RIGIDITY OF THE TIMELIKE MARKED LENGTH SPECTRUM AND LENGTH-TWIST COORDINATES OF SINGULAR DE-SITTER TORI

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ABSTRACT. In this paper, we study the closed timelike geodesics of de-Sitter tori with one singularity and prove their uniqueness in their free homotopy class. We introduce the notion of timelike marked length spectrum of such a torus, and establish its rigidity with respect to the lengths of two homotopy classes of intersection number one. We also construct length-twist coordinates on the deformation space of de-Sitter tori with one singularity.

1. INTRODUCTION

Singular de-Sitter tori form a new class of geometric structures which constitute the Lorentzian analog of hyperbolic surfaces. They are modelled on the two-dimensional *de-Sitter space* $\mathbf{dS}^2$, which is the *space of oriented geodesics of the hyperbolic plane* $\mathbf{H}^2$ endowed with its natural geometry, which happens to be *Lorentzian* and of constant non-zero curvature (a projective model of $\mathbf{dS}^2$ is described in Paragraph 2.1). De-Sitter geometry is in this sense dual to hyperbolic geometry: it is *the geometry of hyperbolic geodesics*. The present paper contributes to the description of the *deformation space of singular de-Sitter tori* through the lengths of their geodesics.

A natural isometry invariant of a closed hyperbolic surface is given by the length of the unique geodesic representative in a given free homotopy class. This forms the *marked length spectrum* of the hyperbolic structure which is famously *rigid*. More precisely, the lengths of a *finite* number of geodesics entirely determine a hyperbolic structure up to isotopy [KF96]. Unlike the Riemannian case, existence and uniqueness of geodesics of non-zero length does not hold in any free homotopy class of a singular de-Sitter torus. We will however see that *closed timelike geodesics* exist and are unique in any eligible free homotopy class of a de-Sitter torus with a unique singularity (Proposition A). This defines the *timelike marked length spectrum* of such a torus. In this paper, we show that *the lengths of a pair of simple closed timelike geodesics of intersection number one determine up to isotopy a de-Sitter structure with a unique singularity of fixed angle* (Theorem B).

1.1. **De-Sitter tori and their lightlike foliations.**

An important ingredient of de-Sitter geometry is the pair of transverse one-dimensional foliations of $\mathbf{dS}^2$ defined by the traces of (weak) stable and unstable horocycles in the space of geodesics of $\mathbf{H}^2$. These foliations are called lightlike, and they define on any surface locally modelled on $\mathbf{dS}^2$ a

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lightlike bi-foliation $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$. The existence of these foliations imposes a strong topological restriction on a closed (and orientable) Lorentzian surface: it has to be homeomorphic to the two-torus $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, due to the Poincaré-Hopf theorem. A Lorentzian version of the Gauß-Bonnet formula moreover imposes an additional constraint linking the topology and the curvature of a Lorentzian surface, which forbids the existence of a regular de-Sitter metric on the torus. To investigate closed Lorentzian surfaces of non-zero constant curvature, it is thus necessary to introduce *singularities*. The natural local notion of singularities was described in previous works [BBS11, p.160], inspired by the classical definition of a conical Riemannian singularity. The first examples of *singular de-Sitter tori* were given thereafter in [MM24], where the global study of singular Lorentzian surfaces was initiated through the lens of the lightlike foliations dynamics. In addition to their Lorentzian nature and their "negative-curvature flavour", some aspects of singular de-Sitter tori are reminiscent of dilation surfaces [BGT25]. The first-return maps of their lightlike foliations are however not affine but *homographic interval exchange transformations* [MM24, §4].

The lightlike bi-foliation of a de-Sitter torus can be interpreted as the *conformal class* of the underlying Lorentzian metric. The classical *uniformization* question translates then as follows in the Lorentzian setting: *is any bi-foliation of the torus the lightlike bi-foliation of a unique singular de-Sitter structure?* This question was answered positively in [MM24] for a unique singularity and in the case of *minimal* foliations. This gave a partial description of the *deformation space* $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ of de-Sitter structures of area $\theta > 0$ and with a unique singularity on the torus $\mathbf{T}^2$. However, the dynamics of the lightlike foliations does not capture the full geometry of singular de-Sitter tori. There exists indeed non-empty open subsets of $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ in which the lightlike foliations are pairwise isotopic (and have closed leaves). One of the goals of the present work is to provide a description of singular de-Sitter tori complementary to that of [MM24], focusing no longer on lightlike foliations but on *timelike geodesics*. In particular, we will distinguish singular de-Sitter structures having the same lightlike dynamics.

1.2. Uniqueness of closed geodesics in singular de-Sitter tori.

Important objects associated with geometric structures on surfaces are their *geodesics*, the first natural question being that of existence: *does every free homotopy class contain a closed geodesic ?* In a singular de-Sitter torus, geodesics have different *signatures* according to the sign of the metric: they are said *timelike* if the metric is negative on the direction of the geodesic. The existence question therefore needs to be specialized to characterize homotopy classes containing geodesics of a given signature. This question is answered in [MM24, Appendix A]: in a singular de-Sitter torus, the free homotopy classes admitting a timelike geodesic representative are entirely characterized by the lightlike foliations dynamics. More precisely, the lifts to $\mathbb{R}^2$ of the lightlike foliations of a singular de-Sitter structure g on $\mathbf{T}^2$ are asymptotic to two oriented lines called the *projective asymptotic cycles*, which delimit a timelike cone $\mathcal{C}^g \subset \mathbb{R}^2$. A free homotopy class $a \in \pi_1(\mathbf{T}^2) \subset \mathbb{R}^2$ contains a closed timelike geodesic if and only if $a \in \mathcal{C}^g$, in which case a is called a *timelike homotopy class* (see Paragraph 3.1.2 for more details).

This existence result raises a second natural question inspired by the hyperbolic case: *is any closed timelike geodesic unique in its free homotopy class ?* In the present paper, we answer this question positively for de-Sitter tori having a unique singularity and distinct lightlike asymptotic cycles, which are called *class A*.

Proposition A. *Any timelike free homotopy class of a class A de-Sitter torus with a single singularity contains a unique geodesic. Moreover, the latter is a multiple of a simple closed geodesic, and it maximizes the length among causal curves within its free homotopy class.*

This result points out a first "negative-curvature behaviour" of singular de-Sitter tori.

1.3. **Timelike marked length spectrum of singular de-Sitter tori and its rigidity.** Proposition A allows the definition of the *timelike marked length spectrum* of a de-Sitter structure g with a unique singularity on $\mathbf{T}^2$, as the map $\mathcal{L}^g$ sending a timelike homotopy class $\mathsf{a} \in \mathcal{C}^g \cap \pi_1(\mathbf{T}^2)$ to the length

$$\mathcal{L}^g(\mathsf{a}) \coloneqq L^g(\gamma^g_{\mathsf{a}})$$

of the unique closed timelike geodesic γ^g_{a} in the free homotopy class a. By construction, $\mathcal{L}^g$ only depends on the isotopy class g of the de-Sitter structure g, and gives thus rise to a map

$$\mathsf{g} \in \mathsf{Def}_\theta(\mathbf{T}^2, 0) \mapsto \mathcal{L}^{\mathsf{g}}.$$

This is the natural analog of the classical *marked length spectrum* of Riemannian manifolds of negative curvature. It is natural to ask wether these lengths entirely determine the metric up to isotopy, in which case the marked length spectrum is said *rigid*. For negative curvature Riemannian manifolds, the question of the rigidity of the marked length spectrum was popularized by Burns-Katok in the form of a conjecture [BK85], which motivated numerous works providing partial answers [Ota90; Cro90; HP97; BL18; GL19]. The case of hyperbolic surfaces is however completely distinct from the one of surfaces with variable negative curvature. A *finite* number of lengths is indeed sufficient to describe a hyperbolic metric [KF96], while an *infinite* number of them is necessary with variable curvature. This simply reflects the fact that the deformation space of hyperbolic metrics is *finite dimensional*, contrarily to the one of variable negative curvature metrics.

In the de-Sitter case, the domain $\mathcal{C}^{\mathsf{g}} \cap \pi_1(\mathbf{T}^2)$ of the marked length spectrum $\mathcal{L}^{\mathsf{g}}$ depends on the structure g, and is exactly equivalent to the asymptotic cycles of the lightlike foliations. According to the uniqueness results of [MM24], the domain of $\mathcal{L}^{\mathsf{g}}$ alone already characterizes $\mathsf{g} \in \mathsf{Def}_\theta(\mathbf{T}^2, 0)$ when both lightlike foliations are minimal. To complement this description, one could hope to characterize a de-Sitter structure with only a part of its timelike marked length spectrum. The following result shows that two timelike lengths are enough to entirely determine a class A de-Sitter structure with a unique singularity.

Theorem B. *Let g_1 and g_2 be two class A de-Sitter structures on the torus of equal areas and having a unique singularity. If g_1 and g_2 give the same lengths to a common pair of timelike free homotopy classes of intersection number one, then they are isotopic.*

1.4. **Length-twist coordinates on the deformation space of singular de-Sitter tori.** As with any deformation space of geometric structures, one may want to have natural coordinates on $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ expressing geometrical parameters of the de-Sitter structures. On the classical Teichmüller space of hyperbolic structures on a surface, this is for instance furnished by the well-known *Fenchel-Nielsen coordinates*. In the de-Sitter case that we are interested in, the Ehresman-Thurston principle shows that $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ is locally homeomorphic to a two-dimensional character variety. We are thus looking for two natural geometric quantities describing $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$, which explains why *two lengths* are sufficient to describe a singular de-Sitter structure in Theorem B. In contrast to the Riemannian case, the usual specificity of the Lorentzian signature prevents to describe the full deformation space by length-twist coordinates. We fix a primitive homotopy class $\mathsf{a} \in \pi_1(\mathbf{T}^2)$, and restrict to the open subset $\mathsf{Def}_\theta(\mathbf{T}^2, 0)^A_\mathsf{a}$ formed by class A de-Sitter structures for which a is a timelike class. The length function

$$\mathcal{L}_\mathsf{a} \colon \mathsf{g} \in \mathsf{Def}_\theta(\mathbf{T}^2, 0)^A_\mathsf{a} \mapsto \mathcal{L}^\mathsf{g}(\mathsf{a}) \in \mathbb{R}^*_+$$

of a is well defined on $\mathsf{Def}_\theta(\mathbf{T}^2, 0)^A_\mathsf{a}$, and gives us our first coordinate. Observe that Theorem B can be rephrased by saying that $\mathcal{L}_\mathsf{a} \times \mathcal{L}_\mathsf{b}$ is injective on $\mathsf{Def}_\theta(\mathbf{T}^2, 0)^A_\mathsf{a} \cap \mathsf{Def}_\theta(\mathbf{T}^2, 0)^A_\mathsf{b}$.

Choosing a second primitive class b to obtain a basis (a, b) of $\pi_1(\mathbf{T}^2)$, we can give a meaning to a second coordinate $\Theta_{\mathsf{a},\mathsf{b}} \colon \mathsf{Def}_\theta(\mathbf{T}^2, 0)^A_\mathsf{a} \to \mathbb{R}$ measuring the *twist* around the geodesic in the homotopy class a with respect to the longitude defined by b. The following result shows that the two previously defined geometrical quantities provide global coordinates on $\mathsf{Def}_\theta(\mathbf{T}^2, 0)^A_\mathsf{a}$.

Theorem C. *For any basis* (a, b) *of* $\pi_1(\mathbf{T}^2)$*, the map*

$$\mathcal{L}_\mathsf{a} \times \Theta_{\mathsf{a},\mathsf{b}} \colon \mathsf{Def}_\theta(\mathbf{T}^2, 0)^A_\mathsf{a} \to \mathbb{R}^*_+ \times \mathbb{R}$$

is a global homeomorphism.

These length-twist coordinates give a natural real-analytic atlas on $\mathsf{Def}_\theta(\mathbf{T}^2, 0)^A$ (see Corollary 4.5 and Remark 4.6 for more details).

1.5. **Methods and perspectives.** We highlight in this text some fundamental geometrical properties of two-dimensional de-Sitter geometry, which do not appear to have been previously exploited and will play an important role in this paper. We also introduce new elementary tools for the study of singular de-Sitter tori. Our hope is to illustrate that while de-Sitter geometry is fundamentally distinct from hyperbolic geometry (essentially due to the existence of invariant foliations), many efficient methods are however available which conceptually replace the usual "toolbox" of a hyperbolic geometer. We wish thereby to motivate the further investigation of these structures.

The study of singular de-Sitter surfaces remains in its infancy, and it is our opinion that the present work has opened many questions. The first broad direction concerns singular de-Sitter structures on *surfaces of genus higher than two*. We refer for instance to [MM24, Remark 4.5] for a discussion of the new singularities to be introduced on such surfaces (in order to introduce singularities of the lightlike foliations). In their case even the existence of closed timelike geodesics is generally not ensured, which is the subject of an ongoing work. The coincidence of de-Sitter geometry and non-trivial topology is likely to make the study of the lengths of closed timelike geodesics particularly interesting, concerning the rigidity question studied in the present paper as well as the *asymptotic counting* of such geodesics. Another promising perspective for these questions is the consideration of *singular Lorentzian surfaces with variable curvature of constant sign*, which are the Lorentzian counterpart of Riemannian surfaces with negative curvature.[1] In a future work, we wish to investigate the links of such structures with dynamical properties of their timelike geodesic flow.

We focus in the present paper on de-Sitter tori with a *unique* singularity. The main reason for this is the possibility for a de-Sitter torus with multiple singularities to contain distinct freely homotopic closed timelike geodesics which “separate” the singularities. Outside of this difficulty, Lemma 4.3 holds and yields partial coordinates on the deformation space, by arguments similar to Paragraph 4.2. The investigation of the specific behaviours appearing in presence of multiple singularities will be the object of a forthcoming work.

1.6. **Organization of the paper.** We give in Section 2 an overview of de-Sitter tori, sufficient for the purposes of this text. In Section 3, we introduce the notions concerning closed timelike geodesics and prove Proposition A. We define the twist coordinate and prove Theorem C in Section 4. We finally prove Theorem B in Section 5.

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2. Singular de-Sitter tori

2.1. **De-Sitter space.** We denote by $\mathbb{R}^{1,2} = (\mathbb{R}^3, q_{1,2})$ the three-dimensional *Minkowski space*, endowed with the Lorentzian quadratic form $q_{1,2}(x,y,z) = x^2 + y^2 - z^2$. A vector $u \in \mathbb{R}^{1,2}$ is respectively called:

– *spacelike* if $q_{1,2}(u) > 0$, *timelike* if $q_{1,2}(u) < 0$,
– *definite* if u is timelike or spacelike,
– *lightlike* if $q_{1,2}(u) = 0$,
– *causal* is $q_{1,2}(u) \leq 0$, and *anticausal* is $q_{1,2}(u) \geq 0$.

Half-lines of $\mathbb{R}^{1,2}$ are named accordingly. A plane $P \subset \mathbb{R}^{1,2}$ is called:

– *timelike* if $q_{1,2}|_P$ has *Lorentzian* signature $(-,+)$, *i.e.* contains two lightlike lines;
– *lightlike* if $q_{1,2}|_P$ is degenerated, *i.e.* contains a unique lightlike line;
– *spacelike* if $q_{1,2}|_P$ has *Euclidean* signature $(+,+)$, *i.e.* contains no lightlike line.

The timelike cone of $\mathbb{R}^{1,2}$ has two connected components, and vectors in the component of $(0,0,1)$ (respectively $(0,0,-1)$) are called *future* (resp. *past*) timelike vectors. The choice of such a future connected component in the timelike cone is called a *time-orientation*.

2.1.1. *De-Sitter space, lightlike foliations and their orientations.* We denote by $\mathbf{P}^+(\mathbb{R}^{1,2})$ (homeomorphic to $\mathbf{S}^2$) the space of half-lines of $\mathbb{R}^{1,2}$. The two-dimensional *de Sitter space* is the quadric

$$\mathbf{dS}^2 := \{d \in \mathbf{P}^+(\mathbb{R}^{1,2}) \text{ spacelike}\} \subset \mathbf{P}^+(\mathbb{R}^{1,2})$$

endowed with the *Lorentzian metric* $\boldsymbol{g}$ induced by the restriction to $\mathbf{dS}^2$ of $q_{1,2}$, which has constant curvature 1. This makes the de-Sitter space $\mathbf{dS}^2$ the Lorentzian analog of the *hyperboloid model of the hyperbolic plane*[1]:

$$\mathbf{H}^2 \coloneqq \{d \in \mathbf{P}^+(\mathbb{R}^{1,2}) \text{ timelike future}\} \subset \mathbf{P}^+(\mathbb{R}^{1,2}).$$

Each timelike plane of $\mathbb{R}^{1,2}$ inherits a time-orientation (whose future cone is contained within the future cone of $\mathbb{R}^{1,2}$). We endow $\mathbf{dS}^2$ with the induced future timelike and spacelike cones, and with the orientation coming from that of $\mathbb{R}^3$ (with the exterior normal rule). The *lightlike foliations* $\mathcal{F}_{\alpha/\beta}$ of $\mathbf{dS}^2$ are the one tangent to its two lightlike line fields. They are named and oriented compatibly with the orientation and the time-orientation of $\mathbf{dS}^2$ as indicated in Figure 2.1.

2.1.2. *Geodesics in the projective de-Sitter model.* The model $\mathbf{dS}^2$ of the de-Sitter space is *projective* in the sense that the geodesics of $\mathbf{dS}^2$ are precisely the connected components of the intersections of projective lines of $\mathbf{P}^+(\mathbb{R}^{1,2})$ with $\mathbf{dS}^2$. Let

$$\partial^{\pm}_{\infty}\mathbf{dS}^2 \coloneqq \{d \in \mathbf{P}^+(\mathbb{R}^{1,2}) \text{ lightlike future/past}\} \subset \mathbf{P}^+(\mathbb{R}^{1,2})$$

denote the *future* (resp. past) *boundary of* $\mathbf{dS}^2$ *at infinity*, which we endow with the orientation compatible with the one of $\mathbf{dS}^2$. The geodesics of $\mathbf{dS}^2$ are described as follows.

(1) *Timelike projective lines* are the one intersecting each of the two boundaries $\partial^{\pm}_{\infty}\mathbf{dS}^2$ in two points. Their intersections with $\mathbf{dS}^2$ have two connected components which are opposite *timelike geodesics*.
(2) *Lightlike projective lines* are the lines L tangent to $\partial^{\pm}_{\infty}\mathbf{dS}^2$, *i.e.* intersecting each of the two boundaries at a unique point $L^{\pm}$. Their intersections with $\mathbf{dS}^2$ have two connected components.
    - The component converging to $\partial^{-}_{\infty}\mathbf{dS}^2$ in the future is an α *lightlike geodesic*.
    - The component converging to $\partial^{+}_{\infty}\mathbf{dS}^2$ in the future is a β *lightlike geodesic*.
(3) *Spacelike projective lines* are the one disjoint from $\partial^{\pm}_{\infty}\mathbf{dS}^2$, and coincide with *spacelike geodesics* of $\mathbf{dS}^2$.

The complement of any spacelike geodesic s of $\mathbf{dS}^2$ in $\mathbf{P}^+(\mathbb{R}^{1,2})$ is the union of a future affine chart $\mathcal{A}$ and of a past affine chart, in which $\partial^{\pm}_{\infty}\mathbf{dS}^2$ is a round circle, and of which $\mathbf{H}^2$ (respectively its past copy) is the interior and $\mathbf{dS}^2_s \coloneqq \mathbf{dS}^2 \cap \mathcal{A}$ the exterior. The traces in $\mathcal{A}$ of timelike, lightlike and spacelike geodesics of $\mathbf{dS}^2$ are the intervals in $\mathbf{dS}^2_s$ of the affine lines of $\mathcal{A}_s$ which respectively intersect, are tangent, and are disjoint from the future boundary $\partial^{+}_{\infty}\mathbf{dS}^2$. We refer to [Nur25, Figure 5 p.13] for a representation of such an affine chart, and for a thorough comparison of different models of $\mathbf{dS}^2$.

2.1.3. *De-Sitter plane and isometries.* The universal cover of the de-Sitter space, homeomorphic to $\mathbb{R}^2$, is denoted by $\widetilde{\mathbf{dS}}^2$ and called the *de-Sitter plane*. We endow $\widetilde{\mathbf{dS}}^2$ with the pullback of the metric of $\mathbf{dS}^2$. The automorphism group of the universal cover $\widetilde{\mathbf{dS}}^2 \to \mathbf{dS}^2$ is isomorphic to $\mathbb{Z}$ and generated by the action of a (closed) spacelike geodesic of $\mathbf{dS}^2$ on the universal cover.

[1]Note that if g is a Lorentzian metric of constant curvature 1 on a surface, then $-g$ is also a Lorentzian metric, and of constant curvature -1. Unlike the Riemannian case, there are therefore *only two local constant curvature Lorentzian geometries in dimension* 2 (up to anti-isometries and scaling by a constant factor): the linear model of curvature zero (the Minkowski plane $\mathbb{R}^{1,1}$), and the model of non-zero curvature (the de-Sitter space $\mathbf{dS}^2$).

The connected component of the identity in the stabiliser of $q_{1,2}$ in $\mathrm{SL}_3(\mathbb{R})$ is denoted by $\mathrm{SO}^0(1,2)$. It is the group of isometries of $\mathbf{dS}^2$ (and of $\mathbf{H}^2$) preserving both its orientation and its time-orientation (see for instance [MM24, Lemma 2.2]).

2.1.4. *Identification with the space of oriented geodesics of* $\mathbf{H}^2$*.* Any $p \in \mathbf{dS}^2$ is contained in a unique α (respectively β) lightlike geodesic p_α (resp. p_β) of $\mathbf{dS}^2$, having a past limit point $p_\alpha^\infty \in \partial_\infty^+ \mathbf{dS}^2$ (resp. a future limit point $p_\beta^\infty \in \partial_\infty^+ \mathbf{dS}^2$). This defines two $\mathrm{SO}^0(1,2)$-equivariant projections

$$\pi_{\alpha/\beta} \colon p \in \mathbf{dS}^2 \mapsto p_{\alpha/\beta}^\infty \in \partial_\infty^+ \mathbf{dS}^2$$

whose fibers are precisely the α (respectively β) lightlike geodesics of $\mathbf{dS}^2$. Note that $p_\alpha^\infty \neq p_\beta^\infty$, namely

$$(\pi_\alpha \times \pi_\beta)(p) \in (\partial_\infty^+ \mathbf{dS}^2)^{(2)} := (\partial_\infty^+ \mathbf{dS}^2)^2 \setminus \left\{(d,d) \,\middle|\, d \in \partial_\infty^+ \mathbf{dS}^2\right\}.$$

Moreover the $\mathrm{SO}^0(1,2)$-equivariant map

$$\pi_\alpha \times \pi_\beta \colon \mathbf{dS}^2 \to (\partial_\infty^+ \mathbf{dS}^2)^{(2)} \tag{2.1}$$

is a bijection. Identifying $\partial_\infty^+ \mathbf{dS}^2$ with the boundary of $\mathbf{H}^2$, $(\partial_\infty^+ \mathbf{dS}^2)^{(2)}$ is the space of oriented geodesics of $\mathbf{H}^2$ which is therefore identified with $\mathbf{dS}^2$ in a $\mathrm{SO}^0(1,2)$-equivariant way. The geodesic of $\mathbf{H}^2$ containing $(\pi_\alpha \times \pi_\beta)(p)$ and oriented from p_α^∞ to p_β^∞ is simply the intersection with $\mathbf{H}^2$ of the orthogonal of p in $\mathbb{R}^{1,2}$ (oriented compatibly with p and with the orientation of $\mathbb{R}^{1,2}$).

Remark 2.1. One can check that $\boldsymbol{g}$ is the unique pseudo-Riemannian metric of $\mathbf{dS}^2$ (up to scaling by a constant factor) which is invariant by $\mathrm{SO}^0(1,2)$ (see [MM24, §2.3] for instance). The de-Sitter geometry studied in this article is therefore the unique isometry-invariant geometry of the space of oriented geodesics of the hyperbolic plane.

2.2. **Singular de-Sitter tori.** We introduce in this paragraph all the notions about singular de-Sitter tori needed in this text. We refer to [MM24, §3 and §4] for more details and for proofs of the claims appearing in this paragraph.

2.2.1. *De-Sitter surfaces and holonomy morphism.* A *de-Sitter surface* is a surface locally modelled on $\mathbf{dS}^2$, in a sense made rigorous by the following notion of (G,X)*-structure*.

Definition 2.2. A $\mathbf{dS}^2$*-atlas* on an oriented topological surface S is an atlas of orientation-preserving topological charts $\varphi_i \colon U_i \to \mathbf{dS}^2$ from connected open subsets $U_i \subset S$ to $\mathbf{dS}^2$ (called $\mathbf{dS}^2$*-charts*), whose transition maps $\varphi_j \circ \varphi_i^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ equal the restriction of an element of $\mathrm{SO}^0(1,2)$. A $\mathbf{dS}^2$*-structure* on S is a maximal $\mathbf{dS}^2$-atlas, and a $\mathbf{dS}^2$*-surface* is an oriented surface endowed with a $\mathbf{dS}^2$-structure.

As in the case of hyperbolic surfaces, one can show that the data of a $\mathbf{dS}^2$-structure on S is exactly equivalent to that of a time-oriented Lorentzian metric of constant curvature 1 (see [MM24, Proposition-Definition 2.5]).

The restriction of the universal covering map to sufficiently small open subsets endows the universal cover $\tilde{S}$ of a $\mathbf{dS}^2$-surface S with an induced $\mathbf{dS}^2$-structure. For instance, $\widetilde{\mathbf{dS}}^2$

has a natural $\mathbf{dS}^2$-structure. The fundamental group $\pi_1(S)$ of S acts on $\tilde{S}$ by isometries, and this action is encoded through the *holonomy morphism*

$$\mathrm{hol}\colon \pi_1(S) \to \mathrm{SO}^0(1,2)$$

of S. The latter is characterized by the existence of a *developing map* $\delta\colon \tilde{S} \to \mathbf{dS}^2$ whose restriction to any sufficiently small subset is a $\mathbf{dS}^2$-chart, and which is equivariant with respect to the holonomy morphism hol.

2.2.2. *Singular de-Sitter surfaces and their lightlike foliations.* We now define the local model of singularities, originally introduced in [BBS11, p.160] and inspired from classical Riemannian conical singularities. Let $\mathsf{o} \in \widetilde{\mathbf{dS}}^2$ be a fixed base-point whose stabilizer in $\mathrm{SO}^0(1,2)$ is denoted by $\{a^u\}_{u\in\mathbb{R}}$. We fix $\theta > 0$. Any future timelike half-geodesic $\gamma \subset \widetilde{\mathbf{dS}}^2$ emanating from o bounds together with $a^\theta(\gamma)$ an open sector D_θ of $\widetilde{\mathbf{dS}}^2$ of angle θ indicated in Figure 2.1. The complement $\widetilde{\mathbf{dS}}^2 \setminus D_\theta$ is a $\mathbf{dS}^2$-surface with two timelike geodesic boundary components γ and $a^\theta(\gamma)$ and a conical point o. The quotient

$$\widetilde{\mathbf{dS}}^2_\theta \coloneqq (\widetilde{\mathbf{dS}}^2 \setminus D_\theta)/\sim_\theta$$

by the identification $\gamma \ni x \sim_\theta a^\theta(x) \in a^\theta(\gamma)$ of the boundary components is the *standard* $\mathbf{dS}^2$*-cone* of angle θ. It bears a marked point $\mathsf{o}_\theta \in \widetilde{\mathbf{dS}}^2_\theta$ which is the projection of o. The identification being made by isometries, $\widetilde{\mathbf{dS}}^2_\theta \setminus \{\mathsf{o}_\theta\}$ has a natural $\mathbf{dS}^2$-structure induced by that of $\widetilde{\mathbf{dS}}^2$. Note that $\widetilde{\mathbf{dS}}^2_\theta$ remains homeomorphic to $\mathbb{R}^2$. A similar construction defines the local singularity for $-\theta$, but we will only need positive singularities in this text as we will see in Proposition 2.7. The local singularity can be described in various other ways as detailed in [MM24, §3.1].

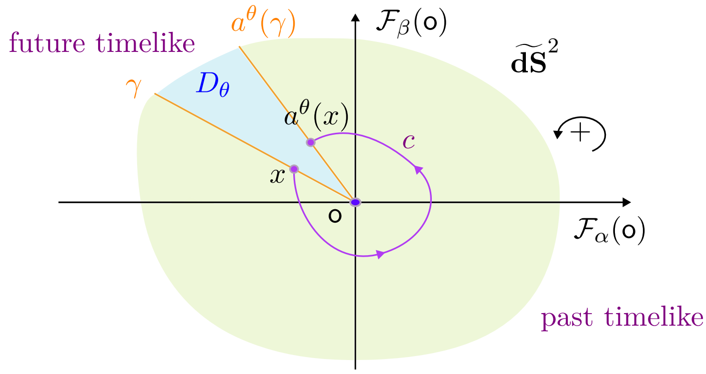

FIGURE 2.1. Local de-Sitter singularity of angle θ.

By the very definition of $\widetilde{\mathbf{dS}}^2_\theta$, the holonomy of a small curve c around o_θ equals a^θ (see Figure 2.1, and [MM24, §3.1.3] for more details). Since $a^\theta \neq \mathrm{id}$, this shows that the $\mathbf{dS}^2$-structure of $\widetilde{\mathbf{dS}}^2_\theta \setminus \{\mathsf{o}_\theta\}$ cannot be extended to o_θ. Indeed if it could, then the curve c would be homotopically trivial in the $\mathbf{dS}^2$-surface $\widetilde{\mathbf{dS}}^2_\theta$, and its holonomy would thus be trivial as well. While this argument may be unclear to a reader not used to the notion of (G,X)-structures, it partially explains the importance of the holonomy for singular $\mathbf{dS}^2$-structures that we now define.

Definition 2.3 ([MM24, Definition 3.16]). A *singular* $\mathbf{dS}^2$*-structure* g with *singular points* $\Sigma \subset S$ on an oriented surface S is a $\mathbf{dS}^2$-structure on $S \setminus \Sigma$ whose singular points are *standard singularities* in the following sense. For any $x \in \Sigma$, there exists $\theta_x \in \mathbb{R}^*$ (the *angle at* x) and a homeomorphism φ (called a *singular* $\mathbf{dS}^2$*-chart* at x) from an open neighbourhood $U \subset S$ of x to an open neighbourhood $V \subset \widetilde{\mathbf{dS}}^2_\theta$ of o_θ such that:

(1) $U \cap \Sigma = \{x\}$,
(2) $\varphi(x) = \mathsf{o}_\theta$,
(3) and φ is an isometry in restriction to $S \setminus \Sigma$.

Points $x \in S \setminus \Sigma$ are said *regular*. A *singular* $\mathbf{dS}^2$*-structure with geodesic boundary* on an oriented surface S with boundary is a singular $\mathbf{dS}^2$-structure on the interior of S, of which the boundary is geodesic (for the induced Lorentzian metric).

The following statement, proved in [MM24, Lemma 3.5], gives a useful way to know if a potential singular point is actually regular or not.

Lemma 2.4 ([MM24, Lemma 3.5]). *Let S be a singular* $\mathbf{dS}^2$*-surface of holonomy morphism* hol*. Let $x \in S$ be a standard singularity of angle θ, and c_x be the homotopy class of a small closed curve around x. Then* $\mathrm{hol}(c_x)$ *is conjugated to* a^θ*. In particular, x is a regular point if and only if* $\mathrm{hol}(c_x) = \mathrm{id}$.

An important property of a singular $\mathbf{dS}^2$-structure g on a surface S is that the lightlike foliations induced by g outside of the singularities extend on S to two transverse *topological* one-dimensional foliations (see [MM24, Lemma 3.24]). These foliations are still called the *lightlike foliations of* S, and are denoted by $(\mathcal{F}^g_\alpha, \mathcal{F}^g_\beta)$.

2.2.3. *Lorentzian angles and Gauß-Bonnet formula.* We will use in the sequel of the text a notion of Lorentzian angles that we now introduce following [BN84].

Definition 2.5 ([BN84]). Let P be an oriented plane endowed with a Lorentzian scalar product $\langle \cdot\,, \cdot\rangle$. For $X, Y \in P$, we denote $\mathsf{or}(X,Y) = 1$ (respectively -1) if (X,Y) is a positively (resp. negatively) oriented basis, and $\mathsf{or}(X,Y) = 0$ if (X,Y) are linearly dependent. Then for (X,Y) two unit timelike vectors belonging to the same quadrant of P, the Lorentzian *angle from X to Y* is defined by

$$(\!(X,Y)\!) := \mathsf{or}(X,Y)\,\mathrm{arcosh}\,|\langle X\,,Y\rangle|$$

with $\mathrm{arcosh}\colon [1\,;+\infty[\; \to \mathbb{R}^+$ the inverse hyperbolic cosine function. This definition is extended to any pair (X,Y) of unit timelike vectors by the relation

$$(\!(X,Y)\!) = (\!(X,-Y)\!)\,.$$

For a and b any two timelike (half-)geodesics of $\mathbf{dS}^2$ emanating from a point $p \in \mathbf{dS}^2$, we will denote by $(\!(a,b)\!)_p = (\!(\mathrm{T}_p a, \mathrm{T}_p b)\!)$ the angle from a to b at p.

Note that (2.5) is well-defined since $|\langle X\,,Y\rangle| \geq 1$ according to the Lorentzian Cauchy-Schwartz inequality. Furthermore for any two unit timelike vectors X, Y, the relations

$$(\!(-X,-Y)\!) = (\!(-X,Y)\!) = -\,(\!(Y,X)\!) = (\!(X,Y)\!) \tag{2.2}$$

follow easily from the definition (see [BN84, Lemma 1]). This notion of angles allows to prove the following Gauß-Bonnet formula for simply connected domains, due to Birman-Nomizu [BN84].

Proposition 2.6 ([BN84, p.80]). *Let $P \subset \widetilde{\mathbf{dS}}^2$ be a compact subset whose boundary is a polygon with timelike geodesic edges $(E_1, \dots, E_n)$ $(n \geq 3)$ of respective endpoints $(v_1, \dots, v_n)$. Let $\nu_i = (\!(E_i, E_{i+1})\!)_{v_i}$ denote the exterior angle at v_i for $i = 1, \dots, n$ (with $E_{n+1} \coloneqq E_1$). The area of P equals the sum of the exterior angles:*

$$\mathcal{A}(P) = \sum_{i=1}^{n} \nu_i.$$

Using this relation, the following Gauß-Bonnet formula was obtained for singular $\mathbf{dS}^2$-surfaces with geodesic boundary in [MM24, Proposition 3.28].

Proposition 2.7 ([MM24, Proposition 3.28]). *Let S be a compact singular $\mathbf{dS}^2$-surface with timelike geodesic boundary and n singularities of angles $(\theta_1, \dots, \theta_n)$. The area of S equals*

$$\mathcal{A}(S) = \sum_{i=1}^{n} \theta_i.$$

If S has a unique singularity x, x has thus a positive angle equal to the area of S.

2.2.4. *Examples of singular de-Sitter tori.* We conclude this section with examples of $\mathbf{dS}^2$-tori with a unique singularity. Recall first that $\mathbf{dS}^2$ identifies $\mathrm{SO}^0(1,2)$-equivariantly with $(\partial_\infty^+ \mathbf{dS}^2)^{(2)}$ according to (2.1). Since $\partial_\infty^+ \mathbf{dS}^2$ can itself be projectively identified with $\mathbb{R}\mathbf{P}^1$, $\mathbf{dS}^2$ is eventually identified with

$$\mathrm{dS}^2 \coloneqq (\mathbb{R}\mathbf{P}^1)^{(2)},$$

equivariantly with respect to an isomorphism from $\mathrm{SO}^0(1,2)$ to $\mathrm{PSL}_2(\mathbb{R})$ [MM24, Remark 2.2]. We work in this paragraph with the model dS^2 that we endow with its unique $\mathrm{PSL}_2(\mathbb{R})$-invariant Lorentzian metric of constant curvature 1 [MM24, §2.3]. We also identify $\mathbb{R}\mathbf{P}^1$ to $\mathbb{R} \cup \{\infty\}$, on which $\mathrm{PSL}_2(\mathbb{R})$ acts by homographies.

For any fixed $\theta > 0$, $x \in \,]1\,;\infty[$ and $y \in \,]0\,;1[$, there exists in dS^2 a unique polygon $\mathcal{L}_{\theta,x,y}$ with lightlike edges and area θ as indicated in Figure 2.2 [MM24, §4.3]. Any $x' \in \,]1\,;\infty[$ and $y' \in \,]0\,;y_+[$ moreover define unique identifications of the edges of $\partial\mathcal{L}_{\theta,x,y}$ by isometries in $\mathrm{PSL}_2(\mathbb{R})$ as indicated in Figure 2.2.[2] The quotient of $\mathcal{L}_{\theta,x,y}$ by these edges identifications is a $\mathbf{dS}^2$-torus $T_{x,y,x',y'}$ of area θ and with at most three singularities (at the projections of the vertices of $\mathcal{L}_{\theta,x,y}$). According to [MM24, Lemma 4.10], there exists unique parameters $(x'(x,y), y'(x,y))$ so that the (projection of the) purple point of coordinates $(1,0)$ is the unique singularity of $T_{x,y,x'(x,y),y'(x,y)}$. The $\mathbf{dS}^2$-torus

$$\mathcal{T}_{\theta,x,y} \coloneqq T_{x,y,x'(x,y),y'(x,y)}$$

having area θ, its unique singularity has thus angle θ according to Proposition 2.7.

[2]This is due to the fact that $\mathrm{PSL}_2(\mathbb{R})$ acts simply transitively on proper α (respectively β) lightlike segments of dS^2.

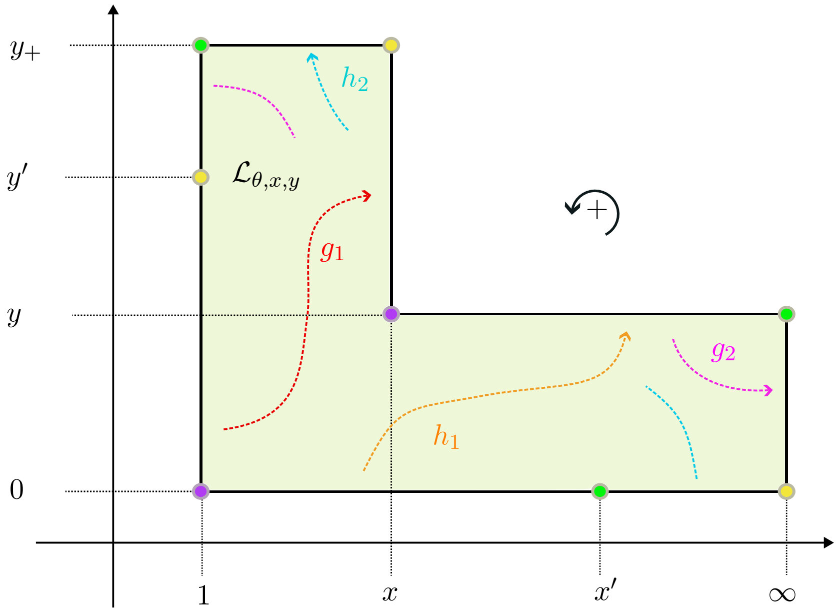

FIGURE 2.2. A class A singular $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,x,y}$.

We actually described all the de-Sitter structures that will be studied in this text, which are the *class A* structures to be defined below.[3] According to [MM24, Theorem 9.6], any class A $\mathbf{dS}^2$-torus of area θ with a unique singularity is indeed isometric to one of the tori $\mathcal{T}_{\theta,x,y}$ that we just constructed.

3. Timelike geodesics representatives in free homotopy classes

3.1. **Closed timelike geodesics.** We introduce in this paragraph the primary object of interest of this article: *closed geodesics* of singular de-Sitter surfaces.

Definition 3.1. A *geodesic loop* of a singular de-Sitter surface (S, Σ) is the image of a closed continuous curve $\gamma\colon [0\,;1] \to S$ such that $\gamma|_{]0;1[}$ is a parametrized geodesic of the regular de-Sitter surface $S^* := S \setminus \Sigma$. A *closed geodesic* is a geodesic loop such that $\gamma(0) = \gamma(1) \in S^*$ and $\gamma'(1) = \gamma'(0)$. A *multiple* of a closed geodesic γ is the closed geodesic $n\gamma$ obtained by travelling n times γ for some $n \in \mathbb{N}^*$.

Beware that geodesic loops and closed geodesics are unparametrized by default, and that a closed geodesic is $\mathcal{C}^\infty$ in this paper (which is sometimes called a $\mathcal{C}^1$-closed geodesic in the literature). We also stress that *closed geodesics do not contain any singularity.*

3.1.1. *Asymptotic cycles and class A structures.* In contrast to the Riemannian case, geodesics have different *signatures* in Lorentzian surfaces. We would like to translate this geometric distinction homologically, into a necessary and sufficient condition for a free homotopy class to contain a causal curve. To this end, we first need to materialize

[3]Note that it is not known yet if there exists de-Sitter tori with a unique singularity which are not class A.

homologically the lightlike foliations. This will be in this text the role of the *oriented projective asymptotic cycle* of a lightlike foliation $\mathcal{F}_{\alpha/\beta}$ of a de-Sitter structure on the two-torus $\mathbf{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, which is a half-line

$$A^+(\mathcal{F}_{\alpha/\beta}) \in \mathbf{P}^+(\mathrm{H}_1(\mathbb{R}, \mathbf{T}^2))$$

in the homology. Its *non-oriented projective asymptotic cycle* is the line $A(\mathcal{F}_{\alpha/\beta}) := \mathbb{R}A^+(\mathcal{F}_{\alpha/\beta})$. We will be interested in this article with singular de-Sitter structures g whose lightlike foliations have distinct non-oriented projective asymptotic cycles $A(\mathcal{F}^g_\alpha) \neq A(\mathcal{F}^g_\beta)$, which are called *class A structures.*

Remark 3.2. The lightlike foliations of a class A structure g are suspensions of circle homeomorphisms (see [MM24, Lemma 6.6]). Let $S \subset \mathbf{T}^2$ be any simple closed curve transverse to $\mathcal{F}^g_{\alpha/\beta}$. The first-return map $P_{\alpha/\beta}$ of $\mathcal{F}^g_{\alpha/\beta}$ on S is a homeomorphism of the circle S which describes entirely $\mathcal{F}^g_{\alpha/\beta}$ "locally", and whose main topological conjugacy invariant is the *rotation number* $\rho(P_{\alpha/\beta}) \in \mathbf{S}^1$. Observe however that the image of $\mathcal{F}^g_{\alpha/\beta}$ by a Dehn twist around S is not isotopic to $\mathcal{F}^g_{\alpha/\beta}$, while keeping the same first-return map. One therefore refines the rotation number of $P_{\alpha/\beta}$ to a *global topological isotopy invariant* of $\mathcal{F}^g_{\alpha/\beta}$, detecting such Dehn twists, which is its oriented projective asymptotic cycle $A(\mathcal{F}_{\alpha/\beta})$. We refer to [MM24, §5.2] for more details.

Since the asymptotic cycles are isotopy invariant, the asymptotic cycle of the isotopy class g of a singular de-Sitter structure g is well-defined.

3.1.2. *Timelike homotopy classes and existence of closed timelike geodesics.* Thanks to asymptotic cycles, we can now interpretate the three signatures of Lorentzian geometry for homotopy classes.

Definition 3.3. For any (isotopy class of) singular de-Sitter structure g of $\mathbf{T}^2$,

$$\mathcal{C}^{\mathsf{g}}_+ := \mathrm{Int}(\mathrm{Conv}(A^+(\mathcal{F}^{\mathsf{g}}_\beta) \cup -A^+(\mathcal{F}^{\mathsf{g}}_\alpha))) \subset \mathrm{H}_1(\mathbb{R}, \mathbf{T}^2)$$

is the *future timelike cone in homology* of g, which is the convex hull of the half-lines $A^+(\mathcal{F}^{\mathsf{g}}_\beta)$ and $-A^+(\mathcal{F}^{\mathsf{g}}_\alpha)$. We also denote by $\mathcal{C}^{\mathsf{g}}_+ := -\mathcal{C}^{\mathsf{g}}_-$ the past timelike cone, and by $\mathcal{C}^{\mathsf{g}} := \mathcal{C}^{\mathsf{g}}_+ \cup \mathcal{C}^{\mathsf{g}}_-$ the *timelike cone in homology.* We define likewise the *spacelike cone in homology* $\mathcal{C}^{\mathsf{g}}_{space}$ of g.

Remarks 3.4.
(1) One may note that $A^+(\mathcal{F}^{\mathsf{g}}_\alpha)$ and $A^+(\mathcal{F}^{\mathsf{g}}_\beta)$ are the oriented lightlike lines of a unique (up to positive scalar multiplication) Lorentzian quadratic form on $\mathrm{H}_1(\mathbb{R}, \mathbf{T}^2)$, of which $\mathcal{C}^{\mathsf{g}}$ is the timelike cone.
(2) If g is a class B structure *i.e.* $A(\mathcal{F}^{\mathsf{g}}_\alpha) = A(\mathcal{F}^{\mathsf{g}}_\beta)$, then either $\mathcal{C}^{\mathsf{g}}$ or $\mathcal{C}^{\mathsf{g}}_{space}$ is empty.

The following existence result is proved in [MM24, Appendix A]. It is a reformulation of Proposition A.8, Corollary A.11 and Theorem A.17 of this article.

Theorem 3.5 ([MM24, Appendix A])**.** *Let g be a class A de-Sitter structure of $\mathbf{T}^2$ with a unique singularity. Then a free homotopy class c of closed curves contains a closed timelike geodesic which maximizes the length among causal curves in c, if and only if c belongs to the timelike cone $\mathcal{C}^g$ in homology.*

3.2. **Uniqueness of closed timelike geodesics.** We make the naive observation that for any base-point $x \in \mathbf{T}^2$, the natural map sending a homotopy class $a \in \pi_1(\mathbf{T}^2, x)$ relative to x to the free homotopy class containing a is an isomorphism.[4] We therefore identify through these maps the group of free homotopy classes with the fundamental groups of $\mathbf{T}^2$, which we denote by $\pi_1(\mathbf{T}^2) \simeq \mathbb{Z}^2$. A free homotopy class $a \in \pi_1(\mathbf{T}^2)$ is said *primitive* if it is not a non-trivial multiple, *i.e.* if it belongs to $\pi_1(\mathbf{T}^2)_* \simeq \mathbb{Z}^2_* := \mathbb{Z}^2 \setminus \mathbb{N}^*\mathbb{Z}^2$. We recall that a free homotopy class of $\pi_1(\mathbf{T}^2)$ is primitive if and only if it contains a *simple* closed curve (*i.e.* a closed curve without self-intersections). We will denote by $[\gamma]$ the free homotopy class of a closed curve γ.

Unlike the case of hyperbolic surfaces, we need to show that closed geodesics in primitive homotopy classes are simple *before* proving the uniqueness.

Lemma 3.6. *Let γ be a definite closed geodesic of a class A de-Sitter torus T with a single singularity. If the free homotopy class of γ in $\pi_1(T)$ is primitive, then γ is simple.*

Proof. We can assume that γ is timelike to fix ideas, the arguments being the same in the spacelike case by inverting the metric. Assume by contradiction that the timelike closed geodesic $\gamma \subset T$ is not simple. Since γ is freely homotopic to a simple closed curve by assumption, it must then contain a non-trivial embedded 1-gon or 2-gon according to [HS85, Theorem 2.7 p.92]. A 1-gon in γ would be a homotopically trivial timelike closed curve, which is forbidden by [MM24, Corollary A.6]. There exists therefore a closed topological disk D in T whose boundary is the union of two distinct simple timelike geodesic segments a and b, such that $a \cap b = \partial a = \partial b$. Assume that D contains the unique singularity p of T, of positive angle θ. Since p is the unique singularity of D and D is simply connected, any singular $\mathbf{dS}^2$-chart at p extends to an open neighborhood U of D, to give a topological embedding $\varphi\colon U \to \widetilde{\mathbf{dS}}^2_\theta$ which is a $\mathbf{dS}^2$-chart outside of p and such that $\varphi(p) = \mathsf{o}_\theta$. This is due to [MM24, Lemma 3.5] in the one hand, and on the other hand to the fact that any class A $\mathbf{dS}^2$-torus is isometric to a torus $\mathcal{T}_{\theta,x,y}$ as described in Figure 2.2, according to [MM24, Theorem 9.6.(1)]. If D does not contain the singularity, then such a $\mathbf{dS}^2$-chart φ also exist and we denote it in the same way with $\theta = 0$. The image by φ of ∂D gives two distinct simple timelike geodesic segments a_0 and b_0 in $\widetilde{\mathbf{dS}}^2_\theta \setminus \{\mathsf{o}_\theta\}$, which meet two times at their extremities. But one easily observe that two distinct geodesics in $\widetilde{\mathbf{dS}}^2$ or $\widetilde{\mathbf{dS}}^2_\theta \setminus \{\mathsf{o}_\theta\}$ meet at most one (see for instance [MM24, Figure A.1]). This contradiction concludes the proof. ☐

3.2.1. *Asymptotic points of timelike geodesics in the de-Sitter space.* Any timelike geodesic l of $\mathbf{dS}^2$ has a unique *future limit point* $l^+ \in \partial^+_\infty \mathbf{dS}^2$ (respectively *past limit point* $l^- \in \partial^-_\infty \mathbf{dS}^2$) in $\mathbf{P}^+(\mathbb{R}^{1,2})$. Moreover $l^+ \neq -l^-$ and l is uniquely described by (l^+, l^-): $(l_1^+, l_1^-) = (l_2^+, l_2^-)$ if and only if $l_1 = l_2$. In other words, the space of timelike geodesics of $\mathbf{dS}^2$ identifies with $\mathbf{dS}^2$ (by taking the orthogonal in $\mathbb{R}^{1,2}$), hence with the space of oriented geodesics of $\mathbf{H}^2$.

Let L_1 and L_2 be two timelike planes of $\mathbb{R}^{1,2}$, and $L_i^\pm$ be the four corresponding timelike geodesics of $\mathbf{dS}^2$. The three possible configurations of L_1 and L_2 are the following.

(1) $L_1 \cap L_2$ *is lightlike*, and contains thus one future and one past half-line $(L_1 \cap L_2)^\pm$.

[4] Because $\pi_1(\mathbf{T}^2, x) \simeq \mathbb{Z}^2$ is abelian.

  - The closure in $\mathbf{P}^+(\mathbb{R}^{1,2})$ of one of the components L_1^+ (respectively L_1^-) of L_1 meets the closure of one of the components L_2^+ (resp. L_2^-) of L_2 along $(L_1 \cap L_2)^+$ (resp. $(L_1 \cap L_2)^-$). The timelike geodesics $L_1^\pm$ and $L_2^\pm$ of $\mathbf{dS}^2$ are said *future asymptotic* (resp. *past asymptotic*).
  - The closure of L_1^+ does not meet L_2^- (resp. of L_1^- does not meet L_2^+).

(2) $L_1 \cap L_2$ *is spacelike*, and we denote by $(L_1 \cap L_2)^\pm$ its two half-lines.[5]
  - One of the components L_1^+ (respectively L_1^-) meets the corresponding L_2^+ (resp. L_2^-) in $\mathbf{dS}^2$ along $(L_1 \cap L_2)^+$ (resp. $(L_1 \cap L_2)^-$).
  - The closure of L_1^+ does not meet L_2^- (resp. of L_1^- does not meet L_2^+).

(3) $L_1 \cap L_2$ *is timelike*. Then the four connected component's closures of L_1 and L_2 are pairwise disjoint.

In hyperbolic surfaces, the uniqueness of geodesics in their free homotopy class follows from the classical fact that two geodesics of the hyperbolic plane which remain at a bounded distance must be equal. In the de-Sitter space $\mathbf{dS}^2$, we will replace this argument by a phenomenon of *spacelike connectedness* between asymptotic timelike geodesics.

Lemma 3.7. *Let l_1, l_2 be two future timelike geodesics of $\mathbf{dS}^2$ such that for any $p_i \in l_i$, there exists a spacelike geodesic segment joining $[p_1\,; +\infty[_{l_1}$ and $[p_2\,; +\infty[_{l_2}$. Then l_1 and l_2 are future asymptotic. The obvious analogous claim holds in the past.*

Proof. Let us denote by L_i the timelike projective line of $\mathbf{P}^+(\mathbb{R}^{1,2})$ containing l_i. We observe first that $L_1 \neq L_2$. Indeed if $L_1 = L_2$ but $l_1 \neq l_2$ by contradiction, then with $s_0 \subset \mathbf{dS}^2$ a spacelike geodesic intersecting each l_i at some point p_i, one notice that any spacelike geodesic s_1 intersecting $]p_1\,; +\infty[_{l_1}$ would not intersect $]p_2\,; +\infty[_{l_2}$, which would contradict our assumption. We now reason by contraposition, assuming that l_1 and l_2 are *not* future asymptotic. Since $L_1 \neq L_2$, there exists a lightlike projective line L of $\mathbf{P}^+(\mathbb{R}^{1,2})$ which intersects l_1 and l_2 at respective points p_1 and p_2. One check then easily that any spacelike geodesic $s \subset \mathbf{dS}^2$ which intersects $]p_1\,; +\infty[_{l_1}$ will not intersect $]p_2\,; +\infty[_{l_2}$. This concludes the proof of the contrapositive of the lemma. ☐

3.2.2. *Proof of Proposition A.* We first show the uniqueness of closed geodesics in primitive homotopy classes.

Lemma 3.8. *Let γ_1 and γ_2 be two definite closed geodesics of a class A de-Sitter torus T with a single singularity, having primitive free homotopy classes in $\pi_1(T)$. If γ_1 and γ_2 are freely homotopic in T, then $\gamma_1 = \gamma_2$ as unparametrized geodesics.*

Proof. Denoting by $\mathsf{0} \in T$ the singularity, the fundamental group of $T_* := T \setminus \{\mathsf{0}\}$ (based at any point) is a free group on two generators, and the inclusion $T_* \subset T$ induces a morphism $\psi\colon \pi_1(T_*) \to \pi_1(T) \simeq \mathbb{Z}^2$ which is the abelianization. The homotopy classes of simple closed curves in T_* are exactly the *primitive* elements of $\pi_1(T_*)$ (namely the ones that can be completed to form a basis of the free group $\pi_1(T_*)$), and two primitive homotopy classes of $\pi_1(T_*)$ belong to the same free homotopy class of curves in T_* if and only if they are conjugated in $\pi_1(T_*)$. Since γ_1 and γ_2 are simple closed curves according to Lemma 3.6, their homotopy classes are primitive in $\pi_1(T_*)$. Since they are freely homotopic in T by assumption, they have moreover the same image by the abelianization ψ. A Theorem

[5]Observe that the sign $\pm$ is here arbitrary and has no meaning of time-orientation.

of Nielsen [Nie17] shows then that the homotopy classes of γ_1 and γ_2 in T_* are conjugated (see [OZ81, Corollary 3.2 p.20]), *i.e.* that γ_1 and γ_2 are freely homotopic in T_*.

As in the proof of Lemma 3.6, we can assume that γ_1 and γ_2 are timelike to fix ideas, up to inverting the metric. Since γ_1 and γ_2 are freely homotopic in T_*, they are contained in a closed annulus $A \subset T_*$ bounded by two closed timelike curves (freely homotopic to γ_1 and γ_2). The proof of the existence Theorem of closed definite geodesics shows the existence of a spacelike geodesic segment $\eta \subset A$ joining γ_1 and γ_2. More precisely, [MM24, Lemma A.15, Proposition A.16 and Theorem A.17] show the existence of such a spacelike locally maximizing segment $\eta \subset A$, and [MM24, Proposition A.10] shows that η is a geodesic segment (since it does not contain any singularity). We fix a connected component $\tilde{A} \subset \tilde{T}_*$ of the lift of the annulus $A \subset T_*$ in the universal cover $\tilde{T}_*$. This topological band contains one lift $\tilde{\gamma}_i$ of each of the two closed timelike geodesics γ_i. Let $\delta\colon \tilde{T}_* \to \mathbf{dS}^2$ be a developing map of the $\mathbf{dS}^2$-structure of T_*. The anti-Riemannian metrics of γ_1 and γ_2 coming from the metric of T_* are complete since the γ_i's are compact, and $\tilde{\gamma}_i$ is therefore complete as well. The restriction of δ is a local isometry from the complete timelike geodesic $\tilde{\gamma}_i$ to a timelike geodesic l_i of $\mathbf{dS}^2$, and such a map has to be surjective onto l_i. Indeed for any $p \in \tilde{\gamma}_i$ and any $q' \in l_i$ in the future of $\delta(p)$, the distance r from $\delta(p)$ to q' along l_i is finite. There exists thus by completeness of $\tilde{\gamma}_i$ a point $q \in \tilde{\gamma}_i$ at distance r in the future of p, and we have then $\delta(q) = q'$ which shows that $\delta(\tilde{\gamma}_i) = l_i$.

Let $p_1' = \delta(p_1) \in l_1$ and $p_2' = \delta(p_2) \in l_2$ be any two points. There exists then a lift $\tilde{\eta} \subset \tilde{A}$ of η going from the future interval $[p_1\,; +\infty[_{\tilde{\gamma}_1}$ of $\tilde{\gamma}_1$ starting at p_1 to the future interval $[p_2\,; +\infty[_{\tilde{\gamma}_2}$ of $\tilde{\gamma}_2$ starting at p_2. The image of $\tilde{\eta}$ by δ is a spacelike geodesic segment of $\mathbf{dS}^2$ going from $[p_1'\,; +\infty[_{l_1}$ to $[p_2'\,; +\infty[_{l_2}$. Since such a geodesic segment exists for any two $p_1' \in l_1$ and $p_2' \in l_2$, Lemma 3.7 shows that l_1 and l_2 are future asymptotic. The obvious analogous reasoning shows that l_1 and l_2 are past asymptotic, and therefore that $l_1 = l_2$ since the future and past limit points of a timelike geodesic of $\mathbf{dS}^2$ entirely determine it.

Since the de-Sitter structure of T is class A, T is isometric to a singular de-Sitter torus $\mathcal{T}_{\theta,x,y}$ described in Figure 2.2 according to [MM24, Theorem 9.6.(1)]. The lightlike polygon $\mathcal{L}_{\theta,x,y}$ being a fundamental domain for T, its interior lifts in $\tilde{T}_*$ to an open set $\widetilde{\mathcal{L}}$ on which the developing map δ is injective. We can choose a lift $\widetilde{\mathcal{L}}$ intersecting the band $\tilde{A}$, hence containing open intervals of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Since we have shown that $\delta(\tilde{\gamma}_1) = \delta(\tilde{\gamma}_2)$ and $\delta|_{\widetilde{\mathcal{L}}}$ is injective, this shows that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ admit a common open interval, and therefore that $\tilde{\gamma}_1 = \tilde{\gamma}_2$. Hence $\gamma_1 = \gamma_2$, which concludes the proof of the lemma. □

We can now conclude the proof of Proposition A.

Proof of Proposition A. As previously, we can assume that γ is timelike to fix ideas. If the timelike free homotopy class c is primitive, the claim is a consequence of the existence Theorem 3.5, and of Lemmas 3.6 and 3.8. Assume now that $c = nc_0$ with c_0 a primitive class and $n \geq 2$. The existence being given by the primitive case, it suffices to show that any geodesic γ in the free homotopy class c has to be a multiple. Indeed, such a γ is then of the form $\gamma = n\gamma_0$ with γ_0 in the primitive class c_0, and the uniqueness of γ in the class c follows thus from the one of γ_0 in the primitive class c_0 (which was already proved). Let us assume by contradiction that γ is not a multiple. It is then a (general position) closed geodesic with excess self-intersection points in its free homotopy class, and γ must thus contain a singular 1-gon or 2-gon according to [HS85, Theorem 4.2]. Since a singular 1-gon would give a homotopically trivial timelike closed curve forbidden by [MM24, Corollary A.6], γ eventually contains a singular 2-gon. As in the proof of Lemma 3.6, this situation leads to two distinct timelike geodesics in $\widetilde{\mathbf{dS}}^2$ having two intersection points, which is impossible. This contradiction concludes the proof of the theorem. ☐

4. Length-twist coordinates on the deformation space of singular de-Sitter tori

In this section, we define coordinates in the *deformation space* $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ which are analogous to the classical Fenchel-Nielsen coordinates in the Teichmüller space.

We denote by $\mathsf{Def}_\theta(\mathbf{T}^2,0)$ (respectively $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$) the space of (resp. class A) singular de-Sitter structures on $\mathbf{T}^2$ of fixed area $\theta > 0$ and with a unique singular point at $0 \in \mathbf{T}^2$, quotiented by the group $\mathrm{Homeo}^0(\mathbf{T}^2,0)$ of homeomorphisms of $\mathbf{T}^2$ isotopic to the identity relative to 0 (see [MM24, §6.1] for more details on $\mathsf{Def}_\theta(\mathbf{T}^2,0)$ and its topology). We will denote by g a singular de-Sitter structure and by $\mathsf{g} \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$ its isotopy class. The quotient $\mathrm{PMod}(\mathbf{T}^2,0)$ of homeomorphisms of $\mathbf{T}^2$ fixing 0 by $\mathrm{Homeo}^0(\mathbf{T}^2,0)$ is called the *modular group of* $(\mathbf{T}^2,0)$, and naturally acts on $\mathsf{Def}_\theta(\mathbf{T}^2,0)$.

We will denote by $L(\gamma)$ the length of any causal curve γ in a $\mathbf{dS}^2$-surface (the context preventing any confusion regarding the surface under consideration).

4.1. de-Sitter tori as identification space of rectangles with two timelike edges.

As in the hyperbolic case, the first step is to uniquely characterize a polygon in the de-Sitter space by the length of its sides.

Lemma 4.1. *For any $(k,l) \in (\mathbb{R}_+^*)^2$, there exists up to action of $\mathrm{SO}^0(1,2)$ a unique rectangle $\mathcal{R}_{\theta,k,l}$ in $\mathbf{dS}^2$ satisfying the following conditions.*

- *The oriented boundary of $\mathcal{R}_{\theta,k,l}$ is formed of four geodesic segments which are consecutively future α lightlike, future timelike, past α lightlike, and past timelike.*
- *The left (respectively right) timelike edge has length k (resp. l).*

Proof. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two such rectangles. We name the vertices and edges of our rectangles as indicated on the reference rectangle $\mathcal{R}_{\theta,k,l}$ on the left-hand side of Figure 4.1. Since $\mathrm{SO}^0(1,2)$ acts transitively on $\mathbf{dS}^2$, we can assume without loss of generality that $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same the bottom left vertex $x \in \mathbf{dS}^2$. Since $\mathrm{Stab}_{\mathrm{SO}^0(1,2)}(x)$ moreover acts simply transitively on timelike directions at x, we can also assume without loss of generality that the left timelike eddge J_1 and J_2 of $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same direction at x. Since those edges have by assumption the same length k, we eventually have $J_1 = J_2 = J$. If the open right timelike edges $\mathrm{Int}(I_1)$ and $\mathrm{Int}(I_2)$ were disjoint, then $\mathcal{R}_1$ and $\mathcal{R}_2$ would not have the same area, contradicting our assumption. Therefore, $\mathrm{Int}(I_1)$ and $\mathrm{Int}(I_2)$ intersect at a point z indicated on the right-hand side of Figure 4.1. The β-leaf of z intersects the α-leaf of y_1' (respectively of y_1) at a point z_+ (resp. z_-). The point z defines together with y_1' and y_2' (resp. with y_1 and y_2) a triangle T_2 (resp. T_1). The rectangles $\mathcal{R}_1$ and $\mathcal{R}_2$ are the respective union (disjoint outside of edges) of a common subrectangle $\mathcal{R}_0$ and of the triangles T_1 and T_2. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same area, T_1 and T_2 have thus the same area. There exists a unique orientation-preserving isometry φ of $\mathbf{dS}^2$ which reverses the time-orientation, fixes z and preserves each non-oriented geodesic through z. Since the angles θ between the timelike edges of T_1 and $\varphi(T_2)$ are equal and their only non-timelike edge is α lightlike, one of the two triangles T_1 and $\varphi(T_2)$ is contained in the other. These triangles having the same area, this shows that $T_1 = \varphi(T_2)$, *i.e.* that the right-hand side of Figure 4.1 has a central symmetry with respect to z. Since I_1 and I_2 have the same length by assumption, we have thus $L([z\,;y_1]_{I_1}) = \frac{L(I_1)}{2} = \frac{L(I_2)}{2} = L([z\,;y_2]_{I_2})$ (with $[p\,;q]_I$ the segment from p to q on an oriented geodesic I), and therefore $y_1 = y_2$. Let indeed denote by a^θ the unique isometry fixing z and sending the direction $[z\,;y_2]_{I_2}$ on the direction $[z\,;y_1]_{I_1}$. Then $a^\theta(y_2) = y_1$ since $L([z\,;y_1]_{I_1}) = L([z\,;y_2]_{I_2})$, hence a^θ fixes $z_- \in \mathcal{F}_\beta(z) \setminus \{z\}$. Since $\mathrm{Stab}_{\mathrm{SO}^0(1,2)}(z)$ acts simply transitively on $\mathcal{F}_\beta(z) \setminus \{z\}$, this shows that $a^\theta = \mathrm{id}$ hence that $y_1 = y_2$. Therefore $y_1' = y_2'$ as well, which shows that $\mathcal{R}_1 = \mathcal{R}_2$ and concludes the proof. □

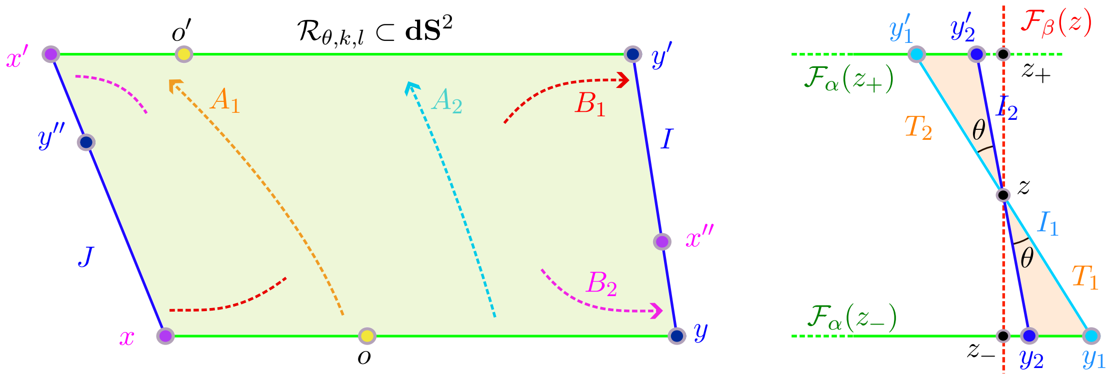

FIGURE 4.1. Rectangle $\mathcal{R}_{\theta,k,l}$ of $\mathbf{dS}^2$ with area θ, α lightlike horizontal edges and timelike vertical edges of lengths (k, l).

We will denote $\mathcal{R}_{\theta,l} := \mathcal{R}_{\theta,l,l}$ when both timelike sides of the rectangle are equal.

Remark 4.2. The choice of points (y'', x'', o, o') on the boundary of $\mathcal{R}_{\theta,l}$ defines identifications of the resulting eight edges of $\partial\mathcal{R}_{\theta,l}$ by isometries (A_1, A_2, B_1, B_2) of $\mathrm{SO}^0(1,2)$, as indicated in Figure 4.1. The quotient of $\mathcal{R}_{\theta,l}$ by such identifications is a $\mathbf{dS}^2$-torus of area θ and with at most three singular points (y, x, o), henceforth called an *identification space* of $\mathcal{R}_{\theta,l}$. We can analogously construct a de-Sitter annulus with (piecewise) geodesic timelike boundary as an *identification space of the lightlike edges of* $\mathcal{R}_{\theta,k,l}$.

The de-Sitter annuli with timelike geodesic boundary and containing a single singularity are basic building blocks for singular de-Sitter tori.

Lemma 4.3. *Let γ_1 and γ_2 be two simple closed timelike geodesics of a class A singular de-Sitter torus T.*

(1) *If $\gamma_1 = \gamma_2$ and T contains a unique singularity, then T is isometric to an identification space of the rectangle $\mathcal{R}_{\theta,l}$, with l the length of $\gamma_1 = \gamma_2$ and θ the area of T.*

(2) *Assume that* γ_1 *and* γ_2 *are freely homotopic, disjoint and bound an open annulus* A *of area* θ *containing a single singularity. Then the singular de-Sitter sub-annulus* $\mathrm{Cl}(A) \subset T$ *with timelike geodesic boundary is isometric to an identification space of the lightlike edges of* $\mathcal{R}_{\theta,l_1,l_2}$*, with* l_i *the length of* γ_i*.*

Proof. We consider the case (2), the proof of (1) being analogous. Let p be the unique singularity of A. Since T is class A, the α leaf of p describes an arc δ in A joining γ_1 to γ_2. Let R be the de-Sitter surface with geodesic boundary obtained by cuting A along δ. Since the interior and the timelike boundaries of A do not contain any singularity, the same proof *mutatis mutandis* than the rectangular case in [MM24, Proposition 9.2] shows that R is isometric to the rectangle $\mathcal{R}_{\theta,l_1,l_2} \subset \mathbf{dS}^2$, and that A is an identification space of $\mathcal{R}_{\theta,l_1,l_2}$. □

4.2. **Length-twist coordinates in** $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$**.** Let $(\mathsf{a}, \mathsf{b}) \in (\pi_1(\mathbf{T}^2)_*)^2$ be a fixed pair of primitive homotopy classes of algebraic intersection number $\hat{i}(\mathsf{a}, \mathsf{b}) = 1$. We can now conclude the construction of coordinates in the open subset

$$\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A_{\mathsf{a}} := \left\{ \mathsf{g} \in \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0}) \;\middle|\; \mathsf{a} \in \mathcal{C}^{\mathsf{g}} \right\}$$

of class A structures of which a is a timelike class. Proposition A already gave us a *length function of* a on $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A_{\mathsf{a}}$, defined as

$$\mathcal{L}_{\mathsf{a}} \colon \mathsf{g} \in \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A_{\mathsf{a}} \mapsto L^g(\gamma^g_{\mathsf{a}}) \in \mathbb{R}^*_+$$

with γ^g_{a} the unique simple closed timelike geodesic of g in the free homotopy class a. Since the lightlike foliations of a class A structure are suspensions, the α-lightlike leaf of the singularity $\mathsf{0}$ has a first intersection with the transverse closed curve γ^g_{a} in the future (respectively in the past) at a point denoted by x_+ (resp. x_-). We refer to [MM24, Proof of Lemma 9.9] for more details on this claim. The concatenation of the α-lightlike segment $[x_+ \,; x_-]_\alpha$ and of the future segment $[x_- \,; x_+]_{\gamma^g_{\mathsf{a}}}$ on the oriented geodesic γ^g_{a} is a closed curve, which can be written as

$$[x_+ \,; x_-]_\alpha \cdot [x_- \,; x_+]_{\gamma^g_{\mathsf{a}}} = \mathsf{b} + n_{\mathsf{a},\mathsf{b}}(\mathsf{g}) \cdot \mathsf{a}$$

for a unique $n_{\mathsf{a},\mathsf{b}}(\mathsf{g}) \in \mathbb{Z}$ (depending only on the isotopy class g). We adopt the convention that the segment $]x_- \,; x_+[_{\gamma^g_{\mathsf{a}}}$ is empty if $x_- = x_+$ (*i.e.* if the α-lightlike leaf of $\mathsf{0}$ is closed).

Definition 4.4. The *twist coordinate around* a *with longitude* b of $\mathsf{g} \in \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A_{\mathsf{a}}$ is the real number

$$\Theta_{\mathsf{a},\mathsf{b}}(\mathsf{g}) := n_{\mathsf{a},\mathsf{b}}(\mathsf{g}) + \frac{L^g\left(]x_- \,; x_+[_{\gamma^g_{\mathsf{a}}}\right)}{\mathcal{L}_{\mathsf{a}}(\mathsf{g})} \in \mathbb{R}.$$

We can now conclude the proof of Theorem C.

Proof of Theorem C. We recall from Lemma 4.1 that the rectangle $\mathcal{R}_{\theta,l}$ is unique up to the action of $\mathrm{SO}^0(1,2)$, and we henceforth fix once and for all such a rectangle. The points (x, y, x', y') are thus fixed. Since $\mathrm{SO}^0(1,2)$ acts simply transitively on future timelike segments of the same finite length of $\mathbf{dS}^2$, and also on proper future α lightlike segments of $\mathbf{dS}^2$, the choice of points (y'', x'', o, o') on $\partial\mathcal{R}_{\theta,l}$ entirely determines the identifications (A_1, A_2, B_1, B_2) of the Figure 4.1. The timelike edges $[x \,; y'']_J$ and $[x'' \,; y']_I$ having moreover the same length, x'' is actually determined by y''. In the end, the three points

(y'', o, o') on $\partial\mathcal{R}_{\theta,l}$ entirely determine a $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,l}(y'', o, o')$ with at most three singularities at the respective projections x, y and o of the marked points on the boundary (named by the same letters on Figure 4.1). The holonomy of a small positively oriented closed curve $\gamma_{x/y/o}$ around one of those points is respectively equal to:

$$\mathrm{hol}(\gamma_x) = A_1^1 B_2^{-1} B_1, \mathrm{hol}(\gamma_y) = B_2 B_1^{-1} A_2 \text{ and } \mathrm{hol}(\gamma_o) = A_2^{-1} A_1.$$

We refer to [MM24, Proposition 4.1] for more details concerning the computation of these holonomies. Therefore, x and y are regular points of $\mathcal{T}_{\theta,l}(y'', o, o')$ if and only if

$$A_1 = B_2^{-1} B_1 \text{ and } A_2 = B_1 B_2^{-1} \tag{4.1}$$

according to Lemma 2.4. In other words, under the condition of having a unique singular point at $o \in \mathcal{T}_{\theta,l}(y'', o, o')$, $y'' \in \partial\mathcal{R}_{\theta,l}$ entirely determines (B_1, B_2) hence (A_1, A_2) according to (4.1), which in return determines (o, o').

Let γ denote the timelike simple geodesic loop of $\mathcal{T}_{\theta,l}(y'', o, o')$ defined by the projection of a timelike edge (I or J) of $\mathcal{R}_{\theta,l}$. We have shown so far that $y'' \in \partial\mathcal{R}_{\theta,l}$ determines a unique $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,l,y''}$ of area θ for which γ has no singularity, *i.e.* is a simple closed geodesic. Moreover, γ has length l and $\mathcal{T}_{\theta,l,y''}$ has a unique singularity at o. Let a denote the free homotopy class of γ, and b denote the free homotopy class of the simple closed curve of $\mathcal{T}_{\theta,l,y''}$ obtained by projecting a simple arc of $\mathcal{R}_{\theta,l}$ intersecting its boundary only at y and y'' and oriented from I to J. There exists, up to pre-composition by a homeomorphism of $\mathbf{T}^2$ isotopic to the identity relative to $\mathsf{0}$, a unique homeomorphism $\Phi\colon \mathbf{T}^2 \to \mathcal{T}_{\theta,l,y''}$ sending $\mathsf{0}$ on o and the homotopy basis (a, b) on (a, b). This well-known fact is due to a result of Epstein [Eps66] (see also [BCLR20, Proposition 1.6, Theorem 2] for more details). We henceforth denote by $\mathsf{g}_{\theta,l,y''} \in \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A$ the pull-back of the $\mathbf{dS}^2$-structure of $\mathcal{T}_{\theta,l,y''}$ by Φ on $\mathbf{T}^2$. Let $D_\mathsf{a} \in \mathrm{PMod}(\mathbf{T}^2, \mathsf{0})$ denote the *positive Dehn twist* around a, namely the unique mapping class satisfying $D_\mathsf{a}(\mathsf{a}) = \mathsf{a}$ and $D_\mathsf{a}(\mathsf{b}) = \mathsf{b} + \mathsf{a}$. Then for any $n \in \mathbb{Z}$, the length-twist coordinates of $(D_\mathsf{a}^n)_* \mathsf{g}_{\theta,l,y''} \in \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A$ are given by

$$\mathcal{L}_\mathsf{a}((D_\mathsf{a}^n)_* \mathsf{g}_{\theta,l,y''}) = l \text{ and } \Theta_{\mathsf{a},\mathsf{b}}((D_\mathsf{a}^n)_* \mathsf{g}_{\theta,l,y''}) = n + \frac{L([x\,; y'']_J)}{l}, \tag{4.2}$$

by the very definition of Φ. This shows the surjectivity of the length-twist map

$$\mathcal{L}_\mathsf{a} \times \Theta_{\mathsf{a},\mathsf{b}}\colon \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A_\mathsf{a} \to \mathbb{R}_+^* \times \mathbb{R}.$$

Let now g_1 and g_2 be two points of $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A_\mathsf{a}$ having the same length-twist coordinates $\mathcal{L}_\mathsf{a}(\mathsf{g}_i) = l$ and $\Theta_{\mathsf{a},\mathsf{b}}(\mathsf{g}_i) = n + u$, with $n \in \mathbb{Z}$ and $u \in [0\,; 1[$. Lemma 4.3 shows that $(D_\mathsf{a}^{-n})_* \mathsf{g}_1$ and $(D_\mathsf{a}^{-n})_* \mathsf{g}_2$ are respectively isometric to structures $\mathsf{g}_{\theta,l,y_1}$ and $\mathsf{g}_{\theta,l,y_2}$, and (4.2) shows that $y_1 = y_2$ since $\Theta_{\mathsf{a},\mathsf{b}}(\mathsf{g}_1) = \Theta_{\mathsf{a},\mathsf{b}}(\mathsf{g}_2)$. This shows that $\mathsf{g}_1 = \mathsf{g}_2$, hence the injectivity of $\mathcal{L}_\mathsf{a} \times \Theta_{\mathsf{a},\mathsf{b}}$. This map being clearly continuous, it remains to show that its inverse is continuous as well.

We fix the bottom-left vertex x of $\mathcal{R}_{\theta,l}$ as well as the future timelike half-geodesic $d \subset \mathbf{dS}^2$ containing J. Let p_r denote the unique point of d at distance r from x. For any $l \in \mathbb{R}_+^*$, there exists a unique rectangle $\mathcal{R}_{\theta,l}$ of area θ having $[x\,; p_l]_d$ as left timelike edge J. The map

$$(l, u) \in \mathbb{R}_+^* \times [0\,; 1[\mapsto \mathsf{g}_{\theta,l,p_{ul}} \in \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^A_\mathsf{a}$$

is continuous and is a local inverse of $\mathcal{L}_\mathsf{a} \times \Theta_{\mathsf{a},\mathsf{b}}$, which concludes the proof. □

Theorem C gives a new proof of the following result, proved in [MM24, Theorem E] by using "lightlike coordinates".

Corollary 4.5. $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ *is a Hausdorff topological surface.*

Proof. Since $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ is covered by the $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A_{\mathsf{a}}$ for $\mathsf{a}\in\pi_1(\mathbf{T}^2)_*$, Theorem C shows that the length-twist coordinates provide a topological atlas of $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$. Each $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A_{\mathsf{a}}$ is moreover Hausdorff as it is globally homeomorphic to $\mathbb{R}^*_+\times\mathbb{R}$. Let $\mathsf{g}_1\neq\mathsf{g}_2\in\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$. If $\mathcal{C}^{\mathsf{g}_1}\cap\mathcal{C}^{\mathsf{g}_2}\neq\varnothing$, g_1 and g_2 belong to a common $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A_{\mathsf{a}}$ in which they can thus be separated by open subsets. If $\mathcal{C}^{\mathsf{g}_1}\cap\mathcal{C}^{\mathsf{g}_2}=\varnothing$, then the pairs of oriented projective asymptotic cycles $\mathcal{A}(\mathsf{g}_1):=(A^+(\mathcal{F}^{\mathsf{g}_1}_\alpha),A^+(\mathcal{F}^{\mathsf{g}_1}_\beta))$ and $\mathcal{A}(\mathsf{g}_2)$ are distinct. The asymptotic cycle map

$$\mathcal{A}\colon \mathsf{Def}_\theta(\mathbf{T}^2,0)\to(\mathbf{P}^+(\mathrm{H}_1(\mathbb{R},\mathbf{T}^2)))^2$$

being continuous according to [MM24, Lemma 6.5], g_1 and g_2 can then be separated in $\mathsf{Def}_\theta(\mathbf{T}^2,0)$ by the pre-images of open subsets separating $\mathcal{A}(\mathsf{g}_1)$ and $\mathcal{A}(\mathsf{g}_2)$. This shows that $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ is Hausdorff and concludes the proof. □

Remark 4.6. The coordinates $\mu_{\theta,x,y}$ constructed on $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ from lightlike informations in [MM24, Lemma 9.4] (see also [MM24, Figure 4.2 and §6.2]) and the length-twist coordinates $\mathcal{L}_{\mathsf{a}}\times\Theta_{\mathsf{a},\mathsf{b}}$ that we just constructed, have real-analytical coordinate changes. Consequently, both of these atlases define the same natural real-analytic structure on $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$.

The unique simple closed timelike geodesic of $\mathcal{T}_{\theta,x,y}$ in a given timelike free homotopy class c is indeed the projection of a disjoint union of timelike arcs δ_c in $\mathcal{L}_{\theta,x,y}$, whose finite number is locally constant in c. One may now observe that these arcs vary real-analytically in (x,y). This shows that both the length and the twist coordinates are real-analytic in (x,y).

5. Rigidity of two timelike lengths

We prove in this section Theorem B. Let g_1 and g_2 be two class A de-Sitter structures of $\mathbf{T}^2$ of equal area θ and having a unique singularity, giving the same lengths

$$(\mathcal{L}_{\mathsf{a}}(\mathsf{g}_1),\mathcal{L}_{\mathsf{b}}(\mathsf{g}_1))=(\mathcal{L}_{\mathsf{a}}(\mathsf{g}_2),\mathcal{L}_{\mathsf{b}}(\mathsf{g}_2))=:(l,k)$$

to a common basis of timelike homotopy classes (a,b).

Step 1. With γ_{a} the unique geodesic of g_1 in the free homotopy class a, we first observe that g_2 *is a twist of* g_1 *around* γ_{a} in the following sense.

Let A be the singular $\mathbf{dS}^2$-annulus with two timelike geodesic boundary components $\gamma^\pm_{\mathsf{a}}$ obtained by cuting the singular $\mathbf{dS}^2$-torus $T=(\mathbf{T}^2,g_1)$ along γ_{a}. We name the left (respectively right) boundary component of A by γ^-_{a} (resp. γ^+_{a}) with respect to the orientation of A. The definition of A from T comes with a unique map $\iota\colon\gamma^+_{\mathsf{a}}\to\gamma^-_{\mathsf{a}}$ between the boundary components such that the quotient $\bar{A}$ of A by the identification $p\in\gamma^+_{\mathsf{a}}\sim\iota(p)\in\gamma^-_{\mathsf{a}}$ identifies with the original singular $\mathbf{dS}^2$-torus T. We denote by a_A the free homotopy class of γ^-_{a} in A. To encode the isotopy class g_1, we need to also keep track of the free homotopy class b in A. Let b_A be the unique class of simple arcs $\delta\colon[0\,;1]\to A$ going from points $\delta(0)\in\gamma^+_{\mathsf{a}}$ to their image $\iota(\delta(0))=\delta(1)\in\gamma^-_{\mathsf{a}}$ modulo free homotopy fixing the base points, such that the simple closed curve $\bar{\delta}$ obtained in $\bar{A}\equiv T$

by projecting any arc δ in b_A is freely homotopic to b. Identifying γ_{a}^- with $\mathbb{R}/l\mathbb{Z}$ through a unit speed parametrization, we denote by R_u the rotation $p \in \gamma_{\mathsf{a}}^- \mapsto p+u \in \gamma_{\mathsf{a}}^-$ for any $u \in [0\,;l[$. The quotient T_u of A by the identification $p \in \gamma_{\mathsf{a}}^+ \sim R_u \circ \iota(p) \in \gamma_{\mathsf{a}}^-$ of its boundary components is a $\mathbf{dS}^2$-torus of area θ with a unique singular point. It is endowed with the primitive free homotopy class a_u induced by a_A. The concatenation of any arc δ in the class b_A with the segment $[\delta(1)\,;\delta(1)+u]_{\gamma_{\mathsf{a}}^-}$ of γ_{a}^- projects in T_u to a simple closed curve whose free homotopy class does not depend on δ and is denoted by b_u. We denote by $(\mathsf{g}_1)_u$ the point of $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ defined by the pullback of the singular $\mathbf{dS}^2$-structure of T_u by any homeomorphism from $\mathbf{T}^2$ to T_u sending (a,b) to $(\mathsf{a}_u,\mathsf{b}_u)$ and 0 to the unique singular point of T_u.

Since $\mathcal{L}_{\mathsf{a}}(\mathsf{g}_1) = \mathcal{L}_{\mathsf{a}}(\mathsf{g}_2) = l$, g_1 and g_2 are identification spaces of the same rectangle $\mathcal{R}_{\theta,l}$ according to Lemma 4.3, the timelike edges I and J of $\mathcal{R}_{\theta,l}$ projecting to the unique simple closed timelike geodesic of g_1 (respectively g_2) in the class a. There exists thus $n \in \mathbb{Z}$ and $u \in [0\,;l[$ such that

$$\mathsf{g}_2 = (D_{\mathsf{a}}^n)_*(\mathsf{g}_1)_u,$$

with D_{a} the positive Dehn twist around a. Note that, possibly exchanging the roles of g_1 and g_2, we can assume without loss of generality that $n \geq 0$. Identifying henceforth g_2 and $(D_{\mathsf{a}}^n)_*(\mathsf{g}_1)_u$, the unique simple closed timelike geodesic of g_2 in the free homotopy class b appears in A as a simple timelike geodesic arc $\delta\colon [0\,;1] \to A$ going from $p_+ := \delta(0) \in \gamma_{\mathsf{a}}^+$ to $p_- := \delta(1) \in \gamma_{\mathsf{a}}^-$. Since the projection of δ in $(\mathbf{T}^2, (D_{\mathsf{a}}^n)_*(\mathsf{g}_1)_u)$ belongs to the free homotopy class b, the concatenation

$$\delta \cdot [p_-\,;p_- + u]_{\gamma_{\mathsf{a}}^-}$$

belongs to the free homotopy class $\mathsf{b}_A - n\mathsf{a}_A$ of arcs in A. We also denote by $\gamma_{\mathsf{b}}\colon [0\,;1] \to A$ the arc of A in the class b_A induced by the unique simple closed timelike geodesic of T in the class b. Let E be the universal cover of A endowed with the singular $\mathbf{dS}^2$-structure induced by A. This is a singular $\mathbf{dS}^2$-surface homeomorphic to a band $[0\,;1]\times\mathbb{R}$, with two timelike geodesic components $\tilde{\gamma}_{\mathsf{a}}^- \simeq \{0\} \times \mathbb{R}$ and $\tilde{\gamma}_{\mathsf{a}}^+ \simeq \{1\} \times \mathbb{R}$ which are the universal covers of the geodesic boundary components $\gamma_{\mathsf{a}}^\pm$ of A. Fixing a lift $\tilde{p}_+ \in \tilde{\gamma}_{\mathsf{a}}^+$ of p_+ in E, we consider the lift $\tilde{\delta}$ of δ starting from $\tilde{p}_+$ and denote by $\tilde{p}_-$ its endpoint.

Step 2. *Assume by contradiction that $n \neq 0$.* The arc δ has then n intersection points with $\gamma_{\mathsf{b}} \setminus \gamma_{\mathsf{a}}^-$, which we denote by $(p_1,\dots,p_n)$ in the increasing order in which they are met on δ with its future orientation. With $\tilde{p}_i \in \tilde{\delta}$ the respective lifts of the p_i's on $\tilde{\delta}$, let $\tilde{\gamma}_{\mathsf{b}}^i$ be the lift of γ_{b} passing through $\tilde{p}_i$, and $q_i \in \tilde{\gamma}_{\mathsf{a}}^+$ be the starting point of $\tilde{\gamma}_{\mathsf{b}}^i$. These constructions are illustrated on the right-hand side of Figure 5.1 for $n=3$. Observe that it may be that $p_+ \in \gamma_{\mathsf{b}}$ *i.e.* that $p_+ = p_1$, in which case $\tilde{p}_+ = \tilde{p}_1 = q_1$ (this situation is illustrated on the left-hand side of Figure 5.1). Since a is a timelike free homotopy class of g_1, there exists according to [MM24, Proposition A.8.(1)] a simple closed timelike curve σ_i of A passing through p_i and avoiding the singularity in the free homotopy class a_A.[6] Let $\tilde{\sigma}_i$ be the lift of σ_i starting at $\tilde{p}_i$, and $\tilde{p}_i'$ be the endpoint of $\tilde{\sigma}_i$.

For any $i = 1,\dots,n-1$, the three future timelike segments $\tilde{\sigma}_i$, $[\tilde{p}_i'\,;\tilde{p}_{i+1}]_{\tilde{\gamma}_{\mathsf{b}}^{i+1}}$ and $[\tilde{p}_i\,;\tilde{p}_{i+1}]_{\tilde{\delta}}$ bound a closed triangle $P_i \subset E$ containing at most one singularity (in its

[6] Though the latter claim is not explicitly stated in this way, it is a byproduct of the proof of [MM24, Proposition A.8.(1)].

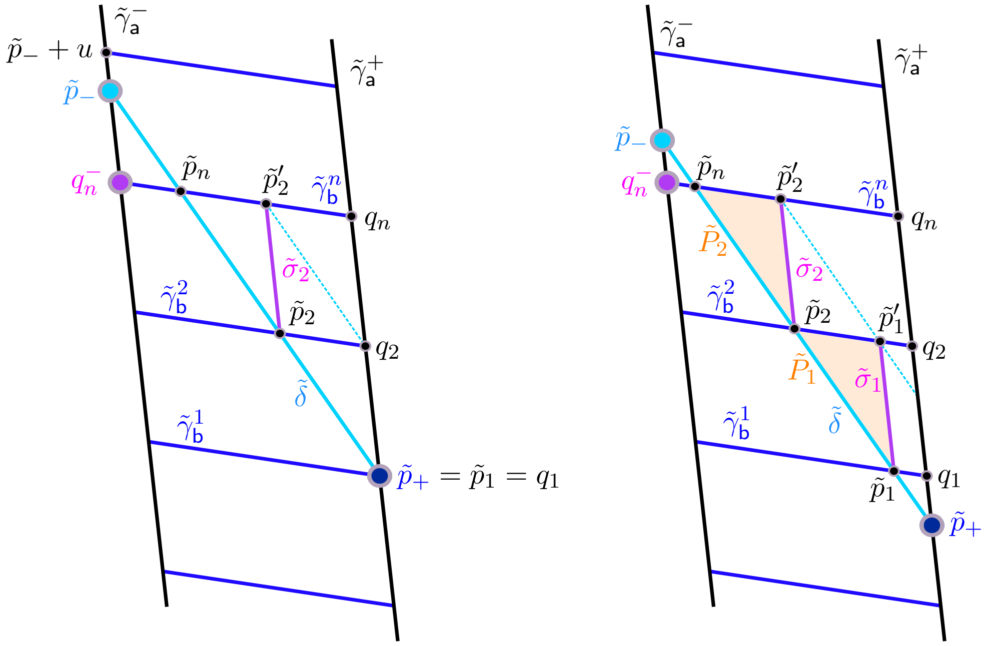

FIGURE 5.1. Lifts of the arcs δ, γ_{b} and $\gamma_{\mathsf{a}}^{\pm}$ in the universal cover E of the annulus A (for $n = 3$).

interior), illustrated in Figure 5.1. There exists therefore an embedding of an open neighbourhood U_i of $\tilde{P}_i$ in $\widetilde{\mathbf{dS}}^2$ or in $\widetilde{\mathbf{dS}}^2_\theta$. This is a consequence of [MM24, Lemma 3.5], and of the fact that any class A $\mathbf{dS}^2$-torus is isometric to a torus $\mathcal{T}_{\theta,x,y}$ described in Figure 2.2 according to [MM24, Theorem 9.6.(1)]. In particular, any future timelike geodesic segment in U_i from x to y maximizes the length of future causal curves in U_i going from x to y according to Proposition [MM24, Proposition A.12]. Since $\tilde{\sigma}_i \cdot [\tilde{p}'_i\,;\tilde{p}_{i+1}]_{\tilde{\gamma}_{\mathsf{b}}^{i+1}}$ is a future causal curve in U_i from $\tilde{p}_i$ to $\tilde{p}'_{i+1}$, we have therefore the following reverse triangle inequality:

$$L\left([\tilde{p}_i\,;\tilde{p}_{i+1}]_{\tilde{\delta}}\right) \geq L\left([\tilde{p}'_i\,;\tilde{p}_{i+1}]_{\tilde{\gamma}_{\mathsf{b}}^{i+1}}\right) + L\left(\tilde{\sigma}_i\right) > L\left([\tilde{p}'_i\,;\tilde{p}_{i+1}]_{\tilde{\gamma}_{\mathsf{b}}^{i+1}}\right). \tag{5.1}$$

The second strict inequality is due to the fact that $L(\tilde{\sigma}_i) > 0$ since it is a non-trivial timelike curve. Let $q_n^- \in \tilde{\gamma}_{\mathsf{a}}^-$ denote the endpoint of $\tilde{\gamma}_{\mathsf{b}}^n$. Then by the same arguments as before, the three timelike segments $[\tilde{p}_n\,;q_n^-]_{\tilde{\gamma}_{\mathsf{b}}^n}$, $[q_n^-\,;\tilde{p}_-]_{\tilde{\gamma}_{\mathsf{a}}^-}$ and $[\tilde{p}_n\,;\tilde{p}_-]_{\tilde{\delta}}$ bound a closed triangle $\tilde{P}_n \subset E$ containing at most one singularity, in which timelike geodesic segments maximize thus the causal lengths. Therefore

$$L\left([\tilde{p}_n\,;\tilde{p}_-]_{\tilde{\delta}}\right) \geq L\left([\tilde{p}_n\,;q_n^-]_{\tilde{\gamma}_{\mathsf{b}}^n}\right) + L\left([q_n^-\,;\tilde{p}_-]_{\tilde{\gamma}_{\mathsf{a}}^-}\right) > L\left([\tilde{p}_n\,;q_n^-]_{\tilde{\gamma}_{\mathsf{b}}^n}\right) \tag{5.2}$$

as previously. If $p_+ \notin \gamma_{\mathsf{b}}$ *i.e.* $\tilde{p}_1 \neq \tilde{p}_+$, we also have to consider the triangle $\tilde{P}_0 \subset E$ bounded by the timelike segments $[\tilde{p}_+ ; q_1]_{\tilde{\gamma}_{\mathsf{a}}^+}$, $[q_1 ; \tilde{p}_1]_{\tilde{\gamma}_{\mathsf{b}}^1}$ and $[\tilde{p}_+ ; \tilde{p}_1]_{\tilde{\delta}}$ (this case is illustrated on the right-hand side of Figure 5.1). This yields as previously the inequality

$$L\left([\tilde{p}_+ ; \tilde{p}_1]_{\tilde{\delta}}\right) \geq L\left([q_1 ; \tilde{p}_1]_{\tilde{\gamma}_{\mathsf{b}}^1}\right) + L\left([\tilde{p}_+ ; q_1]_{\tilde{\gamma}_{\mathsf{a}}^+}\right) > L\left([q_1 ; \tilde{p}_1]_{\tilde{\gamma}_{\mathsf{b}}^1}\right). \tag{5.3}$$

Observe now that $L\left([\tilde{p}'_i ; \tilde{p}_{i+1}]_{\tilde{\gamma}_{\mathsf{b}}^{i+1}}\right) = L\left([p_i ; p_{i+1}]_{\gamma_{\mathsf{b}}}\right)$ for $i = 1, \ldots, n-1$, $L\left([\tilde{p}_n ; q_n^-]_{\tilde{\gamma}_{\mathsf{b}}^n}\right) = L\left([p_n ; \gamma_{\mathsf{b}}(1)]_{\gamma_{\mathsf{b}}}\right)$ and $L\left([q_1 ; \tilde{p}_1]_{\tilde{\gamma}_{\mathsf{b}}^1}\right) = L\left([\gamma_{\mathsf{b}}(0) ; p_1]_{\gamma_{\mathsf{b}}}\right)$. Therefore, the sum of inequalities (5.1), (5.2) and (5.3) gives

$$\mathcal{L}_{\mathsf{b}}(\mathsf{g}_2) = L(\delta) > L(\gamma_{\mathsf{b}}) = \mathcal{L}_{\mathsf{b}}(\mathsf{g}_1)$$

which contradicts our original assumption. This contradiction shows that *n=0*.

Step 3, case A: $\underline{p_+ \in \gamma_{\mathsf{b}}}$, *i.e.* $\delta(0) = \gamma_{\mathsf{b}}(0)$. Then since

$$\delta(1) + u = \iota(\delta(0)) = \gamma_{\mathsf{b}}(1), \tag{5.4}$$

the timelike segments $\tilde{\delta}$, $[\tilde{p}_- ; \tilde{p}_- + u]_{\tilde{\gamma}_{\mathsf{a}}^-}$ and $\tilde{\gamma}_{\mathsf{b}}$ bound in E a triangle to which we can apply our previous argument. This shows that $L(\tilde{\delta}) \geq L(\tilde{\gamma}_{\mathsf{b}}) + L\left([\tilde{p}_- ; \tilde{p}_- + u]_{\tilde{\gamma}_{\mathsf{a}}^-}\right)$, hence that $u = 0$ since $L(\delta) = L(\gamma_{\mathsf{b}})$.

Step 3, case B: $\underline{p_+ \notin \gamma_{\mathsf{b}}}$. In this case, the arcs δ and γ_{b} bound together with $\gamma_{\mathsf{a}}^{\pm}$ two rectangles with timelike edges in A, only of them containing the unique singularity of A (in its interior). Without lost of generality, we assume that the closed rectangle $P \subset A$ "above γ_{b}" does not contain the singularity. This situation is represented in Figure 5.2.

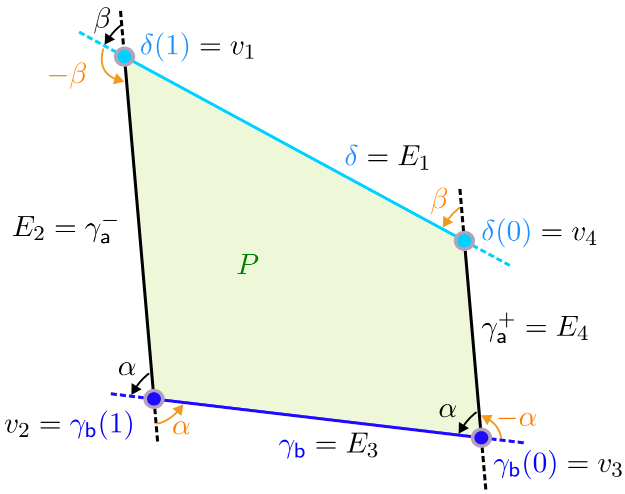

FIGURE 5.2. Rectangle $P \subset \mathbf{dS}^2$ bounded by δ, γ_{b} and $\gamma_{\mathsf{a}}^{\pm}$, and its exterior angles at vertices v_i.

We finally make use of an additional property of δ and γ_{b} that we had not mentioned until now. Since γ_{b} derives from a simple closed geodesic of T, its angles with γ_{a}^- and with γ_{a}^+ are equal:

$$\alpha := \left(\left(\gamma_{\mathsf{a}}^+, \gamma_{\mathsf{b}}\right)\right)_{\gamma_{\mathsf{b}}(0)} = \left(\left(\gamma_{\mathsf{a}}^-, \gamma_{\mathsf{b}}\right)\right)_{\gamma_{\mathsf{b}}(1)}.$$

In the same way, δ projects by construction to a simple closed geodesic of $(\mathsf{g}_1)_u = \mathsf{g}_2$. Since the gluing $R_u \circ \iota \colon \gamma_{\mathsf{a}}^+ \to \gamma_{\mathsf{a}}^-$ is made by isometries, the angles of δ with γ_{a}^- and γ_{a}^+ are also equal:

$$\beta := \left(\left(\gamma_{\mathsf{a}}^+, \delta\right)\right)_{\delta(0)} = \left(\left(\gamma_{\mathsf{a}}^-, \delta\right)\right)_{\delta(1)}.$$

The exterior angles at $v_1 = \delta(1)$, $v_2 = \gamma_{\mathsf{b}}(1)$, $v_3 = \gamma_{\mathsf{b}}(0)$ and $v_4 = \delta(0)$ are thus respectively equal to $\nu_1 = -\beta$, $\nu_2 = -\alpha$, $\nu_3 = \beta$ and $\nu_4 = \alpha$ according to the relations (2.2). They are indicated in orange in Figure 5.2. The Gauss-Bonnet formula of Proposition 2.6 implies then that the area of P vanishes, namely that P has empty interior. Since δ and γ_{b} have non-empty interior, this shows that $\delta = \gamma_{\mathsf{b}}$, hence that $u = 0$ according to (5.4). Finally $n = u = 0$ in both cases *i.e.* $\mathsf{g}_1 = \mathsf{g}_2$, which concludes the proof.

Bibliography

[BBS11] T Barbot, F Bonsante, and JM Schlenker. Collisions of particles in locally AdS spacetimes. I. Local description and global examples. *Communications in Mathematical Physics* 308.1 (2011), pp. 147–200. DOI: 10.1007/s00220-011-1318-6.

[BCLR20] F Béguin, S Crovisier, and F Le Roux. Fixed point sets of isotopies on surfaces. *Journal of the European Mathematical Society (JEMS)* 22.6 (2020), pp. 1971–2046. DOI: 10.4171/JEMS/960.

[BGT25] A Boulanger, S Ghazouani, and G Tahar. Closed geodesics in dilation surfaces. *Geometry & Topology* 29.4 (2025), pp. 2217–2250. DOI: 10.2140/gt.2025.29.2217.

[BK85] K Burns and A Katok. Manifolds with non-positive curvature. *Ergodic Theory and Dynamical Systems* 5 (1985), pp. 307–317. DOI: 10.1017/S0143385700002935.

[BL18] A Bankovic and CJ Leininger. Marked-length-spectral rigidity for flat metrics. *Transactions of the American Mathematical Society* 370.3 (2018), pp. 1867–1884. DOI: 10.1090/tran/7005.

[BN84] GS Birman and K Nomizu. The Gauss-Bonnet theorem for 2-dimensional spacetimes. *Michigan Mathematical Journal* 31 (1984), pp. 77–81. DOI: 10.1307/mmj/1029002964.

[Cro90] CB Croke. Rigidity for surfaces of non-positive curvature. *Commentarii Mathematici Helvetici* 65.1 (1990), pp. 150–169. DOI: 10.1007/BF02566599.

[Eps66] DBA Epstein. Curves von 2-manifolds and isotopies. *Acta Mathematica* 115 (1966), pp. 83–107. DOI: 10.1007/BF02392203.

[GL19] C Guillarmou and T Lefeuvre. The marked length spectrum of Anosov manifolds. *Annals of Mathematics. Second Series* 190.1 (2019), pp. 321–344. DOI: 10.4007/annals.2019.190.1.6.

[HP97] S Hersonsky and F Paulin. On the rigidity of discrete isometry groups of negatively curved spaces. *Commentarii Mathematici Helvetici* 72.3 (1997), pp. 349–388. DOI: 10.1007/s000140050022.

[HS85] J Hass and P Scott. Intersections of curves on surfaces. *Israel Journal of Mathematics* 51 (1985), pp. 90–120. DOI: 10.1007/BF02772960.

[KF96] F Klein and R Fricke. *Vorlesungen über die Theorie der elliptischen Modulfunctionen.* Leipzig, B.G. Teubner, 1896.

[MM24] M Mion-Mouton. *Rigidity of singular de-Sitter tori with respect to their light-like bi-foliation*. 2024. DOI: 10.48550/ARXIV.2410.03260.

[Nie17] J Nielsen. Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Erzeugenden. *Mathematische Annalen* 78.1-4 (1917), pp. 385–397. DOI: 10.1007/BF01457113.

[Nur25] P Nurowski. The Beltrami–de Sitter model: Penrose's CCC, Radon transform and a hidden G_2 symmetry. *Classical and Quantum Gravity* 42.22 (2025), p. 225007. DOI: 10.1088/1361-6382/ae19c3.

[Ota90] JP Otal. Le spectre marqué des longueurs des surfaces à courbure négative. (The spectrum marked by lengths of surfaces with negative curvature). *Annals of Mathematics. Second Series* 131.1 (1990), pp. 151–162. DOI: 10.2307/1971511.

[OZ81] RP Osborne and H Zieschang. Primitives in the Free Group on Two Generators. *Inventiones mathematicae* 63 (1981), pp. 17–24.

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