RIGIDITY OF SINGULAR DE-SITTER TORI WITH RESPECT TO THEIR LIGHTLIKE BI-FOLIATION

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Abstract. In this paper, we introduce a natural notion of constant curvature Lorentzian surfaces with conical singularities, and provide a large class of examples of such structures. We moreover initiate the study of their global rigidity, by proving that de-Sitter tori with a single singularity of a fixed angle are determined by the topological equivalence class of their lightlike bi-foliation. While this is reminiscent of Troyanov's uniformization results on Riemannian surfaces with conical singularities, the rigidity comes from topological dynamics in the Lorentzian case.

1. Introduction

A Lorentzian metric on a surface induces a pair $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ of *lightlike foliations*, and the Poincaré-Hopf theorem therefore implies that the torus is the only closed and orientable Lorentzian surface. An analog of the Gauß-Bonnet formula shows moreover that the only constant curvature Lorentzian metrics on the torus are actually *flat* (see [Ave63, Che63]). It is natural to try to widen this class of geometries, in order to obtain structures locally modelled on the Lorentzian analogue of the hyperbolic plane, the *de-Sitter space* $\mathbf{dS}^2$ (wich is introduced in Paragraph 2.3 below). This is not possible on a closed surface without removing some points, and a natural way to do this is to proceed as in the Riemannian case, by concentrating all the curvature in finitely many points where the metric has *conical singularities* as they appeared in [BBS11].

The first goal of this paper is to introduce this natural class of *singular constant curvature Lorentzian surfaces*, to provide examples of such structures, and to initiate their study by proving some of their fundamental properties. The second and main goal is to investigate in the de-Sitter case the relations of these geometrical objects with associated dynamical ones: their pair of lightlike foliations.

1.1. Singular de-Sitter surfaces.

The Lorentzian conical singularities studied in the present paper are defined analogously to the Riemannian ones, and correspond to the *space-like singularities of degree 1* already appearing in [BBS11, p.160]. The connected component of the identity in the isometry group of $\mathbf{dS}^2$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, acts transitively on $\mathbf{dS}^2$, and the stabilizer of a point $\mathsf{o} \in \mathbf{dS}^2$ in $\mathrm{PSL}_2(\mathbb{R})$ is a one-parameter hyperbolic group $A = \{a^\theta\}_{\theta\in\mathbb{R}} \subset \mathrm{PSL}_2(\mathbb{R})$. Analogously to the Riemannian case, a natural way to describe a conical singularity in the de-Sitter space is to choose a non-trivial isometry $a^\theta \in A$ and a future timelike or spacelike geodesic ray γ emanating from o, to consider the sector from γ to $a^\theta(\gamma)$ in $\mathbf{dS}^2$, and to glue its two boundary components by a^θ. This construction is illustrated in Figure 3.1 below, and is detailed in Paragraph 3.1.5. The resulting identification space $\mathbf{dS}^2_\theta = \mathbf{dS}^2_*/\sim$ is a surface with a marked point o_θ which is the projection of o, endowed on $\mathbf{dS}^2_\theta \setminus \{\mathsf{o}_\theta\}$ with a natural locally $\mathbf{dS}^2$ Lorentzian metric coming from the one of $\mathbf{dS}^2$ (since the gluing was made by isometries). The local model of a *standard singularity of angle θ* is by definition a neighbourhood of o_θ in $\mathbf{dS}^2_\theta$, and a *singular $\mathbf{dS}^2$-surface* is an orientable surface bearing a locally $\mathbf{dS}^2$ Lorentzian metric, outside of a discrete set of points which are standard singularities (see Definition 3.16). The cut-and-paste construction of $\mathbf{dS}^2_\theta$ can also be realized on a *lightlike* half-geodesic (see Paragraph 3.1.1), and we use in practice the latter characterization. Standard singularities are also defined in the case of zero curvature (*i.e.* for the Minkowski space), and are illustrated in Figure 3.1 below.

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To the best of our knowledge, singular constant curvature Lorentzian surfaces did not appear so far in the literature as an object of independent interest, and in particular no examples appeared yet on closed surfaces. One of the purposes of this work is to construct many examples, and to set the ground for the future investigation of singular constant curvature Lorentzian surfaces. To this end, we furnish in Proposition 4.3 a general method to construct a large class of examples, and we carefully prove in Paragraphs 3.1 and 3.2 many structural properties of singular constant curvature Lorentzian surfaces. An important point of view on singular Riemannian surfaces is the one of *metric length spaces*, and a natural Lorentzian counterpart of the latter notion was introduced in [KS18] under the name of *Lorentzian length spaces*. Singular constant curvature Lorentzian surfaces appear as natural candidates to illustrate such a notion, and furnish indeed a large class of examples of Lorentzian length spaces, apparently new in the literature. We refer to Appendix D for more details on this subject.

1.2. **Geometric rigidity of lightlike bifoliations.** Let S_1 and S_2 be two closed, connected and orientable surfaces, endowed with constant curvature Riemannian metrics. The classical notion of conformal diffeomorphism can be generalized to a notion of *quasi-conformal* homeomorphism from S_1 to S_2, which can be formulated by an *elliptic* Partial Differential Equation (see [Ber77] for more details). Therefore, a general result of *elliptic regularity* shows that any quasi-conformal homeomorphism of constant distorsion 1 is actually a smooth conformal diffeomorphism (see [IT92, pp.20-21] and [Fol99, Theorem 9.26 pp.307-308]). Since S_1 and S_2 are global quotients of constant curvature models, one moreover observes that the conformal group of S_i equals its isometry group, unless S_i is isometric to $\mathbf{S}^2$. In conclusion: *any quasi-conformal homeomorphism of constant distorsion* 1 *between Riemannian surfaces of constant non-positive curvature is a smooth isometry.* Note that this fact is essentially an analytical phenomenon.

Let us investigate what is left of the latter statement for (regular) constant curvature Lorentzian surfaces S_1 and S_2. We first recall that such S_1 and S_2 must be homeomorphic to tori and of constant curvature 0, according to the discussion opening this article. We observe then that two Lorentzian metrics on a surface are conformal, if and only if they have identical lightlike bi-foliations. Therefore, a conformal diffeomorphism from S_1 to S_2 is nothing but a *smooth equivalence* between their lightlike bi-foliations, *i.e.* a diffeomorphism $f\colon S_1 \to S_2$ such that $f(\mathcal{F}_\alpha^{S_1}(x)) = \mathcal{F}_\alpha^{S_2}(f(x))$ and $f(\mathcal{F}_\beta^{S_1}(x)) = \mathcal{F}_\beta^{S_2}(f(x))$ for any $x \in S_1$, while respecting the orientations. The natural topological analogue of the latter being a *topological equivalence* between the lightlike bi-foliations (*i.e.* a homeomorphism f satisfying the same assumptions), the previous Riemannian result eventually raises the following question. *Is any topological equivalence between the lightlike bi-foliations of two flat Lorentzian tori of equal area a smooth isometry?* Contrary to the Riemannian case, we show now that the answer is not always positive, and surprisingly depends on the *topological dynamics* of the lightlike foliations.

As in the Riemannian case, the completeness of flat Lorentzian tori (due to [Car89]) first shows that the flat Lorentzian tori S_1 and S_2 are isometric to the Lorentzian metrics $\bar{q}_i$ induced on $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by two Lorentzian quadratic forms q_1 and q_2 on $\mathbb{R}^2$. The lightlike bi-foliations of $\bar{q}_1$ and $\bar{q}_2$ being *linear*, if they are topologically equivalent, they are actually equivalent by an *affine transformation of* $\mathbf{T}^2$ induced by some integer matrix $A \in \mathrm{GL}_2(\mathbb{Z})$. We can therefore replace q_2 by its pullback A^*q_2 so that $\bar{q}_1$ and $\bar{q}_2$ are conformal, showing that $q_1 = q_2$ since they also have the same area. In conclusion, S_1 and S_2 are isometric. If the lightlike foliations are moreover *minimal i.e.* have all their leaves dense (equivalently if the isotropic lines of the q_i's are irrational), one can show that the conformal group of $(\mathbf{T}^2, \bar{q}_i)$ equals its isometry group.[1] Any topological equivalence between the lightlike bi-foliations of S_1 and S_2 is then an isometry. But on the contrary if both isotropic lines of the q_i's are rational, then the lightlike bi-foliation is conjugated to the product foliation of $\mathbf{S}^1 \times \mathbf{S}^1$. Any pair of circle homeomorphisms then induces a topological equivalence between the lightlike bi-foliations of S_1 and S_2, showing the existence of such equivalences which are not smooth, hence even more so non-isometric.

We retain from the previous discussion that the rigidity of the lightlike bi-foliations of flat Lorentzian tori does neither rely on analysis nor really on dynamics, but merely reduces to a

[1] This is for instance a consequence of [MM25, Corollary B].

purely linear phenomenon. We prove in Lemma 3.24 that the lightlike foliations extend at the singularities to define on any singular constant curvature Lorentzian surface a *topological* bi-foliation, which we still call the *lightlike bi-foliation* (in particular, the torus remains thus the only closed and orientable surface bearing a constant curvature Lorentzian metric with standard singularities[2]). Interestingly, those topological foliations are not linear as they are not even smooth but only *piecewise smooth*. This is one of our motivation for the class of singular constant curvature Lorentzian surfaces, which induce in particular a singular projective structure on the surface, and a transverse singular projective structure on each of their lightlike foliations. This suggests that any rigidity of such bi-foliations should be a purely non-linear phenomenon. The first goal of this paper is to exhibit such a rigidity in the case of a unique singularity. Note that, according to Gauß-Bonnet formula (3.7), a constant curvature Lorentzian torus with a unique singularity has non-zero curvature, which explains the focus on singular $\mathbf{dS}^2$-structures in the present paper. Singular Minkoswki tori will be independently investigated in a future work.

Theorem A. *Let S_1 and S_2 be two singular $\mathbf{dS}^2$-tori having a unique singularity of the same angle and minimal lightlike foliations. Then any topological equivalence between the lightlike bi-foliations of S_1 and S_2 is an isometry.*

In particular, any topological equivalence between the lightlike bi-foliations of $\mathbf{dS}^2$-tori with one singularity of the same angle is therefore smooth. This may be fomulated as a *geometric rigidity* result for this class of lightlike bi-foliations (we refer the reader to the very pleasant presentation of the general problem of geometric rigidity for dynamical systems given in [Gha21, p.468]). Note that the condition of equal angles at the singularities is a necessary condition for the existence of both an isometry (because it is an isometry invariant according to Corollary 3.19), and of a smooth equivalence between the lightlike foliations (for the size of the break of the first-return map derivative is determined by the angle, see Lemma 3.24).

We finally observe that, while a homeomorphism preserving the timelike cones between regular Lorentzian manifolds *of dimensions at least three* is automatically smooth (*i.e.* is a classical conformal diffeomorphism) according to a result of Hawking [Haw14, Lemma 19], this purely local phenomenon vanishes in dimension two. There is thus no local reason for such a "topological conformal transformation" between Lorentzian surfaces to be smooth, but Theorem A shows that, for global reasons, any such map between $\mathbf{dS}^2$-tori with one singularity is actually smooth *and even isometric.*[3]

1.3. Global description of the deformation space in terms of asymptotic cycles.

Klein-Poincaré uniformization theorem proves that any conformal class on a closed orientable surface S contains a Riemannian metric of constant curvature (which can be seen to be unique). In the same way, the seminal work of Troyanov [Tro86, Tro91] proves that for any fixed set of singularities and angles on S, any conformal class contains a unique Riemannian metric of a given curvature having the prescribed singularities (with necessary conditions relating the angles, the constant curvature and the Euler characteristic of the surface, given by the Gauß-Bonnet formula). These results may be roughly summarized as answering positively the following vague question: *does any conformal class contains a constant curvature metric, and if such is it unique?*

From a geometrical point of view, the present paper may be seen as a contribution to the same general question of uniformization, in the setting of singular $\mathbf{dS}^2$-structures of the torus having a unique singularity of angle θ at $0 \in \mathbf{T}^2$. The *deformation space* of such structures is denoted by $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ (and is properly introduced in Definition 6.1), and our goal is to propose a global description of $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$. However contrary to the Riemannian case, the description is not done here in terms of conformal structures, as the relevant invariant in the Lorentzian setting is a *topological dynamical invariant* of bi-foliations: the *projective asymptotic cycle*. The latter is introduced later in Paragraph 5.2, and can be seen as a global counterpart of the rotation number

[2] The study of singular Lorentzian metrics on higher genus surfaces requests the introduction of other types of singularities, which will be the subject of a future work (see Remark 4.5 for more details).

[3] This contrasts with Lorentzian manifolds of dimension at least three, for which the conformal group of the flat model (the Einstein universe) is *essential*, *i.e.* does not preserve any metric in the conformal class. See the recent preprint [DFM+25] and references therein for more details on the related *Lorentzian Lichnerowicz Conjecture*.

of the first-return map on a section.[4] The projective asymptotic cycles of the lightlike foliations being isotopy invariant, they are well-defined for an isotopy class in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ (see Lemma 6.2). We first show that the rigidity result of Theorem A is non-empty, with the following existence and uniqueness result.

Theorem B. *Let $A^+_\alpha \neq A^+_\beta \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$ be a positive pair of distinct irrational half-lines, and $\theta \in \mathbb{R}^*_+$. Then there exists in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ a unique point whose lightlike foliations have oriented projective asymptotic cycles $A^+(\mathcal{F}_\alpha) = A^+_\alpha$ and $A^+(\mathcal{F}_\beta) = A^+_\beta$. In particular, $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are minimal suspensions.*

The *positivity* of (A^+_α, A^+_β) is a necessary condition coming from the orientations conventions introduced in Figure 3.1 (see Definition 7.2 and Remark 7.3). The main question investigated in this paper may now be roughly summarized as follows: *to which extent is the map*

$$\mathcal{A}\colon [\mu] \in \mathsf{Def}_\theta(\mathbf{T}^2, 0) \mapsto (A^+(\mathcal{F}^{[\mu]}_\alpha), A^+(\mathcal{F}^{[\mu]}_\beta)) \in \{\text{positive pairs of } \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))^2\} \tag{1.1}$$

bijective? This is in a sense a counterpart of Troyanov's description [Tro86, Tro91], where the deformation space of Riemannian metrics with prescribed conical singularities is shown to identify with the one of conformal structures (namely with the Teichmüller space). Contrary to the Riemannian case, the asymptotic cycles map $\mathcal{A}$ defined in (1.1) is however *not* globally injective, as it may be observed at the level of the first-return map of the foliations. Indeed, any small enough perturbation of a circle homeomorphism T having rational rotation number as well as non-periodic orbits, has the same rotation number than T.[5] Theorems B, C and D show however the surjectivity of $\mathcal{A}$, as well as its injectivity on large parts of $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$.

Theorem C. *Let $\theta \in \mathbb{R}^*_+$ and $c_\alpha \neq c_\beta \in \pi_1(\mathbf{T}^2)$ be a positive pair of distinct primitive elements. Then there exists in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ a unique point $[\mu]$ for which $\mathcal{F}_\alpha(0)$ and $\mathcal{F}_\beta(0)$ are closed and $([\mathcal{F}_\alpha(0)], [\mathcal{F}_\beta(0)]) = (c_\alpha, c_\beta)$. Moreover, $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are suspensions, and $(\mathbf{T}^2, [\mu])$ is isometric to a $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,x}$.*

The $\mathbf{dS}^2$-tori $\mathcal{T}_{\theta,x}$ are introduced below in Proposition 4.8.

Theorem D. *Let $\theta \in \mathbb{R}^*_+$, $c_\alpha \in \pi_1(\mathbf{T}^2)$ be a primitive element and $A^+_\beta \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$ be an irrational half-line such that (c_α, A_β) is positive. Then there exists in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ a unique point $[\mu]$ such that:*

(1) $\mathcal{F}_\alpha(0)$ is closed and $[\mathcal{F}_\alpha(0)] = c_\alpha$;
(2) and $A^+(\mathcal{F}_\beta) = A^+_\beta$.

Moreover, $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are suspensions, $\mathcal{F}_\beta$ is minimal, and $(\mathbf{T}^2, [\mu])$ is isometric to a $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,x}$. The obvious analogous statement holds when exchanging the roles of the α and β-foliations.

Theorems A, B, C and D advertise the general idea that closed singular constant curvature Lorentzian surfaces are much more rigid than their Riemannian counterparts. This rigidity finds its origin in the existence of the two lightlike foliations (such a preferred pair of transverse foliations does not exist for singular Riemannian surfaces).

As emphasized by an anonymous referee, we finally note that, the angle being determined by the area according to Gauß-Bonnet formula (3.7), the renormalization of the Lorentzian metrics yields a natural identification between the deformation spaces $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ of distinct angles.

[4] For the readers more used to (Riemannian) hyperbolic surfaces, it may also be useful to observe that the analogue of asymptotic cycles for higher genus surfaces, are the isotopy classes of projective measured foliations.

[5] This argument is incomplete in this form, since such deformations have *a priori* no reason to correspond to singular $\mathbf{dS}^2$-structures. However, by using arguments similar to those of Lemma 9.9, one can indeed perform such a perturbation inside $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$. In a future work in collaboration with Florestan Martin-Baillon, we will give more details on open subsets of $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ with stationnary rational asymptotic cycles.

1.4. **Methods, and strategies of the main proofs.** In [Tro86, Tro91], Troyanov translates the existence, in a given conformal class, of a unique constant curvature Riemannian metric with suitable singularities, into the existence of a unique solution for a Partial Differential Equation involving the Laplacian. Using the well-behaved properties of the latter, he proves his results by relying mainly on analytical methods. Contrary to the Riemannian one, the Lorentzian Laplacian is a *hyperbolic* differential operator and not anymore an elliptic one, which makes his use more difficult. Moreover, the phenomena that we wish to highlight in this work are by nature dynamical, the geometric rigidity expressed by Theorem A coming from the *topological dynamics* of the lightlike foliations. For this reason, we use in this text a constant interaction of geometrical and dynamical methods. The former should seem relatively familiar to the readers used to classical types of locally homogeneous singular geometric structures on surfaces (for instance translation or dilation surfaces). The latter comes from one-dimensional dynamics (namely piecewise Möbius interval exchange maps and their associated circle homeomorphisms) and are used in connection with the lightlike foliations through their first-return maps.

Our first concern in this paper is to construct examples satisfying the dynamical properties requested in Theorem B. Using identification spaces of polygons, this task eventually relies on the simultaneous realization of pairs of rotation numbers for a two-parameter family of pairs of Möbius interval exchange maps.

The first step of the proof of Theorem D is geometrical. We reduce the statement to the investigation of a one-parameter family of singular $\mathbf{dS}^2$-tori introduced in Paragraph 4.2, which are identification spaces of lightlike rectangles of $\mathbf{dS}^2$, illustrated in Figure 4.1 below. The uniqueness claim is translated in this way in Proposition 7.5 into a statement about a one-parameter family of circle maps, the first-return maps of the β-lightlike foliation on the closed α-leaf. In the end, the statement eventually follows from an important fact of one-dimensional dynamics: the rotation number of a monotonic one-parameter family of circle homeomorphisms increases *strictly* at irrational points (see Lemma B.1). This scheme of proof may serve as a paradigm for the geometrico-dynamical arguments used in the present paper, and for the efficiency of their interactions. Geometrical statements then become natural consequences of dynamical ones, once suitably translated.

The general strategy to prove Theorem A is then to show that two structures μ_1 and μ_2 with topologically equivalent and minimal lightlike foliations admit arbitrarily close *surgeries* $\mu_{1,n}$ and $\mu_{2,n}$, having a closed α-leaf at the singularity and identical irrational asymptotic cycles of their β-foliations. Once such suitable surgeries are constructed, one can rely on Theorem D to prove that $[\mu_{1,n}] = [\mu_{2,n}]$ in the deformation space. Since the latter sequence converges by construction both to $[\mu_1]$ and to $[\mu_2]$, this shows that $[\mu_1] = [\mu_2]$.

1.5. **Perspectives on multiple singularities.** The strategy of proof of Theorem A persists for any number of singularities. The first and main geometrical tool developed in this paper to implement this strategy is indeed the construction of suitable *surgeries* in Paragraph 8.2, which is done in full generality. The existence of simple closed timelike geodesics is known for regular Lorentzian manifolds (see for instance [Tip79, Gal86, Suh13]), and we prove in Appendix A that the usual tools and arguments remain available for singular constant curvature Lorentzian surfaces. This allows us to obtain simple closed timelike geodesics in their case, and to use them to realize the surgeries.

It is actually the proof of Theorem D and more precisely the one of the dynamical Lemma B.1 which fails for $n \geq 2$ singularities, and this is the only reason why the present paper focuses mainly on the case of a single singularity. Indeed, the rough description that we gave previously hided a fundamental aspect of the proof of Theorem D: after the geometrical reduction to identification spaces of polygons, the number of parameters of the resulting family of circle maps is equal to the number of singularities of the initial structure. And while the strict monotonicity of the rotation number at irrational points is easily shown for a *one*-parameter family, essentially everything can happen for generic *two*-parameter families of circle maps. This crucial difference between one-parameter and multiple parameter families of deformations is mainly due to the naive but fundamental observation that the rotation number is itself a *one*-dimensional invariant. The investigation of the rigidity of $\mathbf{dS}^2$-tori with multiple singularities requests therefore a new method to handle this dynamical difficulty, which is the content of a work in progress in collaboration with Selim Ghazouani.

Lastly, we emphasize that in all the examples of singular $\mathbf{dS}^2$-tori constructed in this text, the lightlike foliations have distinct asymptotic cycles (they are said *class A*). We do not know if there exists a singular $\mathbf{dS}^2$-structure on $\mathbf{T}^2$, whose lightlike foliations have the same asymptotic cycles. We actually construct and describe in this paper the whole subset $\mathsf{Def}_\theta(\mathbf{T}^2,0)^{\mathrm{A}}$ of class A structures, as the following result summarizes (a more detailed statement is proved below in Theorem 9.6).

Theorem E. $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ *is a connected component of* $\mathsf{Def}_\theta(\mathbf{T}^2,0)$ *and a Hausdorff topological surface. Moreover,* $\mathcal{A}$ *is a proper map from* $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ *to positive pairs of* $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))^2$.

1.6. **Connection with the smoothness of conjugacies for circle diffeomorphisms with breaks.** As we see in Lemma 3.24, the first-return maps of lightlike foliations in a singular $\mathbf{dS}^2$-surface are not only continuous but are actually *circle diffeomorphisms with breaks*. While this may appear as a technical detail, this regularity actually gives a crucial dynamical information on the first-return map T. Indeed, the seminal work of Denjoy [Den32] implies then that T does not have an exceptional minimal set, and is thus topologically conjugated to a rigid rotation of the circle if it has an irrational *rotation number*. Since T is piecewise smooth, it is natural to wonder at this point if T is actually *smoothly* conjugated to a rotation. But as naive as it may seem, this question is an old and deep one which remains still open in its full generality. Herman showed in [Her79] that a $\mathcal{C}^\infty$ circle diffeomorphism is $\mathcal{C}^\infty$-conjugated to a rigid rotation if its irrational rotation number is *Diophantine*. The latter condition is necessary, as Arnol'd showed in [Arn64] the existence of minimal circle diffeomorphisms for which the latter conjugation is *never* $\mathcal{C}^\infty$. Since these founding works, the research on this subject never stopped to be intensively active and we do not pretend to cover its vast literature. The problem remains unsolved for general circle diffeomorphisms with breaks, about which the optimal result up to date appears in [KKM17] to the best of our knowledge, and answers the question in the case of a single singularity.

Theorem A happens to be similar in its philosophy to the problem of smoothness of the conjugacy to a rigid rotation for a circle diffeomorphism with breaks. Indeed, while any two minimal smooth bi-foliations with the same asymptotic cycles are topologically conjugated according to [AGK03, Theorem 1], they are in general *not* smoothly conjugated. Indeed this is already not true for individual foliations, since we saw previously that their first-return maps are themselves not necessarily smoothly conjugated. In contrast, Theorem A shows that any topological equivalence between lightlike bi-foliations of $\mathbf{dS}^2$-tori with a unique singularity, is smooth. This connection between singular $\mathbf{dS}^2$-structures on the torus and circle diffeomorphisms with breaks is one of our motivations for this subject, and we wish to investigate it more precisely in a future work.

1.7. **Organization of the paper.** Basic definitions and properties of singular constant curvature Lorentzian surfaces are introduced and proved in Section 3. Section 4 is then concerned with the construction of such structures, and we give in Proposition 4.3 a general existence result of surfaces obtained as identification spaces of polygons with lightlike geodesic edges. In the remainder of Section 4, we study thoroughly the properties of a one-parameter and of a two-parameter family of $\mathbf{dS}^2$-tori with one singularity. This allows us to conclude in Paragraph 7.3 the proof of the existence parts of Theorems B, C and D (we prove a more refined statement given in Theorem 7.1). The proofs of Theorems A, B, C and D is concluded in Section 10. Theorem E is refined and proved in Theorem 9.6. We also construct in Paragraph 8.2 a family of surgeries, and prove in Appendix A the existence of simple closed definite geodesics (both results being obtained in the general setting of singular constant curvature Lorentzian surfaces). We prove in Appendix B the main technical results used on the rotation number (which are mostly classical). Lastly, we show in Appendix C that holonomies of lightlike foliations are piecewise Möbius, and explain in Appendix D how singular constant curvature Lorentzian surfaces may be interpreted as Lorentzian length spaces.

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Some usual notation and a standing assumption. If X is a space endowed with an equivalence relation $\sim$, then we usually denote by $\pi\colon X \to X/\sim$ the canonical projection onto the quotient, and also use the notation $[x] = \pi(x) \in X/\sim$ for $x \in X$. For any subset P of a topological space X, we denote by $\mathrm{Int}(P)$ the interior of P, by $\mathrm{Cl}(P)$ its closure and by ∂P its boundary.

All the surfaces (and any other manifolds) considered in this text are assumed to be connected, orientable and boundaryless, unless explicitly stated otherwise.

2. Constant curvature Lorentzian surfaces

As a preparation to consider singular surfaces, we first recall in this preliminary section the necessary background on regular Lorentzian surfaces that are used throughout the text, and fix some notations and conventions.

2.1. **Lorentzian surfaces, time and space-orientation, and lightlike foliations.** A quadratic form is said *Lorentzian* if it is non-degenerate and of signature $(1,n) = (-,+,\dots,+)$. A *Lorentzian metric* of class $\mathcal{C}^k$ on a manifold M is a $\mathcal{C}^k$ field μ of Lorentzian quadratic forms on the tangent bundle of M. Usually, we denote by $g = g_\mu$ the bilinear form associated to μ, so that $\mu(u) = g(u,u)$. Observe that if μ is a Lorentzian metric on a surface S, then $-\mu$ is also a Lorentzian metric on S.

Any Lorentzian vector space (V,q) (or tangent space of a Lorentzian manifold) is decomposed according to the sign of q, $u \in V$ being called:

1. *spacelike* if $q(u) > 0$,
2. *timelike* if $q(u) < 0$,
3. *lightlike* if $q(u) = 0$,
4. *causal* is $q(u) \leq 0$,
5. and *definite* if it is timelike or spacelike.

These denominations of *signatures* of vectors in Lorentzian tangent spaces are used in the natural compatible way for line fields and curves.

A *time-orientation* on a Lorentzian surface (S,μ) is a continuous choice among one of the two connected components of the cone $\mu_x^{-1}(\mathbb{R}_-) \setminus \{0\}$ of non-zero timelike vectors, which is called the *future* cone. We also talk without distinction of the associated *future causal* cone, closure of the future timelike one, and use the obvious similar notion of *space-orientation* in a Lorentzian surface (namely a continuous choice among one of the two connected components of $\mu_x^{-1}(\mathbb{R}_+) \setminus \{0\}$). Not any Lorentzian surface bears a time-orientation, and it is said *time-orientable* if it does. An orientable Lorentzian surface is time-orientable if and only if it is space-orientable.

Any Lorentzian surface S bears locally two (unique) *lightlike line fields*, which are globally well-defined if and only if S is oriented. In the latter case, they give rise to two *lightlike foliations* on the surface, of which we always choose an ordering $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ (defined in Paragraph 2.5 for the surfaces studied in this text). This ordered pair of foliations is called the *lightlike bi-foliation* of the surface, and the lightlike leaves are simply the lightlike geodesics of the metric. If S is furthermore time-oriented, then these lightlike foliations are themselves orientable. We always use the convention for which the orientation of the lightlike bi-foliation $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ is both compatible with the orientation of S and with its time-orientation, as illustrated in Figure 3.1 below. In other words with these conventions, a time-orientation and an ordering $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ of the lightlike foliations of an oriented Lorentzian surface S induce a space-orientation of S and an orientation of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$.

We call *quadrant at* $x \in S$ the four connected components of $\mathrm{T}_xS \setminus \{\mu^{-1}(0)\}$, or of $D \setminus (\mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x))$ for D a disk around x small enough for $(x, D, I_\alpha, I_\beta)$ to be topologically equivalent to $(0,]0\,;1[^2,]0\,;1[\times\{0\}, \{0\}\times]0\,;1[)$, with $I_{\alpha/\beta}$ the respective connected components of $D \cap \mathcal{F}_{\alpha/\beta}(x)$ containing x.

2.2. **The Minkowski space.** The flat model space of Lorentzian metrics is the Minkowski space $\mathbb{R}^{1,n}$, *i.e.* the vector space $\mathbb{R}^{n+1}$ endowed with a Lorentzian quadratic form $q_{1,n}$. In this text we are interested in Lorentzian surfaces, and we thus focus now on the Minkoswki plane $\mathbb{R}^{1,1}$ that we endow with the quadratic form $q_{1,1}(x, y) = xy$ and the induced left-invariant Lorentzian metric $\mu_{\mathbb{R}^{1,1}}$. We fix on $\mathbb{R}^{1,1}$ the standard orientation of $\mathbb{R}^2$, and the time-orientation (respectively space-orientation) for which the set of future timelike (resp. spacelike) vectors is the top left quadrant $\{(u, v) \mid u < 0, v > 0\}$ (resp. top right quadrant $\{(u, v) \mid u > 0, v > 0\}$).

The connected component of the identity in the orthogonal group of $q_{1,1}$ is the subgroup

$$\mathrm{SO}^0(1,1) \coloneqq \left\{a^t_{\mathbb{R}^{1,1}} \;\middle|\; t \in \mathbb{R}\right\} \subset \mathrm{SL}_2(\mathbb{R}) \text{ with } a^t_{\mathbb{R}^{1,1}} \coloneqq \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}. \tag{2.1}$$

Since $q_{1,1}$ is by construction preserved by translations, the subgroup $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1)$ of affine transformations preserves $q_{1,1}$ and its time-orientation, and equals in fact the group $\mathrm{Isom}^0(\mathbb{R}^{1,1})$ of orientation and time-orientation preserving isometries of $\mathbb{R}^{1,1}$. In particular, $\mathrm{Isom}^0(\mathbb{R}^{1,1})$ acts transitively on $\mathbb{R}^{1,1}$ with stabilizer $\mathrm{SO}^0(1,1)$ at $0 = (0, 0)$, which induces a $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1)$-equivariant identification of $\mathbb{R}^{1,1}$ with the homogeneous space $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1)/\mathrm{SO}^0(1,1)$.

2.3. **The de-Sitter space.** We now introduce the Lorentzian homogeneous space of non-zero constant curvature. We denote by $[S]$ the projection of $S \subset \mathbb{R}^{n+1} \setminus \{0\}$ in the projective space $\mathbb{R}\mathbf{P}^n$, by (e_i) the standard basis of $\mathbb{R}^n$, and use the identification

$$\varphi_0 \colon \begin{cases} t \in \mathbb{R} & \mapsto \hat{t} \coloneqq [t : 1] \in \mathbb{R}\mathbf{P}^1 \setminus [e_1] \\ \infty & \mapsto \hat{\infty} \coloneqq [e_1] \end{cases} \tag{2.2}$$

between $\mathbb{R} \cup \{\infty\}$ and $\mathbb{R}\mathbf{P}^1$. Since any pair of distinct points of $\mathbb{R}\mathbf{P}^1$ is contained in the image U of the map $\varphi \coloneqq g \circ \varphi_0|_{\mathbb{R}} \colon \mathbb{R} \to U$ for some $g \in \mathrm{PSL}_2(\mathbb{R})$, the set

$$\mathbf{dS}^2 \coloneqq (\mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^1) \setminus \Delta \text{ with } \Delta \coloneqq \left\{(p, p) \;\middle|\; p \in \mathbb{R}\mathbf{P}^1\right\}$$

is covered by the domains of maps of the form

$$\phi \colon (p, q) \in (U \times U) \setminus \Delta \mapsto (\varphi^{-1}(p), \varphi^{-1}(q)) \in \mathbb{R}^2 \setminus \{\text{diagonal}\} \tag{2.3}$$

which we call *affine charts of* $\mathbf{dS}^2$. The transition map between any two such affine charts is by construction of the form $(x, y) \in I^2 \setminus \{\text{diagonal}\} \mapsto (g(x), g(y)) \in \mathbb{R}^2$, with $I \subset \mathbb{R}$ some interval, and g abusively denoting the homography

$$g(t) \coloneqq \frac{at + b}{ct + d} \text{ associated to } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}), \tag{2.4}$$

characterized by the relation $g\left(\hat{t}\right) = \widehat{g(t)}$. A direct computation shows that the Lorentzian metric

$$\mu^0_{\mathbf{dS}^2} \coloneqq \frac{4}{|x - y|^2} dxdy \tag{2.5}$$

on $\mathbb{R}^2 \setminus \{\text{diagonal}\}$ is preserved by the transition maps $g \times g$ (2.4) between affine charts of $\mathbf{dS}^2$, which allows the following.

Definition 2.1. $\boldsymbol{\mu}$ is defined as the Lorentzian metric of $\mathbf{dS}^2$ equaling $\phi^*\mu^0_{\mathbf{dS}^2}$ on the domain of any affine chart ϕ of the form (2.3). The Lorentzian surface $(\mathbf{dS}^2, \boldsymbol{\mu})$ is called the *de-Sitter space.*

We endow $\mathbb{R}\mathbf{P}^1$ with the $\mathrm{PSL}_2(\mathbb{R})$-invariant orientation induced by the standard one of $\mathbb{R}$ through the identification (2.2), and $\mathbf{dS}^2 \subset \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^1$ with the orientation induced by the one of $\mathbb{R}\mathbf{P}^1$. We also endow $\mathbf{dS}^2$ with the time-orientation (respectively space-orientation) for which the set of future timelike (resp. spacelike) vectors is the top left quadrant $\{(u, v) \mid u < 0, v > 0\}$

(resp. top right quadrant $\{(u,v) \mid u > 0, v > 0\}$), in a tangent space endowed with the coordinates coming from an affine chart (2.3).

By construction, $\boldsymbol{\mu}$ is invariant by the diagonal action $g(x,y) \coloneqq (g(x),g(y))$ of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbf{dS}^2$. This action is moreover transitive and the stabilizer of $\mathsf{o} \coloneqq ([e_1],[e_2]) \in \mathbf{dS}^2$ is the diagonal group

$$A \coloneqq \left\{ a^t_{\mathbf{dS}^2} \;\middle|\; t \in \mathbb{R} \right\}, \text{ with } a^t_{\mathbf{dS}^2} \coloneqq \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \tag{2.6}$$

hence $\mathbf{dS}^2$ is identified with $\mathrm{PSL}_2(\mathbb{R})/A$ in a $\mathrm{PSL}_2(\mathbb{R})$-equivariant way. Note that the projection $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ induces an isomorphism from $\mathrm{SO}^0(1,1)$ defined in (2.1) with A.

We now give another (more usual) description of the de-Sitter space. The quadratic form $q_{1,2}$ of the Minkowki space $\mathbb{R}^{1,2}$ equips (by restriction to its tangent bundle) the quadric

$$\mathrm{dS}^2 \coloneqq \left\{ x \in \mathbb{R}^3 \;\middle|\; q_{1,2}(x) = 1 \right\}$$

with a Lorentzian metric μ_{dS^2} of sectional curvature constant equal to 1 (see for instance [O'N83, Proposition 4.29]), and the Lorentzian surface $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$ is the two-dimensional *hyperboloid model of the de-Sitter space*. Observe that endowing dS^2 with the restriction of the quadratic form $q_{2,1} \coloneqq -q_{1,2}$ defines a Lorentzian metric of constant curvature equal to -1. In other words, the de-Sitter and anti-de-Sitter spaces are anti-isometric in dimension 2 and have thus the same geometry.

Lemma 2.2. *(1)* $\mathrm{PSL}_2(\mathbb{R})$ *is the subgroup of isometries of* $(\mathbf{dS}^2, \boldsymbol{\mu})$ *preserving both its orientation and time-orientation.*
(2) $(\mathbf{dS}^2, \boldsymbol{\mu})$ *has constant curvature equal to* 1*, and is isometric to* $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$.

Proof. (1) This claim follows from the facts that $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathbf{dS}^2$, that the stabilizer of points in $\mathrm{PSL}_2(\mathbb{R})$ realize all linear isometries (*i.e.* that $a \in A \mapsto \mathrm{D}_{\mathsf{o}} a \in \mathrm{O}(\mathrm{T}_{\mathsf{o}}\mathbf{dS}^2, \boldsymbol{\mu}_{\mathsf{o}})$ is surjective), and that the one-jet determines pseudo-Riemannian isometries (a local isometry defined on a connected open subset, fixing a point x and of trivial differential at x, is the identity).
(2) One checks that the stabilizer in $\mathrm{SO}^0(1,2)$ of a point of dS^2 is a one-parameter hyperbolic subgroup, which gives an identification between dS^2 and $\mathrm{PSL}_2(\mathbb{R})/A$, equivariant with respect to some isomorphism between $\mathrm{SO}^0(1,2)$ and $\mathrm{PSL}_2(\mathbb{R})$. This yields two $\mathrm{PSL}_2(\mathbb{R})$-invariant Lorentzian metrics on $\mathrm{PSL}_2(\mathbb{R})/A$, respectively coming from the identifications with $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$ and $(\mathbf{dS}^2, \boldsymbol{\mu})$. But up to multiplication by a constant, $\mathfrak{sl}_2/\mathfrak{a}$ admits a unique Lorentzian quadratic form which is invariant by the adjoint action of A, and $\mathrm{PSL}_2(\mathbb{R})/A$ admits therefore a unique $\mathrm{PSL}_2(\mathbb{R})$-invariant Lorentzian metric up to multiplication by a constant. A direct computation shows that the sectional curvature of the metric $\mu^0_{\mathbf{dS}^2}$ defined in (2.5) is constant equal to 1 (see for instance the formula [O'N83, Chapter 5, Exercise 8.(b) p.150]), hence that $(\mathbf{dS}^2, \boldsymbol{\mu})$ is isometric to $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$. □

Remark 2.3. We emphasize that $\mathcal{C} \coloneqq \mathbf{P}^+(q_{1,2}^{-1}(0)) = \left\{ l \subset \mathbb{R}^{1,2} \;\middle|\; \text{null half-line} \right\}$ can be naturally interpreted as the *conformal boundary* of dS^2, and that this interpretation yields a concrete identification of dS^2 with $\mathbf{dS}^2$ where each $\mathbb{R}\mathbf{P}^1$ appears as a connected component of $\mathcal{C}$. We refer to Lemma C.1 for more details.

2.4. **Lorentzian $(\mathbf{G},\mathbf{X})$-surfaces.** We are interested in this paper in the Lorentzian surfaces locally modelled on one of the two formerly introduced homogeneous spaces. Denoting henceforth by $(\mathbf{G},\mathbf{X})$ one of the pairs $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1), \mathbb{R}^{1,1})$ or $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$, we use in this text the convenient language of $(\mathbf{G},\mathbf{X})$-structures that we now introduce.

Definition 2.4. A $(\mathbf{G},\mathbf{X})$-*atlas* on an oriented topological surface S is an atlas of orientation-preserving $\mathcal{C}^0$-charts $\varphi_i \colon U_i \to \mathbf{X}$ from connected open subsets $U_i \subset S$ to $\mathbf{X}$, whose transition maps $\varphi_j \circ \varphi_i^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ equal on every connected component of their domain the restriction of an element of $\mathbf{G}$ (henceforth, we assume that any two domains of any atlas have a connected intersection). A $(\mathbf{G},\mathbf{X})$-*structure* is a maximal $(\mathbf{G},\mathbf{X})$-atlas, and a $(\mathbf{G},\mathbf{X})$-*surface* is an oriented surface endowed with a $(\mathbf{G},\mathbf{X})$-structure. A $(\mathbf{G},\mathbf{X})$-*morphism* between two $(\mathbf{G},\mathbf{X})$-surfaces is a map which reads in any connected $(\mathbf{G},\mathbf{X})$-chart as the restriction of an element of $\mathbf{G}$.

Convention 2.5. All along this paper, $\mathbf{X}$ is considered solely with the action of the group $\mathbf{G}$. In order to make the text lighter, we thus drop henceforth $\mathbf{G}$ from our notations, and talk simply of $\mathbf{X}$-chart, $\mathbf{X}$-structure, $\mathbf{X}$-surface and $\mathbf{X}$-morphism.

For any $\mathbf{X}$-structure on a surface S, each covering $\pi\colon S'\to S$ of S is induced with the unique $\mathbf{X}$-structure for which π is a $\mathbf{X}$-morphism. In particular, $\pi_1(S)$ acts on the universal cover $\tilde{S}$ by $\mathbf{X}$-morphisms of its $\mathbf{X}$-structure. Moreover for any $\mathbf{X}$-morphism f from a connected open subset $U\subset\tilde{S}$ to $\mathbf{X}$, there exists a unique extension

$$\delta\colon\tilde{S}\to\mathbf{X}\tag{2.7}$$

of f to a $\mathbf{X}$-morphism defined on $\tilde{S}$, and such a map is called a *developing map* of S. For any developing map δ, there exists furthermore a group morphism

$$\rho\colon\pi_1(S)\to\mathbf{G}\tag{2.8}$$

with respect to which δ is equivariant, entirely determined by δ and called the *holonomy morphism* associated to δ. Such a pair (δ,ρ) associated to the $\mathbf{X}$-structure of S is moreover unique up to the action

$$g\cdot(\delta,\rho)\coloneqq(g\circ\delta,g\rho g^{-1})$$

of $\mathbf{G}$. Reciprocally any $\mathbf{G}$-orbit of such local diffeomorphisms (2.7) equivariant for some morphism (2.8) defines a unique compatible $\mathbf{X}$-structure on S. We refer the reader to [Thu97, CEG87] for more details on $(\mathbf{G},\mathbf{X})$-structures.

The core idea of $\mathbf{X}$-surfaces is that any $\mathbf{G}$-invariant geometric object on $\mathbf{X}$ gives rise to a corresponding object on any $\mathbf{X}$-surface. Let $\varepsilon_{\mathbf{X}}$ denote the constant sectional curvature of $\mathbf{X}$.

Proposition-Definition 2.6. *On any orientable surface S, $\mathbf{X}$-structures are in equivalence with time-oriented Lorentzian metrics of constant curvature $\varepsilon_{\mathbf{X}}$ in the following way.*

1. *For any $\mathbf{X}$-structure on S, there exists a unique Lorentzian metric for which $(\mathbf{G},\mathbf{X})$-charts are local isometries. The latter metric is time-oriented and has constant curvature $\varepsilon_{\mathbf{X}}$.*
2. *Conversely, any time-oriented Lorentzian metric of constant curvature $\varepsilon_{\mathbf{X}}$ on S is induced by a unique $\mathbf{X}$-structure.*
3. *Moreover under this correspondence, the $\mathbf{X}$-morphisms between $\mathbf{X}$-surfaces are exactly their orientation-preserving and time-orientation-preserving isometries between connected open subsets.*

Proof of Proposition 2.6. (1) Since $\mathbf{G}$ preserves the time-orientation of $\mathbf{X}$, the Lorentzian metric induced by a $\mathbf{X}$-structure is time-oriented, and of constant curvature $\varepsilon_{\mathbf{X}}$.
(2) Let μ be a time-oriented Lorentzian metric on S of constant sectional curvature $\varepsilon_{\mathbf{X}}$. Then it is locally isometric to $\mathbf{X}$ according to [O'N83, Corollary 8.15], and there exists thus an atlas of local isometric charts of S to $\mathbf{X}$ preserving both orientation and time-orientation. We claim that the transition maps of such an atlas and between two such atlases are restrictions of elements of $\mathbf{G}$, which prove the claim. This is essentially due to the analog of the *Liouville theorem* for $(\mathbf{G},\mathbf{X})$, claiming that any orientation and time-orientation preserving local isometry between two connected open subsets of $\mathbf{X}$, is the restriction of an element of $\mathbf{G}$.
(3) Liouville theorem proves in particular the last claim. □

We denote henceforth by the same letter μ a $\mathbf{X}$-structure on an orientable surface S and its induced Lorentzian metric.

2.5. Lightlike α and β-foliations of X-surfaces.

We now describe the lightlike foliations of our models.

Definition 2.7. We call α and β-*foliation* and denote by $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ the foliations of $\mathbf{dS}^2$ (respectively $\mathbb{R}^{1,1}$) whose leaves are the respective fibers of the second and first projections of $\mathbf{dS}^2\subset\mathbb{RP}^1\times\mathbb{RP}^1$ to $\mathbb{RP}^1$ (resp. the horizontal and vertical affine lines of $\mathbb{R}^{1,1}$). We call and denote in the same way the lightlike foliations induced by the latter on any $\mathbf{dS}^2$-surface (resp. $\mathbb{R}^{1,1}$-surface).

In other words, the α-leaves (resp. β-leaves) of $\mathbf{dS}^2$ read as horizontal (resp. vertical) lines in any affine chart (2.3) (hence the denomination to match the one for $\mathbb{R}^{1,1}$). Observe that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbf{dS}^2$ (respectively of $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1)$ on $\mathbb{R}^{1,1}$) preserve both the α and the β-foliation, which induce thus indeed foliations on any $\mathbf{dS}^2$-surface (resp. $\mathbb{R}^{1,1}$-surface).

We endow the lightlike leaves of $\mathbf{dS}^2$ with the $\mathrm{PSL}_2(\mathbb{R})$-invariant orientation induced by the one of $\mathbb{R}\mathbf{P}^1$, and the lightlike leaves $\mathbb{R}\times\{b\}$ and $\{a\}\times\mathbb{R}$ of $\mathbb{R}^{1,1}$ with the $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1)$-invariant one induced by $\mathbb{R}$. This further induces an orientation on the lightlike foliations of any $\mathbf{X}$-surface, compatible with its orientation, time-orientation and space-orientation as illustrated by Figure 3.1 below. The lightlike leaves of $\mathbf{dS}^2$ and $\mathbb{R}^{1,1}$ are embeddings of $\mathbb{R}$, and we denote by $\mathcal{F}^{+*}_\alpha(p)$ and $\mathcal{F}^{-*}_\alpha(p)$ the *half α-leaves*, *i.e.* the two connected components of $\mathcal{F}_\alpha(p)\setminus\{p\}$ emanating respectively in the positive and negative directions, by $\mathcal{F}^{+}_\alpha(p)$ and $\mathcal{F}^{-}_\alpha(p)$ their closures, and accordingly for $\mathcal{F}^{\pm}_\beta(p)$. Note that the lightlike leaves are the lightlike geodesics of the underlying Lorentzian metric, and have as such a natural affine parametrization.

2.6. **Cyclic order, intervals of a circle and rectangles of $\mathbf{dS}^2$.** The circles $\mathbb{R}\mathbf{P}^1$ and $\mathbf{S}^1$ inherit from their orientation a $\mathrm{PSL}_2(\mathbb{R})$-invariant *cyclic ordering*, *i.e.* a partition of triplets $(x_1,x_2,x_3) \in (\mathbb{R}\mathbf{P}^1)^3$ (respectively $(\mathbf{S}^1)^3$) between *positive* and *negative* ones which is invariant by cyclic permutations, exchanged by transpositions and defined in the following way. Any n-tuple ($n\geq 3$) of pairwise distinct points of $\mathbb{R}\mathbf{P}^1$ has an ordering $(x_1,\dots,x_n)$, unique up to the n cyclic permutations $(1,\dots,n)^k$ for $1\leq k\leq n$, such that for any $1\leq i\leq n-1$, the positively oriented injective path of $\mathbb{R}\mathbf{P}^1$ from x_i to x_{i+1} does not meet any of the x_j for $j\notin\{i,i+1\}$. In this case $(x_1,\dots,x_n)$ is said to be *positively cyclically ordered*, and two n-tuples $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ are said to have *the same cyclic order* if there exists a permutation σ such that $(x_{\sigma(1)},\dots,x_{\sigma(n)})$ and $(y_{\sigma(1)},\dots,y_{\sigma(n)})$ are both positive. For any $x,y\in\mathbb{R}\mathbf{P}^1$, we denote

$$[x\,;y] \coloneqq \{x,y\} \cup \left\{ z\in\mathbb{R}\mathbf{P}^1 \;\middle|\; (x,z,y) \text{ is positively cyclically ordered} \right\} \subset \mathbb{R}\mathbf{P}^1$$

with $[x\,;y]=\{x\}$ if $x=y$, and adopt the same notation for any oriented topological circle. For any $p=(x_p,y_p), q=(x_q,y_q)\in\mathbf{dS}^2$ such that $q\in\mathcal{F}^+_\alpha(p)$ (respectively $q\in\mathcal{F}^+_\beta(p)$) we denote

$$[p\,;q]_\alpha \coloneqq [x_p\,;x_q]\times\{y_p\}, [p\,;q]_\beta \coloneqq \{x_p\}\times[y_p\,;y_q],$$

with obvious corresponding notations in $\mathbb{R}^{1,1}$ and for (half-)open intervals. More generally in any $\mathbf{X}$-surface, $[p\,;q]_{\alpha/\beta}$ denotes the segment of the oriented leaf $\mathcal{F}_{\alpha/\beta}(p)$ from p to q.

Definition 2.8. For any four distinct points $A,B,C,D\in\mathbf{dS}^2$ such that $(x_A,y_A)=A=\mathcal{F}^-_\alpha(B)\cap\mathcal{F}^-_\beta(D)$ and $(x_C,y_C)=C=\mathcal{F}^+_\beta(B)\cap\mathcal{F}^+_\alpha(D)$,

$$\mathcal{R}_{ABCD}=\mathcal{R}_{(x_A,x_C,y_A,y_C)} \coloneqq [x_A\,;x_C]\times[y_A\,;y_C]$$

is called a *rectangle of* $\mathbf{dS}^2$ *with lightlike boundary.*

Note that by convention, the rectangles that we consider are non-degenerated (*i.e.* have distinct edges), and that we name the vertices of a rectangle $\mathcal{R}_{ABCD}$ of $\mathbf{dS}^2$ in the positive cyclic order by starting with its "bottom-left" vertex A. The area of an orientable surface S for the area form induced by a Lorentzian metric μ (which, by definition, gives volume 1 to a direct orthogonal basis of norms $(1,-1)$ for μ), is denoted by $\mathcal{A}_\mu(S)$.

Lemma 2.9. *Two rectangles of* $\mathbf{dS}^2$ *with lightlike boundaries are in the same orbit under* $\mathrm{PSL}_2(\mathbb{R})$ *if and only if they have the same area.*

Proof. For any rectangle $\mathcal{R}_{(x_A,x_C,y_A,y_C)}$, (y_A,y_C,x_A) is a positively cyclically ordered triplet of $\mathbb{R}\mathbf{P}^1$, and we can thus assume without loss of generality that $\mathcal{R}_{(x_A,x_C,y_A,y_C)}=\mathcal{R}_{(\hat{1},\hat{t},\hat{\infty},\hat{0})}$. Since $t\in\,]1\,;+\infty[\,\mapsto\mathcal{A}_\mu(\mathcal{R}_{(\hat{1},\hat{t},\hat{\infty},\hat{0})})\in\mathbb{R}^*_+$ is bijective, two rectangles have the same area if and only if the 4-tuples defining them have the same cross-ratio, which happens if and only if they are in the same orbit under $\mathrm{PSL}_2(\mathbb{R})$. ☐

3. Singular constant curvature Lorentzian surfaces

This section is devoted to define and prove the fundamental notions and properties concerning singular constant curvature Lorentzian surfaces.

3.1. The local model of standard singularities.

We first define in this subsection the local singularities that are considered in this text, and prove some of their fundamental properties. They already appeared with another name in [BBS11, §3.3], the specific relationship between the two denominations being explained in Remark 3.11.

Convention 3.1. $(\mathbf{G},\mathbf{X})$ denotes henceforth either the pair $(\mathbb{R}^{1,1}\rtimes\mathrm{SO}^0(1,1),\mathbb{R}^{1,1})$ or the pair $(\mathrm{PSL}_2(\mathbb{R}),\mathbf{dS}^2)$, $\boldsymbol{\mu}$ the Lorentzian metric of $\mathbf{X}$, and $g_{\boldsymbol{\mu}}$ its associated bilinear form. We also fix the base-point $\mathsf{o}\in\mathbf{X}$ respectively equal to $(0,0)$ or $([e_1],[e_2])$, denote by $A=\{a^t\}_{t\in\mathbb{R}}$ its stabilizer in $\mathbf{G}$, and fix the parametrization $a^t:=a^t_{\mathbf{X}}$ of A respectively defined in (2.1) and (2.6). This choice of parametrization is crucial for the correspondence (3.7) between angles and areas given below by Gauß-Bonnet formula, and does not matter apart from there. A direct computation shows that $t\in\mathbb{R}\mapsto a^t=a^t_{\mathbf{X}}$ is the unique isomorphism such that for any unit timelike vector $u\in\mathrm{T}_{\mathsf{o}}\mathbf{X}$ (*i.e.* $\boldsymbol{\mu}(u)=-1$):

(1) for any $t>0$: $(u,\mathrm{D}_{\mathsf{o}}a^t(u))$ is a negatively oriented basis;
(2) for any $t\in\mathbb{R}$, denoting by cosh the hyperbolic cosine function:

$$g_{\boldsymbol{\mu}}(u,\mathrm{D}_{\mathsf{o}}a^t(u))=-\cosh(t). \tag{3.1}$$

3.1.1. *Standard singularities as identification spaces.* We denote by $\mathbf{X}_*$ the surface with boundary and one conical point obtained from $\mathbf{X}$ by cutting it along $\mathcal{F}^{+*}_\alpha(\mathsf{o})$. The interior of $\mathbf{X}_*$ is identified with $\mathbf{X}\setminus\mathcal{F}^+_\alpha(\mathsf{o})$, its conical point o' with o, and its two boundary components are "upper" and "lower" embeddings $\iota_\pm\colon\mathcal{F}^+_\alpha(\mathsf{o})\to\mathbf{X}_*$ of $\mathcal{F}^+_\alpha(\mathsf{o})$ with $\iota_\pm(\mathsf{o})=\mathsf{o}'$. Furthermore $\mathbf{X}_*$ is endowed with an action of the diagonal subgroup A for which the embeddings $\iota_\pm$ are equivariant.

For $\theta\in\mathbb{R}$, we introduce the equivalence relation generated by the relations $\iota_+(x)\sim_\theta\iota_-(a^\theta(x))$ for any $x\in\mathcal{F}^{+*}_\alpha(\mathsf{o})$, and we denote by

$$\pi_\theta\colon\mathbf{X}_*\to\mathbf{X}_\theta=\mathbf{X}_*/\sim_\theta \tag{3.2}$$

the canonical projection onto the topological quotient of $\mathbf{X}_*$ by $\sim_\theta$. This identification space is illustrated in Figure 3.1.

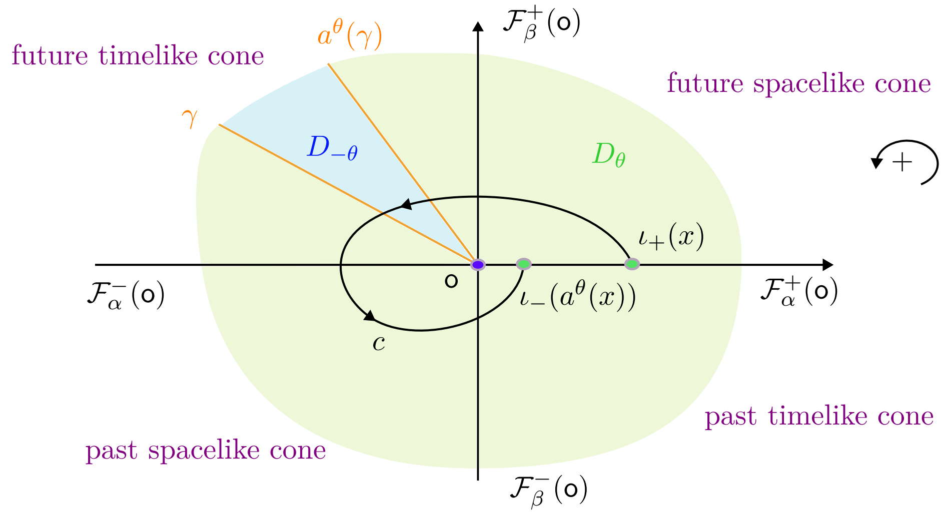

Figure 3.1. Standard singularity, quadrants and orientations conventions.

We define $\mathsf{o}_\theta:=\pi_\theta(\mathsf{o}')$ and endow $\mathbf{X}_\theta\setminus\{\mathsf{o}_\theta\}$ with its *standard* $\mathbf{X}$*-structure* defined by the following atlas.

(1) For any open set $U\subset\mathbf{X}\setminus\mathcal{F}^+_\alpha(\mathsf{o})$, we consider the chart $\varphi_{\pi_\theta(U)}\colon\pi_\theta(U)\to U$ satisfying $\varphi_{\pi_\theta(U)}\circ\pi_\theta|_U=\mathrm{id}|_U$.

(2) Let $U \subset \mathbf{X} \setminus \{\mathsf{o}\}$ be an open set such that $U \setminus \mathcal{F}^+_\alpha(\mathsf{o})$ has two respectively up and down connected components U_+ and U_-, and $a^\theta(U) \cap U = \varnothing$. Then we consider the open set $V = \pi_\theta(U_+ \cup \iota_+(U \cap \mathcal{F}^+_\alpha(\mathsf{o})) \cup a_\theta(U_-))$ of $\mathbf{X}_\theta$, and the chart $\varphi_V \colon V \to U$ satisfying:
   - $\varphi_V \circ \pi_\theta = \mathrm{id}$ in restriction to $U_+ \cup \iota_+(U \cap \mathcal{F}^+_\alpha(\mathsf{o}))$,
   - and $\varphi_V \circ \pi_\theta = a^{-\theta}$ in restriction to $a^\theta(U_-)$.

Definition 3.2. The *standard* $\mathbf{X}$*-cone of angle* θ is the oriented topological surface $\mathbf{X}_\theta$ endowed with its marked point o_θ, its standard $\mathbf{X}$-structure on $\mathbf{X}_\theta \setminus \{\mathsf{o}_\theta\}$ and its associated Lorentzian metric denoted by $\boldsymbol{\mu}_\theta$.

Note that our definition makes sense for $\theta = 0$, and that in this case $\mathbf{X}_0 = \mathbf{X}$.

Remark 3.3. The standard cones that we have introduced do not exhaust the natural geometric singularities, and we refer to Remark 4.5 for a discussion of other kind of examples. However these singularities are the *dynamically natural* ones: they are essentially the only ones at which the lightlike foliations extend to two continuous foliations, in a sense made more precise in Lemma 3.5. The existence of these continuous foliations is our main motivation for considering this specific type of singularities, and is the subject of the next paragraph.

3.1.2. *Lightlike foliations at a standard singularity.* To investigate the behaviour of the lightlike foliations at the singularity, we consider a continuous chart of $\mathbf{X}_\theta$ at o_θ defined as follows. Let $\exp_{\mathsf{o}} \colon \mathrm{T}_{\mathsf{o}}\mathbf{X} \to \mathbf{X}$ denote the exponential chart of $\mathbf{X}$ at o, and $d_\nu \subset \mathrm{T}_{\mathsf{o}}\mathbf{X}$ be the open half-line making a positive euclidean angle $\nu \in [0\,;2\pi[$ with d_0, where $\exp_{\mathsf{o}}(d_0) \subset \mathcal{F}^+_\alpha(\mathsf{o})$. Note that $a^\theta \circ \exp_{\mathsf{o}} = \exp_{\mathsf{o}} \circ \mathrm{D}_{\mathsf{o}} a^\theta$, hence with $\theta' \in \mathbb{R}$ characterized by $\mathrm{D}_{\mathsf{o}} a^\theta(u) = e^{-2\theta'} u$ for $u \in \mathrm{T}_{\mathsf{o}}\mathcal{F}_\alpha(\mathsf{o})$, we have $\iota_+(\exp_{\mathsf{o}}(u)) \sim_\theta \iota_-(\exp_{\mathsf{o}}(e^{-2\theta'} u))$. With D an open disk centered at 0 in $\mathrm{T}_{\mathsf{o}}\mathbf{X}$, we consider the open neighbourhood

$$U \coloneqq \iota_+ \circ \exp_{\mathsf{o}}(d_0 \cap D) \cup \bigcup_{\nu \in]0;2\pi[} \exp_{\mathsf{o}}(e^{-\frac{\nu}{\pi}\theta'}(d_\nu \cap D))$$

of o' in $\mathbf{X}_*$, so that $V = \pi_\theta(U)$ is an open neighbourhood of o_θ in $\mathbf{X}_\theta$. We define then a map $\psi_\theta \colon V \to D$, for any $\nu \in [0\,;2\pi[$ and $u \in e^{-\frac{\nu}{\pi}\theta'}(d_\nu \cap D)$, by

$$\psi_\theta \circ \pi_\theta(\exp_{\mathsf{o}}(u)) = e^{\frac{\nu}{\pi}\theta'} u.$$

In the above equation for $p \in \mathcal{F}^+_\alpha(\mathsf{o})$, we abusively denoted $\iota_+(p)$ simply by p. It is easily checked that ψ_θ is a homeomorphism from V to D.

Proposition 3.4. *The lightlike foliations of* $\mathbf{X}_\theta \setminus \{\mathsf{o}_\theta\}$ *extend uniquely to two topological one-dimensional foliations on* $\mathbf{X}_\theta$*, that we call the* lightlike foliations of $\mathbf{X}_\theta$ *and continue to denote by* $\mathcal{F}_\alpha$ *and* $\mathcal{F}_\beta$*. Moreover for any small enough open neighbourhoods* I *and* J *of* o_θ *in* $\mathcal{F}_\alpha(\mathsf{o}_\theta)$ *and* $\mathcal{F}_\beta(\mathsf{o}_\theta)$*,*

$$\Phi \colon (x, y) \in I \times J \mapsto \mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$$

is a homeomorphism onto its image, restricting outside of o_θ *to a* $\mathcal{C}^\infty$*-diffeomorphism onto its image. The continuous* α *and* β*-foliations are thus* transverse *in the sense that* Φ *defines a simultaneous* $\mathcal{C}^0$ *foliated chart.*

Proof. Since $\psi_\theta(\pi_\theta(\iota_+(\mathcal{F}^{+*}_\alpha(\mathsf{o})) \cup \mathcal{F}^{-*}_\alpha(\mathsf{o}))) = \mathbb{R}\cdot d_0 \setminus \{0\}$ and $\psi_\theta(\pi_\theta(\mathcal{F}^{+*}_\beta(\mathsf{o}) \cup \mathcal{F}^{-*}_\beta(\mathsf{o}))) = \mathbb{R}\cdot d_\beta \setminus \{0\}$ where $\exp_{\mathsf{o}}(\mathbb{R}\cdot d_\beta) = \mathcal{F}_\beta(\mathsf{o})$, the only possible definition of the α and β-leaves of o_θ for it to define a foliation with continuous leaves, is: $\mathcal{F}_\alpha(\mathsf{o}_\theta) = \psi_\theta^{-1}((\mathbb{R} \times \{0\}) \cap D)$ and $\mathcal{F}_\beta(\mathsf{o}_\theta) = \psi_\theta^{-1}((\{0\} \times \mathbb{R}) \cap D)$. This makes $\mathcal{F}_\alpha(\mathsf{o}_\theta)$ and $\mathcal{F}_\beta(\mathsf{o}_\theta)$ two topological 1-manifolds. Now for any small enough open neighbourhoods I and J of o_θ in $\mathcal{F}_\alpha(\mathsf{o}_\theta)$ and $\mathcal{F}_\beta(\mathsf{o}_\theta)$, and any $(x, y) \in I \times J$: $\mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$ is a single point which we denote by $[x, y]$. Moreover for $x, x' \in \mathcal{F}_\alpha(\mathsf{o}_\theta)$, $x \neq x'$ implies $\mathcal{F}_\beta(x) \cap \mathcal{F}_\beta(x') = \varnothing$, and similarly for $y \neq y' \in \mathcal{F}_\beta(\mathsf{o}_\theta)$. Therefore $\Phi \colon (x, y) \in I \times J \mapsto [x, y]$ is an injective map from $I \times J$ to the topological surface $\mathbf{X}_\theta$, which is clearly continuous, and $\Phi(\mathsf{o}_\theta, \mathsf{o}_\theta) = \mathsf{o}_\theta$. By Brouwer's invariance of domain theorem, Φ is thus a homeomorphism onto its image U, which is an open neighbourhood of o_θ. Observe moreover that Φ is a $\mathcal{C}^\infty$-diffeomorphism onto its image on restriction to any small enough open subset of $\mathbf{X}_\theta \setminus \{\mathsf{o}_\theta\}$, since it is so in $\mathbf{X}$. Furthermore $\Phi(\{x\} \times J)$ contains an open neighbourhood of x in $\mathcal{F}_\beta(x)$, and $\Phi(I \times \{y\})$ an open neighbourhood of y in $\mathcal{F}_\alpha(y)$. The restriction of Φ to suitable subsets defines thus a simultaneous continuous foliated chart for the α and β-foliations, which concludes the proof. □

3.1.3. *Characterization of standard singularities and their angles by developing maps and holonomy morphisms.* We now characterize the singularity o_θ of $\mathbf{X}_\theta$ among the $\mathbf{X}$-structures of a punctured disk. Let us call *slit neighbourhood of* $\mathbf{X}$ an open set of the form $U' = U \setminus \mathcal{F}^+_\alpha(p)$ for U an open neighbourhood of a point $p \in \mathbf{X}$.

Lemma 3.5. *Let D be an oriented topological disk, $x \in D$, and $D^* \coloneqq D \setminus \{x\}$ be endowed with a $\mathbf{X}$-structure. Let R denote the* positive generator of $\pi_1(D^*)$, i.e. *the homotopy class of a positively oriented closed loop around x generating $\pi_1(D^*)$. Then the following properties (1) and (2) are equivalent.*

- *(1) There exists $\theta \in \mathbb{R}$, and a homeomorphism φ from an open neighbourhood U of x to an open neighbourhood of o_θ in $\mathbf{X}_\theta$, such that: $\varphi(x) = \mathsf{o}_\theta$, and φ is a $\mathbf{X}$-morphism in restriction to $U^* = U \setminus \{x\}$.*
- *(2)*
  - *(a) The lightlike foliations of D^* extend uniquely to two continuous 1-dimensional foliations of D;*
  - *(b) and there exists an open disk $U \subset D$ containing x, and a $\mathbf{X}$-isomorphism ψ from $U' = U \setminus \mathcal{F}^+_\alpha(x)$ to a slit neighbourhood of o.*

Furthermore property (1) for $\theta \in \mathbb{R}$ is equivalent to (2).(a) and (2).(b) together with:

(2).(c) $\rho(R) = a^\theta$, with ρ the holonomy morphism associated to the developing map extending the lift of a $\mathbf{X}$-morphism ψ like in (2).(b).

In particular, there exists at most one $\theta \in \mathbb{R}$ for which the equivalent properties (1) and (2) can be satisfied for θ.

Definition 3.6. Let $D^* \coloneqq D \setminus \{x\}$ be an oriented topological punctured disk endowed with a $\mathbf{X}$-structure. We say that x is a *standard singularity of angle θ* of D if the equivalent properties (1) and (2).(a)-(c) of Lemma 3.5 are satisfied at x for $\theta \in \mathbb{R}$. A developing map of D^* extending a lift of φ like in (1) (equivalently of ψ like in (2).(b)) and its holonomy morphism are said *compatible at x*.

Remark 3.7. The holonomy of a positively oriented loop around a singularity is well defined only *up to conjugacy*, and for $\theta \in \mathbb{R}$ and $g \in \mathrm{PSL}_2(\mathbb{R})$: $a^\theta = ga^{-\theta}g^{-1}$ if and only if g is an anti-diagonal matrix. Hence if the angles of singularities were to be simply defined as the latter holonomy conjugacy class, then they would be well-defined only *up to sign*. It is not a surprise that the conjugacy class of the holonomy is not sufficient to determine the germ of a singularity since the latter is generally not sufficient to determine a (G, X)-structure (it only determines it *locally*). This is the reason why we have to take into account the developing map around a standard singularity x to define the sign of its angle.

This sign can however be easily interpretated as follows by developing a positively oriented curve around the singularity. Let E be the universal covering of a punctured singular $\mathbf{X}$-disk $D^* = D \setminus \{x\}$ with a single standard singularity at x, γ be a positively oriented loop around x generating $\pi_1(D^*)$, and $\delta\colon E \to \mathbf{X}$ be a compatible developing map at x. Then with $\tilde{\gamma}\colon \mathbb{R} \to E$ any lift of γ in E, the curve $\delta \circ \tilde{\gamma}\colon \mathbb{R} \to \mathbf{X}$ converges to o at $+\infty$ if $\theta > 0$, and at $-\infty$ if $\theta < 0$.

We present in Lemma 3.14 and Corollary 3.15 other intrinsic characterizations of the angle at a singularity.

Lemma 3.5 implies directly the following results.

Corollary 3.8. *Let $D^* \coloneqq D \setminus \{x\}$ be an oriented punctured disk endowed with a $\mathbf{X}$-structure. If x is a standard singularity of angle 0, equivalently a standard singularity of trivial holonomy, then the $\mathbf{X}$-structure of D^* uniquely extends to D. In other words, x is actually a regular point.*

Corollary 3.9. *Let x be a standard singularity of a $\mathbf{X}$-structure on an oriented punctured disk $D^* \coloneqq D \setminus \{x\}$, $\rho\colon \pi_1(D^*) \to \mathbf{G}$ be a compatible holonomy map at x, and c be a positively oriented loop of D^* whose homotopy class $[c]$ generates $\pi_1(D^*)$. Then x is of angle $\theta \in \mathbb{R}$ if and only if $\rho([c]) = a^\theta$.*

The interpretation of the angle θ of a standard singularity x as the holonomy of a positive closed loop c around it is illustrated in Figure 3.1.

Proof of Lemma 3.5. **(1) for θ ⇒ (2).(a),(b)&(c).** The unique continuous extension of the lightlike foliations follows from Proposition 3.4. The restriction of the map φ of (1) to a slit neighbourhood U' of x is a $\mathbf{X}$-isomorphism to a slit neighbourhood of o_θ which is canonically identified with a slit neighbourhood of o by the projection map π_θ, giving us the desired map ψ. Now let O be an open subset of the universal cover of D^* projecting homeomorphically to U', and δ be the developing map extending a lift of ψ to O. Then δ satisfies $\delta \circ R = a^\theta \circ \delta$ (on the non-empty open subset where this equality is well-defined) by the very definition of $\mathbf{X}_\theta$, which shows that $\rho(R) = a^\theta$ and concludes the proof of this implication.

(2).(a)&(b) ⇒ (1) for some θ. Let $\pi\colon E \to U^* = U \setminus \{x\}$ be the universal covering map of U^*, and $O \subset E$ be an open set such that $\pi|_O$ is a diffeomorphism onto $U' = U \setminus \mathcal{F}^+_\alpha(x)$. The existence of ψ shows that the restriction of the developing map $\delta\colon E \to \mathbf{X}$ to O is an isometry onto $V' = V \setminus \mathcal{F}^+_\alpha(\mathsf{o})$, with V an open neighbourhood of o. The lightlike leaf spaces of V' have the following description:

- the leaf space $\mathcal{L}_\beta$ of the β-foliation of V' is homeomorphic to the non-Hausdorff topological 1-manifold $(L^+ \cup L^-)/\sim$, with $L^\pm$ two copies of $\mathbb{R}$ and $p^- \sim p^+$ for $p \in \mathbb{R}_{<0}$, the special points $0^\pm$ corresponding to the special leaves $J^\pm_\beta := \mathcal{F}^\pm_\beta(\mathsf{o}) \cap V'$;
- the leaf space of the α-foliation of V' has one specific point $J^-_\alpha := \mathcal{F}^-_\alpha(\mathsf{o}) \cap V'$, which is the only α-leaf intersecting none of the leaves $p^\pm \in \mathcal{L}_\beta$ for $p \geq 0$.

Since the lightlike foliations of D^* extend by assumption to continuous foliations of D, we can choose U to be a small enough neighbourhood of x for it to be a trivialization domain of both lightlike foliations of D. The same above description holds then for the lightlike leaf spaces of U' than for the ones of V'. Let us denote by $I^\pm_\beta$, respectively I^-_α the lifts of $\mathcal{F}^\pm_\beta(x) \cap U$, resp. $\mathcal{F}^-_\alpha(x) \cap U$ in O, and by $I^{d/u}_\alpha$ the "down and up" lifts of $\mathcal{F}^+_\alpha(x) \cap U$, so that $\partial O = I^d_\alpha \cup I^u_\alpha$ and $R(I^d_\alpha) = I^u_\alpha$. Then since δ is a simultaneous equivalence between the lightlike foliations, the descriptions of the leaf spaces impose $\delta(I^\pm_\beta) = J^\pm_\beta$, $\delta(I^-_\alpha) = J^-_\alpha$ and $\delta(I^{d/u}_\alpha) =]\mathsf{o}\,;p^{d/u}[_\alpha$ with $p^{d/u} \in \mathcal{F}^{+*}_\alpha(\mathsf{o})$. With ρ the holonomy morphism associated to δ we have thus $\rho(R)(]\mathsf{o}\,;p^d[_\alpha) =]\mathsf{o}\,;p^u[_\alpha$, which shows that $\rho(R)$ fixes o, *i.e.* $\rho(R) = a^\theta$ for some θ, and thus $\delta \circ R = a^\theta \circ \delta$.

We now define a map $\varphi\colon U \to \mathbf{X}_\theta$ by:

- $\varphi(x) = \mathsf{o}_\theta$;
- $\varphi \circ \pi = \pi_\theta \circ \delta$ on O;
- $\varphi \circ \pi = \pi_\theta \circ \iota_+ \circ \delta$ on I^d_α;

and show that φ satisfies the properties of (1). Let W be an open neighbourhood of $p \in I^d_\alpha$ so that $\pi|_W$ is a diffeomorphism onto $\pi(W)$, and $W \setminus I^d_\alpha$ has two connected components $W^\pm$, with $W^+ \subset O$ and $R(W^-) \subset O$. Since $\delta \circ R = a^\theta \circ \delta$, we have $\varphi \circ \pi = \pi_\theta \circ a^\theta \circ \delta$ on W^-, $\varphi \circ \pi = \pi_\theta \circ \iota_+ \circ \delta$ on $I^d_\alpha \cap W$ and $\varphi \circ \pi = \pi_\theta \circ \delta$ on W^+, which shows that φ is a $\mathbf{X}$-morphism to $\mathbf{X}_\theta$ on the neighbourhood of $\pi(p)$.

It thus only remains to show that φ is continuous at x. Our former description shows that $\varphi(\mathcal{F}_{\alpha/\beta}(x) \cap U) = \mathcal{F}_{\alpha/\beta}(\mathsf{o}_\theta)$, and thus that φ induces two maps $\phi_{\alpha/\beta}$ between the respective leaf spaces of the α, resp. β-foliations of U and $\varphi(U) \subset \mathbf{X}_\theta$. These foliations being continuous and transverse, it moreover suffices to show that the maps $\phi_{\alpha/\beta}$ induced by φ between the leaf spaces are continuous at $\mathcal{F}_{\alpha/\beta}(x) \cap U$, to conclude that φ is continuous at x. But our former description of the leaf spaces of the slit neighbourhoods U' and V' showed that $\delta(I^-_\alpha) = J^-_\alpha$, and thus for any sequence L_n of α-leaves contained in U' and converging to $\mathcal{F}^\alpha(x) \cap U$, $\varphi(L_n)$ converges to $\mathcal{F}^-_\alpha(\mathsf{o}_\theta)$, which shows the continuity of ϕ_α at $\mathcal{F}^\alpha(x) \cap U$. In the same way, the fact that $\delta(I^\pm_\beta) = J^\pm_\beta$ shows that ϕ_β is continuous at $\mathcal{F}^\beta(x) \cap U$, which concludes the proof of the second implication.

Uniqueness of θ. If θ_1 and θ_2 both satisfy the equivalent properties (1) and (2), then the holonomy morphism of a developing map extending the lift of a $\mathbf{X}$-isomorphism like in (b) should satisfy $a^{\theta_1} = \rho(R) = a^{\theta_2}$ according to (c) (note that (b) is indeed independent of θ). Hence $\theta_1 = \theta_2$, which concludes the proof of the lemma. ☐

3.1.4. *Standard singularities as quotients.* Let D be an open topological disk around o in $\mathbf{X}$ wich is left invariant by a^θ. For $\mathbf{X} = \mathbb{R}^{1,1}$ one can take $D \coloneqq \mathbb{R}^{1,1} \setminus \{0\}$, and $D \coloneqq \mathbf{dS}^2 \setminus \mathcal{F}_\beta([e_2],[e_1])$ for $\mathbf{X} = \mathbf{dS}^2$. Then $a^\theta|_{D^*}$ is an isometry of $D^* \coloneqq D \setminus \{\mathsf{o}\}$, which lifts to a unique isometry $\tilde{a}^\theta$ of the universal cover E of D^* fixing each lift of the connected components of the punctured lightlike leaves of o. On the other hand, E admits also a preferred isometry R which is the positive generator of its covering automorphism group.

Lemma 3.10. *The group generated by $\tilde{a}^\theta \circ R$ acts properly discontinuously on E, and $E/\langle \tilde{a}^\theta \circ R\rangle$ is $\mathbf{X}$-isomorphic to D^*. More precisely, there is a natural embedding of $E/\langle \tilde{a}^\theta \circ R\rangle$ as the complement of a point o_θ in a topological disk $\bar{E}$, for which o_θ is a standard singularity of angle θ of $\bar{E}$.*

Proof. Any lift $\tilde{\mathcal{F}}_\alpha$ of $\mathcal{F}^{+*}_\alpha(\mathsf{o})$ is an embedding of $\mathbb{R}$ separating $E \simeq \mathbb{R}^2$ in two connected components, and since $\langle R\rangle \simeq \mathbb{Z}$ acts properly discontinuously on E, the images of $\tilde{\mathcal{F}}_\alpha$ by $\langle R\rangle$ are pairwise disjoint and form a discrete set. The complement of $\langle R\rangle \cdot \tilde{\mathcal{F}}_\alpha$ in E is a disjoint union of topological disks, the boundary of each of them being the disjoint union of an upper and a lower translate of $\tilde{\mathcal{F}}_\alpha$, and the closure of any of these connected components is a fundamental domain for the action of $\langle R\rangle$ on E. The important observation is now that by definition, $\langle \tilde{a}^\theta\rangle$ preserves the interior and the boundary of any of these fundamental domains and acts properly on it, which shows that $\tilde{a}^\theta \circ R$ acts indeed properly discontinuously on E.

We add to $E/\langle \tilde{a}^\theta \circ R\rangle$ a point o_θ, with a neighbourhood basis composed of images of sets of the form $U \cup \{o_\theta\}$, for all the $\tilde{a}^\theta \circ R$-invariant open sets $U \subset E$ projecting to punctured neighbourhoods of o in D. This defines a topological disk $\bar{E}$, in which the lightlike foliations of $E/\langle \tilde{a}^\theta \circ R\rangle = \bar{E} \setminus \{o_\theta\}$ extend to two continuous transverse foliations. The complement of $\tilde{\mathcal{F}}_\alpha = \mathcal{F}^{+*}_\alpha(o_\theta)$ in $\bar{E}$ is $\mathbf{X}$-isomorphic to the interior of one of the previously described fundamental domains, themselves isomorphic to the slit neighbourhood $D \setminus \mathcal{F}^{+*}_\alpha(\mathsf{o})$ in $\mathbf{X}$. The result now follows from Lemma 3.5. ∎

Remark 3.11. Lemma 3.10 allows to check that a standard singularity as it is defined in the present paper, corresponds to a *space-like singularity of degree 1* as it is defined in the item (4) of the list appearing in [BBS11, p.160].

3.1.5. *Standard singularities as angle defaults.* It is natural to ask wether the standard Lorentzian singularities that we introduced can be interpretated, as in the Riemannian case, as angle defaults. To this end, we first need to introduce a proper notion of Lorentzian angle, following [BN84].

Definition 3.12 ([BN84])**.** Let P be an oriented plane endowed with a Lorentzian scalar product $\langle \cdot , \cdot \rangle$. For $X, Y \in P$, we denote $\mathsf{or}(X,Y) = 1$ (respectively -1) if (X,Y) is a positively (resp. negatively) oriented basis, and $\mathsf{or}(X,Y) = 0$ if (X,Y) are linearly dependent. Then for (X,Y) two unit timelike vectors belonging to the same quadrant of P, the Lorentzian *angle from X to Y* is defined by

$$((X,Y)) \coloneqq \mathsf{or}(X,Y) \operatorname{arcosh} |\langle X , Y\rangle| \tag{3.3}$$

with $\operatorname{arcosh}\colon [1\,;+\infty[\to \mathbb{R}^+$ the inverse hyperbolic cosine function. This definition is extended to any pair (X,Y) of unit timelike vectors by the relation

$$((X,Y)) = ((X,-Y))\,.$$

Note that (3.3) is well-defined, since $|\langle X , Y\rangle| \geq 1$ according to the Lorentzian Cauchy-Schwartz inequality. Furthermore for any three unit timelike vectors X, Y, Z, the relations

$$\begin{cases} ((-X,-Y)) = ((-X,Y)) = ((X,Y)) \\ ((X,X)) = ((X,-X)) = 0 \\ ((X,Z)) = ((X,Y)) + ((Y,Z)) \end{cases} \tag{3.4}$$

follow easily from the definition (see [BN84, Lemma 1]).

Remark 3.13. Our convention (3.1) on the parametrization $A = (a^t)_t$ is made to satisfy the relation

$$\Big(\Big(u, \mathrm{D}_{\mathsf{o}} a^t(u)\Big)\Big) = -t \tag{3.5}$$

for any unit timelike vector $u \in \mathrm{T}_{\mathsf{o}}\mathbf{X}$ and any $t \in \mathbb{R}$.

Let D be a small disk around o in $\mathbf{X}$, $\gamma \subset \mathbf{X}$ be a half-open future-oriented timelike geodesic starting from o, $\theta > 0$ and $\gamma_\theta := a^\theta(\gamma)$. Then $D \setminus (\gamma \cup \gamma_\theta)$ has two connected components illustrated in Figure 3.1 whose closure are denoted by $D_{\pm\theta}$, with $D_{-\theta}$ contained in the future timelike quadrant of o and D_θ containing the three other quadrants. The angle from $a^\theta(\gamma)$ to γ is equal to $\theta > 0$, and $D_{-\theta}$ is thus the (unique up to isometries) futur timelike sector of angle θ at o. We can now consider the quotient $\bar{D}_\theta$ of D_θ by the relation $\gamma \ni x \sim a^\theta(x) \in \gamma_\theta$ on its boundary (in particular $\mathsf{o} \sim \mathsf{o}$). As we did in Paragraph 3.1.1, we also consider the surface D_* obtained from D by cutting it open along $\gamma \setminus \{\mathsf{o}\}$, with two upper and lower boundary components $\iota_\pm \colon \gamma \to D_*$. We can now form the quotient $\bar{D}_{-\theta}$ of $D_* \cup D_{-\theta}$ by the relation: $\iota_-(x) \sim x \in \gamma$ and $\iota_+(x) \sim a^\theta(x) \in \gamma_\theta$ for $x \in \gamma$. The topological disks $\bar{D}_{\pm\theta}$ have a marked point $o_{\pm\theta}$, image of o, and bear a natural $\mathbf{X}$-structure on $\bar{D}_{\pm\theta} \setminus \{o_{\pm\theta}\}$ which is defined as in Paragraph 3.1.1.

Lemma 3.14. *The point o_θ (respectively $o_{-\theta}$) is a standard singularity of angle $\theta > 0$ (resp. $-\theta$) of $\bar{D}_\theta$ (resp. of $\bar{D}_{-\theta}$).*

A singularity of angle $\theta > 0$ is thus obtained by removing a timelike sector of angle θ, and a singularity of angle $-\theta < 0$ by adding a timelike sector of angle θ. Analogous statements can be given for any two half-geodesics of the same signature and orientation. Defining the spacelike angle by $(\!(u, \mathrm{D}_{\mathsf{o}} a^t(u))\!)_{space} = -t$ for any unit spacelike vector u and $t \in \mathbb{R}$ to match the relation (3.5), one proves indeed in the same way that a singularity of angle $\theta > 0$ (respectively $-\theta$) is obtained by adding (resp. removing) a spacelike sector of angle θ.

Proof of Lemma 3.14. The first important observation is that both D_θ and D_* contain three quadrants of D at o, and thus that the lightlike foliations of $\bar{D}_{\pm\theta} \setminus \{o_{\pm\theta}\}$ extend to two transverse continuous foliations of $\bar{D}_{\pm\theta}$. Let E be the universal cover of $D \setminus \{\mathsf{o}\}$, $\tilde{a}^\theta$ the lift of a^θ fixing each lift of the connected components of the punctured lightlike leaves of o, and R be the positive generator of the covering automorphism group of E. With $\tilde{\gamma} \subset E$ a lift of γ, $E \setminus \{R^{-1}(\tilde{\gamma}), \tilde{a}^\theta(\tilde{\gamma})\}$ has three connected components among which a unique one contains neither $\tilde{\gamma}$ nor $\tilde{a}^\theta \circ R^{-1}(\tilde{\gamma})$, whose closure is denoted by $\tilde{D}_\theta$. We also denote by $\tilde{D}_{-\theta} \subset E$ the lift of $D_{-\theta}$ with boundary $\tilde{\gamma} \cup \tilde{a}^\theta(\gamma)$. It is then easily checked that $\tilde{D}_\theta$ is a fundamental domain for the action of $\langle \tilde{a}^\theta \circ R \rangle$ on E, and the universal covering map induces a natural identification between $E / \langle \tilde{a}^\theta \circ R \rangle$ and $\bar{D}_\theta$. According to Lemma 3.10, o_θ is thus a standard singularity of angle θ of $\bar{D}_\theta \equiv E / \langle \tilde{a}^\theta \circ R \rangle$. In the same way, $R^{-1}(\tilde{D}_{-\theta}) \cup \tilde{D}_\theta \cup \tilde{D}_{-\theta}$ is a fundamental domain for the action of $\langle \tilde{a}^{-\theta} \circ R \rangle$ on E and $E / \langle \tilde{a}^{-\theta} \circ R \rangle$ identifies with $\bar{D}_{-\theta}$, which has thus $o_{-\theta}$ for standard singularity of angle $-\theta$ according to Lemma 3.10. ☐

Using Lemma 3.14, we can now compute the total angle at a singularity of angle $\theta \in \mathbb{R}$. Let e_1 and e_2 be two disjoint timelike half-geodesics of $\mathbf{X}_\theta$ emanating from o_θ. Then since e_i is disjoint from $\mathcal{F}^{+*}_\alpha(\mathsf{o})$, we can identify it through the projection π_θ defined in (3.2) with its representant in $\mathbf{X} \setminus \mathcal{F}^{+*}_\alpha(\mathsf{o}) \equiv \mathbf{X}_\theta \setminus \mathcal{F}^{+*}_\alpha(\mathsf{o}_\theta)$. Denoting by $u_i \in \mathrm{T}_{\mathsf{o}}\mathbf{X}$ the unit timelike vector tangent to e_i at o, we call then

$$(\!(e_1, e_2)\!) := (\!(u_1, u_2)\!) \tag{3.6}$$

the *angle at o_θ from e_1 to e_2*.

Corollary 3.15. *Let $(e_i)_{1 \leq i \leq d+1}$ be a finite number of disjoint timelike half-geodesics of $\mathbf{X}_\theta$ emanating from o_θ, and negatively cyclically ordered with respect to the orientation of $\mathbf{X}_\theta$. Then with $e_{d+2} = e_1$, the total angle at o_θ is equal to θ:*

$$\sum_{i=1}^{d+1} (\!(e_i, e_{i+1})\!) = \theta.$$

Proof. We first assume that $\theta > 0$. Without loss of generality, we can assume that $d \geq 1$ and that at least one of the e_i is in the future timelike quadrant. We denote by e_1 the first of the e_i in the future timelike quadrant when following the negative cyclic order, and by e_n the last one. Let us use Lemma 3.14 to work in the model $\bar{D}_\theta$ of $\mathbf{X}_\theta$, with e_n as cutting geodesic. Then for any $i \neq n$, we denote by $\gamma_i \subset D_\theta \subset \mathbf{X}$ the half-geodesic corresponding to e_i, and by γ_n the lower copy of e_n which is glued to $a^\theta(\gamma_n)$. Using the relations (3.4) satisfied by the Lorentzian angle, we obtain then

$$\sum_{i=1}^{d+1} (\!(e_i, e_{i+1})\!) = (\!(\gamma_1, \gamma_n)\!) + \left(\!\left(a^\theta(\gamma_n), \gamma_{n+1}\right)\!\right) + (\!(\gamma_{n+1}, \gamma_{d+1})\!) + (\!(\gamma_{d+1}, \gamma_1)\!)\,.$$

Indeed $(\!(e_{n-1}, e_n)\!) = (\!(\gamma_{n-1}, \gamma_n)\!)$ while $(\!(e_n, e_{n+1})\!)) = \left(\!\left(a^\theta(\gamma_n), \gamma_{n+1}\right)\!\right)$. Using again the additivity of the angle, we have thus $\sum_{i=1}^{d+1} (\!(e_i, e_{i+1})\!) = \left(\!\left(a^\theta(\gamma_n), \gamma_n\right)\!\right) = \theta$ according to (3.5).

If $\theta < 0$ then we work in the model $\bar{D}_\theta$ of $\mathbf{X}_\theta$, with the upper future geodesic e_n as cutting geodesic along which the future timelike sector D_θ of angle $-\theta > 0$ and boundary $\gamma_n \cup a^{-\theta}(\gamma_n)$ is glued. This time $(\!(e_{n-1}, e_n)\!) = (\!(\gamma_{n-1}, \gamma_n)\!)$ and $(\!(e_n, e_{n+1})\!)) = \left(\!\left(\gamma_n, a^{-\theta}(\gamma_n)\right)\!\right) + (\!(\gamma_n, \gamma_{n+1})\!) = \theta + (\!(\gamma_n, \gamma_{n+1})\!)$, and the same computation than previously using the additivity of the angle gives thus $\sum_{i=1}^{d+1} (\!(e_i, e_{i+1})\!) = \theta$, which concludes the proof of the corollary. $\square$

Corollary 3.15 gives in particular a new intrinsic characterization of the angle of a standard singularity (and especially of its sign).

3.2. **Singular X-surfaces.** We use in this subsection the local model of singularities described in Paragraph 3.1, to define singular **X**-surfaces and to prove some of their fundamental properties.

Definition 3.16. A *singular* **X**-*structure* (Σ, μ) on an oriented topological surface S is the data:

1. of a set $\Sigma \subset S$ of *singular points* in S;
2. and of a **X**-*structure* μ on $S^* := S \setminus \Sigma$ for which any $x \in \Sigma$ is a *standard singularity*, *i.e.* for which there exists $\theta_x \in \mathbb{R}$ (the *angle* at x) and a homeomorphism φ from an open neighbourhood $U \subset S$ of x to an open neighbourhood V of o_{θ_x} in $\mathbf{X}_{\theta_x}$, such that:
   (a) $U \cap \Sigma = \{x\}$,
   (b) $\varphi(x) = \mathsf{o}_{\theta_x}$,
   (c) and φ is a **X**-morphism in restriction to $U \setminus \{x\}$.
   Such a map φ is called a *singular* **X**-*chart at* x.

A *singular* **X**-*surface* (S, Σ) is an oriented topological surface S endowed with a singular **X**-structure of singular set Σ. $S^* = S \setminus \Sigma$ is always endowed with the $\mathcal{C}^\infty$ structure defined by its **X**-structure, and S with a $\mathcal{C}^\infty$ structure extending the one of S^* (see for instance [Hat]). The points of S which are not singular are called *regular*, and S itself is said *regular* if it does not have any singular point (*i.e.* if it is a **X**-surface). If we want to specify them, we denote by Θ the (ordered) set of angles of the (ordered) singularities Σ.

A *singular* **X**-*atlas* (φ_i, U_i) on S is an atlas of $\mathcal{C}^0$-charts $\varphi_i\colon U_i \to V_i$ from connected open subsets U_i of S to either $\mathbf{X}$ (*regular charts*) or some $\mathbf{X}_{\theta_i}$ (*singular charts*), such that:

1. any two distinct singular chart domains are disjoint;
2. regular charts cover $S \setminus \Sigma$, with $\Sigma = \{\varphi^{-1}(\mathsf{o}_{\theta_i}) \mid \varphi \text{ singular chart to } \mathbf{X}_{\theta_i}\}$ the set of *singularities* of the atlas;
3. and the transition map between any two charts is a **X**-morphism (which makes sense since $U_i \cap U_j \cap \Sigma = \varnothing$ for any two distinct chart domains U_i, U_j).

An *isometry* between two singular **X**-surfaces $(S_i, \Sigma_i, \mu_i)_{i=1,2}$ is a homeomorphism $f\colon S_1 \to S_2$ such that:

1. $f(\Sigma_1) = \Sigma_2$;
2. and f is a **X**-morphism in restriction to $S_1 \setminus \Sigma_1$.

The *area* of a singular **X**-surface (S, Σ, μ) is the area of $S \setminus \Sigma$ for μ.

Remark 3.17. Let us say that a time-oriented Lorentzian metric μ of constant sectional curvature $\varepsilon_{\mathbf{X}}$ defined on the complement of a discrete subset Σ of an orientable surface S is *singular*, if it is induced by a singular **X**-structure. Then according to Proposition 2.6, time-oriented singular Lorentzian metrics of constant sectional curvature $\varepsilon_{\mathbf{X}}$ are equivalent to singular **X**-structures.

3.2.1. *First properties of singular* $\mathbf{X}$*-surfaces.* We prove now some elementary but fundamental properties of singular $\mathbf{X}$-surfaces.

Lemma 3.18. *Let (S, Σ) be a singular $\mathbf{X}$-surface.*

(1) Σ is discrete, hence finite if S is closed.
(2) For any singularity $x \in \Sigma$ of angle θ_x, $\rho\colon \pi_1(S \setminus \Sigma) \to \mathbf{G}$ a holonomy representation of S^* compatible at x (see Definition 3.6), and $[\gamma] \in \pi_1(S \setminus \Sigma)$ the homotopy class of a positively oriented loop around x homotopic to x in S: $\rho([\gamma]) = a^{\theta_x}$. In particular, $\rho([\gamma])$ is conjugated to a^{θ_x}.
(3) If S is closed, then the area of (S, Σ) is finite.

Proof. (1) Any singular $\mathbf{X}$-chart contains indeed a unique singularity.
(2) Since x is a standard singularity of angle θ_x, this is a direct consequence of Lemma 3.5.
(3) For any compact measurable subset $K \subset S \setminus \Sigma$, $\mathcal{A}_{\mu_S}(K)$ is finite, and the claim follows thus from the fact that for any compact neighbourhood K of o_θ in $\mathbf{X}_\theta$, the area of $K \setminus \{\mathsf{o}_\theta\}$ equals the one of K and is thus finite. □

We emphasize that the second claim of Lemma 3.18 shows that the singularities and their angles are characterized by μ_S, and are geometrical invariants in the following sense.

Corollary 3.19. *Let $f\colon S_1 \to S_2$ be an isometry between two singular $\mathbf{X}$-surfaces. Then for any singular point x of S_1, $x \in \Sigma_1$ and $f(x) \in \Sigma_2$ have the same angle: $\theta_x = \theta_{f(x)}$.*

Proof. Let $[\gamma] \in \pi_1(S_1 \setminus \Sigma_1)$ be the homotopy class of a positively oriented loop homotopic to x, and $\rho\colon \pi_1(S_1 \setminus \Sigma_1) \to \mathbf{G}$ be a compatible holonomy representation of S_1 at x. Then $[f(\gamma)] \in \pi_1(S_2 \setminus \Sigma_2)$ and the morphism $\rho \circ f_*^{-1}\colon \pi_1(S_2 \setminus \Sigma_2) \to \mathbf{G}$ induced by f has the same properties with respect to $f(x)$, hence $a^{\theta_x} = \rho([\gamma]) = \rho \circ f_*^{-1}([f \circ \gamma]) = a^{\theta_{f(x)}}$, *i.e.* $\theta_x = \theta_{f(x)}$. □

Observe that for any $u \in \mathbb{R}$, a^u preserves the equivalence relation $\sim_\theta$ used to define $\mathbf{X}_\theta$. It induces thus a map on $\mathbf{X}_\theta$ preserving o_θ that we denote by $\bar{a}^u$, characterized by $\bar{a}^u \circ \pi_\theta = \pi_\theta \circ a^u$.

Proposition 3.20. *Let φ be a singular $\mathbf{X}$-chart of $\mathbf{X}_\theta$ at o_θ, or equivalently a homeomorphism between two neighbourhoods of o_θ and fixing o_θ which is an isometry on its complement. Then φ is the restriction of some $\bar{a}^u$.*

Proof. First according to Corollary 3.19, a singular $\mathbf{X}$-chart of $\mathbf{X}_\theta$ at o_θ is indeed a local isometry of $\mathbf{X}_\theta$ fixing o_θ. Denoting $U^* \coloneqq U \setminus \{\mathsf{o}_\theta\}$ we can assume without loss of generality that $\mathcal{F}_\beta(\mathsf{o}_\theta) \cap U^*$ is the union of two down and up connected components $I_- =]x\,; \mathsf{o}_\theta[_\beta$ and $I_+ =]\mathsf{o}_\theta\,; y[_\beta$. The first natural but important observation is that φ preserves both ends of $\mathcal{F}^*_\beta(\mathsf{o}_\theta)$ in the sense that $\varphi(I_-) =]x'\,; \mathsf{o}_\theta[_\beta$ and $\varphi(I_+) =]\mathsf{o}_\theta\,; y'[_\beta$ for some x' and y'. Likewise both ends of $\mathcal{F}^*_\alpha(\mathsf{o}_\theta)$ are preserved, the proof being identical. Indeed $\varphi(I_-)$ and $\varphi(I_+)$ are intervals of β-leaves since $\varphi|_{U^*}$ is a $\mathbf{X}$-morphism, containing furthermore o_θ in their closure since $\varphi(\mathsf{o}_\theta) = \mathsf{o}_\theta$. Hence the only alternative to the above claim is that $\varphi(I_-) =]\mathsf{o}_\theta\,; x'[_\beta$ and $\varphi(I_+) =]y'\,; \mathsf{o}_\theta[_\beta$ for some x' and y'. But since $\varphi(\mathsf{o}_\theta) = \mathsf{o}_\theta$, φ would then reverse the canonical orientation defined on β-leaves by the $\mathbf{X}$-structure of U^* (see Paragraph 2.5), which contradicts the fact that $\varphi|_{U^*}$ is a $\mathbf{X}$-morphism.

With $V = \varphi(U)$, let $\mathcal{U}, \mathcal{V}$ be open neighbourhoods of o in $\mathbf{X}$, so that with $U' \coloneqq \mathcal{U} \setminus \mathcal{F}^+_\alpha(\mathsf{o})$: $U = \pi_\theta(U' \cup \iota_-(U \cap \mathcal{F}^+_\alpha(\mathsf{o})) \cup \iota_+(U \cap \mathcal{F}^+_\alpha(\mathsf{o})))$, and likewise for V and $V' =\coloneqq \mathcal{V} \setminus \mathcal{F}^+_\alpha(\mathsf{o})$. Then the restriction of π_θ to U' and V' is a $\mathbf{X}$-morphism, and $\pi_\theta|_{V'}^{-1} \circ \varphi \circ \pi_\theta|_{U'}$ is thus the restriction of an element $g \in \mathbf{G}$. But our previous claim shows that g is simultaneously in the stabilizer of $\mathcal{F}_\alpha(\mathsf{o})$ and $\mathcal{F}_\beta(\mathsf{o})$ whose intersection is $\mathrm{Stab}(\mathsf{o}) = A$. In other words there exists $u \in \mathbb{R}$ so that $\varphi = \bar{a}^u$ on U^* and thus on U, which concludes the proof. □

In particular, the maps $\bar{a}^u$ preserve each of the timelike, spacelike or causal quadrants, which gives a meaning to such quadrants in the domain of any chart of the singular $\mathbf{X}$-atlas, even at a singularity. For any $\mathbf{X}$-surface (S, Σ), the union of a $\mathbf{X}$-atlas of $S \setminus \Sigma$ with a (small enough) singular $\mathbf{X}$-chart at each singularity defines a singular $\mathbf{X}$-atlas of S. Conversely, any singular $\mathbf{X}$-atlas of S defines of course on S a singular $\mathbf{X}$-structure with the same singularities. The following result follows directly from Proposition 3.20.

Corollary 3.21. *Let S be an oriented topological surface. Then the transition maps between any two singular $\mathbf{X}$-atlases defining the same singular $\mathbf{X}$-structure on S are:*

– *either restrictions of some a^u between two singular charts at the same singularity,*
– *or $\mathbf{X}$-morphisms outside of singularities.*

Two singular $\mathbf{X}$-atlases whose transition maps are of this form are said equivalent*, and singular $\mathbf{X}$-structures are in correspondence with equivalence classes of singular $\mathbf{X}$-atlases.*

3.2.2. *First-return maps, suspensions and regularity of the lightlike foliations.* If T is a homeomorphism of the circle $\mathbf{S}^1$, the vertical foliation of $\mathbf{S}^1 \times [0\,;1]$ of leaves $\{p\} \times [0\,;1]$ induces on the quotient $M_T \coloneqq \mathbf{S}^1 \times [0\,;1]/\{(1,p) \sim (0,T(p))\}$, homeomorphic to a torus, a foliation $\mathcal{F}_T$ called the *suspension of T*. We are interested in this text with lightlike foliations of singular $\mathbf{X}$-structures which are suspensions of circle homeomorphisms, and it happens that the dynamics of a circle homeomorphism T, hence of its suspension, is highly dependent of the regularity of T. Indeed, circle homeomorphisms can in general have pathological behaviours by admitting *exceptional minimal sets* (see [HH86, Chapter I §5]), but the seminal work of Herman [Her79] showed that regular enough circle homeomorphisms behave nicely. In this paragraph we give the main technical properties of the lightlike foliations of a singular $\mathbf{X}$-surface, and show in particular that if they are suspensions of a circle homeomorphism T, then T is a $\mathcal{C}^2$ diffeomorphism with breaks.

Definition 3.22. A homeomorphism $f\colon I = [a\,;b] \to J$ between two intervals of $\mathbb{R}$ is an *orientation-preserving $\mathcal{C}^k$-diffeomorphism with breaks* ($1 \leq k \leq \infty$) if there exists a finite number of points $a = x_0 < \cdots < x_N = b$ in I such that for any $1 \leq i \leq N$:

(1) $f|_{]x_{i-1};x_i[}$ is an orientation-preserving $\mathcal{C}^k$-diffeomorphism onto its image,
(2) for any $1 \leq l \leq k$, the ℓ^{th} derivative of f has finite limites from above at x_{i-1} and from below at x_i,
(3) $f'_+(x_i) \coloneqq \lim\limits_{t \to x_i^+} f'(t)$ and $f'_-(x_i) \coloneqq \lim\limits_{t \to x_i^-} f'(t)$ are > 0.

If $f'_+(x_i) \neq f'_-(x_i)$, then x_i is a *break point* of f, and $f'_+(x_i)/f'_-(x_i)$ is the *size of the break*. A homeomorphism of $\mathbf{S}^1$ is a *$\mathcal{C}^k$-diffeomorphism with breaks* if it is a $\mathcal{C}^k$-diffeomorphism with breaks in restriction to any interval of $\mathbf{S}^1$.

The following naive observation is going to be useful to us soon.

Lemma 3.23. *Let two consecutive intervals $[a\,;b]$ and $[b\,;c]$ of $\mathbb{R}$ be endowed with $\mathcal{C}^\infty$-structures $\mathcal{C}^0$-compatible with the topology of $\mathbb{R}$, and $\varphi\colon [a\,;c] \to I \subset \mathbb{R}$ be a homeomorphism. Then for any $1 \leq k \leq \infty$, the following are equivalent.*

(1) φ restricts on $[a\,;b]$ and $[b\,;c]$ to $\mathcal{C}^k$-diffeomorphisms with breaks, and $\lim\limits_{t \to b^\pm} \varphi'(t) > 0$.
(2) In a $\mathcal{C}^\infty$-structure of $[a\,;c]$ which is $\mathcal{C}^\infty$-compatible with the structures of both of its subintervals, φ is a $\mathcal{C}^k$-diffeomorphism with breaks.

Let $\mathcal{F}$ be an oriented topological one-dimensional foliation on a surface S, I and J be two *transversals* of $\mathcal{F}$, *i.e.* one-dimensional topological submanifolds transverse to $\mathcal{F}$ in a foliation chart, and $x \in I$ be such that $\mathcal{F}(x) \cap J \neq \varnothing$. Then by transversality, $\mathcal{F}(x)$ has a *first intersection point* (with respect to the orientation of $\mathcal{F}$) denoted by $H(x)$ with J, and there exists an open neighbourhood I' of x in I such that $H(y) \in J$ is well-defined for any $y \in I'$. The map $H\colon I' \to J$ obtained in this way is a homeomorphism onto its image (which is an open neighbourhood of $H(x)$), and is called the *holonomy of $\mathcal{F}$ from I to J*. We refer to [CLN85, §IV.1] for more details on the notion of holonomy of foliations. A *section* of $\mathcal{F}$ is a simple closed curve γ in S transverse to $\mathcal{F}$ and intersecting all of its leaves. In this case, if the holonomy of $\mathcal{F}$ from γ to itself is well-defined, it is called the *first-return map* of $\mathcal{F}$ on γ and be denoted by $P^\gamma_{\mathcal{F}}$ (in reference to Poincaré). We recall that a homeomorphism (respectively a foliation) of a manifold M is said *minimal* if all of its orbits (resp. leaves) are dense in M.

Lemma 3.24. *Let (S,Σ) be a singular $\mathbf{X}$-surface.*

(1) The lightlike foliations of $S \setminus \Sigma$ extend uniquely to two one-dimensional continuous foliations on S, still denoted by $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$.

(2) *There exists at any point of S a simultaneous $\mathcal{C}^0$ foliation chart for $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ (in the sense of Proposition 3.4).*

Let $\mathcal{F}$ be one of the lightlike foliations of S.

(3) *Let $T_1, T_2 \subset S$ be two small $\mathcal{C}^\infty$ transversals of $\mathcal{F}$ such that $T_1 \cap \Sigma = \{x\}$ and $T_2 \subset S \setminus \Sigma$ intersects $\mathcal{F}(x)$, and $H \colon T_1 \to T_2$ be the holonomy of $\mathcal{F}$ from T_1 to T_2. Then H is a $\mathcal{C}^\infty$-diffeomorphism with breaks.*

(4) *If S is homeomorphic to $\mathbf{T}^2$ and $\mathcal{F}$ is $\mathcal{C}^0$-conjugated to the suspension of an orientation-preserving homeomorphism H of $\mathbf{S}^1$, then H is $\mathcal{C}^0$-conjugated to a $\mathcal{C}^\infty$-diffeomorphism with breaks of $\mathbf{S}^1$, and has no exceptional minimal set. If H has moreover an irrational rotation number $\rho \in \mathbf{S}^1$, then H is $\mathcal{C}^0$-conjugated to the rotation $R_\rho \colon x \in \mathbf{S}^1 \mapsto x + \rho \in \mathbf{S}^1$ and is thus minimal. In particular $\mathcal{F}$ is then $\mathcal{C}^0$-equivalent to the corresponding linear foliation of $\mathbf{T}^2$ and is thus minimal.*

The notion of rotation number is introduced in Proposition-Definition 5.1.

Proof of Lemma 3.24. (1) follows directly from Proposition 3.4, using singular $\mathbf{X}$-charts at the singularities.
(2) follows from Proposition 3.4 at the singularities and from the $\mathbf{X}$-charts at regular points. Indeed the affine charts (2.3) are simultaneous foliated charts of the lightlike foliations of $\mathbf{X}$.
(3) Without loss of generality, we can assume that $S = \mathbf{X}_\theta$, $x = \mathsf{o}_\theta$, $\mathcal{F} = \mathcal{F}_\alpha$, and that $T_1 = \mathcal{F}_\beta(\mathsf{o}_\theta)$ and $T_2 = \mathcal{F}_\beta(p)$ with $p \in \mathcal{F}^+_\alpha(\mathsf{o}_\theta)$. These reductions being done, and since the $\mathcal{C}^\infty$-structure of S is by definition compatible with the $\mathbf{X}$-structure of $S \setminus \Sigma$, Lemma 3.23 shows that it is sufficient to check that the restriction of H to the closure of each component of $\mathcal{F}_\beta(\mathsf{o}_\theta) \setminus \{\mathsf{o}_\theta\}$ is a $\mathcal{C}^\infty$-diffeomorphism with breaks, with a positive limit of the derivative at o_θ from below and above. We do it for $\mathcal{F}^+_\beta(\mathsf{o}_\theta)$, the case of the other component being analogous. According to Proposition 3.4, for I and J small open neighbourhoods of o_θ in $\mathcal{F}_\alpha(\mathsf{o}_\theta)$ and $\mathcal{F}_\beta(\mathsf{o}_\theta)$, the map $(x, y) \in I \times J \mapsto \mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$ defines outside of o_θ a smooth diffeomorphism onto a punctured open neighbourhood of o_θ in $\mathbf{X}_\theta$. Since the holonomy H reads in this chart as the identity of the vertical factor J, it extends on the closure I^+ of the upper component to a $\mathcal{C}^\infty$-diffeomorphism whose derivative has a positive limit at o_θ, which allows to conclude thanks to Lemma 3.23.
(4) Since Σ is finite and $\mathcal{F}$ is by assumption a suspension, there exists a $\mathcal{C}^\infty$ section $T \subset S \setminus \Sigma$ of $\mathcal{F}$. The first-return map $H \colon T \to T$ of $\mathcal{F}$ on T is then well-defined, and is according to (3) a $\mathcal{C}^2$-diffeomorphisms with breaks as a composition of such homeomorphisms. The two last claims follow then from Denjoy Theorem [Den32] (see also [Her79, Théorème VI.5.5 p.76]): if an orientation-preserving homeomorphism T of $\mathbf{S}^1$ is a $\mathcal{C}^2$-diffeomorphism with breaks, then it has no exceptional minimal set. If T has moreover irrational rotation number ρ, then it is $\mathcal{C}^0$-conjugated to the rotation R_ρ.[6] □

Corollary 3.25. *Any closed connected orientable surface which bears a singular $\mathbf{X}$-structure, is homeomorphic to a torus.*

Proof. According to [HH86, Theorem 2.4.6], any closed connected orientable surface bearing a topological foliation is indeed homeomorphic to a torus. □

This corollary shows the necessity of introducing *branched covers* of the standard singularities to obtain singular $\mathbf{X}$-structures on higher-genus surfaces.

3.2.3. *Gauß-Bonnet formula.* The standard Riemannian Gauß-Bonnet formula has a natural counterpart for singular constant curvature Lorentzian surfaces, which imposes a relation between the singularities and the area of a singular $\mathbf{X}$-torus. We recall that $\varepsilon_{\mathbf{X}}$ denotes the constant sectional curvature of $\mathbf{X}$: $\varepsilon_{\mathbb{R}^{1,1}} = 0$ and $\varepsilon_{\mathbf{dS}^2} = 1$. For future use, we prove the Gauß-Bonnet formula for singular $\mathbf{X}$-surfaces with geodesic boundary.

Definition 3.26. A *singular $\mathbf{X}$-structure with geodesic boundary* (Σ, μ) on an oriented topological surface S with boundary is the data of a set Σ of singular points in the interior of S, and of a

[6] Note that this theorem of Denjoy holds more generally for the so-called *class P homeomorphisms*, of which $\mathcal{C}^2$-diffeomorphisms with breaks are specific examples.

time-oriented singular Lorentzian metric μ of constant curvature $\varepsilon_{\mathbf{X}}$ on S of singular set Σ, and for which the boundary of S is geodesic.

Proposition 3.27 (Gauß-Bonnet formula)**.** *Let S be a compact and connected orientable surface endowed with a singular $\mathbf{X}$-structure with timelike geodesic boundary of area $\mathcal{A}(S) \in \mathbb{R}_+^*$, having $n \in \mathbb{N}^*$ singularities of angles $(\theta_1, \dots, \theta_n) \in \mathbb{R}^n$. Then:*

$$\varepsilon_{\mathbf{X}}.\mathcal{A}(S) = \sum_{i=1}^{n} \theta_i. \tag{3.7}$$

In particular, we have the following consequences.

(1) A compact singular $\mathbb{R}^{1,1}$-surface S with timelike geodesic boundary cannot have a single singularity: if it is not regular, then it has at least two singularities (which have opposite signs if there are exactly two singularities).
(2) The area of a compact singular $\mathbf{dS}^2$-surface with timelike geodesic boundary is entirely determined by the angles at its singularities.
(3) If a compact singular $\mathbf{dS}^2$-surface with timelike geodesic boundary has a single singularity x, then x has a positive angle equal to the area $\mathcal{A}(S) \in \mathbb{R}_+^*$ of S.

Proof. Let us denote by Σ the singular set of S, and by $S^* = S \setminus \Sigma$ the $\mathbf{X}$-surface associated to S. A general topological fact ensures that S admits a finite triangulation *subordinate* to any given covering, *i.e.* each of which triangle is contained in an open set of the chosen covering. Let us choose a singular $\mathbf{X}$-atlas of S, each of which chart domain is a normal convex neighbourhood of any of its points. Around a singular point of S, we use a natural generalization in the singular setting of the usual notion of normal convex neighbourhood, introduced in Proposition A.12 below. This allows us to consider a finite triangulation of S whose vertices contain all the singularities of S, and whose edges interiors are geodesic. A slight deformation of such a triangulation ensures that all of its edges are transverse to any given smooth foliation $\mathcal{F}$ of S. Since the edges are compact and in finite number, their tangent lines can even be assumed to avoid any small enough cone around the line bundle tangent to $\mathcal{F}$. But by taking the image of the singular $\mathbf{X}$-structure μ of S by a suitable diffeomorphism f, we can assume the spacelike cone $\mathcal{C}^{space}$ to be as narrow as we want, namely arbitrarily close to a foliation $\mathcal{F}$ tangent to the interior of $\mathcal{C}^{space}$ (this is achieved by pushing μ by a large iterate of an Anosov diffeomorphism of S having $\mathcal{F}$ as unstable foliation, and whose stable line bundle avoids $\mathcal{C}^{space}$). There exists then a triangulation of S whose edges interiors are timelike geodesics of the singular $\mathbf{X}$-structure $f_*\mu$. By taking the preimage of the latter triangulation by f, we obtain a finite triangulation $\mathcal{T}$ of S whose vertices $\mathcal{V}$ contain all the singularities of μ, and whose edges are timelike geodesics of μ.

Formula (3.7) follows from a Lorentzian counterpart of the Gauß-Bonnet formula, proved in [BN84, Theorem p.80] for compact subsets of regular Lorentzian surfaces having piecewise smooth timelike boundaries, and which takes into account the angles at the breaking points (see also [Ave63, Che63] for analogous formulae in any signatures and dimensions and with intrisic proofs, but in the boundaryless setting). The first step is to write the Gauß-Bonnet formula for triangles. The three edges of any triangle $T \in \mathcal{F}$ of the triangulation $\mathcal{T}$ are oriented to match the orientation of ∂T induced by the one of S, and are denoted by (T_1, T_2, T_3) in the positive cyclic order in which they are met when following the orientation of ∂T. Denoting by (T^1, T^2, T^3) the vertices of T with T_i going from T^i to T^{i+1} (and $T^4 = T^1$), let $\alpha(T^i, T) = (\!(T_{i-1}, T_i)\!)$ be the angle at T^i from T_{i-1} to T_i (with $T_0 = T_3$) defined in (3.6). The formula proved in [BN84, Theorem p.80] translates then in our setting as:

$$\varepsilon_{\mathbf{X}}\mathcal{A}(T) = \sum_{i=1}^{3} \alpha(T^i, T). \tag{3.8}$$

In the other hand for any interior vertex $v \in \mathcal{V} \cap \mathrm{Int}(S)$, denoting by $\mathcal{F}_v$ the set of triangles containing v as a vertex, we proved in Corollary 3.15 that the total angle at v satisfies

$$\sum_{T \in \mathcal{F}_v} \alpha(v, T) = \theta_v \tag{3.9}$$

with θ_v the angle of the singularity at v. Since the boundary is timelike geodesic and contains no singularity, the total angle around a vertex $v \in \mathcal{V} \cap \partial S$ on the boundary of S is equal to 0 according the second relation of (3.4). Summing the areas (3.8) of our triangulation's faces, we obtain thus the expected formula

$$\begin{aligned}\varepsilon_{\mathbf{X}}\mathcal{A}(S) &= \sum_{T\in\mathcal{F}} \varepsilon_{\mathbf{X}}\mathcal{A}(T) \\ &= \sum_{T\in\mathcal{F}} \sum_{i=1}^{3} \alpha(T^i, T) \\ &= \sum_{v\in\mathcal{V}} \sum_{T\in\mathcal{F}_v} \alpha(v, T) \\ &= \sum_{v\in\mathcal{V}} \theta_v\end{aligned}$$

by using the relation (3.9) at the last step, which concludes the proof of the proposition. □

4. Constructions of singular $\mathbf{dS}^2$-tori

In this section, we present some constructions of singular $\mathbf{dS}^2$-tori, and investigate two specific families of $\mathbf{dS}^2$-tori with one singularity. The existence results from Theorem B, C and D are proved later in Paragraph 7.3 by using these two families.

We fix for this whole section a positive angle $\theta \in \mathbb{R}_+^*$, and recall that according to the Gauß-Bonnet formula (3.7) in Proposition 3.27, a singular $\mathbf{dS}^2$-torus having a single singularity x has area θ, if and only if x has angle θ. We also identify in the whole section $\mathbb{R}\mathbf{P}^1$ with $\mathbb{R} \cup \{\infty\}$ and elements of $\mathrm{PSL}_2(\mathbb{R})$ with their associated homography of $\mathbb{R} \cup \{\infty\}$, as defined in (2.2) and (2.4).

4.1. Gluings of polygons in $\mathbf{dS}^2$.

Let us denote by $y_\theta := 1 - e^{-\frac{\theta}{2}} \in]0\,;1[$ the unique number such that $\mathcal{A}_{\boldsymbol{\mu}}(\mathcal{R}_{(1,\infty,0,y_\theta)}) = \theta$. According to Lemma 2.9, $\mathcal{R}_\theta := \mathcal{R}_{(1,\infty,0,y_\theta)}$ is, up to the action of $\mathrm{PSL}_2(\mathbb{R})$, the unique rectangle with lightlike edges and area θ in $\mathbf{dS}^2$. Our goal is to define a quotient of $\mathcal{R}_\theta$ with a single singularity, which *a posteriori* necessarily has angle θ by Gauß-Bonnet formula (3.7). A first easy way to do this is to consider the unique elements $g = g_\theta$ and h_θ of $\mathrm{PSL}_2(\mathbb{R})$ such that $g(1, 0, y_\theta) = (\infty, 0, y_\theta)$ and $h_\theta(1, \infty, 0) = (1, \infty, y_\theta)$ in the sense that:

$$g(1) = \infty, g(0) = 0, g(y_\theta) = y_\theta \text{ and } h_\theta(1) = 1, h_\theta(\infty) = \infty, h_\theta(0) = y_\theta, \tag{4.1}$$

and to form the quotient of $\mathcal{R}_\theta$ by gluing its edges through g and h_θ (see Figure 4.1). The gluing being made by isometries, the $\mathbf{dS}^2$-torus obtained in this way has, as sought, a unique singularity at the class of the vertices. However by such a construction, both lightlike leaves of the singularity are always closed. To obtain a structure with a minimal lightlike foliation, it is thus necessary to consider another type of gluing.

4.1.1. *Suspension of homographic interval exchange transformations.* Inspired from the constructions of translation surfaces as "suspensions" of (classical) interval exchange transformations, a natural idea to obtain minimal lightlike foliations is to keep gluing the β-edges of $\mathcal{R}_\theta$ through g, but to glue its two α-edges through a *homographic interval exchange transformation* (*HIET*) with two components of the closed α-leaf. Such a map is a bijection of an interval I of $\mathbb{R}\mathbf{P}^1$ exchanging the components of two partitions of I called *top* and *bottom* partitions, and which is homographic on each component of the top partition (*i.e.* equals the restriction of an element of $\mathrm{PSL}_2(\mathbb{R})$). The notion of HIET is both a natural generalization of the ones of (classical) IET and affine IET, and a restriction of the notion of *generalized interval exchange transformation* (*GIET*). We refer the reader to the excellent [Yoca, Yocb] for more informations on theses notions (which are however not needed in this text).

For any $x, x' \in]1\,;\infty[$, we introduce the following subintervals of $I = [1\,;\infty[$:

$$I_1^t = [1\,;x'[, I_2^t = [x'\,;\infty[, I_1^b = [1\,;x[, I_2^b = [x\,;\infty[, \tag{4.2}$$

delimiting a *top* partition $I = I_1^t \sqcup I_2^t$ and a *bottom* partition $I = I_1^b \sqcup I_2^b$ of I. By three-transitivity of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{R}\mathbf{P}^1$, there exists a unique pair h_1, h_2 of elements of $\mathrm{PSL}_2(\mathbb{R})$ such that $h_1(0) = h_2(0) = y_\theta$, $h_1(I_1^t) = I_2^b$ and $h_2(I_2^t) = I_1^b$, and we define a HIET $E\colon I \to I$ by:

$$E|_{I_1^t} = h_1|_{I_1^t}, E|_{I_2^t} = h_2|_{I_2^t}. \tag{4.3}$$

The condition $h_1(0) = h_2(0) = y_\theta$ ensures that E glues the α-edges of $\mathcal{R}_\theta$ to one another. We now "suspend" E, obtaining the quotient $\mathcal{T}_{\theta,E}$ of the rectangle $\mathcal{R}_\theta$ by the following edges identifications:

$$\begin{cases} [1\,;\infty[\,\times\{0\} \ni (p,0) \sim (E(p), y_\theta) \in [1\,;\infty[\,\times\{y_\theta\}, \\ \{1\}\times[0\,;y_\theta] \ni (1,p) \sim (\infty, g(p)) \in \{\infty\}\times[0\,;y_\theta]. \end{cases}$$

These gluings, illustrated in Figure 4.1, give us a first family of examples of singular $\mathbf{dS}^2$-tori. Vertices of $\mathcal{R}_\theta$ of the same color indicate points identified in the quotient $\mathcal{T}_{\theta,E}$. To prevent any confusion, we emphasize that the denominations of top and bottom partitions are the usual ones in the literature of GIET's which is the reason why we used them, but that they do *not* correspond to their positions in the Figure 4.1: the top partition corresponds to the lower interval and the bottom one to the upper interval.

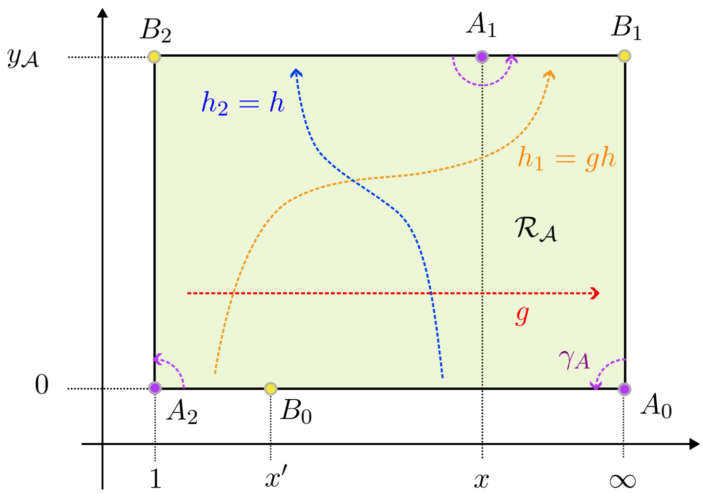

FIGURE 4.1. $\mathbf{dS}^2$-torus with one singularity and a closed α-lightlike leaf.

Proposition 4.1. *For any $\theta \in \mathbb{R}_+^*$ and $x, x' \in\,]1\,;\infty[$, $\mathcal{T}_{\theta,E}$ is homeomorphic to $\mathbf{T}^2$ and the $\mathbf{dS}^2$-structure of the interior of $\mathcal{R}_\theta$ extends to a unique singular $\mathbf{dS}^2$-structure on $\mathcal{T}_{\theta,E}$. The latter has area θ, and the α-leaf of $\overline{(\infty,0)}$ is closed. The unique (potentially) singular points of $\mathcal{T}_{\theta,E}$ are $\overline{(\infty,0)}$ and $\overline{(x',0)}$, and the holonomies of small positively oriented loops around them are:*

1. *holonomy around $(\infty,0) = h_2^{-1}h_1g^{-1}$,*
2. *holonomy around $(x',0) = h_1^{-1}gh_2$.*

Proof. Let us denote by $\pi\colon \mathcal{R}_\theta \to \mathcal{T}_{\theta,E}$ the canonical projection, and $\overline{(a,b)} = \pi(a,b)$ for $(a,b) \in \mathcal{R}_\theta$. We first observe that the gluing of the edges are well-defined for the quotient to be topologically a torus, as a Euler characteristic computation directly shows. The edges being moreover identified by elements of $\mathrm{PSL}_2(\mathbb{R})$, the $\mathbf{dS}^2$-structure of $\pi(\mathrm{Int}(\mathcal{R}_\theta))$ for which $\pi|_{\mathrm{Int}(\mathcal{R}_\theta)}$ is a $\mathbf{dS}^2$-morphism extends to a $\mathbf{dS}^2$-structure of area θ on the complement of the vertices, *i.e.* on $\mathcal{T}_{\theta,E}\setminus\{\overline{(\infty,0)}, \overline{(x',0)}\}$. Lastly, observe that the lightlike foliations of $\pi(\mathrm{Int}(\mathcal{R}_\theta))$ clearly extend to two transverse continuous foliations of $\mathcal{T}_{\theta,E}$.

The top and bottom partitions (4.2) of $[1\,;\infty[$ define associated partitions of the α and β boundary parts of $\mathcal{R}_\theta$, that we call *edges*, and their extremities are called *vertices*. Let us detail in the specific case of $A = \overline{(\infty,0)} \in \mathcal{T}_{\theta,E}$ a general "recipe" to compute the holonomy around any vertex P of $\mathcal{T}_{\theta,E}$, illustrated in Figure 4.1. First of all, note that each vertex P is associated with a positively cyclically ordered periodic orbit $(P_0, P_1, \ldots, P_d)$, which has length 2 for A. A small positively oriented closed loop γ_P around P defines indeed a cyclic ordering on the (finite) equivalence class of P for $\sim$, describing in which order the points are met in $\mathcal{R}_\theta$ when following γ_P. For instance in the case of A if we start with $A_0 = (\infty, 0)$, then we successively meet $A_1 = (x, y_\theta)$, $A_2 = (1, 0)$ and finally come back to A_0. Moreover at each step P_i, $i \geq 1$ of this periodic orbit, γ_P meets in $\mathcal{T}_{\theta,E}$ an interval of a lightlike half-leaf emenating from P which corresponds both to a top edge $e^t_{P_i}$ and to a bottom edge $e^b_{P_i}$ of $\mathcal{R}_\theta$, having respectively P_{i-1} and P_i as one of their extremities. These are for instance $e^t_{A_1} = [x' \,; \infty] \times \{0\}$ (A_0 as right extremity) and $e^b_{A_1} = [1\,; x] \times \{y_\theta\}$ (A_1 as right extremity) for $P_i = A_1$. These edges are then identified in the quotient by some $f_{P_i} \in \mathrm{PSL}_2(\mathbb{R})$, characterized by $f_{P_i}(e^b_{P_i}) = e^t_{P_i}$ (for instance $f_{A_1} = h_2^{-1}$ in our example $P_i = A_1$). Lastly, each point P_i of the periodic orbit $(P_0, P_1, \dots, P_d)$ contributes for a certain sequence Q_{P_i} of *quadrants* around P, ordered as they are met by γ_P. For instance for A, $Q_{A_0} = \text{timelike future}$, $Q_{A_1} = (\text{past spacelike}, \text{past timelike})$ and $Q_{A_2} = \text{future spacelike}$. We say that *the identification of the quadrants around P is standard*, if the sequence $(Q_{P_0}, \dots, Q_{P_d})$ equals the *standard sequence*: $(\text{timelike future}, \text{past spacelike}, \text{past timelike}, \text{future spacelike})$, up to cyclic permutations.

Fact 4.2. *Let us assume that the identification of the quadrants around a vertex P is standard. Then P is a standard singularity of $\mathcal{T}_{\theta,E}$. Moreover with ρ the holonomy morphism associated to the developing map extending the section $s\colon \pi(\mathrm{Int}(\mathcal{R}_\theta)) \to \mathrm{Int}(\mathcal{R}_\theta)$ of π, we have:*

$$\rho(\gamma_P) = f_{P_1} f_{P_2} \dots f_{P_d} f_{P_0} \in \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{R})}(P_0). \tag{4.4}$$

Proof. For the sake of clarity, we write the proof in the specific case of A, but it is formally identical in any situation. We define $\varphi_0 = s$ as a $\mathbf{dS}^2$-chart on $\pi(U_0)$, with U_0 a small neighbourhood of A_0 in $\mathcal{R}_\theta$. Now let U_1 be a small neighbourhood of A_1 in $\mathcal{R}_\theta$, and φ_1 be a $\mathbf{dS}^2$-chart defined on a neighbourhood of $\overline{\pi(U_1)}$ in $\mathcal{T}_{\theta,E} \setminus \{\overline{(\infty,0)}, \overline{(x',0)}\}$, and agreeing with φ_0 on a neighbourhood of $\overline{(\infty,0)}$ in $\pi(]1\,;\infty] \times \{0\})$. Then $\varphi_1 = g_{A_1} \circ s$ on $\pi(U_1)$ with $g_{A_1} \in \mathrm{PSL}_2(\mathbb{R})$ agreeing with $f_{A_1} = h_2^{-1}$ on a neighbourhood of A_1 in $[1\,; x] \times \{y_\theta\}$. The naive but important observation is now that if $g, g' \in \mathrm{PSL}_2(\mathbb{R})$ have the same action on a non-empty open lightlike interval, then $g = g'$. Indeed, it is sufficient to check this for $g, g' \in \mathrm{Stab}(\mathsf{o})$, for which this claim simply follows from the fact that a non-trivial element of $\mathrm{Stab}(\mathsf{o})$ has a non-trivial action on any non-empty open lightlike interval of extremity o. This shows that $g_{A_1} = f_{A_1}$, *i.e.* that $\varphi_1 = f_{A_1} \circ s$ on $\pi(U_1)$.

Continuing in the same way, we conclude that if U_2 is a neighbourhood of A_2 in $\mathcal{R}_\theta$, and φ_2 a $\mathbf{dS}^2$-chart defined on a neighbourhood of $\overline{\pi(U_2)}$ and agreeing with φ_1 on the suited α-interval, then $\varphi_2 = f_{A_1} \circ f_{A_2} \circ s$ on $\pi(U_2)$. To understand this relatively counter-intuitive order in the compositions, observe first that $f_{A_2} \circ s|_{\pi(U_2)}$ and $s|_{\pi(U_1)}$ glue together to define a $\mathbf{dS}^2$-chart on a punctured neighbourhood of $\overline{(1,0)}$ in $\pi([1\,; x'] \times \{0\})$, hence that $f_{A_1} \circ f_{A_2} \circ s$ and $f_{A_1} \circ s = \varphi_1$ agree on the intersection of their domains.

In the end $\varphi_3 = f_{A_1} \circ f_{A_2} \circ f_{A_0} \circ \varphi_0$, and the maps φ_i for $i = 0, \dots, 3$ agree on the intersection of their domains. They glue thus together to give a $\mathbf{dS}^2$-isomorphism ψ from a slit neighbourhood $U' = U \setminus \mathcal{F}_\alpha(\overline{(\infty,0)})$ of $\overline{(\infty,0)}$ to a slit neighbourhood of $(\infty,0) = \mathsf{o}$ in $\mathbf{dS}^2$. This map satisfies the hypotheses of Lemma 3.5.(2), and we conclude thus that $\overline{(\infty,0)} = A$ is a standard singularity of the $\mathbf{dS}^2$-structure of $\mathcal{T}_{\theta,E} \setminus \{\overline{(1,0)}, \overline{(x',0)}\}$, and that $\rho(\gamma_A) = f_{A_1} \circ f_{A_2} \circ f_{A_0} \in \mathrm{Stab}(\mathsf{o})$. □

Fact 4.2 shows our claim for the vertices $\overline{(\infty,0)}$ and $\overline{(x',0)}$, and concludes thus the proof of the proposition. □

4.1.2. *Identification spaces of lightlike polygons are singular* $\mathbf{X}$*-tori.* To clarify our exposition, avoid unnecessary notations and rather emphasize the main ideas, we chose to focus on the constructions of singular $\mathbf{dS}^2$-tori that are developed in the sequel of the text in the case of one singularity. However, the same formal proof than the one of Fact 4.2 offers a general way of constructing singular $\mathbf{X}$-tori, and proves the following result. We refer to the proof of Proposition 4.1 for the definition of a *standard identification of quadrants around a vertex*, and of the related notions appearing in the statement below. We call *lightlike polygon* a compact connected subset of $\mathbf{X}$, homeomorphic to a closed disk and whose boundary is a finite union of lightlike geodesic segments. We also denote by $(\mathbf{G}, \mathbf{X})$ the pair $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ or $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1), \mathbb{R}^{1,1})$.

Proposition 4.3. *Let $\mathcal{P}$ be a lightlike polygon of $\mathbf{X}$, whose boundary is endowed with:*

(1) a decomposition into an even number of edges which are segments of lightlike leaves,
(2) and pairwise identifications between these edges by elements of $\mathbf{G}$.

Assume that the identification of the quadrants around each vertex is standard. Then the quotient of $\mathcal{P}$ by the edges identifications is a torus endowed with a unique singular $\mathbf{X}$-structure compatible with the one of $\mathcal{P}$. This singular $\mathbf{X}$-torus has the same area than $\mathcal{P}$, and the holonomies at the vertices are given by the formula (4.4).

Remark 4.4. As emphasized by an anonymous referee, the condition appearing in Proposition 4.3 of standard identification at each vertex, although having been formulated geometrically, is in fact purely topological. It is indeed satisfied if and only if the identification space is homeomorphic to a torus. This may for instance be observed by affinely embedding the polygon in question in the euclidean plane $\mathbb{R}^2$ as a polygon with horizontal and vertical edges, and by noticing that we can then think of our edges identifications as standard (isometric) IET, since this has no repercussion on our purely combinatorial concern. Our identification space is now a closed translation surface, which is homeomorphic to a torus if and only if the angle is 2π at each vertex. But coming back to our initial Lorentzian setting, the latter condition is seen to be equivalent to standard identification at each vertex.

Remark 4.5. Proposition 4.3 could be stated more generally: the quotient of any connected lightlike polygon of $\mathbf{X}$ whose boundary is endowed with an even partition into edges, by any pairwise identifications of the edges by elements of $\mathbf{G}$, is endowed with a natural $\mathbf{X}$-structure on the complement of the vertices. But these vertices are *not* standard singularities as studied in this text when the identification of quadrants around them is not standard. For instance, non-standard singularities do not see four lightlike half-leaves emanating from them, and in particular the lightlike foliations do not extend to topological foliations at non-standard singularities. This should however not exclude the attention for such examples, particularly interesting ones arising for instance when the lightlike foliations have themselves standard singularities at the singularities of the metric (for instance when they are the stable and unstable foliations of a pseudo-Anosov map). In conclusion, Proposition 4.3 allows the construction of closed Lorentzian surfaces of any genera, with singular points which are not the one studied in the present text, and that will be studied in a future work.

Lastly, Lemma 3.14 shows that standard singularities do not need to be constructed from lightlike geodesics, and that definite geodesics work just as well. A natural analog to Proposition 4.3 can therefore be stated and proved in the same way for any polygon of $\mathbf{X}$ having a geodesic boundary endowed with a partition into an even number of edges and pairwise identifications between them by elements of $\mathbf{G}$.

Remark 4.6. Proposition 4.3 proves in particular the existence of singular $\mathbb{R}^{1,1}$-tori or *singular flat tori*, and offers a way to construct a large family of them. The investigation of singular flat tori will be considered in a future work.

Henceforth, we come back to the homogeneous model space $(\mathbf{G}, \mathbf{X}) = (\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$, and investigate thoroughly two families of $\mathbf{dS}^2$-tori with a single singularity.

4.2. **A one-parameter family of $\mathbf{dS}^2$-tori with one singularity having a closed leaf.** We now apply Proposition 4.1 to obtain a first one-parameter family of $\mathbf{dS}^2$-tori. For any $x \in \,]1\,;\infty]$ and $x' \in [1\,;\infty[$, let $h = h_{(x,x')}$ be the unique element of $\mathrm{PSL}_2(\mathbb{R})$ such that

$$h(x', \infty, 0) = (1, x, y_\theta), \tag{4.5}$$

i.e. $h = h_2$ in the notations of Proposition 4.1. Proposition 4.1 and Corollary 3.8 indicate us that $\overline{(x', 0)} \in \mathcal{T}_{\theta,E}$ is regular if and only if $h_1 = gh_2 = gh$, or equivalently if:

$$gh(1, x', 0) = (x, \infty, y_\theta). \tag{4.6}$$

Since $gh(x', 0) = (\infty, y_\theta)$ is automatically satisfied according to the equations (4.5) and (4.1), the regularity of $\overline{(x', 0)} \in \mathcal{T}_{\theta,E}$ is eventually equivalent to $gh(1) = x$.

Lemma 4.7. *$gh(1) = x$ if and only if $x' = \frac{x}{x-1}$. Moreover, g and h are hyperbolic.*

Proof. The last claim follows from a direct observation of the dynamics of g and h on $\mathbb{R}\mathbf{P}^1$. With $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the definition of g reads: $c+d=0$, $b=0$, $ay_\theta+b = y_\theta(cy_\theta+d)$, *i.e.* $y_\theta(cy_\theta - c - a) = 0$ and thus $a = c(y_\theta - 1)$. Hence $g = (1-y_\theta)^{-1/2}\left(\begin{smallmatrix} -(1-y_\theta) & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and $g(t) = (y_\theta - 1)\frac{t}{t-1}$. Now if $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the definition of h reads: $ax'+b = cx'+d$, $a = cx$, $b = dy_\theta$, hence $d = \frac{cx'(x-1)}{(1-y_\theta)}$ and thus
$$h(t) = \frac{x(1-y_\theta)t + x'(x-1)y_\theta}{(1-y_\theta)t + x'(x-1)}.$$
A direct computation shows that $x - gh(1) = ((1+e^{\frac{\theta}{2}}(-1+x))(x(-1+x')-x'))/(e^{\frac{\theta}{2}}(-1+x)(-1+x'))$. Since $x > 1 > 1-e^{-\frac{\theta}{2}}$, this quantity vanishes if and only if $x(-1+x')-x' = 0$ *i.e.* $x' = x/(x-1)$, which concludes the proof. $\square$

We now fix $x \in [1\,;\infty]$ and denote:

(1) $x' = x'_x \coloneqq \frac{x}{x-1} \in [1\,;\infty]$ (with $x'_\infty = 1$ and $x'_1 = \infty$),

(2) and $h = h_x \coloneqq h_{(x,x'_x)}$ if $x > 1$, extended by $h_1 \coloneqq g^{-1}h_\infty$ for $x = 1$.

The equations (4.5) and (4.6) show that $\lim_{x\to 1} gh_x = h_\infty$, hence that $\lim_{x\to 1} h_x = \lim_{x\to 1} g^{-1}(gh_x) = h_1$, so that the maps
$$x \in [1\,;\infty] \mapsto h_x \in \mathrm{PSL}_2(\mathbb{R}) \text{ and } x \in [1\,;\infty] \mapsto gh_x \in \mathrm{PSL}_2(\mathbb{R})$$
are continuous. Using the top and bottom partitions of $I = [1\,;\infty[$ defined in (4.2), we consider the HIET $E = E_x \colon I \to I$ defined by
$$E_x|_{I_1^t} = gh_x|_{I_1^t} \colon I_1^t \to I_2^b \text{ and } E_x|_{I_2^t} = h_x|_{I_2^t} \colon I_2^t \to I_1^b, \tag{4.7}$$
and denote by $\mathcal{T}_{\theta,x} \coloneqq \mathcal{T}_{\theta,E_x}$ the suspension of E_x defined in Proposition 4.1 and illustrated in Figure 4.1. Note that $E_1 = E_\infty$ is simply the restriction of h_∞ to I, so that $\mathcal{T}_{\theta,1} = \mathcal{T}_{\theta,\infty}$. The following result summarizes the construction, and is a reformulation of Proposition 4.1 in the case $x' = \frac{x}{x-1}$.

Proposition 4.8. *For any $\theta \in \mathbb{R}_+^*$ and $x \in [1\,;\infty]$, $\mathcal{T}_{\theta,x}$ is homeomorphic to $\mathbf{T}^2$ and the $\mathbf{dS}^2$-structure of the interior of $\mathcal{R}_\theta$ extends to a unique singular $\mathbf{dS}^2$-structure on $\mathcal{T}_{\theta,x}$. The latter has area θ, and its unique singular point $\overline{(1,0)} = \overline{(\infty,0)}$ has a closed α-leaf and angle θ.*

Remark 4.9. Of course, one can realize the symmetric construction to obtain a quotient of $\mathcal{R}_\theta$ with this time the β-leaf of $\overline{(\infty,0)}$ being closed. This is done by gluing the α-edges of $\mathcal{R}_\theta$ by the restriction of h_θ defined in (4.1), and its β-edges by a HIET with two components of $J = \{1\} \times [0\,;y_\theta]$ with top and bottom partitions
$$J_1^t = [0\,;y'[,\, J_2^t = [y'\,;y_\theta[,\, J_1^b = [0\,;y[,\, J_2^b = [y\,;y_\theta[.$$
These $\mathbf{dS}^2$-tori of area θ, with one singularity at $\overline{(\infty,0)}$ whose β-leaf is closed, are denoted by $\mathcal{T}_{\theta,*,y}$.

4.3. **A two-parameter family of $\mathbf{dS}^2$-tori with one singularity.** Our goal being to eventually construct singular $\mathbf{dS}^2$-tori with one singularity both of whose lightlike foliations are minimal, we should first make sure that both leaves of the singularity are non-closed. To this end we fix $0 < y \leq y_\theta$ and $1 < x \leq \infty$, and we apply Proposition 4.3 to the "L-shaped polygon"
$$\mathcal{L}_{\theta,x,y} \coloneqq \mathcal{R}_{(1,\infty,0,y_+)} \setminus\,]x\,;\infty] \times]y\,;y_+] \subset \mathbf{dS}^2 \tag{4.8}$$
of area θ illustrated in Figure 4.2. The point
$$y_+ = y_{+(x,y)} \coloneqq \frac{-x + e^{\frac{\theta}{2}}(x-y)}{-1 + e^{\frac{\theta}{2}}(x-y)} \in [y_\theta\,;1[$$
is determined by (x,y), and is the unique one so that $\mathcal{A}_{\boldsymbol{\mu}}(\mathcal{L}_{\theta,x,y}) = \theta$. We emphasize that, contrary to lightlike rectangles, the orbit space of L-shaped polygons of area θ under the action of $\mathrm{PSL}_2(\mathbb{R})$ is not trivial but two-dimensional, and is parametrized by (x,y).

4.3.1. *A pair of HIETs.* As we previously did for the rectangle $\mathcal{R}_\theta$, we want to glue the edges of $\mathcal{L}_{\theta,x,y}$ through HIETs of the intervals $I = [1\,;\infty[$ and $J = [0\,;y_+[$ exchanging the two components of their top and bottom partitions defined by

$$\begin{cases} I_1^t = [1\,;x'[, I_2^t = [x'\,;\infty[,\ I_1^b = [1\,;x[, I_2^b = [x\,;\infty[, \\ J_1^t = [0\,;y'[, J_2^t = [y'\,;y_+[,\ J_1^b = [0\,;y[, J_2^b = [y\,;y_+[, \end{cases}$$

for $x' \in [1\,;\infty]$ and $y' \in [0\,;y_+]$. We denote by $h_1 = h_{1(x,x',y)}$ and $h_2 = h_{2(x,x',y)}$ the unique elements of $\mathrm{PSL}_2(\mathbb{R})$ realizing the gluing of the α-edges of $\mathcal{L}_{\theta,x,y}$ according to these partitions, characterized by

$$h_1(I_1^t \times \{0\}) = I_2^b \times \{y\} \text{ and } h_2(I_2^t \times \{0\}) = I_1^b \times \{y_+\}$$

or equivalently by

$$h_1(1, x', 0) = (x, \infty, y) \text{ and } h_2(x', \infty, 0) = (1, x, y_+). \tag{4.9}$$

We denote in the same way by (g_1, g_2) the elements of $\mathrm{PSL}_2(\mathbb{R})$ realizing the gluing of the β-edges and illustrated in Figure 4.2.

We can then form the quotient of $\mathcal{L}_{\theta,x,y}$ by these gluings as described in Proposition 4.3, and compute the holonomy around the vertices of $\mathcal{L}_{\theta,x,y}$. Formula (4.4) indicate us that $C = \overline{(1, y')}$ and $B = \overline{(x', 0)}$ are regular points in the quotient if and only if

$$g_1 = h_2 h_1 h_2^{-1} \text{ and } g_2 = h_1 h_2^{-1}.$$

These two relations impose two equations on (x, y, x', y'), given by the following lemma which follows from direct computations similar to the ones detailed in Lemma 4.7.

Lemma 4.10. *(1) $h_1 h_2^{-1}$ and h_2 are hyperbolic.*

(2) $h_2 h_1 h_2^{-1}(0) = y$ if and only if $x' = \frac{x}{e^{\frac{\theta}{2}}(y-1)+x}$ $(= 1$ if $x = \infty)$.

(3) $\frac{x}{e^{\frac{\theta}{2}}(y-1)+x} \in \,]1\,;\infty[$ if and only if $y > 1 - e^{-\frac{\theta}{2}}x$.

(4) If $x' = \frac{x}{e^{\frac{\theta}{2}}(y-1)+x}$ and $y > 1 - e^{-\frac{\theta}{2}}x$, then $h_2 h_1^{-1}(0) = \frac{x+e^{\frac{\theta}{2}}x(y-1)}{1+e^{\frac{\theta}{2}}x(y-1)+y(x-1)} \in [0\,;y_+[$.

We thus fix henceforth $x \in \,]1\,;\infty]$ and $y \in \,]1 - e^{-\frac{\theta}{2}}x\,;y_\theta[$, and define

$$\begin{cases} x' = x'_{(x,y)} := \frac{x}{e^{\frac{\theta}{2}}(y-1)+x}, \\ h_1 = h_{1(x,y)} := h_{1\left(x,x'_{(x,y)},y\right)}, h_2 = h_{2(x,y)} := h_{2\left(x,x'_{(x,y)},y\right)}, \\ y' := h_2 h_1^{-1}(0) \\ g_1 := h_2 h_1 h_2^{-1}, g_2 := h_1 h_2^{-1}. \end{cases} \tag{4.10}$$

Then according to Lemma 4.10.(3) and (4): $x' \in [1\,;\infty]$ and $y' \in [0\,;y_+[$. Moreover according to Lemma 4.10.(2) and the definition of h_1 and h_2 in (4.9) we have

$$g_1(1, 0, y') = (x, y, y_+) \text{ and } g_2(1, y', y_+) = (\infty, 0, y). \tag{4.11}$$

This allows us to define a pair $E = E_{x,y} \colon I \to I$ and $F = F_{x,y} \colon J \to J$ of HIET with two components by

$$\begin{cases} E_{x,y}|_{I_1^t} = h_{1(x,y)}|_{I_1^t} \colon I_1^t \to I_2^b \text{ and } E_{x,y}|_{I_2^t} = h_{2(x,y)}|_{I_2^t} \colon I_2^t \to I_1^b, \\ F_{x,y}|_{J_1^t} = g_{1(x,y)}|_{J_1^t} \colon J_1^t \to J_2^b \text{ and } F_{x,y}|_{J_2^t} = g_{2(x,y)}|_{J_2^t} \colon J_2^t \to J_1^b. \end{cases} \tag{4.12}$$

4.3.2. *Gluing of the L-shaped polygon.* We can now form the quotient $\mathcal{T}_{\theta,x,y}$ of $\mathcal{L}_{\theta,x,y}$ by the following edges identifications, given by E and F and illustrated in Figure 4.2:

$$\begin{cases} [1\,;x'[\times \{0\} \ni (p,0) \sim (h_1(p), y) \in [x\,;\infty[\times \{y\}, [x'\,;\infty[\times \{0\} \ni (p,0) \sim (h_2(p), y_+) \in [1\,;x[\times \{y_+\}, \\ \{1\} \times [0\,;y'[\ni (1,p) \sim (x, g_1(p)) \in \{x\} \times [y\,;y_+[, \{1\} \times [y'\,;y_+[\ni (1,p) \sim (\infty, g_2(p)) \in \{\infty\} \times [0\,;y[. \end{cases}$$

The following result summarizes this construction, and follows from Proposition 4.3.

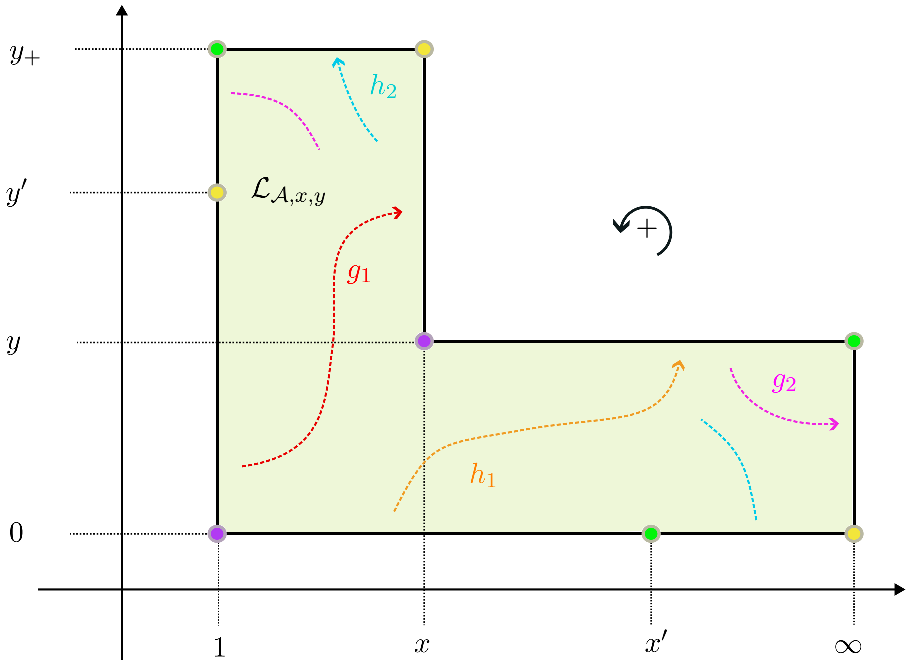

FIGURE 4.2. $\mathbf{dS}^2$-torus with one singularity and two minimal foliations.

Proposition 4.11. *For any $\theta \in \mathbb{R}_+^*$ and (x,y) in*

$$\mathcal{D} := \left\{(x,y) \in [1\,;\infty] \times]0\,;y_\theta] \;\middle|\; y > 1 - e^{-\frac{\theta}{2}}x\right\} \cup (\{\infty\} \times [0\,;y_\theta]) \cup ([1\,;\infty] \times \{y_\theta\}), \tag{4.13}$$

$\mathcal{T}_{\theta,x,y}$ is homeomorphic to $\mathbf{T}^2$ and the $\mathbf{dS}^2$-structure of the interior of $\mathcal{L}_{\theta,x,y}$ extends to a unique singular $\mathbf{dS}^2$-structure on $\mathcal{T}_{\theta,x,y}$. The latter has area θ, $\overline{(1,0)}$ is its unique singular point and it has angle θ.

4.3.3. *At the boundary of the domain.* Let us investigate what happens on the four edges of the boundary of the domain $\mathcal{D}$ where our parameters (x,y) take their values.

Edge 1: if $x \in [1\,;\infty]$ and $y = y_\theta$. Then $y_+ = y = y_\theta$ hence $\mathcal{L}_{\theta,x,y_\theta} = \mathcal{R}_\theta$, $y' = 0$, $F := g_2|_J$, and $\mathcal{T}_{\theta,x,y_\theta}$ is simply the quotient $\mathcal{T}_{\theta,x}$ constructed in Paragraph 4.2.

Edge 2: if $x = \infty$ and $y \in [0\,;y_\theta]$. Then $y_+ = y_\theta$ hence $\mathcal{L}_{\theta,\infty,y} = \mathcal{R}_\theta$, $x' = 1$, $E := h_2|_I$, and $\mathcal{T}_{\theta,\infty,y}$ is an example of the form $\mathcal{T}_{\theta,*,y}$ described in Remark 4.9.

Edge 3: if $x \in]e^{\frac{\theta}{2}}\,;\infty[$ and $y = 0$. Then $y' = y_+ \in]0\,;1[$ and the polygon $\mathcal{L}_{\theta,\infty,y}$ is degenerated. Since $x'_{x,0} = \frac{x}{x-e^{\frac{\theta}{2}}} \in]1\,;\infty[$ according to (4.10), $E_{x,0}$ and $F_{x,0} = g_1|_J$ are well-defined identifications between edges. We now show that $\mathcal{T}_{\theta,x,0}$ is actually simply a quotient of the rectangle $\mathcal{R}_{(1,x,0,y_+)}$ of area θ by suitable edges identifications, and is therefore a well-defined singular $\mathbf{dS}^2$-torus with a single singularity of angle θ at $\overline{(1,0)}$. Observe that the β edges of $\mathcal{L}_{\theta,x,0}$ are simply identified by $F = g_1$, and that we therefore only have to translate the identifications of the α edges of $\mathcal{L}_{\theta,x,0}$ into suitable identifications of α egdes of $\mathcal{R}_{(1,x,0,y_+)}$.

Since $[x'\,;\infty] \times \{0\}$ is identified through h_2 to $[1\,;x] \times \{y_+\}$ and $[x\,;h_1(x)] \times \{0\}$ through h_1 to $[1\,;x] \times \{0\}$, $]h_1(x)\,;x'[\times \{0\}$ is the only subset of $\mathcal{L}_{\theta,x,0}$ which *may* not be identified in the quotient to a subset of $\mathcal{R}_{(1,x,0,y_+)}$, and this happens only if $h_1(x) \in]x\,;x'[$. But since $]h_1(x)\,;x'[\times \{0\}$ is itself identified through h_1 to $]h_1^2(x)\,;\infty[\times \{0\}$, whose subinterval $[x'\,;\infty[\times \{0\}$ is identified through h_2 to $[1\,;x[\times \{y_+\}$, the only possible problematic subset is actually $]h_1^2(x)\,;x'[\times \{0\}$, which exist only if $h_1^2(x) \in]x\,;x'[$. In the end, the only possible case for $\mathcal{T}_{\theta,x,0}$ not to be an identification space of the rectangle $\mathcal{R}_{(1,x,0,y_+)}$ is for the sequence $h_1^n(x)$ to be contained in $]x\,;x'[$. But a direct observation of the definition (4.9) of h_1 shows that for $y = 0$, $h_{1(x,0)}$ is a parabolic or hyperbolic transformation without fixed point in $[1\,;\infty]$, and that h_1 is strictly increasing on $[1\,;\infty]$, hence that there exists a smallest $n_0 \in \mathbb{N}$ for which $h_1^{n_0}(x) \in [x'\,;\infty]$.

It is then easily checked that $\mathcal{T}_{\theta,x,0}$ is equal to the quotient of $\mathcal{R}_{(1,x,0,y_+)}$ by the identifications

$$\begin{cases} [1\,;x[\times\{0\}\ni(p,0)\sim(\tilde{E}(p),y_+)\in[1\,;x[\times\{y_+\}, \\ \{1\}\times[0\,;y_+]\ni(1,p)\sim(x,g_1(p))\in\{x\}\times[0\,;y_+], \end{cases}$$

with $\tilde{E}=\tilde{E}_{x,0}$ the HIET of $[1\,;x]$ defined by

$$E|_{[1;h_1^{-n_0}(x')[}=h_2h_1^{n_0+1}|_{[1;h_1^{-n_0}(x')[},\, E|_{[h_1^{-n_0}(x');x[}=h_2h_1^{n_0}|_{[h_1^{-n_0}(x');x[}.$$

The holonomy of $[h_1^{-n_0}(x'),0]\in\mathcal{T}_{\theta,x,0}$ is furthermore equal to $(h_2h_1^{n_0+1})^{-1}g_1(h_2h_1^{n_0})$ according to formula (4.4), hence to id since $g_1=h_2h_1h_1^{-1}$ according to (4.10). In the end, $\mathcal{T}_{\theta,x,0}$ is indeed a singular $\mathbf{dS}^2$-torus with a single singularity of angle θ at $\overline{(1,0)}$, and is isometric to a singular $\mathbf{dS}^2$-torus of the form $\mathcal{T}_{\theta,\tilde{x}}$. In particular, the α-leaf of the singularity is closed.

Edge 4: if $x\in\,]1\,;e^{\frac{\theta}{2}}]$ and $y=1-e^{-\frac{\theta}{2}}x$. The L-shaped polygon $\mathcal{L}_{\theta,x,1-e^{-\frac{\theta}{2}}x}$ is well-defined and non-degenerated, but $x'=\infty$ while $x\neq 1$. The whole edge $[1\,;x]\times\{y_+\}$ of non-empty interior is thus identified to a same point $\overline{(\infty,0)}$ in the quotient $\mathcal{T}_{\theta,x,1-e^{-\frac{\theta}{2}}x}$, which is therefore *not* a well-defined singular $\mathbf{dS}^2$-torus.[7]

5. From rotation numbers to asymptotic cycles

We would like to prescribe the dynamics of the lightlike foliations of the $\mathbf{dS}^2$-tori constructed in Section 4. Those dynamics are entirely described by a one-dimensional invariant, the *asymptotic cycle*, introduced in Paragraph 5.2. This section presents the basic notions about circle homeomorphisms and torus foliations which are needed in this paper, and may be skipped by specialists of those subjects.

5.1. Rotation numbers.

As we are going to see later, the suspensions are essentially described by a simple scalar invariant of circle homeomorphisms that we introduce now: the *rotation number.*

5.1.1. *From HIET to circle homeomorphisms and rotation numbers.* We see the circle as the additive group $\mathbf{S}^1=\mathbb{R}/\mathbb{Z}$, denote by $\pi\colon\mathbb{R}\to\mathbf{S}^1$ the canonical projection when we need it, and also use the notation $\overline{x}\coloneqq\pi(x)\in\mathbf{S}^1$ for $x\in\mathbb{R}$. We endow $\mathbf{S}^1$ with the orientation induced by the one of $\mathbb{R}$, for which a continuous map $f\colon I\to\mathbf{S}^1$, I being an interval of $\mathbb{R}$, is *non-decreasing* if for any lift $F\colon I\to\mathbb{R}$ of f, F is non-decreasing. In the same way a continuous map $f\colon\mathbf{S}^1\to\mathbf{S}^1$ is non-decreasing if any lift $F\colon\mathbb{R}\to\mathbb{R}$ of f is so. We adopt the natural analogous definitions for non-increasing, and strictly increasing or decreasing maps.

Any HIET E of an interval $I=[a\,;b[\,\subset\mathbb{RP}^1$ with one or two components naturally induces a bijection E of the quotient $\mathbf{S}^1_I\coloneqq[a\,;b]/\{a\sim b\}$, defined by

$$\forall p\in I, \mathsf{E}(\overline{p})=\overline{E(p)}.$$

$\mathbf{S}^1_I$ is homeomorphic to the circle $\mathbf{S}^1$ and bears a natural orientation induced by the one of I, and it is moreover easily checked that E is an *orientation-preserving homeomorphism* of $\mathbf{S}^1_I$ (since the HIET E exchanges at most two components).

If $f\in\mathrm{Homeo}^+(\mathbf{S}^1)$ is an orientation-preserving homeomorphism of the circle, then any lift $F\colon\mathbb{R}\to\mathbb{R}$ of f is a strictly increasing homeomorphism of $\mathbb{R}$ commuting with every integer translation $T_n\colon x\in\mathbb{R}\mapsto x+n\in\mathbb{R}$ ($n\in\mathbb{Z}$). Following [Her79] and the literature, we denote by $\mathrm{D}(\mathbf{S}^1)$ the subgroup of all such homeomophisms of $\mathbb{R}$, *i.e.* of all the lifts of elements of $\mathrm{Homeo}^+(\mathbf{S}^1)$ to $\mathbb{R}$. The *translation number* of $F\in\mathrm{D}(\mathbf{S}^1)$ is the asymptotic average amount by which F translates the points of $\mathbb{R}$. We refer to [Her79, II.2 p.20] and [dFG22, §2.1] for a proof of the following classical results.

Proposition-Definition 5.1. *Let $f,g\in\mathrm{Homeo}^+(\mathbf{S}^1)$ and $F\in\mathrm{D}(\mathbf{S}^1)$ be any lift of f.*

[7] This quotient is greatly more singular than the singularities that we defined. For instance, infinitely many negative β-leaves emanate from $\overline{(\infty,0)}$.

(1) The limit

$$\tau(F) = \lim_{n\to\pm\infty} \frac{F^n(x) - x}{n} \tag{5.1}$$

exists for any $x \in \mathbb{R}$, is independent of x, and is uniform on $\mathbb{R}$. It is called the translation number *of F.*

(2) If $G = F + d$ is another lift of f ($d \in \mathbb{Z}$), then $\tau(G) = \tau(F) + d$, and

$$\rho(f) = \overline{\tau(F)} \in \mathbf{S}^1$$

is called the rotation number *of f.*

(3) The maps $F \in \mathrm{D}(\mathbf{S}^1) \to \tau(F) \in \mathbb{R}$ and $f \in \mathrm{Homeo}^+(\mathbf{S}^1) \to \rho(f) \in \mathbf{S}^1$ are continuous for the compact-open topology.

(4) Moreover ρ is a conjugacy invariant: $\rho(g \circ f \circ g^{-1}) = \rho(f)$.

(5) If f and g commute, then $\rho(f \circ g) = \rho(f) + \rho(g)$.

The following simple observation is useful to us all along this text.

Lemma 5.2. *Let C be an oriented topological circle and $f \in \mathrm{Homeo}^+(C)$. Then for any orientation-preserving homeomorphisms $\varphi_1, \varphi_2 \colon C \to \mathbf{S}^1$: $\rho(\varphi_1 \circ f \circ \varphi_1^{-1}) = \rho(\varphi_2 \circ f \circ \varphi_2^{-1})$. This common number is still called the* rotation number of f *and be denoted by $\rho(f) \in \mathbf{S}^1$.*

Proof. Since $\varphi_2 \circ f \circ \varphi_2^{-1} = \varphi \circ (\varphi_1 \circ f \circ \varphi_1^{-1}) \circ \varphi^{-1}$ with $\varphi = \varphi_2 \circ \varphi_1^{-1} \in \mathrm{Homeo}^+(\mathbf{S}^1)$, the claim follows from Proposition 5.1.(4). □

5.1.2. *Rotation numbers as cyclic ordering of the orbits.* For $\theta \in \mathbf{S}^1$, we say that a sequence $(p_n)_{n\in\mathbb{Z}}$ in $\mathbf{S}^1$ is *of cyclic order $\theta \in \mathbf{S}^1$* if it is cyclically ordered as an orbit of the rotation

$$R_\theta \colon x \in \mathbf{S}^1 \mapsto x + \theta \in \mathbf{S}^1,$$

i.e. if for any $(n_1, n_2, n_3) \in \mathbb{Z}^3$: the three points $(p_{n_1}, p_{n_2}, p_{n_3}) \in (\mathbf{S}^1)^3$ are pairwise distinct and positively cyclically ordered if and only if $(R_\theta^{n_1}(0), R_\theta^{n_2}(0), R_\theta^{n_3}(0)) = (n_1\theta, n_2\theta, n_3\theta)$ are such in $\mathbf{S}^1$. We henceforth assume every rational $\frac{p}{q} \in \mathbb{Q}$ to be *written in reduced form*, *i.e.* such that:

- either $\frac{p}{q} = 0$ and then $(p, q) = (0, 1)$;
- or $p \in \mathbb{Z}^*$, $q \in \mathbb{N}^*$ and p, q are coprimes.

We refer to [dFG22, §1.1] and [dMvS93, §II.2.1.2] for a proof of the following classical results.

Proposition 5.3. *Let $T \in \mathrm{Homeo}^+(\mathbf{S}^1)$.*

(1) $\rho(T) = \frac{p}{q} \in \mathbb{Q}$ if and only if there exists a periodic orbit of T of cyclic order $\frac{p}{q}$. Moreover if this is the case, then any periodic orbit of T is of this form, and has thus in particular minimal period q. In particular, $\rho(T) = 0$ if, and only if T has a fixed point.

(2) $\rho(T) = \theta \in \mathbb{R} \setminus \mathbb{Q}$ if and only if any orbit of T is of cyclic order θ.

5.2. **Projective asymptotic cycles.** Our goal is to prove the existence of singular $\mathbf{dS}^2$-tori whose lightlike foliations are prescribed in terms of an invariant which is in a sense a global version of the rotation number of the first-return map: the *projective asymptotic cycle*. The notion of *asymptotic cycle* was introduced by Schwartzman in [Sch57]. It associates to any suitable orbit O of a topological flow on a closed manifold M, an element of the first homology group of M which is in a sense the "best approximation of O by a closed loop in homology". This notion has a natural projective counterpart for the leaves of an oriented topological one-dimensional foliation $\mathcal{F}$, that we now quickly describe, referring to [Sch57, Yan85] for more details.

We consider an auxiliary smooth Riemannian metric μ on M, the induced metric and its induced distance $d_{\mathcal{F}}$ on the leaves of $\mathcal{F}$. For $x \in M$ and $T \in \mathbb{R}$ we denote by $\gamma_{T,x}$ the closed curve on M obtained by: first following $\mathcal{F}(x)$ from x in the positive direction until the unique point $y \in \mathcal{F}(x)$ such that $d_{\mathcal{F}}(x, y) = T$, and then closing the curve by following the minimal geodesic of μ from y to x. Following [Sch57, Yan85], we then define the *oriented projective asymptotic cycle* of $\mathcal{F}$ at x as the half-line

$$A^{\pm}_{\mathcal{F}}(x) := \mathbb{R}^+ \left(\lim_{T\to+\infty} \frac{1}{T}[\gamma_{T,p}]\right) \in \mathbf{P}^+(\mathrm{H}_1(M, \mathbb{R})) \tag{5.2}$$

in the first homology group of M, if this limit exists and is non-zero. Note that the orientation of $A^+_{\mathcal{F}}(x)$ obviously depends of the orientation of the foliation $\mathcal{F}$, and is reversed when the orientation of $\mathcal{F}$ is. We also denote by $A_{\mathcal{F}}(x) = \mathbb{R}A^+_{\mathcal{F}}(x)$ the unoriented *projective asymptotic cycle*. This line (if it exists) is by definition constant on leaves, does not depend on the auxiliary Riemannian metric, and is moreover natural with respect to any homeomorphism f:

$$A^+_{f_*\mathcal{F}}(f(x)) = f_*(A^+_{\mathcal{F}}(x)). \tag{5.3}$$

In particular, any homeomorphism isotopic to the identity acts trivially on projective asymptotic cycles. For these properties of aymptotic cycles, we refer to [Sch57, Theorem p.275] proving the equivalence between the geometric interpretation (5.2) and the equivariant definition.

In the case of foliations on the torus, asymptotic cycles are described by the following result which is a reformulation of [Yan85, Theorem 6.1 and Theorem 6.2]. We identify henceforth $\mathrm{H}_1(\mathbf{T}^2, \mathbb{R})$ with $\mathbb{R}^2$ through the isomorphism induced by the covering map $\mathbb{R}^2 \to \mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and we say that a line in $\mathrm{H}_1(\mathbf{T}^2, \mathbb{R})$ is *rational* if it passes through a point of the lattice $\mathrm{H}_1(\mathbf{T}^2, \mathbb{Z}) = \mathbb{Z}^2$.

Proposition 5.4 ([Yan85])**.** *Let $\mathcal{F}$ be an oriented topological one-dimensional foliation of $\mathbf{T}^2$, which is the suspension of a $\mathcal{C}^\infty$ circle diffeomorphism with breaks.*

(1) $A^+_{\mathcal{F}}(x)$ exists at any $x \in \mathbf{T}^2$. It is moreover constant on $\mathbf{T}^2$ and is denoted by $A^+(\mathcal{F})$ (respectively $A(\mathcal{F}) = \mathbb{R}A^+(\mathcal{F})$ for the unoriented asymptotic cycle).

(2) If $\mathcal{F}$ has a closed leaf F, then $A^+(\mathcal{F})$ is equal to the homology class $[F]$ of F, and is in particular rational.

(3) If $\mathcal{F}$ is the linear oriented foliation induced by a half-line $l \subset \mathbb{R}^2$, then $A^+(\mathcal{F}) = l$.

Being given a finite foliated atlas of a topological foliation $\mathcal{F}$ of $\mathbf{T}^2$, let us say that a topological foliation $\mathcal{F}'$ is *ε-close to $\mathcal{F}$* if it admits a foliated atlas with the same charts domains, and whose charts are ε-close to those of $\mathcal{F}$ for the compact-open topology (with respect to a given metric).

Definition 5.5. The space of topological foliations of $\mathbf{T}^2$ is endowed with the *$\mathcal{C}^0$-topology*, for which a basis of open neighbourhoods of $\mathcal{F}$ is given by the foliations ε-close to $\mathcal{F}$.

We refer to Paragraph 6.1, where a similar topology is defined, for more details. The asymptotic cycle enjoys the same continuity property than the rotation number.

Proposition 5.6. *The map $\mathcal{F} \mapsto A^+(\mathcal{F}) \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$ is continuous for the $\mathcal{C}^0$-topology on oriented topological foliations of $\mathbf{T}^2$.*

The above "folklore" result is best proved by using the original equivariant definition of [Sch57]. We are going to apply later the notion of projective asymptotic cycle to lightlike foliations of singular $\mathbf{dS}^2$-structures which are suspensions of circle homeomorphisms. According to Lemma 3.24, these foliations are topologically equivalent to suspensions of $\mathcal{C}^\infty$-diffeomorphisms with breaks and have thus no exceptional minimal set. It is useful to have in mind a rough classification of such suspensions, that we summarize in the following statement. Those results are well-known, and are for instance proved in [HH86, §4]. We recall that a foliation (respectively a homeomorphism) is said *minimal* if all its leaves (resp. orbits) are dense.

Proposition 5.7. *Let $\mathcal{F}$ be a topological foliation of $\mathbf{T}^2$. Then if $\mathcal{F}$ has closed leaves, all of them are freely homotopic, and every non-closed leaf is past- and future-asymptotic to one of these closed leaves. Moreover:*

(1) either $\mathcal{F}$ has at least one Reeb component, and in this case $\mathcal{F}$ has a closed leaf;

(2) or $\mathcal{F}$ is a suspension.

Assume now that $\mathcal{F}$ is the suspension of a $\mathcal{C}^\infty$ circle diffeomorphism T with breaks. Then one of the two following exclusive situations arise.

(1) Either T has rational rotation number, and then $\mathcal{F}$ has closed leaves, all of which are freely homotopic, and every non-closed leaf is past- and future-asymptotic to one of these closed leaves.

(2) Or T has irrational rotation number, and then $\mathcal{F}$ is minimal and topologically equivalent to the linear foliation induced by its asymptotic cycle $A(\mathcal{F})$.

We emphasize the following consequence for singular $\mathbf{X}$-tori, thanks to Lemma 3.24.

Corollary 5.8. *If a lightlike foliation of a singular $\mathbf{X}$-torus has irrational asymptotic cycle, then it is minimal.*

The link between the rotation number of the first-return map and the asymptotic cycle, is given by the following result.

Proposition 5.9. *Let (a,b) be a basis of $\pi_1(\mathbf{T}^2)$, and γ be an oriented simple closed curve in the free homotopy class b. Let $\mathcal{F}$ be an oriented topological foliation which is a suspension transverse to γ, and $t \in [0\,;1[$ be the rotation number of the first-return map of $\mathcal{F}$ on γ. Then there exists $n \in \mathbb{Z}$ such that $A^+(\mathcal{F}) = \mathbb{R}^+(a+(t+n)b)$.*

Proposition 5.9 is proved by using Proposition 5.7, and has the following useful consequences.

Corollary 5.10. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two oriented topological foliations of $\mathbf{T}^2$ having the same oriented projective asymptotic cycles, and γ_1, γ_2 be freely homotopic oriented sections of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then the first-return maps on γ_1 and γ_2 have the same rotation number:*

$$\rho(P^{\gamma_1}_{\mathcal{F}_1}) = \rho(P^{\gamma_2}_{\mathcal{F}_2}).$$

The next result state that conversely, the rotation number of the first-return map is locally equivalent to the oriented asymptotic cycle.

Corollary 5.11. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two oriented topological foliations of $\mathbf{T}^2$ such that $A^+(\mathcal{F}_1) = A^+(\mathcal{F}_2)$, and γ_1, γ_1 be two freely homotopic oriented sections of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then for any oriented foliations $\mathcal{F}'_1$ and $\mathcal{F}'_2$ respectively sufficiently close to $\mathcal{F}_1$ and $\mathcal{F}_2$:*

$$\rho(P^{\gamma_1}_{\mathcal{F}'_1}) = \rho(P^{\gamma_2}_{\mathcal{F}'_2}) \Rightarrow A^+(\mathcal{F}'_1) = A^+(\mathcal{F}'_2).$$

Proof. We fix a basis (a,b) of $\pi_1(\mathbf{T}^2) \equiv \mathbb{Z}^2$. If $A^+(\mathcal{F}_1) = A^+(\mathcal{F}_2) =: l$, then there exists a neighbourhood U of l in $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ containing at most one of the half-lines $\{\mathbb{R}^+[a+(u+n)b]\}_{n\in\mathbb{Z}}$ for any $u \in [0\,;1[$. Since the oriented asymptotic cycle vary continuously with the foliation according to Proposition 5.6, for any oriented foliations $\mathcal{F}'_1$ and $\mathcal{F}'_2$ respectively sufficiently close to $\mathcal{F}_1$ and $\mathcal{F}_2$, $A^+(\mathcal{F}'_1)$ and $A^+(\mathcal{F}'_2)$ are contained in U. Therefore, Proposition 5.9 shows that $\rho(P^{\gamma_1}_{\mathcal{F}'_1}) = \rho(P^{\gamma_2}_{\mathcal{F}'_2})$ implies $A^+(\mathcal{F}'_1) = A^+(\mathcal{F}'_2)$, which concludes the proof of the corollary. ☐

6. Deformation space, markings and asymptotic cycles map

We now want to deduce, from the singular $\mathbf{dS}^2$-tori constructed in Section 4, parameter families of singular $\mathbf{dS}^2$-structures on a *fixed* torus $\mathbf{T}^2$. To achieve this process sometimes described as a *marking*, we first have to introduce a suited deformation space to work in.

6.1. Definition of the deformation space.

For any oriented surface S and any set $\Theta = \{\theta_i\}_i$ of angles $\theta_i \in \mathbb{R}$, we denote by $\mathcal{S}(S,\Theta)$ the set of singular $\mathbf{dS}^2$-structures on S whose singular points angles are given by Θ. We endow $\mathcal{S}(S,\Theta)$ with the usual topology on $(\mathbf{G},\mathbf{X})$-structures, defined as follows (see [CEG87, §1.5] for more details).

Let (S,Σ,μ) be a singular $\mathbf{dS}^2$-surface of singular $\mathbf{dS}^2$-atlas $(\varphi_i \colon U_i \to X_i)_i$, where $X_i = \mathbf{dS}^2$ if φ_i is a regular chart, and $X_i = \mathbf{dS}^2_{\theta_i}$ at a singular point x_i of angle θ_i. Let $(U'_i)_i$ be a shrinking of $(U_i)_i$, *i.e.* an open covering of $\mathbf{T}^2$ such that $\overline{U'_i} \subset U_i$ for each i, and assume moreover that for any singular chart $\varphi_i \colon U_i \to X_i$, U'_i contains the unique singular point x_i of U_i. Note that the $\overline{U'_i}$ for singular charts are pairwise disjoint, since the associated U_i are such and $\overline{U'_i} \subset U_i$. Lastly, let $\mathcal{V}_i$ be for any i an open neighbourhood of $\varphi_i|_{U'_i}$ in the compact-open topology of $\mathrm{C}(U'_i, X_i)$, small enough so that for any singular chart φ_i of angle θ_i, $\mathsf{o}_{\theta_i} \in \psi(U'_i)$ for any $\psi \in \mathcal{V}_i$.

Definition 6.1. The set $\mathcal{S}(S,\Theta)$ of singular $\mathbf{dS}^2$-structures of angles Θ on an oriented surface S is endowed with the topology for which the sets of the form

$$\left\{\mu' \in \mathcal{S}(S,\Theta) \text{ defined by a singular } \mathbf{dS}^2\text{-atlas } \psi_i \colon U'_i \to X_i \;\middle|\; \psi_i \in \mathcal{V}_i\right\}$$

form a sub-basis of the topology, for any initial singular $\mathbf{dS}^2$-structure $(\Sigma,\mu) \in \mathcal{S}(S,\Theta)$ on S, and any choice of shrinking $(U'_i)_i$ and of compact-open neighbourhoods $\mathcal{V}_i$ as above. We denote by $\mathcal{S}(S,\Sigma,\Theta)\subset\mathcal{S}(S,\Theta)$ the subspace of singular $\mathbf{dS}^2$-structures on S of (ordered) singular set Σ with (ordered) angles Θ.

Let $\mu\in\mathcal{S}(S,\Sigma,\Theta)$ be a singular $\mathbf{dS}^2$-structure of singular $\mathbf{dS}^2$-atlas (φ_i,U_i). If f is an orientation-preserving homeomorphism of S acting as the identity on Σ, then the singular $\mathbf{dS}^2$-structure $f^*\mu\in\mathcal{S}(S,\Sigma,\Theta)$ is defined by the singular $\mathbf{dS}^2$-atlas $(\varphi_i\circ f,f^{-1}(U_i))$, so that f is an isometry from $(S,f^*\mu)$ to (S,μ). This defines a right action of the subgroup $\mathrm{Homeo}^+(S,\Sigma)$ of orientation-preserving homeomorphisms of S acting as the identity on Σ, on each $\mathcal{S}(S,\Sigma,\Theta)$.

The *deformation space* of singular $\mathbf{dS}^2$-structures on S with singular set Σ of angles Θ, denoted by $\mathsf{Def}_\Theta(S,\Sigma)$, is defined as the quotient of $\mathcal{S}(S,\Sigma,\Theta)$ by the subgroup $\mathrm{Homeo}^0(S,\Sigma)\subset\mathrm{Homeo}^+(S,\Sigma)$ of homeomorphisms of S isotopic to the identity relative to Σ.

We recall that a $f\in\mathrm{Homeo}^+(S,\Sigma)$ is said *isotopic to the identity relative to* Σ, if there exists a continuous family $t\in[0\,;1]\mapsto f_t\in\mathrm{Homeo}^+(S,\Sigma)$ such that $f_0=f$ and $f_1=\mathrm{id}_S$. The quotient $\mathrm{PMod}(S,\Sigma)$ of $\mathrm{Homeo}^+(S,\Sigma)$ by $\mathrm{Homeo}^0(S,\Sigma)$ is called the *pure mapping class group* of (S,Σ), and acts naturally on $\mathsf{Def}_\Theta(S,\Sigma)$. The quotient of this action is the *moduli space* of $\mathbf{dS}^2$-structures on S with singular set Σ of angles Θ.

6.2. **Definition of the markings.** Let $a_{\mathcal{R}}$ (respectively $b_{\mathcal{R}}$) be a continuous path in $\mathcal{R}_\theta$ going from $a_{\mathcal{R}}(0)=(1,y_\theta)$ to $a_{\mathcal{R}}(1)=(\infty,y_\theta)$ (resp. from $b_{\mathcal{R}}(0)=(x',0)$ to $b_{\mathcal{R}}(1)=(1,y_\theta)$), and such that $a_{\mathcal{R}}(]0\,;1[)\subset\mathrm{Int}(\mathcal{R}_\theta)$ (resp. $b_{\mathcal{R}}(]0\,;1[)\subset\mathrm{Int}(\mathcal{R}_\theta)$). Then the respective projections of $a_{\mathcal{R}}$ and $b_{\mathcal{R}}$ define two closed loops in $\mathcal{T}_{\theta,x}$, whose homotopy classes are respectively denoted by a and b and do not depend on the choice of $a_{\mathcal{R}}$ and $b_{\mathcal{R}}$ (satisfying the above conditions). Since $a_{\mathcal{R}}$ and $b_{\mathcal{R}}$ can be chosen to intersect only at their extremities,

$$m_x\coloneqq(a,b)_x$$

is moreover a basis of $\pi_1(\mathcal{T}_{\theta,x})$. In the same way, with $a_{\mathcal{L}}$ (respectively $b_{\mathcal{L}}$) a continuous path in $\mathcal{L}_{\theta,x,y}$ going from $a_{\mathcal{L}}(0)=(1,y')$ to $a_{\mathcal{L}}(1)=(\infty,0)$ (resp. from $b_{\mathcal{L}}(0)=(x',0)$ to $b_{\mathcal{L}}(1)=(1,y_+)$), and such that $a_{\mathcal{L}}(]0\,;1[)\subset\mathrm{Int}(\mathcal{L}_\theta)$ (resp. $b_{\mathcal{L}}(]0\,;1[)\subset\mathrm{Int}(\mathcal{L}_\theta)$), the respective projections of $a_{\mathcal{L}}$ and $b_{\mathcal{L}}$ define two closed loops in $\mathcal{T}_{\theta,x,y}$. Their homotopy classes are respectively denoted by a and b, do not depend on the choices of $a_{\mathcal{L}}$ and $b_{\mathcal{L}}$, and

$$m_{x,y}\coloneqq(a,b)_{x,y} \tag{6.1}$$

is moreover a basis of $\pi_1(\mathcal{T}_{\theta,x,y})$ since $a_{\mathcal{L}}$ and $b_{\mathcal{L}}$ can be chosen to intersect only at their extremities.

We lastly denote by $\mathsf{0}=[0,0]$ the origin of $\mathbf{T}^2=\mathbb{R}^2/\mathbb{Z}^2$, and fix a basis

$$\mathsf{m}=(\mathsf{a},\mathsf{b})$$

of $\pi_1(\mathbf{T}^2)$ inducing the positive orientation of $\mathbf{T}^2$.

Lemma 6.2. *Up to pre-composition by homeomorphisms of $\mathbf{T}^2$ isotopic to the identity relative to $\mathsf{0}$, there exists:*

(1) for any fixed $x\in[1\,;\infty]$, a unique homeomorphism $M_x\colon\mathbf{T}^2\to\mathcal{T}_{\theta,x}$ such that $M_x(\mathsf{0})=\overline{(1,0)}$ and whose action in homotopy sends m to m_x;

(2) for any fixed $(x,y)\in\mathcal{D}$, a unique homeomorphism $M_{x,y}\colon\mathbf{T}^2\to\mathcal{T}_{\theta,x,y}$ such that $M_{x,y}(\mathsf{0})=\overline{(1,0)}$ and whose action in homotopy sends m to $m_{x,y}$.

For any fixed $x\in[1\,;\infty]$ (respectively $(x,y)\in\mathcal{D}$), all such homeomorphisms M_x (resp. $M_{x,y}$) define thus a unique point $[M_x^*\mathcal{T}_{\theta,x}]$ (resp. $[M_{x,y}^*\mathcal{T}_{\theta,x,y}]$) in $\mathsf{Def}_\theta(\mathbf{T}^2,\mathsf{0})$ which is denoted by

$$\mu_{\theta,x}\ \textit{(resp. } \mu_{\theta,x,y}\textit{)}.$$

Proof. The existence being clear, we only have to prove that a homeomorphism of $\mathbf{T}^2$ fixing $\mathsf{0}$ and acting trivially in homotopy, is isotopic to the identity relative to $\mathsf{0}$. This fact is well-known but we outline here the proof for sake of completeness. First, for a homeomorphism f of $\mathbf{T}^2$ fixing $\mathsf{0}$ and with h the restriction of f to $\mathbf{T}^2\setminus\{\mathsf{0}\}$, f is isotopic to $\mathrm{id}_{\mathbf{T}^2}$ relative to $\mathsf{0}$ if and only h is isotopic to $\mathrm{id}_{\mathbf{T}^2\setminus\{\mathsf{0}\}}$ (see for instance [BCLR20, Proposition 1.6]). Then, h is isotopic to $\mathrm{id}_{\mathbf{T}^2\setminus\{\mathsf{0}\}}$ if and only if it is homotopic to $\mathrm{id}_{\mathbf{T}^2\setminus\{\mathsf{0}\}}$, due to a result of Epstein in [Eps66] (see also [BCLR20, Theorem 2]). Lastly, h is homotopic to $\mathrm{id}_{\mathbf{T}^2\setminus\{\mathsf{0}\}}$ if and only if it acts trivially on $\pi_1(\mathbf{T}^2\setminus\{\mathsf{0}\})$

(see [BCLR20, Theorem 2 and §2.2]). But if f acts trivially on $\pi_1(\mathbf{T}^2)$, then h acts trivially on $\pi_1(\mathbf{T}^2 \setminus \{0\})$, which concludes the proof. $\square$

We use the obvious symmetric definition for the markings $\mu^*_{\theta,y}$ of the tori $\mathcal{T}_{\theta,*,y}$ introduced in Remark 4.9.

Proposition 6.3. *The maps*

$$\mu_\theta \colon x \in [1\,;\infty] \mapsto \mu_{\theta,x} \in \mathsf{Def}_\theta(\mathbf{T}^2,0), \mu^*_\theta \colon y \in [0\,;y_\theta] \mapsto \mu^*_{\theta,y} \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$$
$$\text{and } \mu_\theta \colon (x,y) \in \mathcal{D} \mapsto \mu_{\theta,x,y} \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$$

are continuous.

Proof. This follows from the continuity of the gluing maps (h_1, h_2) (respectively (h_1, h_2, g_1, g_2)) in x (resp. in (x,y)). $\square$

Remark 6.4. Let $\rho\colon \pi_1(\mathbf{T}^2 \setminus \{0\}) \to \mathrm{PSL}_2(\mathbb{R})$ be the holonomy representation of a point of $\mathsf{Def}_\theta(\mathbf{T}^2,0)$. Since $\pi_1(\mathbf{T}^2 \setminus \{0\})$ is a free group $\langle a,b\rangle$ in two generators, ρ lifts to a representation of $\mathbb{F}_2$ into $\mathrm{SL}_2(\mathbb{R})$, and it can be checked that $\mathrm{tr}(\rho(aba^{-1}b^{-1})) > 2$. Singular $\mathbf{dS}^2$-tori give therefore a geometric interpretation to such representations, which where thoroughly studied in the seminal work [Gol03]. The geometrization of such representations by singular $\mathbf{dS}^2$-tori will be the content of a future work in collaboration with Florestan Martin-Baillon.

6.3. **Asymptotic cycle map and class A structures.**

We identify henceforth $\mathrm{H}_1(\mathbf{T}^2,\mathbb{R})$ with $\mathbb{R}^2$ through the isomorphism induced by the covering map $\mathbb{R}^2 \to \mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, identify $\pi_1(\mathbf{T}^2)$ with its image $\mathbb{Z}^2$ in $\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}) \equiv \mathbb{R}^2$, and endow $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ with the orientation induced by the one of $\mathbf{T}^2$.

Lemma 6.5. *The map $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2,0) \mapsto (\mathcal{F}^\mu_\alpha, \mathcal{F}^\mu_\beta)$ is continuous for the $\mathcal{C}^0$-topology on the space of topological foliations, and the map*

$$\mathcal{A}\colon [\mu] \in \mathsf{Def}_\theta(\mathbf{T}^2,0) \mapsto (A^+(\mathcal{F}^\mu_\alpha), A^+(\mathcal{F}^\mu_\beta)) \in (\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R})))^2 \tag{6.2}$$

is well-defined, continuous and $\mathrm{PMod}(\mathbf{T}^2,0)$*-equivariant.*

Proof. The first claim follows from the fact that the topology of $\mathsf{Def}_\theta(\mathbf{T}^2,0)$ is induced by the $\mathcal{C}^0$-topology on singular $\mathbf{dS}^2$-atlases, which yield foliated atlases of the lightlike foliations, defining itself the $\mathcal{C}^0$-topology of the space of topological foliations. The projective asymptotic cycles of the lightlike foliations of a point $[\mu] \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$ in the deformation space are well-defined since homeomorphisms isotopic to the identity act trivially on projective asymptotic cycles according to (5.3), and the latter relation also shows the equivariance of $\mathcal{A}$. The continuity of $\mathcal{A}$ follows from the continuity of the asymptotic cycle in the foliation with respect to the $\mathcal{C}^0$-topology (see Proposition 5.6). $\square$

We say, following [Suh13], that a pair $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ of transverse topological foliations is *class A* if their projective (non-oriented) asymptotic cycles are distinct: $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$; and that it is *class B* otherwise. We say that a singular $\mathbf{X}$-surface S is *class A*, respectively class B, if its lightlike bi-foliation is so. We thank an anonymous referee for informing us of the existence of the following fact.

Lemma 6.6. *Let $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ be a class A topological oriented bi-foliation of $\mathbf{T}^2$. If one of the foliations has irrational asymptotic cycle, we assume that it is minimal. Then both foliations are suspensions.*

Proof. The statement being clear if both foliations have irrational asymptotic cycle, we assume for a contradiction that $\mathcal{F}_\alpha$ has a closed Reeb component R. Note that $\mathbf{T}^2$ cannot be reduced to the unique Reeb component R since $\mathcal{F}_\alpha$ is oriented. If $\mathcal{F}_\beta$ has irrational asymptotic cycle, then it is by assumption minimal. It admits thus a leaf F entering $\mathrm{Int}(R)$, which its dense and has thus to meet the non-empty open subset $\mathbf{T}^2 \setminus R$. But since F is transverse to $\mathcal{F}_\beta$, the existence of such a curve entering and exiting the Reeb component R is impossible. Assume now that $\mathcal{F}_\beta$ has rational asymptotic cycle, *i.e.* admits a closed leaf F. Then since $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$, F has non-zero algebraic intersection number with each of the boundary curves of R. This shows again that F is a curve transverse to $\mathcal{F}_\alpha$ which has to enter and to exit the Reeb component R. This second contradiction concludes the proof of the lemma. □

Remark 6.7. A little more work would in fact prove that under the same assumption (satisfied by lightlike bi-foliations of singular $\mathbf{X}$-surfaces), the lift to $\mathbb{R}^2$ of a class A topological bi-foliation is isotopic to the product bi-foliation of the plane by horizontal and vertical lines.

Lemma 6.8. *Let S be a singular $\mathbf{dS}^2$-torus, and F be a closed leaf of a lightlike foliation of S, containing at most one singular point. Then the transversal holonomy of F is non-trivial on both sides.*

Proof. We fix $p \in F$, and choose it to be the only singular point of F if it contains any. Without loss of generality, we can assume that F is a closed leaf of $\mathcal{F}_\beta$, and we prove that the holonomy of F is non-trivial on the right. We choose as a one-sided transversal of $\mathcal{F}_\beta$ on the right a one-sided α-interval $T =]p\,;p_+[_\alpha$, and denote by $H\colon T \to T$ the holonomy of $\mathcal{F}_\beta$. Since the set Σ of singular points of S is discrete and p is the unique singular point of F, we can furthermore assume T small enough so that for any $q \in T$, the β-segment from q to $H(q)$ does not contain any singular point. We fix a $\mathbf{dS}^2$-chart φ defined on a punctured upper-right quadrant of p, which extends to a singular $\mathbf{dS}^2$-chart sending p to $p_0 = (x_0, y_0) \in \mathbf{dS}^2$. In particular, we can assume φ to be defined on the transversal T. The punctured β-leaf $F \setminus \{p\}$ is developed by $\mathbf{dS}^2$-charts compatible with φ to an interval $]p_0\,;p_1[_\beta \subset \mathbf{dS}^2$, with $p_1 = (x_0, y_1)$. We denote by $g \in \mathrm{PSL}_2(\mathbb{R})$ the corresponding holonomy of the homotopy class of the closed leaf F in $\pi_1(S \setminus \Sigma)$ (with Σ the singular set of S), and observe that $(g(x_0), g(y_0)) = (x_0, y_1)$. The leaf F being not restricted to a point, $y_1 \neq y_0$ and thus $g \neq \mathrm{id}$. This forces the homography g of $\mathbb{R}\mathbf{P}^1$ to be non-trivial on the right of its fixed point x_0, which concludes the proof of the lemma since the holonomy satisfies the relation $H \circ \varphi^{-1}(x_0, y) = \varphi^{-1}(x_0, g(y))$. □

Corollary 6.9. *The subsets of class A and of class B structures are both unions of connected components of* $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$.

Proof. The condition $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$ of class A structures being open by Lemma 6.5, the set of class A structures is open. In the other hand according to Lemma A.7, if a structure μ is class B then its lightlike α and β foliations respectively have closed leaves F_α and F_β, such that F_α is freely homotopic to $\pm[F_\beta]$. The holonomy of these closed leaves are moreover non-trivial on both sides according to Lemma 6.8.

Observe now that if a topological foliation $\mathcal{F}$ of $\mathbf{T}^2$ has a closed leaf F whose holonomy is non-trivial on both sides, then any foliation $\mathcal{F}'$ which is sufficiently $\mathcal{C}^0$-close to $\mathcal{F}$, still contains a closed leaf which is homotopic to F. A $\mathcal{C}^1$-version of this classical claim is for instance proved in [HH86, Chapter I §6], and we give here a quick proof for the convenience of the reader. Let $H\colon T \to T$ be the holonomy of $\mathcal{F}$ on a small interval transverse to $\mathcal{F}$ meeting F only at p. Then for any foliation $\mathcal{F}'$ sufficiently $\mathcal{C}^0$-close to $\mathcal{F}$: T remains transverse to $\mathcal{F}'$, and the (germ of the) holonomy $H'\colon T \to T$ of $\mathcal{F}'$ is as $\mathcal{C}^0$-close to H as we want. In particular if H' is sufficiently $\mathcal{C}^0$-close to H, then H' admits a fixed point $p' \in T$ close to p, hence $\mathcal{F}'$ admits a closed leaf homotopic to F.

Therefore any small deformation of μ contains two closed α and β-lightlike leaves respectively homotopic to F_α and F_β, and remains therefore class B. This shows that the subset of class B structures is open. Since class A and B structures form a partition of all singular $\mathbf{dS}^2$-structures in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$, this shows in the end that the set of class A (respectively class B) structures is both open and closed, *i.e.* is a union of connected components of $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$. □

We study in this article the subset $\mathsf{Def}_\theta(\mathbf{T}^2, 0)^{\mathrm{A}}$ of class A singular $\mathbf{dS}^2$-structures.

7. Realization of asymptotic cycles: existence results

In this section, we conclude the proofs of the existence results from Theorem B, C and D. More precisely, we prove the following.

Theorem 7.1. *Let $\theta \in \mathbb{R}_+^*$, $c_\alpha \neq c_\beta \in \pi_1(\mathbf{T}^2)$ be two distinct primitive elements and $A_\alpha \neq A_\beta \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ be two distinct irrational rays, such that (c_α, c_β), (c_α, A_β) and (A_α, A_β) are positive. Then there exists on $\mathbf{T}^2$ a singular $\mathbf{dS}^2$-structure having a unique singularity of angle θ at $\mathsf{0} = [0,0]$, whose lightlike foliations are suspensions of circle homeomorphisms, and satisfy moreover any of the following properties.*

1. *$\mathcal{F}_\alpha(\mathsf{0})$ and $\mathcal{F}_\beta(\mathsf{0})$ are closed leaves of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$, and $([\mathcal{F}_\alpha(\mathsf{0})], [\mathcal{F}_\beta(\mathsf{0})]) = (c_\alpha, c_\beta)$. We can moreover assume that either $\mathcal{F}_\alpha(\mathsf{0})$ or $\mathcal{F}_\beta(\mathsf{0})$ is the unique closed leaf of its foliation. If (c_α, c_β) is a basis of $\pi_1(\mathbf{T}^2)$, we can even assume that both $\mathcal{F}_\alpha(\mathsf{0})$ and $\mathcal{F}_\beta(\mathsf{0})$ are the unique closed leaves of their foliations.*
2. *$([\mathcal{F}_\alpha(\mathsf{0})], A^+(\mathcal{F}_\beta)) = (c_\alpha, A_\beta)$ (in particular, $\mathcal{F}_\beta$ is minimal), and $\mathcal{F}_\alpha(\mathsf{0})$ is the unique closed leaf of $\mathcal{F}_\alpha$. The analogous claim holds with $(A^+(\mathcal{F}_\alpha), [\mathcal{F}_\beta(\mathsf{0})]) = (A_\alpha, c_\beta)$.*
3. *$(A^+(\mathcal{F}_\alpha), A^+(\mathcal{F}_\beta)) = (A_\alpha, A_\beta)$ (in particular, $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are both minimal).*

Definition 7.2. An element $a \in \pi_1(\mathbf{T}^2)$ is *primitive* if it cannot be written as $a = b^k$ with $b \in \pi_1(\mathbf{T}^2)$ and $k \geq 2$ (equivalently if a is represented by a simple closed curve of $\mathbf{T}^2$). We denote by $[\gamma]$ the homotopy class of a curve γ in $\pi_1(\mathbf{T}^2)$. A half-line $l \in \mathbb{R}\mathbf{P}^1_+ := \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ is *rational* if $l = \mathbb{R}a$ with $a \in \pi_1(\mathbf{T}^2) \equiv \mathrm{H}_1(\mathbf{T}^2,\mathbb{Z}) \subset \mathrm{H}_1(\mathbf{T}^2,\mathbb{R})$, and *irrational* otherwise.

A pair $(x,y) \in (\mathbb{R}\mathbf{P}^1_+)^2$ is said *positive* if $y \in]x\,;-x[$ (in particular $\mathbb{R}x \neq \mathbb{R}y$), where $\mathrm{H}_1(\mathbf{T}^2,\mathbb{R})$ and $\mathbb{R}\mathbf{P}^1_+$ are endowed with the orientation induced by the one of $\mathbf{T}^2$, and $]x\,;-x[$ is the interval from x to $-x$ in the oriented circle $\mathbb{R}\mathbf{P}^1_+$. The open subset of positive pairs of $(\mathbb{R}\mathbf{P}^1_+)^2$ is denoted by $(\mathbb{R}\mathbf{P}^1_+)^{(2)}$.

We recall that according to Proposition 3.27, the positive angles are the only ones which can be realized by a single singularity of a $\mathbf{dS}^2$-torus, hence the necessary condition $\theta \in \mathbb{R}_+^*$ in Theorem 7.1. The positivity of asymptotic cycles is also necessary according to the following remark.

Remark 7.3. Since our $\mathbf{dS}^2$-charts are assumed to be orientation-preserving, the orientation conventions in $\mathbf{dS}^2$ described in Figure 3.1 impose that:

$$\mathcal{A}(\mathsf{Def}_\theta(\mathbf{T}^2,\mathsf{0})^{\mathrm{A}}) \subset (\mathbb{R}\mathbf{P}^1_+)^{(2)}.$$

7.1. **Rotation numbers and asymptotic cycles of the one-parameter family.** Before starting the study of the asymptotic cycle map, we first come back to the HIET that we suspended in Paragraph 4.2, and show existence results for their rotation numbers. We use the notations of the Paragraph 4.2.

For any $x \in [1\,;\infty]$, we consider the orientation-preserving homeomorphism E_x of $\mathbf{S}^1_I := [1\,;\infty]/\{1 \sim \infty\}$ induced by the HIET E_x of $I = [1\,;\infty[$ defined in (4.7). Note that when x converges to 1, x'_x converges to ∞ and gh_x to $h_\infty = gh_1$, since

$$gh_x(1, x'_x, 0) = (x, \infty, y_\theta).$$

Hence E_x converges to $\mathsf{E}_1 = \mathsf{E}_\infty$ for the compact-open topology of $\mathrm{Homeo}^+(\mathbf{S}^1_I)$ when $x \to 1$, and the map

$$\mathsf{E}\colon [x] \in \mathbf{S}^1_I \mapsto \mathsf{E}_x \in \mathrm{Homeo}^+(\mathbf{S}^1_I) \tag{7.1}$$

is therefore continuous. Let $\{g^t\}_{t\in\mathbb{R}} \subset \mathrm{PSL}_2(\mathbb{R})$ denote the one-parameter hyperbolic subgroup containing g, parametrized so that $g = g^1$ (with g defined by (4.1)).

Lemma 7.4. *Let $x_1 \leq x_2 \in [1\,;\infty]$.*

1. *$h_{x_1}^{-1} g h_{x_1} g^{-1} = h_{x_2}^{-1} g h_{x_2} g^{-1}$.*
2. *There exists a unique $\tau \in [0\,;1]$ such that $x_2 = g^\tau(x_1)$, and $h_{x_2} = g^\tau h_{x_1}$.*
3. *Moreover $E_{x_2} = S_\tau \circ E_{x_1}$, with S_τ the HIET defined by*

$$\begin{cases} \forall p \in [1\,; E_{x_1}(x'_2)[,\, S_\tau(p) = g^\tau(p) \in [g^\tau(1)\,;\infty[, \\ \forall p \in [E_{x_1}(x'_2)\,;\infty[,\, S_\tau(p) = g^{\tau-1}(p) \in [1\,; g^\tau(1)[. \end{cases}$$

Proof. (1) According to Proposition 4.1, the holonomy around $\overline{(\infty,0)}$ in $\mathcal{T}_{\theta,x_i}$ is equal to $h_{x_i}^{-1} g h_{x_i} g^{-1}$ (for a developing map compatible at $\overline{(\infty,0)}$, see Lemma 3.5), hence $h_{x_1}^{-1} g h_{x_1} g^{-1} = a^\theta = h_{x_2}^{-1} g h_{x_2} g^{-1}$.

Note that this extends to the case $x_1 = 1$ since by definition of h_1 we have $h_1^{-1}gh_1g^{-1} = (h_\infty^{-1}g)g(g^{-1}h_\infty)g^{-1} = h_\infty^{-1}gh_\infty g^{-1}$.
(2) According to (1), $hgh^{-1} = g$ with $h = h_{x_2}h_{x_1}^{-1}$. Hence h is in the centralizer of $g = g^1$ in $\mathrm{PSL}_2(\mathbb{R})$, which is equal to $\{g^t\}_t$. Now if $h_{x_2} = g^\tau h_{x_1}$ we obtain directly from (4.5) that $x_2 = g^\tau(x_1)$. Moreover $g^1(1) = \infty$ according to (4.1), and thus $\tau \in [0\,;1]$ since $x_1, x_2 \in [1\,;\infty]$.
(3) Indeed for any $p \in [1\,;x_1[$, $E_{x_1}^{-1}(p) = h_1^{-1}(p) \in [x_1'\,;\infty[$, and $x_2' < x_1'$ hence $E_{x_2} \circ E_{x_1}^{-1}(p) = h_2h_1^{-1}(p) = g^\tau(p) \in [g^\tau(1)\,;x_2[$. Note that $gh_1(x_2') \in]x_1\,;\infty]$, so that for $p \in [x_1\,;gh_1(x_2')[$, $E_{x_1}^{-1}(p) = h_1^{-1}g^{-1}(p) \in [1\,;x_2'[$ and $E_{x_2} \circ E_{x_1}^{-1}(p) = gh_2h_1^{-1}g^{-1}(p) = g^\tau(p) \in [x_2\,;\infty[$. Lastly for $p \in [gh_1(x_2')\,;\infty[$, $E_{x_1}^{-1}(p) = h_1^{-1}g^{-1}(p) \in [x_2'\,;x_1'[$, and thus $E_{x_2} \circ E_{x_1}^{-1}(p) = g^\tau h_1h_1^{-1}g^{-1}(p) = g^{\tau-1}(p) \in [x_2\,;\infty[$. ☐

Proposition 7.5. *The map $\overline{x} \in \mathbf{S}^1_I \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1$ is continuous, non-decreasing, and has degree one (in particular, it is surjective). Moreover it is strictly increasing at any x for which $\rho(\mathsf{E}_x) \in \mathbb{R}\setminus\mathbb{Q}$. In particular for any $u \in \mathbb{R}\setminus\mathbb{Q}$, there exists a unique $x \in \mathbf{S}^1_I$ such that $\rho(\mathsf{E}_x) = u$. Lastly, for any $r \in \mathbb{Q}$ there exists $x \in [1\,;\infty]$ such that the orbit of $\overline{(1,0)}$ under E_x is periodic and of cyclic order r.*

Proof. The continuity of $x \in [1\,;\infty] \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1$ follows from the continuity of E (see (7.1)) and of the rotation number itself (see for instance [Her79, Proposition 2.7]), for the compact-open topology of $\mathrm{Homeo}^+(\mathbf{S}^1_I)$. Note that both E_1 and E_∞ have $\overline{1} \in \mathbf{S}^1_I$ as a fixed point, and thus that $\rho(\mathsf{E}_1) = \rho(\mathsf{E}_\infty) = 0 \in \mathbf{S}^1$. By the intermediate value theorem, there exists a parameter $x_0 \in]1\,;\infty[$ for which $x'_{x_0} = x_0$, satisfying $\mathsf{E}_{x_0}(\overline{1}) \neq \overline{1}$ and $\mathsf{E}^2_{x_0}(\overline{1}) = \overline{1}$, *i.e.* $\rho(\mathsf{E}_{x_0}) = \frac{1}{2}$. In particular, $x \in [1\,;\infty] \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1$ is not constant.

According to Lemma 7.4.(3), we have moreover $\mathsf{E}_{g^\tau(1)} = S_\tau \circ \mathsf{E}_1$ with $\tau \in [0\,;1] \mapsto S_\tau \in \mathrm{Homeo}^+(\mathbf{S}^1_I)$ a continuous map such that $\tau \in [0\,;1] \mapsto S_\tau(p) \in \mathbf{S}^1_I$ is strictly increasing for any $p \in \mathbf{S}^1_I$. According to Lemma B.1.(2), $x \in [1\,;\infty] \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1$ is thus non-decreasing. But since it is also not constant and attains the same value 0 at 1 and ∞, it is actually surjective according to the Intermediate value theorem. Moreover for any $x \in [1\,;x_0[$, $x' > x$ implies $\rho(\mathsf{E}_x) \in [0\,;\frac{1}{2}[$, and for any $x \in]x_0\,;\infty]$, $x' < x$ implies $\rho(\mathsf{E}_x) \in]\frac{1}{2}\,;0]$. The latter claims are for instance a consequence of Fact B.2. The map $\overline{x} \in \mathbf{S}^1_I \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1$ has thus degree one. It is strictly increasing at any x for which $\rho(\mathsf{E}_x) \in \mathbb{R}\setminus\mathbb{Q}$ according to Lemma B.1.(4), which forbids any element of $\mathbb{R}\setminus\mathbb{Q}$ to have more than one pre-image in $\mathbf{S}^1_I$ since the map also has degree one. By surjectivity, there exists $\overline{x} \in \mathbf{S}^1_I$ such that $\rho(\mathsf{E}_x)$ is irrational, and since E_x is a $\mathcal{C}^\infty$-diffeomorphism with breaks it is then minimal according to Denjoy theorem (see also Lemma 3.24.(4)). The existence of periodic orbits of any rational cyclic order under the maps E_x for $\overline{(1,0)}$ follows then from Lemma B.1.(5), which concludes the proof of the proposition. ☐

We now begin the study of the asymptotic cycle map $\mathcal{A}$ defined in (6.2), by describing the image under $\mathcal{A}$ of the one-parameter family $\mu_{\theta,x}$. For any $u \in \mathrm{H}_1(\mathbf{T}^2,\mathbb{R})$, we henceforth denote $[u] \coloneqq \mathbb{R}^+u \in \mathbb{R}\mathbf{P}^1_+ = \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$. However, to avoid burdeing the notations and since no confusion is possible in this case, for $u, v \in \mathrm{H}_1(\mathbf{T}^2,\mathbb{R})\setminus\{0\}$ we simply denote by $[u\,;v]$ the interval from $[u]$ to $[v]$ in the oriented circle $\mathbb{R}\mathbf{P}^1_+$.

Lemma 7.6. *The continuous map*

$$\mathcal{A} \circ \mu_\theta \colon [1\,;\infty] \to [\mathsf{a}] \times [\mathsf{a}+\mathsf{b}\,;\mathsf{b}]$$

is surjective and non-decreasing, and strictly increasing at irrational points. For any primitive element $c \in \pi_1(\mathbf{T}^2)$ there exists $x \in [1\,;\infty]$ such that $\mathcal{F}^{\mu_{\theta,x}}_\beta(0)$ is closed and homotopic to c. The obvious analogous claims hold with the opposite monotonicity for

$$\mathcal{A} \circ \mu^*_\theta \colon [0\,;y_\theta] \to [\mathsf{a}\,;\mathsf{a}+\mathsf{b}] \times [\mathsf{b}].$$

Proof. We detail the proof for $\mu_{\theta,x}$, the case of $\mu^*_{\theta,y}$ being formally identical. By definition, $\mathcal{F}^{\mu_{\theta,x}}_\alpha(0)$ is closed and homotopic to a for any x, hence $A^+(\mathcal{F}^{\mu_{\theta,x}}_\alpha) = [\mathsf{a}]$ as claimed. On the other hand by our choice of markings, the closed curve $\mathcal{F}^{\mu_{\theta,1}}_\beta(0)$ is homotopic to $\mathsf{a}+\mathsf{b}$ and $\mathcal{F}^{\mu_{\theta,\infty}}_\beta(0)$ is homotopic to b, hence $A^+(\mathcal{F}^{\mu_{\theta,1}}_\beta) = [\mathsf{a}+\mathsf{b}]$ and $A^+(\mathcal{F}^{\mu_{\theta,\infty}}_\beta) = [\mathsf{b}]$. The first-return map of $\mathcal{F}^{\mu_{\theta,x}}_\beta$

on $\mathcal{F}_\alpha^{\mu_{\theta,x}}(0)$ is equal to E_x^{-1}, with E_x the homeomorphism of the circle $[1\,;\infty]/\{1\sim\infty\}$ (naturally identified with $\mathcal{F}_\alpha^{\mu_{\theta,x}}(0) = ([1\,;\infty]/\{1\sim\infty\})\times\{0\}$) introduced in Paragraph 7.1. According to Proposition 5.9 we have thus $A^+(\mathcal{F}_\beta^{\mu_{\theta,x}}) = [(1-\rho(\mathsf{E}_x))\mathsf{a}+\mathsf{b}]$. Moreover $\mathcal{F}_\beta^{\mu_{\theta,x}}(0)$ is closed and homotopic to $c\in\pi_1(\mathbf{T}^2)$ if and only if $[1]$ is periodic under E_x^{-1}, of the appropriate cyclic order $q\in\mathbb{Q}$ corresponding to c. The claims follow then from the properties of $x\in[1\,;\infty]\mapsto\rho(\mathsf{E}_x)\in\mathbf{S}^1$ proved in Proposition 7.5. □

Remark 7.7. For any primitive element $u\in\pi_1(\mathbf{T}^2)$, let us denote by D_u the *positive* (respectively *negative*) *Dehn twist* around u, *i.e.* the unique element of $\mathrm{PMod}(\mathbf{T}^2,0)$ whose action in homotopy satisfies $D_u(u)=u$, and $D_u(v)=u+v$ (respectively $D_u(v)=u-v$) for any v such that (u,v) is a positive (resp. negative) basis of $\pi_1(\mathbf{T}^2)$. Lemma 7.6 shows then that $\mu_{\theta,\infty}=(D_{-\mathsf{a}})_*\mu_{\theta,1}$. In particular, $\mu_{\theta,x}$ is not a closed loop but a segment in $\mathsf{Def}_\theta(\mathbf{T}^2,0)$.

Definition 7.8. We henceforth denote

$$\mathcal{R}^\alpha_{\theta,\mathsf{a},\mathsf{b}} \coloneqq \mu_\theta([1\,;\infty]) \text{ and } \mathcal{R}^\beta_{\theta,\mathsf{a},\mathsf{b}} \coloneqq \mu_\theta^*([0\,;y_\theta]).$$

7.2. **Asymptotic cycles of the two-parameter family.** We deduce the image of $\mu_{\theta,x,y}$ under $\mathcal{A}$ from the easier description of the image of the boundary of the domain $\mathcal{D}$ of μ_θ. We saw indeed in Paragraph 4.3.3 that three of the four boundary egdes of $\mathcal{D}$ are copies of the one-parameter families already studied in the previous Paragraph 7.1. More precisely:

Edge 1: $\mu_{\theta,x,y_\theta}=\mu_{\theta,x}$;
Edge 2: $\mu_{\theta,\infty,y}=\mu^*_{\theta,y}$;
Edge 3: and for any $x\in\,]e^{\frac{\theta}{2}}\,;\infty]$, $\mu_{\theta,x,0}=f_*\mu_{\theta,\tilde{x}}$ for some $\tilde{x}\in[1\,;\infty]$ and $f\in\mathrm{PMod}(\mathbf{T}^2,0)$.

One easily checks that the closed curve $\mathcal{F}_\alpha^{\mu_{\theta,x,0}}(0)$ is homotopic to $\mathsf{a}+\mathsf{b}$ by our choice of markings, and that if $x'_{(x,y)}=x$, then $\mathcal{F}_\beta^{\mu_{\theta,x,0}}(0)$ is closed and homotopic to $\mathsf{a}+2\mathsf{b}$. A direct computation shows that for any $y\in[0\,;y_\theta]$,

$$x(y)\coloneqq 1+e^{\frac{\theta}{2}}(1-y)\in\,]1\,;\infty[$$

is the unique point of $[1\,;\infty]$ satisfying $x'_{(x(y),y)}=x(y)$. The integer $n_0\in\mathbb{N}$ appearing in the description of Paragraph 4.3.3 is constant equal to 0 on the subinterval $x\in[x(0)\,;\infty]$, which shows that the corresponding sub-edge is the translation of the one-parameter family $\mu_{\theta,x}$ by the Dehn twist around b:

$$\textbf{Edge 3':}\ \{\mu_{\theta,x,0}\}_{x\in[x(0);\infty]}=(D_{\mathsf{b}})_*\{\mu_{\theta,x}\}_{x\in[1;\infty]}. \tag{7.2}$$

Our two-parameter family is undefined on the fourth edge of the domain, which makes the description of the image of $\mathcal{A}$ more difficult technically. To bypass this issue, we consider a smaller domain by taking as a new fourth edge the curve $\{\mu_{\theta,x(y),y}\}_{y\in[0;y_\theta]}$, on which $\mathcal{F}_\beta^{\mu_{\theta,x(y),y}}(0)$ is closed, and homotopic to $\mathsf{a}+2\mathsf{b}$. The latter claim follows easily from the observation that a segment contained in $\mathcal{L}_{\theta,x,y}$ and joining $(x',0)$ to (∞,y), defines in the marked $\mathbf{dS}^2$-torus $(\mathbf{T}^2,\mu_{\theta,x,y})$ a closed curve freely homotopic to $\mathsf{a}+\mathsf{b}$. Observe now that in restriction to $\{\mu_{\theta,x(y),y}\}_{y\in[0;y_\theta]}$, since the edge $[x'\,;\infty]\times\{0\}$ is glued to $[1\,;x]\times\{y_+\}$ with $x'=x$, $\mathcal{T}_{\theta,x(y),y}$ is actually isometric to a torus of the form $\mathcal{T}_{\theta,*,y}$. More precisely, with $y_0\in[0\,;y_\theta]$ the unique point such that $y'_{y_0}=y_0$ for the gluings of $\mathcal{T}_{\theta,*,y}$, we have:

$$\textbf{Edge 4':}\ \{\mu_{\theta,x(y),y}\}_{y\in[0;y_\theta]}=(D_{\mathsf{a}+\mathsf{b}})_*\{\mu^*_{\theta,y}\}_{y\in[0;y_0]}.$$

Since $\mathcal{F}_\alpha^{\mu^*_{\theta,0}}(0)$ is homotopic to $\mathsf{a}+\mathsf{b}$ and $\mathcal{F}_\alpha^{\mu^*_{\theta,y_0}}(0)$ to $2\mathsf{a}+\mathsf{b}$, $\mathcal{A}(\{\mu^*_{\theta,y}\}_{y\in[0;y_0]})=[\mathsf{a}+\frac{\mathsf{b}}{2}\,;\mathsf{a}+\mathsf{b}]\times[\mathsf{b}]$ by Lemma 7.6, hence $\mathcal{A}(\{\mu_{\theta,x(y),y}\}_{y\in[0;y_\theta]})=[\mathsf{a}\,;\mathsf{a}+\mathsf{b}]\times[\frac{\mathsf{a}}{2}+\mathsf{b}]$. We lastly introduce the

$$\textbf{Edge 1':}\ \{\mu_{\theta,x,y_\theta}\}_{x\in[x(y_\theta);\infty]},$$

satisfying $\mathcal{A}(\{\mu_{\theta,x,y_\theta}\}_{x\in[x(y_\theta);\infty]})=[\mathsf{a}]\times[\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}]$ since $x'_{(x,y_\theta)}\leq x$ for any $x\in[x(y_\theta)\,;\infty]$ (this is for instance a consequence of Fact B.2). The subdomain

$$\mathcal{E}\coloneqq\{(x,y)\in[1\,;\infty]\times[0\,;y_\theta]\mid x\geq x(y)\}=\left\{(x,y)\in[1\,;\infty]\times[0\,;y_\theta]\,\middle|\,x'_{(x,y)}\leq x\right\}\subset\mathcal{D}$$

is bounded by the edges that we previously described. With

$$\mathcal{R} := [\mathsf{a}\,;\mathsf{a}+\mathsf{b}] \times \left[\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}\right] \subset (\mathbb{R}\mathbf{P}^1_+)^{(2)},$$

$\partial\mathcal{E}$ and $\partial\mathcal{R}$ are two oriented topological circles which divide into four edges mapped to each other under $\mathcal{A}\circ\mu_\theta$ according to Lemma 7.6:

Edge 1': $[x(y_\theta)\,;\infty] \times \{y_\theta\}$ maps to $[\mathsf{a}] \times [\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}]$;
Edge 2: $\{\infty\} \times [0\,;y_\theta]$ maps to $[\mathsf{a}\,;\mathsf{a}+\mathsf{b}] \times [\mathsf{b}]$;
Edge 3': $\{(x(y),y))\}_{y\in[0;y_\theta]}$ maps to $[\mathsf{a}+\mathsf{b}] \times [\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}]$;
Edge 4': and $[x(0)\,;\infty] \times \{0\}$ maps to $[\mathsf{a}\,;\mathsf{a}+\mathsf{b}] \times [\frac{\mathsf{a}}{2}+\mathsf{b}]$.

We summarize the results obtained so far in this paragraph as follows.

Lemma 7.9. *The continuous map*

$$\mathcal{A}\circ\mu_\theta \colon \partial\mathcal{E} \to \partial\mathcal{R}$$

is orientation-preserving and has degree one (in particular, it is surjective).

Using the description of the image of the boundary of $\mathcal{E}$, we are now able to prove that:

Corollary 7.10. $\mathcal{A}\circ\mu_\theta(\mathcal{E}) = \mathcal{R}$.

Proof. Let γ be the oriented simple closed curve of $(\mathbf{T}^2,\mu_{\theta,x,y})$ freely homotopic to a and transverse to $\mathcal{F}_\beta$, obtained by projecting a simple path of $\operatorname{Int}(\mathcal{L}_{\theta,x,y})$ going from $(1,y')$ to $(\infty,0)$ and transverse to $\mathcal{F}_\beta$. The first projection of $\mathbf{dS}^2$ induces an identification ι of γ with the circle $\mathbf{S}^1_I = [1\,;\infty]/\{1\sim\infty\}$, and we recall that the HIET E of $I=[1\,;\infty[$ (4.12) induces a homeomorphism E of $\mathbf{S}^1_I$. By definition of the gluings of $\mathcal{T}_{\theta,x,y}$, the first-return map P^γ_β of $\mathcal{F}_\beta$ on γ is then conjugated by ι to E^{-1}: $\iota\circ P^\gamma_\beta = \mathsf{E}^{-1}\circ\iota$. However $x'\leq x$ for any $(x,y)\in\mathcal{E}$, hence $\rho(\mathsf{E}_{(x,y)})\in[\frac{1}{2}\,;1[$ according to Fact B.2 and therefore $A^+(\mathcal{F}^{\mu_{\theta,x,y}}_\beta)\in[\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}]$ according to Proposition 5.9. The same kind of reasoning shows that the first-return map of $\mathcal{F}_\alpha$ on a simple closed curve freely homotopic to b and transverse to $\mathcal{F}_\alpha$ is conjugated to F^{-1}, hence that $A^+(\mathcal{F}^{\mu_{\theta,x,y}}_\alpha)\in[\mathsf{a}\,;\mathsf{a}+\mathsf{b}]$. In the end, $\mathcal{A}\circ\mu_\theta(\mathcal{E})\subset\mathcal{R}$.

We recall from Lemma 7.9 that the restriction of $\mathcal{A}\circ\mu_\theta$ to $\partial\mathcal{E}$ is a degree one map between the circles $\mathcal{E}$ and $\partial\mathcal{R}$. We are thus left to show that a continuous map f from a closed topological disk D to itself, and whose restriction to ∂D is a degree one map from ∂D to itself, is actually surjective. Assume by contradiction that $D\setminus f(D)$ is non-empty, so that the closed loop $\gamma := f|_{\partial D}$ is non-homotopically trivial in $f(D)$. But γ being a restriction of f, it is homotopic to a constant loop within $f(D)$, which is a contradiction. This concludes the proof that $\mathcal{A}\circ\mu_\theta(\mathcal{E})=\mathcal{R}$. ☐

Remark 7.11. Observe that the α and the β foliations do not play symmetric roles in the definition of the identification space $\mathcal{T}_{\theta,x,y}$. In the same way that we did with $\mu^*_{\theta,y}$, we can however exchange the roles of α and β, and consider the obvious symmetric two-parameter family $\mu^*_{\theta,x,y}$, defined on a symmetric domain $\mathcal{D}^*$. The restriction of μ^*_θ to the sub-domain $\mathcal{E}^*$ corresponding to $\mathcal{E}$ satisfies of course the conclusion analogous to Corollary 7.10, namely that $\mathcal{A}\circ\mu^*_\theta(\mathcal{E}^*)=\mathcal{R}^*$, with

$$\mathcal{R}^* := \left[\mathsf{a}\,;\mathsf{a}+\frac{\mathsf{b}}{2}\right] \times [\mathsf{a}+\mathsf{b}\,;\mathsf{b}] \subset (\mathbb{R}\mathbf{P}^1_+)^{(2)}.$$

Definition 7.12. We henceforth denote

$$\mathcal{L}_{\theta,\mathsf{a},\mathsf{b}} := \mu_\theta(\mathcal{D}) \cup \mu^*_\theta(\mathcal{D}^*).$$

7.3. **Conclusion of the proof of Theorem 7.1.** We can now harvest the fruits of our previous descriptions to conclude the proof of the existence Theorem 7.1. We have made most of the work, and the only remaining observation to be made is that the rectangles $\mathcal{R}$ and $\mathcal{R}^*$ realized by the two-parameter families are sufficient to reach the whole $(\mathbb{R}\mathbf{P}^1_+)^{(2)}$ with the help of the mapping class group action.

Lemma 7.13. $\operatorname{PMod}(\mathbf{T}^2,0)\cdot(\mathcal{R}\cup\mathcal{R}^*) = (\mathbb{R}\mathbf{P}^1_+)^{(2)}$.

Proof. Since $D^n_{\mathsf{a}+\mathsf{b}}[\mathsf{a}] = [\mathsf{a} + \frac{n}{n+1}\mathsf{b}]$ and $D^n_{\mathsf{a}+\mathsf{b}}[\mathsf{b}] = [\frac{n}{n+1}\mathsf{a} + \mathsf{b}]$ for any $n \in \mathbb{N}$, we already have $\cup_{n\in\mathbb{N}} D^n_{\mathsf{a}+\mathsf{b}}(\mathcal{R} \cup \mathcal{R}^*) = \mathcal{R}_0 \coloneqq ([\mathsf{a}\,;\mathsf{a}+\mathsf{b}] \times [\mathsf{a}+\mathsf{b}\,;\mathsf{b}]) \setminus \{([\mathsf{a}+\mathsf{b}], [\mathsf{a}+\mathsf{b}])\}$. It is thus sufficient to show that any $(x, y) \in (\mathbb{R}\mathbf{P}^1_+)^{(2)}$ is in $\mathrm{PMod}(\mathbf{T}^2, 0) \cdot \mathcal{R}_0$. If x is rational, then since $\mathrm{PMod}(\mathbf{T}^2, 0)$ acts transitively on $\mathbb{R}\mathbf{P}^1_+$, we can assume without loss of generality that $x = [\mathsf{a}]$. Since $y \in \,]\mathsf{a}\,; -\mathsf{a}[$ and $[\mathsf{a} + \mathsf{b}\,; \mathsf{b}]$ is a fundamental domain of the action of D_{a} on $]\mathsf{a}\,; -\mathsf{a}[$, there exists $n \in \mathbb{Z}$ such that $D^n_{\mathsf{a}}(y) \in [\mathsf{a}+\mathsf{b}\,;\mathsf{b}]$, hence $(x, y) \in \mathrm{PMod}(\mathbf{T}^2, 0)\cdot\mathcal{R}_0$. If y is rational, we conclude in the same way.

Let now x and y be both irrational, and x be the limit of an increasing sequence of rational elements $[u_n] \in \mathbb{R}\mathbf{P}^1_+$, with $u_n \in \pi_1(\mathbf{T}^2)$ a primitive element. For any n, the set of half-lines of the form $[v]$ with (u_n, v) a positive basis of $\pi_1(\mathbf{T}^2)$ is an orbit O_n of the Dehn twist around u_n. Since this orbit accumulate on $[u_n]$ for any n and is constituted of rational points on the first hand, and since x and y are both irrational on the other hand, there exists finally n such that: $x, y \in \,]u_n\,; -u_n[$, and the interval $]x\,; y[$ contains a point v_n of the orbit O_n. Without loss of generality, we can assume that $u_n = \mathsf{a}$ and that $[\mathsf{a} + \mathsf{b}] = [v_n] \in \,]x\,; y[$, *i.e.* that $x \in \,]\mathsf{a}\,; \mathsf{a} + \mathsf{b}[$ and $y \in \,]\mathsf{a} + \mathsf{b}\,; -\mathsf{a}[$. Since $[\mathsf{a} + \mathsf{b}\,; \mathsf{b}]$ is a fundamental domain of the action of D_{a} on $]\mathsf{a}\,; -\mathsf{a}[$ and $[\mathsf{a}]$ is an attractive fixed point of D_{a}, there exists $k \in \mathbb{N}$ such that $D^k_{\mathsf{a}}(y) \in [\mathsf{a} + \mathsf{b}\,; \mathsf{b}]$. But $D_{\mathsf{a}}(]\mathsf{a}\,; \mathsf{a} + \mathsf{b}]) \subset \,]\mathsf{a}\,; \mathsf{a} + \mathsf{b}[$, hence $D^k_{\mathsf{a}}(x, y) \in \mathcal{R}_0$, which concludes the proof of the lemma. □

Conclusion of the proof of Theorem 7.1. (1) It is clear from the dynamics of g and h_1 that $\mathcal{F}^{\mu_{\theta,1}}_\alpha(0)$ (respectively $\mathcal{F}^{\mu_{\theta,1}}_\alpha(0)$) is the unique closed α-leaf (resp. β-leaf) of the torus $(\mathbf{T}^2, \mu_{\theta,1})$, and that $\mathcal{F}_\alpha(0)^{\mu_{\theta,x}}$ is the unique closed α-leaf for any x. By acting with $\mathrm{PMod}(\mathbf{T}^2, 0)$ on $(\mathsf{a}, \mathsf{b}) = ([\mathcal{F}^{\mu_{\theta,1}}_\alpha(0)], [\mathcal{F}^{\mu_{\theta,1}}_\alpha(0)])$ one obtains any basis of $\pi_1(\mathbf{T}^2)$, which proves the claim if (c_α, c_β) is a basis. If it is not a basis, then we can assume without loss of generality that $c_\alpha = \mathsf{a}$. Since (c_α, c_β) is positive, $c_\beta \in \,]\mathsf{a}\,; -\mathsf{a}[$, and we can thus assume that $c_\beta \in [\mathsf{a} + \mathsf{b}\,; \mathsf{b}]$ since $[\mathsf{a} + \mathsf{b}\,; \mathsf{b}]$ is a fundamental domain for the action of D_{a} on $]\mathsf{a}\,; -\mathsf{a}[$. The claims follow then from Lemma 7.6.
(2) As before, we can assume without loss of generality that $c_\alpha = \mathsf{a}$ and $A_\beta \in [\mathsf{a} + \mathsf{b}\,; \mathsf{b}]$, and the claims follow then from Lemma 7.6 since we saw in (1) that $\mathcal{F}^{\mu_{\theta,x}}_\alpha(0)$ is the unique closed leaf of the α foliation and is homotopic to a.
(3) This last claim is a direct consequence of Corollary 7.10, Remark 7.11 and Lemma 7.13. □

8. Surgeries of singular constant curvature Lorentzian surfaces

8.1. **Geodesics and affine circles.** Denoting by $(\mathbf{G}, \mathbf{X})$ the pair $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ or $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1, 1), \mathbb{R}^{1,1})$, we define in this subsection the natural notion of geodesics in a singular $\mathbf{X}$-surface.

8.1.1. *Geodesics of* $\mathbf{X}$. On an oriented topological one-dimensional manifold, we call:

(1) *affine structure* an $(\mathrm{Aff}^+(\mathbb{R}), \mathbb{R})$-structure, with $\mathrm{Aff}^+(\mathbb{R}) \simeq \mathbb{R}^*_+ \rtimes \mathbb{R}$ the group of (orientation-preserving) affine transformations $\lambda\,\mathrm{id} + u \colon x \mapsto \lambda x + u$ of $\mathbb{R}$ (with $\lambda \in \mathbb{R}^*_+$ and $u \in \mathbb{R}$);
(2) and *translation structure* a $(\mathbb{R}, \mathbb{R})$-structure (which induces obviously an affine structure);

the charts of both structures being assumed to be orientation-preserving homeomorphisms. An *affine automorphism* is of course a $(\mathrm{Aff}^+(\mathbb{R}), \mathbb{R})$-morphism of affine structures. As for any affine connection, the geodesic of $\mathbf{X}$ have a natural affine structure given by parametrizations satisfying the geodesic equation, and its definite geodesics even have a natural translation structure given by constant speed parametrizations. For $\mathbf{X} = \mathbb{R}^{1,1}$, the affinely parametrized geodesics are simply the affinely parametrized affine segments.

Lemma 8.1. *Let γ be a geodesic of $\mathbf{X}$.*

(1) The stabilizer of γ in $\mathbf{G}$ acts transitively on γ. It is moreover:
  (a) a one-parameter group if γ is timelike, which is hyperbolic for $\mathbf{X} = \mathbf{dS}^2$;
  (b) a one-parameter group if γ is spacelike, which is elliptic for $\mathbf{X} = \mathbf{dS}^2$
  *(c) and a two-dimensional group if γ is lightlike, which is parabolic (*i.e. *conjugated to a triangular subgroup) for $\mathbf{X} = \mathbf{dS}^2$.*

(2) There exists for any $x \in \gamma$ a one-parameter subgroup (g^t) stabilizing γ and acting freely at x, and $t \in \mathbb{R} \mapsto g^t(x) \in \gamma$ is then an affine parametrization of an open subset of γ.

(3) *Let $\varphi\colon I \to J$ be an affine transformation between two non-empty open intervals of γ, which is a translation if γ is definite. Then there exists a unique $g \in \mathbf{G}$ such that $g|_I = \varphi$.*

Proof. (1) For $\mathbf{X} = \mathbf{dS}^2$ we can work with the hyperboloid model dS^2. The stabilizer of a plane $P \subset \mathbb{R}^{1,2}$ is also the one of its orthogonal for $q_{1,2}$, which is respectively spacelike, timelike and lightlike in the three above cases. Straightforward computations show then that these stabilizers are of the announced form and act transitively (observe that $\mathrm{Stab}_{\mathrm{SO}^0(1,2)}(\gamma)$ preserves each connected component of $P \cap \mathrm{dS}^2$).
(2) This fact follows easily from the identification of $\mathbf{X}$ with the homogeneous space $\mathbf{G}/A$.
(3) The action of $\mathrm{Stab}_{\mathbf{G}}(\gamma)$ defines a subgroup of affine transformations of γ, which is according to (1) a one-dimensional subgroup of translations in the definite case, and a two-dimensional subgroup in the lightlike case. This observation shows that the announced affine transformations of γ are indeed induced by elements of $\mathbf{G}$, which proves the existence.

For $x = (p, q) \in \mathbf{dS}^2$, let denote $x^{\mathrm{opp}} := (q, p) \in \mathbf{dS}^2$.

Fact 8.2. *Let $x \neq y \in \mathbf{X}$ such that $y \neq x^{\mathrm{opp}}$ if $\mathbf{X} = \mathbf{dS}^2$, and $g_1, g_2 \in \mathbf{G}$ such that: $g_1(x) = g_2(x)$ and $g_1(y) = g_2(y)$. Then $g_1 = g_2$.*

Proof. This claim follows from the straightforward observation that with $A = \mathrm{Stab}_{\mathbf{G}}(\mathsf{o})$ and $x \neq o$, $x \neq o^{\mathrm{opp}}$ if $\mathbf{X} = \mathbf{dS}^2$: $a \in A \mapsto a(x)$ is injective. □

Fact 8.2 shows the uniqueness, which concludes the proof of the lemma. □

8.1.2. *Affine structures of lightlike leaves in singular* **X***-surfaces.* Any timelike or spacelike geodesic avoiding the singularities of a singular $\mathbf{X}$-surface has a natural translation structure, given by the future-oriented and unit speed parametrizations. In the other hand, while the lightlike leaves of a $\mathbf{X}$-structure have a natural affine structure, one can wonder wether a lightlike leaf F of a singular $\mathbf{X}$-surface (S, Σ) has a well-defined affine structure, extending the one of each connected component of $F \setminus \Sigma$. It turns out that the affine structure of $\mathcal{F}_\alpha(\mathsf{o}_\theta) \setminus \{\mathsf{o}_\theta\}$ in the standard cone $\mathbf{X}_\theta$ has two natural extensions to the whole α-lightlike leaf $\mathcal{F}_\alpha(\mathsf{o}_\theta)$:

(1) an *upper affine structure*, for which the map $\pi_\theta \circ (\mathrm{id} \cup \iota_+)\colon \mathcal{F}_\alpha(\mathsf{o}) \to \mathcal{F}_\alpha(\mathsf{o}_\theta)$ is declared to be an affine map at o_θ;
(2) and a *lower affine structure*, for which $\pi_\theta \circ (\mathrm{id} \cup \iota_-)\colon \mathcal{F}_\alpha(\mathsf{o}) \to \mathcal{F}_\alpha(\mathsf{o}_\theta)$ is an affine map.

Note that while these two charts are compatible with the affine structure of each connected component of $\mathcal{F}_\alpha(\mathsf{o}_\theta) \setminus \{\mathsf{o}_\theta\}$, they are *not* compatible with one another. Indeed the transition map between them is the identity on the left interval but is the restriction of an homothety on the right one, and such a map is not affine.

Definition 8.3. The affine structure of any α-lightlike leaf in a singular $\mathbf{X}$-surface is defined as the one given by the previous lower affine structure (2) in any chart of the singular $\mathbf{X}$-atlas.

Note that this definition makes sense since singular $\mathbf{X}$-surfaces are oriented, and lightlike leaves admit thus two-sided neighbourhoods. It is moreover compatible with the affine structure of lightlike leaves away from singularities.

8.1.3. *Affine structures of closed geodesics.* The easiest example of affine circle is given by the natural translation structure of $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$. For any $\mu \in \mathbb{R}_+^*$, $\mathbb{R}_+^*/\langle \mu\,\mathrm{id}\rangle$ gives in the other hand an example of affine circle which is not induced by a translation structure. Those two types of affine circles are in fact the only ones.

Lemma 8.4. *An affine circle C is either isomorphic to $\mathbb{R}/\mathbb{Z}$, or to $\mathbb{R}_+^*/\langle \mu\,\mathrm{id}\rangle$ for some $\mu \in \mathbb{R}_+^*$. Moreover:*

- *the affine automorphisms of $\mathbb{R}/\mathbb{Z}$ are the translations;*
- *the affine automorphisms of $\mathbb{R}_+^*/\langle \mu\,\mathrm{id}\rangle$ are induced by homotheties $\lambda\,\mathrm{id}$, $\lambda \in \mathbb{R}_+^*$.*

In both cases $\mathrm{ev}_x\colon \varphi \in \mathrm{Aff}^+(C) \mapsto \varphi(x) \in C$ is a homeomorphism for any $x \in C$, and we endow the circle $\mathrm{Aff}^+(C)$ with the orientation induced by C through any of the identifications ev_x.

Proof. With E the universal cover of C and γ a generator of its covering automorphism group, an affine structure on C is determined by a pair (δ, g), with $g = \lambda \,\mathrm{id} + u \in \mathrm{Aff}^+(\mathbb{R})$ and $\delta \colon E \to \mathbb{R}$ an orientation-preserving local homeomorphism such that $\delta \circ \gamma = g \circ \delta$. In particular δ is globally injective, and g has thus no fix point on the g-invariant interval $I = \delta(E)$. Up to the action of $\mathrm{Aff}^+(\mathbb{R})$, we can assume that I is either $\mathbb{R}$ or $\mathbb{R}_+^*$. In the first case $\lambda \neq 1$ would imply that $g = \lambda \,\mathrm{id} + u$ has a fixed point on $\mathbb{R}$, hence $\lambda = 1$ and g is a translation. The latter can moreover be assumed to be $\mathrm{id} + 1$ up to conjugation by $\mathrm{Aff}^+(\mathbb{R})$, proving that C is isomorphic to $\mathbb{R}/\mathbb{Z}$. In the second case, the fact that $g = \lambda \,\mathrm{id} + u$ preserves $\mathbb{R}_+^*$ shows that $u = 0$, hence that C is isomorphic to some $\mathbb{R}_+^*/\langle \mu \,\mathrm{id} \rangle$, which proves the first claim.

The second claim of the lemma follows from the fact that affine automorphisms of C are induced by the affine automorphisms of $\delta(E)$ that normalize the holonomy group $\langle g \rangle$.

The last claim follows then from a direct observation. □

Closed timelike and spacelike geodesics in singular $\mathbf{X}$-surfaces which avoid the singularities have a translation structure and are thus isomorphic to $\mathbb{R}/\mathbb{Z}$. In the other hand, it is easy to check that the closed lightlike geodesics passing through the singular point of the singular $\mathbf{dS}^2$-tori $\mathcal{T}_{\theta,x}$ introduced in Proposition 4.8 are isomorphic to some affine circle $\mathbb{R}_+^*/\langle \mu \,\mathrm{id} \rangle$.

8.2. **Construction of the surgeries.** In this subsection we introduce a useful notion of surgery for singular $\mathbf{X}$-surfaces, $(\mathbf{G}, \mathbf{X})$ denoting as before the pair $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ or $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1), \mathbb{R}^{1,1})$. If it is well-defined, then we denote by

$$P^{\gamma}_{\alpha/\beta} \colon \gamma \to \gamma$$

the first-return map of the lightlike foliation $\mathcal{F}_{\alpha/\beta}$ on a simple closed geodesic γ. It is characterized by the fact that for any $x \in \gamma$, $P^{\gamma}_{\alpha/\beta}(x)$ is the first intersection point of $\mathcal{F}_{\alpha/\beta}(x)$ with γ starting from x (for the orientation of $\mathcal{F}_{\alpha/\beta}$).

The topology of the space $\mathcal{S}(S, \Sigma, \Theta)$ of singular $\mathbf{X}$-structures on a torus S with singular points Σ and angles Θ was introduced in Definition 6.1, and we use the notations of this definition. We endow this space with a distance d defined as follows. Let $(\varphi_i \colon U_i \to X_i)_i$ be a finite singular $\mathbf{dS}^2$-atlas of $\mu \in \mathcal{S}(S, \Sigma, \Theta)$ (where $X_i = \mathbf{dS}^2$ if φ_i is a regular chart and $X_i = \mathbf{dS}^2_{\theta_i}$ at a singular point of angle θ_i) and $\mathcal{U}' = (U_i')_i$ be a shrinking of $(U_i)_i$ as in Definition 6.1. Then with d_i a fixed distance on X_i and $d_i^{\infty}(f, g) = \max\limits_{x \in U_i} d_i(f(x), g(x))$ the associated uniform distance on continuous maps from U_i' to X_i, for any $\mu' \in \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$ defined by a singular $\mathbf{dS}^2$-atlas $\mathcal{A}' = (\psi_i \colon U_i' \to X_i)_i$, we define:

$$d(\mu', \mu) = \min \left(1, \inf \left\{ \max_i \, d_i^{\infty}(\varphi_i|_{U_i'}, \psi_i) \;\middle|\; \mathcal{A}' \text{ atlas for } \mu' \text{ defined on } \mathcal{U}' \right\} \right). \tag{8.1}$$

Proposition 8.5. *Let (S, Σ, μ) be a closed singular $\mathbf{X}$-surface of angles Θ, and let $\gamma \subset S$ be a simple closed curve, which is either a definite geodesic avoiding the singular set or a lightlike leaf. Then for any surjective, continuous and orientation-preserving map $u \in [0\,;1] \mapsto T_u \in \mathrm{Aff}^+(\gamma)$, which is injective on $[0\,;1[$ and such that $T_0 = \mathrm{id}_{\gamma} = T_1$, there exists a continuous family*

$$u \in [0\,;1] \mapsto [\mu_{T_u}] \in \mathsf{Def}_{\Theta}(S, \Sigma)$$

of surgeries of μ around γ with respect to T_u, *satisfying the following conditions.*

1. *$[\mu_{\mathrm{id}_{\gamma}}] = [\mu]$, and $[\mu_{T_1}] = (D_{[\gamma]})_*[\mu]$ with D_{γ} the positive Dehn twist around γ.*
2. *There exists a continuous lift $u \in [0\,;1] \mapsto \mu_{T_u} \in \mathcal{S}(S, \Sigma, \Theta)$ of $[\mu_{T_u}]$.*
3. *For any $T \in \mathrm{Aff}^+(\gamma)$, μ_T can be chosen to coincide with μ outside of a tubular neighbourhood of γ as small as one wants.*
4. *γ remains a simple closed geodesic of μ_T with the same signature and affine structure.*
5. *If the first-return map $P^{\gamma}_{\alpha,\mu} \colon \gamma \to \gamma$ of the α-foliation of μ is well-defined on γ, then the first-return map of $\mathcal{F}_{\alpha}^{\mu_T}$ is also well-defined on γ and is equal to*

$$P^{\gamma}_{\alpha,\mu_T} = P^{\gamma}_{\alpha,\mu} \circ T.$$

Alternatively if $P^{\gamma}_{\beta,\mu}\colon \gamma \to \gamma$ is well-defined then P^{γ}_{β,μ_T} is well-defined as well, and the surgery can be chosen to satisfy

$$P^{\gamma}_{\beta,\mu_T} = P^{\gamma}_{\beta,\mu} \circ T.$$

(6) *Assume that γ is a timelike geodesic. Then there exists a constant $C > 0$, such that for any surgery μ_T of μ around γ having a closed lightlike leaf $\mathcal{F}$ and for any affine transformation $U \in \mathrm{Aff}^+(\mathcal{F})$, the surgery $(\mu_T)_U$ of μ_T around $\mathcal{F}$ with respect to U satisfies*

$$d(\mu_T, (\mu_T)_U) \leq C \max_{x\in\mathcal{F}} L([x\,;U(x)]_{\mathcal{F}}). \tag{8.2}$$

In the previous inequality, $L([x\,;y]_\gamma)$ denotes the length of the segment $[x\,;y]_\gamma$ of γ from x to y, with respect to a fixed Riemannian metric on S.

Proof of Proposition 8.5. Without loss of generality, we can assume that γ is a timelike geodesic (avoiding the singular set) or a lightlike leaf, up to replacing the Lorentzian metric by its opposite. We endow γ with its future orientation, and fix a parametrization $u \in [0\,;1] \mapsto T_u \in \mathrm{Aff}^+(\gamma)$ of its group of affine automorphisms satisfying the statement.

(a) Unmarked surgeries. The first step is to construct the most intuitive notion of surgery that one could imagine, with respect to some affine automorphism $T \in \mathrm{Aff}^+(\gamma)$. Let S_* denote the annulus with boundary obtained by cutting S along the simple closed curve γ. We denote by $\iota\colon S \setminus \gamma \to \mathrm{Int}(S_*)$ the natural identification of the interior of S_* with $S \setminus \gamma$, and endow S_* with the orientation induced by S. We also denote by $\iota_\pm\colon \gamma \to \gamma_\pm$ the natural identifications of γ with the two boundary components $\gamma_\pm$ of S_*, where γ_+ is the "left" boundary component and γ_- the "right" one, when γ is oriented upwards. More precisely with γ'_- the derivative of γ_- and $\gamma'^{\perp}_-$ its normal exterior to S_*, we assume that $(\gamma'_-, \gamma'^{\perp}_-)$ defines the positive orientation of S_*. We can now introduce the equivalence relation generated by the relations $\iota_+(x) \sim_T \iota_-(T(x))$ for any $x \in \gamma$, and the associated identification space

$$\pi_T\colon S_* \to S_T := S_*/\sim_T\,.$$

With $\bar\iota_+ := \pi_T \circ \iota_+\colon \gamma \to \gamma_T := \pi_T \circ \iota_+(\gamma)$ and $\bar\iota := \pi_T \circ \iota\colon S \setminus \gamma \to S_T \setminus \gamma_T$, we endow $S_T \setminus \gamma_T$ with the unique singular $\mathbf{X}$-structure for which $\bar\iota$ is an isometry. Now for any $x \in \gamma \setminus \Sigma$, there exists two $\mathbf{X}$-charts $\varphi\colon U \to \mathbf{X}$ and $\psi\colon V \to \mathbf{X}$ with $x \in U$ and $T(x) \in V$, such that $U \cap \gamma$ and $V \cap \gamma$ are connected, and such that $U \setminus \gamma$ and $V \setminus \gamma$ have two left and right connected components $U_\pm$ and $V_\pm$ (denoted in a way compatible with our notations for the boundary components $\gamma_\pm$ of S_*). According to Lemma 8.1, we can moreover assume that $T(U \cap \gamma) = V \cap \gamma$ and that $\varphi|_{U\cap\gamma} = \psi \circ T|_{U\cap\gamma}$, possibly post-composing ψ by the suitable element of $\mathbf{G}$. Note that this is possible since T is a translation if γ is a timelike geodesic, according to Lemma 8.4. Then $W = \bar\iota(U_+) \cup \bar\iota_+(\gamma \cap U) \cup \bar\iota(V_-)$ is a neighbourhood of $\bar\iota_+(x)$ in S_T, and the map $\phi\colon W \to \mathbf{X}$ defined by

$$\begin{cases} \phi \circ \bar\iota|_{U_+} & = \varphi|_{U_+} \\ \phi \circ \bar\iota_+|_{\gamma\cap U} & = \varphi|_{\gamma\cap U} \\ \phi \circ \bar\iota|_{V_-} & = \psi|_{V_-} \end{cases} \tag{8.3}$$

is a homeomorphism onto its image. The transition maps of ϕ with every chart of the $\mathbf{X}$-atlas of $S_T \setminus \gamma_T$ having values in $\mathbf{G}$, we can define a singular $\mathbf{X}$-atlas on $S_T \setminus \bar\iota_+(\gamma \cap \Sigma)$ which extends the one of $S_T \setminus \gamma_T$, by declaring all maps ϕ defined as in (8.3) as $\mathbf{X}$-charts. Moreover, any chart at $\bar\iota_+(x)$ which is compatible with the $\mathbf{X}$-atlas of $S_T \setminus \gamma_T$ must coincide with such a chart ϕ on the left and right sides $\bar\iota(U_+) \cup \bar\iota(V_-)$ of its domain, hence must coincide with ϕ by continuity. In conclusion, the singular $\mathbf{X}$-structure μ^0_T of $S_T \setminus \bar\iota_+(\gamma \cap \Sigma)$ that we defined is the only one which extends the singular $\mathbf{X}$-structure of $S_T \setminus \gamma_T$, and is in particular well-defined. If γ is a timelike geodesic avoiding the singularities, then $S_T \setminus \bar\iota_+(\gamma \cap \Sigma) = S_T$ and the construction is finished.

If γ is a closed lightlike leaf of μ, we have to check that the singular $\mathbf{X}$-structure μ^0_T of $S_T \setminus \bar\iota_+(\gamma\cap\Sigma)$ is indeed a singular $\mathbf{X}$-structure of S_T, and that it has in addition the same singularities and angles than μ on $\gamma \cap \Sigma$. We can assume without loss of generality that γ is an α-lightlike closed leaf. Let $\varphi\colon U \to \mathbf{X}_\theta$ be a singular $\mathbf{X}$-chart at a singularity $x \in \gamma$ of angle θ, and with

$y := T(x)$ and θ' the angle at y, let $\psi\colon V \to \mathbf{X}_{\theta'}$ be a chart of the singular $\mathbf{X}$-atlas of μ at y. As before, we assume that $U \cap \gamma$ and $V \cap \gamma$ are connected, that $U \setminus \gamma$ and $V \setminus \gamma$ have two left and right connected components $U_\pm$ and $V_\pm$[8], and we also assume that $(U \cap \gamma) \setminus \{x\}$ has two past and future connected components $(U \cap \gamma)_-$ and $(U \cap \gamma)_+$. We denote by $\varphi_\pm\colon \mathrm{Cl}(U_\pm) \to \mathbf{X}$ the maps such that

$$\begin{cases} \pi_\theta \circ \varphi_+|_{U_+} & = \varphi|_{U_+} \\ \pi_\theta \circ \iota_\pm \circ \varphi_\pm|_{\gamma\cap U} & = \varphi|_{\gamma\cap U} \\ \pi_\theta \circ \varphi_-|_{U_-} & = \varphi|_{U_-}, \end{cases}$$

and adopt the analog notations for $\psi_\pm\colon \mathrm{Cl}(V_\pm) \to \mathbf{X}$. Here, we use the notations of Paragraph 3.1.1 concerning the definition of standard singularities. Note that by definition of $\mathbf{X}_\theta$, we have

$$\varphi_-|_{(\gamma\cap U)_-} = \varphi_+|_{(\gamma\cap U)_-} \text{ and } \varphi_-|_{(\gamma\cap U)_+} = a^\theta \circ \varphi_+|_{(\gamma\cap U)_+}. \tag{8.4}$$

Now since $\varphi_-|_{\gamma\cap U}$ and $\psi_-|_{\gamma\cap V}$ are affine according to our Definition 8.3 of the affine structures on lightlike leaves, we can assume according to Lemma 8.1 that $T(\gamma \cap U) = \gamma \cap V$, and that $\varphi_-|_{\gamma\cap U} = \psi_- \circ T|_{\gamma\cap U}$. According to (8.4), we have thus

$$\psi_- \circ T|_{(\gamma\cap U)_-} = \varphi_+|_{(\gamma\cap U)_-} \text{ and } \psi_- \circ T|_{(\gamma\cap U)_+} = a^\theta \circ \varphi_+|_{(\gamma\cap U)_+}. \tag{8.5}$$

With $W_* := U_+ \cup \iota_+(\gamma \cap U) \cup \iota_-(\gamma \cap V) \cup V_- \subset S_*$, let consider the map $\Phi\colon W_* \to \mathbf{X}_*$ defined by

$$\begin{cases} \Phi \circ \iota|_{U_+} & = \varphi_+|_{U_+} \\ \Phi \circ \iota_+|_{\gamma\cap U} & = \iota_+ \circ \varphi_+|_{\gamma\cap U} \\ \Phi \circ \iota_-|_{\gamma\cap V} & = \iota_- \circ \psi_-|_{\gamma\cap V} \\ \Phi \circ \iota|_{V_-} & = \psi_-|_{V_-}. \end{cases}$$

According to (8.5), we have $\Phi(\iota_+(p)) = \Phi(\iota_-(T(p)))$ for any $p \in (\gamma \cap U)_-$, and $\Phi(\iota_+(p)) \sim_\theta \Phi(\iota_-(T(p)))$ for any $p \in (\gamma \cap U)_+$. Therefore, Φ induces a map $\phi\colon W \to \mathbf{X}_\theta$, defined on the neighbourhood $W := \pi_T(W_*)$ of $\bar{x} := \bar{\iota}_+(x)$ in S_T and characterized by $\phi \circ \pi_T = \pi_\theta \circ \Phi$, which is a homeomorphism onto its image and such that $\phi(\bar{x}) = \mathsf{o}_\theta$. Moreover $\phi|_{W\setminus\{\bar{x}\}}$ is a $\mathbf{X}$-morphism since $\varphi_\pm$ and $\psi_\pm$ are $\mathbf{X}$-charts. This proves that $\bar{x}$ is a singularity of angle θ of μ^0_T and concludes our construction.

We emphasize that γ_T remains a geodesic of μ^0_T with the same signature than γ, and that $\bar{\iota}_+$ is by construction an affine isomorphism between γ and γ_T.

(b) Marking the surgeries. The only drawback of this intuitive construction, is that we actually constructed a family (S_{T_u}, μ_{T_u}) of singular $\mathbf{X}$-tori and not a family of structures defined on the same initial surface S. To this end, we now pullback these structures on S thanks to a prescribed family of homeomorphisms. We first choose a one-sided neighbourhood K of γ on the right, which we henceforth implicitly identify topologically with $[0\,;1] \times \gamma$ in such a way that $K \cap \mathcal{F}_\alpha(x) = [0\,;1] \times \{x\}$ for any $x \in \gamma$. We can then define a homeomorphism $f_u\colon S_{T_u} \to S$ by

$$f_u(s, x) := (s, T^{-1}_{(1-s)u}(x))$$

for any $(s,x) \in [0\,;1] \times \gamma \equiv K$, and $f_u \circ \bar{\iota}|_{S\setminus K} = \mathrm{id}|_{S\setminus K}$. The map $u \in [0\,;1] \mapsto f_u \in \mathrm{Homeo}^+(S_{T_u}, S)$ obviously satisfies the following properties:

$$\begin{cases} u \mapsto f_u \circ \bar{\iota}|_{S\setminus\gamma} & \text{is continuous} \\ f_u \circ \bar{\iota}|_{S\setminus K} & = \mathrm{id}|_{S\setminus K} \\ \max\limits_{x\in S} d_S(f_u \circ \bar{\iota}(x), x) & \leq \max\limits_{x\in\gamma} L([x\,;T_u(x)]_\gamma), \end{cases} \tag{8.6}$$

with d_S the distance induced on S by a fixed Riemannian metric and $L([x\,;y]_\gamma)$ the length of an interval $[x\,;y]_\gamma$ of γ for this Riemannian metric. We can now define $\mu_{T_u} := (f_u)_* \mu^0_{T_u} \in \mathcal{S}(S, \Sigma, \Theta)$,

[8] Note that for the convenience of the reader, our current conventions are compatible with the ones of the definition of standard singularities in Paragraph 3.1.1.

so that the map $u \in [0\,;1] \mapsto \mu_{T_u}$ satisfies the properties (2) and (3) of the statement.[9] We proved in Paragraph **(a)** of the proof that $\bar{\iota}_+$ is an affine isomorphism, showing that γ remains a geodesic of μ_{T_u} with the same affine structure than μ, *i.e.* that μ_{T_u} satisfies the property (4) of the statement. We also proved in Paragraph **(a)** that μ_{T_u} has the same singularities and angles than μ. The relations $[\mu_{\mathrm{id}_\gamma}] = \mu$ and $[\mu_{T_1}] = D^\gamma_*[\mu]$ being direct consequences of the definition of $[\mu_{T_u}]$, we have proved the properties (1) to (4) of the statement.

(c) First-return maps of lightlike foliations in the surgeries. We described here the construction for the α-foliation, and in the case where γ is either a simple closed timelike geodesic or a closed β-leaf. Note that in both of these cases the leaves of $\mathcal{F}_\alpha$ leave γ "from the right" (namely from the copy $\gamma_- \subset S_*$), while the leaves of $\mathcal{F}_\beta$ leave γ "from the left" when γ is a spacelike or α-lightlike closed geodesic. For this reason, the latter cases are formally identical, but the appropriate orientation modifications have to be made in the definition of the marked surgeries $[\mu_{T_u}]$ at the step **(b)**.

With $H_1^{\mu/\mu_{T_u}} \colon \{0\} \times \gamma \to \{1\} \times \gamma$ the respective holonomies of $\mathcal{F}^\mu_\alpha$ and $\mathcal{F}^{\mu_{T_u}}_\alpha$ from the left to the right boundary components of K, we observe that $H_1^{\mu_{T_u}} = H_1^\mu \circ T_u$ by definition of f_u. Since $f_u \circ \bar{\iota}|_{S\setminus K} = \mathrm{id}|_{S\setminus K}$, the holonomies H_2 of the α-foliations from the right boundary component $\{1\}\times\gamma$ of K to γ satisfy in the other hand $H_2^{\mu_{T_u}} = H_2^\mu$. The first-return maps $P^\gamma_\alpha = H_2 \circ H_1$ satisfy thus the expected relation $P^\gamma_{\alpha,\mu_{T_u}} = P^\gamma_{\alpha,\mu} \circ T_u$, which proves the property (5) of the statement.

(d) Bounding the size of the surgeries. We lastly prove the estimate (8.2) on the surgery ν_U of $\nu \coloneqq \mu_T$ around a closed lightlike leaf $\mathcal{F}$. By construction ν_U coincides with ν outside of the one-sided neighbourhood K. Denoting by f the homeomorphism described in (8.6), we have to prove that $d(\nu|_K, \nu_U|_K) \leq C \max_{x\in\mathcal{F}} L([x\,;U(x)]_\mathcal{F})$ for some constant $C > 0$. It is sufficient to prove this claim for any small enough surgery ν_U of ν, since the inequality follows then for further surgeries by triangular inequality. With $(\varphi_i \colon U_i \to X_i)_i$ a finite singular $\mathbf{X}$-atlas of ν and $(U'_i)_i$ a shrinking of $(U_i)_i$ as in (8.1), we can thus assume that $f(U'_i) \subset U_i$. Note that $\varphi_i \circ (f \circ \bar{\iota})^{-1}$ is a singular $\mathbf{X}$-atlas of ν_U. By finiteness of the atlas and continuity of the φ_i's, there exists a constant $C > 0$ such that $d^\infty_i(\varphi_i|_{U'_i}, \varphi_i \circ (f\circ\bar{\iota})^{-1}|_{U'_i}) \leq C d^\infty_S(\mathrm{id}|_{U'_i}, f\circ\bar{\iota}|_{U'_i})$ for any i and f, and therefore $d(\nu|_K, \nu_U|_K) \leq C d^\infty_S(\mathrm{id}_S, f\circ\bar{\iota})$. Since f satisfies $d^\infty_S(\mathrm{id}_S, f\circ\bar{\iota}) \leq \max_{x\in\gamma} L([x\,;T_u(x)]_\gamma)$ according to (8.6), we obtain $d(\nu, \nu_U) \leq C \max_{x\in\gamma} L([x\,;T_u(x)]_\gamma)$ as expected, which proves property (6) and concludes the proof of the proposition. □

9. Local and global topology of the deformation space

9.1. Realization of singular $\mathbf{dS}^2$-tori by L-shaped polygons.

In what follows, all the graphs are assumed to be finite.

Definition 9.1. A graph C embedded in a singular $\mathbf{dS}^2$-surface S is said *lightlike* if any vertex of C has degree at least 2, and any edge is a connected subset of a lightlike leaf. It is *L-shaped* if:

1. $S \setminus C$ is a topological disk.
2. any singularity of S is a vertex of C,
3. C has at most 3 vertices, and the oriented boundary of the surface $S \setminus\!\setminus C$ obtained from cutting S along C is a lightlike *L-shaped polygon* as illustrated in Figure 4.2.[10]

A *rectangular* graph is a specific sort of L-shaped lightlike graph satisfying the above conditions (1) and (2), having at most two vertices, and such that the oriented boundary of $S \setminus\!\setminus C$ is a lightlike rectangle as illustrated in Figure 4.1.[11] Note that the vertices addressed here are the

[9]Let a be a simple closed curve in S based at a point $o \in \gamma$, and such that $a \cap \gamma = o$ and the basis $([a], [\gamma])$ of $\pi_1(S)$ is positively oriented. Composing a with the past-oriented segment of γ from o to $T_{-u}(o)$ defines a simple closed curve in S_{T_u}. One can then observe that the isotopy class of f_u relative to Σ is characterized as the homeomorphisms $f \colon S_{T_u} \to S$ so that $f_*([a_u], [\gamma_{T_u}]) = ([a], [\gamma])$, and that this relation therefore uniquely characterizes the point $[\mu_{T_u}] \in \mathsf{Def}_\Theta(S, \Sigma)$ in the deformation space.

[10]Namely the successive union of a positive α-segment, a positive β-segment, a negative α-segment, a positive β-segment, a negative α-segment, and a negative β-segment.

[11]Namely the successive union of a positive α-segment, a positive β-segment, a negative α-segment and a negative β-segment.

ones of the graph in the identification space S, and not the ones of the rectangle. A L-shaped (respectively rectangular) lightlike graph in a singular $\mathbf{dS}^2$-torus S induces a marking (a,b) of $\pi_1(S)$, which is defined in the same way than the markings introduced in Paragraph 6.2.

We use in the following proposition the notations introduced in Definitions 7.8 and 7.12 for the one and two-parameter families $\mathcal{R}^{\alpha/\beta}_{\theta,\mathsf{a},\mathsf{b}}$ and $\mathcal{L}_{\theta,\mathsf{a},\mathsf{b}}$.

Proposition 9.2. *Let $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2, 0)$ admit a rectangular (respectively L-shaped) lightlike graph of induced marking (a,b). Then $\mu \in \mathcal{R}^{\alpha}_{\theta,\mathsf{a},\mathsf{b}}$ or $\mu \in \mathcal{R}^{\beta}_{\theta,\mathsf{a},\mathsf{b}}$ depending if the α-leaf or the β-leaf of the singularity is closed (respectively $\mu \in \mathcal{L}_{\theta,\mathsf{a},\mathsf{b}}$).*

Proof. Let $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2, 0)$ admit a L-shaped lightlike graph $\bar{C}$. An easy adaptation of the proof shows the claim in the case of a rectangular graph. We endow $\mathbb{R}^2$ with the $\mathbb{Z}^2$-invariant singular $\mathbf{dS}^2$-structure $\tilde{\mu}$ for which the universal covering $\pi\colon \mathbb{R}^2 \to \mathbf{T}^2$ is a local isometry, and denote by $\tilde{C} = \pi^{-1}(\bar{C})$ the lift of $\bar{C}$. This is an embedded graph in $\mathbb{R}^2$ satisfying properties (2) and (3) of Definition 9.1 for $S = \mathbb{R}^2$, and such that each connected component of $\mathbb{R}^2 \setminus \tilde{C}$ is a topological disk. We denote by E the closure of one of these connected components, and by C the subgraph of $\tilde{C}$ which is the boundary of E. Then E is a fundamental domain for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$, and $(\mathbf{T}^2, \mu)$ is thus isometric to the quotient of E by the identifications of the edges of C by suitable elements of $\mathbb{Z}^2$. Note that any edge of $\bar{C}$ has two lifts in C, hence C has an even number of edges.

(a) Injectivity of the developing map on a fundamental domain. Since the singularities $\bar{\Sigma}$ of μ are by assumption contained in $\bar{C}$, the singularities $\tilde{\Sigma} = \pi^{-1}(\bar{\Sigma})$ of $\tilde{\mu}$ are contained in $\tilde{C}$, and with $\Sigma = \tilde{\Sigma} \cap C$, we have $\pi(\Sigma) = \bar{\Sigma}$. In particular $E^* := E \setminus \Sigma$ is contained in $\mathbb{R}^2 \setminus \tilde{\Sigma}$, and with U a simply connected open neighbourhood of E^* contained in $\mathbb{R}^2 \setminus \tilde{\Sigma}$, there exists a $\mathbf{dS}^2$-morphism

$$\delta\colon U \to \mathbf{dS}^2,$$

which is the developing map of the $\mathbf{dS}^2$-structure of U. Note that U is a topological disk, as is any connected and simply connected open subset of the plane.

Fact 9.3. *The developing map δ extends to a continuous map D from a neighbourhood $\mathcal{U}$ of E to $\mathbf{dS}^2$. There exists moreover a lightlike L-shaped polygon E_0 in $\mathbf{dS}^2$, a decomposition of the boundary of E_0 into a graph C_0 whose edges are segments of lightlike leaves, and a subset Σ_0 of the vertices of C_0, such that:*

(1) $D(E) \subset E_0$,
(2) $D(\Sigma) = \Sigma_0$ and D is a graph morphism from C to C_0,
(3) D is injective in restriction to C.

Proof. By assumption, any vertex of $\tilde{C}$ has degree at least 2, and since any edge is a segment of lightlike leave, the vertices also have degree at most 4 inside $\tilde{C}$ (in the maximal case, segments of the four lightlike half-leaves emanate from a vertex). But C being the boundary of E hence a topological circle, any vertex of C has of course degree exactly 2 inside C. Now we endow the circle $C = \partial E$ with the orientation induced by the one of E, fix $v \in \Sigma$ a singular vertex of C, and denote by e_-, e_+ the two (closed) edges of C of extremity v ($e_- \neq e_+$ since v has degree 2), e_+ being met after e_- in the positive orientation of C. Up to a cyclic permutation of the quadrants, the three following situations are the only one that can arise.

(1) e_- is a segment of the α-leaf of v denoted by $[x_-\,; v]_\alpha$, going from x_- to v for the positive orientation of C. Similarly, e_+ is a segment of the β-leaf of v of the form $[v\,; x_+]_\beta$. Moreover, v admits an open neighbourhood $Q_v \subset E^* \cup \{v\}$ in E which is a small timelike future quadrant, and such that $Q_v \cap \Sigma = \{v\}$.
(2) e_- is an α-segment $[x_-\,; v]_\alpha$, e_+ an α-segment $[v\,; x_+]_\alpha$, and v admits an open neighbourhood $Q_v \subset E^* \cup \{v\}$ in E which is the union of a small timelike future quadrant and of a small future spacelike quadrant.
(3) e_- is an α-segment $[x_-\,; v]_\alpha$, e_+ a β-segment $[x_+\,; v]_\beta$, and v admits an open neighbourhood $Q_v \subset E^* \cup \{v\}$ in E which is the union of a small timelike future quadrant, a small future spacelike quadrant and a small past timelike quadrant.

Note that the segments $e_\pm$ are endowed with two orientations, respectively induced by the one of $C = \partial E$ and by the lightlike foliations. These two orientations coincide for $[x_- ; v]_\alpha$ in the three above cases and for $[v ; x_+]_\beta$ and $[v ; x_+]_\alpha$ in cases (1) and (2), but they are opposite for $[x_+ ; v]_\beta$ in case (3).

Since v is a standard singularity, denoting by $Q_\mathsf{o} \subset \mathbf{dS}^2$ the union of quadrants at o corresponding to Q_v, $Q_v^* := Q_v \setminus \{v\}$ is isometric to $Q_\mathsf{o}^* := Q_\mathsf{o} \setminus \{\mathsf{o}\}$. Namely, there exists an isometry φ from a neighbourhood $V \subset U$ of Q_v^* in $\mathbb{R}^2$ to a neighbourhood V_0 of Q_o^* in $\mathbf{dS}^2$, such that $\varphi(Q_v^*) = Q_0^*$ (see Lemma 3.5). Since $\delta|_V$ is another $\mathbf{dS}^2$-morphism from V to $\mathbf{dS}^2$, there exists moreover $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $\delta|_V = g \circ \varphi$. Hence $\delta(Q_v^*) = g(Q_0^*) = Q_{v_0}^*$, with Q_{v_0} the union of quadrants at $v_0 := g(\mathsf{o})$ corresponding to Q_v. In particular, this shows that $\delta|_V$ extends to an injective continuous map D_v from a neighbourhood $W \subset \mathbb{R}^2$ of Q_v to a neighbourhood $W_0 \subset \mathbf{dS}^2$ of Q_{v_0}, sending v to v_0.

We can now glue together these maps D_v, to define a map D from a neighbourhood $\mathcal{U}$ of E to $\mathbf{dS}^2$. Since δ is a local diffeomorphism, it is injective in restriction to any open edge of C, and D is thus injective in restriction to any closed edge since the lightlike leaves of $\mathbf{dS}^2$ are embeddings of $\mathbb{R}$. By construction, $C_0 := D(C)$ is a lightlike L-shaped closed loop in $\mathbf{dS}^2$, and we define a decomposition of C_0 by stating that D is a graph morphism (which makes sense since D is injective in restriction to any edge). A simple but important observation is now that any lightlike L-shaped closed loop in $\mathbf{dS}^2$ is simple, *i.e.* without any self-intersection. Since E is moreover always on the same side of C by definition of its orientation (namely on the left), $D(E)$ is always on the same side of C_0, hence $D(E)$ is contained in the (unique) lightlike L-shaped polygon E_0 of $\mathbf{dS}^2$ bounded by C_0.

We know at this stage that $D|_C$ is a continuous map from the topological circle $C = \partial E$ to the topological circle C_0, which is locally injective hence a local homeomorphism. But since the oriented graph C contains only one positively travelled α-segment, $D|_C$ cannot have degree > 1. Therefore $D|_C$ is injective, which concludes the proof of the fact. $\square$

Now since the continuous map $D|_E \colon E \to E_0$ is locally injective and injective in restriction to ∂E, $D|_E$ is injective according to [MO63, Theorem 1 p.75] (see also Definition 3 p.74 therein). And since δ is a local diffeomorphism, D is actually injective in restriction to a small enough neighbourhood $\mathcal{U} \subset \mathbb{R}^2$ of E, and is thus a homeomorphism from $\mathcal{U}$ to a neighbourhood $\mathcal{U}_0$ of E_0 in $\mathbf{dS}^2$ according to Brouwer's invariance of domain theorem. In particular, $D(E)$ is a compact subset of E_0 of boundary ∂E_0, *i.e.* $D(E) = E_0$.

(b) Edges identifications. Recall that $C = \partial E$ has an even number of edges denoted by $\{(e_i^t, e_i^b)\}_i$, and that $(\mathbf{T}^2, \mu)$ is isometric to the quotient $\mathcal{E}$ of E by the identification of each e_i^t with the corresponding e_i^b through a translation T_{u_i} (where $u_i \in \mathbb{Z}^2$ and $T_{u_i}(e_i^t) = e_i^b$). Since integral translations are isometries of $\tilde{\mu}$, there exists moreover unique elements $g_i \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\delta \circ T_{u_i} = g_i \circ \delta$$

in restriction to a connected neighbourhood of e_i^t. Since D is a graph morphism according to Fact 9.3, we can define a decomposition of C_0 associated to the one of C by $f_i^t = D(e_i^t)$ and $f_i^b = D(e_i^b)$. We have then $g_i(f_i^t) = f_i^b$, and we can thus form the quotient $\mathcal{E}_0$ of E_0 by these edges identifications, given by Proposition 4.3. By construction, D induces then an isometry from $\mathcal{E} \simeq (\mathbf{T}^2, \mu)$ to $\mathcal{E}_0$.

By acting by $\mathrm{PSL}_2(\mathbb{R})$, we can assume without loss of generality that E_0 is a lightlike L-shaped polygon $\mathcal{L}_{\theta,x,y}$ as defined in (4.8). Since μ has a single singularity, Lemma 4.10 shows moreover that the gluing of the edges is the one of $\mathcal{T}_{\theta,x,y}$ defined in Proposition 4.11 and illustrated in Figure 4.2. Therefore, $\mathcal{E}_0 \simeq (\mathbf{T}^2, \mu)$ is isometric to a point of $\mu_\theta(\mathcal{D})$. Likewise if μ was assumed to be rectangular, then we can assume without loss of generality that E_0 is the lightlike rectangle $\mathcal{R}_\theta$. Since μ has a single singularity, Lemma 4.7 shows that the gluing of the edges is the one of $\mathcal{T}_{\theta,x}$ or $\mathcal{T}_{\theta,y,*}$ defined in Proposition 4.8 and Remark 4.9 and illustrated in Figure 4.1, hence that $(\mathbf{T}^2, \mu)$ is isometric to a point of $\mu_\theta([1 ; \infty])$ or $\mu_\theta^*([0 ; y_\theta])$. This concludes the proof of the proposition. $\square$

An important consequence of this proposition is the following.

Lemma 9.4. *The map* $(x,y) \in \mathcal{D} \mapsto \mu_{\theta,x,y} \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$ *is a homeomorphism onto its image. The same claim holds for the map* $(x,y) \in \mathcal{D}^* \mapsto \mu^*_{\theta,x,y} \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$.

Proof. We first show that this map is injective, and consider to this end (x_1,y_1) and (x_2,y_2) in $\mathcal{D}$ such that $\mu_{\theta,x_1,y_1} = \mu_{\theta,x_2,y_2}$ in $\mathsf{Def}_\theta(\mathbf{T}^2,0)$ (the proof is identical for $\mu^*_{\theta,x,y}$). Recall that the marking of $\mu_{\theta,x,y}$ is defined by the respective homotopy classes a and b of the piecewise lightlike simple closed curves $\gamma^a = [\overline{(1,0)}\,;\overline{(\infty,0)}]_\alpha \cdot [\overline{(1,0)}\,;\overline{(1,y')}]^{-1}_\beta$ and $\gamma^b = [\overline{(1,0)}\,;\overline{(x',0)}]^{-1}_\alpha \cdot [\overline{(1,0)}\,;\overline{(1,y_+)}]_\beta$ at $\overline{(1,0)}$. The equality $\mu_{\theta,x_1,y_1} = \mu_{\theta,x_2,y_2}$ is then equivalent to the existence of an isometry ϕ from $\mathcal{T}_{\theta,x_1,y_1}$ to $\mathcal{T}_{\theta,x_2,y_2}$, sending $(a_{x_1,y_1}, b_{x_1,y_1})$ to $(a_{x_2,y_2}, b_{x_2,y_2})$.

Fact 9.5. *Let γ_1 and γ_2 be two homotopic simple closed curves of $\mathcal{T}_{\theta,x,y}$ passing through $\overline{(1,0)}$, and of the form $\gamma_i = \alpha_i\beta_i^{-1}$ with α_i (respectively β_i) a positive α (resp. β) segment starting from $\overline{(1,0)}$. Then $\gamma_1 = \gamma_2$ as non-parametrized curves.*

Proof. Possibly exchanging 1 and 2, we can assume without loss of generality that α_2 is longer than α_1, namely that $\alpha_2 = \alpha_1\alpha_2'$ with α_2' a positive α-segment (possibly *trivial*, *i.e.* reduced to a point).

Case 1: β_1 is longer, *i.e.* $\beta_1 = \beta_2\beta_1'$ with β_1' a positive β-segment. Then $\gamma_1\gamma_2^{-1} = \alpha_1\beta_1'^{-1}\alpha_2'^{-1}\alpha_1^{-1}$ is homotopically trivial, hence $\beta_1'^{-1}\alpha_2'^{-1}$ is also homotopically trivial. Since $\beta_1'^{-1}\alpha_2'^{-1}$ is a past anticausal curve, this contradicts Corollary A.6 unless $\beta_1'^{-1}\alpha_2'^{-1}$ is trivial. Therefore β_1' and α_2' are both trivial, hence $\alpha_2 = \alpha_1$ and $\beta_1 = \beta_2$ which proves the claim in this case.

Case 2: β_2 is longer, *i.e.* $\beta_2 = \beta_1\beta_2'$ with β_2' a positive β-segment. As before, $\gamma_1\gamma_2^{-1} = \alpha_1\beta_2'\alpha_2'^{-1}\alpha_1^{-1}$ and thus $\beta_2'\alpha_2'^{-1}$ are then homotopically trivial. Since $\beta_2'\alpha_2'^{-1}$ is a future causal curve, this forces β_2' and α_2' to be trivial according to Corollary A.6, hence $\beta_2 = \beta_1$ and $\alpha_2 = \alpha_1$ which concludes the proof. □

Since ϕ is an isometry, it sends the unique singularity $\overline{(1,0)}$ of $\mathcal{T}_{\theta,x_1,y_1}$ to the unique singularity $\overline{(1,0)}$ of $\mathcal{T}_{\theta,x_2,y_2}$, and sends any α (respectively β) lightlike segment to an α (resp. β) lightlike segment while preserving its orientation. Since $\phi_*[\gamma_1^a] = [\gamma_2^a]$ and $\phi_*[\gamma_1^b] = [\gamma_2^b]$ in homotopy, Fact 9.5 shows then that $\phi(\gamma_1^a) = \gamma_2^a$ and $\phi(\gamma_1^b) = \gamma_2^b$. Therefore ϕ sends the α (respectively β) segments of γ_1^a and γ_1^b to the corresponding segments of γ_2^a and γ_2^b, and induces thus an isometry from $\mathcal{L}_{\theta,x_1,y_1}$ to $\mathcal{L}_{\theta,x_2,y_2}$. The latter is the restriction of some $g \in \mathrm{PSL}_2(\mathbb{R})$ which preserves $(1,0)$ and $(\infty,0)$, hence $g = \mathrm{id}$, which shows that $(x_1,y_1) = (x_2,y_2)$ and concludes the proof of the injectivity.

The map μ_θ being continuous according to Proposition 6.3, there only remains to show that it is open. Let $(x_0,y_0) \in \mathcal{D}$. Since the lightlike foliations vary continuously with the metric, any small enough deformation μ of μ_{θ,x_0,y_0} induces an arbitrarily small deformation of the L-shaped lightlike graph defined by (x_0,y_0), into a lightlike graph which remains L-shaped and of induced marking (a,b). Therefore, any $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$ sufficiently close to μ_{θ,x_0,y_0} is according to Proposition 9.2 of the form $\mu_{\theta,x,y}$ with $(x,y) \in \mathcal{D}$. Since the holonomy varies continuously with $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$, the pair (g_1,h_1) varies continuously with $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$, and the relations $x = h_1(1)$ and $y = g_1(0)$ eventually show that (x,y) varies continuously with μ. In the end, any $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$ sufficiently close to μ_{θ,x_0,y_0} is of the form $\mu_{\theta,x,y}$ with (x,y) arbitrarily close to (x_0,y_0). This shows that μ_θ is open and concludes the proof of the lemma. □

9.2. **The deformation space is Hausdorff.** We henceforth use the notations introduced in Paragraphs 7.1 and 7.2 for the one and two-parameter families $\mathcal{R}^{\alpha/\beta}_{\theta,\mathsf{a},\mathsf{b}}$ and $\mathcal{L}_{\theta,\mathsf{a},\mathsf{b}}$. The main goal of this subsection is to show the following result.

Theorem 9.6.
(1) $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A = \mathrm{PMod}(\mathbf{T}^2,0)\cdot(\mu_\theta(\mathcal{E}) \cup \mu^*_\theta(\mathcal{E}^*))$, *and* $\mathrm{PMod}(\mathbf{T}^2,0)\cdot\mathcal{R}^\alpha_{\theta,\mathsf{a},\mathsf{b}}$ *(respectively* $\mathrm{PMod}(\mathbf{T}^2,0)\cdot\mathcal{R}^\beta_{\theta,\mathsf{a},\mathsf{b}}$*) is the subset of* $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ *for which the α-leaf (resp. the β-leaf) of the singularity is closed.*
(2) $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ *is a connected component of* $\mathsf{Def}_\theta(\mathbf{T}^2,0)$.
(3) $\mathcal{A}$ *is a proper map from* $\mathsf{Def}_\theta(\mathbf{T}^2,0)^A$ *to* $(\mathbb{R}\mathbf{P}^1_+)^{(2)}$.

(4) $\mathsf{Def}_\theta(\mathbf{T}^2,\mathsf{0})^A$ *is a Hausdorff topological surface.*

We now prove a series of four results, of which Theorem 9.6 is an easy consequence. The statements below may seem technical at a first sight, but their proofs are relatively easy, and similar arguments are repeated. To warm ourselves up, we begin with an investigation of the case where the singularity has one or two closed lightlike leaves.

Corollary 9.7. *Let* $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2,\mathsf{0})^A$ *be such that* $\mathcal{F}^\mu_\alpha(\mathsf{0})$ *is closed and homotopic to* a*, and*

(1) either $\mathcal{F}^\mu_\beta(\mathsf{0})$ *is closed and homotopic to* b*;*
(2) or $A^+(\mathcal{F}^\mu_\beta) \in]\mathsf{a}+\mathsf{b}\,;\mathsf{b}[$.

Then $\mu \in \mathcal{R}^\alpha_{\theta,\mathsf{a},\mathsf{b}}$*. If* $\mathcal{F}^\mu_\beta(\mathsf{0})$ *is closed and homotopic to* b*, then under the obvious corresponding assumptions we have* $\mu \in \mathcal{R}^\beta_{\theta,\mathsf{a},\mathsf{b}}$*.*

Proof. In the first case, $\mathcal{F}_\alpha(\mathsf{0})$ and $\mathcal{F}_\beta(\mathsf{0})$ define a rectangular graph of induced marking (a,b), hence $\mu \in \mathcal{R}^\alpha_{\theta,\mathsf{a},\mathsf{b}}$ according to Proposition 9.2. In the second case, $\mathcal{F}_\beta(\mathsf{0})$ has a first-return point x on $\mathcal{F}_\alpha(\mathsf{0})$, and the segment $[\mathsf{0}\,;x]_\beta$ together with $\mathcal{F}_\alpha(\mathsf{0})$ define a rectangular lightlike graph. Its induced marking is $(\mathsf{a},\mathsf{b}-n\mathsf{a})$ for some $n \in \mathbb{Z}$, and according to Proposition 9.2 we have then $\mu \in (D^n_\mathsf{a})_*\mathcal{R}^\alpha_{\theta,\mathsf{a},\mathsf{b}}$, hence $A^+(\mathcal{F}^\mu_\beta) \in [[(1+n)\mathsf{a}+\mathsf{b}]\,;[n\mathsf{a}+\mathsf{b}]]$ according to Lemma 7.6. Since $A^+(\mathcal{F}^\mu_\beta) \in]\mathsf{a}+\mathsf{b}\,;\mathsf{b}[$, this shows that $n=0$ and concludes the proof. $\square$

We recall that $\mathcal{R}$ is the rectangle

$$\mathcal{R} = [\mathsf{a}\,;\mathsf{a}+\mathsf{b}] \times \left[\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}\right] \subset (\mathbb{R}\mathbf{P}^1_+)^{(2)}.$$

Corollary 9.8. *Let* $\mu \in \mathsf{Def}_\theta(\mathbf{T}^2,\mathsf{0})^A$ *be such that* $\mathcal{F}^\mu_\alpha(\mathsf{0})$ *and* $\mathcal{F}^\mu_\beta(\mathsf{0})$ *are closed.*

(1) If $\mathcal{A}(\mu) \in \partial\mathcal{R}$*, then* $\mu \in \mu_\theta(\partial\mathcal{E})$*.*
(2) If $\mathcal{A}(\mu) \in \mathrm{Int}(\mathcal{R})$*, then* $\mu \in \mu_\theta(\mathrm{Int}(\mathcal{E}))$*.*

The obvious analogous claims hold for $\mathcal{R}^*$*,* μ^*_θ *and* $\mathcal{E}^*$*.*

Proof. Assume first that $\mathcal{A}(\mu) \in \partial\mathcal{R}$. If $\mathcal{A}(\mu) \in [\mathsf{a}\,;\mathsf{a}+\mathsf{b}] \times [\mathsf{b}]$ or $\mathcal{A}(\mu) \in [\mathsf{a}] \times [\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}]$, then μ is in the corresponding edge of $\mu_\theta(\partial\mathcal{E})$ according to Corollary 9.7. If $\mathcal{A}(\mu) \in [\mathsf{a}\,;\mathsf{a}+\mathsf{b}] \times [\frac{\mathsf{a}}{2}+\mathsf{b}]$, respectively $\mathcal{A}(\mu) \in [\mathsf{a}+\mathsf{b}] \times [\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}]$, then $D^{-1}_{\mathsf{a}+\mathsf{b}} \cdot \mathcal{A}(\mu) \in [\mathsf{a}+\frac{\mathsf{b}}{2}\,;\mathsf{a}+\mathsf{b}] \times [\mathsf{b}]$, resp. $D^{-1}_\mathsf{b} \cdot \mathcal{A}(\mu) \in [\mathsf{a}] \times [\frac{\mathsf{a}}{2}+\mathsf{b}\,;\mathsf{b}]$. Corollary 9.7 shows then that $(D^{-1}_{\mathsf{a}+\mathsf{b}})_*\mu$, resp. $(D^{-1}_\mathsf{b})_*\mu$ is in $\{\mu^*_{\theta,y}\}_{y\in[0;y_0]}$, resp. $\{\mu_{\theta,x}\}_{x\in[1;\infty]}$, and μ is thus in the corresponding edge 4' or 3' of $\mu_\theta(\partial\mathcal{E})$ (see Paragraph 7.2 for more details).

Assume now that $\mathcal{A}(\mu) \in \mathrm{Int}(\mathcal{R})$. Note in particular that $\mathcal{F}^\mu_\alpha(\mathsf{0})$ and $\mathcal{F}^\mu_\beta(\mathsf{0})$ intersect then more than once. We saw in the proof of Corollary 9.7 that the closed curves $\mathcal{F}^\mu_\alpha(\mathsf{0})$ and $\mathcal{F}^\mu_\beta(\mathsf{0})$ define a rectangular lightlike graph. But since its induced marking is in general different from (a,b), this only gives us $\mu \in \mathrm{PMod}(\mathbf{T}^2,\mathsf{0}) \cdot (\mathcal{R}^\alpha_{\theta,\mathsf{a},\mathsf{b}} \cup \mathcal{R}^\beta_{\theta,\mathsf{a},\mathsf{b}})$. To refine this description and show that μ actually belongs to $\mu_\theta(\mathrm{Int}(\mathcal{E}))$, it is sufficient according to Proposition 9.2 to use $\mathcal{F}^\mu_\alpha(\mathsf{0})$ and $\mathcal{F}^\mu_\beta(\mathsf{0})$ to define another lightlike graph, which is this time L-shaped but has (a,b) as induced marking. Such a graph is obtained as follows. Let p_α (respectively p_β) be the first of the finitely many points of $(\mathcal{F}_\alpha(\mathsf{0}) \cap \mathcal{F}_\beta(\mathsf{0})) \setminus \{\mathsf{0}\}$ on the positively oriented segment $\mathcal{F}_\alpha(\mathsf{0}) \setminus \{\mathsf{0}\}$ (resp. $\mathcal{F}_\beta(\mathsf{0}) \setminus \{\mathsf{0}\}$). It is then easily checked that the segments $[\mathsf{0}\,;p_\beta]_\alpha$ and $[\mathsf{0}\,;p_\alpha]_\beta$ define a L-shaped lightlike graph of induced marking m. Moreover, there is a unique isometric identification of $[1\,;\infty] \times \{0\} \subset \mathbf{dS}^2$ with $[\mathsf{0}\,;p_\beta]_\alpha$, in which $(x',0)$ identifies with p_α, and $(x,0)$ identifies with the first of the points of $(\mathcal{F}_\alpha(\mathsf{0}) \cap \mathcal{F}_\beta(\mathsf{0})) \setminus \{\mathsf{0}\}$ on the negatively oriented segment $\mathcal{F}_\beta(\mathsf{0}) \setminus \{\mathsf{0}\}$. In particular $x' \leq x$, which shows that $\mu \in \mu_\theta(\mathcal{E})$. Since $\mathcal{A} \circ \mu_\theta(\partial\mathcal{E}) \subset \partial\mathcal{R}$ according to Lemma 7.9 and $\mathcal{A}(\mu) \in \mathrm{Int}(\mathcal{R})$ by assumption, we have thus $\mu \in \mu_\theta(\mathrm{Int}(\mathcal{E}))$ which concludes the proof. $\square$

Having noticed that the case of a closed lightlike leaf at the singularity is easily described, we now use the surgeries introduced in Proposition 8.5 to construct adapted deformations, allowing us to close a lightlike leaf at the singularity while controling the asymptotic cycles.

Lemma 9.9. *Let* $\mu \in \mathcal{A}^{-1}(\mathrm{Int}(\mathcal{R}))$ *(respectively* $\mu \in \mathcal{A}^{-1}(\partial\mathcal{R})$*). Then there exists a continuous path* $t \in [0\,;1] \mapsto \mu(t) \in \mathsf{Def}_\theta(\mathbf{T}^2,\mathsf{0})$ *starting from* $\mu=\mu(0)$*, and such that:*

(1) $\mathcal{A}(\mu([0;1])) \subset \mathrm{Int}(\mathcal{R})$ *(resp.* $\mathcal{A}(\mu([0;1])) \subset \partial\mathcal{R}$*);*
(2) both lightlike leaves of the singularity are closed for $\mu(1)$*.*

The same claim holds for $\mathcal{R}^*$*.*

Proof. We write the proof for $\mathcal{R}$, the case of $\mathcal{R}^*$ being identical. Note first that $\mathcal{A}^{-1}(\mathcal{R}) \subset \mathsf{Def}_\theta(\mathbf{T}^2,0)^{\mathrm{A}}$ since $\mathcal{R} \subset (\mathbb{R}\mathbf{P}^1_+)^{(2)}$. Let now $\mu \in \mathcal{A}^{-1}(\mathcal{R})$. It will moreover be clear along the proof, by following the construction of $\mu(t)$, that $\mathcal{A}(\mu([0;1])) \subset \mathrm{Int}(\mathcal{R})$ (respectively $\mathcal{A}(\mu([0;1])) \subset \partial\mathcal{R}$) if $\mu \in \mathcal{A}^{-1}(\mathrm{Int}(\mathcal{R}))$ (resp. $\mu \in \mathcal{A}^{-1}(\partial\mathcal{R})$) in the first place.

Case 1: $\mathcal{F}^\mu_\alpha(0)$ or $\mathcal{F}^\mu_\beta(0)$ is closed (we write the proof if $\mathcal{F}^\mu_\alpha(0)$ is closed, the other case being formally identical). Since μ is class A, $\mathcal{F}^\mu_\beta$ is a suspension according to Lemma 6.6. The closed curve $\mathcal{F}^\mu_\alpha(0)$ being transverse to $\mathcal{F}^\mu_\beta$, it has thus to intersect all the leaves of $\mathcal{F}^\mu_\beta$ (we thank an anonymous referee for informing us of the existence of this fact). The first-return map $P_{\beta,\mu}$ of $\mathcal{F}^\mu_\beta$ on $\mathcal{F}^\mu_\alpha(0)$ is therefore well-defined. The former claim is clear if $\mathcal{F}^\mu_\beta$ is minimal. If $A(\mathcal{F}^\mu_\beta)$ is rational, then any of its closed leaves F_β is homologically independent from $\mathcal{F}^\mu_\alpha(0)$: $\mathbb{R}[F_\beta] = A^+(\mathcal{F}^\mu_\beta)$ is distinct from $\mathbb{R}[\mathcal{F}^\mu_\alpha(0)] = A^+(\mathcal{F}^\mu_\alpha)$ since μ is class A. Therefore F_β has non-zero algebraic intersection number with $\mathcal{F}^\mu_\alpha(0)$, and in particular intersect it. Any other leaf of $\mathcal{F}^\mu_\beta$ is moreover future and past asymptotic to a closed leaf F_β of $\mathcal{F}^\mu_\beta$ by Proposition 5.7, and it intersects therefore also $\mathcal{F}^\mu_\alpha(0)$ since F_β does.

Proposition 8.5.(5) yields then a continuous family $s \in [0;1] \mapsto \mu_s \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$ of surgeries of μ around $\mathcal{F}^\mu_\alpha(0)$ such that $\mathcal{F}^{\mu_s}_\alpha(0) = \mathcal{F}^\mu_\alpha(0)$, and whose first-return map of $\mathcal{F}^{\mu_s}_\beta$ on $\mathcal{F}^\mu_\alpha(0)$ equals $P_{\beta,\mu_s} = P_{\beta,\mu} \circ T_s$, with $s \in [0;1]/\{0\sim 1\} \mapsto T_s \in \mathrm{Aff}^+(\mathcal{F}^\mu_\alpha(0))$ a continuous and degree one map. Moreover $\mu_1 = (D_{[\mathcal{F}^\mu_\alpha(0)]})_*\mu$ according to Proposition 8.5.(1) and the map $s \in [0;1] \mapsto \mathcal{A}(\mu_s) \in [\mathcal{F}^\mu_\alpha(0)] \times [A^+(\mathcal{F}^\mu_\beta); A^+(\mathcal{F}^\mu_\beta) + [\mathcal{F}^\mu_\alpha(0)]]$ is therefore surjective according to Lemma B.1.(3) and Proposition 5.9. In particular, there exists $s_1 \in\,]0;1[$ such that $A^+(\mathcal{F}^{\mu_{s_1}}_\beta)$ is irrational, and $\mathcal{A}(\mu_s) \in \mathrm{Int}(\mathcal{R})$ (resp. $\mathcal{A}(\mu_s) \in \partial\mathcal{R}$) for any $s \in [0;s_1]$. Lemma B.1.(5) and Proposition 5.9 show then the existence of $s_2 \in\,]s_1;1[$ such that $\mathcal{F}^{\mu_{s_2}}_\beta(0)$ is closed, and $\mathcal{A}(\mu_s) \in \mathrm{Int}(\mathcal{R})$ (resp. $\mathcal{A}(\mu_s) \in \partial\mathcal{R}$) for any $s \in [0;s_2]$. This shows the claim in the first case.

Case 2: $\mathcal{F}^\mu_\alpha$ (resp. $\mathcal{F}^\mu_\beta$) has a closed leaf that we denote by F_α. As in Case 1, F_α intersects all the leaves of $\mathcal{F}^\mu_\beta$, and the first-return map $P_{\beta,\mu}$ of $\mathcal{F}^\mu_\beta$ on F_α is thus well-defined. Proposition 8.5.(5) yields then a continuous family $s \in [0;1] \mapsto \mu_s \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$ of surgeries of μ around F_α, such that F_α remains a closed α-leaf of μ_s, and whose first-return map of $\mathcal{F}^{\mu_s}_\beta$ on F_α equals $P_{\beta,\mu_s} = P_{\beta,\mu} \circ T_s$, with $s \in [0;1]/\{0\sim 1\} \mapsto T_s \in \mathrm{Aff}^+(F_\alpha)$ a continuous and degree one map. As in Case 1, this shows the existence of $s_1 \in\,]0;1[$ such that $A^+(\mathcal{F}^{\mu_{s_1}}_\beta)$ is irrational, and $\mathcal{A}(\mu_s) \in \mathrm{Int}(\mathcal{R})$ (resp. $\mathcal{A}(\mu_s) \in \partial\mathcal{R}$) for any $s \in [0;s_1]$. Lemma B.1.(5) shows then the existence of $s_2 \in\,]s_1;1[$ such that $\mathcal{F}^{\mu_{s_2}}_\beta(0)$ is closed, and $\mathcal{A}(\mu_s) \in \mathrm{Int}(\mathcal{R})$ (resp. $\mathcal{A}(\mu_s) \in \partial\mathcal{R}$) for any $s \in [0;s_2]$. Since μ_{s_2} satisfies the assumptions of Case 1, we can now compose the path μ_s of surgeries around F_α that we just constructed, with the path of surgeries around $\mathcal{F}^{\mu_{s_2}}_\beta(0)$ given by Case 1, which shows the claim in the second Case.

Case 3: $\mathcal{F}^\mu_\alpha$ and $\mathcal{F}^\mu_\beta$ are both minimal. Note that in this case, $\mathcal{A}(\mu) \in \mathrm{Int}(\mathcal{R})$. According to Theorem A.1, μ admits then a simple closed timelike geodesic γ avoiding the singularity (since it is class A). Since $\mathcal{F}^\mu_\alpha$ is minimal, the first-return map $P^\gamma_{\alpha,\mu}$ of $\mathcal{F}^\mu_\alpha$ on γ is well-defined, and as before Proposition 8.5.(5) gives a continuous family $s \in [0;1] \mapsto \mu_s \in \mathsf{Def}_\theta(\mathbf{T}^2,0)$ of surgeries of μ around γ. We denote by x the first intersection point of $\mathcal{F}^\mu_\alpha(0)$ with γ. According to Lemmas B.1.(5) and 8.4, there exists $s_1 \in\,]0;1[$ for which the orbit of x for $P^\gamma_{\alpha,\mu} \circ T_{s_1}$ is periodic, and which is small enough for $\mathcal{A}(\mu_s)$ to be in $\mathrm{Int}(\mathcal{R})$ for any $s \in [0;s_1]$ (this is allowed by the continuity of $\mathcal{A}$ and μ_s, since $\mathrm{Int}(\mathcal{R})$ is open). Since $\mathcal{F}^{\mu_{s_1}}_\alpha(0)$ is closed, μ_{s_1} satisfies the assumptions of the Case 1. We can thus compose the path of surgeries that we just constructed with the one furnished by the Case 1, to show our claim in this last case. This concludes the proof of the lemma. □

An important consequence of Lemma 9.9 is the following result, which may be seen as a first step towards the injectivity of $\mathcal{A}$: we control the "size" of preimages of particular subsets.

Corollary 9.10. *$\mathcal{A}^{-1}(\mathrm{Int}(\mathcal{R})) = \mu_\theta(\mathrm{Int}(\mathcal{E}))$, and $\mathcal{A}^{-1}(\mathcal{R}) = \mu_\theta(\mathcal{E})$. The obvious analogous claims hold for $\mathcal{R}^*$, μ_θ^* and $\mathcal{E}^*$.*

Proof. We detail the proof in the case of $\mathcal{R}$, the one of $\mathcal{R}^*$ being formally the same. We first observe that since μ_θ is a homeomorphism onto its image according to Lemma 9.4, we have: $\mu_\theta(\partial\mathcal{E}) = \partial(\mu_\theta(\mathrm{Int}\,\mathcal{E}))$. Let $\mu \in \mathcal{A}^{-1}(\mathrm{Int}(\mathcal{R}))$, and $\mu\colon [0\,;1] \to \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$ be the continuous path given by Lemma 9.9. Since $\mathcal{A}(\mu([0\,;1])) \subset \mathrm{Int}(\mathcal{R})$ and $\mathcal{A}(\mu_\theta(\partial\mathcal{E})) \subset \partial\mathcal{R}$, we observe that $\mu([0\,;1])$ does not intersect $\mu_\theta(\partial\mathcal{E}) = \partial(\mu_\theta(\mathrm{Int}\,\mathcal{E}))$. Since both lightlike leaves of the singularity of $\mu(1)$ are closed and $\mathcal{A}(\mu(1)) \in \mathrm{Int}(\mathcal{R})$, Corollary 9.8 shows that $\mu(1) \in \mu_\theta(\mathrm{Int}(\mathcal{E}))$. Since $\mu([0\,;1])$ is path-connected and does not intersect $\partial(\mu_\theta(\mathrm{Int}\,\mathcal{E}))$, this shows that $\mu([0\,;1]) \subset \mu_\theta(\mathrm{Int}(\mathcal{E}))$, hence that $\mu = \mu(0) \in \mu_\theta(\mathrm{Int}(\mathcal{E}))$ which concludes the proof of the first claim.

Let now $\mu \in \mathcal{A}^{-1}(\partial\mathcal{R})$, and $\mu\colon [0\,;1] \to \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$ be the continuous path given by Lemma 9.9. Since both lightlike leaves of the singularity of $\mu(1)$ are closed and $\mathcal{A}(\mu(1)) \in \partial\mathcal{R}$, Corollary 9.8 shows that $\mu(1) \in \mu_\theta(\partial\mathcal{E})$. Let C be the associated L-shaped (or rectangle) lightlike graph of $\mu(1)$. We observe now that the concatenated surgeries of Lemma 9.9 constituting $\mu(t)$ and going backward from $\mu(1)$ to μ, transform C into a L-shaped lightlike graph of induced marking (a, b). This shows that $\mu = \mu(0) \in \mathcal{L}_{\theta,\mathsf{a},\mathsf{b}}$ according to Proposition 9.2, hence that $\mu \in \mu_\theta(\mathcal{E})$ since $\mathcal{A}(\mu) \in \mathcal{R}$, which concludes the proof. □

Proof of Theorem 9.6. (1) We recall that $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}} = \mathcal{A}^{-1}((\mathbb{R}\mathbf{P}^1_+)^{(2)})$, and that $(\mathbb{R}\mathbf{P}^1_+)^{(2)} = \mathrm{PMod}(\mathbf{T}^2, \mathsf{0}) \cdot (\mathcal{R} \cup \mathcal{R}^*)$ according to Lemma 7.13. Hence $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}} = \mathrm{PMod}(\mathbf{T}^2, \mathsf{0}) \cdot \mathcal{A}^{-1}(\mathcal{R} \cup \mathcal{R}^*) = \mathrm{PMod}(\mathbf{T}^2, \mathsf{0}) \cdot (\mu_\theta(\mathcal{E}) \cup \mu_\theta^*(\mathcal{E}^*))$ according to Corollary 9.10, which proves the first claim. The other claims are direct consequences of Corollary 9.7.
(2) We already know from Corollary 6.9 that $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}}$ is a union of connected components, hence only have to show that $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}} = \mathrm{PMod}(\mathbf{T}^2, \mathsf{0}) \cdot (\mu_\theta(\mathcal{E}) \cup \mu_\theta^*(\mathcal{E}^*))$ is connected. We note first that $\mu_\theta(\mathcal{E})$ and $\mu_\theta^*(\mathcal{E}^*)$ are connected as the images of the connected spaces $\mathcal{E}$ and $\mathcal{E}^*$ by the continuous maps μ_θ and μ_θ^*. Since $\mu_\theta(\mathcal{E})$ and $\mu_\theta^*(\mathcal{E}^*)$ intersect, $\mathcal{C} \coloneqq \mu_\theta(\mathcal{E}) \cup \mu_\theta^*(\mathcal{E}^*)$ is also connected. It follows easily from Remark 7.7 that any $f \in \mathrm{PMod}(\mathbf{T}^2, \mathsf{0})$ can be written as $f = f_n \dots f_1$, where the f_k are Dehn twists such that $f_{k+1} \dots f_1(\mathcal{C})$ and $f_k \dots f_1(\mathcal{C})$ intersect along their boundary for any k. This shows that $\mathcal{C} \cup_{k=1}^n f_k \dots f_1(\mathcal{C})$ is connnected and thus that any point of $f(\mathcal{C})$ can be joined to $\mathcal{C}$ by a continuous path. Since this was done for any $f \in \mathrm{PMod}(\mathbf{T}^2, \mathsf{0})$, $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}} = \mathrm{PMod}(\mathbf{T}^2, \mathsf{0}) \cdot \mathcal{C}$ is connected wich concludes the proof of the claim.
(3) Let $K \subset (\mathbb{R}\mathbf{P}^1_+)^{(2)}$ be compact. There exists then $f_1, \dots, f_n \in \mathrm{PMod}(\mathbf{T}^2, \mathsf{0})$ such that $K \subset \cup_{k=1}^n f_k(\mathcal{R} \cup \mathcal{R}^*)$. According to Corollary 9.10, we have then $\mathcal{A}^{-1}(K) \subset \cup_{k=1}^n f_k(\mu_\theta(\mathcal{E}) \cup \mu_\theta^*(\mathcal{E}^*))$. Since $\mu_\theta(\mathcal{E})$ and $\mu_\theta^*(\mathcal{E}^*)$ are compact as the images of the compact sets $\mathcal{E}$ and $\mathcal{E}^*$ by the continuous maps μ_θ and μ_θ^*, this shows that $\mathcal{A}^{-1}(K)$ is compact and proves the properness.
(4) Since $\mu_\theta(\mathcal{E})$ and $\mu_\theta^*(\mathcal{E}^*)$ are homeomorphic to closed disks according to Lemma 9.4, the first claim of the Theorem shows that $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}} = \mathrm{PMod}(\mathbf{T}^2, \mathsf{0}) \cdot (\mu_\theta(\mathcal{E}) \cup \mu_\theta^*(\mathcal{E}^*))$ is a topological surface. We prove now that it is Hausdorff. Let $\mu \neq \mu'$ in $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}}$. If $\mathcal{A}(\mu) \neq \mathcal{A}(\mu')$, let U and U' be disjoint open neighbourhoods of $\mathcal{A}(\mu)$ and $\mathcal{A}(\mu')$. Since $\mathcal{A}$ is a continuous map, $\mathcal{A}^{-1}(U)$ and $\mathcal{A}^{-1}(U')$ are then disjoint open neighbourhoods of μ and μ'. Assume now that $\mathcal{A}(\mu) = \mathcal{A}(\mu')$. Possibly translating μ and μ' by the same element of $\mathrm{PMod}(\mathbf{T}^2, \mathsf{0})$ and exchanging the roles of α and β, we can assume without loss of generality that $\mathcal{A}(\mu) = \mathcal{A}(\mu') \in \mathcal{R}$. Corollary 9.10 shows then that μ and μ' belong to $\mu_\theta(\mathcal{E})$. The latter being Hausdorff, $\mu \neq \mu'$ admit separating open neighbourhoods in $\mu_\theta(\mathcal{E})$, which concludes the proof. □

We emphasize that we do not know yet wether $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}}$ equals $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$ or not.

10. Rigidity of singular $\mathbf{dS}^2$-tori

10.1. **Proof of the uniqueness part of Theorem C.** The existence part was proved in Theorem 7.1. Let $\mu_1, \mu_2 \in \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$ have their lightlike leaves at $\mathsf{0}$ closed and homotopic:

$$([\mathcal{F}^{\mu_1}_\alpha(\mathsf{0})], [\mathcal{F}^{\mu_1}_\beta(\mathsf{0})]) = ([\mathcal{F}^{\mu_2}_\alpha(\mathsf{0})], [\mathcal{F}^{\mu_2}_\beta(\mathsf{0})]) =: (c_\alpha, c_\beta). \tag{10.1}$$

Without loss of generality, we can assume that either $(c_\alpha, c_\beta) = (\mathsf{a}, \mathsf{b})$, or $c_\alpha = \mathsf{a}$ and $[c_\beta] \in]\mathsf{a}+\mathsf{b}\,;\mathsf{b}[$. According to Corollary 9.7, there exists then $x_1, x_2 \in [1\,;\infty[$ such that $\mu_1 = \mu_{\theta,x_1}$ and $\mu_2 = \mu_{\theta,x_2}$. There only remains to show that $x_1 = x_2$ to conclude the proof of Theorem C.

The first return map of $\mathcal{F}_\beta^{\mu_{\theta,x_i}}$ on $\mathcal{F}_\alpha^{\mu_{\theta,x_i}}(0)$ being $\mathsf{E}_{x_i}^{-1}$ (see the proof of Lemma 7.6), we can translate the fact that $\mathcal{F}_\beta^{\mu_{\theta,x_1}}(0)$ and $\mathcal{F}_\beta^{\mu_{\theta,x_2}}(0)$ are closed and homotopic in terms of orbits of the E_{x_i}'s: $[1] \in \overline{[1\,;\infty]} := [1\,;\infty]/\{1 \sim \infty\}$ is periodic under E_{x_1} and E_{x_2}, say of minimal period $q \in \mathbb{N}^*$, and of the same cyclic order on the circle $\overline{[1\,;\infty]}$. If $(c_\alpha, c_\beta) = (\mathsf{a}, \mathsf{b})$, then $[1]$ is a fixed point of E_{x_1} and E_{x_2}, hence $x_1 = x_2$ since $x \in [1\,;\infty] \mapsto x'_x$ is strictly decreasing. We can therefore assume without loss of generality that $x_1, x_2 \in]1\,;\infty[$ and that $q \geq 2$. For $p \in \overline{[1\,;\infty]}$, let us denote:

(1) $l(p) = a$ if $p \in [1\,;x'_i[$, equivalently if $\mathsf{E}_{x_i}(p) = gh_{x_i}(p)$;
(2) and $l(p) = b$ if $p \in [x'_i\,;\infty[$, equivalently if $\mathsf{E}_{x_i}(p) = h_{x_i}(p)$.

Then with $l_1 = l([1])$ and $l_{k+1} = l(l_k([1]))$, the word $w = l_q \dots l_1$ in the letters a and b is the *coding* of the periodic orbit of $[1]$ under E_{x_i}, and is equivalent to its cyclic ordering. In other words, the respective codings of $[1]$ under E_{x_1} and E_{x_2} are equal to a common word $w = l_q \dots l_1$, characterized by

$$\mathsf{E}_{x_i}^k([1]) = w_k(gh, h)([1]) \tag{10.2}$$

for any $1 \leq k \leq q$, where $w_k = l_k \dots l_1$ and $v(A, B) \in \mathrm{PSL}_2(\mathbb{R})$ is obtained for any $A, B \in \mathrm{PSL}_2(\mathbb{R})$ from a word v in the letters a and b by replacing a by A and b by B.

According to Lemma 7.4 there exists $T \in [0\,;1]$ such that $x_2 = g^T(x_1)$ and $h_{x_2} = g^T h_{x_1}$, and we thus only have to show that $T = 0$. From now on we denote $h := h_{x_1}$ to simplify notations, and work in $\mathbb{R} \cup \{\infty\}$ identified with $\mathbb{R}\mathbf{P}^1$ (in the same $\mathrm{PSL}_2(\mathbb{R})$-equivariant way (2.2) than usually). The equalities (10.2) translate then as:

$$\begin{cases} w(gh, h)(1) = w(g^{T+1}h, g^T h)(1) = 1 \\ \forall k \in \{1, \dots, q-1\} : w_k(gh, h)(1) \text{ and } w_k(g^{T+1}h, g^T h)(1) \in]1\,;\infty[. \end{cases} \tag{10.3}$$

Fact 10.1. *For any $k \in \{1, \dots, q\}$, the map $s \in [0\,;T] \mapsto w_k(g^{s+1}h, g^s h)(1)$ is strictly increasing and has values in $[1\,;\infty[$.*

Fact 10.1 concludes the proof of our claim, and thus of Theorem C. Indeed the map $s \in [0\,;T] \mapsto w_q(g^{s+1}h, g^s h)(1) = w(g^{s+1}h, g^s h)(1)$ is in particular strictly increasing, but has according to (10.3) the same value 1 at $s = 0$ and $s = T$ which implies $T = 0$.

Proof of Fact 10.1. We prove the claim by induction on k.

Case $k = 1$. Then $w_1 = l_1 = a$ and since $gh(1) \in]1\,;\infty[$, $s \in \mathbb{R} \mapsto w_1(g^{s+1}h, g^s h)(1) = g^{s+1}h(1)$ is strictly increasing in $\mathbb{R} \cup \{\infty\}$. Since $g^{T+1}h(1) \in]1\,;\infty[$ as well according to (10.3), we have thus $g^{s+1}h(1) \in]1\,;\infty[$ for any $s \in [0\,;T]$ by the intermediate values Theorem.

From $k \in \{1, \dots, q-1\}$ to $k+1$. Then $w_{k+1}(g^{s+1}h, g^s h)(1) = l_{k+1}(g, \mathrm{id}) g^s h(\alpha(s))$ for $s \in [0\,;T]$, with $\alpha \colon s \in [0\,;1] \mapsto w_k(g^{s+1}h, g^s h)(1)$ a strictly increasing map having values in $[1\,;\infty[$ by induction. Since h is orientation-preserving, $s \in [0\,;T] \mapsto h \circ \alpha(s)$ is strictly increasing as well. The dynamics of h show moreover that its attractive and repulsive fixed points respectively satisfy $h_+ \in]y_\theta\,;1[$ and $h_- \in]\infty\,;0[$, and the attractive and repulsive fixed points of g are on the other hand 0 and y_θ. We have thus $h \circ \alpha([0\,;T]) \subset]h_+\,;\infty[\subset [y_\theta\,;0]$, and denoting $G(s, p) := g^s(p)$ for any $(s, p) \in \mathbb{R} \times]y_\theta\,;0[$ we have: $\frac{\partial G}{\partial s}(s, p) > 0$ due to the dynamics of g, and $\frac{\partial G}{\partial p}(s, p) > 0$ due to the fact that g^s is orientation-preserving. Therefore:

$$\frac{d}{ds} g^s h(\alpha(s)) = \frac{d}{ds} G(s, h(\alpha(s))) = \frac{\partial G}{\partial s}(s, h(\alpha(s))) + \frac{d}{ds} h(\alpha(s)) \frac{\partial G}{\partial p}(s, h(\alpha(s)))$$

is strictly positive for any $s \in [0\,;T]$ as a sum of strictly positive terms. Therefore $s \in [0\,;T] \mapsto w_{k+1}(g^{s+1}h, g^s h)(1) = l_{k+1}(g, \mathrm{id}) g^s h(\alpha(s))$ is strictly increasing, since g is orientation-preserving. Since $w_{k+1}(gh, h)(1)$ and $w_{k+1}(g^{T+1}h, g^T h)(1)$ are moreover in $[1\,;\infty[$ according to (10.3), we have $w_{k+1}(g^{s+1}h, g^s h)(1) \in [1\,;\infty[$ for any $s \in [0\,;T]$, which concludes the proof of the fact. $\square$

10.2. **Proof of the uniqueness part of Theorem D.** The existence part is given by Theorem 7.1. Let $\mu_1, \mu_2 \in \mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$ have their α-leaves at $\mathsf{0}$ closed, and satisfy:

$$([\mathcal{F}_\alpha^{\mu_1}(\mathsf{0})], A^+(\mathcal{F}_\beta^{\mu_1})) = ([\mathcal{F}_\alpha^{\mu_2}(\mathsf{0})], A^+(\mathcal{F}_\beta^{\mu_2})) =: (c_\alpha, A_\beta) \tag{10.4}$$

with A_β irrational. Without loss of generality, we can assume that $c_\alpha = \mathsf{a}$ and $A_\beta \in]\mathsf{a} + \mathsf{b}\,;\mathsf{b}[$. According to Corollary 9.7, there exists then $x_1, x_2 \in]1\,;\infty[$ such that $\mu_1 = \mu_{\theta,x_1}$ and $\mu_2 = \mu_{\theta,x_2}$. Since $x \in [1\,;\infty] \mapsto A^+(\mathcal{F}_\beta^{\mu_{\theta,x}})$ is non-decreasing and strictly increasing at irrational points according to Lemma 7.6, this shows that $x_1 = x_2$ which conclude the proof of Theorem D.

10.3. **Proof of Theorem A.** We first show how Theorem A is deduced from the uniqueness part of Theorem B. Let (S_1, μ_1) and (S_2, μ_2) be two closed singular $\mathbf{dS}^2$-surfaces having a unique singularity of the same angle $\theta \in \mathbb{R}_+^*$ and minimal lightlike foliations, and let f be a topological equivalence between their lightlike bifoliations. Without loss of generality we can assume that $S_1 = S_2 = \mathbf{T}^2$. The singular $\mathbf{dS}^2$-structures $\mu_1' \coloneqq f^*\mu_2$ and μ_1 of $\mathbf{T}^2$ share then the same minimal lightlike bi-foliation $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$, and have the same singularity x with the same angle. According to Theorem B, there exists thus a homeomorphism g of $\mathbf{T}^2$ isotopic to the identity relatively to x, such that $\mu_1' = g^*\mu_1$. In particular g preserves then the minimal bi-foliation $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$, and is thus the identity according to [MM25, Corollary B] (see also [AGK03]). Therefore $f^*\mu_2 = \mu_1' = \mu_1$, *i.e.* f is an isometry from S_1 to S_2 as claimed.

10.4. **Proof of the uniqueness part of Theorem B.** The existence part was proved in Theorem 7.1. Let now S_1 and S_2 be two closed singular $\mathbf{dS}^2$-surfaces having a unique singularity of the same angle $\theta \in \mathbb{R}_+^*$, and minimal lightlike bifoliations with the same oriented projective asymptotic cycles

$$A^+(\mathcal{F}_{\alpha/\beta}^{\mu_1}) = A^+(\mathcal{F}_{\alpha/\beta}^{\mu_2}).$$

Without loss of generality we can assume that $S_1 = S_2 = \mathbf{T}^2$, and up to translations of $\mathbf{T}^2$ we can moreover assume that $\mathsf{0}$ is the unique singularity of both μ_1 and μ_2, without changing the equality of asymptotic cycles. According to [AGK03, Theorem 1] (see also [MM25, Theorem A]), the equality of asymptotic cycles implies the existence of a homeomorphism f of $\mathbf{T}^2$, isotopic to the identity relatively to $\mathsf{0}$, and sending $(\mathcal{F}_\alpha^{\mu_1}, \mathcal{F}_\beta^{\mu_1})$ on $(\mathcal{F}_\alpha^{\mu_2}, \mathcal{F}_\beta^{\mu_2})$. We can therefore assume that $(\mathcal{F}_\alpha^{\mu_1}, \mathcal{F}_\beta^{\mu_1}) = (\mathcal{F}_\alpha^{\mu_2}, \mathcal{F}_\beta^{\mu_2})$. Note that μ_1 and μ_2 are class A according to Lemma A.7. According to Theorem A.1, μ_1 and μ_2 admit then freely homotopic simple closed timelike geodesics γ_1 and γ_2 avoiding the singularity. Our goal is to show the following approximation result.

Proposition 10.2. *Let μ_1, μ_2 be two singular $\mathbf{dS}^2$-structures on $\mathbf{T}^2$:*

- *having $\mathsf{0}$ as unique singularity of the same angle θ;*
- *admitting freely homotopic simple closed timelike geodesics γ_1 and γ_2 avoiding the singularity;*
- *and whose lightlike bi-foliations are minimal, and have the same asymptotic cycles denoted by $A_{\alpha/\beta}^+ \coloneqq A^+(\mathcal{F}_{\alpha/\beta}^{\mu_1}) = A^+(\mathcal{F}_{\alpha/\beta}^{\mu_2})$.*

Then there exists sequences $\nu_{1,n}, \nu_{2,n}$ of singular $\mathbf{dS}^2$-structures in $\mathcal{S}(\mathbf{T}^2, \mathsf{0}, \theta)$ respectively converging to μ_1 and μ_2, and such that for any n:

(1) $\mathcal{F}_\alpha^{\nu_{1,n}}(\mathsf{0})$ and $\mathcal{F}_\alpha^{\nu_{2,n}}(\mathsf{0})$ are closed and freely homotopic;
(2) and $A^+(\mathcal{F}_\beta^{\nu_{1,n}}) = A^+(\mathcal{F}_\beta^{\nu_{2,n}}) = A_\beta^+$.

We first show how to conclude the proof of Theorem B with the help of Proposition 10.2. Since the α-leaves $\mathcal{F}_\alpha^{\nu_{1,n}}(\mathsf{0})$ and $\mathcal{F}_\alpha^{\nu_{2,n}}(\mathsf{0})$ are closed and freely homotopic in the one hand, and the β-foliations are minimal with identical irrational oriented projective asymptotic cycles $A^+(\mathcal{F}_\beta^{\nu_{1,n}}) = A^+(\mathcal{F}_\beta^{\nu_{2,n}})$ in the other hand, Theorem D shows that $[\nu_{1,n}] = [\nu_{2,n}]$ in the deformation space $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$. The same sequence $[\nu_{1,n}] = [\nu_{2,n}]$ converges thus both to $[\mu_1]$ and to $[\mu_2]$ in the connected component $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}}$ of $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$. Since $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})^{\mathrm{A}}$ is Hausdorff according to Theorem 9.6.(4), this shows that $[\mu_1] = [\mu_2]$ and concludes the proof of Theorem B. ☐

Proof of Proposition 10.2. We denote by x_i the first intersection point of $\mathcal{F}^{\mu_i}_\alpha(\mathsf{0})$ with γ_i. Since $\mathcal{F}^{\mu_i}_\alpha$ and $\mathcal{F}^{\mu_i}_\beta$ are both assumed minimal, the first-return maps $P^{\gamma_i}_{\alpha/\beta,\mu_i}\colon \gamma_i \to \gamma_i$ are well-defined, and moreover have the same rotation numbers

$$\rho(P^{\gamma_1}_{\alpha/\beta,\mu_1}) = \rho(P^{\gamma_2}_{\alpha/\beta,\mu_2})$$

according to Corollary 5.10, since γ_1 and γ_2 are freely homotopic. According to Lemmas B.1.(5) and 8.4, there exists thus a sequence $r_n \in \mathbf{S}^1$ of rationals converging to $\rho(P^{\gamma_1}_{\alpha,\mu_1}) = \rho(P^{\gamma_2}_{\alpha,\mu_2}) \in [\mathbb{R} \setminus \mathbb{Q}]$ and sequences $T_{i,n} \in \mathrm{Aff}^+(\gamma_i)$ of affine transformations of γ_i converging uniformly to id_{γ_i}, such that for $i = 1$ and 2 and for any n: the orbit of x_i for $P^{\gamma_i}_{\alpha,\mu_i} \circ T_{i,n}$ is periodic and of rational cyclic order r_n. Proposition 8.5 yields then a surgery $\mu_{i,n} = (\mu_i)_{T_{i,n}}$ of μ_i around the geodesic γ_i with respect to $T_{i,n}$ such that:

1. $\mu_{i,n}$ has a unique singularity of angle θ at $\mathsf{0}$;
2. γ_i remains a timelike simple closed geodesic of $\mu_{i,n}$;
3. the first-return map of $\mathcal{F}^{\mu_{i,n}}_\alpha$ on γ_i is well-defined and equals the circle homeomorphism

$$P^{\gamma_i}_{\alpha,\mu_{i,n}} = P^{\gamma_i}_{\alpha,\mu_i} \circ T_{i,n}. \tag{10.5}$$

Possibly exchanging the direction of the surgeries and passing to a subsequence, we can moreover assume that $T_{i,n}$ converges uniformly *and monotonically to* id_{γ_i} *from above, i.e.* that for any $x \in \gamma_i$, $(T_{i,n}(x))_n$ is decreasing for the orientation of γ_i and converges uniformly to x. Therefore:

$$\lim \mu_{i,n} = \mu_i \tag{10.6}$$

according to Proposition 8.5. Hence $\mathcal{F}^{\mu_{i,n}}_{\alpha/\beta}$ converges to $\mathcal{F}^{\mu_i}_{\alpha/\beta}$, and in particular $A^+(\mathcal{F}^{\mu_{i,n}}_{\alpha/\beta})$ converges to $A^+(\mathcal{F}^{\mu_i}_{\alpha/\beta})$. Moreover according to (10.5) and by construction of $T_{i,n}$, the respective orbits of x_1 and x_2 for $P^{\gamma_1}_{\alpha,\mu_{1,n}}$ and $P^{\gamma_2}_{\alpha,\mu_{2,n}}$ are periodic and of the same rational cyclic order r_n, hence $\rho(P^{\gamma_1}_{\alpha,\mu_{1,n}}) = \rho(P^{\gamma_2}_{\alpha,\mu_{2,n}}) = r_n$ according to Proposition 5.3. In particular, the α-lightlike leaves $\sigma_{1,n} \coloneqq \mathcal{F}^{\mu_{1,n}}_\alpha(\mathsf{0})$ and $\sigma_{2,n} \coloneqq \mathcal{F}^{\mu_{2,n}}_\alpha(\mathsf{0})$ are thus closed. For any large enough n, Corollary 5.11 shows moreover that $\rho(P^{\gamma_1}_{\alpha,\mu_{1,n}}) = \rho(P^{\gamma_2}_{\alpha,\mu_{2,n}})$ implies

$$A^+(\mathcal{F}^{\mu_{1,n}}_\alpha) = A^+(\mathcal{F}^{\mu_{2,n}}_\alpha),$$

since γ_1 and γ_2 are freely homotopic and $\mathcal{F}^{\mu_{1,n}}_\alpha, \mathcal{F}^{\mu_{2,n}}_\alpha$ close enough. In particular the closed α-lightlike leaves $\sigma_{1,n}$ and $\sigma_{2,n}$ are thus freely homotopic, since $A^+(\mathcal{F}^{\mu_{i,n}}_\alpha) = [\sigma_{i,n}]$ according to Proposition 5.4.

We now perform on $\mu_{i,n}$ a second surgery around $\sigma_{i,n}$, allowing us to keep the closed α-leaves $\sigma_{i,n}$ unchanged while modifying the asymptotic cycle of the β-foliation until recovering the original one of $\mathcal{F}^{\mu_i}_\beta$.

Lemma 10.3. *Let μ be a singular $\mathbf{dS}^2$-structure on $\mathbf{T}^2$, with $\mathsf{0}$ as unique singular point of angle θ, and whose lightlike foliations are minimal. Let γ be a simple closed timelike geodesic of μ, and $T_n \in \mathrm{Aff}^+(\gamma)$ be a sequence converging uniformly and monotonically to id_γ from above, and such that $\sigma_n \coloneqq \mathcal{F}^{\mu_n}_\alpha(\mathsf{0})$ is closed for any n, with $\mu_n \coloneqq \mu_{T_n}$ the surgery of μ around γ with respect to T_n given by Proposition 8.5. Then there exists a sequence $S_n \in \mathrm{Aff}^+(\sigma_n)$ such that:*

(1) S_n converges uniformly and monotonically to the identity from above, in the sense that:

$$\lim \max_{x\in\sigma_n} L([x\,;S_n(x)]_{\sigma_n}) = 0 \tag{10.7}$$

with $L([a\,;b]_{\sigma_n})$ the length of intervals $[a\,;b]_{\sigma_n}$ of the oriented curve σ_n for a fixed Riemannian metric on $\mathbf{T}^2$;

(2) $A^+(\mathcal{F}^{\nu_n}_\beta) = A^+(\mathcal{F}^{\mu}_\beta)$, with $\nu_n \coloneqq (\mu_n)_{S_n}$ the surgery of μ_n around σ_n with respect to S_n given by Proposition 8.5.

Let us temporarily admit this statement and conclude thanks to it the proof of Proposition 10.2. Denoting by $S_{i,n} \in \mathrm{Aff}^+(\sigma_{i,n})$ the affine transformations given by Lemma 10.3 and by $\nu_{i,n}$ the surgery $(\mu_{i,n})_{S_{i,n}}$, the limit (10.7) shows that $\lim d(\nu_{i,n}, \mu_{i,n}) = 0$ according to Proposition 8.5, with d the distance on $\mathcal{S}(\mathbf{T}^2, \mathsf{0}, \theta)$ defined in (8.1). We finally conclude that $\nu_{i,n}$ converges to μ_i in $\mathcal{S}(\mathbf{T}^2, \mathsf{0}, \theta)$, since $\mu_{i,n}$ does so according to (10.6). Since the closed α-leaf $\sigma_{i,n}$ is unchanged during the surgery given by Proposition 8.5, the α-leaves $\mathcal{F}_{\alpha}^{\nu_{1,n}}(0)=\sigma_{1,n}$ and $\mathcal{F}_{\alpha}^{\nu_{2,n}}(0)=\sigma_{2,n}$ remain closed and homotopic. Moreover $A^{+}(\mathcal{F}_{\beta}^{\nu_{1,n}})=A^{+}(\mathcal{F}_{\beta}^{\nu_{2,n}})=A^{+}_{\beta}$ by assumption on the $S_{i,n}$, which concludes the proof of Proposition 10.2. □

The last step in the proof of Theorem B is thus the:

Proof of Lemma 10.3. Note that our assumption on T_n implies that μ_n converges to μ according to Proposition 8.5, hence that $\mathcal{F}^{\mu_n}_{\alpha/\beta}$ converges to $\mathcal{F}^{\mu}_{\alpha/\beta}$ according to Lemma 6.5. We also emphasize that it may help the reader, to understand and picture some arguments in the coming proof, to keep in mind that the intial bi-foliation $(\mathcal{F}^{\mu}_{\alpha},\mathcal{F}^{\mu}_{\beta})$ can be assumed to be a *linear* bi-foliation according to [AGK03, Theorem1] (see also [MM25, Theorem A]).

Step 1: existence of S_n satisfying $A^{+}(\mathcal{F}_{\beta}^{(\mu_n)_{S_n}})=A^{+}(\mathcal{F}^{\mu}_{\beta})$. While the first-return map $P^{\sigma_n}_{\beta,\mu}$ is well-defined since $\mathcal{F}^{\mu}_{\beta}$ is minimal, we first check that a surgery around σ_n allows us to modify the asymptotic cycle of $\mathcal{F}^{\mu_n}_{\beta}$, since:

Fact 10.4. *Possibly passing to a subsequence, σ_n is a section of $\mathcal{F}^{\mu_n}_{\beta}$, and the first-return map $P^{\sigma_n}_{\beta,\mu_n}$ is thus well-defined.*

Proof. Since $\mathcal{F}^{\mu}_{\alpha/\beta}$ are minimal, μ is class A according to Lemma A.7. Since μ_n converges to μ and being class A is an open property, this shows that μ_n is class A for any large enough n, hence that $\mathcal{F}^{\mu_n}_{\beta}$ is a suspension according to Lemma 6.6. The simple closed curve σ_n is transverse to $\mathcal{F}^{\mu_n}_{\beta}$, and its homotopy class $[\sigma_n]$ satisfies $\mathbb{R}^{+}[\sigma_n]=A^{+}(\mathcal{F}^{\mu_n}_{\alpha})\neq A^{+}(\mathcal{F}^{\mu_n}_{\beta})$ since μ_n is class A. As we already showed in the case 1 of the proof of Lemma 9.9, this shows that σ_n intersect all the leaves of $\mathcal{F}^{\mu_n}_{\beta}$, which concludes the proof of the fact. □

We fix henceforth a Riemannian metric on $\mathbf{T}^2$ and denote by $L([a\,;b]_C)$ the length of an interval $[a\,;b]_C$ of a curve C. We emphasize that we are really interested along the proof in the lengths of intervals of curves and not only in the mere distance between points, and that we moreover pay attention to the orientation along those curves: $\lim L([x\,;x_n]_C)=0$ means that x_n converges to x *from the right* along the curve C. Although the length of the closed α-curves σ_n is not bounded, we first show that the distance between the first-return maps $P^{\sigma_n}_{\beta,\mu_n}$ and $P^{\sigma_n}_{\beta,\mu}$ converges to 0 in the following specific sense. The main reason for this convergence, is that the closed curve γ around which the surgery μ_n is performed, is *fixed*.

Fact 10.5. $\lim\limits_{n\to\infty}\max\limits_{x\in\sigma_n} L([P^{\sigma_n}_{\beta,\mu_n}(x)\,;P^{\sigma_n}_{\beta,\mu}(x)]_{\sigma_n})=0.$

Proof. Assume for a contradiction that there exists $k_n\in\mathbb{N}$ strictly increasing, $x_n\in\sigma_{k_n}$ and $\varepsilon>0$, such that

$$L([P^{\sigma_{k_n}}_{\beta,\mu_{k_n}}(x_n)\,;P^{\sigma_{k_n}}_{\beta,\mu}(x_n)]_{\sigma_{k_n}})\geq\varepsilon \tag{10.8}$$

for any k. To simplify the notations, we henceforth assume that $k_n=n$ which does not change the argument. By compactness we can assume without loss of generality that x_n converges to a point $x\in\mathbf{T}^2$.

Observe first that with y_n the first intersection point of $\mathcal{F}^{\mu}_{\beta}(x)$ with σ_n: $L([x\,;y_n]_{\beta,\mu})$ is non-increasing, hence bounded. The rough idea is that since σ_n is constituted of a non-decreasing number of segments of $\mathcal{F}^{\mu}_{\alpha}$ glued together, it cuts more and more often the β-foliation, decreasing the time a β-segment takes to meet σ_n again. Indeed, since T_n converges uniformly to id_{γ} from above on the timelike geodesic γ, our orientation conventions show that for $m\geq n$, $\mathcal{F}^{\mu}_{\beta}(x)$ has to meet σ_m at some point y' before it meets σ_n at y_n. If the β-segment $[x\,;y']_{\beta,\mu}$ does not meet σ_m before y', then $y'=y_m$, showing that $L([x\,;y_m]_{\beta,\mu})=L([x\,;y']_{\beta,\mu})\leq L([x\,;y_n]_{\beta,\mu})$, since y' is before y_n on $\mathcal{F}^{\mu}_{\beta}(x)$. If $[x\,;y']_{\beta,\mu}$ meets σ_m before y', then it is even better: $[x\,;y_m]_{\beta,\mu}$ is shorter than $[x\,;y']_{\beta,\mu}$, hence $L([x\,;y_m]_{\beta,\mu})\leq L([x\,;y']_{\beta,\mu})\leq L([x\,;y_n]_{\beta,\mu})$ again.

Since $L([x\,;y_n]_{\beta,\mu})$ is bounded and x_n converges to x, we can assume $[x_n\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\beta,\mu}$ to be arbitrarily close to $[x\,;y_n]_{\beta,\mu}$ by continuity of the foliation $\mathcal{F}^{\mu}_{\beta}$, hence $L([x_n\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\beta,\mu})$ is bounded as well. If the β-segment $[x_n\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\beta,\mu}$ did not intersect γ, then $P^{\sigma_n}_{\beta,\mu}(x_n)=P^{\sigma_n}_{\beta,\mu_n}(x_n)$

by definition of the surgery μ_n, which contradicts (10.8). Since $[x_n\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\beta,\mu}$ is arbitrarily close to $[x\,;y_n]_{\beta,\mu}$ and of bounded length, and since the curve γ around which the surgery μ_n is performed is fixed, there exists $d\in\mathbb{N}^*$ such that for any n sufficiently large $[x_n\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\beta,\mu}$ intersects γ in a finite and bounded number $d_n\leq d$ of points $(z^1_n,\dots,z^{d_n}_n)$ increasingly ordered on $\mathcal{F}^{\mu}_{\beta}(x_n)$. There are moreover sequences $(p^i_n)_{i=1,\dots,d_n+1}$ and $(q^i_n)_{i=1,\dots,d_n+1}$ such that: $p^1_n=x_n$, q^i_n is the first intersection point of $\mathcal{F}^{\mu}_{\beta}(p^i_n)$ with γ (hence $q^1_n=z^1_n$), $p^{i+1}_n:=T_n(q^i_n)$, and $q^{d_n+1}_n=P^{\sigma_n}_{\beta,\mu_n}(x_n)$. Since T_n converges uniformly to id_γ from above, p^{i+1}_n is above q^i_n on the timelike curve γ, and for any $\eta>0$ there exists N such that for any $n\geq N$ and $i=1,\dots,d_n$:

$$L([q^i_n\,;p^{i+1}_n]_\gamma)\leq\eta. \tag{10.9}$$

Note that our orientation conventions reverse the monotonicity, since "moving in the future on γ is equivalent to moving in the past on σ_n". Since $[x_n\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\beta,\mu}$ is arbitrarily close to $[x\,;y_n]_{\beta,\mu}$ and of bounded length, and by continuity of $\mathcal{F}^{\mu}_{\beta}$, there exists $\eta>0$ and $\eta_{d_n}>0$ such that (10.9) for $i=d_n$ together with $L([q^d_n\,;z^{d_n}_n]_{\alpha,\mu})\leq\eta_{d_n}$, imply that $L([q^{d_n+1}_n\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\alpha,\mu})=L([P^{\sigma_n}_{\beta,\mu_n}(x_n)\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\sigma_n})<\varepsilon$ for any sufficiently large n. But possibly reducing $d_n\leq d$ times η, at every step i, if $\eta_i>0$ is known there exists $\eta_{i-1}>0$ such that (10.9) together with $L([q^i_n\,;z^i_n]_{\alpha,\mu})\leq\eta_{i-1}$ implies $L([q^i_n\,;z^i_n]_{\alpha,\mu})\leq\eta_i$ for any sufficiently large n. The condition being satisfied at the first step since $q^1_n=z^1_n$, a finite recurrence shows the existence of $\eta>0$ small enough to ensure that $L([P^{\sigma_n}_{\beta,\mu_n}(x_n)\,;P^{\sigma_n}_{\beta,\mu}(x_n)]_{\sigma_n})<\varepsilon$ for any sufficiently large n. This contradicts our initial assumption and concludes the proof of the fact. □

For $S\in\mathrm{Aff}^+(\sigma_n)$ let us denote by $(\mu_n)_S$ the surgery of μ_n around the closed α-leaf σ_n with respect to S given by Proposition 8.5, such that:

1. $(\mu_n)_S$ has a unique singularity of angle θ at $\mathsf{0}$;
2. $\mathcal{F}^{(\mu_n)_S}_{\alpha}(\mathsf{0})=\mathcal{F}^{\mu_n}_{\alpha}(\mathsf{0})=\sigma_n$;
3. the first-return map of $\mathcal{F}^{(\mu_n)_S}_{\beta}$ on σ_n is well-defined and equal to the circle homeomorphism $P^{\sigma_n}_{\beta,(\mu_n)_S}=P^{\sigma_n}_{\beta,\mu_n}\circ S$.

Since we will eventually consider at the end of the proof first-return maps on the fixed simple closed curve γ to be able to obtain asymptotic estimates, we henceforth alternate between σ_n and γ in our analysis. Note that while γ is not anymore a geodesic of $(\mu_n)_S$, it remains however a section of $\mathcal{F}^{(\mu_n)_S}_{\beta}$ since it is a section of $\mathcal{F}^{\mu_n}_{\beta}$, and the first-return map $P^{\gamma}_{\beta,(\mu_n)_S}$ is therefore well-defined. Let $\sigma(t)$ be a fixed affine parametrisation of $\mathcal{F}^{\mu}_{\alpha}(\mathsf{0})$ starting from $\mathsf{0}=\sigma(0)$. Let $\sigma_n\colon[0\,;l_n]/\{0\sim l_n\}\xrightarrow{\sim}\sigma_n$ be the unique simple affine parametrisation of σ_n starting from $\mathsf{0}=\sigma_n(0)$ and coinciding with σ on an interval $[0\,;\varepsilon]$, $\varepsilon>0$. Then we denote by $t\in[0\,;l_n]/\{0\sim l_n\}\mapsto S^t_n\in\mathrm{Aff}^+(\sigma_n)$ the parametrisation such that $S^t_n(\mathsf{0})=\sigma_n(t)$. According to Lemma 8.4,

$$t\in[0\,;l_n]/\{0\sim l_n\}\mapsto P^{\sigma_n}_{\beta,(\mu_n)_{S^t_n}}(x)=P^{\sigma_n}_{\beta,\mu_n}\circ S^t_n(x)\in\sigma_n \tag{10.10}$$

is then a continuous, degree one and strictly increasing map for any $x\in\sigma_n$, and $t\in[0\,;l_n]\mapsto(\mu_n)_{S^t_n}$ is moreover continuous according to Proposition 8.5. This moreover shows that

$$t\in[0\,;l_n]\mapsto P^{\gamma}_{\beta,(\mu_n)_{S^t_n}}\in\mathrm{Homeo}^+(\gamma) \tag{10.11}$$

is continuous, and that

$$t\in[0\,;l_n]/\{0\sim l_n\}\mapsto P^{\gamma}_{\beta,(\mu_n)_{S^t_n}}(x)\in\gamma \tag{10.12}$$

is a continuous, degree one and strictly decreasing map for any $x\in\gamma$, since the holonomy of $\mathcal{F}^{\mu_n}_{\beta}$ induces homeomorphisms from small intervals of $\mathcal{F}^{\mu_n}_{\alpha}$ to small intervals of γ.

Note that our orientations conventions described in Figure 3.1 induce a reversal of the direction of the perturbation, wether it is observed on the first-return map on σ_n in (10.10) or on the first-return map on γ in (10.12). To say it roughly: "moving in the future on σ_n is equivalent to moving in the past on γ". Due to this change of orientation, the continuous maps $t\in[0\,;l_n]\mapsto\rho(P^{\gamma}_{\beta,(\mu_n)_{S^t_n}})\in\mathbf{S}^1$ and $t\in[0\,;l_n]\mapsto A^+(\mathcal{F}^{(\mu_n)_{S^t_n}}_{\beta})\in\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ are non-increasing according to Lemma B.1.(2) (the topological circle $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ being endowed with the orientation induced by the one of $\mathbf{T}^2$). On the other hand, $A^+(\mathcal{F}_\beta^{\mu_n})$ is decreasing to the irrational half-line $A^+(\mathcal{F}_\beta^{\mu})$ since T_n is assumed to converge to id_γ from above. In conclusion for any large enough n, $A^+(\mathcal{F}_\beta^{(\mu_n)_{S_n^t}})$ is slightly above $A^+(\mathcal{F}_\beta^{\mu})$ at $t=0$ and is non-increasing with t. The distance of $A^+(\mathcal{F}_\beta^{(\mu_n)_{S_n^t}})$ to $A^+(\mathcal{F}_\beta^{\mu})$ on the circle $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ is thus non-increasing.

Since $t\in[0\,;l_n]/\{0\sim l_n\}\mapsto P^\gamma_{\beta,(\mu_n)_{S_n^t}}(x)\in\gamma$ is surjective for any $x\in\gamma$ according to (10.12), the map $t\in[0\,;l_n]/\{0\sim l_n\}\mapsto\rho(P^\gamma_{\beta,(\mu_n)_{S_n^t}})\in\mathbf{S}^1$ is also surjective according to Lemma B.1.(3). There exists thus a smallest time $t_n\in[0\,;l_n]$ satisfying

$$\rho(P^\gamma_{\beta,\nu_n})=\rho(P^\gamma_{\beta,\mu}) \tag{10.13}$$

with $S_n:=S_n^{t_n}\in\mathrm{Aff}^+(\sigma_n)$ and $\nu_n:=(\mu_n)_{S_n}$. According to Proposition 5.9, this implies that

$$A^+(\mathcal{F}_\beta^{\nu_n})=D^k_{[\gamma]}(A^+(\mathcal{F}_\beta^{\mu})) \tag{10.14}$$

for some $k\in\mathbb{Z}$, with $D_{[\gamma]}$ the positive Dehn twist around γ. Note that $[\gamma]$ is an attractive fixed point of $D_{[\gamma]}$, and that $A^+(\mathcal{F}_\beta^{\mu_n})\in\,]D^{-1}_{[\gamma]}(A^+(\mathcal{F}_\beta^{\mu}))\,;A^+(\mathcal{F}_\beta^{\mu})]$ for any large enough n. Hence by definition of t_n, we have $A^+(\mathcal{F}_\beta^{(\mu_n)_{S_n^s}})\in\,]D^{-1}_{[\gamma]}(A^+(\mathcal{F}_\beta^{\mu}))\,;A^+(\mathcal{F}_\beta^{\mu})]$ for any $s\in[0\,;t_n[$, and therefore (10.14) actually implies

$$A^+(\mathcal{F}_\beta^{\nu_n})=A^+(\mathcal{F}_\beta^{\mu}), \tag{10.15}$$

which was our initial goal. Note that for any n, denoting $F_n(t)=\rho(P^\gamma_{\beta,(\mu_n)_{S_n^t}})$ we have:

$$F_n([0\,;t_n])=[\rho(P^\gamma_{\beta,\mu})\,;\rho(P^\gamma_{\beta,\mu_n})] \tag{10.16}$$

since t_n is the smallest time where the equality (10.13) is satisfied. Let $t_0>0$ be the first intersection time of $\sigma(t)$ with γ.

Fact 10.6. *For any large enough n: $t_n<t_0$.*

Proof. Let η the closed curve formed by the concatenation of the segments $[0\,;\sigma(t_0)]_\alpha$ and $[\sigma(t_0)\,;0]_\gamma$. Since the Dehn twist $D_{[\eta]}$ has a north-south dynamics on $\mathbb{R}\mathbf{P}^1_+$ with attractive and repulsive fixed points $\mathbb{R}^+[\pm\eta]$, and since $A^+(\mathcal{F}_\beta^{\mu_n})$ converges to $A^+(\mathcal{F}_\beta^{\mu})$ from the right, we have $A^+(\mathcal{F}_\beta^{\mu_n})\in[A^+(\mathcal{F}_\beta^{\mu})\,;D_{[\eta]}(A^+(\mathcal{F}_\beta^{\mu}))[$ for any large enough n. Therefore $D_{[\eta]}(A^+(\mathcal{F}_\beta^{\mu_n}))\in[D_{[\eta]}(A^+(\mathcal{F}_\beta^{\mu}))\,;[\eta][$, and in particular $A^+(\mathcal{F}_\beta^{\mu})\notin[D_{[\eta]}(A^+(\mathcal{F}_\beta^{\mu_n}))\,;-[\sigma_n]]$. In the other hand since $S_n^t(0)=\sigma_n(t)$, we have $A^+(\mathcal{F}_\beta^{(\mu_n)_{S_n^t}})\in[D_{[\eta]}(A^+(\mathcal{F}_\beta^{\mu_n}))\,;-[\sigma_n]]$ for any large enough n and $t\in[t_0\,;l_n]$ by definition of the surgery $(\mu_n)_{S_n^t}$. This implies in particular $A^+(\mathcal{F}_\beta^{(\mu_n)_{S_n^t}})\neq A^+(\mathcal{F}_\beta^{\mu})$ for any large enough n and $t\in[t_0\,;l_n]$, which shows our claim. □

Step 2: convergence of S_n to the identity. To conclude the proof of Lemma 10.3, it remains now to control the size of the surgery ν_n around σ_n, by proving the limit (10.7) that we recall for the convenience of the reader:

$$\lim\max_{x\in\sigma_n}L([x\,;S_n(x)]_{\sigma_n})=0. \tag{10.17}$$

We proceed by contradiction and assume thus that the limit (10.17) does not hold. There exists then $\varepsilon_1>0$, a strictly increasing sequence $k_n\in\mathbb{N}$ (assumed to be equal to n to simplify notations, which does not change the argument), and points $x_n\in\sigma_{k_n}=\sigma_n$, such that for all n:

$$L([x_n\,;S_n(x_n)]_{\sigma_n})\geq\varepsilon_1. \tag{10.18}$$

Denoting $P^{\sigma_n}_{\beta,\mu_n}=P^{\sigma_n}_{\beta,\mu}\circ U_n$ so that $P^{\sigma_n}_{\beta,\nu_n}=P^{\sigma_n}_{\beta,\mu}\circ U_n\circ S_n$, it is important to note at this point that U_n is *not* an affine transformation of σ_n, since the computation of U_n involves the holonomy of $\mathcal{F}_\beta^{\mu}$ between γ and segments of leaves of $\mathcal{F}_\alpha^{\mu}$, which is not affine but only *projective*. Therefore, while U_n converges to the identity since $\mathcal{F}_\beta^{\mu_n}$ converges to $\mathcal{F}_\beta^{\mu}$, we are now comparing maps U_n and S_n of σ_n which are *not* in the same one-parameter group of $\mathrm{Homeo}^+(\sigma_n)$, and this is what makes the proof of (10.17) more technical than expected. Since $P^{\sigma_n}_{\beta,\mu_n}$ converges to $P^{\sigma_n}_{\beta,\mu}$ from below according to Fact 10.5, we would like to infer that for any large enough n, $P^{\sigma_n}_{\beta,\nu_n}$ pushes every point x above $P^{\sigma_n}_{\beta,\mu}(x)$ by a distance bounded from below. This would show that $\rho(P^{\sigma_n}_{\beta,\nu_n}) \neq \rho(P^{\sigma_n}_{\beta,\mu})$ according to Lemma B.1.(4), contradicting (10.15) according to Corollary 5.10, and concluding thus the proof. The only possible phenomenon preventing us to apply this argument straightforwardly this way, and forcing us to be more cautious, is that some points x may be moved by $P^{\sigma_n}_{\beta,\nu_n}$ above $P^{\sigma_n}_{\beta,\mu}(x)$ while some other may move between $P^{\sigma_n}_{\beta,\mu_n}(x)$ and $P^{\sigma_n}_{\beta,\mu}(x)$. But since all of them are in any case pushed above $P^{\sigma_n}_{\beta,\mu_n}(x)$ which itself uniformly approaches $P^{\sigma_n}_{\beta,\mu}(x)$ from below, the uniform lower bound (10.18) allows us to apply the same argument on the limit, and to conclude by continuity of the rotation number. We now implement this strategy as follows.

Let us consider the first intersection point q_n (respectively r_n) of $\mathcal{F}^{\mu_n}_\beta(x_n)$ (resp. $\mathcal{F}^{\mu_n}_\beta(S_n(x_n))$) with γ in the future. By compactness, we can assume without loss of generality that x_n converges to a point $x \in \mathbf{T}^2$ and that q_n and r_n converge in γ, by taking subsequences and relabelling them. In particular for any large enough n, the intervals $[x_n\,;S_n(x_n)]_{\sigma_n}$ of σ_n are plaques of the foliation $\mathcal{F}^\mu_\alpha$ contained in the domain U of a given foliated chart of $\mathcal{F}^\mu_\alpha$ around x. Since $\mathcal{F}^{\mu_n}_\beta$ converges to $\mathcal{F}^\mu_\beta$, and since the holonomy of $\mathcal{F}^\mu_\beta$ induces a homeomorphism between the plaques of the foliation $\mathcal{F}^\mu_\alpha$ in U and the fixed timelike curve γ, we infer then from (10.18) the existence of $\varepsilon_2 > 0$ such that

$$L([r_n\,;q_n]_\gamma) \geq \varepsilon_2 \tag{10.19}$$

for any n. With p_n the first intersection point of $\mathcal{F}^{\nu_n}_\beta(x_n)$ with γ in the past, observe that $P^\gamma_{\beta,\nu_n}(p_n)$ is not necessarily equal to r_n. Indeed if the β-segment $]S_n(x_n)\,;r_n]_{\beta,\mu_n}$ meets σ_n, then S_n twists again in the future the leaf $\mathcal{F}^{\mu_n}_\beta$ after exiting at $S_n(x_n)$. But the important observation is that any further twist push in the same direction: the future of σ_n. Our orientation conventions ensure thus that $(P^\gamma_{\beta,\nu_n}(p_n), r_n, q_n, P^\gamma_{\beta,\mu_n}(p_n))$ is in any case a positively oriented quadruplet of the future-oriented timelike curve γ. Consequently, (10.19) implies that for any n:

$$L([P^\gamma_{\beta,\nu_n}(p_n)\,;P^\gamma_{\beta,\mu_n}(p_n)]_\gamma) \geq \varepsilon_2. \tag{10.20}$$

Since $\mathcal{F}^{\mu_n}_\beta$ $\mathcal{C}^0$-converges to $\mathcal{F}^\mu_\beta$, P^γ_{β,μ_n} converges to $P^\gamma_{\beta,\mu}$ for the compact-open topology on $\mathrm{Homeo}^+(\gamma)$. On the other hand, since $t_n \in [0\,;t_0]$ is bounded according to Fact 10.6, we may assume according to the Arzelà-Ascoli theorem that P^γ_{β,ν_n} converges to some continuous map $P_\infty\colon \gamma \to \gamma$ (by passing to a subsequence). Note that while P_∞ is not necessarily a homeomorphism, it remains an orientation-preserving *endomorphism* of γ, *i.e.* by definition a continuous, degree-one and orientation-preserving self-map of γ. According to [PJM82, Appendix Lemma 3] and [NPT83, Chapter III Proposition 3.3], Proposition-Definition 5.1 defining the rotation number extends to endomorphisms of γ, and the rotation number remains moreover continuous on $\mathrm{End}^+(\gamma)$. The equality (10.13) yields thus

$$\rho(P_\infty) = \rho(P^\gamma_{\beta,\mu}) \tag{10.21}$$

at the limit. Up to taking a subsequence, we can assume that $p_n \in \gamma$ converges to a point $p \in \gamma$, and the uniform bound (10.20) becomes then

$$L([P_\infty(p)\,;P^\gamma_{\beta,\mu}(p)]_\gamma) \geq \varepsilon_2 > 0. \tag{10.22}$$

by uniform convergence of P^γ_{β,μ_n} and P^γ_{β,ν_n} to $P^\gamma_{\beta,\mu}$ and P_∞. For any n, $G_n\colon s \in [0\,;1] \mapsto P^\gamma_{\beta,(\mu_n)_{S_n^{st_n}}} \in \mathrm{Homeo}^+(\gamma)$ is according to (10.11) a continuous one-parameter family from $(G_n)_0 = P^\gamma_{\beta,\mu_n}$ to $(G_n)_1 = P^\gamma_{\beta,\nu_n}$, and $s \in [0\,;1] \mapsto (G_n)_s(y)$ is moreover non-increasing for any $y \in \gamma$ according to (10.12). Since $t_n \in [0\,;t_0]$ is bounded, possibly passing to a subsequence, these continuous maps G_n uniformly converge to a continuous map $G\colon [0\,;1] \mapsto G_t \in \mathrm{End}^+(\gamma)$ such that $G_0 = P^\gamma_{\beta,\mu}$, $G_1 = P_\infty$ and $t \mapsto G_t(y)$ is non-increasing for any $y \in \gamma$. Moreover (10.16) shows that $t \in [0\,;1] \mapsto \rho(G_t) \in \mathbf{S}^1$ is not surjective, while (10.22) shows that $G_1(p) \neq G_0(p)$. The proof of Lemma B.1.(4) holds now without any modification for circle endomorphisms G_t and shows thus that $\rho(P_\infty) \neq \rho(P^\gamma_{\beta,\mu})$, which contradicts (10.21). This contradiction eventually shows that the limit (10.17) holds, and concludes the proof of the lemma. □

Appendix A. Simple closed definite geodesics in singular constant curvature Lorentzian surfaces

The main goal of this appendix is to prove the existence of simple closed timelike geodesics in any de-Sitter torus having a unique singularity. More precisely, we prove the following existence result which is a direct consequence of Proposition A.8, Theorem A.17 and Corollary A.11 proved below.

Theorem A.1. *Let μ_1 and μ_2 be two class A singular $\mathbf{dS}^2$-structures on a torus, having a unique singularity, and identical oriented lightlike bi-foliations. Then μ_1 and μ_2 admit freely homotopic simple closed timelike geodesics avoiding the singularity, which are not null-homotopic.*

This appendix being entirely independent from the rest of the paper, the reader may choose to use this result as a "black-box" in a first reading, and to come back to its proof later on. We emphasize that in all this appendix, what we call a *simple closed* timelike geodesic avoiding singularities is a curve with periodic derivative, *i.e.* a curve whose lift in the tangent bundle is simple closed.

This result is well-known for regular Lorentzian surfaces, see for instance [Tip79, Gal86, Suh13], and we show here that it remains valid in our singular setting. While it is *a priori* not clear that the usual tools and results of Lorentzian geometry can be used in our singular setting, the goal of this appendix is precisely to show that this toolbox persists in the setting of singular $\mathbf{X}$-surfaces, which may have an independent interest in the future for their further study. Notions and results of this section are well-known in the classical setting of regular Lorentzian manifolds, and their proofs are mainly adapted from [Min19] or [BEE96]. We essentially follow the proof of [Tip79] to show Theorem A.1, with slight adaptations more suited to our setting. The main idea is to prove the existence of a simple closed timelike curve which *maximizes* the Lorentzian length, which is the extremal property of Lorentzian timelike geodesics in contrast with Riemannian ones.

The main subtelty and novelty of the result is contained in Corollary A.11, where we highlight a surprising and interesting phenomenon, specific to the singular setting. Indeed, locally maximizing timelike curves avoid the positive singularities, while locally maximizing spacelike curves avoid the negative ones. This is the only reason why Theorem A.1 is specific to the case of a unique singularity: in this case, the singularity is avoided by a simple closed locally maximizing timelike curve.

We work in this section in the general setting of singular $\mathbf{X}$-surfaces, $(\mathbf{G}, \mathbf{X})$ denoting as usual the pair $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ or $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1), \mathbb{R}^{1,1})$.

A.1. Timelike curves and causality notions.

In a Lorentzian surface (S,g), we call *anticausal* the tangent vectors and the curves which are causal for the Lorentzian metric $-g$. The following definition is identical to the classical one, to the exception of condition (1) handling the singular points.

Definition A.2. In a singular $\mathbf{X}$-surface (S,Σ), a *timelike* (respectively *causal*, *spacelike*, *anticausal*) curve is a continuous curve $\sigma\colon [a\,;b] \to S$ such that:

1. for any $t_0 \in [a\,;b]$, there exists $\varepsilon > 0$ and a singular $\mathbf{X}$-chart domain U containing $\gamma(t_0)$, such that $\gamma|_{]t_0-\varepsilon;t_0[} \subset U^-$ and $\gamma|_{]t_0;t_0+\varepsilon[} \subset U^+$, with U^- and U^+ the past and future timelike (resp. spacelike, causal, anticausal) quadrants in U;
2. σ is locally Lipschitz;
3. $\sigma'(t)$ is almost everywhere non-zero, future-directed and timelike (resp. causal, spacelike, anticausal).

We emphasize that timelike, causal, spacelike and anticausal curves are in particular always assumed to be relatively compact and future-oriented, unless explicitly stated otherwise. They are moreover not *trivial* (*i.e.* reduced to a point), and $\sigma^{-1}(\Sigma)$ is discrete according to (1), hence finite. S is always endowed with an auxiliary $\mathcal{C}^\infty$ Riemannian metric h and its induced distance d, with respect to which the Lipschitz conditions are considered. Note that σ is compact and locally Lipschitz, hence Lipschitz. A locally Lipschitz function being almost everywhere differentiable according to Rademacher's Theorem, $\sigma'(t)$ is almost everywhere defined which gives sense to the condition (3). *Past* timelike, causal, spacelike and anticausal curves are defined as future-oriented curves of the same signature travelled in the opposite direction.

Definition A.3. In a singular $\mathbf{X}$-surface S, we denote for $x \in S$ by:

(1) $I^+(x)$ (respectively $I^-(x)$) the set of points that can be reached from x by a timelike (resp. past timelike) curve;
(2) $J^+(x)$ (respectively $J^-(x)$) the set of points that can be reached from x by a causal (resp. past causal) curve.

We denote $I^+_S(x)$ and likewise for the other notions, to specify that the curves are assumed to be contained in S. An open set U of a singular $\mathbf{X}$-surface S is *causally convex* if there exists no causal curve of S which intersects U in a disconnected set. S is said *strongly causal* if any point of S admits arbitrarily small causally convex open neighbourhoods. In particular S is then *causal*, *i.e.* admits no non-trivial closed causal curves. S is *globally hyperbolic* if it is strongly causal, and if for any $p, q \in S$, the *causal diamond* $J^+(p) \cap J^-(q)$ is relatively compact.

Observe that in the domain U of any chart of the singular $\mathbf{X}$-atlas containing x and of future and past timelike quadrants U^+ and U^-, $I^\pm_U(x) = U^\pm$. This is classical in the regular Lorentzian setting (see for instance [Min19, Theorem 2.9 p.29]) and follows from our definition of timelike and causal curves at a singular point. Observe moreover that a $\mathbf{X}$-structure on $\mathbb{R}^2$ has no closed lightlike leaves, as a consequence of the classical Poincaré-Hopf theorem for topological foliations proved for instance in [HH86, Theorem 2.4.6]. The following result is well-known for regular Lorentzian metrics on $\mathbb{R}^2$, and we give here a quick argument using the Haefliger-Reeb theorem on foliations of the plane.

Lemma A.4. *Let F be a lightlike leaf of a singular $\mathbf{X}$-surface homeomorphic to $\mathbb{R}^2$. Then a timelike (respectively spacelike) curve, or a lightlike leaf distinct from F, intersects F at most once.*

Proof. Let $\mathbb{R}^2$ be endowed with a singular $\mathbf{X}$-structure, and assume that F is an α-leaf. Since two distinct leaves of the same foliation obviously not meet, it is sufficient for lightlike foliations to prove the claim for a β-leaf. Let thus $\sigma\colon I \to \mathbb{R}^2$ be an injective and lightlike, or locally injective and timelike curve, defined on an interval $I \subset \mathbb{R}$. Denoting by V the space of leaves of the α-foliation of $\mathbb{R}^2$, σ induces a continuous curve $\bar{\sigma}\colon I \to V$, which is strictly monotonous since σ is locally injective and transverse to $\mathcal{F}_\alpha$. According to Haefliger-Reeb theorem [HR57, Proposition 1 p.121] (see [HR22, Proposition 3 p.14] for an english translation), V is a 1-dimensional (possibly non-Hausdorff) simply connected topological manifold, and therefore $\bar{\sigma}$ cannot be closed. This shows that σ does not meet F more than once, and concludes the proof of the lemma. ☐

Lemma A.4 implies in particular that for any α-lightlike (respectively β-lightlike) leaf F of a singular $\mathbf{X}$-structure on $\mathbb{R}^2$ and for any $x \in F$, there exists a transversal T to the α-foliation (resp. β-foliation) intersecting F only at x. It suffices indeed to take for T a timelike curve through x. This means by definition that the lightlike leaves of a singular $\mathbf{X}$-structure on $\mathbb{R}^2$ are *proper*.

Corollary A.5. *Any singular $\mathbf{X}$-surface homeomorphic to $\mathbb{R}^2$ is strongly causal.*

Proof. Assume by contradiction that a singular $\mathbf{X}$-structure on $\mathbb{R}^2$ is not strongly causal. Then there exists a point $x \in \mathbb{R}^2$, a chart domain U of the singular $\mathbf{X}$-atlas containing x, and a causal curve starting from x, leaving U and returning to it. It is easy to deform this curve to a timelike curve σ with the same properties. We can moreover choose the boundary of U to be the union of lightlike segments, and denote by I one of these segments which is first met by σ when it leaves U. We can then clearly extend σ if necessary, for it to be a timelike curve intersecting I twice. This contradicts Lemma A.4 and concludes the proof. ☐

Corollary A.6. *A singular $\mathbf{X}$-surface of universal cover homeomorphic to $\mathbb{R}^2$ does not admit any non-trivial null-homotopic closed causal or anti-causal curve.*

Proof. It is sufficient to treat causal curves by symmetry. But a non-trivial null-homotopic closed causal curve would lift to a non-trivial closed causal curve of a singular $\mathbf{X}$-structure on $\mathbb{R}^2$, contradicting Corollary A.5. $\square$

We recall that for $S \simeq \mathbf{T}^2$ a closed singular $\mathbf{X}$-surface, a line l in $\mathrm{H}_1(S,\mathbb{R}) \simeq \mathbb{R}^2$ is said *rational* if it passes through $\mathrm{H}_1(S,\mathbb{Z}^2) \simeq \mathbb{Z}^2$ and *irrational* otherwise, and that S is *class A* if the projective asymptotic cycles of its α and β lightlike foliations are distinct: $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$, and is *class B* otherwise.

Lemma A.7. *A closed singular $\mathbf{X}$-surface S is class B if, and only if both of its lightlike foliations have closed leaves which are freely homotopic up to orientation, and it is class A otherwise. In particular, if one of the lightlike foliations has irrational projective asymptotic cycle, then S is class A.*

Proof. If the lightlike foliations have closed leaves which are not freely homotopic up to orientation, then since two primitive element $c_\alpha \neq \pm c_\beta$ of $\pi_1(S)$ are not proportional in $\mathrm{H}_1(\mathbf{T}^2,\mathbb{R})$, the projective asymptotic cycles are distinct according to Lemma 5.4 and S is thus class A. If only one of the lightlike foliations has a closed leaf, then it has a rational projective asymptotic cycle while the other lightlike foliation has an irrational cycle, hence $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$.

If none of the lightlike foliations have closed leaves, then none of them has a Reeb component, hence both of them is a suspension of a homeomorphism according to Proposition 5.7, having irrational rotation number. The latter is a $\mathcal{C}^\infty$ diffeomorphism with breaks, and is thus minimal according to Lemma 3.24.(4). Hence $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ is a pair of transverse and minimal foliations of $\mathbf{T}^2$. According to [AGK03, Theorem 1 p.458] (see also [MM25, Theorem A]), such a minimal bi-foliation of $\mathbf{T}^2$ is topologically (simultaneously) conjugated to a linear bi-foliation. Since two transverse linear foliations have distinct asymptotic cycles, this shows that $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$ (see also [AGK03, Step 1 of the Proof of Theorem 1 p.460] for a direct argument), and concludes the proof of the lemma. $\square$

Proposition A.8. *Let μ_1 and μ_2 be two class A singular $\mathbf{X}$-structures on $\mathbf{T}^2$ having identical oriented lightlike bi-foliations. Then for any $x \in \mathbf{T}^2$ we have the following.*

(1) μ_1 and μ_2 admit freely homotopic simple closed timelike (respectively spacelike) curves passing through x which are not null-homotopic.

(2) Let a be a simple closed timelike curve of μ_1 (respectively μ_2). Then the minimal number of intersection points of any simple closed spacelike curve with a is:

  (a) 2 if $A^+(\mathcal{F}^{\mu_i}_{\alpha/\beta}) = \mathbb{R}^+ c_{\alpha/\beta}$, with $c_{\alpha/\beta} \in \pi_1(\mathbf{T}^2)$ two primitive classes of algebraic intersection number equal to 1;

  (b) and 1 otherwise.

Proof. The oriented projective asymptotic cycles of the lightlike foliations of a class A singular $\mathbf{X}$-surface $(\mathbf{T}^2,\mu)$ delimit an open *timelike cone*

$$\mathcal{C}_\mu = \mathrm{Int}(\mathrm{conv}(A^+(\mathcal{F}_\beta) \cup (-A^+(\mathcal{F}_\alpha)))) \subset \mathrm{H}_1(\mathbf{T}^2,\mathbb{R}) \tag{A.1}$$

in the homology, and likewise an open *spacelike cone* $\mathcal{C}^{\mathrm{space}}_\mu = \mathrm{Int}(\mathrm{conv}(A^+(\mathcal{F}_\alpha) \cup A^+(\mathcal{F}_\beta)))$.
(1) We identify the action of $\pi_1(\mathbf{T}^2)$ on the universal cover $\pi\colon \mathbb{R}^2 \to \mathbf{T}^2$ with the translation action of $\mathbb{Z}^2$, and endow $\mathbb{R}^2$ with the induced singular $\mathbf{X}$-structures $\tilde{\mu}_1 \coloneqq \pi^*\mu_1$ and $\tilde{\mu}_2 \coloneqq \pi^*\mu_2$ and with a $\mathbb{Z}^2$-invariant auxiliary complete Riemannian metric. With $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ the common lightlike foliations of $\tilde{\mu}_1$ and $\tilde{\mu}_2$, the half-leaves $\mathcal{F}^+_\beta(p)$ and $\mathcal{F}^-_\alpha(p)$ are for any $p \in \mathbb{R}^2$ proper embeddings of $\mathbb{R}^+$. They intersect furthermore only at p according to Lemma A.4, and delimit thus a closed subset $C_p \subset \mathbb{R}^2$ of boundary $\mathcal{F}^-_\alpha(p) \cup \mathcal{F}^+_\beta(p)$ containing all the timelike curves emanating from p. On the other hand there exists a constant $K > 0$ such that for any $p \in \mathbb{R}^2$, $\mathcal{F}_\alpha(p)$ and $\mathcal{F}_\beta(p)$ are respectively contained in the K-neighbourhoods of the affine lines $p + A(\mathcal{F}_\alpha)$ and $p + A(\mathcal{F}_\beta)$. This property follows from the equivalence between asymptotic cycles and winding numbers described in [Sch57, p. 278], which is also very well explained in [Suh13, §3.1]. In particular, there exists p_0 in the timelike cone $\mathcal{C} \coloneqq \mathcal{C}_{\mu_1} = \mathcal{C}_{\mu_2}$ in homology defined in (A.1), such that with $\mathcal{C}' \coloneqq p_0 + \mathcal{C}$: $x + \mathcal{C}' \subset \mathrm{Int}(C_x)$ for any $x \in \mathbb{R}^2$. We fix henceforth $x \in \mathbb{R}^2$ and $c \in \mathcal{C}'$, and we have then

$x+c \in \mathrm{Int}(C_x)$, and in particular $x+c \notin \mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x)$. Moreover the half-leaves $\mathcal{F}^-_\beta(x+c)$ and $\mathcal{F}^-_\alpha(x)$ intersect, at a unique point y according to Lemma A.4, and $y \notin \{x, x+c\}$ since $x+c \notin \mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x)$.

Let $\tilde{\nu}$ denote the curve from x to $x+c$ defined in $\mathbb{R}^2$ by following $\mathcal{F}^-_\alpha(x)$ from x to y and then $\mathcal{F}^+_\beta(y)$ from y to $x+c$. Then by construction, $\tilde{\nu}$ is a piecewise lightlike and a causal curve of $\tilde{\mu}_1$ and $\tilde{\mu}_2$, and it is furthermore contained in the closure of the cone $C_x \subset \mathbb{R}^2$. In particular, $\tilde{\nu}$ is *not* entirely contained in a lightlike leaf $\mathcal{F}_\alpha(x)$ or $\mathcal{F}_\beta(x+c)$ since $y \notin \{x, x+c\}$. Let ν denote the projection of $\tilde{\nu}$ to $\mathbf{T}^2$, which a piecewise lightlike and causal closed curve of μ_1 and μ_2 passing through $\bar{x} := \pi(x)$. Since the causal curve ν is not entirely contained in a single lightlike leaf, it can be slightly deformed to a closed timelike curve σ of μ_1 and μ_2, passing through $\bar{x}$ and homotopic to ν. Note that the condition of being timelike depends only on the lightlike bi-foliation, and that ν can therefore indeed be deformed to a curve σ which is timelike both for μ_1 and for μ_2.

Let $t = \sup\left\{ s \in [0\,;1] \;\middle|\; \sigma|_{[0;s[} \text{ is injective}\right\}$ (note that $t > 0$ since timelike curves are locally injective) so that $\sigma(t)$ is the first self-intersection point of σ with itself, and let $u \in [0\,;t[$ denote the unique time for which $\sigma(t) = \sigma(u)$. If $u = 0$, *i.e.* $\sigma(t) = \sigma(u) = \sigma(0)$, then we define $\gamma := \sigma|_{[0;t]}$. If $u \neq 0$, then we define σ_1 as the curve constituted by $\sigma|_{[0;u]}$ followed by $\sigma|_{[t;1]}$, and repeat the process on σ_1. Using for instance Fact A.14 to be proved below, there exists $\varepsilon > 0$ such that for any $s \in [0\,;1]$, $\sigma|_{]s-\varepsilon;s+\varepsilon[}$ is injective. Therefore this process finishes in a finite number of steps by compactness of σ, and yields a simple closed subcurve γ of σ passing through $\bar{x} \in \mathbf{T}^2$. This simple closed timelike curve γ of μ_1 and μ_2 passing through $\bar{x}$ cannot be null-homotopic according to Corollary A.6, which concludes the proof of the claim.

(2) Let $\mathcal{C}'$ be the sub-cone of the future spacelike cone $\mathcal{C}^{\mathrm{space}}$ in homology introduced in the proof of (1), such that $p + \mathcal{C}' \subset \mathrm{Int}(C^{\mathrm{space}}_p)$ for any $p \in \mathbb{R}^2$ with $C^{\mathrm{space}}_p \subset \mathbb{R}^2$ the closed subset of boundary $\mathcal{F}^+_\alpha(p) \cup \mathcal{F}^+_\beta(p)$ in the future of p. Then in the case (b) (respectively (a)), there exists a free homotopy class $c \in \pi_1(\mathbf{T}^2)$, contained in $\mathcal{C}'$ and of algebraic intersection number $\hat{i}(c, [a]) = 1$ (resp. $\hat{i}(c, [a]) = 2$) with $[a]$. The proof of the first claim of the proposition yields moreover a closed spacelike curve σ through $x = a(0)$ in the free homotopy class c. Since σ and a intersect only transversally and with a positive sign according to our orientations conventions (see Figure 3.1), $\hat{i}([\sigma], [a]) = 1$ (res. $\hat{i}([\sigma], [a]) = 2$) implies moreover that σ and a intersect only at x (resp. at two points). With γ the simple closed subcurve of σ through x constructed in the first part of the proof, a and γ intersect thus again only at $x = a(0) = \sigma(0)$ (resp. at most two points). In case (a), since $\hat{i}(c', [a]) \geq 2$ for any $c' \in \mathcal{C}^{\mathrm{space}}$, a and γ intersect indeed at two points, which concludes the proof of the claim. □

A.2. **Lorentzian length, time-separation and extremal curves.** We define the *Lorentzian length* of a causal curve $\gamma\colon [0\,;l] \to S$ in a singular $\mathbf{X}$-surface (S, Σ) by

$$L(\gamma) := \int_0^l \sqrt{-\mu_S(\gamma'(t))}\,dt \in [0\,;+\infty].$$

Similarly, we define the length of an anticausal curve by $L^+(\gamma) := \int_0^l \sqrt{\mu_S(\gamma'(t))}dt$. Causal curves being almost everywhere differentiable (see Paragraph A.1 for more details), this quantity is well-defined and moreover independent of the (locally Lipschitz) parametrization of γ thanks to the change of variable formula. An important remark to keep in mind for this whole paragraph is that singular points do not play any role in the length of a causal curve γ in S. Indeed since $\gamma^{-1}(\Sigma)$ is finite, γ is the concatenation of a finite number n of *regular pieces*, namely the connected components γ_i of $\gamma \cap S^*$ with $S^* := S \setminus \Sigma$, and we have

$$L(\gamma) = \sum_{i=1}^n L(\gamma_i), \tag{A.2}$$

the lengths appearing in the right-hand finite sum being computed in the regular Lorentzian surface S^*. The Lorentzian length allows us to define on $S \times S$ the *time-separation function* by

$$\tau_S(x,y) := \sup_\sigma L_S(\sigma) \in [0\,;+\infty], \tag{A.3}$$

the sup being taken on all future causal curves in S going from x to y if such a curve exists (*i.e.* if $y \in J^+(x)$), and by $\tau_S(x,y) = 0$ otherwise. We also define the similar notion of *space-separation function* $\tau_S^+(x,y) \coloneqq \sup_\sigma L_S^+(\sigma)$, the sup being taken on all future anticausal curves from x to y, and extended to $\tau_S^+(x,y) = 0$ if no such curve exists. To avoid any confusion we emphasize that, on the contrary to τ_S, the Lorentzian length $L(\gamma)$ computed in any open subset $U \subset S$ of course agrees with the one computed in S, which is why we do not bother to specify S in the notation $L(\gamma)$.

Lemma A.9. *Let $y \in J^+(x)$ and $z \in J^+(y)$, then $\tau_S(x,z) \geq \tau_S(x,y) + \tau_S(y,z)$.*

Proof. The same exact proof than in the regular setting (see for instance [Min19, Theorem 2.32]) works in our case, and we repeat it here for the reader to get a grasp of the Lorentzian specificities. If $\tau(x,y)$ or $\tau(y,z)$ is infinite, then using concatenations of causal curves from x to y and from y to z, one easily constructs causal curves of arbitrarily large lengths going from x to z, which proves the inequality (with equality). Assume now that $\tau(x,y)$ and $\tau(y,z)$ are both finite, let $\varepsilon > 0$ and γ, σ be causal curves respectively going from x to y and from y to z such that $L(\gamma) \geq \tau_S(x,y) - \varepsilon$ and $L(\sigma) \geq \tau_S(y,z) - \varepsilon$. Then the causal curve ν equal to the concatenation of γ and σ goes from x to z, hence $\tau_S(x,z) \geq L(\nu) = L(\gamma) + L(\sigma) \geq \tau_S(x,y) + \tau_S(y,z) - 2\varepsilon$ by the definition of τ_S, which proves the claim by letting ε converge to 0. $\square$

The above reverse triangle inequality is a way to explain the so-called *twin "paradox"* (see for instance [O'N83, Example 22 p.173] for more details). It is important to keep in mind that all the usual inequalities, suprema and infima encountered in Riemannian geometry when dealing with lengths of curves and geodesics are exchanged in Lorentzian geometry for causal and anticausal curves, as the reverse triangle inequality of Lemma A.9 already showed. The best way to understand this phenomenon (confusing at first sight), is for the reader to explicitly check in the case of the Minkoswki plane $\mathbb{R}^{1,1}$ that timelike geodesics realize the maximal length of a causal curve between two points. A future causal curve $\gamma\colon I \to S$ is said to be *locally maximizing* if for any $t \in I$ there exists a connected neighbourhood $I_t = [a_t\,; b_t]$ of t in I and a connected open neighbourhood U_t of $\gamma(t)$ in S, such that $\gamma(I_t) \subset U_t$ and

$$L(\gamma|_{I_t}) = \tau_{U_t}(\gamma(a_t), \gamma(b_t)).$$

If $I = [a\,; b]$ and $L(\gamma) = \tau_S(\gamma(a), \gamma(b))$, then we say that the causal curve γ is *maximizing*. Similarly, a future anticausal curve is locally maximizing if the equality $L^+(\gamma|_{I_t}) = \tau_{U_t}^+(\gamma(a_t), \gamma(b_t))$ is satisfied in a suited neighbourhood of any point. We now analyse the behaviour of locally maximizing causal and anticausal curves at the neighbourhood of a singularity.

Proposition A.10. *Let S be a singular* $\mathbf{X}$*-surface.*

(1) *A future causal curve $\gamma\colon I \to S$ is locally maximizing if and only if it is either an interval of a lightlike leaf, or it satisfies the following conditions.*
   (a) *γ is a timelike geodesic (up to reparametrization) outside of the singularities.*
   (b) *γ does not meet any singularity of positive angle.*
   (c) *Let x be any singularity of negative angle θ met by γ, $\varphi\colon U \to \mathbf{X}_\theta$ be a singular chart at x, γ_+ (respectively γ_-) be the future (resp. past) interval of $\gamma \cap U$, and γ^0 be the geodesic segment of $\mathbf{X}$ through o containing $\varphi(\gamma_-)$. Then $\varphi(\gamma_+)$ belongs to the future closed timelike sector of angle θ delimited by γ^0 and $\bar{a}^{-\theta}(\gamma^0)$, called the* shadow at x.
(2) *Any maximizing causal curve is locally maximizing.*
(3) *A future anticausal curve $\gamma\colon I \to S$ is locally maximizing if and only if it is either an interval of a lightlike leaf, or it satisfies the following conditions.*
   (a) *γ is a spacelike geodesic (up to reparametrization) outside of the singularities.*
   (b) *γ does not meet any singularity of negative angle.*
   (c) *In a singular chart $\varphi\colon U \to \mathbf{X}_\theta$ at any singularity of positive angle met by γ, $\varphi(\gamma_+)$ belongs to the future closed spacelike sector of angle θ delimited by γ^0 and $\bar{a}^{-\theta}(\gamma^0)$.*

Note that according to Proposition 3.20, the conditions (1).(c) and (2).(c) make sense since at a given singularity x, they do not depend on the chosen singular chart at x.

Proof of Proposition A.10. (1) Outside of the singularities, the fact that causal curves are locally maximizing if and only if they are geodesic (up to reparametrization) is a classical fact concerning regular Lorentzian manifolds, and is for instance proved in [Min19, Theorem 2.9 and 2.20]. In particular their signature is fixed, and lightlike curves remain in the same lightlike foliation. We now treat the case of singularities, and assume that γ is locally maximizing.

The result being local, we can assume that $S \subset \mathbf{X}_\theta$ and that γ is maximizing. We observe first that if γ is timelike somewhere outside of the singularities, then it cannot become lightlike when crossing a singularity, or else there would exist a longer timelike curve (avoiding the singularity) which would contradict the maximality. Likewise, a lightlike curve cannot become timelike, and cannot neither switch to the other lightlike foliation. We can therefore assume henceforth without loss of generality that γ is timelike.

We denote by $\gamma_\pm$ the future and past components of $\gamma \setminus \{\mathsf{o}_\theta\}$, and by γ^0 the projection in $\mathbf{X}_\theta$ of the geodesic of $\mathbf{X}$ through o containing γ_-. We first assume by contradiction that γ meets o_θ with $\theta > 0$, and illustrate this situation by the Figure A.1 below. The geodesic γ^0 separates the future timelike quadrant in two sectors: a lower open sector $\mathcal{S}^1$ under γ^0, and an upper half-closed sector $\mathcal{S}^2$ over γ^0 containing γ^0. In the first case where $\gamma^1_+ \subset \mathcal{S}^1$, any point $x \in \gamma_-$ is joined to some point $y^1 \in \gamma^1_+$ sufficiently close to o_θ, by a timelike geodesic $\tilde{\gamma}^1$ drawn in red in Figure A.1 which avoids the singularity o_θ. In the second case where $\gamma^2_+ \subset \mathcal{S}^2$, by taking into account the gluing of points $\iota_+(p) \sim_\theta \iota_-(a^\theta(p))$ along $\mathcal{F}^+_\alpha(\mathsf{o})$ which takes place in $\mathbf{X}_\theta$, any point $x \in \gamma_-$ is also joined to a point $y^2 \in \gamma^2_+$ sufficiently close to o_θ, by a red timelike geodesic $\tilde{\gamma}^2$ avoiding o_θ. Observe that such a timelike geodesic $\tilde{\gamma}^2$ avoiding o_θ and joining x to y^2 exists even in the case where $\gamma^2_+ \subset \gamma^0$ thanks to the gluing along $\mathcal{F}^+_\alpha(\mathsf{o})$. We emphasize that the existence of such timelike geodesics $\tilde{\gamma}^i$ is easily checked by using an affine chart of $\mathbf{X}$ where every geodesic is an affine interval, and that such affine charts are used in the Figures A.1 and A.2. Now according to the reverse triangle inequality of Lemma A.9, the red timelike geodesics $\tilde{\gamma}^i$ from x to y^i are longer than the segment of γ from x to y^i, which contradicts the fact that γ is maximizing. This shows that γ has to satisfy the condition (1).(b).

We now assume that γ meets o_θ with $\theta < 0$, and illustrate this situation by the Figure A.2 below. Denoting by $\bar{a}^\theta$ the isometry of $\mathbf{X}_\theta$ induced by a^θ introduced in Proposition 3.20, we consider the image $\bar{a}^{-\theta}(\gamma^0)$ of γ^0, which separates together with γ^0 the future timelike quadrant in three sectors: an open sector $\mathcal{S}^1$ under γ^0, an open sector $\mathcal{S}^2$ above $\bar{a}^{-\theta}(\gamma^0)$, and a closed sector $\mathcal{S}^0$ of angle θ between γ^0 and $\bar{a}^{-\theta}(\gamma^0)$. Let assume by contradiction that γ does not satisfy the condition (1).(c). In other words, either $\gamma^1_- \subset \mathcal{S}^1$, or $\gamma^2_- \subset \mathcal{S}^2$. Then in these two cases, the same arguments than before show that any point $x \in \gamma_-$ is joined to some point $y^i \in \gamma^i_+$ sufficiently close to o_θ, by a timelike geodesic $\tilde{\gamma}^i$ drawn in red in Figure A.2 which avoids the singularity o_θ. Again, the reverse triangle inequality shows then than the red timelike geodesics $\tilde{\gamma}^i$ are longer than the segment of γ, which contradicts the maximality of γ and eventually shows that it has to satisfy the condition (1).(c). This concludes the proof of the direct implication.

We now consider a causal curve γ satisfying the conditions of the statement, and prove that it is locally maximizing. Since the classical case of regular Lorentzian manifolds already ensures that γ is locally maximizing at any regular point, we only have to show that a causal curve $\gamma \subset \mathbf{X}_\theta$ passing through o_θ with $\theta < 0$ and such that $\gamma_+ \subset \mathcal{S}^0$, is locally maximizing. We recall first that any longer causal curve $\tilde{\gamma}$ has to be piecewise geodesic, *i.e.* to remain a timelike geodesic outside of o_θ. Observe now that if a timelike piecewise geodesic $\tilde{\gamma}$ coincides with γ_- until o_θ and passes through o_θ, then if $\tilde{\gamma}_+$ is distinct from γ_+, it does not meet γ_+ again. Likewise, it is clear by using an affine chart of $\mathbf{dS}^2$ that any timelike piecewise geodesic $\tilde{\gamma}$ starting from γ_- and going strictly below γ^0 cannot meet γ_+ again. Lastly due to the gluing along $\mathcal{F}^+_\alpha(\mathsf{o})$, any timelike piecewise geodesic $\tilde{\gamma}$ starting from γ_- and going strictly above γ_- cannot meet γ_+ again neither, since in the limit case $\gamma_+ \subset \bar{a}^{-\theta}(\gamma^0)$, while the geodesic of $\mathbf{X}$ containing $\tilde{\gamma}$ is sent strictly above $\bar{a}^{-\theta}(\gamma^0)$ by the gluing. Therefore, there does not exist any longer causal curve joining two points of γ, which eventually shows that γ is locally maximizing, and concludes the proof of the first part of the proposition.

(2) Let $\gamma\colon [a\,;b] \to S$ be a maximizing causal curve. For any $a < t < b$ we have:

$$L(\gamma|_{[a;t]}) + L(\gamma|_{[t;b]}) = L(\gamma) = \tau_S(\gamma(a), \gamma(b)) \geq \tau_S(\gamma(a), \gamma(t)) + \tau_S(\gamma(t), \gamma(b)) \tag{A.4}$$

according to the reverse triangular inequality (Lemma A.9). Since on the other hand $L(\gamma|_{[a;t]}) \leq \tau_S(\gamma(a), \gamma(t))$ and $L(\gamma|_{[t;b]}) \leq \tau_S(\gamma(t), \gamma(b))$ by definition of τ_S, both of the latter inequalities have to be equalities to match (A.4). Applying twice this argument to $a_t \in [a\,;b]$ and then $b_t \in [a_t\,;b]$ we obtain $L(\gamma|_{[a_t;b_t]}) = \tau_S(\gamma(a_t), \gamma(b_t)) \geq \tau_{U_t}(\gamma(a_t), \gamma(b_t))$, the latter inequality following from the definition of τ as a supremum. On the other hand $L(\gamma|_{[a_t;b_t]}) \leq \tau_{U_t}(\gamma(a_t), \gamma(b_t))$ by definition of τ_{U_t}, hence $L(\gamma|_{[a_t;b_t]}) = \tau_{U_t}(\gamma(a_t), \gamma(b_t))$, *i.e.* γ is locally maximizing.
(3) The anticausal case follows from the same arguments. □

Corollary A.11. *Let S be a closed singular $\mathbf{dS}^2$-surface of class A admitting a unique singularity. Then any locally maximizing timelike curve avoids the singularity, and is a timelike geodesic (up to reparametrization).*

Proof. According to the Gauß-Bonnet formula in Proposition 3.27, the unique singularity x of S has a positive angle. Proposition A.10 shows then that any locally maximizing timelike curve avoids x, and is a geodesic. □

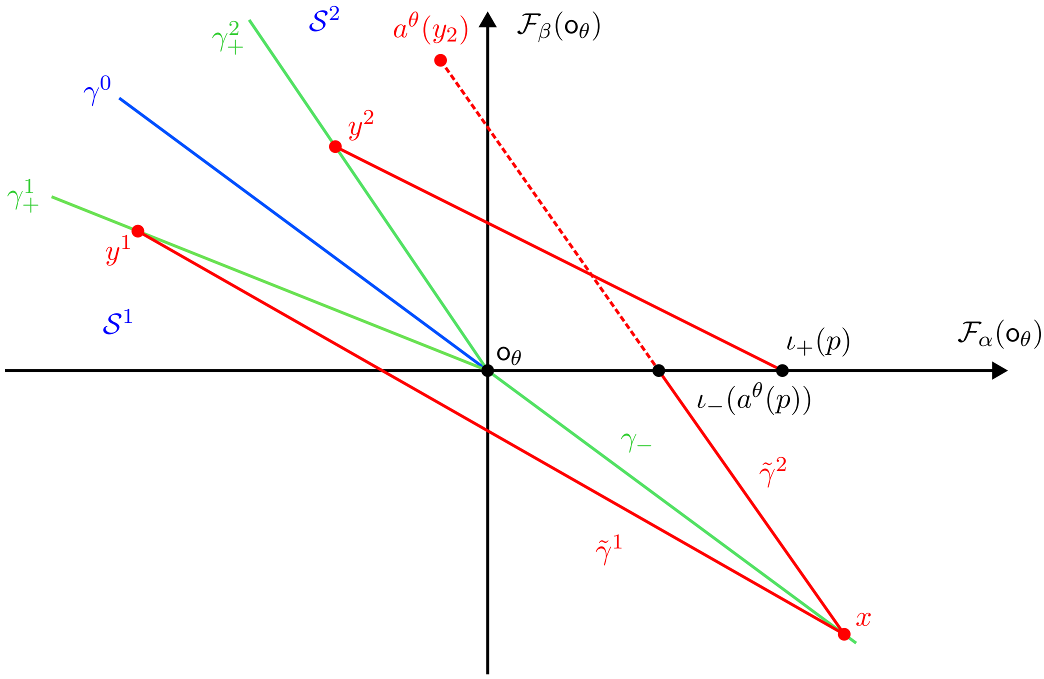

FIGURE A.1. Maximizing timelike curves avoid positive singularities.

Proposition A.12. *Any point $x \in S$ admits a connected open neighbourhood U homeomorphic to a disk, and such that:*

1. *(1) U is the domain of a chart of the singular $\mathbf{X}$-atlas centered at x;*
2. *(2) U is the domain of a simultaneous foliated $\mathcal{C}^0$-chart of the lightlike foliations;*
3. *(3) with I_α and I_β the connected components of $\mathcal{F}_\alpha(x) \cap U$ and $\mathcal{F}_\beta(x) \cap U$ containing x, $U \setminus (I_\alpha \cup I_\beta)$ has four connected components, called the* quadrants of U at x*;*
4. *(4) for any two points $y \neq z \in U$, one of the following two exclusive situations arise:*
   - *(a) either y and z are causally related, and then there exists a unique causal segment $[y\,;z]_U \subset U$ of endpoints y and z which is maximizing in U, and $[y\,;z]_U$ is moreover disjoint from (at least) one of the open quadrants at x;*
   - *(b) or y and z are related by a spacelike curve, and then there exists a unique spacelike segment $[y\,;z]_U \subset U$ of endpoints y and z which is maximizing in U, and $[y\,;z]_U$ is moreover disjoint from (at least) one of the open quadrants at x.*

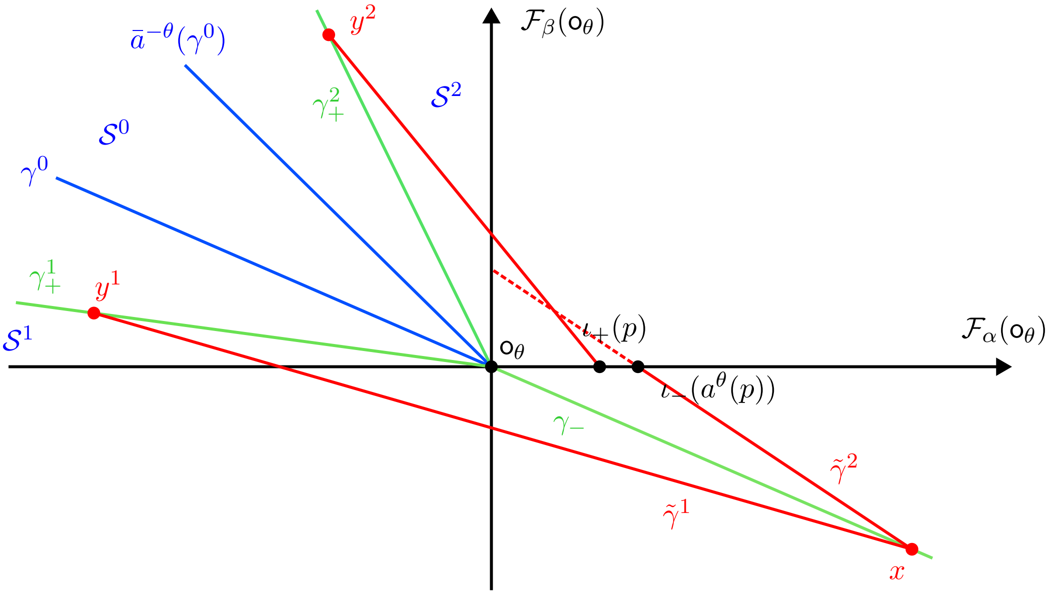

FIGURE A.2. Shadow for maximizing timelike curves at a negative singularity.

Such an U is called a normal convex neighbourhood of x*. Moreover quadrants are themselves* convex*,* i.e. *if y, z are in a same open quadrant Q of U at x, then $[y\,;z]_U \subset Q$.*

Proof. This claim is proved in $\mathbf{X}$, and thus on $S \setminus \Sigma$, by using standard normal convex neighbourhoods. At the neighbourhood of a singular point, it follows directly from Proposition A.10. □

The following result is well-known in the classical setting of regular Lorentzian manifolds, where it is a particular case of the *Limit curve theorems*. We give here the main arguments of its proof to make it clear that it persists in our singular setting, refering for instance to [Min19, §2.11 and Theorem 2.53] for more details.

Lemma A.13. *Let γ_n be a sequence of causal curves in a globally hyperbolic singular $\mathbf{X}$-surface S joining two points x and y. The (γ_n) have then uniformly bounded arclength with respect to a fixed Riemannian metric h on S. Let σ_n denote the reparametrization of γ_n by h-arclength. Then there exists a causal curve σ from x to y and a subsequence σ_{n_k} of σ_n converging to σ in the $\mathcal{C}^0$-topology. Moreover* $\limsup L(\sigma_{n_k}) \leq L(\sigma) < +\infty$*.*

Proof. The first important and classical fact is:

Fact A.14. *For any relatively compact normal convex neighbourhood U of a $\mathbf{X}$-surface S (not necessarily globally hyperbolic), causal curves contained in U are equi-Lipschitz, of uniformly bounded Riemannian length, and leave U in a uniform bounded time. Namely for any Riemannian metric h on U, there exists a constant $K > 0$ and a time-function f such that for any causal curve γ in U:*

(1) γ may be reparametrized by f to be K-Lipschitz;
(2) with this reparametrization, γ leaves U in a time bounded by K;
(3) and the h-arclength of γ is bounded by K.

Proof. We explain the main ideas leading to these properties for a causal curve γ contained in a relatively compact normal convex neighbourhood U of $p \in S^*$, and refer to [BEE96, p.75] and [Min19, Theorem 1.35, Remark 1.36 and Theorem 2.12] for more details. Denoting by g the Lorentzian metric of S^*, let $x = (x_1, x_2)$ be coordinates on U such that $g_p(\partial x_1, \partial x_1) = -1$, $g_p(\partial x_2, \partial x_2) = 1$ and $g_p(\partial x_1, \partial x_2) = 0$. Then there exists $\varepsilon > 0$ such that, possibly shrinking U further around p, the timelike cones of the Lorentzian metric $-(1+\varepsilon)dx_1^2 + dx_2^2$ of U strictly contain the causal cones of g (this is indeed true at p by assumption, hence on a neighbourhood of p by continuity of g). Introducing the Riemannian metric $h = dx_1^2 + dx_2^2$ on U and $K_0 := \sqrt{2+\varepsilon} > 0$, this inclusion translates as $\|u\|_h < K_0 dx_1(u)$ for any g-causal vector u, hence as

$$\int_0^t \|\gamma'(t)\|_h < K_0(x_1(\gamma(t)) - x_1(\gamma(0))) \tag{A.5}$$

for any causal curve $\gamma \subset U$ by integration. This last inequality shows that the h-arclengths of causal curves contained in U for h is uniformly bounded, that x_1 is strictly increasing over them, hence that they leave U in a uniformly bounded time when reparametrized by x_1, and that they are moreover equi-Lipschitz for this reparametrization. Note that for any function f sufficiently close to x_1, the causal curves in U retain these uniform properties when reparametrized by f (possibly changing the constants).

To conclude the proof we only have to argue that these properties persist on the neighbourhood of a singular point p. We first consider normal convex neighbourhoods U^- and U^+ contained in S^*, respectively avoid the future and past timelike quadrants at p, and such that $U := U^- \cup U^+ \cup \{p\}$ is a neighbourhood of p. We next choose coordinates (x_1, x_2) on U so that x_1 is sufficiently close to the respective functions $x_1^\pm$ of the previous discussion on the neighbourhoods $U^\pm$, for the uniform properties to be satisfied. Property (1) of Definition A.2 implies then that x_1 is strictly increasing on any causal curve γ in U, hence that γ leaves U in uniformly bounded time. When reparametrized by x_1, the causal curves of U are moreover clearly equi-Lipschitz and of uniformly bounded length for a fixed Riemannian metric, since the inequality (A.5) does not take into account the singular point p. This concludes the proof of the fact. □

We now come back to the proof of the lemma and fix a Riemannian metric h on S. Since S is strongly causal and $J^+(x) \cap J^-(y)$ relatively compact by global hyperbolicity, we can cover $J^+(x) \cap J^-(y)$ by a finite number of normal convex neighbourhoods U_i which are causally convex. Since the causal curves γ_n join x to y, they are contained in $J^+(x) \cap J^-(y)$. We reparametrize then each γ_n in U_i thanks to the Fact A.14, obtaining in this way an equi-Lipshitz family. Since each of the γ_n meets a given U_i at most once by causal convexity, since the h-arclengths of the $\gamma_n|_{U_i}$ are uniformly bounded for any i according to Fact A.14, and since the covering $(U_i)_i$ is finite, the h-arclength of the γ_n is in the end uniformly bounded.

In particular, the sequence of causal curves $\sigma_n \colon [0\,;a_n] \to S$ obtained by reparametrizing the γ_n by h-arclength remains equi-Lipschitz (because the changes of parametrizations are themselves equi-Lipschitz by boundedness of the arclengths). The sequence (a_n) being bounded, we can moreover assume by passing to a subsequence that it converges to some $a \in \,]0\,;+\infty[$. We now extend the σ_n to *future inextendible* causal curves $\nu_n \colon \mathbb{R}^+ \to S$, *i.e.* such that $\nu_n(t)$ has no limit when $t \to +\infty$. One easily proves using Fact A.14 that the h-arclength of the ν_n is infinite, and we can therefore reparametrize them by h-arclength on $[a_n\,;\infty]$, obtaining in this way an equi-Lipschitz family $\eta_n \colon \mathbb{R}^+ \to S$ of causal curves.

For any $m \in \mathbb{N}$, we can now apply the Arzelà-Ascoli theorem to $(\eta_n|_{[0;m]})_n$. This shows that a subsequence of $(\eta_n|_{[0;m]})_n$ uniformly converges to a continuous curve η_∞^m in S, which is Lipschitz as a uniform limit of equi-Lipschitz curves. By a diagonal argument, we conclude to the existence of a subsequence $(\eta_{n_k})_k$ and of a continuous curve $\eta_\infty \colon \mathbb{R}^+ \to S$ obtained as the union of the η_∞^m, such that $(\eta_{n_k}|_I)_k$ uniformly converges to $\eta_\infty|_I$ for any compact interval $I \subset \mathbb{R}^+$. It is moreover easy to show that η_∞ is a causal curve as a uniform limit of such curves (see for instance [Min19, top of p.46]). With σ the restriction of η_∞ to $[0\,;a]$, the subsequence $(\sigma_{n_k})_k$ uniformly converges to σ, which proves the second claim.

Lastly the proof that $\limsup L(\gamma_{n_k}) \leq L(\sigma)$ given in [Min19, Theorem 2.41] works without any variation in our singular setting, using the decomposition (A.2) of the length into the ones of its regular pieces. This concludes the proof of the lemma. □

A.3. **Conclusion of the proof of Theorem A.1.** Let S be a closed singular $\mathbf{X}$-surface of class A, $\bar{b}$ be a simple closed spacelike curve in S, and $\pi_C \colon C \to S$ be the $\mathbb{Z}$-covering of S for which $\pi_{C*}(\pi_1(C))$ is generated by $[\bar{b}]$, endowed with the singular $\mathbf{X}$-structure induced by S. Note that S is homeomorphic to $\mathbf{T}^2$, and C to a cylinder $\mathbf{S}^1 \times \mathbb{R}$.

Lemma A.15. *C is a globally hyperbolic singular $\mathbf{X}$-surface.*

Proof. Since S is class A, the lightlike bi-foliation of the universal cover $\Pi\colon \tilde{S} \to C$ of C is topologically equivalent to the product bi-foliation of $\mathbb{R}^2$ by horizontal and vertical lines (see Remark 6.7). For any $x, y \in \tilde{S}$, the causal diamond $J^+(x) \cap J^-(y)$ of $\tilde{S}$ is thus compact, and the causal diamonds of C are therefore compact as well.

Assume now for a contradiction that C is not strongly causal. Then there exists in $\tilde{S}$ a causal curve γ starting from a point x and arriving arbitrarily close to x', with x' the image of x by the automorphism of Π induced by the closed curve $\bar{b}$. Denoting by B the inextendible lift of $\bar{b}$ to $\tilde{S}$, let I be a neighbourhood of x' on B which does not contain x. Then since the lightlike bi-foliation of $\tilde{S}$ is a product-bi-foliation, there exists a neighbourhood U of x' such that $U \cap B \subset I$, $U \setminus B$ has two upper and lower connected components $U^{\pm}$, and for any $p \in U^+$: a past-oriented causal curve starting from p and meeting B has to meet the interval I. We can assume that γ arrives in U. If it arrives in U^-, we can extend γ to a causal curve meeting I. If it arrives in U^+, then the property of U ensures that γ meets I in the past. In any case, γ is a causal curve of $\tilde{S} \simeq \mathbb{R}^2$ which meets the spacetime curve B at least twice: once at x, and once on $I \not\ni x$. This contradicts Lemma A.4 and concludes the proof. □

Let $\bar{a}$ be a closed timelike curve of S intersecting $\bar{b}$ at a point $\bar{x} = \bar{a}(0) = \bar{b}(0)$, and of algebraic intersection number $\hat{\imath}([\bar{b}], [\bar{a}]) = 1$ with $\bar{b}$. In particular $([\bar{a}], [\bar{b}])$ is a basis of $\pi_1(S) \simeq \mathbb{Z}^2$. We fix a lift $x_1 \in \pi_C^{-1}(\bar{x})$ of $\bar{x}$ in C, and denote by $a\colon [0\,;1] \to C$ and $b_1\colon [0\,;1] \to C$ the lifts of $\bar{a}$ and $\bar{b}$ starting from $x_1 = a(0) = b_1(0)$. By definition of C we have $b_1(1) = x_1$, *i.e.* b_1 is a simple closed curve in C. On the other hand a is a simple segment but is not closed, and $x_2 \coloneqq a(1) = R(x_1)$ with R the positive generator of the covering automorphism group of π_C induced by $[\bar{a}]$. We denote by $b_2\colon [0\,;1] \to C$ the lift of $\bar{b}$ starting from x_2, so that $b_2 = R \circ b_1$. For $p \in b_1$ we denote by $\mathcal{S}_p$ the set of causal curves of C from p to $R(p)$ which are *causally freely homotopic to a*, *i.e.* freely homotopic to a with endpoints fixed and through causal curves. The following result is a version of the classical Avez-Seifert theorem (see for instance [Min19, Theorem 4.123]), suitably adapted to our setting.

Proposition A.16. *The function*

$$F\colon p \in b_1 \mapsto \sup_{\sigma \in \mathcal{S}_p} L(\sigma) \in [0\,;\infty[\tag{A.6}$$

has finite values, is continuous, and moreover for any $p \in b_1$ there exists $\sigma \in \mathcal{S}_p$ such that $L(\sigma) = F(p)$.

Proof. We fix on C a complete Riemannian metric and endow C with its induced distance. Let $p \in b_1$ and $\sigma_n \in \mathcal{S}_p$ be a sequence of causal curves such that $\lim L(\sigma_n) = F(p)$. Since C is globally hyperbolic according to Lemma A.15, there exists according to Lemma A.13 a subsequence σ_{n_k} converging to a causal curve σ from p to $R(p)$. For any normal convex neighbourhood U, there exists $\varepsilon_U > 0$ and $V \subset U$ such that for any causal curve $\gamma \subset V$, all the causal curves ε_U-close to γ are contained in U and causally homotopic to γ. Since $J^+(p) \cap J^-(R(p))$ is compact by global hyperbolicity and contains any curve of $\mathcal{S}_p$, we can cover $J^+(p) \cap J^-(R(p))$ by a finite number of normal convex neighbourhoods V as before, and we conclude to the existence of $\varepsilon > 0$ such that for any $\gamma \in \mathcal{S}_p$, any causal curve ε-close to γ is causally homotopic to γ. Hence for any large enough k, σ is causally homotopic to $\sigma_{n_k} \in \mathcal{S}_p$, and therefore $\sigma \in \mathcal{S}_p$. Hence $L(\sigma) \leq F(p)$ by definition of F, and since $F(p) = \lim L(\sigma_{n_k}) \leq L(\sigma)$ according to Lemma A.13, this shows that $F(p) = L(\sigma) < +\infty$ and proves the first and third claims.

The proof that F is lower semi-continuous is a straightforward adaptation of [Min19, Theorem 2.32], to which we refer for more details. Let $p \in b_1$, $\varepsilon > 0$ be such that $0 < 3\varepsilon < F(p)$ and $\gamma \in \mathcal{S}_p$ so that $L(\gamma) > F(p) - \varepsilon > 0$. We slightly modify γ for it to be timelike and still satisfy the latter inequality. We choose then $p' \in \gamma$ close enough to p so that $L(\gamma|_{[p;p']}) < \varepsilon$, and $q' \in \gamma$ close enough to $R(p)$ so that $L(\gamma|_{[q';R(p)]}) < \varepsilon$, hence $L(\gamma|_{[p';q']}) > F(p) - 3\varepsilon > 0$. If p' and q' are close enough to p and $R(p)$, then the respective past and future timelike quadrants U and V of normal convex neighbourhoods of p' and q' are neighbourhoods of p and $R(p)$, $I \coloneqq U \cap b_1$ is a neighbourhood of p in b_1, and $R(I)$ is a neighbourhood of $R(p)$ in b_2. We recall that $[a\,;b]_U \subset U$ denotes the unique geodesic contained in U going from $a \in U$ to $b \in J^+(a) \cap U$. For any $x \in I$, let γ_x denote the causal curve going from x to $R(x)$ formed by first following the geodesic $[x\,;p']_U \subset U$, then $\gamma|_{[p';q']}$ and finally $[q'\,;R(x)]_V \subset V$. This curve γ_x is freely causally homotopic to $\gamma \in \mathcal{S}_p$, hence $\gamma_x \in \mathcal{S}_p$ and $F(x) \geq L(\gamma_x) \geq L(\gamma|_{[p';q']}) > F(p) - 3\varepsilon$. This proves the lower semi-continuiuty of F.

Assume now by contradiction that F is not upper semi-continuous, *i.e.* that there exists $p_n \to p$ in b_1 and $\varepsilon > 0$ such that $F(p_n) \geq F(p) + 2\varepsilon$ for any n. Then with $\gamma_n \in \mathcal{S}_{p_n}$ such that $L(\gamma_n) \geq F(p_n) - \varepsilon$, since p_n converges to p and $R(p_n)$ to $R(p)$, Lemma A.13 shows the existence of a causal curve γ from p to $R(p)$ to which a subsequence $(\gamma_{n_k})_k$ converges. Indeed with $p' \in I^-(p)$ and $q' \in I^+(R(p))$ sufficiently close to p and $R(p)$, there exists for any large enough n timelike geodesics γ_n^- and γ_n^+ respectively from p' to p_n and from $R(p_n)$ to q', contained in normal convex neighbourhoods of p' and q'. We can now directly apply Lemma A.13 to the sequence of causal curves formed by following γ_n^-, γ_n and then γ_n^+, and restrict the obtained limit curve to its segment γ from p to $R(p)$. According to Lemma A.13 and by assumption on $L(\gamma_n)$ and $F(p_n)$, we have then $L(\gamma) \geq \limsup L(\gamma_{n_k}) \geq \limsup F(p_{n_k}) - \varepsilon \geq F(p) + \varepsilon$. But the argument of the first paragraph of this proof shows that $\gamma \in \mathcal{S}_p$, and this last inequality contradicts thus the definition of $F(p)$. This concludes the proof of the upper semi-continuity, hence the one of the lemma. ☐

We can finally conclude the proof of Theorem A.1 thanks to the following result.

Theorem A.17. *Let μ be a singular $\mathbf{X}$-structure of class A on $\mathbf{T}^2$. Then any simple closed timelike (resp. spacelike) curve of μ admits a freely homotopic simple closed timelike (resp. spacelike) curve which is locally maximizing.*

Proof. We prove the claim for a simple closed timelike curve a, and the proof follows then in the spacelike case by inverting the Lorentzian metric μ. Let b be a simple closed spacelike curve of μ minimizing the number of intersection points with a. If $\hat{i}(\mathsf{b},\mathsf{a}) > 1$, let $\pi_S\colon S \to \mathbf{T}^2$ be the finite covering of $(\mathbf{T}^2,\mu)$ satisfying $\pi_{S*}(\pi_1(S)) = \langle[\mathsf{b}],[\mathsf{a}]\rangle$. Note that for any lifts $\bar{a}$ and $\bar{b}$ of a and b in S: $\hat{i}(\bar{b},\bar{a}) = 1$.

We now use the notations introduced before Proposition A.16 for the $\mathbb{Z}$-covering $\pi_C\colon C \to S$ of S such that $\pi_{C*}(\pi_1(C)) = \langle[\bar{b}]\rangle$, for the lifts a, b_i and x_i ($i = 1,2$) of $\bar{a}$, $\bar{b}$ and $\bar{x}$, and for the covering automorphism R induced by the action of $[\bar{a}]$. With this setup, we want to find a simple timelike geodesic segment $\gamma\colon [0\,;l] \to C$ freely homotopic to a, such that $\gamma(0) \in b_1$ and $\gamma(l) = R(\gamma(0)) \in b_2$. According to Proposition A.16, the function F defined in (A.6) is continuous and finite on the compact set b_1, and reaches thus its maximum at a point $p_0 \in b_1$. There exists moreover according to the same proposition a causal curve $\gamma \in \mathcal{S}_{p_0}$ such that

$$L(\gamma) = F(p_0) = \sup_{p\in b_1}\sup_{\sigma\in\mathcal{S}_p} L(\sigma). \tag{A.7}$$

In particular, note that $L(\gamma) \geq L(a) = L(\bar{a}) > 0$.

We now prove that $\gamma\colon [0\,;1] \to C$ is locally maximizing. Indeed let $t \in [0\,;1]$, U be a normal convex neighbourhood of $\gamma(t)$ and $I = [a\,;b]$ be a connected neighbourhood of t in $[0\,;1]$ such that $\gamma(I) \subset U$. Then the unique geodesic segment $[\gamma(a)\,;\gamma(b)]_U$ of U from $\gamma(a)$ to $\gamma(b)$ is (future) timelike, and homotopic to $\gamma|_I$ through causal curves while fixing the extremities. In other words the curve ν obtained by concatenating $\gamma|_{[0;a]}$, $[\gamma(a)\,;\gamma(b)]_U$ and $\gamma|_{[b;1]}$ is in $\mathcal{S}_{p_0}$, and thus $L(\nu) \leq L(\gamma)$ according to (A.7). But on the other hand $L([\gamma(a)\,;\gamma(b)]_U) = \tau_U(\gamma(a),\gamma(b))$ since $[\gamma(a)\,;\gamma(b)]_U$ is maximizing in U, and thus $\tau_U(\gamma(a),\gamma(b)) \geq L(\gamma|_{[a;b]})$ by definition, hence $L(\nu) \geq L(\gamma)$. The latter inequality is therefore an equality, which imposes $\tau_U(\gamma(a),\gamma(b)) = L(\gamma|_{[a;b]})$. This proves that γ is locally maximizing, hence that $\bar{\gamma} = \pi_C \circ \gamma\colon [0\,;l] \to S$ and $\pi_S \circ \bar{\gamma}\colon [0\,;l] \to \mathbf{T}^2$ are locally maximizing as well.

Since C is strongly causal according to Lemma A.15, it contains in particular no closed timelike curve, and γ is thus injective. Furthermore, $\gamma(]0\,;l[)$ is contained in the interior of the unique compact connected annulus E of C bounded by b_1 and b_2 (as we have already seen in the second part of the proof of Lemma A.15), and in particular $\gamma(]0\,;l[)$ is thus disjoint from $b_1 \cup b_2$. Since $\pi_C\colon C \to S$ is injective in restriction to $\mathrm{Int}(E)$ and $\pi_C(\gamma(0)) = \pi_C(\gamma(l))$, this proves that $\bar{\gamma} = \pi_C \circ \gamma\colon [0\,;l] \to S$ is a simple closed timelike curve of S, freely homotopic to $\bar{a}$ (since γ is freely homotopic to a). At this point $\pi_S \circ \bar{\gamma}$ is a closed and locally maximizing timelike curve of $\mathbf{T}^2$, freely homotopic to our original closed timelike curve a. As seen in Proposition A.8, if the covering $\pi_S\colon S \to \mathbf{T}^2$ is non-trivial, then $\hat{i}(\mathsf{b},\mathsf{a})=2$, π_S is of degree 2, and there exists two closed lightlike leaves F_α and F_β such that $\hat{i}([F_\alpha],[F_\beta])=1$. Consequently, the homotopy classes $[F_\alpha]$ and $[F_\beta]$ define the same order 2 automorphism of π_S, generating its automorphism group. If $\pi_S\circ\bar{\gamma}$ was not a simple closed curve, there would thus exist in C a lightlike segment going from some point $x\in\gamma(]0\,;l[)$ to another point $y\in\gamma(]0\,;l[)$. But this segment would lift in the universal covering $\tilde{C}\simeq\mathbb{R}^2$ of C to a lightlike leaf intersecting two times the timelike curve lifting γ, which is forbiddden by Lemma A.4. Hence $\pi_S\circ\bar{\gamma}$ is a simple closed curve, which concludes the proof. □

Appendix B. Some classical results on the rotation number

The claims (1) and (2) of Lemma B.1 below are classical, and Selim Ghazouani indicated us that the claims (3) and (4) are also known to specialists of one-dimensional dynamics (related results can for istance be found in [Gha, Chapter 3 and 4]). However we did not find a reference proving these specific results, and we give thus a proof here for sake of completeness.

Lemma B.1. *Let $f\in\mathrm{Homeo}^+(\mathbf{S}^1)$, and $t\in[0\,;1]\mapsto g_t\in\mathrm{Homeo}^+(\mathbf{S}^1)$ be a continuous map such that:*

- $g_0=\mathrm{id}_{\mathbf{S}^1}$,
- *and $t\in[0\,;1]\mapsto g_t(x)\in\mathbf{S}^1$ is non-decreasing for any $x\in\mathbf{S}^1$.*

Then with $f_t\coloneqq g_t\circ f$, the map $t\in[0\,;1]\mapsto\rho(f_t)\in\mathbf{S}^1$ is:

(1) continuous;
(2) and non-decreasing.

Moreover:

(3) Assume that $g_1=\mathrm{id}_{\mathbf{S}^1}$, and that there exists $x_0\in\mathbf{S}^1$ such that $t\in[0\,;1]\mapsto g_t(x_0)\in\mathbf{S}^1$ is surjective. Then $t\in[0\,;1]\mapsto\rho(f_t)\in\mathbf{S}^1$ is surjective.
(4) Assume that f is minimal, and that there exists $x_0\in\mathbf{S}^1$ such that $t\in[0\,;1]\mapsto g_t(x_0)\in\mathbf{S}^1$ is not constant. Then $t\in[0\,;1]\mapsto\rho(f_t)$ is not constant at 0. More precisely for any $\varepsilon>0$ such that $t\in[0\,;\varepsilon]\mapsto\rho(f_t)\in\mathbf{S}^1$ is not surjective and $f_\varepsilon(x_0)\neq f(x_0)$: $\rho(f_\varepsilon)\neq\rho(f)$.
(5) Assume that f is minimal, and that $t\in[0\,;1]\mapsto g_t(x)\in\mathbf{S}^1$ is strictly increasing for any $x\in\mathbf{S}^1$. Then for any $\varepsilon>0$, there exists $\eta>0$ such that for any rational $r\in[\rho(f)\,;\rho(f)+\eta]\subset\mathbf{S}^1$ and any $x\in\mathbf{S}^1$, there exists $t\in[0\,;\varepsilon]$ such that the orbit of x under f_t is periodic and of cyclic order r. In particular $\rho(f_t)=r$.

The obvious analogous statements hold for non-increasing maps, and for a family $t\mapsto f\circ g_t$ of deformations.

Proof. The obvious analogous claims for non-increasing maps $t\mapsto g_t(x)$ follow from the non-decreasing case by interverting orientations. The same claims follow then for the family of deformations $t\mapsto f\circ g_t$ by taking the inverse of $f\circ g_t$, since $\rho(f^{-1})=-\rho(f)$ for any circle homeomorphism.
(1) The continuity follows readily from the ones of the rotation number (see Proposition 5.1) and of $t\mapsto g_t$.
(2) The assumptions on (g_t) ensure the existence of a family of lifts $G_t\in\mathrm{D}(\mathbf{S}^1)$ of g_t such that for any $x\in\mathbb{R}$: $t\mapsto G_t(x)$ is non-decreasing. Let F be a lift of f, and $s\leq t\in[0\,;1]$. Then $G_s\circ F(0)\leq G_t\circ F(0)$ and if we assume that $(G_s\circ F)^n(0)\leq(G_t\circ F)^n(0)$ for some $n\in\mathbb{N}$, then since F and the G_u are strictly increasing and $x\mapsto G_u(x)$ is non-decreasing for any $x\in\mathbb{R}$ we obtain: $(G_s\circ F)^{n+1}(0)\leq G_t(F\circ(G_s\circ F)^n)(0)\leq(G_t\circ F)^{n+1}(0)$. In the end $(G_s\circ F)^n(0)\leq(G_t\circ F)^n(0)$ for any $n\in\mathbb{N}$, which shows that $\tau(G_s\circ F)\leq\tau(G_t\circ F)$ according to (5.1). Hence $u\in[0\,;1]\mapsto\tau(G_u\circ F)\in\mathbb{R}$ is non-decreasing. Since the latter is a lift of the map $u\in[0\,;1]\mapsto\rho(g_u\circ f)\in\mathbf{S}^1$, this proves our claim.
(3) Assume that $F\colon t\in[0\,;1]\mapsto\rho(f_t)$ is not constant. Then there exists $t_0\in]0\,;1]$ such that $F(t_0)\in\mathbf{S}^1\setminus\{\rho(f)\}$, and since F is continuous and non-decreasing according to (1) and (2), and in the other hand $F(1)=\rho(f)$ by assumption on $g_1=\mathrm{id}_{\mathbf{S}^1}$, we obtain $\mathbf{S}^1=[\rho(f)\,;F(t_0)]\cup[F(t_0)\,;\rho(f)]\subset F([0\,;1])$, which proves the claim. It remains now to argue that $F\colon t\in[0\,;1]\mapsto\rho(f_t)$ is not constant, from the existence of $x_0\in\mathbf{S}^1$ such that $t\in[0\,;1]\mapsto g_t(x_0)\in\mathbf{S}^1$ is surjective.

If $\rho(f) \neq 0$, then $x_0 \neq f(x_0)$ but there exists some $t \in [0\,;1]$ such that $f_t(x_0) = x_0$, proving that $\rho(f_t) = 0 \neq \rho(f)$ and thus that F is not constant.

Assume now that $\rho(f) = 0$. Without loss of generality, we can assume that

$$0 = \max\{t \in [0\,;1] \mid \forall s \in [0\,;t], f_s(x_0) = x_0\} \text{ and } 1 = \min\{t \in]0\,;1] \mid f_t(x_0) = x_0\}. \tag{B.1}$$

Since $t \in [0\,;1] \mapsto f_t(x_0) \in \mathbf{S}^1$ is degree one and non-decreasing, $t \in [0\,;1] \mapsto f_t^{-1}(x_0) \in \mathbf{S}^1$ is degree one and non-increasing. Denoting $[t] = t \bmod 1 \in \mathbf{S}^1$, $\alpha\colon t \in [0\,;1] \mapsto ([t], f_t(x_0)) \in \mathbf{T}^2$ and $\beta\colon t \in [0\,;1] \mapsto ([t], f_t^{-1}(x_0)) \in \mathbf{T}^2$ are two simple closed curves of $\mathbf{T}^2$ starting at $([0], x_0)$ and of respective homotopy classes $(1,1)$ and $(1,-1)$ in $\pi_1(\mathbf{T}^2) \equiv \mathbb{Z}^2$. Since they have algebraic intersection number $\hat{i}([\alpha],[\beta]) = -1-1 = -2$, they meet at least twice, hence at least once outside of $([0], x_0)$. By (B.1), such an intersection point is of the form $([t], f_t(x_0))$ with $t \in]0\,;1[$ and $f_t(x_0) = f_t^{-1}(x_0)$, *i.e.* $f_t^2(x_0) = x_0$. Since $t \in]0\,;1[$, (B.1) shows that we have also $x_0 \neq f_t(x_0)$, and therefore $\rho(f_t) = \frac{1}{2} \neq \rho(f) = 0$. In the end $t \in [0\,;1] \mapsto \rho(f_t)$ is not constant, which concludes the proof of the claim.

(4) For any connected subset I of $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$, we denote by $L(I)$ the length of I (with $L(\mathbf{S}^1) = 1$). We fix once and for all $\varepsilon > 0$ such that $t \in [0\,;\varepsilon] \mapsto \rho(f_t) \in \mathbf{S}^1$ is not surjective and $f_\varepsilon(x_0) \neq f(x_0)$. Since $(t,x) \mapsto f_t(x)$ is continuous, there exists then a neighbourhood $I := [x_0^-\,;x_0^+]$ of x_0 in $\mathbf{S}^1$ and a fixed constant $\alpha > 0$, such that for any $x \in I$:

$$L([f(x)\,;f_\varepsilon(x)]) \geq \alpha. \tag{B.2}$$

Since f is moreover minimal, there exists a strictly increasing sequence $n_k \in \mathbb{N}^*$ such that $f^{n_k}(x_0) \in [x_0^-\,;x_0[$ is strictly increasing and converges to x_0. In particular $\lim f^{n_k+1}(x_0) = f(x_0)$, and there exists thus a smallest $K \in \mathbb{N}$ so that

$$L([f^{n_K+1}(x_0)\,;f(x_0)]) < \alpha. \tag{B.3}$$

Since $f^{n_K}(x_0) \in [x_0^-\,;x_0[$ by construction of the n_k's, we have thus

$$L([f^{n_K+1}(x_0)\,;f_\varepsilon \circ f^{n_K-1}(f(x_0))]) \geq \alpha \tag{B.4}$$

according to (B.2).

We now prove by induction that $t \mapsto f_t^m(x)$ is non-decreasing for any $x \in \mathbf{S}^1$ and $m \in \mathbb{N}$. The claim being true by assumption for $m = 1$, let us assume it to be true for some m. Let $u \in [0\,;1] \mapsto F_u \in \mathrm{D}(\mathbf{S}^1)$ be a lift of $u \mapsto f_u$, and let us fix $s \leq t$ in $[0\,;1]$. Since $F_s^m(x) \leq F_t^m(x)$ by assumption and since F_s is order-preserving, we have $F_s^{m+1}(x) \leq F_s \circ F_t^m(x)$. But since $u \mapsto F_u(F_t^m(x))$ is non-decreasing and $s \leq t$, we have $F_s \circ F_t^m(x) \leq F_t^{m+1}(x)$. In the end $F_s^{m+1}(x) \leq F_t^{m+1}(x)$, which concludes the proof of our claim.

Therefore $t \in [0\,;\varepsilon] \mapsto f_t^{n_K-1}(f(x_0))$ is non-decreasing, hence $t \in [0\,;\varepsilon] \mapsto f_\varepsilon \circ f_t^{n_K-1}(f(x_0))$ is non-decreasing as well since f_ε is order-preserving. In the end, $t \in [0\,;\varepsilon] \mapsto L([f^{n_K+1}(x_0)\,;f_\varepsilon \circ f_t^{n_K-1}(f(x_0))]) \in [0\,;1]$ is non-decreasing, showing that

$$L([f^{n_K+1}(x_0)\,;f_\varepsilon^{n_K}(f(x_0))]) \geq \alpha$$

according to (B.4). According to (B.3), we have thus $f(x_0) \in [f^{n_K}(f(x_0))\,;f_\varepsilon^{n_K}(f(x_0))]$. Since $t \in [0\,;\varepsilon] \mapsto f_t^{n_K}(f(x_0))$ is continuous, there exists thus $t_0 \in]0\,;\varepsilon]$ such that $f_{t_0}^{n_K}(f(x_0)) = f(x_0)$. But $f(x_0)$ is then a periodic point of f_{t_0}, and $\rho(f_{t_0})$ is thus rational and in particular distinct from $\rho(f)$. The continuous and non-decreasing map $t \in [0\,;\varepsilon] \mapsto \rho(f_t) \in \mathbf{S}^1$ is thus not constant, and since it is also not surjective by assumption, this shows that $\rho(f_\varepsilon) \neq \rho(f)$ which concludes the proof of the claim.

(5) We begin with a useful general fact. Let $r = \frac{p}{q} \in]0\,;1[$ be a rational number written in reduced form (in particular, $q \geq 2$), and

$$r = [r_1, \dots, r_m] := \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{\cdots + \cfrac{1}{r_m}}}} \tag{B.5}$$

be its continued fraction expansion with $(r_1,\dots,r_m)\in\mathbb{N}^*$. If $m=1$ (hence $r_1\geq 2$), then we denote $I_r^-:=[0\,;r]$ and $I_r^+:=[r\,;\frac{1}{r_1-1}[$. If $m\geq 2$ is odd, we denote $I_r^-:=]0\,;r]$ and $I_r^+:=[r\,;[r_1,\dots,r_{m-1},r_m-1][$, and if m is even $I_r^-:=][r_1,\dots,r_{m-1},r_m-1]\,;r]$ and $I_r^+:=[r\,;1[$. For any finite sequence $(x_1,\dots,x_q)$ of pairwise distinct points of the circle, let us denote $(x_1,\dots,x_q)\sim r$ if $(x_1,\dots,x_q)$ has the same cyclic order than $([0],[r],[2r],\dots,[(q-1)r])$.

Fact B.2. *For any $T\in\mathrm{Homeo}^+(\mathbf{S}^1)$ and any $x\in\mathbf{S}^1$, we have*

$$\left\{(x,T(x),\dots,T^{q-1}(x))\sim r \text{ and } T^q(x)\in\,]T^{k_{q-1}}(x)\,;x]\right\}\Rightarrow\rho(T)\in I_r^- \tag{B.6a}$$

$$\left\{(x,T(x),\dots,T^{q-1}(x))\sim r \text{ and } T^q(x)\in[x\,;T^{k_1}(x)[\right\}\Rightarrow\rho(T)\in I_r^+ \tag{B.6b}$$

with $(0,k_1,\dots,k_{q-1})$ the ordering of $\{0,1,\dots,q-1\}$ for which $(x,T^{k_1}(x),\dots,T^{k_{q-1}}(x))$ is positively cyclically ordered.

Proof. These claims follow from the interpretation of the rotation number in terms of cyclic ordering of the orbits given by Proposition 5.3. More precisely, we now define the sequence (q_n) of times of closest return to x of the orbit $(T^k(x))_k$, and define along the way an associated sequence (a_n) whose continued fraction $[a_1,a_2,\dots,a_n,\dots]$ is equal to $\rho(T)$.

Definition of the sequence $(a_n(\rho(T)))_n$. The time $q_0:=1$ and associated point $x_0:=T(x)$ of the orbit of x is for any circle homeomorphism of non-zero rotation number a trivial closest return time of the orbit of x to itself, which gives therefore no information on the combinatorics of the orbits. The first interesting time of closest return is the largest integer $q_1=a_1\in\mathbb{N}^*$ such that $(x,T(x),\dots,T^{q_1}(x))$ is positively cyclically ordered. The associated point $x_1:=T^{q_1}(x)$ is the first closest return of the orbit of x to itself after the trivial time $q_0:=1$ (note that we may have $q_1=q_0=1$ and thus $x_1=x_0=T(x)$). Since T is order-preserving, $T^{q_1+1}(x)=T(T^{q_1}(x))$ is contained in $[T^{q_1}(x)\,;T(x)]$, and since it cannot be in $[T^{q_1}(x)\,;x]$ by definition of q_1, we have $T^{q_1+1}(x)=T^{q_1}(x_0)\in[x\,;x_0]$. Since T^{q_1} is order-preserving, we have thus $T^{2q_1}(x_0)\in[x_1\,;T^{q_1}(x_0)]$. We can then define $a_2\in\mathbb{N}^*$ as the largest integer such that the decreasing sequence $(x_0,T^{q_1}(x_0),\dots,T^{a_2q_1}(x_0)=T^{a_2q_1+q_0}(x))$ is contained in $[x\,;x_0]$. The second time and point of closest return of the orbit of x to itself are then $q_2:=a_2q_1+q_0=a_2q_1+1$ and $x_2:=T^{q_2}(x)$. By an analogous order reasoning, $T^{(a_2+1)q_1}(x_0)=T^{q_2}(x_1)\in[x_1\,;x]$ and the sequence $T^{kq_2}(x_1)$ is increasing, so that $a_3\in\mathbb{N}^*$ is defined as the largest integer for which $(x_1,T^{q_2}(x_1),\dots,T^{a_3q_2}(x_1))$ is contained in $[x_1\,;x]$. The third time and point of closest return of the orbit of x to itself are then $q_3:=a_3q_2+q_1$ and $x_3:=T^{q_3}(x)$. Note in particular that x_2 is closer to x than x_0, and x_3 closer to x than x_1. If $(a_1,\dots,a_n)$, $(q_1,\dots,q_n)$ and $(x_1,\dots,x_n)$ are defined and n even, $a_{n+1}\in\mathbb{N}^*$ is the largest integer such that the increasing sequence $(x_{n-1},T^{q_n}(x_{n-1}),\dots,T^{a_{n+1}q_n}(x_{n-1}))$ is contained in $[x_{n-1}\,;x]$, $q_{n+1}:=a_{n+1}q_n+q_{n-1}$ and $x_{n+1}:=T^{q_{n+1}}(x)$. Conversely if n is odd, $a_{n+1}\in\mathbb{N}^*$ is the largest integer such that the decreasing sequence $(x_{n-1},T^{q_n}(x_{n-1}),\dots,T^{a_{n+1}q_n}(x_{n-1}))$ is contained in $[x\,;x_{n-1}]$, and q_{n+1},x_{n+1} are defined in the same way. The sequence (x_n) of closest returns to x is thus alternating and converging to x.

Case 1: $\rho(T)$ is irrational. Then it can be checked that the algorithm that we just defined does not stop, *i.e.* that the sequence $(a_n)_n$ is infinite, whatever point x it is applied to. Moreover $\rho(T)$ is then equal to the infinite continued fraction $[a_1,a_2,\dots,a_n,\dots]\in\mathbb{R}\setminus\mathbb{Q}$ (see [Gha, Chapter 3 and 4] or [dMvS93, §I.2.1.2] for a proof of these two facts).

We can now use this description to prove Fact B.2 in the irrational case. Assume that m is even. The condition (B.6a) is then easily seen to be equivalent to: $a_i(\rho(T))=r_i$ for $i=1,\dots,m-1$ and $a_m(\rho(T))=r_m-1$, which implies that $\rho(T)\in I_r^-$. The condition (B.6b) is equivalent to: $a_i(\rho(T))=r_i$ for $i=1,\dots,m-1$ and $a_m(\rho(T))\geq r_m$, which gives less information: $\rho(T)\in[r\,;1]$, *i.e.* $\rho(T)\in I_r^+$. An analogous reasoning proves the fact if m is odd.

Case 2: $\rho(T)=\frac{p'}{q'}=[a_1(\rho(T)),\dots,a_l(\rho(T))]$ is rational. Then the above algorithm always stops. If x is periodic, it stops at the step l with $T^{a_lq_{l-1}}(x_{l-1})=x$. If x is not periodic, then it accumulates on a periodic orbit. In this case the algorithm never finishes the step $l+1$ because $q_l=q'$, hence T^{q_l} has an attractive fixed point towards which the strictly monotonic infinite orbit $(T^{kq_l}(x_{l-1}))_{k\geq 1}$ converges, and a_{l+1} is therefore undefined. But note that in both these cases, the partial entrances of the continued fraction of $\rho(T)$ is still given by the finite dynamical sequence $(a_1,\dots,a_l)$ defined by x: $a_i(\rho(T)) = a_i$ for $1 \leq i \leq l$.

This allows us to use again this description to prove Fact B.2 in the rational case. Assume that m is even. Then condition (B.6a) means that the above algorithm applied on x is well-defined until step m, hence that $l \geq m$, that $a_i = r_i$ for $i = 1,\dots,l-1$ and that $a_l = r_l - 1$. According to our previous description, this shows that $\rho(T) \in I_r^-$. Conversely, the condition (B.6b) shows that $l \geq m$, $a_i = r_i$ for $i = 1,\dots,m-1$ and $a_l \geq r_l$, showing $\rho(T) \in [r\,;1] = I_r^+$. The case of m odd is treated accordingly, which concludes the proof of the fact. □

We now come back to the study of our family $f_t = g_t \circ f$. Since f is minimal, $F\colon t \mapsto \rho(f_t)$ is not constant on a neighbourhood of 0 according to Lemma B.1.(4), and there exists thus by continuity of F some $\eta > 0$ such that $[\rho(f)\,;\rho(f)+\eta] \subset [\rho(f)\,;\rho(f_\varepsilon)]$. Then for any rational $r \in [\rho(f)\,;\rho(f)+\eta]$, there exists because of the continuity and the monotonicity of F some $t_1 \leq t_2 \in \,]0\,;\varepsilon]$ and some small $\varepsilon' > 0$ such that:

- $F(t) \in [\rho(f)\,;r[$ for any $t \in [0\,;t_1[$,
- $F([t_1\,;t_2]) = \{r\}$,
- $F(t) \in \,]r\,;\rho(f)+\eta]$ for any $t \in \,]t_2\,;t_2+\varepsilon']$.

Let $x \in \mathbf{S}^1$, and assume that x is not periodic for $f_{t_1} = g_{t_1} \circ f$.

Assume first that $r = \frac{p}{q} \neq 0$. We claim that $f_{t_1}^q(x)$ is then either in $I_{f_{t_1}}^- := \,]f_{t_1}^{k_{q-1}}(x)\,;x]$ or in $I_{f_{t_1}}^+ := [x\,;f_{t_1}^{k_1}(x)[$. Indeed $(x, f_{t_1}(x), \dots, f_{t_1}^{q-1}(x)) \sim r$ since $\rho(f_{t_1}) = r$, showing that $(f_{t_1}^{k_{q-1}-1}(x), f_{t_1}^{q-1}(x), f_{t_1}^{k_1-1}(x))$ is positively cyclically ordered, and thus $f_{t_1}^q(x) \in \,]f_{t_1}^{k_{q-1}}(x)\,;f_{t_1}^{k_1}(x)[$ since f_{t_1} is order-preserving. Now if $f_{t_1}^q(x) \in I_{f_{t_1}}^+$, then $f_t^q(x) \in I_{f_t}^+$ for any $t \in [0\,;t_1[$ sufficiently close to t_1 (since $t \mapsto f_t^q(x)$ is continuous and non-decreasing), which implies $\rho(f_t) \in I_r^+$ for any such t according to Fact B.2 and contradicts the definition of t_1. Therefore $f_{t_1}^q(x) \in I_{f_{t_1}}^-$. Since $t \mapsto f_t^q(x)$ is continuous and non-decreasing with $\rho(f_t) = r$ for any $t \in [t_1\,;t_2]$, we have thus either $f_t^q(x) = x$ for some $t \in \,]t_1\,;t_2]$, or $f_{t_2}^q(x)$ remains in $]f_{t_2}^{k_{q-1}}(x)\,;x[$. In the latter case, $f_t^q(x) \in I_{f_t}^-$ for any $t \in \,]t_2\,;t_2+\varepsilon']$ sufficiently close to t_2, which implies $\rho(f_t) \in I_r^-$ for such a t according to Fact B.2 and contradicts the definition of t_2. In conclusion, $f_t^q(x) = x$ for some $t \in \,]t_1\,;t_2]$.

We assume now that $\rho(f_{t_1}) = r = [0]$. According to the interpretation of the rotation number in terms of cyclic ordering of the orbits given by Proposition 5.3 and Fact B.2, this is equivalent to say that the sequence $(f_{t_1}^n(x))_{n\in\mathbb{N}}$ is positively cyclically ordered. More precisely, the cyclic monotonicity of $(f_t^n(x))_{n\in\mathbb{N}}$ forces $\rho(f_t)$ to be rational according to Proposition 5.3 and to be zero by Fact B.2, and reciprocally if $(f_t^n(x))_{n\in\mathbb{N}}$ is not cyclically monotonous, then Fact B.2 implies that $\rho(f_t) \neq [0]$. Assume by contradiction that $(f_{t_1}^n(x))_{n\in\mathbb{N}}$ is positively cyclically ordered, hence strictly since $f_{t_1}(x) \neq x$ by assumption. Then since $t \mapsto f_t^n(x)$ is increasing for any n, the sequence $(f_t^n(x))_{n\in\mathbb{N}}$ is strictly positively cyclically ordered for any $t \in [0\,;t_1[$ close enough to t_1. But this implies $\rho(f_t) = [0]$ for such a t as we have seen previously, which contradicts the definition of t_1. Therefore $(f_{t_1}^n(x))_{n\in\mathbb{N}}$ is negatively cyclically ordered, and thus using again that $t \mapsto f_t^n(x)$ is increasing for any n: either $f_t(x) = x$ for some $t \in \,]t_1\,;t_2]$, or $(f_{t_2}^n(x))_{n\in\mathbb{N}}$ remains strictly negatively cyclically ordered. But in the latter case $(f_t^n(x))_{n\in\mathbb{N}}$ is strictly negatively cyclically ordered for any $t \in [t_2\,;t_2+\varepsilon'[$ close enough to t_2, which implies $\rho(f_t) = [0]$ for such a t and contradicts the definition of t_2. In conclusion $f_t(x) = x$ for some $t \in \,]t_1\,;t_2]$, which concludes the proof. □

Appendix C. Holonomies of lightlike foliations are piecewise Möbius

This appendix is entirely independent from the rest of the paper, and is not used anywhere in the text. We first make precise the Remark 2.3, by detailing a natural geometrical identification between $\mathbf{dS}^2$ and its hyperboloid model dS^2, that we see here as the set $\{l \in \mathbf{P}^+(\mathbb{R}^{1,2}) \mid \text{spacelike}\}$ of spacelike half-lines of $\mathbb{R}^{1,2}$. With

$$\mathcal{C} := \left\{ l \in \mathbf{P}^+(\mathbb{R}^{1,2}) \,\middle|\, \text{lightlike and positive} \right\}$$

the ($\mathrm{SO}^0(1,2)$-invariant) positive copy of the conformal boundary of dS^2, we define two $\mathrm{SO}^0(1,2)$-equivariant projections

$$\pi_{\alpha/\beta}\colon l \in \mathrm{dS}^2 \mapsto l_{\alpha/\beta} \in \mathcal{C}$$

whose fibers are the α and β-lightlike foliations of dS^2. Any $l \in \mathrm{dS}^2$ is contained in exactly two null planes $N^l_{\alpha/\beta}$ defining two lightlike geodesics $n^l_{\alpha/\beta}$ containing l (the connected components of $N^l_{\alpha/\beta} \cap \mathrm{dS}^2$ containing l), and we name them in such a way that with $l_{\alpha/\beta} = N^l_{\alpha/\beta} \cap \mathcal{C}$, the positive orientation of n^l_α (respectively n^l_β) goes from l to l_α (resp. l_β). We emphasize that $\pi_\alpha(l) \neq \pi_\beta(l)$ and $l = n^l_\alpha \cap n^l_\beta$ for any $l \in \mathrm{dS}^2$. We can now observe that:

Lemma C.1.

$$l \in \mathrm{dS}^2 \mapsto (\pi_\alpha(l), \pi_\beta(l)) \in \mathcal{C}^2 \setminus \{\textit{diagonal}\}$$

is a $\mathrm{SO}^0(1,2)$*-equivariant bijection, which identifies* dS^2 *with* $\mathbf{dS}^2$ *once* $\mathcal{C}$ *is projectively identified with* $\mathbb{R}\mathbf{P}^1$*.*

We prove now that the holonomies of lightlike foliations in a singular $\mathbf{X}$-surface are piecewise Möbius maps. A *projective structure* on a topological one-dimensional manifold is a $(\mathrm{PSL}_2(\mathbb{R}), \mathbb{R}\mathbf{P}^1)$-structure consisting of orientation preserving charts, and we call *projective* the $(\mathrm{PSL}_2(\mathbb{R}), \mathbb{R}\mathbf{P}^1)$-morphisms between two projective curves. We endow $\mathbb{R}$ with its standard projective structure for which $x \in \mathbb{R} \mapsto [x : 1] \in \mathbb{R}\mathbf{P}^1$ is a global chart, so that projective morphisms between intervals of $\mathbb{R}$ are precisely the (restrictions of) homographies. We recall that geodesics of singular $\mathbf{dS}^2$-surfaces which are lightlike or avoid the singularities have well-defined affine structures (see Paragraph 8.1.2), and observe that these affine structures define in particular a projective structure on geodesics (through the embedding $\mathbb{R} \hookrightarrow \mathbb{R}\mathbf{P}^1$, equivariant for the natural embedding $\mathrm{Aff}^+(\mathbb{R}) \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$).

Definition C.2. A homeomorphism $F\colon I \to J$ between two projective 1-dimensional manifolds is *piecewise projective* if there exists a finite number of points $x_1, \dots, x_N$ in I, called the *singular points* of F, such that F is projective in restriction to any connected component C of $I \setminus \{x_1, \dots, x_N\}$.

Proposition C.3. *Let* $H\colon I \to J$ *be the holonomy of a lightlike foliation between two connected subsets* I *and* J *of geodesics in a singular* $\mathbf{X}$*-surface (*$I = J$ *being allowed) which avoid the singularities. Then* H *is piecewise projective.*

Proof. **Case of $\mathbb{R}^{1,1}$.** In this case, the leaves of the α and β foliations are the affine lines respectively parallel to the vector lines $\mathbb{R}e_1$ and $\mathbb{R}e_2$. On the other hand the affinely parametrized geodesics are the affinely parametrized segments, and the holonomy between them is thus a dilation, *i.e.* an affine and in particular projective transformation.

Case of $\mathbf{dS}^2$. For any geodesic $s \subset \mathbf{dS}^2$ which is not β-lightlike, we claim that the restriction to s of the first projection $\pi_\alpha|_s\colon s \to \mathbb{R}\mathbf{P}^1$ is projective for the affine structure of s (the same proof showing that $\pi_\beta|_s$ is projective if s is not β-lightlike). Indeed according to Lemma 8.1, the stabilizer of s in $\mathrm{PSL}_2(\mathbb{R})$ contains a one-parameter subgroup (g^t) acting transitively on s, and $t \in \mathbb{R} \mapsto g^t(x) \in s$ is an affine parametrization of s for any $x \in s$. The equivariance $\pi_\alpha(g^t(x)) = g^t(\pi_\alpha(x))$ of π_α concludes then the proof of the claim by definition of the projective structure of $\mathbb{R}\mathbf{P}^1$. Observe moreover that $\pi_\alpha|_s$ is injective and defines thus a projective isomorphism onto its image.

Now for any two geodesics s_1, s_2 of $\mathbf{dS}^2$, the holonomy H of $\mathcal{F}_\alpha$ from s_1 to s_2 satisfies by definition the invariance $\pi_\alpha|_{s_2} \circ H = \pi_\alpha|_{s_1}$ on the open subset where this equality is well-defined, showing that H is a projective isomorphism since the $\pi_\alpha|_{s_i}$ are such.

General case. Let (S, Σ) be a singular $\mathbf{X}$-surface. Without loss of generality, we can assume that H is the holonomy of the α foliation between relatively compact connected subsets I and J of geodesics of S. Since Σ is discrete and $\mathcal{F}_\alpha$ continuous, the set I_Σ of points $p \in I$ such that $[p\,; H(p)]_\alpha \cap \Sigma \neq \varnothing$ is discrete in I, hence finite (we denote by $[p\,; H(p)]_\alpha$ the interval of the oriented leaf $\mathcal{F}_\alpha(p)$ from p to $H(p)$). Let C be a connected component of $I \setminus I_\Sigma$. Then for any $x \in C$, we can cover $[x\,; H(x)]_\alpha$ by a finite chain of compatible regular $\mathbf{X}$-charts. This expresses $H|_C$ as a finite composition of holonomies H_i between geodesics which are, for any i, contained in the domain of a given regular $\mathbf{X}$-chart. We proved previously that each H_i is projective, and $H|_C$ is thus projective as a composition of such maps. This shows that H is piecewise projective and concludes the proof. $\square$

Appendix D. Singular constant curvature Lorentzian surfaces as Lorentzian length spaces

We show in this appendix, entirely independent from the rest of the text, that globally hyperbolic singular $\mathbf{X}$-surfaces give examples of the *Lorentzian length spaces* introduced in [KS18].

The latter are natural Lorentzian counterparts of the usual metric length spaces (for which [BH99] is a classical reference), and give a synthetic approach to Lorentzian geometry by forgetting the metric itself and rather looking at its main geometrical byproducts. Existing examples included for now (beyond smooth Lorentzian metrics) the Lorentzian metrics with low regularity, the cone structures [KS18, §5], the so-called "generalized cones" [AGKS21] and some gluing constructions [BR24]. To the best of our knowledge and understanding, the singular constant curvature Lorentzian surfaces as we introduce them here were not considered yet in the literature as examples of Lorentzian length spaces. It seems to us that they provide natural examples, as the constant curvature Riemannian metrics with conical singularities give important examples of metric length spaces.

We quickly describe the relation with Lorentzian length spaces without entering into too much details, most of the technical work beeing done in Appendix A. Until the end of this section, S denotes a singular $\mathbf{X}$-surface endowed with the distance d_S induced by a fixed complete Riemannian metric.

The structure of a *causal space* on a set X is defined in [KS18, Definition 2.1] by a causal relation $\leq$ (formally a reflexive and transitive relation) and a chronological relation $\ll$ (formally a transitive relation contained in $\leq$) on X. We endow of course our singular $\mathbf{X}$-surface S with the chronological and causal relations defined by the timelike and causal futures (see Definition A.3), namely by definition:

(1) $x \leq y$ if and only if $y \in J^+(x)$;
(2) $x \ll y$ if and only if $y \in I^+(x)$.

On a metrizable causal space $(X, d, \leq, \ll)$, a *time-separation* function is then defined as a map $\tau\colon X \times X \to [0\,;+\infty]$ such that $x \not\leq y$ implies $\tau(x,y) = 0$, $\tau(x,y) > 0$ if and only if $x \ll y$, τ satisfies the reverse triangular inequality

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \tag{D.1}$$

for any $x \leq y \leq z$, and τ is lower semi-continuous. The two first conditions are by definition satisfied by the time-separation function τ_S of S defined in (A.3), which also satisfies the reverse triangular inequality (D.1) according to Lemma A.9. Lastly, the lower semi-continuity of τ_S is proved in the same way than the second part of the proof of Proposition A.16, which does not rely on global hyperbolicity (see also [Min19, Theorem 2.32]). $(S, d_S, \leq, \ll, \tau_S)$ is then a *Lorentzian pre-length space* as defined in [KS18, Definition 2.8], and it is moreover automatically causally path connected as defined in [KS18, Definition 2.18, Definition 3.1].

We assume from now on that S is globally hyperbolic in the sense of Definition A.3. In this case the Lorentzian pre-length space $(S, d_S, \leq, \ll, \tau_S)$ satisfies some additional nice properties. Lemma A.13 first shows that S is *causally closed* in the sense that if $p_n \leq q_n$ respectively converge to p and q, then $p \leq q$. It is moreover easy to show that the restriction of τ_S to a normal convex neighbourhood of S (see Proposition A.12) gives a localizing neighbourhood as defined in [KS18, Definition 3.16], hence that $(S, d_S, \leq, \ll, \tau_S)$ is *strongly localizable*.

The last step to Lorentzian length spaces mimics the definition of usual metric length spaces. The τ_S-length of a causal curve $\gamma\colon [a\,;b] \to S$ is defined in [KS18, Definition 2.24] as

$$L_{\tau_S}(\gamma) = \inf\left\{ \sum_{i=0}^{N} \tau_S(\gamma(t_i), \gamma(t_{i+1})) \;\middle|\; N \in \mathbb{N}, a = t_0 < t_1 < \cdots < t_N = b \right\}.$$

Note that our usual notion of causal curve coincides with the one of [KS18, Definition 2.18] according to [KS18, Lemma 2.21]. Using [KS18, Proposition 2.32] and the decomposition (A.2) of the usual Lorentzian length $L(\gamma)$ into the ones of its regular pieces, one easily shows that $L(\gamma) = L_{\tau_S}(\gamma)$. This last equality shows the following.

Proposition D.1. *Any globally hyperbolic singular* $\mathbf{X}$*-surface* S *has a natural structure of a* regular Lorentzian length space $(S, d_S, \leq, \ll, \tau_S)$ *as defined in* [KS18, Definition 3.22].

We recall that according to Proposition A.8, any class A closed singular $\mathbf{X}$-surface admits a simple closed spacelike curve, and that $\mathbb{Z}$-coverings with respect to such curves give according to Lemma A.15 examples of globally hyperbolic singular $\mathbf{X}$-surfaces. Such coverings are regular Lorentzian length spaces according to Proposition D.1.

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