Reductions of path structures and classification of homogeneous structures in dimension three

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Abstract

In this paper we show that if a path structure has non-vanishing curvature at a point then it has a canonical reduction to a $\mathbb{Z}/2\mathbb{Z}$-structure at a neighbourhood of that point (in many cases it has a canonical parallelism). A simple implication of this result is that the automorphism group of a non-flat path structure is of maximal dimension three (a result by Tresse of 1896). We also classify the invariant path structures on three-dimensional Lie groups.

1 Introduction

A path structures on a 3-manifold is a choice of two subbundles T^1 and T^2 in TM such that $T^1 \cap T^2 = \{0\}$ and such that $T^1 \oplus T^2$ is a contact distribution. This geometry gives rise to a Cartan connection on a canonical principal bundle (with structure group, $B_{\mathbb{R}}$, the Borel subgroup of upper triangular matrices in $\mathbf{SL}(3,\mathbb{R})$) which we call Y ([Car], see [IL] for a modern presentation and section 2). There are two curvature functions Q^1 and Q^2 defined on Y which should determine, in certain situations, the path structure up to equivalence. Indeed, when $Q^1 = Q^2 = 0$ the path structure is locally equivalent to the path structure on the model space $\mathbf{SL}(3,\mathbb{R})/B_{\mathbb{R}}$ (see section 2).

A simple way to define a path structure on a 3-manifold is to fix a contact form and two transverse vector fields contained in the kernel of the form. In particular, this defines a parallelism of the 3-manifold. Reciprocally, one might ask whether there exists a canonical parallelism, with transverse vector fields contained in the contact distribution, associated to a path structure. In this case, the automorphism group of the path structure should coincide with the automorphism group of the parallelism.

We show in this paper that if a path structure has non-vanishing curvature at a point then it has a canonical reduction to a $\mathbb{Z}/2\mathbb{Z}$-structure at a neighbourhood of that point (in many cases it has a canonical parallelism).

In section 2 we recall the construction of the Cartan bundle Y and adapted connection to a path structure on a 3-manifold. We also recall the definition of strict path structure, that is, when a contact form is fixed over the manifold and the definition of a Cartan bundle Y^1 and connection adapted to that structure which was used in ([FMMV]) to obtain a classification of compact 3-manifolds with non-compact automorphism group preserving the strict path structure. We also recall in section 3 a natural embedding $Y^1 \to Y$ (see [FMMV] and [FV1]). A parallelism with transverse vector fields contained in the contact distribution naturally defines a strict path structure and an embedding of the manifold M into Y.

In section 4 we prove a canonical reduction of a path structure when the structure is non-flat:

Theorem 1.1 *If the path structure is not flat, there exists a canonical reduction of the fiber bundle Y to a $\mathbb{Z}/2\mathbb{Z}$-structure.*

A more precise theorem is proved in section 4 where we give conditions for the existence of a further reduction to a parallelism. The theorem implies the classical theorem by Tresse that a non-flat path structure has an automorphism group of dimension at most three ([T]).

In section 5 we classify left invariant path structures on three dimensional Lie groups. The results are gathered in tables in section 5.3. We choose for each structure a parallelism (some are canonical) and we compute the curvatures for each of these invariant structures using an embedding of the group in the corresponding Cartan bundle Y (see Proposition 5.1 in section 5.2).

In the last section 5.4 we give a geometric description of the invariant structures on $\mathbf{SL}(2,\mathbb{R})$ involving the type of the contact plane with respect to the Killing metric and a cross-ratio which parametrizes the positions of the one dimensional distributions in the contact plane. Similar descriptions can be made for each of the three dimensional groups.

2 The Cartan connection of a path structure

Path geometries are treated in detail in section 8.6 of [IL] and in [BGH]. Le M be a real three dimensional manifold and TM be its tangent bundle.

Definition 2.1 *A path structure on M is a choice of two subbundles T^1 and T^2 in TM such that $T^1 \cap T^2 = \{0\}$ and such that $T^1 \oplus T^2$ is a contact distribution.*

The condition that $T^1 \oplus T^2$ be a contact distribution means that, locally, there exists a one form $\theta \in T^*M$ such that $\ker \theta = T^1 \oplus T^2$ and $d\theta \wedge \theta$ is never zero.

Flat path geometry is the geometry of real flags in $\mathbb{R}^3$. That is the geometry of the space of all couples (p,l) where $p \in \mathbb{R}P^2$ and l is a real projective line containing p. The space of flags is identified to the quotient

$$\mathbf{SL}(3,\mathbb{R})/B_{\mathbb{R}}$$

where $B_{\mathbb{R}}$ is the Borel group of all real upper triangular matrices. The Maurer-Cartan form on $\mathbf{SL}(3,\mathbb{R})$ is given by a form with values in the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$:

$$\pi = \begin{pmatrix} \varphi + w & \varphi^2 & \psi \\ \omega^1 & -2w & \varphi^1 \\ \omega & \omega^2 & -\varphi + w \end{pmatrix}$$

satisfying the equation $d\pi + \pi \wedge \pi = 0$. That is

$$d\omega = \omega^1 \wedge \omega^2 + 2\varphi \wedge \omega$$

$$
\begin{aligned}
d\omega^1 &= \varphi \wedge \omega^1 + 3w \wedge \omega^1 + \omega \wedge \varphi^1 \\
d\omega^2 &= \varphi \wedge \omega^2 - 3w \wedge \omega^2 - \omega \wedge \varphi^2 \\
dw &= -\frac{1}{2}\varphi^2 \wedge \omega^1 + \frac{1}{2}\varphi^1 \wedge \omega^2 \\
d\varphi &= \omega \wedge \psi - \frac{1}{2}\varphi^2 \wedge \omega^1 - \frac{1}{2}\varphi^1 \wedge \omega^2 \\
d\varphi^1 &= \psi \wedge \omega^1 - \varphi \wedge \varphi^1 + 3w \wedge \varphi^1 \\
d\varphi^2 &= -\psi \wedge \omega^2 - \varphi \wedge \varphi^2 - 3w \wedge \varphi^2 \\
d\psi &= \varphi^1 \wedge \varphi^2 + 2\psi \wedge \varphi.
\end{aligned}
$$

2.1 The coframe bundle Y over the bundle E of contact forms

We recall the construction of the $\mathbb{R}^*$-bundle of contact forms. Define E to be the $\mathbb{R}^*$-bundle of all forms θ on TM such that $\ker\theta = T^1 \oplus T^2$. Remark that this bundle is trivial if and only if there exists a globally defined non-vanishing form θ. Define the set of forms θ^1 and θ^2 on M satisfying

$$
\theta^1(T^1) \neq 0 \text{ and } \theta^2(T^2) \neq 0,
$$

$$
\ker\theta^1_{|\ker\theta} = T^2 \text{ and } \ker\theta^2_{|\ker\theta} = T^1.
$$

Henceforth we fix one such choice, and all others are given by $\theta'^i = a^i\theta^i + v^i\theta$.

On E we define the tautological form ω. That is $\omega_\theta = \pi^*(\theta)$ where $\pi : E \to M$ is the natural projection. We also consider the tautological forms defined by the forms θ^1 and θ^2 over the line bundle E. That is, for each $\theta \in E$ we let $\omega^i_\theta = \pi^*(\theta^i)$. At each point $\theta \in E$ we have the family of forms defined on E:

$$
\omega' = \omega
$$

$$
\omega'^1 = a^1\omega^1 + v^1\omega
$$

$$
\omega'^2 = a^2\omega^2 + v^2\omega
$$

We may, moreover, suppose that

$$
d\theta = \theta^1 \wedge \theta^2 \text{ modulo } \theta
$$

and therefore

$$
d\omega = \omega^1 \wedge \omega^2 \text{ modulo } \omega.
$$

This imposes that $a^1a^2 = 1$.

Those forms vanish on vertical vectors, that is, vectors in the kernel of the map $TE \to TM$. In order to define non-horizontal 1-forms we let θ be a section of E over M and introduce the coordinate $\lambda \in \mathbb{R}^*$ in E. By abuse of notation, let θ denote the tautological form on the section θ. We write then the tautological form ω over E as

$$
\omega_{\lambda\theta} = \lambda\theta.
$$

Differentiating this formula we obtain

$$d\omega = 2\varphi \wedge \omega + \omega^1 \wedge \omega^2 \tag{1}$$

where $\varphi = \frac{d\lambda}{2\lambda}$ modulo $\omega, \omega^1, \omega^2$. Here $\frac{d\lambda}{2\lambda}$ is a form intrinsically defined on E up to horizontal forms. We obtain in this way a coframe bundle satisfying equation 1 over E. The coframes at each point of E are given by

$$\begin{aligned}
\omega' &= \omega \\
\omega'^1 &= a^1\omega^1 + v^1\omega \\
\omega'^2 &= a^2\omega^2 + v^2\omega \\
\varphi' &= \varphi - \frac{1}{2}a^1v^2\omega^1 + \frac{1}{2}a^2v^1\omega^2 + s\omega
\end{aligned}$$

$v^1, v^2, s \in \mathbb{R}$ and $a^1, a^2 \in \mathbb{R}^*$ such that $a^1a^2 = 1$.

Definition 2.2 *We denote by Y the coframe bundle $Y \to E$ given by the set of 1-forms $\omega, \omega^1, \omega^2, \varphi$ as above. Two coframes are related by*

$$(\omega', \omega'^1, \omega'^2, \varphi') = (\omega, \omega^1, \omega^2, \varphi) \begin{pmatrix} 1 & v^1 & v^2 & s \\ 0 & a^1 & 0 & -\frac{1}{2}a^1v^2 \\ 0 & 0 & a^2 & \frac{1}{2}a^2v^1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where and $s, v^1, v^2 \in \mathbb{R}$ and $a^1, a^2 \in \mathbb{R}^*$ satisfy $a^1a^2 = 1$.

The bundle Y can also be fibered over the manifold M. In order to describe the bundle Y as a principal fiber bundle over M observe that choosing a local section θ of E and forms θ^1 and θ^2 on M such that $d\theta = \theta^1 \wedge \theta^2$ one can write a trivialization of the fiber bundle as

$$\begin{aligned}
\omega &= \lambda\theta \\
\omega^1 &= a^1\theta^1 + v^1\lambda\theta \\
\omega^2 &= a^2\theta^2 + v^2\lambda\theta \\
\varphi &= \frac{d\lambda}{2\lambda} - \frac{1}{2}a^1v^2\theta^1 + \frac{1}{2}a^2v^1\theta^2 + s\theta,
\end{aligned}$$

where $v^1, v^2, s \in \mathbb{R}$ and $a^1, a^2 \in \mathbb{R}^*$ such that $a^1a^2 = \lambda$. Here the coframe $\omega, \omega^1, \omega^2, \varphi$ is seen as composed of tautological forms.

The group H acting on the right of this bundle is

$$H = \left\{ \begin{pmatrix} \lambda & v^1\lambda & v^2\lambda & s \\ 0 & a^1 & 0 & -\frac{1}{2}a^1v^2 \\ 0 & 0 & a^2 & \frac{1}{2}a^2v^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } s, v^1, v^2 \in \mathbb{R} \text{ and } a^1, a^2 \in \mathbb{R}^* \text{ satisfy } a^1a^2 = \lambda \right\}.$$

Consider the homomorphism from the Borel group $B \subset \mathbf{SL}(3, \mathbb{R})$ of upper triangular matrices with determinant one into H

$$j : B \to H$$

given by

$$\begin{pmatrix} a & c & e \\ 0 & \frac{1}{ab} & f \\ 0 & 0 & b \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{a}{b} & -a^2 f & \frac{c}{b} & -eb + \frac{1}{2}acf \\ 0 & a^2 b & 0 & -\frac{1}{2}abc \\ 0 & 0 & \frac{1}{ab^2} & -\frac{f}{2b} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One verifies that the homomorphism is surjective so that H is isomorphic to the Borel group of upper triangular matrices in $\mathbf{SL}(3,\mathbb{R})$. Therefore we have constructed a canonical fiber bundle with structure group the Borel group:

Proposition 2.3 *The bundle $Y \to M$ is a principal bundle with structure group H.*

2.2 The connection form on the bundle Y

Here we review the $\mathfrak{sl}(3,\mathbb{R})$-valued Cartan connection defined on the coframe bundle $Y \to E$ as described in [FV1, FV2].

Each point in the coframe bundle Y over E is lifted to a family of tautological forms on T^*Y. This family is then completed to obtain a coframe bundle over Y by an appropriate choice of conditions. As usual, the conditions are essentially curvature conditions and are obtained by differentiating the tautological forms and introducing new linearly independent forms satisfying certain canonical equations. We state the final existence theorem of the adapted Cartan connection:

Theorem 2.4 *There exists a unique $\mathfrak{sl}(3,\mathbb{R})$ valued connection form on the bundle Y*

$$\pi = \begin{pmatrix} \varphi + w & \varphi^2 & \psi \\ \omega^1 & -2w & \varphi^1 \\ \omega & \omega^2 & -\varphi + w \end{pmatrix},$$

whose curvature satisfies

$$\Pi = d\pi + \pi \wedge \pi = \begin{pmatrix} 0 & \Phi^2 & \Psi \\ 0 & 0 & \Phi^1 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\Phi^1 = Q^1 \omega \wedge \omega^2$, $\Phi^2 = Q^2 \omega \wedge \omega^1$ and $\Psi = (U_1 \omega^1 + U_2 \omega^2) \wedge \omega$, for functions Q^1, Q^2, U^1 and U^2 on Y.

Writing the components of the curvature form explicitly we obtain the following equations:

$$d\omega = 2\varphi \wedge \omega + \omega^1 \wedge \omega^2 \tag{2}$$

$$d\omega^1 = \varphi \wedge \omega^1 + 3w \wedge \omega^1 + \omega \wedge \varphi^1 \text{ and } d\omega^2 = \varphi \wedge \omega^2 - 3w \wedge \omega^2 - \omega \wedge \varphi^2 \tag{3}$$

$$d\varphi = \omega \wedge \psi - \frac{1}{2}(\varphi^2 \wedge \omega^1 + \varphi^1 \wedge \omega^2) \tag{4}$$

$$dw + \frac{1}{2}\omega^2 \wedge \varphi^1 - \frac{1}{2}\omega^1 \wedge \varphi^2 = 0, \tag{5}$$

$$\Phi^1 = d\varphi^1 + 3\varphi^1 \wedge w + \omega^1 \wedge \psi + \varphi \wedge \varphi^1 = Q^1 \omega \wedge \omega^2, \tag{6}$$

$$\Phi^2 = d\varphi^2 - 3\varphi^2 \wedge w - \omega^2 \wedge \psi + \varphi \wedge \varphi^2 = Q^2 \omega \wedge \omega^1, \tag{7}$$

$$\Psi := d\psi - \varphi^1 \wedge \varphi^2 + 2\varphi \wedge \psi = (U_1 \omega^1 + U_2 \omega^2) \wedge \omega. \tag{8}$$

where Q^1, Q^2, U^1 and U^2 are functions on Y.

2.3 Transformation properties of the connection

We will need to review the explicit transformation properties of the connection. We compute $Ad_{h^{-1}}\pi$ for an element

$$h = \begin{pmatrix} a & c & e \\ 0 & \frac{1}{ab} & f \\ 0 & 0 & b \end{pmatrix}$$

We have, for a constant h,

$$\tilde{\pi} = Ad_{h^{-1}}\pi.$$

A computation shows that

$$\begin{aligned}
\tilde{\omega} &= \frac{a}{b}\,\omega \\
\tilde{\omega}^1 &= a^2 b\,\omega^1 - a^2 f\,\omega \\
\tilde{\omega}^2 &= \frac{1}{ab^2}\,\omega^2 + \frac{c}{b}\,\omega \\
\tilde{\varphi} &= \varphi - \frac{1}{2}abc\,\omega^1 - \frac{f}{2b}\,\omega^2 + (\frac{1}{2}acf - \frac{e}{b})\,\omega \\
\tilde{w} &= w - \frac{1}{2}abc\,\omega^1 + \frac{f}{2b}\,\omega^2 + \frac{1}{2}acf\,\omega \\
\tilde{\varphi}^1 &= b^2 a\,\varphi^1 - 3abf\,w + baf\,\varphi + bae\,\omega^1 - f^2 a\,\omega^2 - fae\,\omega \\
\tilde{\varphi}^2 &= \frac{1}{ba^2}\,\varphi^2 + \frac{3c}{a}\,w + \frac{c}{a}\,\varphi - bc^2\,\omega^1 + (-\frac{e}{a^2b^2} + \frac{cf}{ab})\,\omega^2 + (-\frac{ce}{ab} + fc^2)\,\omega \\
\tilde{\psi} &= \frac{b}{a}\,\psi + (\frac{2e}{a} - bcf)\,\varphi - bce\,\omega^1 + (-\frac{fe}{ab} + cf^2)\,\omega^2 - cb^2\,\varphi^1 + \frac{f}{a}\,\varphi^2 + 3fbc\,w + (-\frac{e^2}{ab} + fce)\,\omega
\end{aligned} \tag{9}$$

and for the curvature

$$\Pi = d\pi + \pi \wedge \pi = \begin{pmatrix} 0 & \Phi^2 & \Psi \\ 0 & 0 & \Phi^1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\Pi} = Ad_{h^{-1}}\Pi = \begin{pmatrix} 0 & \frac{1}{a^2b}\Phi^2 & \frac{b}{a}\Psi + \frac{f}{a}\Phi^2 - cb^2\Phi^1 \\ 0 & 0 & ab^2\Phi^1 \\ 0 & 0 & 0 \end{pmatrix}.$$

That is,

$$\tilde{\Phi}^1 = ab^2\Phi^1,$$

$$\tilde{\Phi}^2 = \frac{1}{a^2 b}\Phi^2,$$

and

$$\tilde{\Psi} = \frac{b}{a}\Psi + \frac{f}{a}\Phi^2 - cb^2\Phi^1.$$

Recalling that $\Phi^1 = Q^1\omega \wedge \omega^2$, $\Phi^2 = Q^2\omega \wedge \omega^1$ and $\Psi = (U_1\omega^1 + U_2\omega^2) \wedge \omega$, one may write therefore

$$\tilde{Q}^1\tilde{\omega} \wedge \tilde{\omega}^2 = ab^2\, Q^1\omega \wedge \omega^2.$$

But $\tilde{Q}^1\tilde{\omega} \wedge \tilde{\omega}^2 = \tilde{Q}^1\frac{a}{b}\,\omega \wedge \frac{1}{ab^2}\omega^2$ and then

$$\tilde{Q}^1 = ab^5\, Q^1, \tag{10}$$

$$\tilde{Q}^2 = \frac{1}{a^5 b}\, Q^2 \tag{11}$$

and, analogously,

$$\tilde{U}_1 = \frac{b}{a^4}(U_1 - \frac{f}{b}Q^2), \tag{12}$$

$$\tilde{U}_2 = \frac{b^4}{a}(U_2 + abcQ^1). \tag{13}$$

These transformation properties imply that we can define two tensors on Y which are invariant under H and will give rise to two tensors on M. Indeed

$$Q^1\,\omega^2 \wedge \omega \otimes \omega \otimes e_1$$

and

$$Q^2\,\omega^1 \wedge \omega \otimes \omega \otimes e_2,$$

where e_1 and e_2 are duals to ω^1 and ω^2 in the dual frame of the coframe bundle of Y, are easily seen to be H-invariant.

2.4 Bianchi identities

In this section we compute Bianchi identities. They are essential to obtain relations between the curvatures and its derivatives and will be heavily used in the reductions of the path structures.

2.4.1

Equation $d(d\varphi^1) = 0$ obtained differentiating Φ^1 (equation 6) implies

$$dQ^1 - 6Q^1 w + 4Q^1\varphi = S^1\omega + U_2\omega^1 + T^1\omega^2, \tag{14}$$

where we introduced functions S^1 and T^1.

2.4.2

Anagously, equation $d(d\varphi^2) = 0$ obtained differentiating Φ^2 (equation 7) implies

$$dQ^2 + 6Q^2 w + 4Q^2 \varphi = S^2 \omega - U_1 \omega^2 + T^2 \omega^1, \tag{15}$$

where we introduced new functions S^2 and T^2.

2.4.3

Equation $d(d\psi) = 0$ obtained from equation 8 implies

$$dU_1 + 5U_1 \varphi + 3U_1 w + Q^2 \varphi^1 = A\omega + B\omega^1 + C\omega^2 \tag{16}$$

and

$$dU_2 + 5U_2 \varphi - 3U_2 w - Q^1 \varphi^2 = D\omega + C\omega^1 + E\omega^2. \tag{17}$$

2.4.4 Higher order Bianchi identities

If we derive equation 14 and replace the known terms we get

$$dS^1 + 6S^1 \varphi - 6S^1 w - U_2 \varphi^1 + T^1 \varphi^2 + 4Q^1 \psi = X_0 \omega + D\omega^1 + X_2 \omega^2 \tag{18}$$

$$dT^1 + 5T^1 \varphi - 9T^1 w + 5Q^1 \varphi^1 = X_2 \omega - (S^1 - E)\omega^1 + Y_2 \omega^2 \tag{19}$$

In the same way, if we derive equation 15 and replace the known terms we get

$$dS^2 + 6S^2 \varphi + 6S^2 w - T^2 \varphi^1 - U_1 \varphi^2 + 4Q^2 \psi = Y_0 \omega + Y_1 \omega^1 - A\omega^2 \tag{20}$$

$$dT^2 + 5T^2 \varphi + 9T^2 w + 5Q^2 \varphi^2 = Y_1 \omega + X_1 \omega^1 + (S^2 - B)\omega^2 \tag{21}$$

If we differentiate equation 21, and use equations 20, 21 and 15 we get

$$\begin{aligned}
&\omega \wedge (dY_1 + 7Y_1 \varphi + 9Y_1 w - X_1 \varphi^1 + (6S^2 - B)\varphi^2 + 5T^2 \psi + 5(Q^2)^2 \omega^1 - Y_0 \omega^2) + \\
&\omega^1 \wedge (dX_1 + 6X_1 \varphi + 12X_1 w + 12T^2 \varphi^2) - \\
&\omega^2 \wedge (dB + 6B\varphi + 6Bw + T^2 \varphi^1 + 4U_1 \varphi^2 - Q^2 \psi - 2Y_1 \omega^1) = 0,
\end{aligned} \tag{22}$$

and from this we get

$$dX_1 + 6X_1 \varphi + 12X_1 w + 12T^2 \varphi^2 = X_{10} \omega + X_{11} \omega^1 + X_{12} \omega^2 \tag{23}$$

$$dY_1 + 7Y_1 \varphi + 9Y_1 w - X_1 \varphi^1 + (6S^2 - B)\varphi^2 + 5T^2 \psi + 5(Q^2)^2 \omega^1 - Y_0 \omega^2 = Y_{10} \omega + X_{10} \omega^1 + Y_{12} \omega^2 \tag{24}$$

3 Strict path structures

In this section we recall the definition of strict path structures (see [FMMV] and [FV1]). They correspond to path structures with a fixed contact form.

G_1 denotes from now on the subgroup of $\mathbf{SL}(3,\mathbb{R})$ defined by

$$G_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ x & \frac{1}{a^2} & 0 \\ z & y & a \end{pmatrix} \mid a \in \mathbb{R}^*, (x,y,z) \in \mathbb{R}^3 \right\}$$

and $P_1 \subset G_1$ the subgroup defined by

$$P_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix} \right\}.$$

Writing the Maurer-Cartan form of G_1 as the matrix

$$\begin{pmatrix} w & 0 & 0 \\ \theta^1 & -2w & 0 \\ \theta & \theta^2 & w \end{pmatrix}$$

one obtains the Maurer-Cartan equations:

$$d\theta + \theta^2 \wedge \theta^1 = 0$$

$$d\theta^1 - 3w \wedge \theta^1 = 0$$

$$d\theta^2 + 3w \wedge \theta^2 = 0$$

$$dw = 0.$$

Let M be a three-manifold equipped with a path structure $D = E^1 \oplus E^2$. Fixing a contact form $\boldsymbol{\theta}$ such that $\ker \boldsymbol{\theta} = D$ defines a strict path structure. Let $\boldsymbol{R}$ be the Reeb vector field associated to $\boldsymbol{\theta}$. That is $\boldsymbol{\theta}(\boldsymbol{R}) = 1$ and $d\boldsymbol{\theta}(\boldsymbol{R}, \cdot) = 0$. Let $\boldsymbol{X}_1 \in E^1$, $\boldsymbol{X}_2 \in E^2$ be such that $d\theta(\boldsymbol{X}_1, \boldsymbol{X}_2) = 1$. The dual coframe of $(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{R})$ is $(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \boldsymbol{\theta})$, for two 1-forms $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ verifying $d\boldsymbol{\theta} = \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2$.

At any point $x \in M$, one can look at the coframes of the form

$$\theta^1 = a^3 \boldsymbol{\theta}^1(x),\ \theta^2 = \frac{1}{a^3} \boldsymbol{\theta}^2(x),\ \theta = \boldsymbol{\theta}(x)$$

for $a \in \mathbb{R}^*$.

Definition 3.1 *We denote by $\pi : Y^1 \to M$ the $\mathbb{R}^*$-coframe bundle over M given by the set of coframes $(\theta, \theta^1, \theta^2)$ of the above form.*

We will denote the tautological forms defined by $\theta^1, \theta^2, \theta$ using the same letters. That is, we write θ^i at the coframe $(\theta^1, \theta^2, \theta)$ to be $\pi^*(\theta^i)$.

Proposition 3.2 *There exists a unique $\mathfrak{g}_1$-valued Cartan connection on Y^1*

$$\varpi = \begin{pmatrix} \upsilon & 0 & 0 \\ \theta^1 & -2\upsilon & 0 \\ \theta & \theta^2 & \upsilon \end{pmatrix}$$

such that its curvature form is of the form

$$\varpi = \begin{pmatrix} d\upsilon & 0 & 0 \\ \theta\wedge\tau^1 & -2d\upsilon & 0 \\ 0 & \theta\wedge\tau^2 & d\upsilon \end{pmatrix}$$

with $\tau^1\wedge\theta^2 = \tau^2\wedge\theta^1 = 0$.

Observe that the condition $\tau^1\wedge\theta^2 = \tau^2\wedge\theta^1 = 0$ implies that we may write $\tau^1 = \tau^1_2\theta^2$ and $\tau^2 = \tau^2_1\theta^1$. The structure equations are

$$\begin{array}{lcl} d\theta^1 - 3\upsilon\wedge\theta^1 & = & \theta\wedge\tau^1, \\ d\theta^2 + 3\upsilon\wedge\theta^2 & = & \theta\wedge\tau^2, \\ d\theta & = & \theta^1\wedge\theta^2. \end{array} \tag{25}$$

A choice of coframe $(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \boldsymbol{\theta})$ on a strict path structure defines an embedding

$$j : M \to Y^1$$

and therefore

$$\begin{array}{lcl} d\boldsymbol{\theta}^1 - 3j^*\upsilon\wedge\boldsymbol{\theta}^1 & = & \boldsymbol{\theta}\wedge j^*\tau^1, \\ d\boldsymbol{\theta}^2 + 3j^*\upsilon\wedge\boldsymbol{\theta}^2 & = & \boldsymbol{\theta}\wedge j^*\tau^2, \\ d\boldsymbol{\theta} & = & \boldsymbol{\theta}^1\wedge\boldsymbol{\theta}^2. \end{array} \tag{26}$$

3.1 Bianchi identities

In what follows, the equations should be understood as definitions for the coefficients appearing in the right hand terms. Bianchi identities give the following equations:

$$d\upsilon = R\theta^1\wedge\theta^2 + W^1\theta^1\wedge\theta + W^2\theta^2\wedge\theta \tag{27}$$

$$d\tau^1 + 3\tau^1\wedge\upsilon = 3W^2\theta^1\wedge\theta^2 + S^1_1\theta\wedge\theta^1 + S^1_2\theta\wedge\theta^2 \tag{28}$$

$$d\tau^2 - 3\tau^2\wedge\upsilon = 3W^1\theta^1\wedge\theta^2 + S^2_1\theta\wedge\theta^1 + S^2_2\theta\wedge\theta^2 \tag{29}$$

Moreover, we have the relation

$$S^1_1 = S^2_2 = \tau^1_2\tau^2_1.$$

From equation $dd\upsilon = 0$ one obtains that

$$dR = R_0\theta + R_1\theta^1 + R_2\theta^2,$$

$$dW^1 + 3W^1\upsilon = W^1_0\theta + W^1_1\theta^1 + W^1_2\theta^2 \tag{30}$$

and

$$dW^2 - 3W^2\upsilon = W^2_0\theta + W^2_1\theta^1 + W^2_2\theta^2 \tag{31}$$

with

$$R_0 = W^1_2 - W^2_1.$$

Also, writing $dR_0 = R_{00}\theta + R_{01}\theta^1 + R_{02}\theta^2$, one gets

$$dR_1 + 3R_1\upsilon + R_2\tau_1^2\theta - \frac{1}{2}R_0\theta^2 = R_{01}\theta + R_{11}\theta^1 + R_{12}\theta^2 \tag{32}$$

and

$$dR_2 - 3R_2\upsilon + R_1\tau_2^1\theta + \frac{1}{2}R_0\theta^1 = R_{02}\theta + R_{12}\theta^1 + R_{22}\theta^2. \tag{33}$$

We have moreover

$$d\tau_2^1 - 6\tau_2^1\upsilon = 3W^2\theta^1 + S_2^1\theta \mod \theta^2$$

and

$$d\tau_1^2 + 6\tau_1^2\upsilon = -3W^1\theta^2 + S_1^2\theta \mod \theta^1.$$

3.2 The embedding $\iota : Y^1 \to Y$

We recall here the embedding described in [FV1, FV2]. This embedding allows curvatures of path structures to be expressed in terms of curvatures of strict path structures which, in turn, are much easier to compute.

Proposition 3.3 (cf. [FV1]) *There exists a unique equivariant embedding of fiber bundles $\iota : Y^1 \to Y$ such that*

$$\iota^*\omega = \theta, \quad \iota^*\omega^1 = \theta^1, \quad \iota^*\omega^2 = \theta^2, \quad \iota^*\varphi = 0. \tag{34}$$

Moreover

$$\iota^*Q^1 = S_2^1 + \frac{3}{2}R\tau_2^1 + 2W_2^2 - \frac{1}{2}R_{22}$$

and

$$\iota^*Q^2 = -S_1^2 + \frac{3}{2}R\tau_1^2 - 2W_1^1 - \frac{1}{2}R_{11}.$$

Proof. In order to express the curvatures of the path structures, we first compute the pull-back of the connection of the path structure in terms of the connection of the strict structure. The pull back of the structure equations are (here we denote the pull-back of a form by the same letter except for the forms $\omega, \omega^1, \omega^2$ which are given, from the above proposition, as $\theta, \theta^1, \theta^2$):

$$\begin{aligned}
d\theta &= \theta^1 \wedge \theta^2 \\
d\theta^1 &= 3w \wedge \theta^1 + \theta \wedge \varphi^1 \\
d\theta^2 &= -3w \wedge \theta^2 - \theta \wedge \varphi^2 \\
\theta \wedge \psi &= \frac{1}{2}(\varphi^2 \wedge \theta^1 + \varphi^1 \wedge \theta^2) \\
dw &= \frac{1}{2}(-\varphi^2 \wedge \theta^1 + \varphi^1 \wedge \theta^2) \\
Q^1\theta \wedge \theta^2 &= d\varphi^1 + 3\varphi^1 \wedge w + \theta^1 \wedge \psi \\
Q^2\theta \wedge \theta^1 &= d\varphi^2 - 3\varphi^2 \wedge w - \theta^2 \wedge \psi \\
d\psi - \varphi^1 \wedge \varphi^2 &= (U_1\theta^1 + U_2\theta^2) \wedge \theta.
\end{aligned}$$

It follows, by comparing with Proposition 3.2, that

$$\begin{aligned} w &= \upsilon + c\theta \\ \varphi^1 &= \tau^1 - 3c\theta^1 + E^1\theta \\ \varphi^2 &= -\tau^2 - 3c\theta^2 + E^2\theta. \end{aligned}$$

where E^1, E^2 and c are functions on Y^1. We obtain then that

$$\psi = \frac{1}{2}(E^2\theta^1 + E^1\theta^2 + G\theta)$$

where G is a function on Y^1. Substituting the formulas for φ^1, φ^2 and w in the equation for dw we obtain, using equation 27

$$dw = R\theta^1 \wedge \theta^2 + W^1\theta^1 \wedge \theta + W^2\theta^2 \wedge \theta + dc \wedge \theta + cd\theta = \frac{1}{2}(-6c\theta^1 \wedge \theta^2 - E^2\theta \wedge \theta^1 + E^1\theta \wedge \theta^2),$$

which implies, comparing the terms in $\theta^1 \wedge \theta^2$,

$$c = -\frac{R}{4},$$

and then, using $dR = R_0\theta + R_1\theta^1 + R_2\theta^2$, we obtain

$$E^1 = -2W^2 + \frac{R_2}{2}, \quad E^2 = 2W^1 - \frac{R_1}{2}.$$

- Substituting the formulas for φ^1, ψ and w, with the above values of E^1 and c, in the equation $Q^1\theta \wedge \theta^2 = d\varphi^1 + 3\varphi^1 \wedge w + \theta^1 \wedge \psi$, we obtain

$$Q^1\theta \wedge \theta^2 = d(\tau^1 - 3c\theta^1 + E^1\theta) + 3(\tau^1 - 3c\theta^1 + E^1\theta) \wedge (\upsilon + c\theta) + \theta^1 \wedge \frac{1}{2}(E^1\theta^2 + G\theta). \quad (35)$$

Compute, using equations 31 and 33

$$dE^1 \wedge \theta = d(-2W^2 + \frac{R_2}{2}) \wedge \theta = -2(3W^2\upsilon + W^2_1\theta^1 + W^2_2\theta^2) \wedge \theta + \frac{1}{2}(3R_2\upsilon - \frac{1}{2}R_0\theta^1 + R_{12}\theta^1 + R_{22}\theta^2) \wedge \theta,$$

and

$$-3d(c\theta^1) = -3dc \wedge \theta^1 - 3cd\theta^1 = \frac{3}{4}dR \wedge \theta^1 + \frac{3}{4}R(-3\theta^1 \wedge \upsilon + \theta \wedge \tau^1)$$

Recall also equation 28

$$d\tau^1 + 3\tau^1 \wedge \upsilon = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2.$$

Substituting the above expressions into equation 35 we obtain

$$\begin{aligned} Q^1\theta \wedge \theta^2 = & -3\tau^1 \wedge \upsilon + 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2 + \frac{3}{4}dR \wedge \theta^1 + \frac{3}{4}R(-3\theta^1 \wedge \upsilon + \theta \wedge \tau^1) \\ & - 2(3W^2\upsilon + W^2_1\theta^1 + W^2_2\theta^2) \wedge \theta + \frac{1}{2}(3R_2\upsilon - \frac{1}{2}R_0\theta^1 + R_{12}\theta^1 + R_{22}\theta^2) \wedge \theta + E^1\theta^1 \wedge \theta^2 \\ & + 3(\tau^1 - 3c\theta^1 + E^1\theta) \wedge (\upsilon + c\theta) + \theta^1 \wedge \frac{1}{2}(E^1\theta^2 + G\theta), \end{aligned}$$

which, using $3E^1\theta\wedge\upsilon = (-6W^2+\frac{3R_2}{2})\theta\wedge\upsilon$, simplifies to an expression with no terms involving υ:

$$
\begin{aligned}
Q^1\theta\wedge\theta^2 = {} & 3W^2\theta^1\wedge\theta^2 + S^1_1\theta\wedge\theta^1 + S^1_2\theta\wedge\theta^2 + \frac{3}{4}dR\wedge\theta^1 + \frac{3}{4}R\theta\wedge\tau^1 \\
& - 2(W^2_1\theta^1 + W^2_2\theta^2)\wedge\theta + \frac{1}{2}(-\frac{1}{2}R_0\theta^1 + R_{12}\theta^1 + R_{22}\theta^2)\wedge\theta + E^1\theta^1\wedge\theta^2 \\
& + 3(\tau^1 - 3c\theta^1)\wedge c\theta + \theta^1\wedge\frac{1}{2}(E^1\theta^2 + G\theta).
\end{aligned}
$$

Observe now that the coefficient of $\theta^1\wedge\theta^2$ is

$$3W^2 - \frac{3}{4}R_2 + \frac{3}{2}E^1 = 0.$$

Therefore we may simplify to

$$
\begin{aligned}
Q^1\theta\wedge\theta^2 = {} & (S^1_1 + R_0 + 2W^2_1 - \frac{1}{2}R_{12} + \frac{9}{16}R^2 - \frac{1}{2}G)\theta\wedge\theta^1 \\
& + (S^1_2 + \frac{3}{2}R\tau^1_2 + 2W^2_2 - \frac{1}{2}R_{22})\theta\wedge\theta^2
\end{aligned}
$$

and conclude that

$$Q^1 = S^1_2 + \frac{3}{2}R\tau^1_2 + 2W^2_2 - \frac{1}{2}R_{22} \tag{36}$$

- Analogously, substituting the formulas for φ^1, ψ and w, with the above values of E^1 and c, in the equation $Q^2\theta\wedge\theta^1 = d\varphi^2 - 3\varphi^2\wedge w - \theta^2\wedge\psi$ we obtain

$$Q^2\theta\wedge\theta^1 = d(-\tau^2 - 3c\theta^2 + E^2\theta) - 3(-\tau^2 - 3c\theta^2 + E^2\theta)\wedge(\upsilon + c\theta) - \theta^2\wedge\frac{1}{2}(E^2\theta^1 + G\theta). \tag{37}$$

Following the same computations we conclude with the formula

$$Q^2 = -S^2_1 + \frac{3}{2}R\tau^2_1 - 2W^1_1 - \frac{1}{2}R_{11}. \tag{38}$$

□

4 Reductions

In this section we prove the main theorem concerning canonical reductions of a path structure. The motivation behind the reductions is a gap theorem on the possible dimensions of the automorphism group. Indeed, the group of transformations preserving a path structure which is not flat has dimension at most three:

Theorem 4.1 (M. A. Tresse [T]) *The group of transformations of the fiber bundle Y has dimension eight (in the flat case) or at most dimension three.*

A modern proof is contained in M. Mion-Mouton thesis [MM] and a more general result is obtained in [KT].

We will describe in this section reductions of the fiber bundle Y to a parallelism or a $\mathbb{Z}/2\mathbb{Z}$-bundle over the manifold in the case the path structure is not flat. This clearly implies Tresse's theorem.

Theorem 4.2 *If the path structure is not flat, there exists a canonical reduction of the fiber bundle Y to a $\mathbb{Z}/2\mathbb{Z}$-structure. Moreover,*

1. *if $Q^1 \neq 0$ and $Q^2 \neq 0$ and $T_1 \neq 0$ or $T_2 \neq 0$ there exists a further reduction to a parallelism.*
2. *if $Q^1 = 0$ and $Q^2 \neq 0$ and $Y_1 \neq 0$ there exists a further reduction to a parallelism.*

Here T_1, T_2 and Y_1 are the functions on the bundle Y introduced in equations 14, 15 and 20.

The theorem is a consequence of propositions 4.3, 4.4, 4.5, 4.6 and 4.7, where details of the reductions are given. Note that the case $Q^1 \neq 0$ and $Q^2 = 0$ differs from the second case in the theorem by an ordering of the decomposition of the plane field and the result is analogous.

4.1 Reduction of the structure: $Q^1 \neq 0$ and $Q^2 \neq 0$.

Suppose there exists a section of the coframe bundle Y with $Q^1 \neq 0$ and $Q^2 \neq 0$. From equations 10, $ab^5 Q_1 = \tilde{Q}_1$, $a^5 b \tilde{Q}_2 = Q_2$, we can solve for a, b such that $\tilde{Q}_1 = 1$ and $\tilde{Q}_2 = \epsilon$, where $Q_1 Q_2 = \epsilon |Q_1 Q_2|$. Observe that if we only consider coframes on Y satisfying $Q_1 = 1$, $Q_2 = \epsilon$, then $a = b = \pm 1$.

From transformation properties 12:

$$\tilde{U}_1 = \frac{b}{a^4}(U_1 - \frac{f}{b} Q^2) \tag{39}$$

and

$$\tilde{U}_2 = \frac{b^4}{a}(U_2 + abcQ^1),$$

we can choose $c = -\frac{U_2}{abQ^1}$ and $f = \frac{bU_1}{Q^2}$ such that $\tilde{U}_1 = \tilde{U}_2 = 0$. Then, from equations 14 and 15, that is

$$d\tilde{Q}^1 - 6\tilde{Q}^1 \tilde{w} + 4\tilde{Q}^1 \tilde{\varphi} = \tilde{S}^1 \tilde{\omega} + \tilde{U}_2 \tilde{\omega}^1 + \tilde{T}^1 \tilde{\omega}^2$$

and

$$d\tilde{Q}^2 + 6\tilde{Q}^2 \tilde{w} + \tilde{4} Q^2 \tilde{\varphi} = \tilde{S}^2 \tilde{\omega} - \tilde{U}_1 \tilde{\omega}^2 + \tilde{T}^2 \tilde{\omega}^1,$$

we get

$$4\tilde{\varphi} - 6\tilde{w} = \tilde{S}^1 \tilde{\omega} + \tilde{T}^1 \tilde{\omega}^2, \tag{40}$$

$$\epsilon(4\tilde{\varphi} + 6\tilde{w}) = \tilde{S}^2 \tilde{\omega} + \tilde{T}^2 \tilde{\omega}^1. \tag{41}$$

Consider now only the coframes on Y satisfying $Q_1 = 1$, $Q_2 = \epsilon$ and $U_1 = U_2 = 0$. Using formulas 9 with $a = b = \pm 1, c = f = 0$ (these parameters are fixed by the previous conditions) we obtain that $\tilde{w} = w$ and $\tilde{\varphi} = \varphi - e\omega$. Therefore

$$4(\varphi - e\omega) - 6w = \tilde{S}^1\omega + \tilde{T}^1\omega^2$$

and

$$4(\varphi - e\omega) + 6w = \tilde{S}^2\omega + \tilde{T}^2\omega^1.$$

From these equations we observe that φ and w are combinations of ω, ω^1 and ω^2. We also obtain that

$$\tilde{S}^1 = S^1 + 4e$$

and

$$\tilde{S}^2 = S^2 + \epsilon 4e.$$

Proposition 4.3 *Suppose there is a section of Y such that $Q^1 \neq 0$ and $Q^2 \neq 0$. Then there exists a unique $\mathbb{Z}/2\mathbb{Z}$-bundle $Y^{red} \subset Y$ such that on Y^{red}, $Q^1 = 1$, $Q^2 = \epsilon$, $U_1 = U_2 = 0$ and $S^1 + \epsilon S^2 = 0$, where $Q^1Q^2 = \epsilon|Q^1Q^2|$. Moreover, if $T_1 \neq 0$ or $T_2 \neq 0$ there is a further reduction to a parallelism.*

Proof. Observe that $\tilde{S}^1 + \epsilon\tilde{S}^2 = S^1 + \epsilon S^2 + 8e$ and choose the unique e satisfying $\tilde{S}^1 + \epsilon\tilde{S}^2 = 0$. This defines the reduction to a $\mathbb{Z}/2\mathbb{Z}$-bundle $Y^{red} \subset Y$.

From 40 and 41 we get on Y^{red}

$$\varphi = \frac{1}{8}(T^1\omega^2 + \epsilon T^2\omega^1)$$

and

$$w = \frac{1}{12}((\epsilon S^2 - S^1)\omega + \epsilon T^2\omega^1 - T^1\omega^2)$$

Since both forms are invariant on Y^{red}, fixing the sign of T^1 or T^2 determines one of the two coframes. That is, either $(\omega^1, \omega^2, \omega)$ or $(-\omega^1, -\omega^2, \omega)$.

□

The structure equations of a parallelism can be written as follows. From equations 16 and 17 we get on Y^{red}

$$\varphi^1 = \epsilon(A\omega + B\omega^1 + C\omega^2)$$

and

$$\varphi^2 = -(D\omega + C\omega^1 + E\omega^2).$$

Therefore the structure equations are

$$d\omega = \frac{1}{4}(T^1\omega^2 + \epsilon T^2\omega^1) \wedge \omega + \omega^1 \wedge \omega^2, \tag{42}$$

$$d\omega^1 = \frac{1}{8}T^1\omega^1 \wedge \omega^2 + (\frac{1}{4}(\epsilon S^2 - S^1) + \epsilon B)\omega \wedge \omega^1 + \epsilon C\omega \wedge \omega^2 \tag{43}$$

$$d\omega^2 = -\epsilon\frac{1}{8}T^2\omega^1\wedge\omega^2 + C\omega\wedge\omega^1 + (-\frac{1}{4}(\epsilon S^2 - S^1) + E)\omega\wedge\omega^2. \tag{44}$$

As a final observation, other possible reductions to a $\mathbb{Z}^2$-bundle could be made. For instance, imposing the condition $S^1 = 0$ defines such a reduction. In any case, the theorem shows that in the case both Q^1 and Q^2 are non-vanishing, the dimension of the automorphism group is at most three.

4.2 Reduction of the structure: $Q^1 = 0$ and $Q^2 \neq 0$.

Suppose there exists a section of the coframe bundle Y with $Q^1 = 0$ and $Q^2 \neq 0$. From equation 14,

$$dQ^1 - 6Q^1w + 4Q^1\varphi = S^1\omega + U_2\omega^1 + T^1\omega^2,$$

we obtain that $S^1 = U_2 = T^1 = 0$ on Y. Also, from equation 17 we obtain $D = C = E = 0$, and from 18 and 19, $X_0 = X_2 = Y_2 = 0$.

Using the transformation $\tilde{Q}^2 = \frac{1}{a^5b}Q^2$ one can chose an arbitrary function a and then the function b such that $\frac{1}{a^5b}Q^2 = 1$ is determined. One considers now the subbundle with coframes satisfying $Q^2 = 1$. Its structure group satisfies then $a^5 = \frac{1}{b}$. From equation 39, as in the previous section,

$$\tilde{U}_1 = \frac{b}{a^4}(U_1 - \frac{f}{b}Q^2) = \frac{b}{a^4}(U_1 - fa^5), \tag{45}$$

and one can choose f (depending on a) such that $U_1 = 0$.

The structure group of this reduction consists of matrices of the form

$$\begin{pmatrix} a & c & e \\ 0 & a^4 & 0 \\ 0 & 0 & \frac{1}{a^5} \end{pmatrix}$$

Compute now the transformation by a section of this structure group of the pull-back of the connection forms to this subbundle. We have $R_h^*(\pi) = h^{-1}dh + h^{-1}\pi h$. The computation of $h^{-1}\pi h$ is given above in formulae 9. We also compute

$$h^{-1}dh = \begin{pmatrix} a^{-1}da & \star & \star \\ 0 & -a^{-1}da - b^{-1}db & \star \\ 0 & 0 & b^{-1}db \end{pmatrix}. \tag{46}$$

Therefore, taking into account that $b = \frac{1}{a^5}$ (so $b^{-1}db = -5\frac{da}{a}$) and $f = 0$ in 9,

$$\tilde{w} = R_h^*(w) = \frac{1}{2}(a^{-1}da + b^{-1}db) + w - \frac{1}{2}abc\,\omega^1 = -2a^{-1}da + w - \frac{1}{2}ca^{-4}\,\omega^1$$

and

$$\tilde{\varphi} = R_h^*(\varphi) = \frac{1}{2}(a^{-1}da - b^{-1}db) + \varphi - \frac{1}{2}abc\,\omega^1 - \frac{e}{b}\,\omega = 3a^{-1}da + \varphi - \frac{1}{2}ca^{-4}\,\omega^1 - ea^5\,\omega.$$

Observe now that

$$6\tilde{w}+4\tilde{\varphi}=6w+4\varphi-5ca^{-4}\,\omega^1-ea^5\,\omega$$

Replacing equation 15: $dQ^2+6Q^2w+4Q^2\varphi=S^2\omega-U_1\omega^2+T^2\omega^1$, (imposing $Q^2=1$ and $U_1=0$) on the last equation we obtain

$$6\tilde{w}+4\tilde{\varphi}=(T^2-5ca^{-4})\,\omega^1+(S^2-ea^5)\,\omega.$$

We may now choose c and e so that

$$6\tilde{w}+4\tilde{\varphi}=0.$$

That is, $\tilde{T}^2=\tilde{S}^2=0$.

The coframes with $Q^2=1$ and $6w+4\varphi=0$ form a subbundle Y^1 with group

$$H_1=\left\{\begin{pmatrix} a & 0 & 0\\ 0 & a^4 & 0\\ 0 & 0 & a^{-5}\end{pmatrix}\mid a\neq 0\right\}\subset H.$$

From equation 16 we obtain that on Y^1

$$\varphi^1=A\omega+B\omega^1. \tag{47}$$

From equation 20 we obtain that on Y^1

$$4\psi=Y_0\omega+Y_1\omega^1-A\omega^2. \tag{48}$$

Also, from equation 21 we obtain on Y^1

$$5\varphi^2=Y_1\omega+X_1\omega^1-B\omega^2. \tag{49}$$

We showed

Proposition 4.4 *There exists a unique $\mathbb{R}^*$-bundle $Y^1\subset Y$ such that on Y^1, $Q^1=0$, $Q^2=1$, $U_1=0$ and $3w=-2\varphi$. Moreover on Y^1 we have $C=D=E=S^1=S^2=T^1=T^2=U_2=0$ and $X_2=Y_2=0$, also $\varphi^1=A\omega+B\omega^1$, $5\varphi^2=Y_1\omega+X_1\omega^1-B\omega^2$ and $4\psi=Y_0\omega+Y_1\omega^1-A\omega^2$.*

4.2.1 Further reductions of the structure: $Q^1=0$ and $Q^2\neq 0$.

From equations 9, taking into account Proposition 4.4, we obtain on Y^1 the transformations

$$\begin{aligned}\tilde{\omega}&=a^6\,\omega\\ \tilde{\omega}^1&=a^{-3}\,\omega^1\\ \tilde{\omega}^2&=a^9\,\omega^2\\ \tilde{\varphi}^1&=b^2a\,\varphi^1=a^{-9}\,\varphi^1\\ \tilde{\varphi}^2&=\frac{1}{ba^2}\,\varphi^2=a^3\,\varphi^2\\ \tilde{\psi}&=\frac{b}{a}\,\psi=\frac{1}{a^6}\,\psi.\end{aligned} \tag{50}$$

Therefore, from Proposition 4.4, we compute the transformations of X_1, Y_0, Y_1, A and B: For instance, from $\tilde{\varphi}^2 = a^3\,\varphi^2$ we have

$$\tilde{Y}_1 a^6\omega + \tilde{X}_1 a^{-3}\omega^1 - \tilde{B}a^9\omega^2 = 5\tilde{\varphi}^2 = a^3\,5\varphi^2 = a^3(Y_1\omega + X_1\omega^1 - B\omega^2)$$

and comparing the terms of this equality we obtain

$$\begin{aligned} \tilde{X}_1 &= a^6 X_1 \\ \tilde{Y}_1 &= a^{-3} Y_1 \\ \tilde{B} &= a^{-6} B. \end{aligned} \tag{51}$$

Analogously, we also obtain, from $\tilde{\psi} = a^{-6}\,\psi$

$$\tilde{Y}_0 a^6\,\omega + \tilde{Y}_1 a^{-3}\,\omega^1 - \tilde{A}a^9\,\omega^2 = 4\tilde{\psi} = 4\frac{1}{a^6}\,\psi = \frac{1}{a^6}(Y_0\omega + Y_1\omega^1 - A\omega^2)$$

and therefore

$$\begin{aligned} \tilde{Y}_0 &= a^{-12} Y_0 \\ \tilde{A} &= a^{-15} A. \end{aligned} \tag{52}$$

Remark that if any of the coefficients X_1, Y_0, Y_1, A or B is non zero one could reduce the bundle by fixing the function a (only up to a sign if $Y_1 = 0$ and $A = 0$) so that the coefficient be constant equal to one. Now we compute equation 22, taking into account Proposition 4.4, and supposing that $X_1 = Y_0 = Y_1 = A = 0$, we obtain

$$5\omega \wedge \omega^1 = 0,$$

which is a contradiction. We obtained therefore the following

Proposition 4.5 *If $Q^1 = 0$ and $Q^2 \neq 0$ there exists a canonical reduction of the path structure to a $\mathbb{Z}/2\mathbb{Z}$-structure. Moreover, if $Y_1 \neq 0$ or $A \neq 0$, one can reduce further to a parallelism.*

A more refined description of the reduction is obtained by observing that the functions X_1 and Y_1 cannot both vanish.

Proposition 4.6 *On the fiber bundle Y^1 the functions X_1 and Y_1 can not be simultaneously zero.*

Proof. On Y^1, taking account of theorem 4.4, we get from 22

$$\begin{aligned} &\omega \wedge \left(dY_1 + Y_1\varphi + (5 - X_1B)\omega^1 + (\tfrac{3}{2}A - Y_0)\omega^2\right) + \omega^1 \wedge (dX_1 - 2X_1\varphi - 2Y_1\omega^2) \\ &-\omega^2 \wedge \left(dB + 2B\varphi - \tfrac{1}{4}Y_0\omega - \tfrac{1}{4}Y_1\omega^1\right) = 0. \end{aligned}$$

If $X_1 = Y_1 = 0$, we obtain modulo ω^2

$$5\omega \wedge \omega^1 = 0$$

which is a contradiction. □

We can then use the function a to fix $X_1 = \pm 1$ or $Y_1 = 1$. Consider first the case $Y_1 \neq 0$. We can use a section of H^1 to impose $\tilde{Y}_1 = 1$.

Proposition 4.7 *If $Y_1 \neq 0$ we can choose $a = (Y_1)^{\frac{1}{3}}$, and with this choice we reduce Y^1 to an $\{e\}$-bundle Y^2 where $\tilde{Y}_1 = 1$, $\tilde{Q}^2 = 1$, $\tilde{U}_1 = \tilde{T}^2 = \tilde{S}^2 = 0$. Moreover on Y^2 we have $\tilde{Q}^1 = \tilde{S}^1 = \tilde{U}_2 = \tilde{T}^1 = 0$, $3\tilde{w} = -2\tilde{\varphi}$, $\tilde{\varphi}^1 = \tilde{A}\tilde{\omega} + \tilde{B}\tilde{\omega}^1$, $5\tilde{\varphi}^2 = \tilde{\omega} + \tilde{X}_1\tilde{\omega}^1 - \tilde{B}\tilde{\omega}^2$, $4\tilde{\psi} = \tilde{Y}_0\tilde{\omega} + \tilde{\omega}^1 - \tilde{A}\tilde{\omega}^2$, $\tilde{\varphi} = (\tilde{A}\tilde{X}_1 + \frac{1}{5}\tilde{B} + \tilde{Y}_{10})\tilde{\omega} + (\frac{6}{5}\tilde{X}_1\tilde{B} + X_{10} - 5)\tilde{\omega}^1 + (\tilde{Y}_0 + \tilde{Y}_{12} - \frac{\tilde{B}^2}{5})\tilde{\omega}^2$.*

Proof. The choice of a implies that $\tilde{Y}_1 = 1$. The other properties follow from theorem 4.4 and from 24. □

If $Y_1 = 0$, we can use $X_1 \neq 0$ to reduce Y^1 to a $\mathbb{Z}/2\mathbb{Z}$-bundle.

Proposition 4.8 *If $X_1 \neq 0$, $Y_1 = 0$, we can choose $a = \pm|X_1|^{-\frac{1}{6}}$, and with this choice we reduce Y^1 to a $\mathbb{Z}/2\mathbb{Z}$-bundle Y^2 where $\tilde{X}_1 = \epsilon$, with $\epsilon = \pm 1$, $\tilde{Q}^2 = 1$, $\tilde{U}_1 = \tilde{T}^2 = \tilde{S}^2 = \tilde{Y}_1 = 0$. Moreover on Y^2 we have $\tilde{Q}^1 = \tilde{S}^1 = \tilde{U}_2 = \tilde{T}^1 = 0$, $3\tilde{w} = -2\tilde{\varphi}$, $\tilde{\varphi}^1 = \tilde{A}\tilde{\omega} + \tilde{B}\tilde{\omega}^1$, $5\tilde{\varphi}^2 = \epsilon\tilde{\omega}^1 - B\tilde{\omega}^2$, $4\tilde{\psi} = \tilde{Y}_0\tilde{\omega} - \tilde{A}\tilde{\omega}^2$, $-2\epsilon\tilde{\varphi} = \tilde{X}_{10}\tilde{\omega} + \tilde{X}_{11}\tilde{\omega}^1 + \tilde{X}_{12}\tilde{\omega}^2$*

Proof. The choice of a implies that $\tilde{X}_1 = \epsilon$. The other properties follows from theorem 4.4 and from 23. □

5 Homogeneous structures

5.1 Three dimensional Lie groups with path structures

By Tresse's result, all homogeneous path structures which are not flat have a three dimensional group of automorphisms. Therefore they are all locally isomorphic to a left invariant structure on a three dimensional Lie group. In this section we classify three dimensional Lie groups with homogeneous path structures. The results are summarized in Tables in section 5.3.

We follow an analogous scheme to classify homogeneous CR structures (see [FG]). In order to classify the simply connected groups having left invariant path structure it is sufficient to classify three-dimensional Lie algebras $\mathfrak{g}$ admitting a vector subspace $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ with a decomposition $\mathfrak{p} = \mathfrak{e}_1 \oplus \mathfrak{e}_2$.

Given a basis $\{X_1, X_2\}$ of $\mathfrak{p}$, with $X_i \in \mathfrak{e}_i$, define $Y = -[X_1, X_2]$ and consider the map $\mathrm{ad}_Y : \mathfrak{p} \to \mathfrak{g}$ whose matrix in the given basis is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Jacobi identities are computed as the following equations:

$$a_{11} + a_{22} = 0$$

$$a_{31}a_{12} - a_{32}a_{11} = 0$$

$$a_{31}a_{22} - a_{32}a_{21} = 0.$$

Note that in terms of a dual basis $\omega, \omega^1, \omega^2$ we obtain the equations

$$\begin{aligned} d\omega^1 &= a_{11}\omega^1 \wedge \omega + a_{12}\omega^2 \wedge \omega \\ d\omega^2 &= a_{21}\omega^1 \wedge \omega + a_{22}\omega^2 \wedge \omega \\ d\omega &= a_{31}\omega^1 \wedge \omega + a_{32}\omega^2 \wedge \omega + \omega^1 \wedge \omega^2 \end{aligned} \tag{53}$$

A different basis $\{\bar{X}_1, \bar{X}_2, \bar{Y} = -[\bar{X}_1, \bar{X}_2]\}$ for $\mathfrak{g}$, which defines the same path structure, is given by the change of basis matrix

$$P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1\lambda_2 \end{pmatrix},$$

with $\lambda_1, \lambda_2 \in \mathbb{R}^*$. The matrix of $\mathrm{ad}_{\bar{Y}}$ is now $\bar{A} = \lambda_1\lambda_2 P^{-1}AN$, where

$$N = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and therefore

$$\bar{A} = \begin{pmatrix} \lambda_1\lambda_2 a_{11} & {\lambda_2}^2 a_{12} \\ {\lambda_1}^2 a_{21} & \lambda_1\lambda_2 a_{22} \\ \lambda_1 a_{31} & \lambda_2 a_{32} \end{pmatrix}.$$

The goal now is to use the change of basis in order to find normal forms for ad_Y. Observe that the vector space isomorphism

$$X_1 \to X_2, \quad X_2 \to X_1, \quad Y \to -Y$$

changes the path structures. But we might not distinguish them as they correspond to a reordering of the decomposition. In other words,without loss of generality, we allow ourselves to change the order of the decomposition $\mathfrak{p} = \mathfrak{e}_1 \oplus \mathfrak{e}_2$. In this case, the matrix A changes to

$$\bar{A} = \begin{pmatrix} -a_{22} & -a_{21} \\ -a_{12} & -a_{11} \\ a_{32} & a_{31} \end{pmatrix}.$$

There are two cases to consider:

5.1.1 $\mathbf{ad}_Y \mathfrak{p} \subset \mathfrak{p}$

In this case $a_{31} = a_{32} = 0$. The general matrix for ad_Y is

$$\begin{pmatrix} a & b \\ c & -a \\ 0 & 0 \end{pmatrix}.$$

- If $a = b = c = 0$ we obtain the usual invariant path structure on the Heisenberg group.

- If $b = c = 0$ and $a \neq 0$ we may normalize the basis to obtain the normal form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix},$$

which gives the usual invariant path structure on $\mathbf{SL}(2,\mathbb{R})$.

- If $b \neq 0$ (which we can assume also if $c \neq 0$ by a change of order) we normalize the matrix of ad_Y to

$$\begin{pmatrix} a & \pm 1 \\ c & -a \\ 0 & 0 \end{pmatrix}$$

where $a, c \in \mathbb{R}$. One can further normalize:

1. If $a \neq 0$ then we normalize to

$$\begin{pmatrix} 1 & \pm 1 \\ c & -1 \\ 0 & 0 \end{pmatrix}.$$

Here there are two cases. If $c \neq \mp 1$ then the group is simple. Indeed if, $\pm c + 1 > 0$ then it corresponds to $\mathbf{SL}(2,\mathbb{R})$ and if $\pm c + 1 < 0$ it corresponds to $SU(2)$. This can be checked computing the Killing form of the Lie algebra and showing that it is non-degenerate and, moreover, negative exactly when $\pm c + 1 < 0$. Observe also that the case

$$\begin{pmatrix} 1 & -1 \\ c & -1 \\ 0 & 0 \end{pmatrix},$$

with $c < 0$ is equivalent, using a reordering of the vectors X_1 and X_2, to the case

$$\begin{pmatrix} 1 & 1 \\ -c & -1 \\ 0 & 0 \end{pmatrix}.$$

If $c = \mp 1$ then it is solvable. We obtain the Euclidean and Poincaré groups. Indeed, for $c = -1$ we get (making $e_1 = X_1 - X_2,\ e_2 = Y,\ e_3 = X_2$) the usual relations $[e_1, e_2] = 0,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = e_1$ of the Euclidean group. For $c = 1$ we get (making $e_1 = X_1 + X_2,\ e_2 = Y,\ e_3 = X_2$) the usual relations $[e_1, e_2] = 0,\ [e_1, e_3] = e_2,\ [e_2, e_3] = e_1$ of the Poincaré group.

2. If $a = 0$ and $c \neq 0$, we normalize to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which corresponds to the Lie algebra of $SU(2)$, or

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which corresponds to the Lie algebra of $\mathbf{SL}(2,\mathbb{R})$ (make $e_1 = X_1 - X_2,\ e_2 = X_1 + X_2,\ e_3 = Y$ to obtain the usual relations $[e_1, e_2] = 2e_3,\ [e_3, e_1] = -e_1,\ [e_3, e_2] = e_2$).

3. If $a = c = 0$ we may write

$$\begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This corresponds to two solvable groups. One of them (the case the upper right coefficient is $+1$) is the euclidian group (Bianchi VII) which we can recognise making $e_1 = Y, e_2 = X_1, e_3 = X_2$ so the algebra becomes $[e_1, e_2] = 0,\ [e_3, e_1] = e_2,\ [e_3, e_2] = -e_1$. The other case corresponds to the Poincaré group (Bianchi VI).

5.1.2 $\mathrm{ad}_Y \mathfrak{p} \not\subseteq \mathfrak{p}$

Then a_{31} and a_{32} do not vanish at the same time. One can suppose that $a_{31} \neq 0$ by exchanging the vectors X_1 and X_2. Observe now that from

$$a_{31}a_{12} - a_{32}a_{11} = 0$$

$$a_{31}a_{22} - a_{32}a_{21} = 0.$$

and $a_{31} \neq 0$ we have $a_{11}a_{22} - a_{12}a_{21} = 0$. We normalize the matrix of ad_Y as

$$\begin{pmatrix} a & b \\ c & -a \\ 1 & f \end{pmatrix},$$

where $a^2 + bc = 0$, $af - b = 0$ and $cf + a = 0$. We may write then

$$\begin{pmatrix} -cf & -cf^2 \\ c & cf \\ 1 & f \end{pmatrix}.$$

We normalize further:

1. If $f \neq 0$ and $c = 0$ we may normalize to

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

This corresponds to the solvable group obtained by the direct product of the affine group with $\mathbb{R}$: An invariant path structure on the decomposable solvable group (the only group with derived algebra of dimension one which is not central).

2. If $f = 0$ and $c \neq 0$, we may normalize to

$$\begin{pmatrix} 0 & 0 \\ c & 0 \\ 1 & 0 \end{pmatrix}$$

which corresponds to a family of solvable groups (make $e_1 = Y, e_2 = cX_2 + Y, e_3 = X_1$, then we obtain $[e_1, e_2] = 0,\ [e_1, e_3] = e_2,\ [e_2, e_3] = cY + cX_2 + Y = e_2 + ce_1$. The two families of solvable groups appear: Bianchi VI (for $c > -1/4$ and $c \neq 0$) and VII (for $c < -1/4$). Moreover if $c = -1/4$ then we get Bianchi IV.

3. If $f = 0$ and $c = 0$ we obtain another invariant path structure on the decomposable solvable group. The matrix ad_Y is normalized to

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

4. If $f \neq 0$ and $c \neq 0$ and we may normalize to

$$\begin{pmatrix} -c & -c \\ c & c \\ 1 & 1 \end{pmatrix},$$

and a computation shows that for $c > -1/4, c \neq 0$ the group is of type Bianchi VI and for $c < -1/4$ the group is of type VII. In the case $c = -1/4$ we obtain type IV.

5.2 Path structure curvatures of the homogeneous examples

The path geometry of the homogenous examples may be described using strict path structure invariants. Indeed, we showed that there exists parallelisms for each homogeneous three dimensional path structure. In particular there exists strict path structures associated to each of those homogeneous three dimensional path structures. In many cases the parallelism and therefore the strict structures are canonical.

Fix a parallelism $X_1, X_2, Y = -[X_1, X_2]$ of the strict path structure. Recall (see the beginning of section 5.1) that the matrix of ad_Y in this basis is:

$$A = \begin{pmatrix} a & b \\ c & -a \\ e & f \end{pmatrix}$$

and satisfies $af - be = 0$ and $cf + ae = 0$.

The goal of this section is to compute the path structure curvatures for each homogeneous structure.

Proposition 5.1 *The curvatures of the invariant path structures on Lie groups are given by*

$$j^*\iota^*Q^1 = S^1_2 + \frac{3}{2}R\tau^1_2 + 2W^2_2 - \frac{1}{2}R_{22} = -a(\frac{3}{2}b - \frac{1}{3}f^2)$$

and

$$j^*\iota^*Q^2 = -S^2_1 + \frac{3}{2}R\tau^2_1 - 2W^1_1 - \frac{1}{2}R_{11} = -a(\frac{3}{2}c + \frac{1}{3}e^2)$$

Proof.

We obtain a basis of the Lie algebra corresponding to an invariant structure with dual left invariant forms $\theta, \theta^1, \theta^2$ satisfying equations 53

$$\begin{array}{rcl} d\theta^1 & = & a\theta^1 \wedge \theta + b\theta^2 \wedge \theta \\ d\theta^2 & = & c\theta^1 \wedge \theta - a\theta^2 \wedge \theta \\ d\theta & = & e\theta^1 \wedge \theta + f\theta^2 \wedge \theta + \theta^1 \wedge \theta^2 \end{array} \tag{54}$$

The goal now is to identify the strict path geometry invariants of the structures. For that sake we need to obtain the structure equations 25. Note that θ is the fixed contact form.

First, in order to obtain equation $d\theta = \theta^1 \wedge \theta^2$ write θ^1 and θ^2 in terms of a new basis $\theta, \theta^1 - f\theta$ and $\theta^2 + e\theta$. We obtain then, using the same symbols $\theta, \theta^1, \theta^2$ for the new basis, the equations

$$\begin{array}{rcl} d\theta^1 & = & a\theta^1 \wedge \theta + b\theta^2 \wedge \theta - f\theta^1 \wedge \theta^2 \\ d\theta^2 & = & c\theta^1 \wedge \theta - a\theta^2 \wedge \theta + e\theta^1 \wedge \theta^2 \\ d\theta & = & \theta^1 \wedge \theta^2 \end{array} \tag{55}$$

Now we proceed as in section 3 (equations 26) and write the connection and torsion forms imposing the following equations

$$\begin{array}{rcl} d\theta^1 - 3\upsilon \wedge \theta^1 & = & \theta \wedge \tau^1, \\ d\theta^2 + 3\upsilon \wedge \theta^2 & = & \theta \wedge \tau^2, \\ d\theta & = & \theta^1 \wedge \theta^2. \end{array} \tag{56}$$

where we write the pull-back of the forms υ, τ^1 and τ^2 by the embedding j using the same letters. We obtain

$$\begin{array}{rcl} d\theta^1 & = & \theta^1 \wedge (a\theta - f\theta^2) - b\theta \wedge \theta^2 \\ d\theta^2 & = & -\theta^2 \wedge (a\theta + e\theta^1) - c\theta \wedge \theta^1 \\ d\theta & = & \theta^1 \wedge \theta^2 \end{array} \tag{57}$$

Comparing with the structure equations we obtain

$$\begin{array}{rcl} \upsilon & = & -\frac{1}{3}(a\theta + e\theta^1 - f\theta^2) \\ \tau^1 & = & -b\theta^2 \\ \tau^2 & = & -c\theta^1 \end{array} \tag{58}$$

We compute equation 27, the exterior derivative of υ, $d\upsilon = R\theta^1 \wedge \theta^2 + W^1\theta^1 \wedge \theta + W^2\theta^2 \wedge \theta$ and obtain

$$\begin{array}{rcl} R & = & -\frac{1}{3}(a - 2ef) \\ W^1 & = & -\frac{1}{3}(ae - cf) = \frac{2}{3}cf \\ W^2 & = & -\frac{1}{3}(be + af) = -\frac{2}{3}be. \end{array} \tag{59}$$

From

$$d\tau^1 + 3\tau^1 \wedge \upsilon = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2$$

we obtain

$$-bd\theta^2 + 3b\theta^2 \wedge \frac{1}{3}(a\theta + e\theta^1) = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2.$$

That is

$$-b(-\theta^2 \wedge (a\theta + e\theta^1) - c\theta \wedge \theta^1) + 3b\theta^2 \wedge \frac{1}{3}(a\theta + e\theta^1) = 3W^2\theta^1 \wedge \theta^2 + S^1_1\theta \wedge \theta^1 + S^1_2\theta \wedge \theta^2$$

which implies

$$W^2 = -\frac{2}{3}eb, \quad S^1_1 = bc, \quad S^1_2 = -2ab.$$

Analogously, from $d\tau^2 - 3\tau^2 \wedge \upsilon = 3W^1\theta^1 \wedge \theta^2 + S^2_1\theta \wedge \theta^1 + S^2_2\theta \wedge \theta^2$, we obtain

$$-c(\theta^1 \wedge (a\theta - f\theta^2) - b\theta \wedge \theta^2) + 3(-c\theta^1) \wedge \frac{1}{3}(a\theta - f\theta^2) = 3W^1\theta^1 \wedge \theta^2 + S^2_1\theta \wedge \theta^1 + S^2_2\theta \wedge \theta^2,$$

which implies

$$W^1 = \frac{2}{3}cf, \quad S^2_1 = 2ac, \quad S^2_2 = bc.$$

Moreover, we have the relation

$$S^1_1 = S^2_2 = \tau^1_2\tau^2_1.$$

Remark that from equations 30 and 31

$$dW^1 + 3W^1\upsilon = W^1_0\theta + W^1_1\theta^1 + W^1_2\theta^2 \tag{60}$$

and

$$dW^2 - 3W^2\upsilon = W^2_0\theta + W^2_1\theta^1 + W^2_2\theta^2, \tag{61}$$

we obtain

$$-\frac{2}{9}cf(a\theta + e\theta^1 - f\theta^2)$$

$$W^1_0 = -\frac{2}{3}cfa, \quad W^1_1 = -\frac{2}{3}cfe, \quad W^1_2 = \frac{2}{3}cf^2$$

and

$$\frac{2}{9}eb(a\theta + e\theta^1 - f\theta^2)$$

$$W^2_0 = -\frac{2}{3}eba, \quad W^2_1 = -\frac{2}{3}e^2b, \quad W^2_2 = \frac{2}{3}ebf$$

Substituting the expressions obtained above in Proposition 3.3 completes the proof.

□

5.3 Tables of invariant path structures on three dimensional Lie groups

In this section we put together the classification of path structures on the three dimensional Lie groups in the form of two tables. One for solvable Lie groups and the other for the remaining simple groups. In the following, we use the Bianchi classification of three dimensional Lie groups. Recall that the group Bianchi VI_0 is the Poincaré group and Bianchi VII_0 is the Euclidean group. The Bianchi groups of type I and V don't have invariant contact structures and therefore don't appear in the tables. The proof of the classification can be read in the previous section 5.1.

Path structures on three dimensional solvable Lie groups			
	ad_Y	Q^1	Q^2
Heisenberg, Bianchi II	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$	0	0
$Aff(2) \times \mathbb{R}$ Bianchi III	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$	0	0
Bianchi IV	$\begin{pmatrix} 0 & 0 \\ -\frac{1}{4} & 0 \\ 1 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \\ 1 & 1 \end{pmatrix}$	$-\frac{1}{4}(\frac{3}{8} - \frac{1}{3})$	$-\frac{1}{4}(-\frac{3}{8}+\frac{1}{3})$
Bianchi VI	$\begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Bianchi VI_0	0	0
	$\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}$ Bianchi VI_0	$\frac{3}{2}$	$-\frac{3}{2}$
	$\begin{pmatrix} 0 & 0 \\ c & 0 \\ 1 & 0 \end{pmatrix}$, $0 \neq c > -\frac{1}{4}$	0	0
	$\begin{pmatrix} -c & -c \\ c & c \\ 1 & 1 \end{pmatrix}$, $0 \neq c > -\frac{1}{4}$	$c(-\frac{3}{2}c - \frac{1}{3})$	$c(\frac{3}{2}c + \frac{1}{3})$
Bianchi VII	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Bianchi VII_0	0	0
	$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix}$ Bianchi VII_0	$-\frac{3}{2}$	$\frac{3}{2}$
	$\begin{pmatrix} 0 & 0 \\ c & 0 \\ 1 & 0 \end{pmatrix}$, $c < -\frac{1}{4}$	0	0
	$\begin{pmatrix} -c & -c \\ c & c \\ 1 & 1 \end{pmatrix}$, $c < -\frac{1}{4}$	$c(-\frac{3}{2}c - \frac{1}{3})$	$c(\frac{3}{2}c + \frac{1}{3})$

Path structures on three dimensional simple Lie groups			
	ad_Y	Q^1	Q^2
$\mathbf{SL}(2,\mathbb{R})$ Bianchi VIII	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} 1 & 1 \\ c & -1 \\ 0 & 0 \end{pmatrix}, {\scriptstyle c>-1}$	$-\frac{3}{2}$	$-\frac{3}{2}c$
	$\begin{pmatrix} 1 & -1 \\ c & -1 \\ 0 & 0 \end{pmatrix}, {\scriptstyle 0<c<1}$	$\frac{3}{2}$	$-\frac{3}{2}c$
$SU(2)$ Bianchi IX	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	0	0
	$\begin{pmatrix} 1 & 1 \\ c & -1 \\ 0 & 0 \end{pmatrix}, {\scriptstyle c<-1}$	$-\frac{3}{2}$	$-\frac{3}{2}c$
	$\begin{pmatrix} 1 & -1 \\ c & -1 \\ 0 & 0 \end{pmatrix}, {\scriptstyle c>1}$	$\frac{3}{2}$	$-\frac{3}{2}c$

5.4 Natural invariants in the case of $\mathbf{SL}(2,\mathbb{R})$

In this subsection, we relate the left-invariant structures on $\mathbf{SL}(2,\mathbb{R})$ to the geometry of the Killing form of $\mathfrak{sl}(2,\mathbb{R})$ which defines a Lorentzian structure on $\mathbf{SL}(2,\mathbb{R})$.

Recall the parallelism $X_1, X_2, Y = -[X_1, X_2]$ and the matrix of ad_Y in this basis (section 5.1):

$$A = \begin{pmatrix} a & b \\ c & -a \\ e & f \end{pmatrix}$$

satisfying $af - be = 0$ and $cf + ae = 0$. We have

$$\mathrm{ad}_{X_1} = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & -c \\ 0 & -1 & -e \end{pmatrix}, \quad \mathrm{ad}_{X_2} = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & a \\ 1 & 0 & -f \end{pmatrix}, \quad \mathrm{ad}_Y = \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ e & f & 0 \end{pmatrix}.$$

Therefore, the Killing form of the Lie algebra satisfies

$$\kappa(X_1, X_1) = \operatorname{tr} X_1^2 = 2c + e^2, \quad \kappa(X_2, X_2) = \operatorname{tr} X_2^2 = -2b + f^2, \quad \kappa(Y, Y) = \operatorname{tr} Y^2 = 2(a^2 + bc).$$

$$\kappa(X_1, X_2) = -2a + ef, \quad \kappa(X_1, Y) = -ae - cf, \quad \kappa(X_2, Y) = -be + af.$$

5.4.1 Lorentzian geometry of $\mathfrak{sl}(2,\mathbb{R})$

Let us consider the Killing form of $\mathfrak{sl}(2,\mathbb{R})$, $\kappa(u,v)=\frac{1}{2}\mathrm{tr}\,(uv)$ whose associated quadratic form is

$$q(u)=-\det(u). \tag{62}$$

Then, using the standard basis

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix},F=\begin{pmatrix}0&0\\1&0\end{pmatrix},H=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{63}$$

of $\mathfrak{sl}(2,\mathbb{R})$ satisfying the relation $H=[E,F]$, $(H,E+F,E-F)$ is an orthonormal basis of signature $(1,1,-1)$ for q. In particular the choice of this basis identifies $\mathfrak{sl}(2,\mathbb{R})$ to the Lorentzian Minkowski space $\mathbb{R}^{1,2}$. Elements of $\mathfrak{sl}(2,\mathbb{R})$ are thus distinguished according to the sign of q: $u\in\mathfrak{sl}(2,\mathbb{R})$ is called

- *timelike* if $q(u)<0$,
- *lightlike* if $q(u)=0$,
- *spacelike* if $q(u)>0$.

Recall that an element of $\mathfrak{sl}(2,\mathbb{R})$ is called *hyperbolic* (respectively *parabolic*, *elliptic*) if it generates a one-parameter group of the given kind in $\mathbf{SL}(2,\mathbb{R})$. We have then the following straightforward correspondence between the geometry of q and the algebraic types in $\mathfrak{sl}(2,\mathbb{R})$.

Lemma 5.2 *Let $u\in\mathfrak{sl}(2,\mathbb{R})$. Then:*

- *u is timelike $\Leftrightarrow$ u is elliptic;*
- *u is lightlike $\Leftrightarrow$ u is parabolic;*
- *u is spacelike $\Leftrightarrow$ u is hyperbolic.*

Lastly, q is invariant by the adjoint action of $\mathbf{SL}(2,\mathbb{R})$, and more precisely

$$\mathrm{Ad}\colon g\in\mathbf{SL}(2,\mathbb{R})\mapsto Ad_g\in SO^0(q) \tag{64}$$

is surjective and has kernel $\{\pm id\}$, $SO^0(q)$ denoting the connected component of the identity in the stabilizer of q in $\mathrm{GL}(\mathfrak{sl}(2,\mathbb{R}))$.

5.4.2 The moduli space of left-invariant path structures on $\mathbf{SL}(2,\mathbb{R})$

Definition 5.3 *We define $\mathcal{S}$ to be the set of left-invariant path structures on $\mathbf{SL}(2,\mathbb{R})$.*

Our goal is now to extract the natural geometric invariants distinguishing, in the space $\mathcal{S}$, those which are locally isomorphic.

We say that a plane P in $\mathfrak{sl}(2,\mathbb{R})$ is

- *timelike* if $q|_P$ has Lorentzian signature $(1,-1)$, *i.e.* if P contains a timelike line;

- *lightlike* if $q|_P$ has signature $(1,0)$, *i.e.* if it is degenerated, or if P contains exactly one lightlike line;
- *spacelike* if $q|_P$ has euclidean signature $(1,1)$, *i.e.* if P contains no lightlike line.

Each choice of a plane in $\mathfrak{sl}(2,\mathbb{R})$ defines, by translation, a distribution on $\mathbf{SL}(2,\mathbb{R})$. The adjoint action by $\mathbf{SL}(2,\mathbb{R})$ on the set of planes in $\mathfrak{sl}(2,\mathbb{R})$ has three orbits. Two of them (the set of timelike planes and the set of spacelike planes) are open sets and correspond to the two invariant contact distributions in $\mathbf{SL}(2,\mathbb{R})$ and the other one is a closed orbit and corresponds to an integrable distribution which will not be relevant in the sequel.

Consider the projective space $\mathbb{P}(\mathfrak{sl}(2,\mathbb{R}))$. We introduce the two open subsets in $\mathbb{P}(\mathfrak{sl}(2,\mathbb{R}))\times\mathbb{P}(\mathfrak{sl}(2,\mathbb{R}))$,

$$\mathcal{E}_{space} = \{(D_1,D_2)\}|\{D_1 \neq D_2, D_1 \oplus D_2 \text{ spacelike}\}, \tag{65}$$
$$\mathcal{E}_{time} = \{(D_1,D_2)\}|\{D_1 \neq D_2, D_1 \oplus D_2 \text{ timelike}\} \tag{66}$$

as parameter spaces for $\mathcal{S}$. Note that two pairs of distinct points of $\mathbb{P}(\mathfrak{sl}(2,\mathbb{R}))$ are in the same orbit under the adjoint action of $\mathbf{SL}(2,\mathbb{R})$, if and only if the pairs of left-invariant line fields that they generate are related by an element of $\mathbf{SL}(2,\mathbb{R})$. Hence any invariant of adjoint orbits of $\mathbf{SL}(2,\mathbb{R})$ will be relevant for our study. There exists one natural one-parameter such invariant in each of the open subsets $\mathcal{E}_{space}$ and $\mathcal{E}_{time}$.

Definition 5.4 *For $(D_1,D_2)\in\mathcal{E}_{space}$ or $\mathcal{E}_{time}$, pairs with a lightlike line being excluded, define the cross-ratio*

$$cr(D_1,D_2) = \frac{\kappa(X_1,X_2)\kappa(X_2,X_1)}{\kappa(X_1,X_1)\kappa(X_2,X_2)}$$

with X_i a generator of D_i for $i=1,2$.

For any $u\subset\mathfrak{sl}(2,\mathbb{R})$ $\tilde{u}$ denotes the left-invariant vector field of $\mathbf{SL}(2,\mathbb{R})$ generated by u, with an analog notation for left-invariant line or plane fields.

Proposition 5.5 *The map*

$$(D_1,D_2)\in\mathcal{E}_{space}\cup\mathcal{E}_{time}\mapsto(\widetilde{D_1},\widetilde{D_2})\in\mathcal{S} \tag{67}$$

is surjective. Moreover, $(\widetilde{D_1},\widetilde{D_2})$ and $(\widetilde{D'_1},\widetilde{D'_2})$ are locally isomorphic if, and only if one of the following mutually exclusive conditions is satisfied, with $P=D_1\oplus D_2$ and $P'=D'_1\oplus D'_2$.

1. *All the lines D_i and D'_i are lightlike, and in this case the associated path structures are flat.*
2. *Exactly one of the lines (D_1,D_2), respectively (D'_1,D'_2) is lightlike and the others are of the same type. In this case one of the Cartan curvatures is null.*
3. *P and P' are spacelike and $cr(D_1,D_2)=cr(D'_1,D'_2)$.*
4. *P and P' are timelike, D_i of the same type as D'_i for $1\le i\le 2$, none of the lines is lightlike, and $cr(D_1,D_2)=cr(D'_1,D'_2)$. There are three components corresponding to D_1 and D_2 both timelike, both space-like or of different type.*

Proof: Surjectivity follows from the fact that contact distributions arise either from timelike planes or spacelike planes. The closed orbit of lightlike planes is excluded.

Consider vectors X_1 and X_2 generating D_1 and D_2. Let $X_1, X_2, Y = -[X_1, X_2]$ be a basis of the Lie algebra and the matrix of ad_Y in this basis (section 5.1) for $\mathfrak{sl}(2,\mathbb{R})$ be written as:

$$A = \begin{pmatrix} a & b \\ c & -a \\ 0 & 0 \end{pmatrix}.$$

From the computation of the Killing form we obtain $\kappa(X_1, X_1) = 2c$, $\kappa(X_2, X_2) = -2b$ and $\kappa(X_1, X_2) = -2a$. Also $Q^1 = -\frac{3}{2}ab$ and $Q^2 = -\frac{3}{2}ac$. One also have

$$cr(D_1, D_2) = \frac{\kappa(X_1, X_2)\kappa(X_2, X_1)}{\kappa(X_1, X_1)\kappa(X_2, X_2)} = -\frac{a^2}{bc}.$$

The proposition now follows from the following computations:

- If D_1 and D_2 are lightlike, the invariant flag structure is flat.
- If the two lines are orthogonal, one timelike and the other spacelike, the invariant flag structure is flat.
- If $c = 0, b > 0$, (D_1, D_2) is timelike (X_1 lightlike and X_2 timelike) and one can normalize so that $a = 1, b = 1$. One verifies that $Q^1 = -1$ and $Q^2 = 0$.
- If $c = 0, b < 0$, (D_1, D_2) is timelike (X_1 lightlike and X_2 spacelike) and one can normalize so that $a = 1, b = -1$. One verifies that $Q^1 = 1$ and $Q^2 = 0$.
- If $c > 0, b < 0$, (D_1, D_2) is spacelike (if $a^2 + bc < 0$) or timelike (if $a^2 + bc > 0$). One can normalize so that $a = 1, b = -1$. Then $Q^1 = \frac{3}{2}$, $Q^2 = -\frac{3}{2}c$ and $cr(D_1, D_2) = 1/c$.
- If $c > 0, b > 0$, (D_1, D_2) is timelike (X_1 spacelike and X_2 timelike) and one can normalize so that $a = 1, b = 1$. Then $Q^1 = -\frac{3}{2}$, $Q^2 = -\frac{3}{2}c$ and $cr(D_1, D_2) = -1/c < 0$.
- If $c < 0, b > 0$, (D_1, D_2) is timelike (X_1 and X_2 timelike) and one can normalize so that $a = 1, b = 1$. Then $Q^1 = -\frac{3}{2}$, $Q^2 = -\frac{3}{2}c$ and $cr(D_1, D_2) = -1/c > 0$.

To conclude the reverse implication of the Proposition, we use the fact that two homogeneous path structures having the same invariants Q^i are locally isomorphic, as a consequence of the classification of section 5.1 and of the proof of Theorem 4.2. □

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