Reductions of path structures and classification of homogeneous structures in dimension three

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Abstract

In this paper we show that if a path structure has non-vanishing curvature at a point then it has a canonical reduction to a $\mathbb{Z}/2\mathbb{Z}$-structure at a neighbourhood of that point (in many cases it has a canonical parallelism). A simple implication of this result is that the automorphism group of a non-flat path structure is of maximal dimension three (a result by Tresse of 1896). We also classify the invariant path structures on three-dimensional Lie groups.

Keywords: Path structures, Homogeneous spaces, Cartan connections, automorphism group.

1 Introduction

A path structure on a 3-manifold is a choice of two subbundles T^1 and T^2 in TM such that $T^1 \cap T^2 = \{0\}$ and such that $T^1 \oplus T^2$ is a contact distribution. This geometry gives rise to a Cartan connection on a canonical principal bundle (with structure group, $B_{\mathbb{R}}$, the Borel subgroup of upper triangular matrices in $\mathbf{SL}(3,\mathbb{R})$) which we call Y ([Car], see [IL] for a modern presentation and section 2). There are two curvature functions Q^1 and Q^2 defined on Y which should determine, in certain situations, the path structure up to equivalence. Indeed, when $Q^1 = Q^2 = 0$ the path structure is locally equivalent to the path structure on the model space $\mathbf{SL}(3,\mathbb{R})/B_{\mathbb{R}}$ (see section 2).

A simple way to define a path structure on a 3-manifold is to fix a contact form and two transverse vector fields contained in the kernel of the form. In particular, this defines a parallelism of the 3-manifold. Reciprocally, one might ask whether there exists a canonical parallelism, with transverse vector fields contained in the contact distribution, associated to a path structure. In this case, the automorphism group of the path structure should coincide with the automorphism group of the parallelism.

We show in this paper that if a path structure has non-vanishing curvature at a point then it has a canonical reduction to a $\mathbb{Z}/2\mathbb{Z}$-structure at a neighbourhood of that point (in many cases it has a canonical parallelism).

In section 2 we recall the construction of the Cartan bundle Y and adapted connection to a path structure on a 3-manifold. The curvature invariants Q^1 and Q^2 are related to the invariants found by Tresse (see the explicit computation in section 3.5 in [FV2]). We also recall the definition of strict path structure, that is, when a contact form is fixed over the manifold and the definition of a Cartan bundle Y^1 and connection adapted to that structure which was used in ([FMMV]) to obtain a classification of compact 3-manifolds with non-compact automorphism group preserving the strict path structure. We also give details in appendix 6.2 for a natural embedding $Y^1 \to Y$ which is used to compute curvatures of the homogeneous path structures in section 5.

In section 4 we prove a canonical reduction of a path structure when the structure is non-flat:

Theorem 1.1 *If the path structure is not flat, there exists a canonical reduction of the fiber bundle Y to a $\mathbb{Z}/2\mathbb{Z}$-structure.*

A more precise theorem is proved in section 4 where we give conditions for the existence of a further reduction to a parallelism. The theorem implies the classical theorem by Tresse that a non-flat path structure has an automorphism group of dimension at most three ([T]). While locally homogeneous path structures of course realize this upper bound, Tresse's theorem does not, however, answer the question wether a path structure having a three-dimensional automorphism group, is locally homogeneous or not. A one-dimensional subgroup of automorphisms may indeed fix a point, forbidding the orbit of the automorphism group at this point to be open. A direct and important Corollary of Theorem 1.1, proved in section 5.1, shows that this phenomenon does not happen.

Corollary 1.2 *At a point where the curvature is non-zero and the algebra of local Killing fields has dimension at least three, a path structure is locally isomorphic to a left-invariant path structure on a three-dimensional Lie group.*

This is a motivation to classify left invariant path structures on three dimensional Lie groups, which is done in section 5. The results are gathered in tables in section 5.3. We choose for each structure a parallelism (some are canonical) and we compute the curvatures for each of these invariant structures using an embedding of the group in the corresponding Cartan bundle Y (see Proposition 5.2 in section 5.2).

In the last section 5.4 we give a geometric description of the invariant structures on $\mathbf{SL}(2,\mathbb{R})$ involving the type of the contact plane with respect to the Killing metric and a cross-ratio which parametrizes the positions of the one dimensional distributions in the contact plane. Similar descriptions can be made for each of the three dimensional groups.

It is interesting to note that homogeneous CR structures in dimension three were classified by Cartan in [C] (see a presentation analogous to the classification of path structures in the present paper in [FG]). We did not find, however, the analogous classification of path structures in his work. The gap phenomenon which is a consequence of theorem 1.1 is more general (see [KT]) and, in particular, is valid for CR structures. But the detailed reductions obtained in section 4 and the corollary above were not worked out in the case of CR structures as far as we know.

2 The Cartan connection of a path structure

Path geometries are treated in detail in section 8.6 of [IL] and in [BGH]. Le M be a real three dimensional manifold and TM be its tangent bundle.

Definition 2.1 *A path structure on M is a choice of two subbundles T^1 and T^2 in TM such that $T^1 \cap T^2 = \{0\}$ and such that $T^1 \oplus T^2$ is a contact distribution.*

The condition that $T^1 \oplus T^2$ be a contact distribution means that, locally, there exists a one form $\theta \in T^*M$ such that $\ker\theta = T^1 \oplus T^2$ and $d\theta \wedge \theta$ is never zero.

Flat path geometry is the geometry of real flags in $\mathbb{R}^3$. That is the geometry of the space of all couples (p, l) where $p \in \mathbb{R}P^2$ and l is a real projective line containing p. The space of flags is identified to the quotient

$$\mathbf{SL}(3,\mathbb{R})/B_{\mathbb{R}}$$

where $B_{\mathbb{R}}$ is the Borel group of all real upper triangular matrices. The Maurer-Cartan form on $\mathbf{SL}(3,\mathbb{R})$ is given by a form with values in the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$:

$$\pi = \begin{pmatrix} \varphi + w & \varphi^2 & \psi \\ \omega^1 & -2w & \varphi^1 \\ \omega & \omega^2 & -\varphi + w \end{pmatrix}$$

satisfying the equation $d\pi + \pi \wedge \pi = 0$. That is

$$d\omega = \omega^1 \wedge \omega^2 + 2\varphi \wedge \omega$$

$$d\omega^1 = \varphi \wedge \omega^1 + 3w \wedge \omega^1 + \omega \wedge \varphi^1$$
$$d\omega^2 = \varphi \wedge \omega^2 - 3w \wedge \omega^2 - \omega \wedge \varphi^2$$
$$dw = -\frac{1}{2}\varphi^2 \wedge \omega^1 + \frac{1}{2}\varphi^1 \wedge \omega^2$$
$$d\varphi = \omega \wedge \psi - \frac{1}{2}\varphi^2 \wedge \omega^1 - \frac{1}{2}\varphi^1 \wedge \omega^2$$
$$d\varphi^1 = \psi \wedge \omega^1 - \varphi \wedge \varphi^1 + 3w \wedge \varphi^1$$
$$d\varphi^2 = -\psi \wedge \omega^2 - \varphi \wedge \varphi^2 - 3w \wedge \varphi^2$$
$$d\psi = \varphi^1 \wedge \varphi^2 + 2\psi \wedge \varphi.$$

2.1 The coframe bundle Y over the bundle E of contact forms

We recall the construction of the $\mathbb{R}^*$-bundle of contact forms (we follow here [FV1] and [FV2]).

Define E to be the $\mathbb{R}^*$-bundle of all forms θ on TM such that $\ker\theta = T^1 \oplus T^2$. Remark that this bundle is trivial if and only if there exists a globally defined non-vanishing form θ. Define the set of forms θ^1 and θ^2 on M satisfying

$$\theta^1(T^1) \neq 0 \text{ and } \theta^2(T^2) \neq 0,$$

$$\ker\theta^1_{|\ker\theta} = T^2 \text{ and } \ker\theta^2_{|\ker\theta} = T^1.$$

Henceforth we fix one such choice, and all others are given by $\theta'^i = a^i\theta^i + v^i\theta$.

On E we define the tautological form ω. That is $\omega_\theta = \pi^*(\theta)$ where $\pi : E \to M$ is the natural projection. We also consider the tautological forms defined by the forms θ^1 and θ^2

over the line bundle E. That is, for each $\theta \in E$ we let $\omega^i_\theta = \pi^*(\theta^i)$. At each point $\theta \in E$ we have the family of forms defined on E:

$$\omega' = \omega$$

$$\omega'^1 = a^1\omega^1 + v^1\omega$$
$$\omega'^2 = a^2\omega^2 + v^2\omega$$

We may, moreover, suppose that

$$d\theta = \theta^1 \wedge \theta^2 \ \text{ modulo } \theta$$

and therefore

$$d\omega = \omega^1 \wedge \omega^2 \ \text{ modulo } \omega.$$

This imposes that $a^1a^2 = 1$.

Those forms vanish on vertical vectors, that is, vectors in the kernel of the map $TE \to TM$. In order to define non-horizontal 1-forms we let θ be a section of E over M and introduce the coordinate $\lambda \in \mathbb{R}^*$ in E. By abuse of notation, let θ denote the tautological form on the section θ. We write then the tautological form ω over E as

$$\omega_{\lambda\theta} = \lambda\theta.$$

Differentiating this formula we obtain

$$d\omega = 2\varphi \wedge \omega + \omega^1 \wedge \omega^2 \tag{1}$$

where $\varphi = \frac{d\lambda}{2\lambda}$ modulo $\omega, \omega^1, \omega^2$. Here $\frac{d\lambda}{2\lambda}$ is a form intrinsically defined on E up to horizontal forms. We obtain in this way a coframe bundle satisfying equation 1 over E. The coframes at each point of E are given by

$$\omega' = \omega$$
$$\omega'^1 = a^1\omega^1 + v^1\omega$$
$$\omega'^2 = a^2\omega^2 + v^2\omega$$
$$\varphi' = \varphi - \frac{1}{2}a^1v^2\omega^1 + \frac{1}{2}a^2v^1\omega^2 + s\omega$$

$v^1, v^2, s \in \mathbb{R}$ and $a^1, a^2 \in \mathbb{R}^*$ such that $a^1a^2 = 1$.

Definition 2.2 *We denote by Y the coframe bundle $Y \to E$ given by the set of 1-forms $\omega, \omega^1, \omega^2, \varphi$ as above. Two coframes are related by*

$$(\omega', \omega'^1, \omega'^2, \varphi') = (\omega, \omega^1, \omega^2, \varphi) \begin{pmatrix} 1 & v^1 & v^2 & s \\ 0 & a^1 & 0 & -\frac{1}{2}a^1v^2 \\ 0 & 0 & a^2 & \frac{1}{2}a^2v^1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where and $s, v^1, v^2 \in \mathbb{R}$ and $a^1, a^2 \in \mathbb{R}^*$ satisfy $a^1a^2 = 1$.

The bundle Y can also be fibered over the manifold M. In order to describe the bundle Y as a principal fiber bundle over M observe that choosing a local section θ of E and forms θ^1 and θ^2 on M such that $d\theta = \theta^1 \wedge \theta^2$ one can write a trivialization of the fiber bundle as

$$\begin{aligned}
\omega &= \lambda\theta \\
\omega^1 &= a^1\theta^1 + v^1\lambda\theta \\
\omega^2 &= a^2\theta^2 + v^2\lambda\theta \\
\varphi &= \frac{d\lambda}{2\lambda} - \frac{1}{2}a^1v^2\theta^1 + \frac{1}{2}a^2v^1\theta^2 + s\theta,
\end{aligned}$$

where $v^1, v^2, s \in \mathbb{R}$ and $a^1, a^2 \in \mathbb{R}^*$ such that $a^1a^2 = \lambda$. Here the coframe $\omega, \omega^1, \omega^2, \varphi$ is seen as composed of tautological forms. The group H acting on the right of this bundle is

$$H = \left\{ \begin{pmatrix} \lambda & v^1\lambda & v^2\lambda & s \\ 0 & a^1 & 0 & -\frac{1}{2}a^1v^2 \\ 0 & 0 & a^2 & \frac{1}{2}a^2v^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } s, v^1, v^2 \in \mathbb{R} \text{ and } a^1, a^2 \in \mathbb{R}^* \text{ satisfy } a^1a^2 = \lambda \right\}.$$

The bundle $Y \to M$ is a principal bundle with structure group H which can be identified to the Borel group $B \subset \mathbf{SL}(3, \mathbb{R})$ of upper triangular matrices with determinant one via the map

$$j : B \to H$$

given by

$$\begin{pmatrix} a & c & e \\ 0 & \frac{1}{ab} & f \\ 0 & 0 & b \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{a}{b} & -a^2f & \frac{c}{b} & -eb + \frac{1}{2}acf \\ 0 & a^2b & 0 & -\frac{1}{2}abc \\ 0 & 0 & \frac{1}{ab^2} & -\frac{f}{2b} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.2 The connection form on the bundle Y

Here we review the $\mathfrak{sl}(3, \mathbb{R})$-valued Cartan connection defined on the coframe bundle $Y \to E$ as described in [FV1, FV2].

Each point in the coframe bundle Y over E is lifted to a family of tautological forms on T^*Y. This family is then completed to obtain a coframe bundle over Y by an appropriate choice of conditions. As usual, the conditions are essentially curvature conditions and are obtained by differentiating the tautological forms and introducing new linearly independent forms satisfying certain canonical equations. We state the final existence theorem of the adapted Cartan connection:

Theorem 2.3 *There exists a unique $\mathfrak{sl}(3, \mathbb{R})$ valued connection form on the bundle Y*

$$\pi = \begin{pmatrix} \varphi + w & \varphi^2 & \psi \\ \omega^1 & -2w & \varphi^1 \\ \omega & \omega^2 & -\varphi + w \end{pmatrix},$$

whose curvature satisfies

$$\Pi = d\pi + \pi \wedge \pi = \begin{pmatrix} 0 & \Phi^2 & \Psi \\ 0 & 0 & \Phi^1 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\Phi^1 = Q^1\omega \wedge \omega^2$, $\Phi^2 = Q^2\omega \wedge \omega^1$ *and* $\Psi = (U_1\omega^1 + U_2\omega^2) \wedge \omega$, *for functions* Q^1, Q^2, U^1 *and* U^2 *on* Y.

Writing the components of the curvature form explicitly we obtain the following equations:

$$d\omega = 2\varphi \wedge \omega + \omega^1 \wedge \omega^2 \tag{2}$$

$$d\omega^1 = \varphi \wedge \omega^1 + 3w \wedge \omega^1 + \omega \wedge \varphi^1 \text{ and } d\omega^2 = \varphi \wedge \omega^2 - 3w \wedge \omega^2 - \omega \wedge \varphi^2 \tag{3}$$

$$d\varphi = \omega \wedge \psi - \frac{1}{2}(\varphi^2 \wedge \omega^1 + \varphi^1 \wedge \omega^2) \tag{4}$$

$$dw + \frac{1}{2}\omega^2 \wedge \varphi^1 - \frac{1}{2}\omega^1 \wedge \varphi^2 = 0, \tag{5}$$

$$\Phi^1 = d\varphi^1 + 3\varphi^1 \wedge w + \omega^1 \wedge \psi + \varphi \wedge \varphi^1 = Q^1\omega \wedge \omega^2, \tag{6}$$

$$\Phi^2 = d\varphi^2 - 3\varphi^2 \wedge w - \omega^2 \wedge \psi + \varphi \wedge \varphi^2 = Q^2\omega \wedge \omega^1, \tag{7}$$

$$\Psi := d\psi - \varphi^1 \wedge \varphi^2 + 2\varphi \wedge \psi = (U_1\omega^1 + U_2\omega^2) \wedge \omega. \tag{8}$$

where Q^1, Q^2, U^1 and U^2 are functions on Y.

2.3 Bianchi identities

In this section we compute Bianchi identities. They are essential to obtain relations between the curvatures and its derivatives and will be heavily used in the reductions of the path structures.

2.3.1

Equation $d(d\varphi^1) = 0$ obtained differentiating Φ^1 (equation 6) implies

$$dQ^1 - 6Q^1w + 4Q^1\varphi = S^1\omega + U_2\omega^1 + T^1\omega^2, \tag{9}$$

where we introduced functions S^1 and T^1.

2.3.2

Anagously, equation $d(d\varphi^2) = 0$ obtained differentiating Φ^2 (equation 7) implies

$$dQ^2 + 6Q^2w + 4Q^2\varphi = S^2\omega - U_1\omega^2 + T^2\omega^1, \tag{10}$$

where we introduced new functions S^2 and T^2.

2.3.3

Equation $d(d\psi)=0$ obtained from equation 8 implies

$$dU_1+5U_1\varphi+3U_1w+Q^2\varphi^1=A\omega+B\omega^1+C\omega^2 \tag{11}$$

and

$$dU_2+5U_2\varphi-3U_2w-Q^1\varphi^2=D\omega+C\omega^1+E\omega^2. \tag{12}$$

2.3.4 Higher order Bianchi identities

If we derive equation 9 and replace the known terms we get

$$dS^1+6S^1\varphi-6S^1w-U_2\varphi^1+T^1\varphi^2+4Q^1\psi=X_0\omega+D\omega^1+X_2\omega^2 \tag{13}$$

$$dT^1+5T^1\varphi-9T^1w+5Q^1\varphi^1=X_2\omega-(S^1-E)\omega^1+Y_2\omega^2 \tag{14}$$

In the same way, if we derive equation 10 and replace the known terms we get

$$dS^2+6S^2\varphi+6S^2w-T^2\varphi^1-U_1\varphi^2+4Q^2\psi=Y_0\omega+Y_1\omega^1-A\omega^2 \tag{15}$$

$$dT^2+5T^2\varphi+9T^2w+5Q^2\varphi^2=Y_1\omega+X_1\omega^1+(S^2-B)\omega^2 \tag{16}$$

If we differentiate equation 16, and use equations 15, 16 and 10 we get

$$\begin{aligned}&\omega\wedge(dY_1+7Y_1\varphi+9Y_1w-X_1\varphi^1+(6S^2-B)\varphi^2+5T^2\psi+5(Q^2)^2\omega^1-Y_0\omega^2)+\\&\omega^1\wedge(dX_1+6X_1\varphi+12X_1w+12T^2\varphi^2)-\\&\omega^2\wedge(dB+6B\varphi+6Bw+T^2\varphi^1+4U_1\varphi^2-Q^2\psi-2Y_1\omega^1)=0,\end{aligned} \tag{17}$$

and from this we get

$$dX_1+6X_1\varphi+12X_1w+12T^2\varphi^2=X_{10}\omega+X_{11}\omega^1+X_{12}\omega^2 \tag{18}$$

$$dY_1+7Y_1\varphi+9Y_1w-X_1\varphi^1+(6S^2-B)\varphi^2+5T^2\psi+5(Q^2)^2\omega^1-Y_0\omega^2=Y_{10}\omega+X_{10}\omega^1+Y_{12}\omega^2 \tag{19}$$

3 Strict path structures

In this section we recall the definition of strict path structures (see [FMMV] and [FV1]). They correspond to path structures with a fixed contact form.

G_1 denotes from now on the subgroup of $\mathbf{SL}(3,\mathbb{R})$ defined by

$$G_1=\left\{\begin{pmatrix}a&0&0\\x&\frac{1}{a^2}&0\\z&y&a\end{pmatrix}\mid a\in\mathbb{R}^*,(x,y,z)\in\mathbb{R}^3\right\}$$

and $P_1\subset G_1$ the subgroup defined by

$$P_1=\left\{\begin{pmatrix}a&0&0\\0&\frac{1}{a^2}&0\\0&0&a\end{pmatrix}\right\}.$$

Writing the Maurer-Cartan form of G_1 as the matrix

$$\begin{pmatrix} w & 0 & 0 \\ \theta^1 & -2w & 0 \\ \theta & \theta^2 & w \end{pmatrix}$$

one obtains the Maurer-Cartan equations:

$$d\theta + \theta^2 \wedge \theta^1 = 0$$

$$d\theta^1 - 3w \wedge \theta^1 = 0$$

$$d\theta^2 + 3w \wedge \theta^2 = 0$$

$$dw = 0.$$

Let M be a three-manifold equipped with a path structure $D = E^1 \oplus E^2$. Fixing a contact form $\boldsymbol{\theta}$ such that $\ker \boldsymbol{\theta} = D$ defines a strict path structure. Let $\boldsymbol{R}$ be the Reeb vector field associated to $\boldsymbol{\theta}$. That is $\boldsymbol{\theta}(\boldsymbol{R}) = 1$ and $d\boldsymbol{\theta}(\boldsymbol{R}, \cdot) = 0$. Let $\boldsymbol{X}_1 \in E^1$, $\boldsymbol{X}_2 \in E^2$ be such that $d\theta(\boldsymbol{X}_1, \boldsymbol{X}_2) = 1$. The dual coframe of $(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{R})$ is $(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \boldsymbol{\theta})$, for two 1-forms $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ verifying $d\boldsymbol{\theta} = \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2$.

At any point $x \in M$, one can look at the coframes of the form

$$\theta^1 = a^3 \boldsymbol{\theta}^1(x),\ \theta^2 = \frac{1}{a^3}\boldsymbol{\theta}^2(x),\ \theta = \boldsymbol{\theta}(x)$$

for $a \in \mathbb{R}^*$.

Definition 3.1 *We denote by $\pi : Y^1 \to M$ the $\mathbb{R}^*$-coframe bundle over M given by the set of coframes $(\theta, \theta^1, \theta^2)$ of the above form.*

We will denote the tautological forms defined by $\theta^1, \theta^2, \theta$ using the same letters. That is, we write θ^i at the coframe $(\theta^1, \theta^2, \theta)$ to be $\pi^*(\theta^i)$.

Proposition 3.2 *There exists a unique $\mathfrak{g}_1$-valued Cartan connection on Y^1*

$$\varpi = \begin{pmatrix} \upsilon & 0 & 0 \\ \theta^1 & -2\upsilon & 0 \\ \theta & \theta^2 & \upsilon \end{pmatrix}$$

such that its curvature form is of the form

$$\varpi = \begin{pmatrix} d\upsilon & 0 & 0 \\ \theta \wedge \tau^1 & -2d\upsilon & 0 \\ 0 & \theta \wedge \tau^2 & d\upsilon \end{pmatrix}$$

with $\tau^1 \wedge \theta^2 = \tau^2 \wedge \theta^1 = 0$.

Observe that the condition $\tau^1 \wedge \theta^2 = \tau^2 \wedge \theta^1 = 0$ implies that we may write $\tau^1 = \tau^1_2 \theta^2$ and $\tau^2 = \tau^2_1 \theta^1$. The structure equations are

$$\begin{array}{lcl} d\theta^1 - 3\upsilon \wedge \theta^1 & = & \theta \wedge \tau^1, \\ d\theta^2 + 3\upsilon \wedge \theta^2 & = & \theta \wedge \tau^2, \\ d\theta & = & \theta^1 \wedge \theta^2. \end{array} \tag{20}$$

A choice of coframe $(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \boldsymbol{\theta})$ on a strict path structure defines an embedding

$$j : M \to Y^1$$

and therefore

$$\begin{array}{lcl} d\boldsymbol{\theta}^1 - 3j^*\upsilon \wedge \boldsymbol{\theta}^1 & = & \boldsymbol{\theta} \wedge j^*\tau^1, \\ d\boldsymbol{\theta}^2 + 3j^*\upsilon \wedge \boldsymbol{\theta}^2 & = & \boldsymbol{\theta} \wedge j^*\tau^2, \\ d\boldsymbol{\theta} & = & \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2. \end{array} \tag{21}$$

3.1 Bianchi identities

In what follows, the equations should be understood as definitions for the coefficients appearing in the right hand terms. Bianchi identities give the following equations:

$$d\upsilon = R\theta^1 \wedge \theta^2 + W^1\theta^1 \wedge \theta + W^2\theta^2 \wedge \theta \tag{22}$$

$$d\tau^1 + 3\tau^1 \wedge \upsilon = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2 \tag{23}$$

$$d\tau^2 - 3\tau^2 \wedge \upsilon = 3W^1\theta^1 \wedge \theta^2 + S^2_1\theta \wedge \theta^1 + S^2_2\theta \wedge \theta^2 \tag{24}$$

Moreover, we have the relation

$$S^1_1 = S^2_2 = \tau^1_2\tau^2_1.$$

From equation $dd\upsilon = 0$ one obtains that

$$dR = R_0\theta + R_1\theta^1 + R_2\theta^2,$$

$$dW^1 + 3W^1\upsilon = W^1_0\theta + W^1_1\theta^1 + W^1_2\theta^2 \tag{25}$$

and

$$dW^2 - 3W^2\upsilon = W^2_0\theta + W^2_1\theta^1 + W^2_2\theta^2 \tag{26}$$

with

$$R_0 = W^1_2 - W^2_1.$$

Also, writing $dR_0 = R_{00}\theta + R_{01}\theta^1 + R_{02}\theta^2$, one gets

$$dR_1 + 3R_1\upsilon + R_2\tau^2_1\theta - \frac{1}{2}R_0\theta^2 = R_{01}\theta + R_{11}\theta^1 + R_{12}\theta^2 \tag{27}$$

and

$$dR_2 - 3R_2\upsilon + R_1\tau^1_2\theta + \frac{1}{2}R_0\theta^1 = R_{02}\theta + R_{12}\theta^1 + R_{22}\theta^2. \tag{28}$$

We have moreover

$$d\tau^1_2 - 6\tau^1_2\upsilon = 3W^2\theta^1 + S^1_2\theta \mod \theta^2$$

and

$$d\tau^2_1 + 6\tau^2_1\upsilon = -3W^1\theta^2 + S^2_1\theta \mod \theta^1.$$

4 Reductions

In this section we prove the main theorem concerning canonical reductions of a path structure. The motivation behind the reductions is a gap theorem on the possible dimensions of the automorphism group. Indeed, the group of transformations preserving a path structure which is not flat has dimension at most three:

Theorem 4.1 (M. A. Tresse [T]) *The group of transformations of the fiber bundle Y has dimension eight (in the flat case) or at most dimension three.*

A modern proof is contained in M. Mion-Mouton thesis [MM] and a more general result is obtained in [KT].

We will describe in this section reductions of the fiber bundle Y to a parallelism or a $\mathbb{Z}/2\mathbb{Z}$-bundle over the manifold in the case the path structure is not flat. This clearly implies Tresse's theorem.

Theorem 4.2 *If the path structure is not flat, there exists a canonical reduction of the fiber bundle Y to a $\mathbb{Z}/2\mathbb{Z}$-structure. Moreover,*

1. *if $Q^1 \neq 0$, and $Q^2 \neq 0$, and $T_1 \neq 0$ or $T_2 \neq 0$, there exists a further reduction to a parallelism.*
2. *if $Q^1 = 0$ and $Q^2 \neq 0$ and $Y_1 \neq 0$ there exists a further reduction to a parallelism.*

Here Q^1, Q^2, T_1, T_2 and Y_1 are the functions on the bundle Y introduced in equations 6, 7, 9, 10 and 15.

The theorem is a consequence of propositions 4.3, 4.4, 4.5, 4.6 and 4.7, where details of the reductions are given. Note that the case $Q^1 \neq 0$ and $Q^2 = 0$ differs from the second case in the theorem by an ordering of the decomposition of the plane field and the result is analogous.

4.1 Reduction of the structure: $Q^1 \neq 0$ and $Q^2 \neq 0$.

Suppose there exists a section of the coframe bundle Y with $Q^1 \neq 0$ and $Q^2 \neq 0$. From equations 59, $ab^5Q_1 = \tilde{Q}_1$, $a^5b\tilde{Q}_2 = Q_2$, we can solve for a, b such that $\tilde{Q}_1 = 1$ and $\tilde{Q}_2 = \epsilon$, where $Q_1Q_2 = \epsilon|Q_1Q_2|$. Observe that if we only consider coframes on Y satisfying $Q_1 = 1$, $Q_2 = \epsilon$, then $a = b = \pm 1$.

From transformation properties 61:

$$\tilde{U}_1 = \frac{b}{a^4}(U_1 - \frac{f}{b}Q^2) \tag{29}$$

and

$$\tilde{U}_2 = \frac{b^4}{a}(U_2 + abcQ^1),$$

we can choose $c = -\frac{U_2}{abQ^1}$ and $f = \frac{bU_1}{Q^2}$ such that $\tilde{U}_1 = \tilde{U}_2 = 0$. Then, from equations 9 and 10, that is

$$d\tilde{Q}^1 - 6\tilde{Q}^1\tilde{w} + 4\tilde{Q}^1\tilde{\varphi} = \tilde{S}^1\tilde{\omega} + \tilde{U}_2\tilde{\omega}^1 + \tilde{T}^1\tilde{\omega}^2$$

and

$$d\tilde{Q}^2 + 6\tilde{Q}^2\tilde{w} + 4\tilde{Q}^2\tilde{\varphi} = \tilde{S}^2\tilde{\omega} - \tilde{U}_1\tilde{\omega}^2 + \tilde{T}^2\tilde{\omega}^1,$$

we get

$$4\tilde{\varphi} - 6\tilde{w} = \tilde{S}^1\tilde{\omega} + \tilde{T}^1\tilde{\omega}^2, \tag{30}$$

$$\epsilon(4\tilde{\varphi} + 6\tilde{w}) = \tilde{S}^2\tilde{\omega} + \tilde{T}^2\tilde{\omega}^1. \tag{31}$$

Consider now only the coframes on Y satisfying $Q_1 = 1$, $Q_2 = \epsilon$ and $U_1 = U_2 = 0$. Using formulas 58 with $a = b = \pm 1, c = f = 0$ (these parameters are fixed by the previous conditions) we obtain that $\tilde{w} = w$ and $\tilde{\varphi} = \varphi - e\omega$. Therefore

$$4(\varphi - e\omega) - 6w = \tilde{S}^1\omega + \tilde{T}^1\omega^2$$

and

$$4(\varphi - e\omega) + 6w = \tilde{S}^2\omega + \tilde{T}^2\omega^1.$$

From these equations we observe that φ and w are combinations of ω, ω^1 and ω^2. We also obtain that

$$\tilde{S}^1 = S^1 + 4e$$

and

$$\tilde{S}^2 = S^2 + \epsilon 4e.$$

Proposition 4.3 *Suppose there is a section of Y such that $Q^1 \neq 0$ and $Q^2 \neq 0$. Then there exists a unique $\mathbb{Z}/2\mathbb{Z}$-bundle $Y^{red} \subset Y$ such that on Y^{red}, $Q^1 = 1$, $Q^2 = \epsilon$, $U_1 = U_2 = 0$ and $S^1 + \epsilon S^2 = 0$, where $Q^1Q^2 = \epsilon|Q^1Q^2|$. Moreover, if $T_1 \neq 0$ or $T_2 \neq 0$ there is a further reduction to a parallelism.*

Proof. Observe that $\tilde{S}^1 + \epsilon\tilde{S}^2 = S^1 + \epsilon S^2 + 8e$ and choose the unique e satisfying $\tilde{S}^1 + \epsilon\tilde{S}^2 = 0$. This defines the reduction to a $\mathbb{Z}/2\mathbb{Z}$-bundle $Y^{red} \subset Y$.

From 30 and 31 we get on Y^{red}

$$\varphi = \frac{1}{8}(T^1\omega^2 + \epsilon T^2\omega^1)$$

and

$$w = \frac{1}{12}((\epsilon S^2 - S^1)\omega + \epsilon T^2\omega^1 - T^1\omega^2)$$

Since both forms are invariant on Y^{red}, fixing the sign of T^1 or T^2 determines one of the two coframes. That is, either $(\omega^1, \omega^2, \omega)$ or $(-\omega^1, -\omega^2, \omega)$.

□

The structure equations of a parallelism can be written as follows. From equations 11 and 12 we get on Y^{red}

$$\varphi^1 = \epsilon(A\omega + B\omega^1 + C\omega^2)$$

and

$$\varphi^2 = -(D\omega + C\omega^1 + E\omega^2).$$

Therefore the structure equations are

$$d\omega = \frac{1}{4}(T^1\omega^2 + \epsilon T^2\omega^1) \wedge \omega + \omega^1 \wedge \omega^2, \tag{32}$$

$$d\omega^1 = \frac{1}{8}T^1\omega^1 \wedge \omega^2 + (\frac{1}{4}(\epsilon S^2 - S^1) + \epsilon B)\omega \wedge \omega^1 + \epsilon C\omega \wedge \omega^2 \tag{33}$$

$$d\omega^2 = -\epsilon\frac{1}{8}T^2\omega^1 \wedge \omega^2 + C\omega \wedge \omega^1 + (-\frac{1}{4}(\epsilon S^2 - S^1) + E)\omega \wedge \omega^2. \tag{34}$$

As a final observation, other possible reductions to a $\mathbb{Z}^2$-bundle could be made. For instance, imposing the condition $S^1 = 0$ defines such a reduction. In any case, the theorem shows that in the case both Q^1 and Q^2 are non-vanishing, the dimension of the automorphism group is at most three.

4.2 Reduction of the structure: $Q^1 = 0$ and $Q^2 \neq 0$.

Suppose there exists a section of the coframe bundle Y with $Q^1 = 0$ and $Q^2 \neq 0$. From equation 9,

$$dQ^1 - 6Q^1w + 4Q^1\varphi = S^1\omega + U_2\omega^1 + T^1\omega^2,$$

we obtain that $S^1 = U_2 = T^1 = 0$ on Y. Also, from equation 12 we obtain $D = C = E = 0$, and from 13 and 14, $X_0 = X_2 = Y_2 = 0$.

Using the transformation $\tilde{Q}^2 = \frac{1}{a^5b}Q^2$ one can chose an arbitrary function a and then the function b such that $\frac{1}{a^5b}Q^2 = 1$ is determined. One considers now the subbundle with coframes satisfying $Q^2 = 1$. Its structure group satisfies then $a^5 = \frac{1}{b}$. From equation 29, as in the previous section,

$$\tilde{U}_1 = \frac{b}{a^4}(U_1 - \frac{f}{b}Q^2) = \frac{b}{a^4}(U_1 - fa^5), \tag{35}$$

and one can choose f (depending on a) such that $U_1 = 0$.

The structure group of this reduction consists of matrices of the form

$$\begin{pmatrix} a & c & e \\ 0 & a^4 & 0 \\ 0 & 0 & \frac{1}{a^5} \end{pmatrix}$$

Compute now the transformation by a section of this structure group of the pull-back of the connection forms to this subbundle. We have $R_h^*(\pi) = h^{-1}dh + h^{-1}\pi h$. The computation of $h^{-1}\pi h$ is given above in formulae 58. We also compute

$$h^{-1}dh = \begin{pmatrix} a^{-1}da & \star & \star \\ 0 & -a^{-1}da - b^{-1}db & \star \\ 0 & 0 & b^{-1}db \end{pmatrix}. \tag{36}$$

Therefore, taking into account that $b = \frac{1}{a^5}$ (so $b^{-1}db = -5\frac{da}{a}$) and $f = 0$ in 58,

$$\tilde{w} = R_h^*(w) = \frac{1}{2}(a^{-1}da + b^{-1}db) + w - \frac{1}{2}abc\,\omega^1 = -2a^{-1}da + w - \frac{1}{2}ca^{-4}\,\omega^1$$

and

$$\tilde{\varphi} = R_h^*(\varphi) = \frac{1}{2}(a^{-1}da - b^{-1}db) + \varphi - \frac{1}{2}abc\,\omega^1 - \frac{e}{b}\,\omega = 3a^{-1}da + \varphi - \frac{1}{2}ca^{-4}\,\omega^1 - ea^5\,\omega.$$

Observe now that

$$6\tilde{w} + 4\tilde{\varphi} = 6w + 4\varphi - 5ca^{-4}\,\omega^1 - ea^5\,\omega$$

Replacing equation 10: $dQ^2 + 6Q^2w + 4Q^2\varphi = S^2\omega - U_1\omega^2 + T^2\omega^1$, (imposing $Q^2 = 1$ and $U_1 = 0$) on the last equation we obtain

$$6\tilde{w} + 4\tilde{\varphi} = (T^2 - 5ca^{-4})\,\omega^1 + (S^2 - ea^5)\,\omega.$$

We may now choose c and e so that

$$6\tilde{w} + 4\tilde{\varphi} = 0.$$

That is, $\tilde{T}^2 = \tilde{S}^2 = 0$.

The coframes with $Q^2 = 1$ and $6w + 4\varphi = 0$ form a subbundle Y^1 with group

$$H_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^4 & 0 \\ 0 & 0 & a^{-5} \end{pmatrix} \mid a \neq 0 \right\} \subset H.$$

From equation 11 we obtain that on Y^1

$$\varphi^1 = A\omega + B\omega^1. \tag{37}$$

From equation 15 we obtain that on Y^1

$$4\psi = Y_0\omega + Y_1\omega^1 - A\omega^2. \tag{38}$$

Also, from equation 16 we obtain on Y^1

$$5\varphi^2 = Y_1\omega + X_1\omega^1 - B\omega^2. \tag{39}$$

We showed

Proposition 4.4 *There exists a unique $\mathbb{R}^*$-bundle $Y^1 \subset Y$ such that on Y^1, $Q^1 = 0$, $Q^2 = 1$, $U_1 = 0$ and $3w = -2\varphi$. Moreover on Y^1 we have $C = D = E = S^1 = S^2 = T^1 = T^2 = U_2 = 0$ and $X_2 = Y_2 = 0$, also $\varphi^1 = A\omega + B\omega^1$, $5\varphi^2 = Y_1\omega + X_1\omega^1 - B\omega^2$ and $4\psi = Y_0\omega + Y_1\omega^1 - A\omega^2$.*

4.2.1 Further reductions of the structure: $Q^1 = 0$ and $Q^2 \neq 0$.

From equations 58, taking into account Proposition 4.4, we obtain on Y^1 the transformations

$$\begin{aligned}
\tilde{\omega} &= a^6\,\omega \\
\tilde{\omega}^1 &= a^{-3}\,\omega^1 \\
\tilde{\omega}^2 &= a^9\,\omega^2 \\
\tilde{\varphi}^1 &= b^2a\,\varphi^1 = a^{-9}\,\varphi^1 \\
\tilde{\varphi}^2 &= \frac{1}{ba^2}\,\varphi^2 = a^3\,\varphi^2 \\
\tilde{\psi} &= \frac{b}{a}\,\psi = \frac{1}{a^6}\,\psi.
\end{aligned} \tag{40}$$

Therefore, from Proposition 4.4, we compute the transformations of X_1, Y_0, Y_1, A and B: For instance, from $\tilde{\varphi}^2 = a^3\,\varphi^2$ we have

$$\tilde{Y}_1 a^6\omega + \tilde{X}_1 a^{-3}\omega^1 - \tilde{B}a^9\omega^2 = 5\tilde{\varphi}^2 = a^3\,5\varphi^2 = a^3(Y_1\omega + X_1\omega^1 - B\omega^2)$$

and comparing the terms of this equality we obtain

$$\begin{aligned} \tilde{X}_1 &= a^6 X_1 \\ \tilde{Y}_1 &= a^{-3} Y_1 \\ \tilde{B} &= a^{-6} B. \end{aligned} \tag{41}$$

Analogously, we also obtain, from $\tilde{\psi} = a^{-6}\,\psi$

$$\tilde{Y}_0 a^6\,\omega + \tilde{Y}_1 a^{-3}\,\omega^1 - \tilde{A}a^9\,\omega^2 = 4\tilde{\psi} = 4\frac{1}{a^6}\,\psi = \frac{1}{a^6}(Y_0\omega + Y_1\omega^1 - A\omega^2)$$

and therefore

$$\begin{aligned} \tilde{Y}_0 &= a^{-12} Y_0 \\ \tilde{A} &= a^{-15} A. \end{aligned} \tag{42}$$

Remark that if any of the coefficients X_1, Y_0, Y_1, A or B is non zero one could reduce the bundle by fixing the function a (only up to a sign if $Y_1 = 0$ and $A = 0$) so that the coefficient be constant equal to one. Now we compute equation 17, taking into account Proposition 4.4, and supposing that $X_1 = Y_0 = Y_1 = A = 0$, we obtain

$$5\omega \wedge \omega^1 = 0,$$

which is a contradiction. We obtained therefore the following

Proposition 4.5 *If $Q^1 = 0$ and $Q^2 \neq 0$ there exists a canonical reduction of the path structure to a $\mathbb{Z}/2\mathbb{Z}$-structure. Moreover, if $Y_1 \neq 0$ or $A \neq 0$, one can reduce further to a parallelism.*

A more refined description of the reduction is obtained by observing that the functions X_1 and Y_1 cannot both vanish.

Proposition 4.6 *On the fiber bundle Y^1 the functions X_1 and Y_1 can not be simultaneously zero.*

Proof. On Y^1, taking account of theorem 4.4, we get from 17

$$\begin{aligned} &\omega \wedge \big(dY_1 + Y_1\varphi + (5 - X_1 B)\omega^1 + (\tfrac{3}{2}A - Y_0)\omega^2\big) + \omega^1 \wedge (dX_1 - 2X_1\varphi - 2Y_1\omega^2) \\ &-\omega^2 \wedge \big(dB + 2B\varphi - \tfrac{1}{4}Y_0\omega - \tfrac{1}{4}Y_1\omega^1\big) = 0. \end{aligned}$$

If $X_1 = Y_1 = 0$, we obtain modulo ω^2

$$5\omega \wedge \omega^1 = 0$$

which is a contradiction. □

We can then use the function a to fix $X_1 = \pm 1$ or $Y_1 = 1$. Consider first the case $Y_1 \neq 0$. We can use a section of H^1 to impose $\tilde{Y}_1 = 1$.

Proposition 4.7 *If $Y_1 \neq 0$ we can choose $a = (Y_1)^{\frac{1}{3}}$, and with this choice we reduce Y^1 to an $\{e\}$-bundle Y^2 where $\tilde{Y}_1 = 1$, $\tilde{Q}^2 = 1$, $\tilde{U}_1 = \tilde{T}^2 = \tilde{S}^2 = 0$. Moreover on Y^2 we have $\tilde{Q}^1 = \tilde{S}^1 = \tilde{U}_2 = \tilde{T}^1 = 0$, $3\tilde{w} = -2\tilde{\varphi}$, $\tilde{\varphi}^1 = \tilde{A}\tilde{\omega} + \tilde{B}\tilde{\omega}^1$, $5\tilde{\varphi}^2 = \tilde{\omega} + \tilde{X}_1\tilde{\omega}^1 - \tilde{B}\tilde{\omega}^2$, $4\tilde{\psi} = \tilde{Y}_0\tilde{\omega} + \tilde{\omega}^1 - \tilde{A}\tilde{\omega}^2$, $\tilde{\varphi} = (\tilde{A}\tilde{X}_1 + \frac{1}{5}\tilde{B} + \tilde{Y}_{10})\tilde{\omega} + (\frac{6}{5}\tilde{X}_1\tilde{B} + X_{10} - 5)\tilde{\omega}^1 + (\tilde{Y}_0 + \tilde{Y}_{12} - \frac{\tilde{B}^2}{5})\tilde{\omega}^2$.*

Proof. The choice of a implies that $\tilde{Y}_1 = 1$. The other properties follow from theorem 4.4 and from 19. $\square$

If $Y_1 = 0$, we can use $X_1 \neq 0$ to reduce Y^1 to a $\mathbb{Z}/2\mathbb{Z}$-bundle.

Proposition 4.8 *If $X_1 \neq 0$, $Y_1 = 0$, we can choose $a = \pm|X_1|^{-\frac{1}{6}}$, and with this choice we reduce Y^1 to a $\mathbb{Z}/2\mathbb{Z}$-bundle Y^2 where $\tilde{X}_1 = \epsilon$, with $\epsilon = \pm 1$, $\tilde{Q}^2 = 1$, $\tilde{U}_1 = \tilde{T}^2 = \tilde{S}^2 = \tilde{Y}_1 = 0$. Moreover on Y^2 we have $\tilde{Q}^1 = \tilde{S}^1 = \tilde{U}_2 = \tilde{T}^1 = 0$, $3\tilde{w} = -2\tilde{\varphi}$, $\tilde{\varphi}^1 = \tilde{A}\tilde{\omega} + \tilde{B}\tilde{\omega}^1$, $5\tilde{\varphi}^2 = \epsilon\tilde{\omega}^1 - B\tilde{\omega}^2$, $4\tilde{\psi} = \tilde{Y}_0\tilde{\omega} - \tilde{A}\tilde{\omega}^2$, $-2\epsilon\tilde{\varphi} = \tilde{X}_{10}\tilde{\omega} + \tilde{X}_{11}\tilde{\omega}^1 + \tilde{X}_{12}\tilde{\omega}^2$*

Proof. The choice of a implies that $\tilde{X}_1 = \epsilon$. The other properties follows from theorem 4.4 and from 18. $\square$

5 Homogeneous structures

5.1 Three dimensional Lie groups with path structures

By Tresse's result, all homogeneous path structures which are not flat have a three dimensional group of automorphisms. Therefore they are all locally isomorphic to a left invariant structure on a three dimensional Lie group. More precisely, we have the following direct Corollary (Corollary 1.2 of the introduction that we state here again for the convenience of the reader). A *local Killing field* of a path structure at a point p is a vector field defined on a neighbourhood of p, whose flow preserves the path structure.

Corollary 5.1 *At a point where the curvature is non-zero and the algebra of local Killing fields has dimension at least three, a path structure is locally isomorphic to a left-invariant path structure on a three-dimensional Lie group, and the algebra of local Killing fields has dimension exactly three.*

Proof. Let $p \in M^3$ be a point where the curvature of the path structure $\mathcal{L}$ does not vanish, and where the Lie algebra $\mathfrak{g} = \mathfrak{kill}^{loc}_{\mathcal{L}}(p)$ of (germs of) local Killing fields of $\mathcal{L}$ at p has dimension at least three. According to Theorem 4.2, the Cartan bundle Y admits a canonical reduction to a $\mathbb{Z}/2\mathbb{Z}$-sub-bundle Y'. In particular for any $V \in \mathfrak{g}$, the canonical lift $\hat{\varphi}^t_V$ of the flow of V to Y which preserves the Cartan connexion, does preserve the reduction Y'. Hence if $V(p) = 0$, then with $\hat{p} \in Y'$ any point of the fiber of p in Y', the continuous map $t \mapsto \hat{\varphi}^t_V(\hat{p})$ has values in the discrete fiber of p in Y' and is therefore constant. Therefore $\hat{\varphi}^t_V(\hat{p}) = \hat{p}$ for any small enough times t, and since $\hat{\varphi}^t_V$ preserves the Cartan connexion which is a parallelism on Y, this forces $\hat{\varphi}^t_V$ to equal the identity on a neighbourhood of $\hat{p}$ in Y. The flow of V is thus trivial on a neighbourhood of p, *i.e.* $V = 0$ in $\mathfrak{g}$. In the end, the evaluation map $V \in \mathfrak{g} \mapsto V(p)$ is injective. In particular, $\mathfrak{g}$ has dimension exactly three.

With G the simply connected Lie group of Lie algebra $\mathfrak{g}$ and U_0 an open neighbourhood of 0 in $\mathfrak{g}$ on restriction to which the exponential map exp is a diffeomorphism onto its image U', this shows moreover that the differential of the map $Ev_p\colon \exp(V) \in U \mapsto \varphi^1_V(p) \in M^3$ at the identity is injective. Since $\dim \mathfrak{g} \geq 3 = \dim M$, Ev_p is therefore by the inverse mapping theorem a diffeomorphism onto an open neighbourhood V of p, possibly restricting U_0. With $\mathcal{L}_G$ the unique path structure of U such that Ev_p is a path structure isomorphism from $(U, \mathcal{L}_G)$ to $(V, \mathcal{L})$, $\mathcal{L}_G$ is by construction invariant by the left-invariant vector fields of G, which allows to extend $\mathcal{L}_G$ to a left-invariant path structure on G (denoted in the same way). This shows that the restriction of $\mathcal{L}$ to V is isomorphic to the restriction of a left-invariant path structure on a three-dimensional Lie group, which concludes the proof. □

In this section we classify three dimensional Lie groups with homogeneous path structures. The results are summarized in Tables in section 5.3.

We follow an analogous scheme to classify homogeneous CR structures (see [FG]). In order to classify the simply connected groups having left invariant path structure it is sufficient to classify three-dimensional Lie algebras $\mathfrak{g}$ admitting a vector subspace $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ with a decomposition $\mathfrak{p} = \mathfrak{e}_1 \oplus \mathfrak{e}_2$.

Given a basis $\{X_1, X_2\}$ of $\mathfrak{p}$, with $X_i \in \mathfrak{e}_i$, define $Y = -[X_1, X_2]$ and consider the map $\mathrm{ad}_Y : \mathfrak{p} \to \mathfrak{g}$ whose matrix in the given basis is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Jacobi identities are computed as the following equations:

$$a_{11} + a_{22} = 0$$

$$a_{31}a_{12} - a_{32}a_{11} = 0$$

$$a_{31}a_{22} - a_{32}a_{21} = 0.$$

Note that in terms of a dual basis $\omega, \omega^1, \omega^2$ we obtain the equations

$$\begin{aligned} d\omega^1 &= a_{11}\omega^1 \wedge \omega + a_{12}\omega^2 \wedge \omega \\ d\omega^2 &= a_{21}\omega^1 \wedge \omega + a_{22}\omega^2 \wedge \omega \\ d\omega &= a_{31}\omega^1 \wedge \omega + a_{32}\omega^2 \wedge \omega + \omega^1 \wedge \omega^2 \end{aligned} \tag{43}$$

A different basis $\{\bar{X}_1, \bar{X}_2, \bar{Y} = -[\bar{X}_1, \bar{X}_2]\}$ for $\mathfrak{g}$, which defines the same path structure, is given by the change of basis matrix

$$P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1\lambda_2 \end{pmatrix},$$

with $\lambda_1, \lambda_2 \in \mathbb{R}^*$. The matrix of $\mathrm{ad}_{\bar{Y}}$ is now $\bar{A} = \lambda_1\lambda_2 P^{-1}AN$, where

$$N = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and therefore

$$\bar{A} = \begin{pmatrix} \lambda_1\lambda_2 a_{11} & {\lambda_2}^2 a_{12} \\ {\lambda_1}^2 a_{21} & \lambda_1\lambda_2 a_{22} \\ \lambda_1 a_{31} & \lambda_2 a_{32} \end{pmatrix}.$$

The goal now is to use the change of basis in order to find normal forms for ad_Y. Observe that the vector space isomorphism

$$X_1 \to X_2, \quad X_2 \to X_1, \quad Y \to -Y$$

changes the path structures. But we might not distinguish them as they correspond to a reordering of the decomposition. In other words,without loss of generality, we allow ourselves to change the order of the decomposition $\mathfrak{p} = \mathfrak{e}_1 \oplus \mathfrak{e}_2$. In this case, the matrix A changes to

$$\bar{A} = \begin{pmatrix} -a_{22} & -a_{21} \\ -a_{12} & -a_{11} \\ a_{32} & a_{31} \end{pmatrix}.$$

There are two cases to consider:

5.1.1 $\mathbf{ad}_Y\mathfrak{p} \subset \mathfrak{p}$

In this case $a_{31} = a_{32} = 0$. The general matrix for ad_Y is

$$\begin{pmatrix} a & b \\ c & -a \\ 0 & 0 \end{pmatrix}.$$

- If $a = b = c = 0$ we obtain the usual invariant path structure on the Heisenberg group.
- If $b = c = 0$ and $a \neq 0$ we may normalize the basis to obtain the normal form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix},$$

  which gives the usual invariant path structure on $\mathbf{SL}(2, \mathbb{R})$.

- If $b \neq 0$ (which we can assume also if $c \neq 0$ by a change of order) we normalize the matrix of ad_Y to

$$\begin{pmatrix} a & \pm 1 \\ c & -a \\ 0 & 0 \end{pmatrix}$$

  where $a, c \in \mathbb{R}$. One can further normalize:

  1. If $a \neq 0$ then we normalize to

$$\begin{pmatrix} 1 & \pm 1 \\ c & -1 \\ 0 & 0 \end{pmatrix}.$$

Here there are two cases. If $c \neq \mp 1$ then the group is simple. Indeed if, $\pm c+1 > 0$ then it corresponds to $\mathbf{SL}(2,\mathbb{R})$ and if $\pm c + 1 < 0$ it corresponds to $SU(2)$. This can be checked computing the Killing form of the Lie algebra and showing that it is non-degenerate and, moreover, negative exactly when $\pm c + 1 < 0$. Observe also that the case

$$\begin{pmatrix} 1 & -1 \\ c & -1 \\ 0 & 0 \end{pmatrix},$$

with $c < 0$ is equivalent, using a reordering of the vectors X_1 and X_2, to the case

$$\begin{pmatrix} 1 & 1 \\ -c & -1 \\ 0 & 0 \end{pmatrix}.$$

If $c = \mp 1$ then it is solvable. We obtain the Euclidean and Poincaré groups. Indeed, for $c = -1$ we get (making $e_1 = X_1 - X_2,\ e_2 = Y,\ e_3 = X_2$) the usual relations $[e_1, e_2] = 0,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = e_1$ of the Euclidean group. For $c = 1$ we get (making $e_1 = X_1 + X_2,\ e_2 = Y,\ e_3 = X_2$) the usual relations $[e_1, e_2] = 0,\ [e_1, e_3] = e_2,\ [e_2, e_3] = e_1$ of the Poincaré group.

2. If $a = 0$ and $c \neq 0$, we normalize to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which corresponds to the Lie algebra of $SU(2)$, or

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which corresponds to the Lie algebra of $\mathbf{SL}(2,\mathbb{R})$ (make $e_1 = X_1 - X_2,\ e_2 = X_1 + X_2,\ e_3 = Y$ to obtain the usual relations $[e_1, e_2] = 2e_3,\ [e_3, e_1] = -e_1,\ [e_3, e_2] = e_2$).

3. If $a = c = 0$ we may write

$$\begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This corresponds to two solvable groups. One of them (the case the upper right coefficient is $+1$) is the euclidian group (Bianchi VII) which we can recognise making $e_1 = Y, e_2 = X_1, e_3 = X_2$ so the algebra becomes $[e_1, e_2] = 0,\ [e_3, e_1] = e_2,\ [e_3, e_2] = -e_1$. The other case corresponds to the Poincaré group (Bianchi VI).

5.1.2 $\mathrm{ad}_Y\mathfrak{p} \not\subseteq \mathfrak{p}$

Then a_{31} and a_{32} do not vanish at the same time. One can suppose that $a_{31} \neq 0$ by exchanging the vectors X_1 and X_2. Observe now that from

$$a_{31}a_{12} - a_{32}a_{11} = 0$$

$$a_{31}a_{22} - a_{32}a_{21} = 0.$$

and $a_{31} \neq 0$ we have $a_{11}a_{22} - a_{12}a_{21} = 0$. We normalize the matrix of ad_Y as

$$\begin{pmatrix} a & b \\ c & -a \\ 1 & f \end{pmatrix},$$

where $a^2 + bc = 0$, $af - b = 0$ and $cf + a = 0$. We may write then

$$\begin{pmatrix} -cf & -cf^2 \\ c & cf \\ 1 & f \end{pmatrix}.$$

We normalize further:

1. If $f \neq 0$ and $c = 0$ we may normalize to

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

   This corresponds to the solvable group obtained by the direct product of the affine group with $\mathbb{R}$: An invariant path structure on the decomposable solvable group (the only group with derived algebra of dimension one which is not central).

2. If $f = 0$ and $c \neq 0$, we may normalize to

$$\begin{pmatrix} 0 & 0 \\ c & 0 \\ 1 & 0 \end{pmatrix}$$

   which corresponds to a family of solvable groups (make $e_1 = Y, e_2 = cX_2 + Y, e_3 = X_1$, then we obtain $[e_1, e_2] = 0,\ [e_1, e_3] = e_2,\ [e_2, e_3] = cY + cX_2 + Y = e_2 + ce_1$. The two families of solvable groups appear: Bianchi VI (for $c > -1/4$ and $c \neq 0$) and VII (for $c < -1/4$). Moreover if $c = -1/4$ then we get Bianchi IV.

3. If $f = 0$ and $c = 0$ we obtain another invariant path structure on the decomposable solvable group. The matrix ad_Y is normalized to

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

4. If $f \neq 0$ and $c \neq 0$ and we may normalize to

$$\begin{pmatrix} -c & -c \\ c & c \\ 1 & 1 \end{pmatrix},$$

and a computation shows that for $c > -1/4, c \neq 0$ the group is of type Bianchi VI and for $c < -1/4$ the group is of type VII. In the case $c = -1/4$ we obtain type IV.

5.2 Path structure curvatures of the homogeneous examples

The path geometry of the homogenous examples may be described using strict path structure invariants. Indeed, we showed that there exists parallelisms for each homogeneous three dimensional path structure. In particular there exists strict path structures associated to each of those homogeneous three dimensional path structures. In many cases the parallelism and therefore the strict structures are canonical.

Fix a parallelism $X_1, X_2, Y = -[X_1, X_2]$ of the strict path structure, defining an embedding $j : M \to Y^1$ into the strict path structure bundle. Recall (see the beginning of section 5.1) that the matrix of ad_Y in this basis is:

$$A = \begin{pmatrix} a & b \\ c & -a \\ e & f \end{pmatrix}$$

and satisfies $af - be = 0$ and $cf + ae = 0$.

The goal of this section is to compute the path structure curvatures for each homogeneous structure.

Proposition 5.2 *The curvatures of the invariant path structure defined by* $(\mathbb{R}X_1, \mathbb{R}X_2)$ *are given by*

$$j^*\iota^*Q^1 = -a(\frac{3}{2}b - \frac{1}{3}f^2)$$

and

$$j^*\iota^*Q^2 = -a(\frac{3}{2}c + \frac{1}{3}e^2).$$

The embedding $\iota : Y^1 \to Y$ of the strict path structures bundle into the path structure bundle was introduced in Proposition 6.1.

Proof.

We obtain a basis of the Lie algebra corresponding to an invariant structure with dual left invariant forms $\theta, \theta^1, \theta^2$ satisfying equations 43

$$\begin{aligned} d\theta^1 &= a\theta^1 \wedge \theta + b\theta^2 \wedge \theta \\ d\theta^2 &= c\theta^1 \wedge \theta - a\theta^2 \wedge \theta \\ d\theta &= e\theta^1 \wedge \theta + f\theta^2 \wedge \theta + \theta^1 \wedge \theta^2 \end{aligned} \tag{44}$$

The goal now is to identify the strict path geometry invariants of the structures. For that sake we need to obtain the structure equations 20. Note that θ is the fixed contact form.

First, in order to obtain equation $d\theta = \theta^1 \wedge \theta^2$ write θ^1 and θ^2 in terms of a new basis $\theta, \theta^1 - f\theta$ and $\theta^2 + e\theta$. We obtain then, using the same symbols $\theta, \theta^1, \theta^2$ for the new basis, the equations

$$\begin{array}{rcl} d\theta^1 &=& a\theta^1 \wedge \theta + b\theta^2 \wedge \theta - f\theta^1 \wedge \theta^2 \\ d\theta^2 &=& c\theta^1 \wedge \theta - a\theta^2 \wedge \theta + e\theta^1 \wedge \theta^2 \\ d\theta &=& \theta^1 \wedge \theta^2 \end{array} \tag{45}$$

Now we proceed as in section 3 (equations 21) and write the connection and torsion forms imposing the following equations

$$\begin{array}{rcl} d\theta^1 - 3\upsilon \wedge \theta^1 &=& \theta \wedge \tau^1, \\ d\theta^2 + 3\upsilon \wedge \theta^2 &=& \theta \wedge \tau^2, \\ d\theta &=& \theta^1 \wedge \theta^2. \end{array} \tag{46}$$

where we write the pull-back of the forms υ, τ^1 and τ^2 by the embedding j using the same letters. We obtain

$$\begin{array}{rcl} d\theta^1 &=& \theta^1 \wedge (a\theta - f\theta^2) - b\theta \wedge \theta^2 \\ d\theta^2 &=& -\theta^2 \wedge (a\theta + e\theta^1) - c\theta \wedge \theta^1 \\ d\theta &=& \theta^1 \wedge \theta^2 \end{array} \tag{47}$$

Comparing with the structure equations we obtain

$$\begin{array}{rcl} \upsilon &=& -\frac{1}{3}(a\theta + e\theta^1 - f\theta^2) \\ \tau^1 &=& -b\theta^2 \\ \tau^2 &=& -c\theta^1 \end{array} \tag{48}$$

We compute equation 22, the exterior derivative of υ, $d\upsilon = R\theta^1 \wedge \theta^2 + W^1\theta^1 \wedge \theta + W^2\theta^2 \wedge \theta$ and obtain

$$\begin{array}{rcl} R &=& -\frac{1}{3}(a - 2ef) \\ W^1 &=& -\frac{1}{3}(ae - cf) = \frac{2}{3}cf \\ W^2 &=& -\frac{1}{3}(be + af) = -\frac{2}{3}be. \end{array} \tag{49}$$

From

$$d\tau^1 + 3\tau^1 \wedge \upsilon = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2$$

we obtain

$$-bd\theta^2 + 3b\theta^2 \wedge \frac{1}{3}(a\theta + e\theta^1) = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2.$$

That is

$$-b(-\theta^2 \wedge (a\theta + e\theta^1) - c\theta \wedge \theta^1) + 3b\theta^2 \wedge \frac{1}{3}(a\theta + e\theta^1) = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2$$

which implies

$$W^2 = -\frac{2}{3}eb, \quad S^1_1 = bc, \quad S^1_2 = -2ab.$$

Analogously, from $d\tau^2 - 3\tau^2 \wedge \upsilon = 3W^1\theta^1 \wedge \theta^2 + S^2_1\theta \wedge \theta^1 + S^2_2\theta \wedge \theta^2$, we obtain

$$-c(\theta^1 \wedge (a\theta - f\theta^2) - b\theta \wedge \theta^2) + 3(-c\theta^1) \wedge \frac{1}{3}(a\theta - f\theta^2) = 3W^1\theta^1 \wedge \theta^2 + S^2_1\theta \wedge \theta^1 + S^2_2\theta \wedge \theta^2,$$

which implies

$$W^1 = \frac{2}{3}cf, \quad S_1^2 = 2ac, \quad S_2^2 = bc.$$

Moreover, we have the relation

$$S_1^1 = S_2^2 = \tau_2^1\tau_1^2.$$

Remark that from equations 25 and 26

$$dW^1 + 3W^1\upsilon = W_0^1\theta + W_1^1\theta^1 + W_2^1\theta^2 \tag{50}$$

and

$$dW^2 - 3W^2\upsilon = W_0^2\theta + W_1^2\theta^1 + W_2^2\theta^2, \tag{51}$$

we obtain

$$-\frac{2}{9}cf(a\theta + e\theta^1 - f\theta^2)$$

$$W_0^1 = -\frac{2}{3}cfa, \;\; W_1^1 = -\frac{2}{3}cfe, \;\; W_2^1 = \frac{2}{3}cf^2$$

and

$$\frac{2}{9}eb(a\theta + e\theta^1 - f\theta^2)$$

$$W_0^2 = -\frac{2}{3}eba, \;\; W_1^2 = -\frac{2}{3}e^2b, \;\; W_2^2 = \frac{2}{3}ebf$$

Substituting the expressions obtained above in Proposition 6.1 completes the proof.

□

5.3 Tables of invariant path structures on three dimensional Lie groups

In this section we put together the classification of path structures on the three dimensional Lie groups in the form of two tables. One for solvable Lie groups and the other for the remaining simple groups. In the following, we use the Bianchi classification of three dimensional Lie groups. Recall that the group Bianchi VI_0 is the Poincaré group and Bianchi VII_0 is the Euclidean group. The Bianchi groups of type I and V don't have invariant contact structures and therefore don't appear in the tables. The proof of the classification can be read in the previous section 5.1.

Path structures on three dimensional solvable Lie groups			
	ad_Y	Q^1	Q^2
Heisenberg, Bianchi II	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$	0	0
$Aff(2)\times\mathbb{R}$ Bianchi III	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$	0	0
Bianchi IV	$\begin{pmatrix} 0 & 0 \\ -\frac{1}{4} & 0 \\ 1 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \\ 1 & 1 \end{pmatrix}$	$-\frac{1}{4}(\frac{3}{8}-\frac{1}{3})$	$-\frac{1}{4}(-\frac{3}{8}+\frac{1}{3})$
Bianchi VI	$\begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Bianchi VI_0	0	0
	$\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}$ Bianchi VI_0	$\frac{3}{2}$	$-\frac{3}{2}$
	$\begin{pmatrix} 0 & 0 \\ c & 0 \\ 1 & 0 \end{pmatrix}$, $0\neq c>-\frac{1}{4}$	0	0
	$\begin{pmatrix} -c & -c \\ c & c \\ 1 & 1 \end{pmatrix}$, $0\neq c>-\frac{1}{4}$	$c(-\frac{3}{2}c-\frac{1}{3})$	$c(\frac{3}{2}c+\frac{1}{3})$
Bianchi VII	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Bianchi VII_0	0	0
	$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix}$ Bianchi VII_0	$-\frac{3}{2}$	$\frac{3}{2}$
	$\begin{pmatrix} 0 & 0 \\ c & 0 \\ 1 & 0 \end{pmatrix}$, $c<-\frac{1}{4}$	0	0
	$\begin{pmatrix} -c & -c \\ c & c \\ 1 & 1 \end{pmatrix}$, $c<-\frac{1}{4}$	$c(-\frac{3}{2}c-\frac{1}{3})$	$c(\frac{3}{2}c+\frac{1}{3})$

Path structures on three dimensional simple Lie groups			
	ad_Y	Q^1	Q^2
$\mathbf{SL}(2,\mathbb{R})$ Bianchi VIII	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} 1 & 1 \\ c & -1 \\ 0 & 0 \end{pmatrix}$, $c > -1$	$-\frac{3}{2}$	$-\frac{3}{2}c$
	$\begin{pmatrix} 1 & -1 \\ c & -1 \\ 0 & 0 \end{pmatrix}$, $0 < c < 1$	$\frac{3}{2}$	$-\frac{3}{2}c$
$SU(2)$ Bianchi IX	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} 1 & 1 \\ c & -1 \\ 0 & 0 \end{pmatrix}$, $c < -1$	$-\frac{3}{2}$	$-\frac{3}{2}c$
	$\begin{pmatrix} 1 & -1 \\ c & -1 \\ 0 & 0 \end{pmatrix}$, $c > 1$	$\frac{3}{2}$	$-\frac{3}{2}c$

5.4 Natural invariants in the case of $\mathbf{SL}(2,\mathbb{R})$

In this subsection, we relate the left-invariant structures on $\mathbf{SL}(2,\mathbb{R})$ to the geometry of the Killing form of $\mathfrak{sl}(2,\mathbb{R})$ which defines a Lorentzian structure on $\mathbf{SL}(2,\mathbb{R})$.

Recall the parallelism $X_1, X_2, Y = -[X_1, X_2]$ and the matrix of ad_Y in this basis (section 5.1):

$$A = \begin{pmatrix} a & b \\ c & -a \\ e & f \end{pmatrix}$$

satisfying $af - be = 0$ and $cf + ae = 0$. We have

$$\mathrm{ad}_{X_1} = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & -c \\ 0 & -1 & -e \end{pmatrix}, \quad \mathrm{ad}_{X_2} = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & a \\ 1 & 0 & -f \end{pmatrix}, \quad \mathrm{ad}_Y = \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ e & f & 0 \end{pmatrix}.$$

Therefore, the Killing form of the Lie algebra satisfies

$$\kappa(X_1, X_1) = \mathrm{tr}\, X_1^2 = 2c + e^2, \quad \kappa(X_2, X_2) = \mathrm{tr}\, X_2^2 = -2b + f^2, \quad \kappa(Y, Y) = \mathrm{tr}\, Y^2 = 2(a^2 + bc).$$

$$\kappa(X_1, X_2) = -2a + ef, \quad \kappa(X_1, Y) = -ae - cf, \quad \kappa(X_2, Y) = -be + af.$$

5.4.1 Lorentzian geometry of $\mathfrak{sl}(2,\mathbb{R})$

Let us consider the Killing form of $\mathfrak{sl}(2,\mathbb{R})$, $\kappa(u,v)=\frac{1}{2}\mathrm{tr}\,(uv)$ whose associated quadratic form is

$$q(u)=-\det(u). \tag{52}$$

Then, using the standard basis

$$E=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, F=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{53}$$

of $\mathfrak{sl}(2,\mathbb{R})$ satisfying the relation $H=[E,F]$, $(H,E+F,E-F)$ is an orthonormal basis of signature $(1,1,-1)$ for q. In particular the choice of this basis identifies $\mathfrak{sl}(2,\mathbb{R})$ to the Lorentzian Minkowski space $\mathbb{R}^{1,2}$. Elements of $\mathfrak{sl}(2,\mathbb{R})$ are thus distinguished according to the sign of q: $u\in\mathfrak{sl}(2,\mathbb{R})$ is called

- *timelike* if $q(u)<0$,
- *lightlike* if $q(u)=0$,
- *spacelike* if $q(u)>0$.

Recall that an element of $\mathfrak{sl}(2,\mathbb{R})$ is called *hyperbolic* (respectively *parabolic*, *elliptic*) if it generates a one-parameter group of the given kind in $\mathbf{SL}(2,\mathbb{R})$. We have then the following straightforward correspondence between the geometry of q and the algebraic types in $\mathfrak{sl}(2,\mathbb{R})$.

Lemma 5.3 *Let $u\in\mathfrak{sl}(2,\mathbb{R})$. Then:*

- *u is timelike $\Leftrightarrow$ u is elliptic;*
- *u is lightlike $\Leftrightarrow$ u is parabolic;*
- *u is spacelike $\Leftrightarrow$ u is hyperbolic.*

Lastly, q is invariant by the adjoint action of $\mathbf{SL}(2,\mathbb{R})$, and more precisely

$$\mathrm{Ad}\colon g\in\mathbf{SL}(2,\mathbb{R})\mapsto Ad_g\in SO^0(q) \tag{54}$$

is surjective and has kernel $\{\pm id\}$, $SO^0(q)$ denoting the connected component of the identity in the stabilizer of q in $\mathrm{GL}(\mathfrak{sl}(2,\mathbb{R}))$.

5.4.2 The moduli space of left-invariant path structures on $\mathbf{SL}(2,\mathbb{R})$

Definition 5.4 *We define $\mathcal{S}$ to be the set of left-invariant path structures on* $\mathbf{SL}(2,\mathbb{R})$.

Our goal is now to extract the natural geometric invariants distinguishing, in the space $\mathcal{S}$, those which are locally isomorphic.

We say that a plane P in $\mathfrak{sl}(2,\mathbb{R})$ is

- *timelike* if $q|_P$ has Lorentzian signature $(1,-1)$, *i.e.* if P contains a timelike line;

- *lightlike* if $q|_P$ has signature $(1,0)$, *i.e.* if it is degenerated, or if P contains exactly one lightlike line;
- *spacelike* if $q|_P$ has euclidean signature $(1,1)$, *i.e.* if P contains no lightlike line.

Each choice of a plane in $\mathfrak{sl}(2,\mathbb{R})$ defines, by translation, a distribution on $\mathbf{SL}(2,\mathbb{R})$. Note that the adjoint action by $\mathbf{SL}(2,\mathbb{R})$ on the set of planes in $\mathfrak{sl}(2,\mathbb{R})$ has three orbits. Indeed since this action factorizes through the action of $SO^0(1,2)$ on the Minkowski space $\mathbb{R}^{1,2}$ according to (54), it is sufficient to check that $SO^0(1,2)$ has three orbits on the set of planes of $\mathbb{R}^{1,2}$, which the following dichotomy shows:

Timelike planes The restriction of $SO^0(1,2)$ to the open set of timelike lines of $\mathbb{R}^{1,2}$ is conjugated to the action of the group of orientation-preserving isometries on the hyperbolic plane, which is transitive with the stabilizer of any point p acting transitively on unitary tangent vectors at p. This shows in particular that $SO^0(1,2)$ acts transitively on the set of timelike planes.

Spacelike planes The restriction of $SO^0(1,2)$ to the open set of spacelike half-lines of $\mathbb{R}^{1,2}$ is conjugated to the action of the group of orientation and time-orientation preserving isometries on the de-Sitter space, which is transitive with the stabilizer of any point p acting transitively on unitary spacelike tangent vectors at p. This shows in particular that $SO^0(1,2)$ acts transitively on the set of spacelike planes.

Lightlike planes $SO^0(1,2)$ acts transitively on the closed set of lightlike lines of $\mathbb{R}^{1,2}$, and any lightlike line is contained in a unique lightlike plane. This shows that $SO^0(1,2)$ acts transitively on the set of lightlike planes.

The open orbits of timelike and spacelike planes of $\mathfrak{sl}(2,\mathbb{R})$ correspond to the two invariant contact distributions in $\mathbf{SL}(2,\mathbb{R})$. The closed orbit of timelike planes corresponds to an integrable distribution which will not be relevant in the sequel.

Consider the projective space $\mathbb{P}(\mathfrak{sl}(2,\mathbb{R}))$. We introduce the two open subsets in $\mathbb{P}(\mathfrak{sl}(2,\mathbb{R}))\times\mathbb{P}(\mathfrak{sl}(2,\mathbb{R}))$,

$$\mathcal{E}_{space} = \{(D_1,D_2)\}|\{D_1 \neq D_2, D_1 \oplus D_2 \text{ spacelike}\}, \tag{55}$$

$$\mathcal{E}_{time} = \{(D_1,D_2)\}|\{D_1 \neq D_2, D_1 \oplus D_2 \text{ timelike}\} \tag{56}$$

as parameter spaces for $\mathcal{S}$. Note that two pairs of distinct points of $\mathbb{P}(\mathfrak{sl}(2,\mathbb{R}))$ are in the same orbit under the adjoint action of $\mathbf{SL}(2,\mathbb{R})$, if and only if the pairs of left-invariant line fields that they generate are related by an element of $\mathbf{SL}(2,\mathbb{R})$. Hence any invariant of adjoint orbits of $\mathbf{SL}(2,\mathbb{R})$ will be relevant for our study. There exists one natural invariant in each of the open subsets $\mathcal{E}_{space}$ and $\mathcal{E}_{time}$.

Definition 5.5 *For $(D_1,D_2)\in\mathcal{E}_{space}$ or $\mathcal{E}_{time}$, pairs with a lightlike line being excluded, define the cross-ratio*

$$cr(D_1,D_2) = \frac{\kappa(X_1,X_2)\kappa(X_2,X_1)}{\kappa(X_1,X_1)\kappa(X_2,X_2)}$$

with X_i a generator of D_i for $i=1,2$.

For any $u \subset \mathfrak{sl}(2,\mathbb{R})$ $\tilde{u}$ denotes the left-invariant vector field of $\mathbf{SL}(2,\mathbb{R})$ generated by u, with an analog notation for left-invariant line or plane fields.

Proposition 5.6 *The map*

$$(D_1, D_2) \in \mathcal{E}_{space} \cup \mathcal{E}_{time} \mapsto (\widetilde{D_1}, \widetilde{D_2}) \in \mathcal{S} \tag{57}$$

is surjective. Moreover, $(\widetilde{D_1}, \widetilde{D_2})$ *and* $(\widetilde{D_1'}, \widetilde{D_2'})$ *are locally isomorphic if, and only if one of the following mutually exclusive conditions is satisfied, with* $P = D_1 \oplus D_2$ *and* $P' = D_1' \oplus D_2'$.

1. *All the lines* D_i *and* D_i' *are lightlike, and in this case the associated path structures are flat.*

2. *Exactly one of the lines* (D_1, D_2)*, respectively* (D_1', D_2') *is lightlike and the others are of the same type. In this case one of the Cartan curvatures is null.*

3. P *and* P' *are spacelike and* $cr(D_1, D_2) = cr(D_1', D_2')$.

4. P *and* P' *are timelike,* D_i *of the same type as* D_i' *for* $1 \leq i \leq 2$*, none of the lines is lightlike, and* $cr(D_1, D_2) = cr(D_1', D_2')$*. There are three components corresponding to* D_1 *and* D_2 *both timelike, both space-like or of different type.*

PROOF: Surjectivity follows from the fact that contact distributions arise either from timelike planes or spacelike planes. The closed orbit of lightlike planes is excluded.

Consider vectors X_1 and X_2 generating D_1 and D_2. Let $X_1, X_2, Y = -[X_1, X_2]$ be a basis of the Lie algebra and the matrix of ad_Y in this basis (section 5.1) for $\mathfrak{sl}(2,\mathbb{R})$ be written as:

$$A = \begin{pmatrix} a & b \\ c & -a \\ 0 & 0 \end{pmatrix}.$$

From the computation of the Killing form we obtain $\kappa(X_1, X_1) = 2c$, $\kappa(X_2, X_2) = -2b$ and $\kappa(X_1, X_2) = -2a$. Also $Q^1 = -\frac{3}{2}ab$ and $Q^2 = -\frac{3}{2}ac$. One also have

$$cr(D_1, D_2) = \frac{\kappa(X_1, X_2)\kappa(X_2, X_1)}{\kappa(X_1, X_1)\kappa(X_2, X_2)} = -\frac{a^2}{bc}.$$

The proposition now follows from the following computations:

- If D_1 and D_2 are lightlike, the invariant flag structure is flat.

- If the two lines are orthogonal, one timelike and the other spacelike, the invariant flag structure is flat.

- If $c = 0, b > 0$, (D_1, D_2) is timelike (X_1 lightlike and X_2 timelike) and one can normalize so that $a = 1, b = 1$. One verifies that $Q^1 = -1$ and $Q^2 = 0$.

- If $c = 0, b < 0$, (D_1, D_2) is timelike (X_1 lightlike and X_2 spacelike) and one can normalize so that $a = 1, b = -1$. One verifies that $Q^1 = 1$ and $Q^2 = 0$.

- If $c>0, b<0$, (D_1,D_2) is spacelike (if $a^2+bc<0$) or timelike (if $a^2+bc>0$). One can normalize so that $a=1, b=-1$. Then $Q^1=\frac{3}{2}$, $Q^2=-\frac{3}{2}c$ and $cr(D_1,D_2)=1/c$.
- If $c>0, b>0$, (D_1,D_2) is timelike (X_1 spacelike and X_2 timelike) and one can normalize so that $a=1, b=1$. Then $Q^1=-\frac{3}{2}$, $Q^2=-\frac{3}{2}c$ and $cr(D_1,D_2)=-1/c<0$.
- If $c<0, b>0$, (D_1,D_2) is timelike (X_1 and X_2 timelike) and one can normalize so that $a=1, b=1$. Then $Q^1=-\frac{3}{2}$, $Q^2=-\frac{3}{2}c$ and $cr(D_1,D_2)=-1/c>0$.

To conclude the reverse implication of the Proposition, we use the fact that two homogeneous path structures having the same invariants Q^i are locally isomorphic, as a consequence of the classification of section 5.1 and of the proof of Theorem 4.2. □

6 Appendix

6.1 Transformation properties of the adapted connection for path structures

In this section we review the explicit transformation properties of the Cartan connection of the $B_{\mathbb{R}}$ (Borel group) principal bundle associated to a path structure (see [FV1] for an analogous computation in the context of a $B_{\mathbb{C}}$ principal bundle).

We compute $Ad_{h^{-1}}\pi$ for an element

$$h=\begin{pmatrix} a & c & e\\ 0 & \frac{1}{ab} & f\\ 0 & 0 & b\end{pmatrix}$$

We have, for a constant h,

$$\tilde{\pi}=Ad_{h^{-1}}\pi.$$

A computation shows that

$$\begin{aligned}
\tilde{\omega} &= \frac{a}{b}\,\omega\\
\tilde{\omega}^1 &= a^2b\,\omega^1-a^2f\,\omega\\
\tilde{\omega}^2 &= \frac{1}{ab^2}\,\omega^2+\frac{c}{b}\,\omega\\
\tilde{\varphi} &= \varphi-\frac{1}{2}abc\,\omega^1-\frac{f}{2b}\,\omega^2+(\frac{1}{2}acf-\frac{e}{b})\,\omega\\
\tilde{w} &= w-\frac{1}{2}abc\,\omega^1+\frac{f}{2b}\,\omega^2+\frac{1}{2}acf\,\omega\\
\tilde{\varphi}^1 &= b^2a\,\varphi^1-3abf\,w+baf\,\varphi+bae\,\omega^1-f^2a\,\omega^2-fae\,\omega\\
\tilde{\varphi}^2 &= \frac{1}{ba^2}\,\varphi^2+\frac{3c}{a}\,w+\frac{c}{a}\,\varphi-bc^2\,\omega^1+(-\frac{e}{a^2b^2}+\frac{cf}{ab})\,\omega^2+(-\frac{ce}{ab}+fc^2)\,\omega\\
\tilde{\psi} &= \frac{b}{a}\,\psi+(\frac{2e}{a}-bcf)\,\varphi-bce\,\omega^1+(-\frac{fe}{ab}+cf^2)\,\omega^2-cb^2\,\varphi^1+\frac{f}{a}\,\varphi^2+3fbc\,w+(-\frac{e^2}{ab}+fce)\,\omega
\end{aligned}\tag{58}$$

and for the curvature

$$\Pi = d\pi + \pi\wedge\pi = \begin{pmatrix} 0 & \Phi^2 & \Psi \\ 0 & 0 & \Phi^1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\Pi} = Ad_{h^{-1}}\Pi = \begin{pmatrix} 0 & \frac{1}{a^2 b}\Phi^2 & \frac{b}{a}\Psi + \frac{f}{a}\Phi^2 - cb^2\Phi^1 \\ 0 & 0 & ab^2\Phi^1 \\ 0 & 0 & 0 \end{pmatrix}.$$

That is,

$$\tilde{\Phi}^1 = ab^2\Phi^1,$$

$$\tilde{\Phi}^2 = \frac{1}{a^2 b}\Phi^2,$$

and

$$\tilde{\Psi} = \frac{b}{a}\Psi + \frac{f}{a}\Phi^2 - cb^2\Phi^1.$$

Recalling that $\Phi^1 = Q^1\omega\wedge\omega^2$, $\Phi^2 = Q^2\omega\wedge\omega^1$ and $\Psi = (U_1\omega^1 + U_2\omega^2)\wedge\omega$, one may write therefore

$$\tilde{Q}^1\tilde{\omega}\wedge\tilde{\omega}^2 = ab^2\,Q^1\omega\wedge\omega^2.$$

But $\tilde{Q}^1\tilde{\omega}\wedge\tilde{\omega}^2 = \tilde{Q}^1\frac{a}{b}\,\omega\wedge\frac{1}{ab^2}\omega^2$ and then

$$\tilde{Q}^1 = ab^5\,Q^1, \tag{59}$$

$$\tilde{Q}^2 = \frac{1}{a^5 b}\,Q^2 \tag{60}$$

and, analogously,

$$\tilde{U}_1 = \frac{b}{a^4}(U_1 - \frac{f}{b}Q^2), \tag{61}$$

$$\tilde{U}_2 = \frac{b^4}{a}(U_2 + abcQ^1). \tag{62}$$

These transformation properties imply that we can define two tensors on Y which are invariant under H and will give rise to two tensors on M. Indeed

$$Q^1\,\omega^2\wedge\omega\otimes\omega\otimes e_1$$

and

$$Q^2\,\omega^1\wedge\omega\otimes\omega\otimes e_2,$$

where e_1 and e_2 are duals to ω^1 and ω^2 in the dual frame of the coframe bundle of Y, are easily seen to be H-invariant.

6.2 The embedding $\iota : Y^1 \to Y$

We recall here the embedding described in [FV1, FV2]. We give additional details of the computation of the path structure curvatures in terms of invariants of the strict path structure. This embedding allows curvatures of path structures to be expressed in terms of curvatures of strict path structures which, in turn, are much easier to compute.

Proposition 6.1 *There exists a unique equivariant embedding of fiber bundles $\iota : Y^1 \to Y$ such that*

$$\iota^*\omega = \theta, \quad \iota^*\omega^1 = \theta^1, \quad \iota^*\omega^2 = \theta^2, \quad \iota^*\varphi = 0. \tag{63}$$

Moreover

$$\iota^* Q^1 = S^1_2 + \frac{3}{2}R\tau^1_2 + 2W^2_2 - \frac{1}{2}R_{22}$$

and

$$\iota^* Q^2 = -S^2_1 + \frac{3}{2}R\tau^2_1 - 2W^1_1 + \frac{1}{2}R_{11}.$$

Proof. In order to express the curvatures of the path structures, we first compute the pull-back of the connection of the path structure in terms of the connection of the strict structure. The pull back of the structure equations are (here we denote the pull-back of a form by the same letter except for the forms $\omega, \omega^1, \omega^2$ which are given, from the above proposition, as $\theta, \theta^1, \theta^2$):

$$\begin{aligned}
d\theta &= \theta^1 \wedge \theta^2 \\
d\theta^1 &= 3w \wedge \theta^1 + \theta \wedge \varphi^1 \\
d\theta^2 &= -3w \wedge \theta^2 - \theta \wedge \varphi^2 \\
\theta \wedge \psi &= \frac{1}{2}(\varphi^2 \wedge \theta^1 + \varphi^1 \wedge \theta^2) \\
dw &= \frac{1}{2}(-\varphi^2 \wedge \theta^1 + \varphi^1 \wedge \theta^2) \\
Q^1 \theta \wedge \theta^2 &= d\varphi^1 + 3\varphi^1 \wedge w + \theta^1 \wedge \psi \\
Q^2 \theta \wedge \theta^1 &= d\varphi^2 - 3\varphi^2 \wedge w - \theta^2 \wedge \psi \\
d\psi - \varphi^1 \wedge \varphi^2 &= (U_1\theta^1 + U_2\theta^2) \wedge \theta.
\end{aligned}$$

It follows, by comparing with Proposition 3.2, that

$$\begin{aligned}
w &= \upsilon + c\theta \\
\varphi^1 &= \tau^1 - 3c\theta^1 + E^1\theta \\
\varphi^2 &= -\tau^2 - 3c\theta^2 + E^2\theta.
\end{aligned}$$

where E^1, E^2 and c are functions on Y^1. We obtain then that

$$\psi = \frac{1}{2}(E^2\theta^1 + E^1\theta^2 + G\theta)$$

where G is a function on Y^1. Substituting the formulas for φ^1, φ^2 and w in the equation for dw we obtain, using equation 22

$$dw = R\theta^1 \wedge \theta^2 + W^1\theta^1 \wedge \theta + W^2\theta^2 \wedge \theta + dc \wedge \theta + cd\theta = \frac{1}{2}(-6c\theta^1 \wedge \theta^2 - E^2\theta \wedge \theta^1 + E^1\theta \wedge \theta^2),$$

which implies, comparing the terms in $\theta^1 \wedge \theta^2$,

$$c = -\frac{R}{4},$$

and then, using $dR = R_0\theta + R_1\theta^1 + R_2\theta^2$, we obtain

$$E^1 = -2W^2 + \frac{R_2}{2}, \quad E^2 = 2W^1 - \frac{R_1}{2}.$$

- Substituting the formulas for φ^1, ψ and w, with the above values of E^1 and c, in the equation $Q^1\theta \wedge \theta^2 = d\varphi^1 + 3\varphi^1 \wedge w + \theta^1 \wedge \psi$, we obtain

$$Q^1\theta \wedge \theta^2 = d(\tau^1 - 3c\theta^1 + E^1\theta) + 3(\tau^1 - 3c\theta^1 + E^1\theta) \wedge (\upsilon + c\theta) + \theta^1 \wedge \frac{1}{2}(E^1\theta^2 + G\theta). \quad (64)$$

Compute, using equations 26 and 28

$$dE^1 \wedge \theta = d(-2W^2 + \frac{R_2}{2}) \wedge \theta = -2(3W^2\upsilon + W^2_1\theta^1 + W^2_2\theta^2) \wedge \theta + \frac{1}{2}(3R_2\upsilon - \frac{1}{2}R_0\theta^1 + R_{12}\theta^1 + R_{22}\theta^2) \wedge \theta,$$

and

$$-3d(c\theta^1) = -3dc \wedge \theta^1 - 3cd\theta^1 = \frac{3}{4}dR \wedge \theta^1 + \frac{3}{4}R(-3\theta^1 \wedge \upsilon + \theta \wedge \tau^1)$$

Recall also equation 23

$$d\tau^1 + 3\tau^1 \wedge \upsilon = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2.$$

Substituting the above expressions into equation 64 we obtain

$$\begin{aligned} Q^1\theta \wedge \theta^2 = &-3\tau^1 \wedge \upsilon + 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2 + \frac{3}{4}dR \wedge \theta^1 + \frac{3}{4}R(-3\theta^1 \wedge \upsilon + \theta \wedge \tau^1) \\ &- 2(3W^2\upsilon + W^2_1\theta^1 + W^2_2\theta^2) \wedge \theta + \frac{1}{2}(3R_2\upsilon - \frac{1}{2}R_0\theta^1 + R_{12}\theta^1 + R_{22}\theta^2) \wedge \theta + E^1\theta^1 \wedge \theta^2 \\ &+ 3(\tau^1 - 3c\theta^1 + E^1\theta) \wedge (\upsilon + c\theta) + \theta^1 \wedge \frac{1}{2}(E^1\theta^2 + G\theta), \end{aligned}$$

which, using $3E^1\theta \wedge \upsilon = (-6W^2 + \frac{3R_2}{2})\theta \wedge \upsilon$, simplifies to an expression with no terms involving υ:

$$\begin{aligned} Q^1\theta \wedge \theta^2 = &\, 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2 + \frac{3}{4}dR \wedge \theta^1 + \frac{3}{4}R\theta \wedge \tau^1 \\ &- 2(W^2_1\theta^1 + W^2_2\theta^2) \wedge \theta + \frac{1}{2}(-\frac{1}{2}R_0\theta^1 + R_{12}\theta^1 + R_{22}\theta^2) \wedge \theta + E^1\theta^1 \wedge \theta^2 \\ &+ 3(\tau^1 - 3c\theta^1) \wedge c\theta + \theta^1 \wedge \frac{1}{2}(E^1\theta^2 + G\theta). \end{aligned}$$

Observe now that the coefficient of $\theta^1 \wedge \theta^2$ is

$$3W^2 - \frac{3}{4}R_2 + \frac{3}{2}E^1 = 0.$$

Therefore we may simplify to

$$Q^1\theta \wedge \theta^2 = (S_1^1 + R_0 + 2W_1^2 - \frac{1}{2}R_{12} + \frac{9}{16}R^2 - \frac{1}{2}G)\theta \wedge \theta^1$$
$$+ (S_2^1 + \frac{3}{2}R\tau_2^1 + 2W_2^2 - \frac{1}{2}R_{22})\theta \wedge \theta^2$$

and conclude that

$$Q^1 = S_2^1 + \frac{3}{2}R\tau_2^1 + 2W_2^2 - \frac{1}{2}R_{22} \tag{65}$$

- Analogously, substituting the formulas for φ^1, ψ and w, with the above values of E^1 and c, in the equation $Q^2\theta \wedge \theta^1 = d\varphi^2 - 3\varphi^2 \wedge w - \theta^2 \wedge \psi$ we obtain

$$Q^2\theta\wedge\theta^1 = d(-\tau^2 - 3c\theta^2 + E^2\theta) - 3(-\tau^2 - 3c\theta^2 + E^2\theta)\wedge(\upsilon + c\theta) - \theta^2 \wedge \frac{1}{2}(E^2\theta^1 + G\theta). \tag{66}$$

Following the same computations we conclude with the formula

$$Q^2 = -S_1^2 + \frac{3}{2}R\tau_1^2 - 2W_1^1 + \frac{1}{2}R_{11}. \tag{67}$$

□

A similar formula is obtained in section of [FV2] (see proposition 3.5) for the curvature of the path structure adapted bundle in terms of invariants of an enriched structure. We included a proof here because we used the embedding of the strict path structure adapted bundle into the path structure adapted bundle instead. Moreover, conventions are slightly changed. Namely, in the definition of the strict structure bundle, the following changes should be made from the present paper to [FV2]: $\tau_2^1 \to \tau_2^1$, $\tau_2^1 \to \tau_2^1$, $R \to S$, $W^1 \to -C$, $W^2 \to -D$, $S_2^1 \to \tau_{20}^1$ and $S_1^2 \to -\tau_{10}^2$.

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8 Data : "Not available".

References

[B] Bryant, R. : Élie Cartan and geometric duality. Preprint 1998.

[BGH] Bryant, R. , Griffiths, P. , Hsu, L. : Toward a geometry of differential equations. Geometry, topology, and physics, 1-76, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.

[C] Cartan, E.: Sur la gréométrie pseudo-conforme des hypersurfaces de I'espace de deux variables complexes, I, Ann. Math. Pura Appl IV, U (1933), 17-90.

[Car] Cartan, E.: Les espaces généralisés et l'intégration de certaines classes d' équations différentielles, C. R. Acad. Sci. 206 (1938), 1689-1693.

[FG] Falbel, E. , Gorodski, C. : Sub-Riemannian homogeneous spaces in dimensions 3 and 4. Geom. Dedicata 62 (1996), no. 3, 227-252.

[FMMV] Falbel, E. , Mion-Mouton, M. , Veloso, J. M. : Cartan connections and path structures with large automorphism groups. Internat. J. Math. 32 (2021), no. 13.

[FV1] Falbel, E. , Veloso, J. M. : Flag structures on real 3-manifolds. Geom. Dedicata 209 (2020) pg. 149-176.

[FV2] Falbel, E. , Veloso, J. M. : A Global invariant for path structures and second order differential equations. arXiv:2306.17705 (2023)

[IL] Ivey, T. A. , Landsberg, J. M. : Cartan for beginners: differential geometry via moving frames and exterior differential systems. Graduate Studies in Mathematics, 61. American Mathematical Society, Providence, RI, 2003.

[KT] Kruglikov, Boris, The, Dennis : The gap phenomenon in parabolic geometries. Journal für die reine und angewandte Mathematik, 2017.

[MM] Mion-Mouton, M. : Quelques propriétés géométriques et dynamiques globales des structures Lagrangiennes de contact. Strasbourg, 2020.

[T] Tresse, A. : Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre $y'' = w(x, y, y')$. Jablonowskischen gesellschaft zu Leipzig. XXXII. 1896.

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