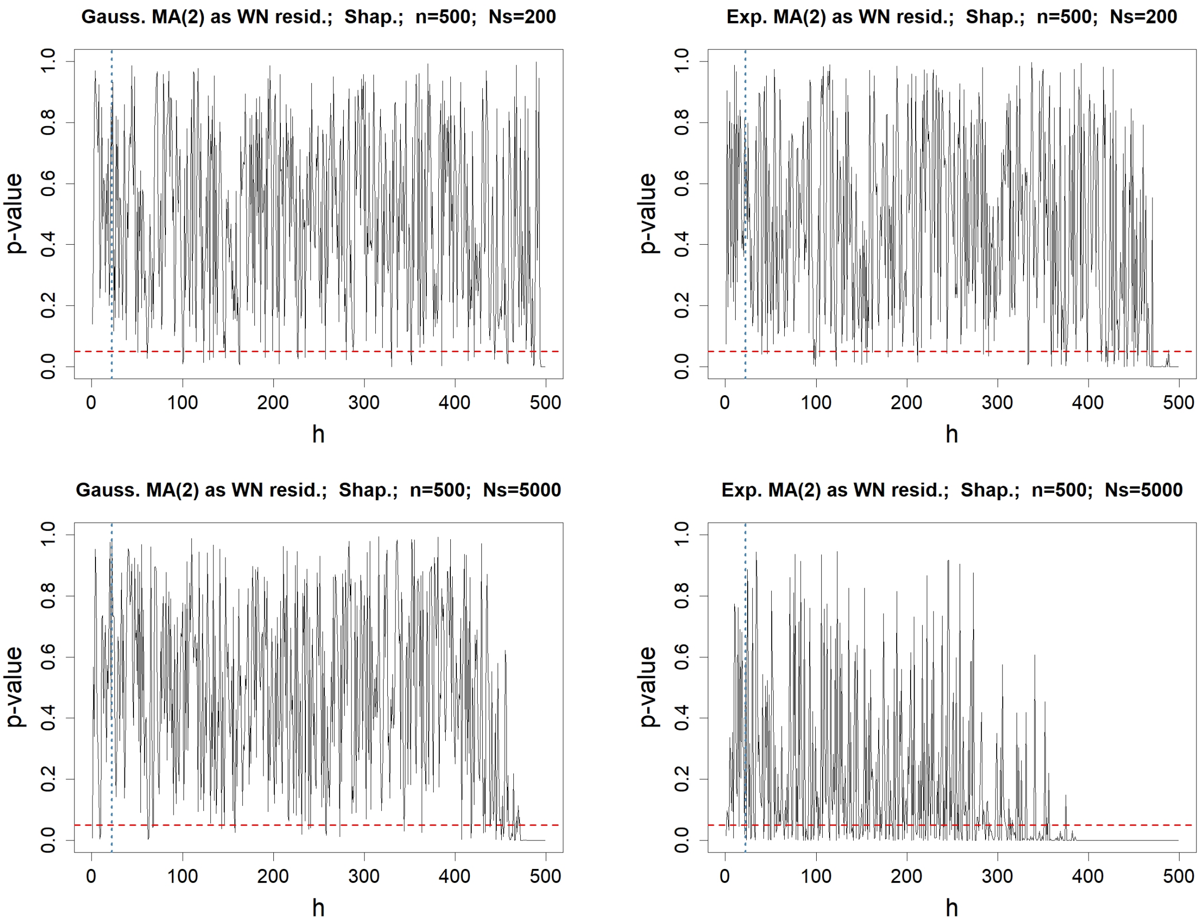

Figure S32.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential underlying-WN process. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

Thus both normality tests, applied to the successive ACFs are sensitive to the fact that the model is misspecified, alerting us to the fact that the residuals probably do not form a white noise.

3.2.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S42 shows that Box-Pierce and Ljung-Box tests have detected that the model was misspecified.

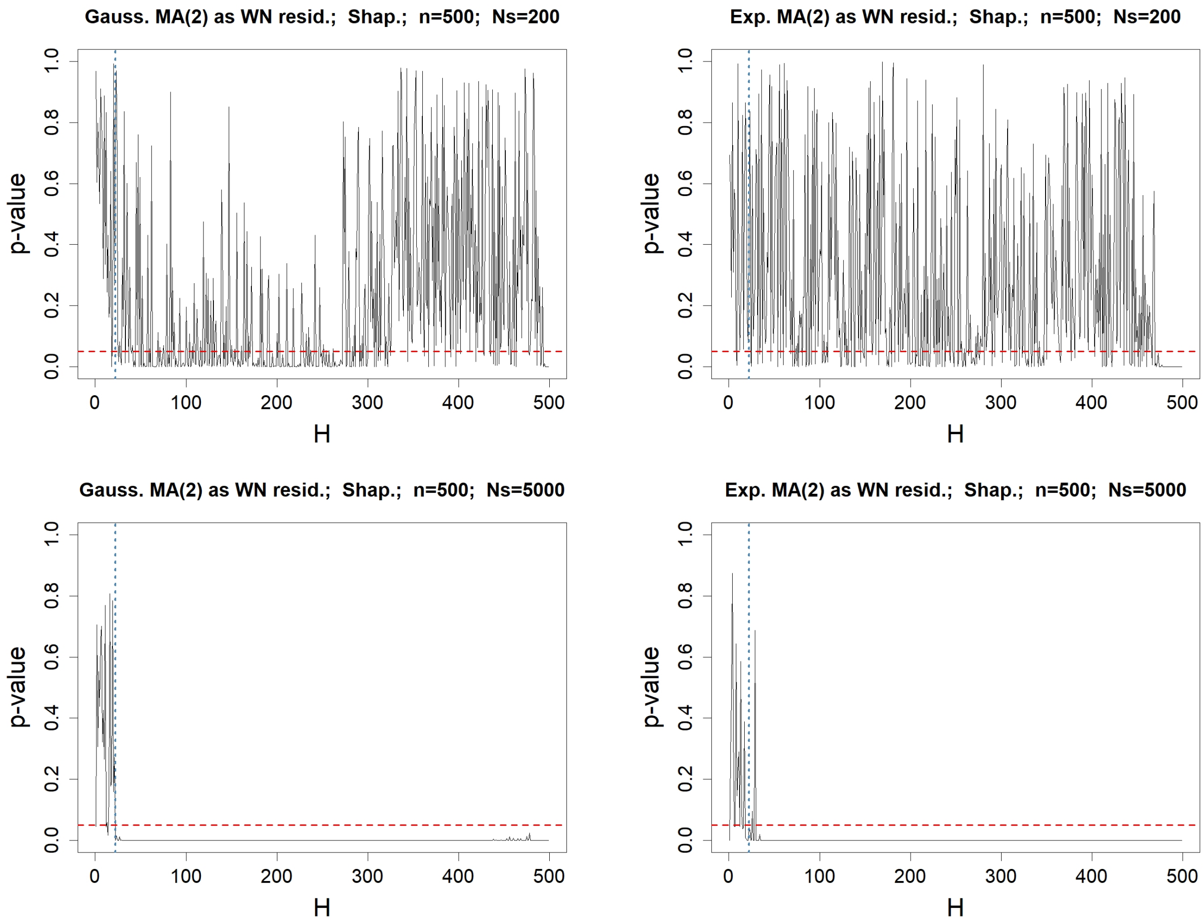

Figure S33.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

3.3. Results for the residuals of simulated ARIMA(0,2,2), estimated as AR(1), with length $n = 500$

We simulate an ARIMA(0,2,2) with equation

$$\Delta^2(Z_t) \;=\; \mathcal{E}_t \;-\; \frac{3}{4}\mathcal{E}_{t-1} \;+\; \frac{1}{8}\mathcal{E}_{t-2}\,,$$

where $(\mathcal{E}_t)_t$ is either a Gaussian- or an Exponential-WN, with length $n = 500$. But instead of considering the convenient ARIMA(0,2,2) model, we estimate the simulated process with a WN model.

3.3.1. Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0,\, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk.

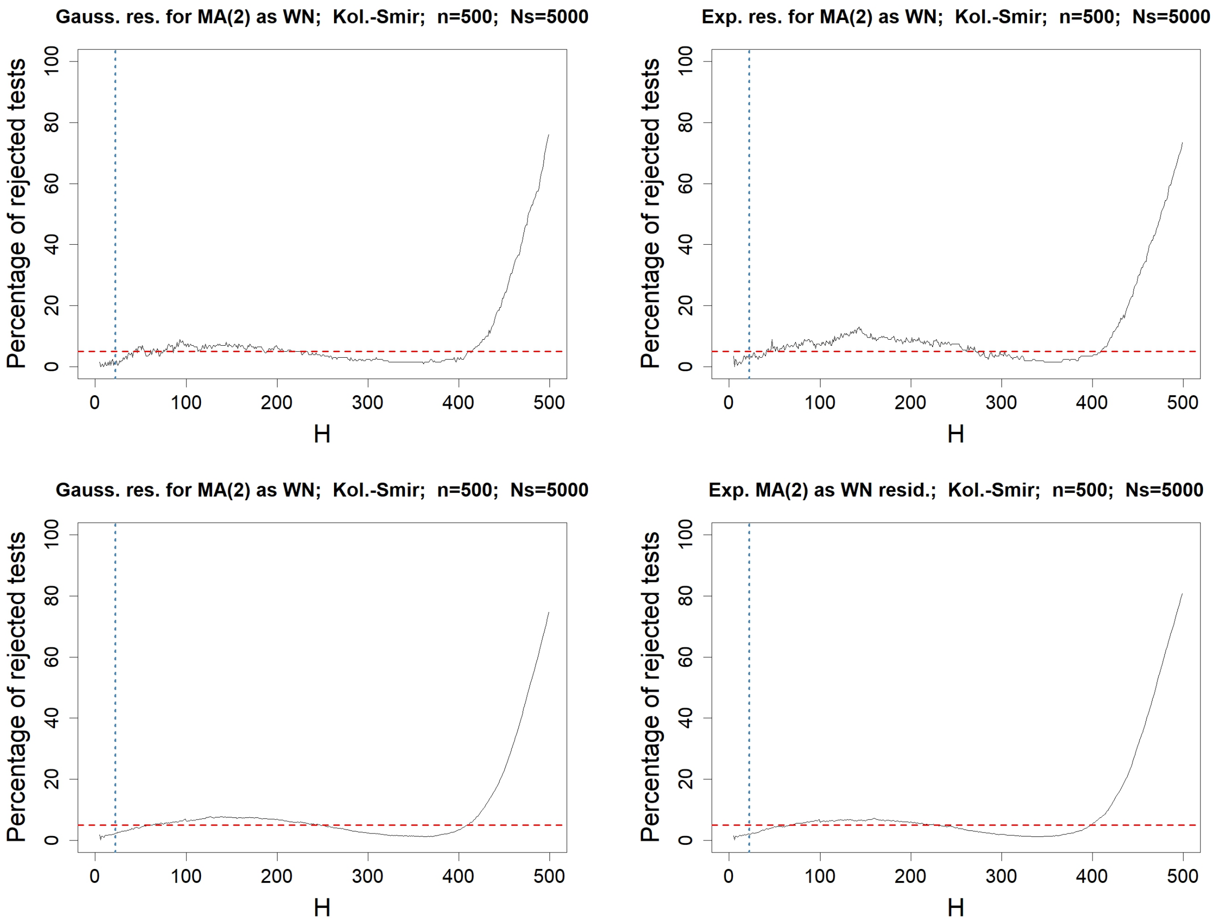

Figure S34.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

Figure S43 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$, whereas Figure S44 gives the p-values provided by Shapiro-Wilk.

Here, we observe a different behavior from the one observed for simulated-WN or for the residuals of well-specified models. Both Kolmogorov-Smirnov and Shapiro tests systematically reject the null hypothesis of normality. This means that the ACF may not follow a normal distribution.

3.3.2. Check for the normality of $SACF(H)$ at a fixed lag H

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S45 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$. Like ACF, SACF do not follow a normal distribution.

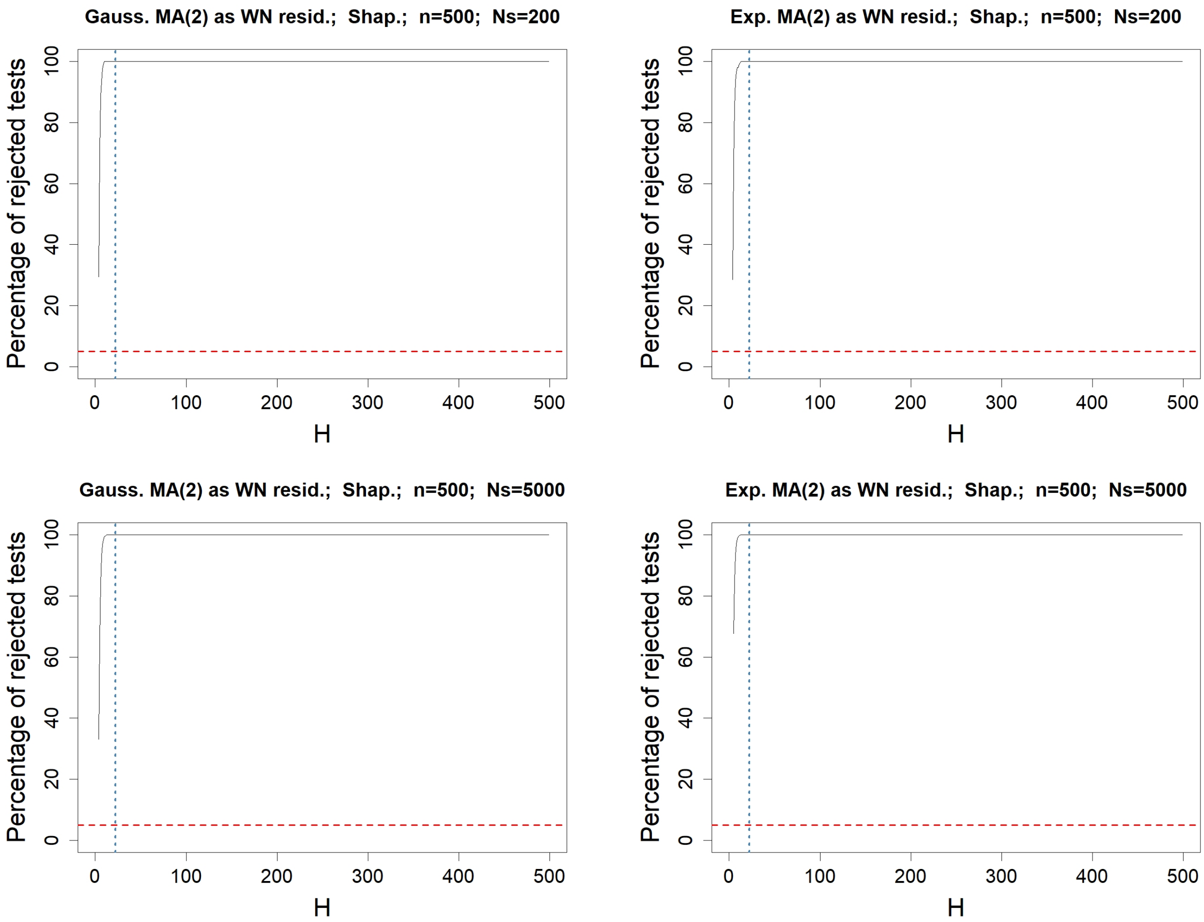

Figure S35.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

3.3.3. Check for $\left(\boldsymbol{\mathcal{A}_H}(\underline{\boldsymbol{\mu}}, \boldsymbol{\Sigma})\right)$

For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words that they are the realizations of independent variables. Let us suppose that this hypothesis is satisfied. First we use Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and next Shapiro-Wilk to test for the normality behavior. Figure S47 gives the percentage of inadequate testing conclusions with Kolmogorov-Smirnov test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of ARIMA(0,2,2) processes estimated by a WN, whereas Figure S46 gives the same percentages with Shapiro-Wilk. In Figure S46 and S47, we observe that both Kolmogorov-Smirnov and Shapiro tests largely reject the normality condition for the successive ACFs, as if they had detected that the residuals were not a WN. Thus both normality tests, applied to the successive ACFs are sensitive to the fact that the model is misspecified, alerting us to the fact that the residuals probably do not form

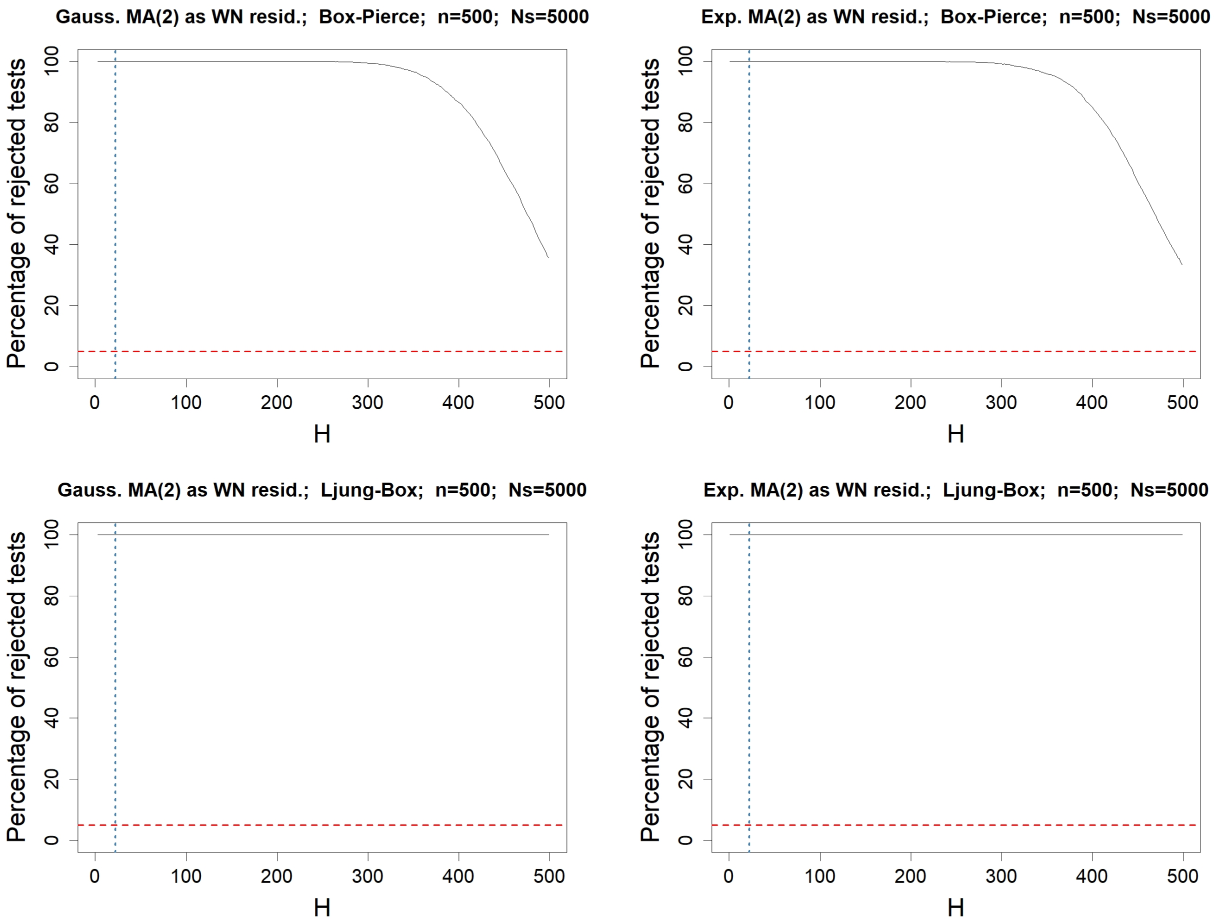

Figure S36.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns MA(2)-simulations associated to Gaussian underlying-WN whereas the right one deals with Exponential ones. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

a white noise.

3.3.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S48 shows that Box-Pierce and Ljung-Box tests have detected that the model was misspecified.

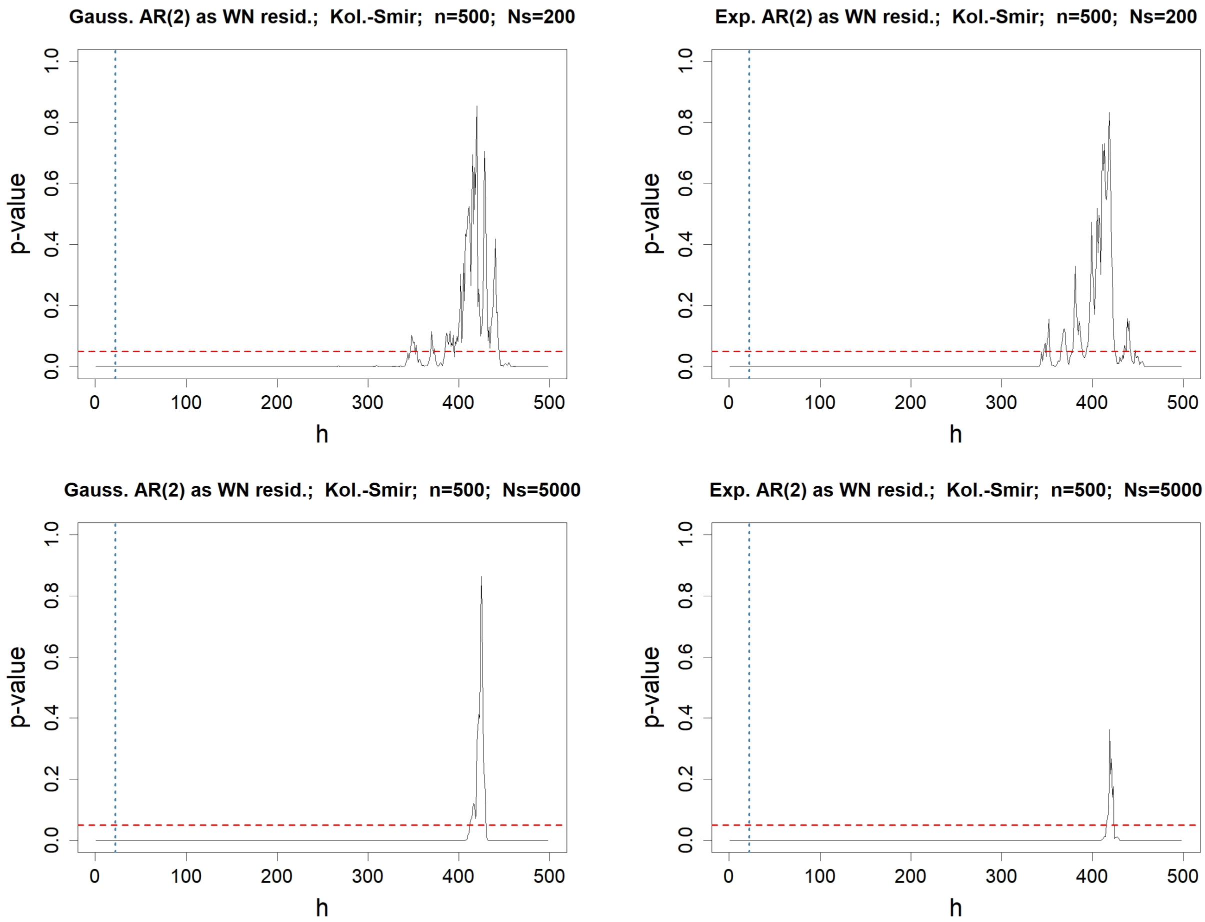

Figure S37.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential underlying-WN process. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

3.4. Results for the residuals of simulated ARIMA(1,1,1), estimated as ARIMA(2,1,2), with length $n = 500$

Here, we simulate N_S ARIMA(1,1,1) processes with equation

$$\Delta(Z_t) - \frac{1}{4}\Delta(Z_{t-1}) = \mathcal{E}_t + \frac{1}{2}\mathcal{E}_{t-1},$$

where $(\mathcal{E}_t)_t$ is either a Gaussian- or an Exponential-WN, with length $n = 500$. We estimate every simulated series as an ARIMA(2,1,2) process, and compute its residuals. All the testing-procedures provide the same results as for the residuals of well-specified models.

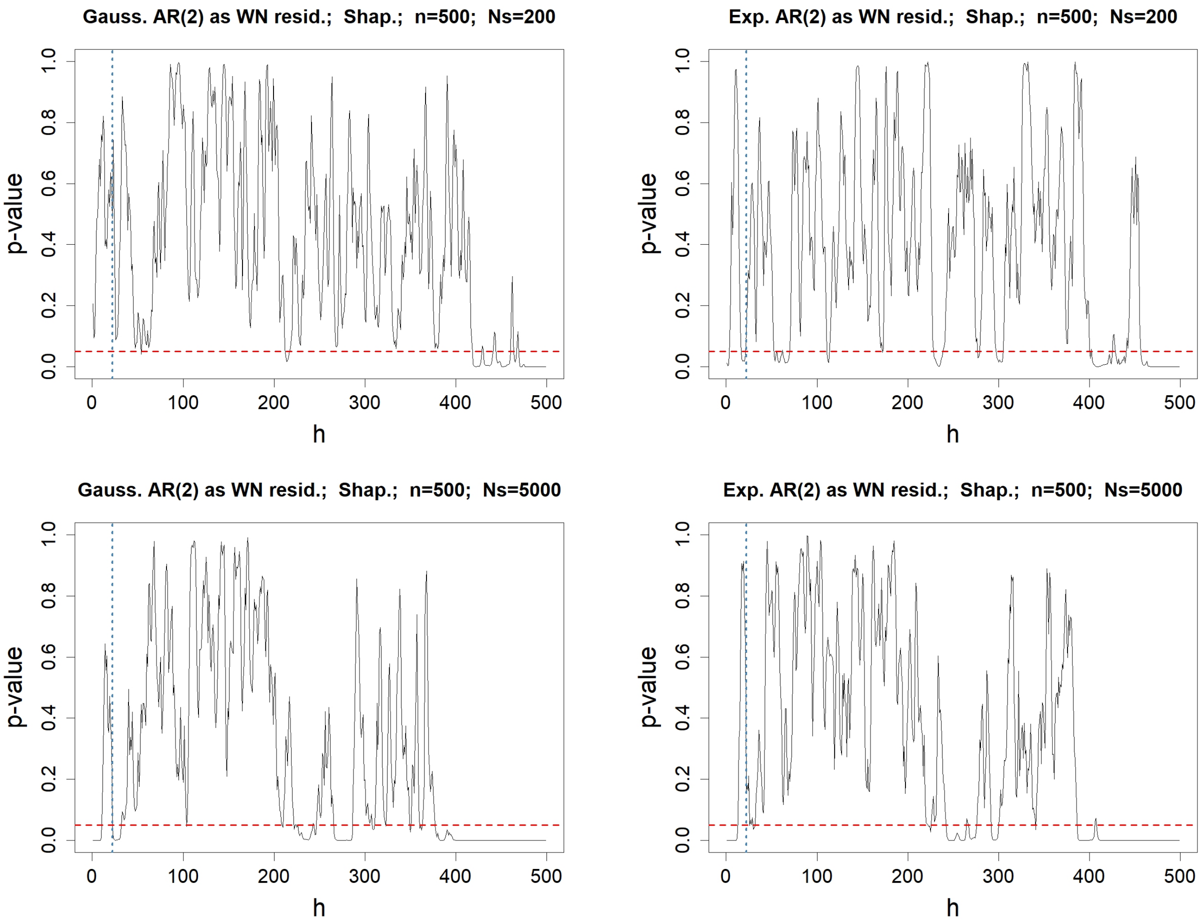

Figure S38.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential underlying-WN process. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

3.4.1. Check for the normality of $\hat{\Xi}(h)$ at a fixed lag h

At a given lag h, using Kolmogorov-Smirnov test, we test for the adequacy of $\hat{\Xi}(h)$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$, and we also merely test for its normality using Shapiro-Wilk. Figure S49 displays the p-values provided by the Kolmogorov-Smirnov test, when applied to the residuals of either $N_S = 200$ or $N_S = 5\,000$ simulations of an ARIMA(1,1,1) process, whereas Figure S50 gives the p-values provided by Shapiro-Wilk.

3.4.2. Check for the normality of $S_{ACF}(H)$ at a fixed lag H

At a given lag H, we test for the normality of the N_S values of $S_{ACF}(H)$. Figure S51 displays the p-values provided by the Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of WN with length $n = 500$.

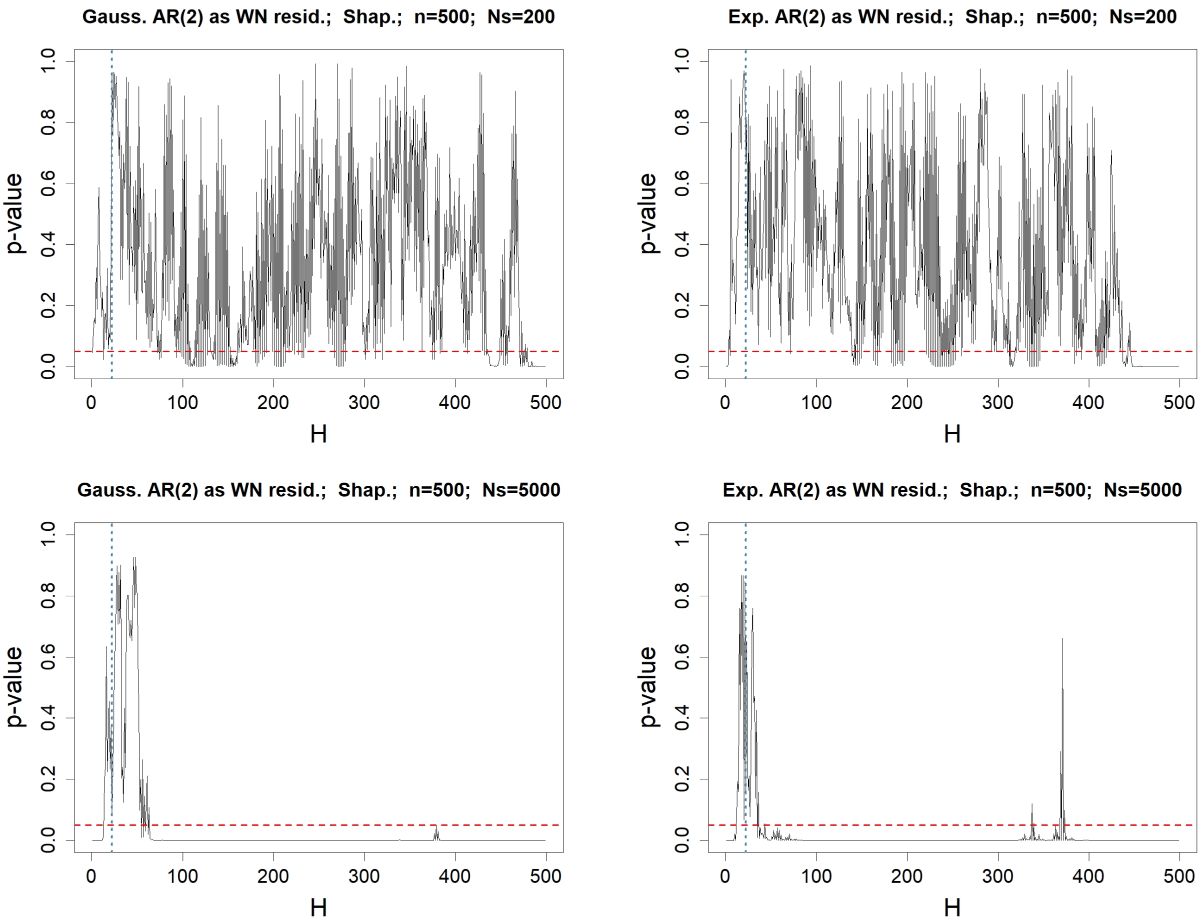

Figure S39.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

3.4.3. Check for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$

For a given simulation, we test the normality of the set $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to n-1. This procedure appears adapted to check for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with a Gaussian vector, but it requires that the successive sample ACF form a sample, in other words that they are the realizations of independent variables. Let us suppose that this hypothesis is satisfied. First we use Shapiro-Wilk to test for the normality behavior, and next Kolmogorov-Smirnov to test for the adequacy of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$ with the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{n}\right)$. Figure S52 gives the percentage of inadequate testing conclusions with Shapiro-Wilk test, when applied to either $N_S = 200$ or $N_S = 5\,000$ simulations of MA(2) with length $n = 500$ and, whereas Figure S53 gives the same percentages with Kolmogorov-Smirnov.

3.4.4. Behavior of portmanteau tests

We explore the reliability of Box-Pierce and Ljung-Box tests on our simulations, by applying these tests on every simulation at lags $H = 1$ to $n-1$, and we compute the percentage of

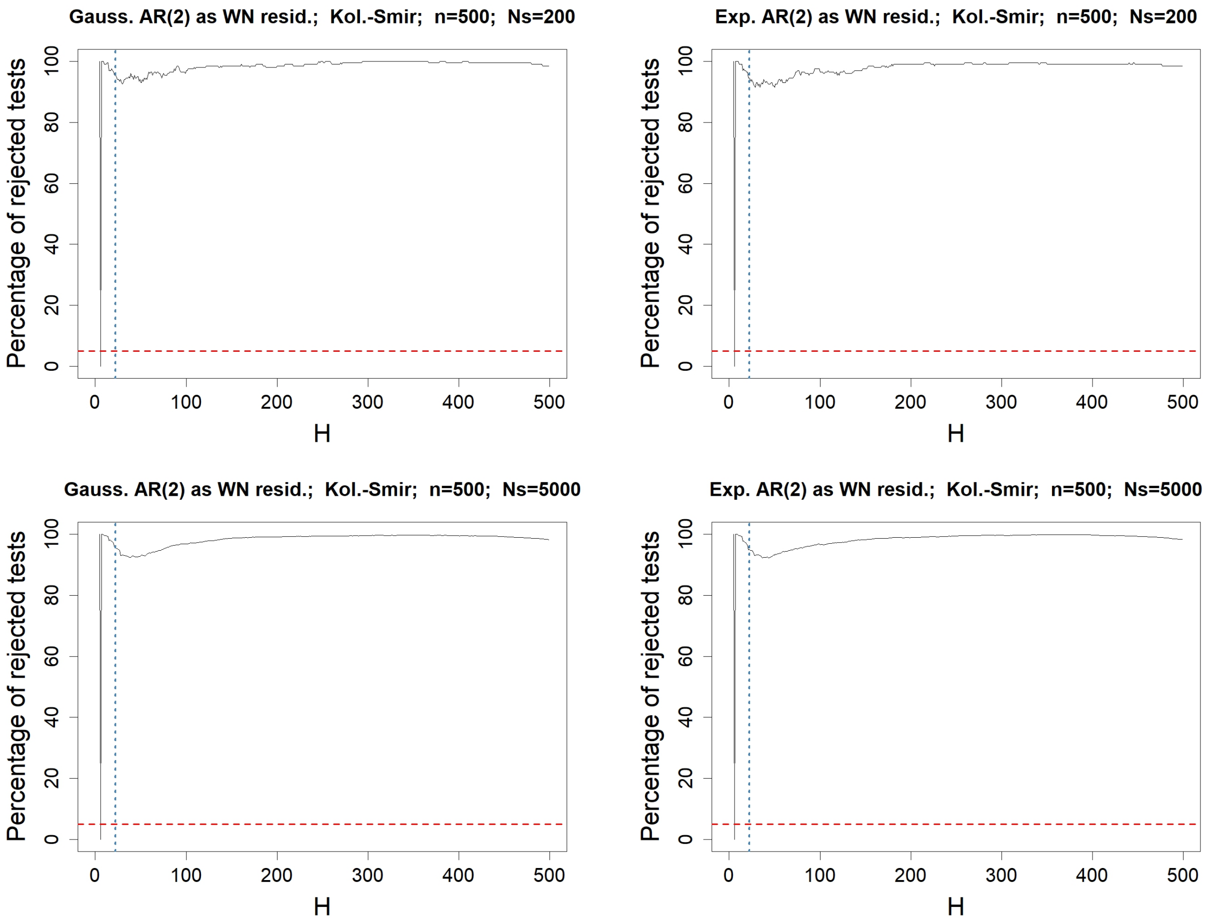

Figure S40.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

unexpected p-values ($< \alpha = 5\%$) among the N_S simulations. Figure S54 shows that Box-Pierce and Ljung-Box tests are not totally accurate. Indeed, Box-Pierce appears to be too conservative, whereas Ljung-Box is too liberal.

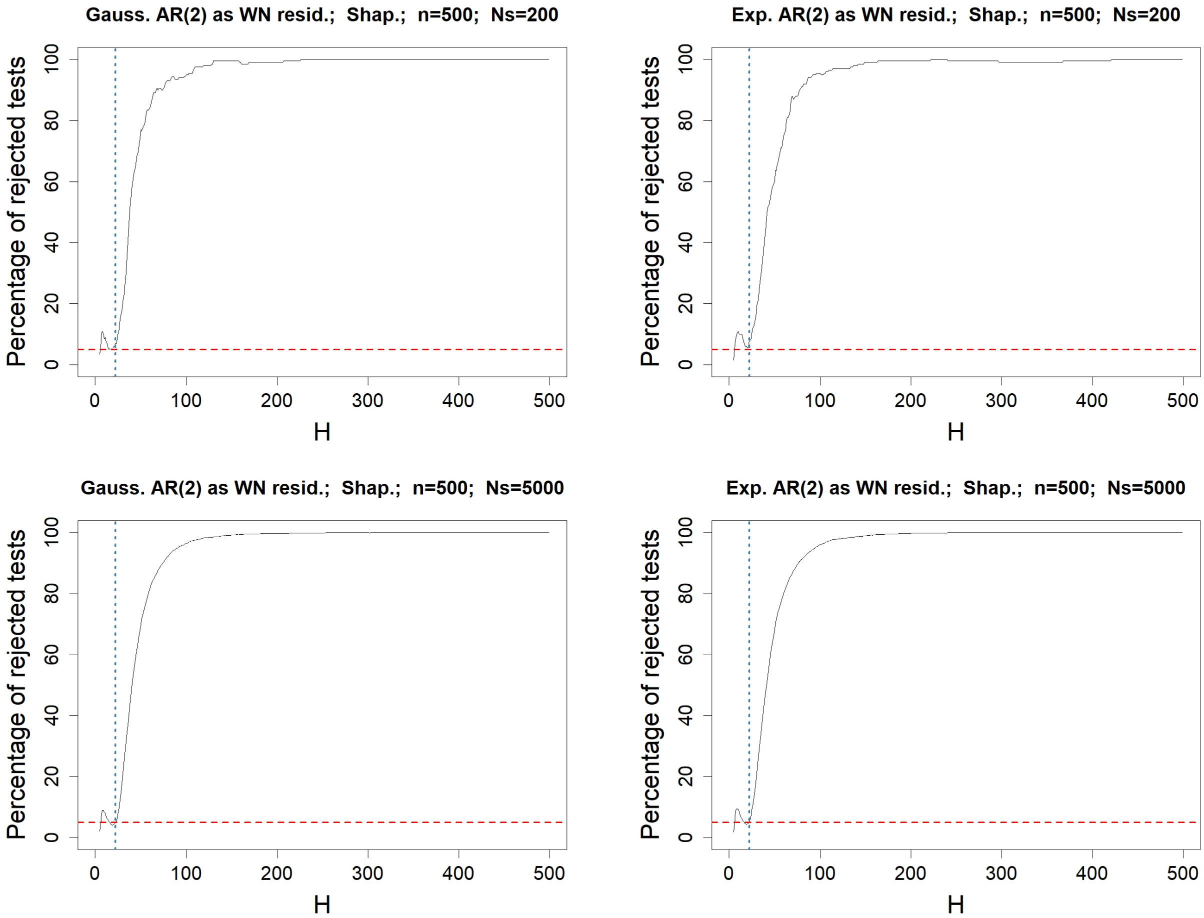

Figure S41.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

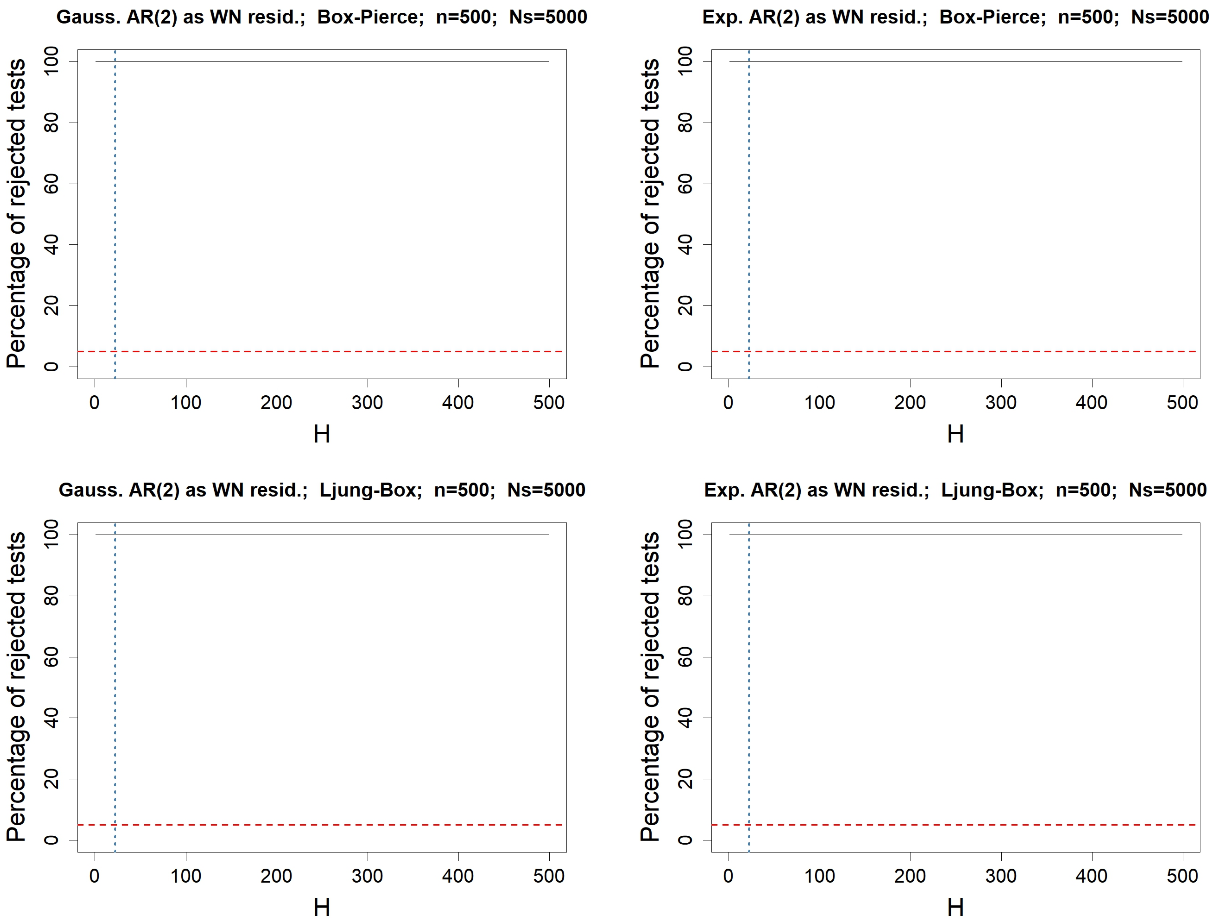

Figure S42.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns AR(2)-simulations associated to Gaussian underlying-WN whereas the right one deals with Exponential ones. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

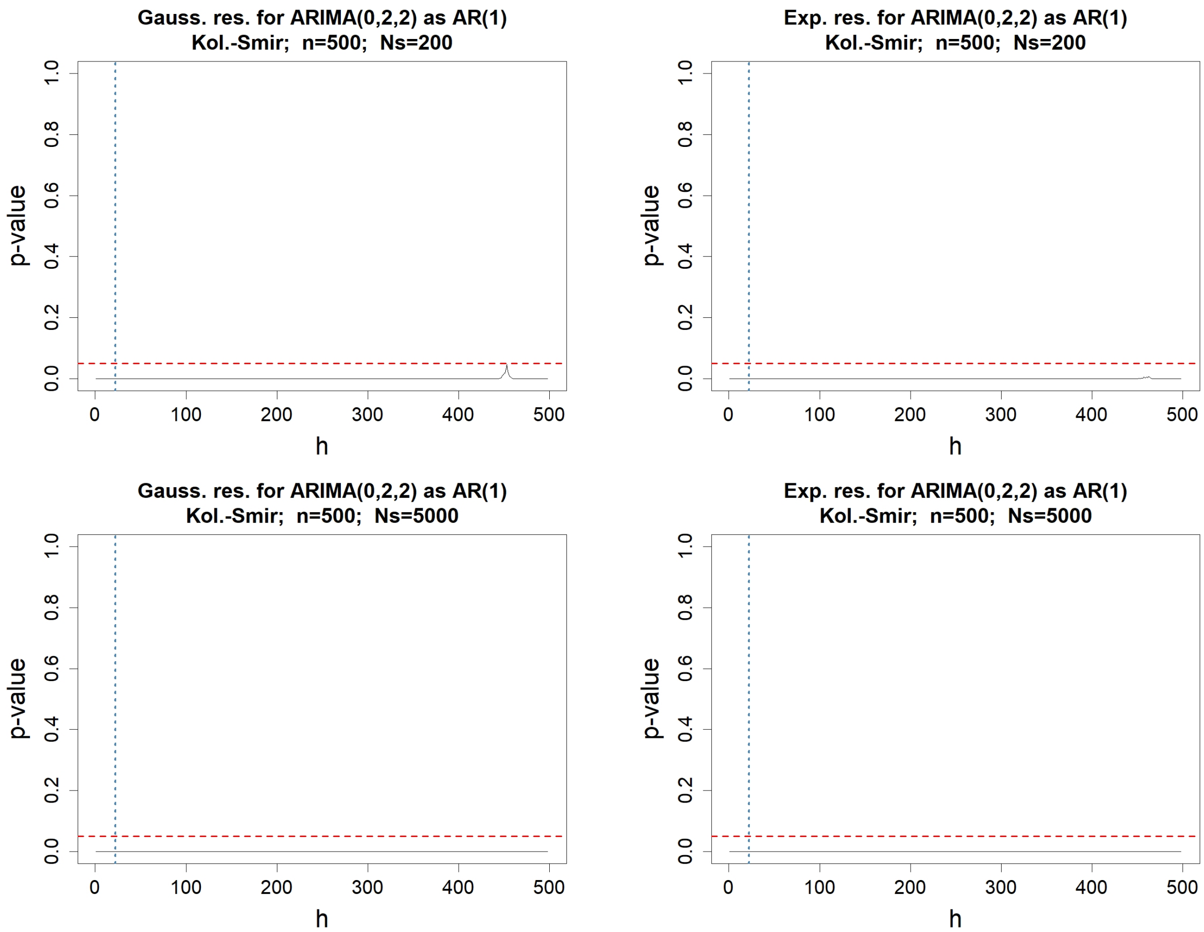

Figure S43.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential underlying-WN process. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

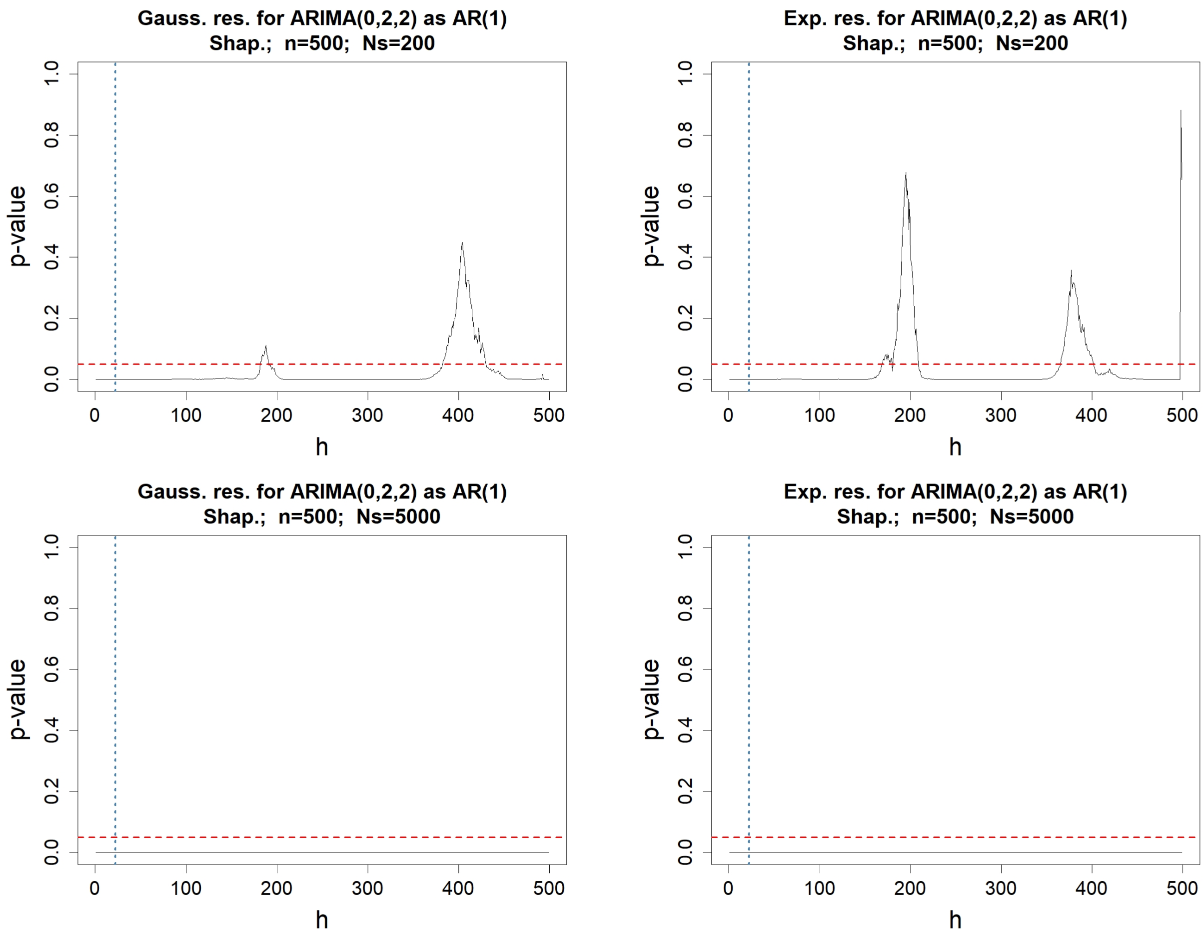

Figure S44.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential underlying-WN process. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

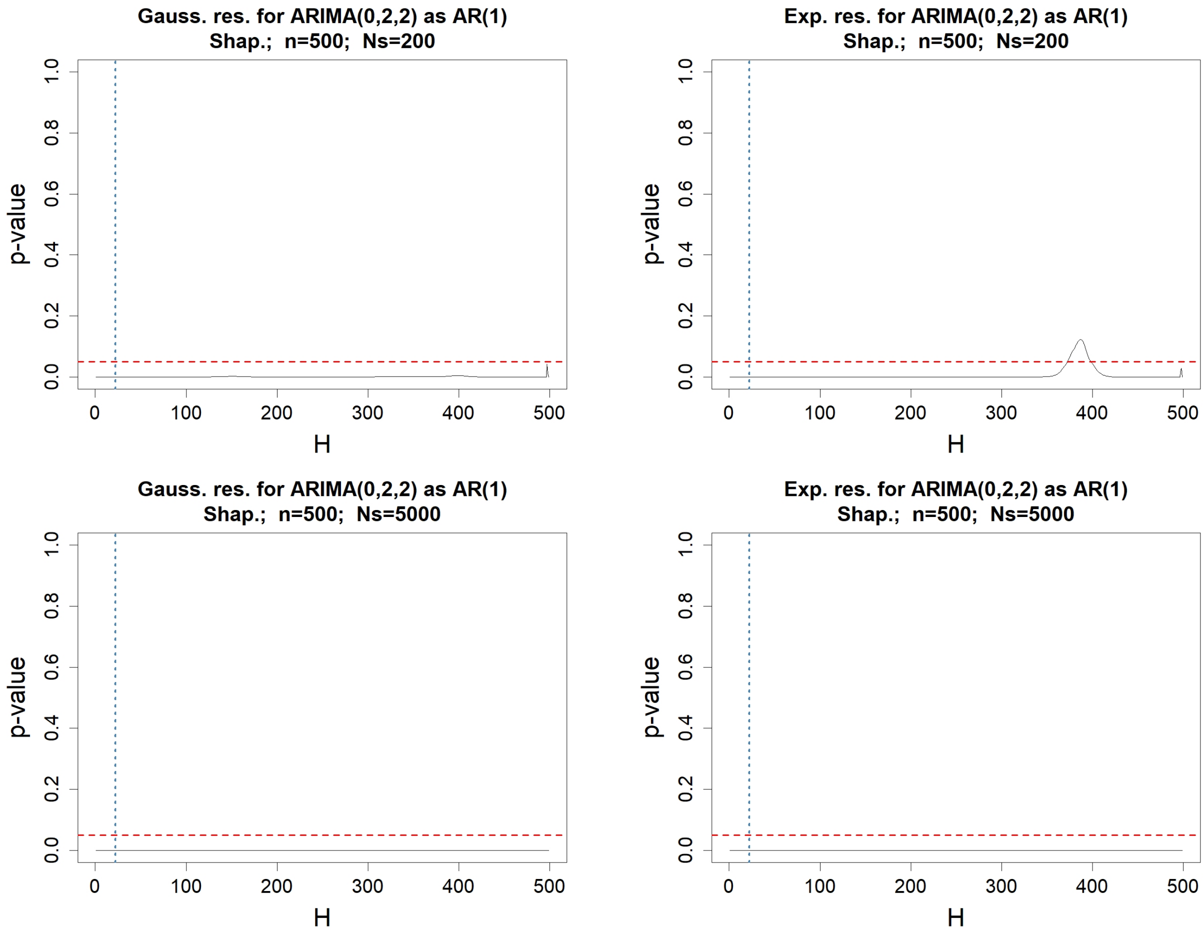

Figure S45.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

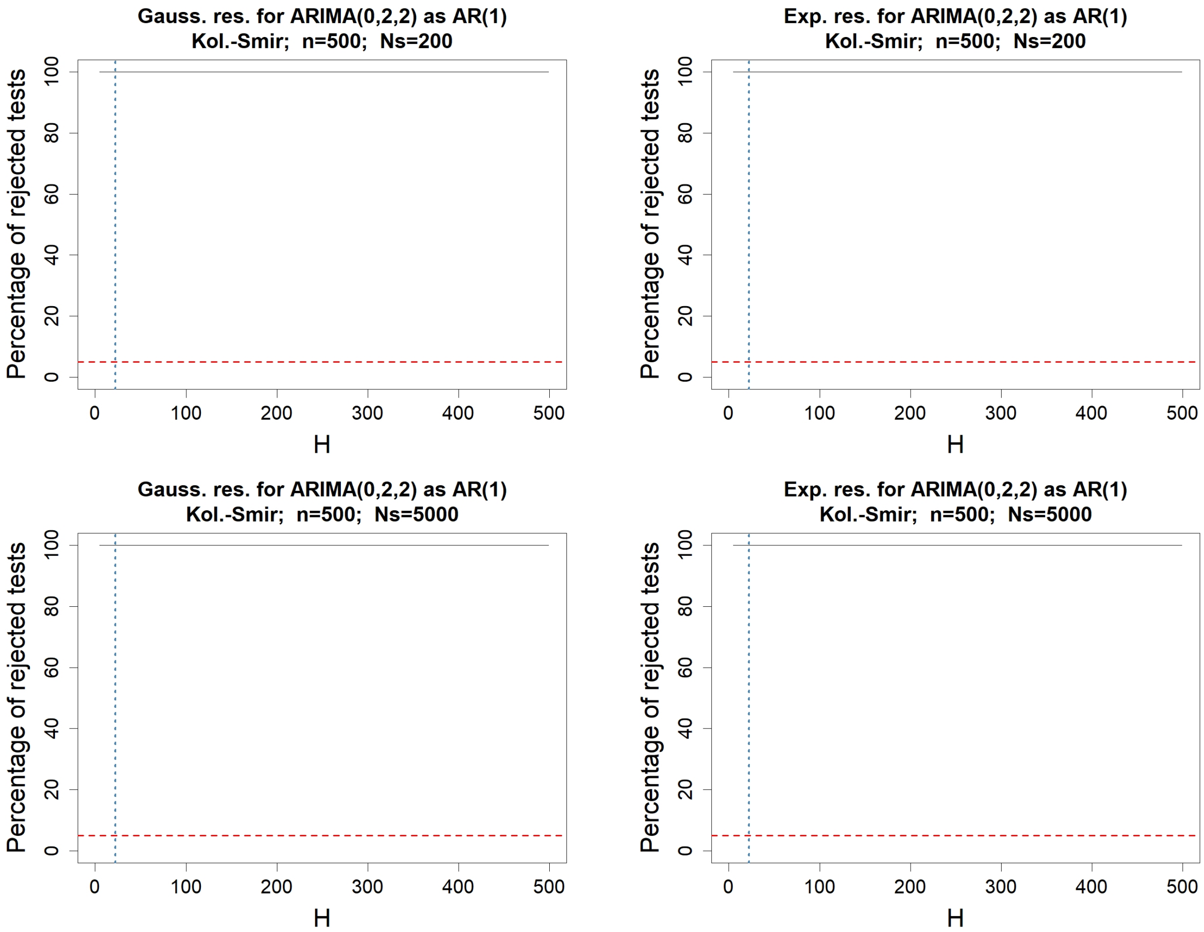

Figure S46.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

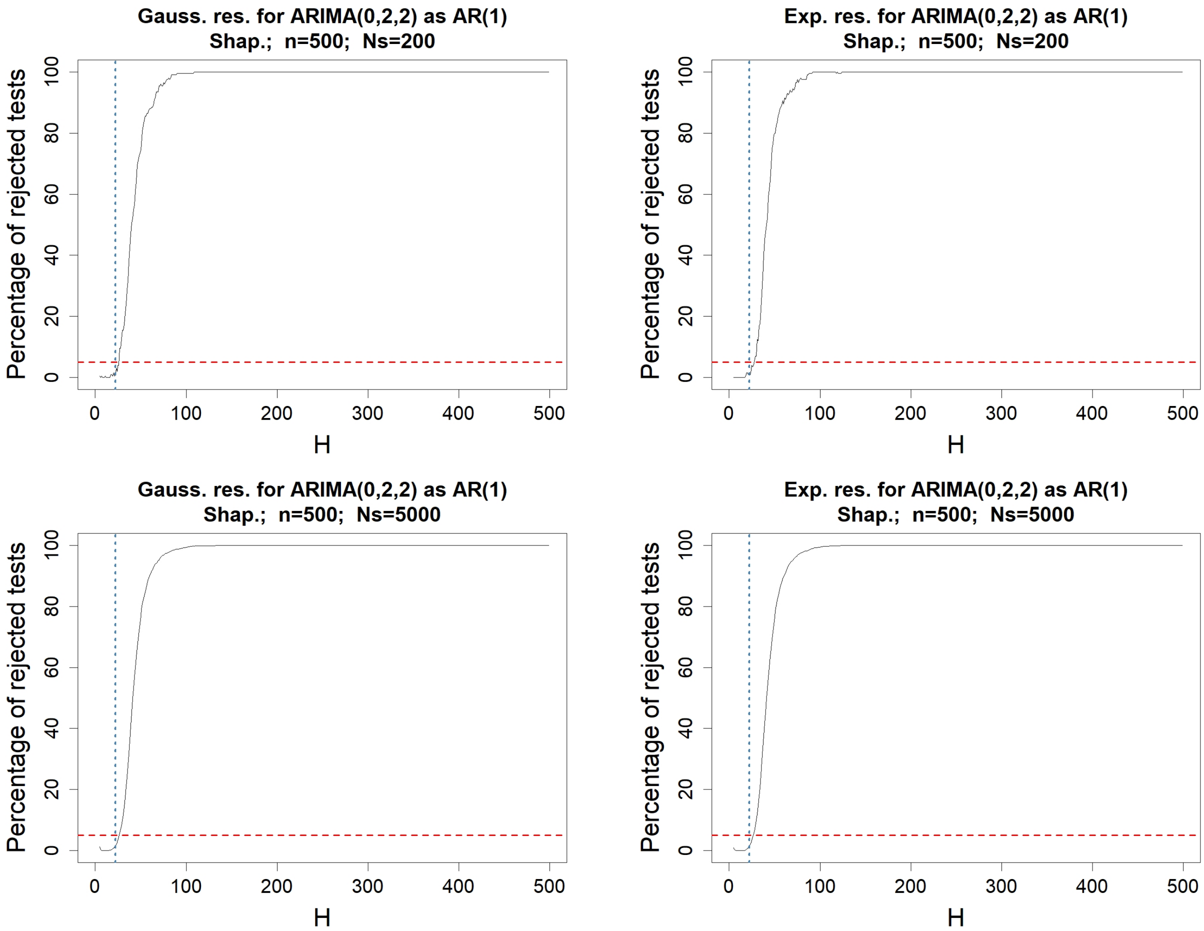

Figure S47.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $(\hat{\rho}(1), \cdots, \hat{\rho}(H))$, with H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian underlying-WN whereas the right one deals with Exponential ones. The length of the simulated WN process is $n = 500$. In the upper figures, the number of simulated WN processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

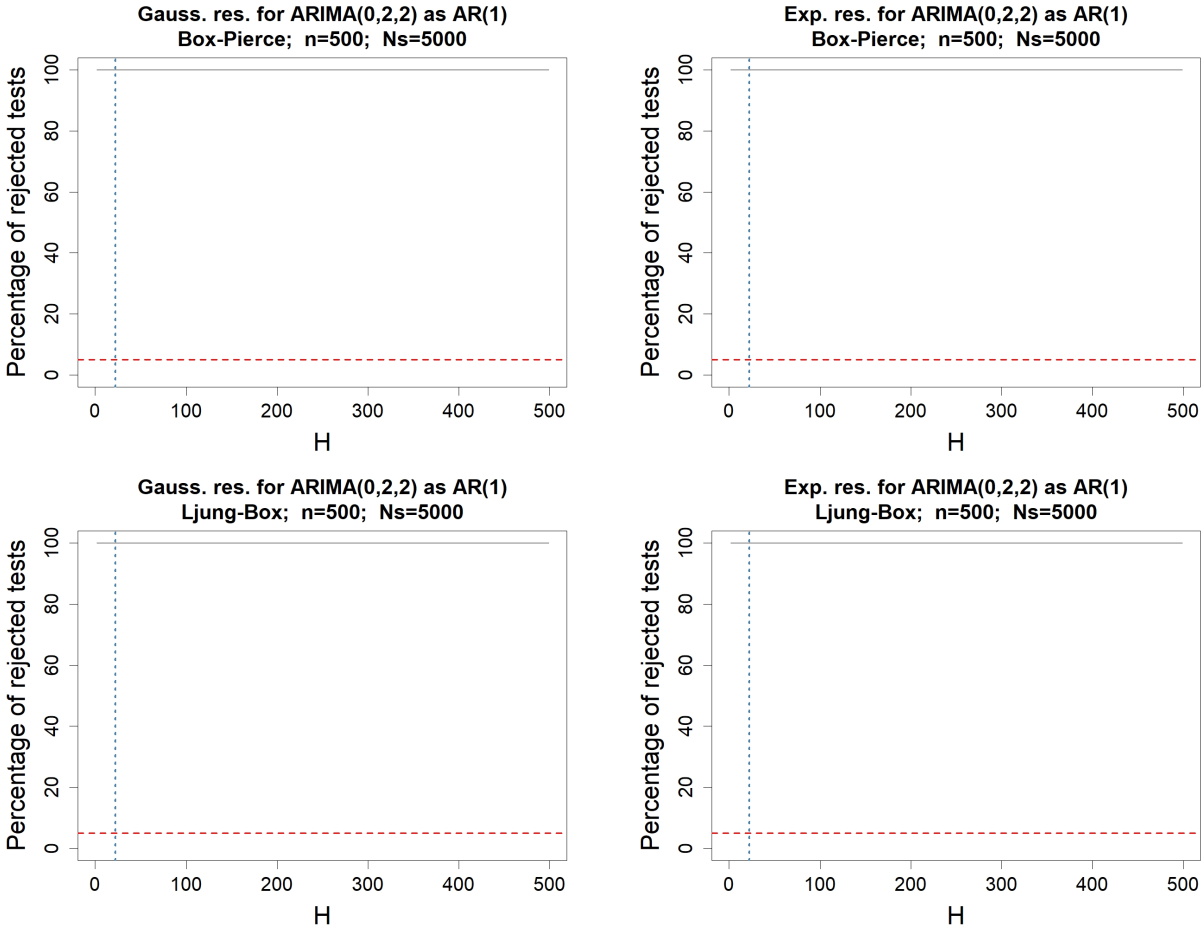

Figure S48.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns ARIMA(0,2,2)-simulations associated to Gaussian underlying-WN whereas the right one deals with Exponential ones. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

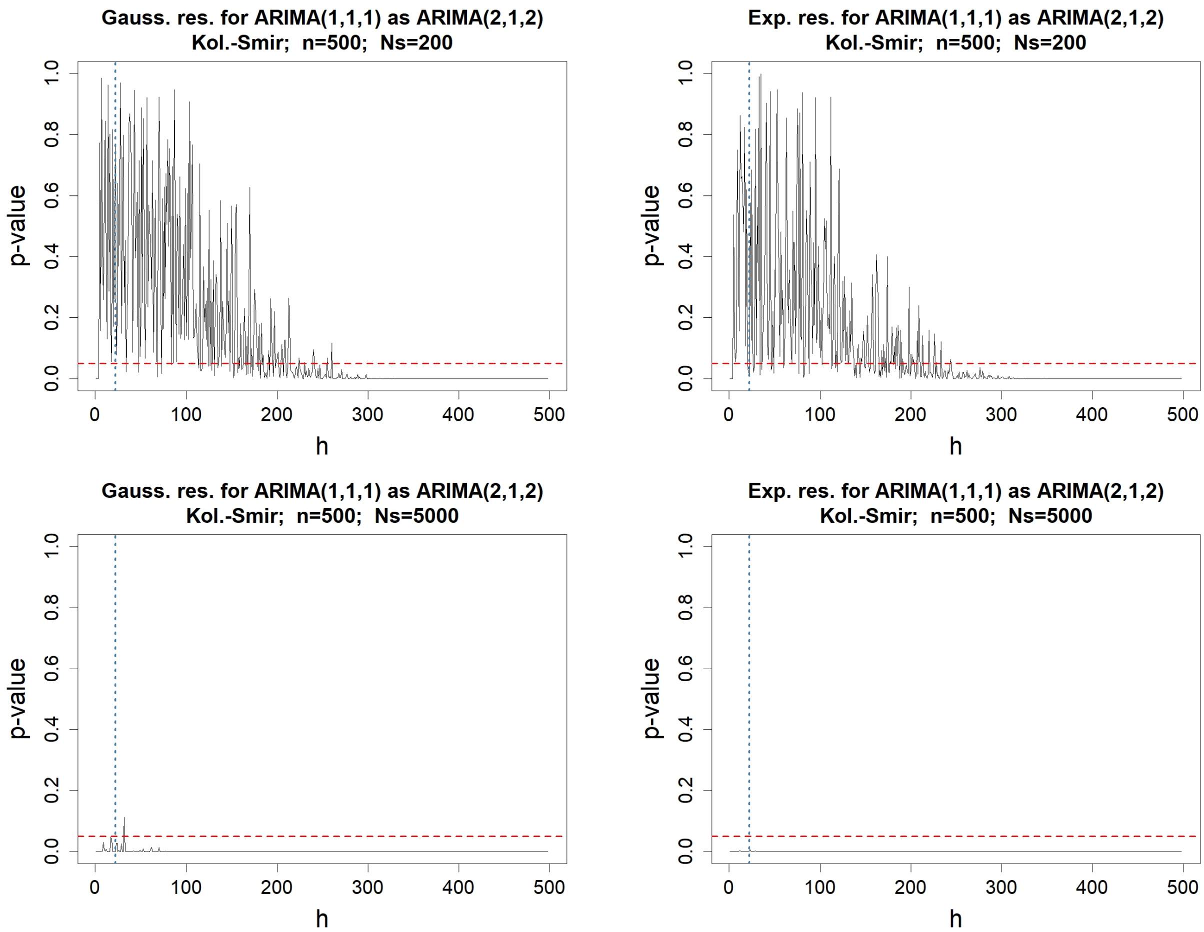

Figure S49.: P-values when testing for the adequacy of the N_S values of $\hat{\rho}(h)$ with $\mathcal{N}\left(0, \frac{1}{n}\right)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns ARIMA(2) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(1,1,1) process is $n = 500$. In the upper figures, the number of simulated ARIMA(1,1,1) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

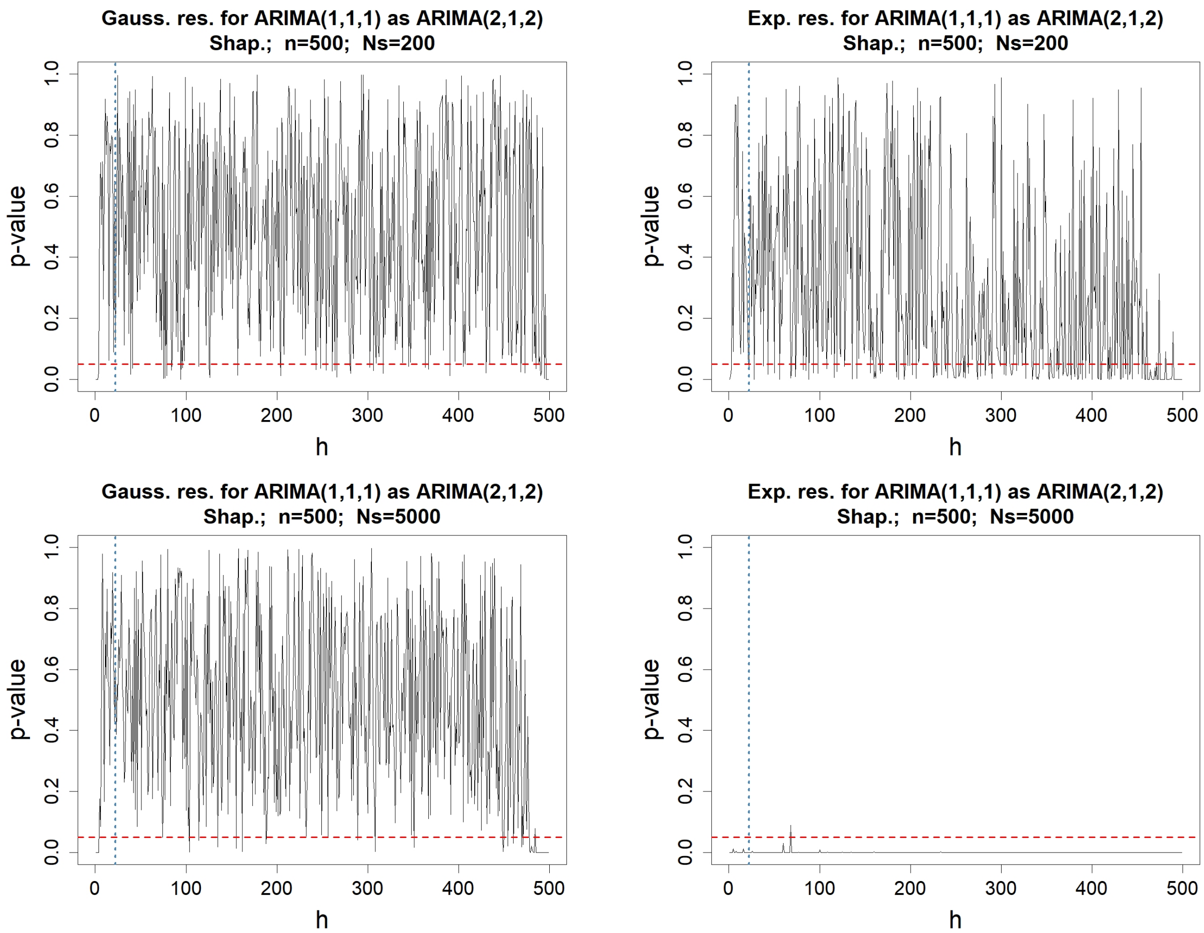

Figure S50.: P-values when testing for the normality of the N_S values of $\hat{\rho}(h)$, for any fixed lag h varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns Gaussian ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(1,1,1) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(1,1,1) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $h = \sqrt{n}$.

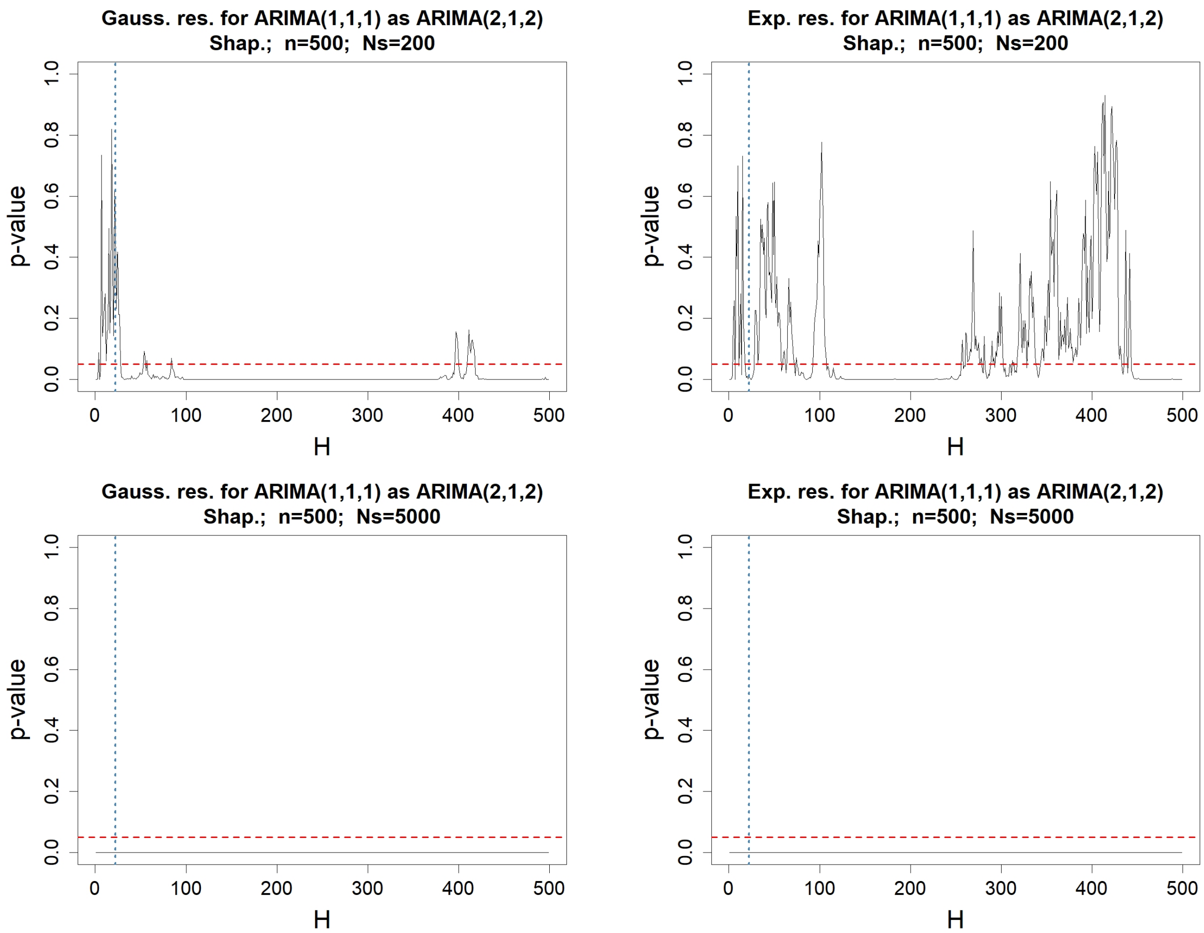

Figure S51.: P-values when testing for the normality of the N_S values of $S_{acf}(H)$, for any fixed lag H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns concerns ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN process. The length of the simulated ARIMA(1,1,1) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(1,1,1) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

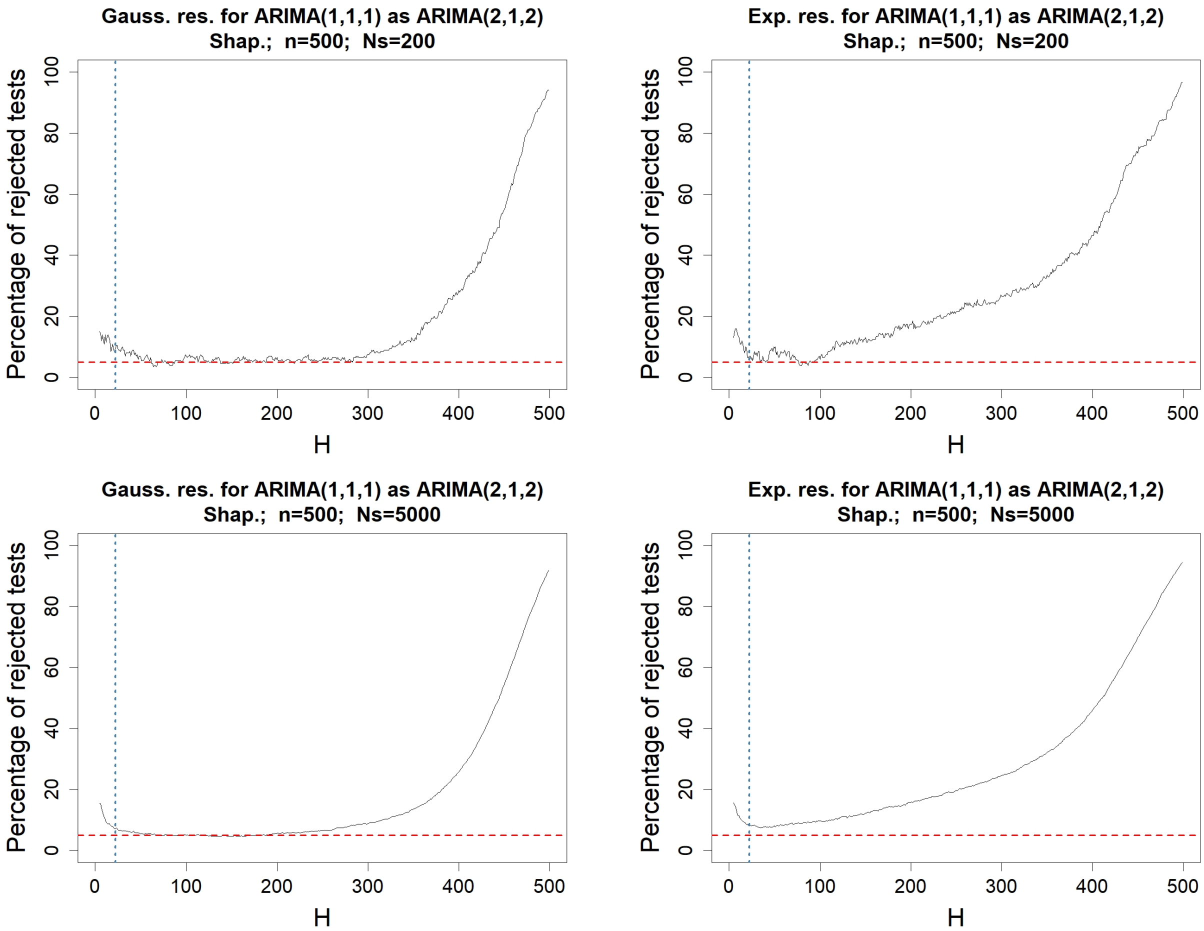

Figure S52.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the N_S simulations, when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with H varying from 1 to $n-1$. The involved normality test is Shapiro-Wilk's. The left column concerns ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential WN processes. The length of the simulated ARIMA(1,1,1) processes is $n = 500$. In the upper figures, the number of simulated ARIMA(1,1,1) processes is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

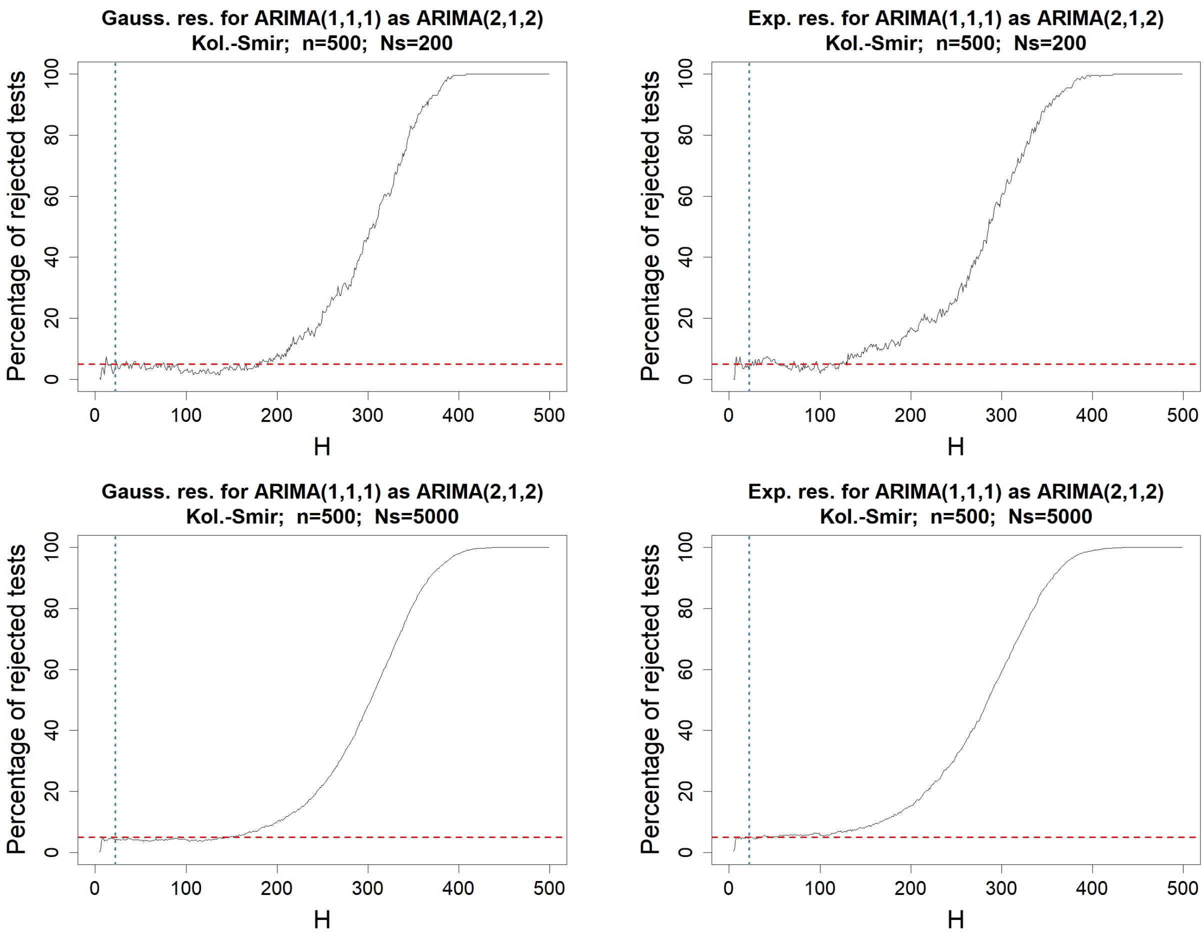

Figure S53.: Percentage of unexpected p-values ($< \alpha = 5\%$) when testing for the normality of $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, with $\mathcal{N}\left(0, \frac{1}{n}\right)$ when H varies from 1 to $n-1$. The involved normality test is Kolmogorov-Smirnov's. The left column concerns ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. In the upper figures, the number of simulated ARIMA(1,1,1) process is $N_S = 200$, whereas it is $N_S = 5\,000$ in the bottom. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.

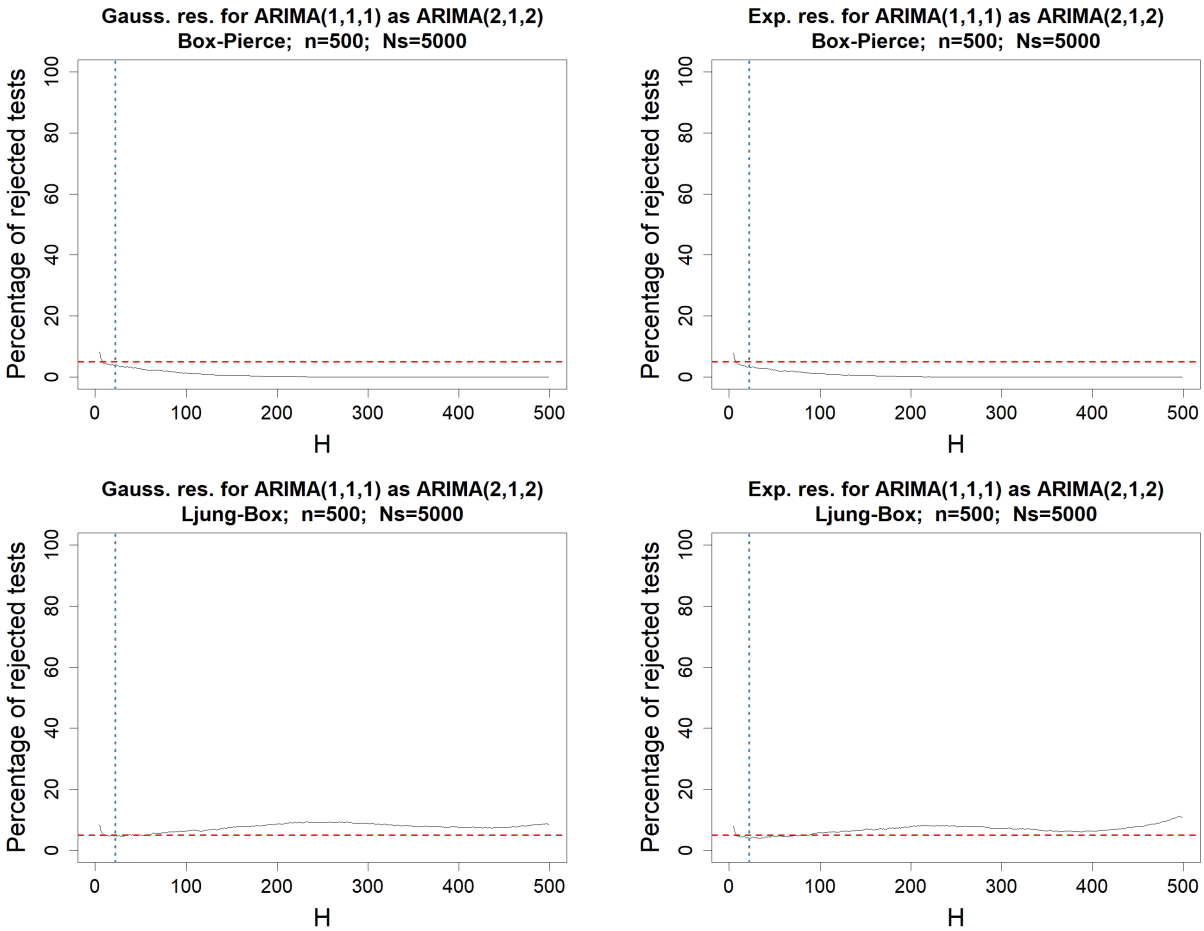

Figure S54.: Percentage of unexpected p-values ($< \alpha = 5\%$) among the $N_S = 5\,000$ when applying portmanteau tests on $\hat{\rho}(1), \cdots, \hat{\rho}(H)$, when H varies from 1 to $n-1$. The involved portmanteau tests are Box-Pierce (upper figures) and Ljung-Box (bottom figures). The left column concerns ARIMA(1,1,1) simulations associated to Gaussian-WN whereas the right one deals with Exponential-WN processes. The red-dotted horizontal line represents 5%, while the blue-dotted vertical line represents $H = \sqrt{n}$.