RIGIDITY OF SINGULAR DE-SITTER TORI WITH RESPECT TO THEIR LIGHTLIKE BI-FOLIATION

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ABSTRACT. In this paper, we introduce a natural notion of constant curvature Lorentzian surfaces with conical singularities, and provide a large class of examples of such structures. We moreover initiate the study of their global rigidity, by proving that de-Sitter tori with a single singularity of a fixed angle are determined by the topological equivalence class of their lightlike bi-foliation. While this result is reminiscent of Troyanov's work on Riemannian surfaces with conical singularities, the rigidity will come from topological dynamics in the Lorentzian case.

1. INTRODUCTION

A Lorentzian metric on a surface induces a pair of *lightlike foliations*, and the Poincaré-Hopf theorem therefore implies that the torus is the only closed and orientable Lorentzian surface. An analog of the Gauß-Bonnet formula shows moreover that the only constant curvature Lorentzian metrics on the torus are actually *flat* (see [Ave63, Che63]). It is then natural to try to widen this class of geometries, in order to obtain structures locally modelled on the *de-Sitter space* $\mathbf{dS}^2$ – the Lorentzian homogeneous space of non-zero curvature, introduced in Paragraph 2.1.3 below. This is not possible on a closed surface without removing some points, and a natural way to do this is to proceed as in the Riemannian case, by concentrating all the curvature in finitely many points where the metric has *conical singularities* as they appeared in [BBS11].

The first goal of this paper is to introduce this natural class of *singular constant curvature Lorentzian surfaces*, to provide examples of such structures, and to initiate their study by proving some of their fundamental properties. The second and main goal is to investigate in the de-Sitter case the relations of these geometrical objects with associated dynamical ones: their pair of lightlike foliations.

1.1. **Singular de-Sitter surfaces.** The Lorentzian conical singularities are defined analogously to the Riemannian ones, and their local definition already appeared in [BBS11]. The connected component of the identity in the isometry group of $\mathbf{dS}^2$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, acts transitively on $\mathbf{dS}^2$, and the stabilizer of a point $\mathsf{o} \in \mathbf{dS}^2$ in $\mathrm{PSL}_2(\mathbb{R})$ is a one-parameter hyperbolic group $A = \{a^\theta\}_{\theta\in\mathbb{R}} \subset \mathrm{PSL}_2(\mathbb{R})$. As in the Riemannian case, a natural way to describe a conical singularity in the de-Sitter space is to choose a non-trivial isometry $a^\theta \in A$ and a geodesic ray γ emanating from o, to consider the sector from γ to $a^\theta(\gamma)$ in $\mathbf{dS}^2$ and to glue its two boundary components by a^θ. For simplicity we choose a *lightlike* half-geodesic $\gamma = \mathcal{F}^+_\alpha(\mathsf{o})$, and a phenomenon specific to the Lorentzian situation then happens: $\mathcal{F}^+_\alpha(\mathsf{o})$ is fixed by a^θ. In other words, the sector described by $a^\theta(\mathcal{F}^+_\alpha(\mathsf{o})) = \mathcal{F}^+_\alpha(\mathsf{o})$ is simply the surface $\mathbf{dS}^2_*$ obtained by cutting $\mathbf{dS}^2$ open along $\mathcal{F}^+_\alpha(\mathsf{o})$. The latter contains two up and down copies $\iota_\pm(\mathcal{F}^+_\alpha(\mathsf{o}))$ of the initial geodesic ray as boundary components, which can be identified by $\iota_+(x) \sim \iota_-(a^\theta(x))$ to obtain a topological surface $\mathbf{dS}^2_\theta = \mathbf{dS}^2_*/\sim$ in the quotient. This identification space has a marked point o_θ which is the projection of o, and the metric of $\mathbf{dS}^2$ induces a natural locally $\mathbf{dS}^2$ Lorentzian metric on $\mathbf{dS}^2_\theta \setminus \mathsf{o}_\theta$ since the gluing was made by isometries. The neighbourhood of o_θ is defined as the local model of a *standard singularity of angle* θ in a locally $\mathbf{dS}^2$ surface, and a *singular* $\mathbf{dS}^2$*-surface* is an orientable surface bearing a locally $\mathbf{dS}^2$ Lorentzian metric outside of a discrete set of points which are standard singularities (see Definition 2.22). We refer to Paragraph 2.2.1 for

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more details on this construction, analogously introduced in the case of zero curvature (namely for the Minkowski space), and illustrated in Figure 2.1 below.

To the best of our knowledge, singular constant curvature Lorentzian surfaces did not appear so far in the literature as an object of independent interest, and in particular no examples appeared yet on closed surfaces. One of the purposes of this work is to construct many examples, and to set the ground for the future investigation of singular constant curvature Lorentzian surfaces. To this end, we furnish in Proposition 3.4 a general method to construct a large class of examples, and we carefully prove in Paragraphs 2.2 and 2.3 many structural properties of singular constant curvature Lorentzian surfaces. An important point of view on singular Riemannian surfaces is the one of *metric length spaces*, and a natural Lorentzian counterpart of the latter notion was introduced in [KS18] under the name of *Lorentzian length spaces*. Singular constant curvature Lorentzian surfaces appear as natural candidates to illustrate such a notion, and we will explain in the Appendix D that they furnish indeed a large class of examples of Lorentzian length spaces, apparently new in the literature.

1.2. **Main results: dynamics of the lightlike foliations and geometric rigidity.** As we will see in Paragraph 2.2.5, one can use any geodesic ray to define a standard singularity. The benefit of using a lightlike ray as we did in Paragraph 1.1, is to naturally observe from the construction that the lightlike foliations $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ of $\mathbf{dS}^2$ extend at the standard singularity o_θ to two transverse (one-dimensional) *topological foliations* of $\mathbf{dS}^2_\theta$ (a result properly proved in Proposition 2.13). Any singular $\mathbf{dS}^2$-structure on a surface induces thus a *lightlike bi-foliation* $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$, and the torus remains therefore the only closed and orientable surface bearing a Lorentzian metric with constant curvature and standard singularities. The study of constant curvature Lorentzian metrics on higher genus surfaces requests the introduction of other types of singularities, which produce *singular* lightlike foliations. They will be the object of a future work, and we refer to Remark 3.6 for a discussion of such examples.

The seminal work of Troyanov [Tro86, Tro91] describes the main global rigidity properties of Riemannian surfaces with conical singularities. Troyanov proves therein that for any fixed set of singularities and angles on a closed orientable surface, any conformal class contains a unique metric of a given curvature having the prescribed singularities (with necessary conditions relating the angles, the constant curvature and the Euler characteristic of the surface, given by the Gauß-Bonnet formula). On the other hand, it is easily checked that two Lorentzian metrics μ_1 and μ_2 on a surface are conformal if, and only if, they have identical lightlike bi-foliations. In the direction of Troyanov's results, it is then natural to investigate the relation of singular constant curvature Lorentzian surfaces to their lightlike bi-foliations. The following theorem is the main result of this paper, and provides an answer to this question for the non-zero curvature in the case of one singularity.

Theorem A. *Let S_1, S_2 be two closed singular $\mathbf{dS}^2$-surfaces having a unique singularity of the same angle. Assume that the lightlike bi-foliations of S_1 and S_2 are minimal and topologically equivalent. Then S_1 and S_2 are isometric.*

We say that a lightlike bi-foliation $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ is *minimal* if both foliations are such, *i.e.* have all of their leaves dense. The lightlike bi-foliations of S_1 and S_2 are said to be *topologically equivalent* if there exists a homeomorphism $f\colon S_1 \to S_2$ which is a simultaneous equivalence of the α and the β oriented foliations, *i.e.* such that $f(\mathcal{F}^{S_1}_\alpha(x)) = \mathcal{F}^{S_2}_\alpha(f(x))$ and $f(\mathcal{F}^{S_1}_\beta(x)) = \mathcal{F}^{S_2}_\beta(f(x))$ while respecting the orientations for any $x \in S_1$.

A crucial difference between Theorem A and Troyanov's work on the Riemannian case should be emphasized at this point: the isometry between the singular $\mathbf{dS}^2$-surfaces is obtained in the present paper from an equivalence which is only *topological* between their lightlike bi-foliations. In particular, we deduce from a topological equivalence between the bi-foliations the existence of a smooth one, which may be seen as a *geometric rigidity* result for this class of bi-foliations (we refer the reader to the very pleasant presentation of the general problem of geometric rigidity for dynamical systems given in [Gha21, p.468]). The former rigidity result would be of little interest without its companion existence result, given by the following theorem.

Theorem B. *Let $A^+_\alpha \neq A^+_\beta \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ be two distinct irrational half-lines and $\theta \in \mathbb{R}^*_+$. Then there exists on $\mathbf{T}^2$ a singular $\mathbf{dS}^2$-structure with a unique singularity of angle θ, and whose lightlike foliations are suspensions of oriented projective asymptotic cycles $A^+(\mathcal{F}_\alpha) = A^+_\alpha$ and $A^+(\mathcal{F}_\beta) = A^+_\beta$. In particular, $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are both minimal.*

The main results of this paper may be seen as a global description of the *deformation space* of singular $\mathbf{dS}^2$-structures of the two-torus having a unique singularity of angle θ at $0 \in \mathbf{T}^2$, denoted by $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ and properly introduced in Definition 3.29. The description is done here in terms of the *projective asymptotic cycles* of the lightlike foliations, which is the main topological invariant of oriented topological foliations on the torus. It can be seen as a global counterpart of the rotation number of the first-return map on a section, and it will be introduced in Paragraph 3.6. The projective asymptotic cycles of the lightlike foliations are well-defined for an isotopy class $[\mu]$ of structures in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ (see Remark 3.30), and the general question investigated in this paper may then be roughly summarized as follows: *to which extent is the map*

$$[\mu] \in \mathsf{Def}_\theta(\mathbf{T}^2, 0) \mapsto (A^+(\mathcal{F}^{[\mu]}_\alpha), A^+(\mathcal{F}^{[\mu]}_\beta)) \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))^2 \tag{1.1}$$

bijective ? This is in a sense a counterpart of Troyanov's description [Tro86, Tro91], where the deformation space of Riemannian metrics with prescribed conical singularities is shown to identify with the one of conformal structures (namely with the Teichmüller space). Contrarily to Troyanov's work, the description is however done in the current paper in terms of a *topological dynamical invariant*: the projective asymptotic cycle.

The map defined in (1.1) is not globally injective, as it may be observed at the level of the first-return map of the foliations. Indeed, any small enough perturbation of a circle homeomorphism T having rational rotation number as well as non-periodic orbits, has the same rotation number than T. We will however prove in the two following results the surjectivity of the map (1.1), as well as its injectivity on large parts of $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$.

Theorem C. *Let $\theta \in \mathbb{R}^*_+$ and $c_\alpha \neq c_\beta \in \pi_1(\mathbf{T}^2)$ be two distinct primitive elements. Then there exists in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ a unique point $[\mu]$ for which $\mathcal{F}_\alpha(0)$ and $\mathcal{F}_\beta(0)$ are closed and $([\mathcal{F}_\alpha(0)], [\mathcal{F}_\beta(0)]) = (c_\alpha, c_\beta)$. Moreover, $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are suspensions, and $(\mathbf{T}^2, [\mu])$ is isometric to a $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,x}$.*

The $\mathbf{dS}^2$-tori $\mathcal{T}_{\theta,x}$ are introduced below in Proposition 3.12.

Theorem D. *Let $\theta \in \mathbb{R}^*_+$, $c_\alpha \in \pi_1(\mathbf{T}^2)$ be a primitive element and $A^+_\beta \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ be an irrational half-line. Then there exists in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ a unique point $[\mu]$ such that:*

(1) $\mathcal{F}_\alpha(0)$ is closed and $[\mathcal{F}_\alpha(0)] = c_\alpha$;
(2) and $A^+(\mathcal{F}_\beta) = A^+_\beta$.

Moreover, $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are suspensions, $\mathcal{F}_\beta$ is minimal, and $(\mathbf{T}^2, [\mu])$ is isometric to a $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,x}$. The obvious analogous statement holds when exchanging the roles of the α and β-foliations.

Theorems A, C and D advertise the general idea that closed singular constant curvature Lorentzian surfaces are much more rigid than their Riemannian counterparts. This rigidity will be a leitmotiv in this text, and finds its origin in the existence of the two lightlike foliations (such a preferred pair of transverse foliations does not exist for singular Riemannian surfaces).

1.3. **Methods, and strategies of the main proofs.** In [Tro86, Tro91], Troyanov translates the existence, in a given conformal class, of a unique constant curvature Riemannian metric with suitable singularities, into the existence of a unique solution for a differential equation involving the Laplacian. Using the well-behaved properties of the latter, he proves his results by relying mainly on analytical methods. Contrarily to the Riemannian one, the Lorentzian Laplacian is not widely studied, and is more importantly a *hyperbolic* differential operator and not anymore an elliptic one, which makes his use less suited to our purpose. Moreover, the phenomena that we wish to highlight in this work are by nature dynamical, the geometric rigidity expressed by Theorem A coming from the *topological dynamics* of the lightlike foliations.

For this reason, we will use in this text a constant interaction of geometrical and dynamical methods. The former will seem relatively familiar to the readers used to more classical types of locally homogeneous geometric structures on surfaces (for instance translation or dilation surfaces), while the latter will come from one-dimensional dynamics (namely piecewise Möbius interval exchange maps and their associated circle homeomorphisms) and will be used in connection with the lightlike foliations through their first-return maps.

Our first concern in this paper is to construct examples satisfying the dynamical properties requested in Theorem B. Using identification spaces of polygons, this task eventually relies on the simultaneous realization of pairs of rotation numbers for a two-parameter family of pairs of Möbius interval exchange maps.

The first step of the proof of Theorem D is geometrical. We reduce the statement to the investigation of a one-parameter family of singular $\mathbf{dS}^2$-tori introduced in Paragraph 3.2, which are identification spaces of lightlike rectangles of $\mathbf{dS}^2$, illustrated in Figure 3.1 below. The uniqueness claim is translated in this way in Proposition 3.23 into a statement about a one-parameter family of circle maps – the first-return maps of the β-lightlike foliation on the closed α-leaf. In the end, the statement eventually follows from an important fact of one-dimensional dynamics: the rotation number of a monotonic one-parameter family of circle homeomorphisms increases *strictly* at irrational points (see Lemma B.1). This scheme of proof may serve as a paradigm for the geometrico-dynamical arguments used in the present paper, and for the efficiency of their interactions – geometrical statements becoming natural consequences of dynamical ones, once suitably translated.

The general strategy to prove Theorem A is then to show that two structures μ_1 and μ_2 with topologically equivalent and minimal lightlike foliations admit arbitrarily close *surgeries* $\mu_{1,n}$ and $\mu_{2,n}$, having a closed α-leaf at the singularity and identical irrational asymptotic cycles of their β-foliations. Once such suitable surgeries are constructed, one can rely on Theorem D to prove that $[\mu_{1,n}] = [\mu_{2,n}]$ in the deformation space. Since the latter sequence converges by construction both to $[\mu_1]$ and to $[\mu_2]$, this shows that $[\mu_1] = [\mu_2]$.

1.4. **Perspectives on multiple singularities and singular flat tori.** The strategy of proof of Theorem A will essentially persist for any number of singularities. The first and main geometrical tool developed in this paper to implement this strategy is indeed the construction of suitable *surgeries* in paragraph 4.4, which is done in full generality. The existence of simple closed timelike geodesics is known for regular Lorentzian manifolds (see for instance [Tip79, Gal86, Suh13]), and we prove in Appendix A that the usual tools and arguments remain available for singular constant curvature Lorentzian surfaces. This allows us to obtain simple closed timelike geodesics in their case, and to use them to realize the surgeries.

It is actually the proof of Theorem D and more precisely the one of the dynamical Lemma B.1 which fails for $n \geq 2$ singularities, and this is the only reason why the present paper focuses mainly on the case of a single singularity. Indeed, the rough description that we gave previously hided a fundamental aspect of the proof of Theorem D: after the geometrical reduction to identification spaces of polygons, the number of parameters of the resulting family of circle maps is equal to the number of singularities of the initial structure. And while the strict monotonicity of the rotation number at irrational points is easily shown for a *one*-parameter family, essentially everything can happen for generic *two*-parameter families of circle maps. This crucial difference between one-parameter and multiple parameter families of deformations is mainly due to the naive but fundamental observation that the rotation number is itself a *one*-dimensional invariant. The investigation of the rigidity of $\mathbf{dS}^2$-tori with multiple singularities requests therefore a new method to handle this dynamical difficulty, which is the content of a work in progress of the author in collaboration with Selim Ghazouani.

We will prove in Proposition 2.32 a version of the Gauß-Bonnet formula, showing in particular that a constant curvature Lorentzian metric on the torus with exactly one singularity necessarily has non-zero curvature. We focused therefore in the present paper on singular $\mathbf{dS}^2$-structures, and not on *flat* ones. Singular flat tori will be independently investigated in a future work.

We lastly emphasize that in all the examples of singular $\mathbf{dS}^2$-tori constructed in this text, both lightlike foliations are suspensions of circle homeomorphisms. The author does not know if there exists a singular $\mathbf{dS}^2$-structure on $\mathbf{T}^2$, one of whose lightlike foliations has a Reeb component.

1.5. **Connection with the smoothness of conjugacies for circle diffeomorphisms with breaks.** As we will see in Lemma 2.30, the first-return maps of lightlike foliations in a singular $\mathbf{dS}^2$-surface are not only continuous but are actually *circle diffeomorphisms with breaks*, and while it may appear as a technical detail, this regularity actually gives a crucial dynamical information on the first-return map T. Indeed, the seminal work of Denjoy [Den32] implies then that T does not have an exceptional minimal set, and is thus topologically conjugated to a rigid rotation of the circle if it has an irrational *rotation number*. Since T is piecewise smooth, it is natural to wonder at this point if T is actually *smoothly* conjugated to a rotation. But as naive as it may seem, this question is an old and deep one which remains still open in its full generality. If T is $\mathcal{C}^\infty$ and its rotation number *Diophantine*, Herman showed in [Her79] that it is $\mathcal{C}^\infty$-conjugated to a rigid rotation, following the initial work of Arnol'd [Arn64] on this question. Since these founding works, the research on this subject never stopped to be intensively active and we do not pretend to cover its vast literature. The problem remains unsolved for general circle diffeomorphisms with breaks, about which the optimal result up to date appears in [KKM17] to the best of our knowledge, and answers the question in the case of a single singularity.

The main rigidity result proved in this paper happens to be similar in its philosophy to the problem of smoothness of the conjugacy to a rigid rotation for a circle diffeomorphism with breaks. Indeed, a topological equivalence between two pairs of foliations forces in Theorem A the existence of a smooth one – hence of a smooth conjugacy between the first-return maps. This connection between singular $\mathbf{dS}^2$-structures on the torus and circle diffeomorphisms with breaks is one of our motivations for this subject, and we wish to investigate it more precisely in a future work.

1.6. **Organization of the paper.** Basic definitions and properties of singular constant curvature Lorentzian surfaces are introduced and proved in Section 2. Section 3 is then concerned with the construction of such structures, and we give in Proposition 3.4 a general existence result of surfaces obtained as identification spaces of polygons with lightlike geodesic edges. In the remainder of Section 3, we study thoroughly the properties of a one-parameter and of a two-parameter family of $\mathbf{dS}^2$-tori with one singularity. This allows us to conclude in Paragraph 3.8 the proof of the existence parts of Theorems B, C and D (we prove actually a more refined statement given in Theorem 3.1). The proof of the uniqueness parts of Theorems A, C and D is concluded in section 4. Along the way, we construct in Paragraph 4.4 a family of surgeries and prove in Appendix A the existence of simple closed definite geodesics, both results being obtained in the general setting of singular constant curvature Lorentzian surfaces. We also prove in Appendix B the main technical result used on the rotation number (which is classical), and in Appendix C that holonomies of lightlike foliations are piecewise Möbius. Lastly, we explain in Appendix D how singular constant curvature Lorentzian surfaces may be interpretated as Lorentzian length spaces.

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Some usual notations and a standing assumption. If X is a space endowed with an equivalence relation $\sim$, then we usually denote by $\pi\colon X \to X/\sim$ the canonical projection onto the quotient, and also use the notation $[x] = \pi(x) \in X/\sim$ for $x \in X$. For any subset P of a topological space X, we denote by $\mathrm{Int}(P)$ the interior of P, by $\mathrm{Cl}(P)$ its closure and by ∂P its boundary.

All the surfaces (and any other manifolds) considered in this text are assumed to be connected, orientable and boundaryless, unless explicitly stated otherwise.

2. Singular constant curvature Lorentzian surfaces

This section is devoted to define and prove the fundamental notions and properties concerning singular constant curvature Lorentzian surfaces.

2.1. **Constant curvature Lorentzian surfaces.**

As a preparation to consider singular structures, we first focus in this subsection on regular ones. We define the main Lorentzian notions that will be used throughout the text, and introduce the two 2-dimensional Lorentzian homogeneous spaces as well as the surfaces modelled on them.

2.1.1. *Lorentzian surfaces, time and space-orientation, and lightlike foliations.* A quadratic form is said *Lorentzian* if it is non-degenerate and of signature $(1,n)=(-,+,\dots,+)$. A *Lorentzian metric* of class $\mathcal{C}^k$ on a manifold M is a $\mathcal{C}^k$ field μ of Lorentzian quadratic forms on the tangent bundle of M. Usually, we will denote by $g=g_\mu$ the bilinear form associated to μ, so that $\mu(u)=g(u,u)$. Observe that if μ is a Lorentzian metric on a surface S, then $-\mu$ is also a Lorentzian metric on S.

Any Lorentzian vector space (V,q) (or tangent space of a Lorentzian manifold) is decomposed according to the sign of q, $u\in V$ being called:

(1) *spacelike* if $q(u)>0$,
(2) *timelike* if $q(u)<0$,
(3) *lightlike* if $q(u)=0$,
(4) *causal* is $q(u)\leq 0$,
(5) and *definite* if it is timelike or spacelike.

These denominations of *signatures* of vectors in Lorentzian tangent spaces will be used in the natural compatible way for line fields and curves.

A *time-orientation* on a Lorentzian surface (S,μ) is a continuous choice among one of the two connected components of the cone $\mu_x^{-1}(\mathbb{R}_-)\setminus\{0\}$ of non-zero timelike vectors, which is called the *future* cone. We will also talk without distinction of the associated *future causal* cone, closure of the future timelike one, and use the obvious similar notion of *space-orientation* in a Lorentzian surface (namely a continuous choice among one of the two connected components of $\mu_x^{-1}(\mathbb{R}_+)\setminus\{0\}$). Not any Lorentzian surface bears a time-orientation, and it is said *time-orientable* if it does. An orientable Lorentzian surface is time-orientable if, and only if it is space-orientable.

Any Lorentzian surface S bears locally two (unique) *lightlike line fields*, which are globally well-defined if, and only if S is oriented. In the latter case, they give rise to two *lightlike foliations* on the surface, of which we always choose an ordering $(\mathcal{F}_\alpha,\mathcal{F}_\beta)$ (defined in paragraph 2.1.5 for the surfaces studied in this text). This ordered pair of foliations will be called the *lightlike bi-foliation* of the surface, and the lightlike leaves are simply the lightlike geodesics of the metric. If S is furthermore time-oriented, then these lightlike foliations are themselves orientable. We will always use the convention for which the orientation of the lightlike bi-foliation $(\mathcal{F}_\alpha,\mathcal{F}_\beta)$ is both compatible with the orientation of S and with its time-orientation, as illustrated in Figure 2.1 below. In other words with these conventions, a time-orientation and an ordering $(\mathcal{F}_\alpha,\mathcal{F}_\beta)$ of the lightlike foliations of an oriented Lorentzian surface S induce a space-orientation of S and an orientation of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$.

We will call *quadrant at* $x\in S$ the four connected components of $\mathrm{T}_xS\setminus\{\mu^{-1}(0)\}$, or of $D\setminus(\mathcal{F}_\alpha(x)\cup\mathcal{F}_\beta(x))$ for D a disk around x small enough for (x,D,I_α,I_β) to be topologically equivalent to $(0,]0\,;1[^2,]0\,;1[\times\{0\},\{0\}\times]0\,;1[)$, with $I_{\alpha/\beta}$ the respective connected components of $D\cap\mathcal{F}_{\alpha/\beta}(x)$ containing x.

2.1.2. *The Minkowski space.* The flat model space of Lorentzian metrics is the Minkowski space $\mathbb{R}^{1,n}$, *i.e.* the vector space $\mathbb{R}^{n+1}$ endowed with a Lorentzian quadratic form $q_{1,n}$. In this text we will be interested in Lorentzian surfaces, and we thus focus now on the Minkoswki plane $\mathbb{R}^{1,1}$ that we endow with the quadratic form $q_{1,1}(x,y)=2xy$ and the induced left-invariant Lorentzian metric $\mu_{\mathbb{R}^{1,1}}$. We fix on $\mathbb{R}^{1,1}$ the standard orientation of $\mathbb{R}^2$, and the time-orientation (respectively space-orientation) for which the set of future timelike (resp. spacelike) vectors is the top left quadrant $\{(u,v)\mid u<0,v>0\}$ (resp. top right quadrant $\{(u,v)\mid u>0,v>0\}$).

The connected component of the identity in the orthogonal group of $q_{1,1}$ is the subgroup

$$\mathrm{SO}^0(1,1) \coloneqq \left\{ a^t \;\middle|\; t \in \mathbb{R} \right\} \subset \mathrm{SL}_2(\mathbb{R}) \text{ with } a^t \coloneqq \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{2.1}$$

Since $q_{1,1}$ is by construction preserved by translations, the subgroup $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,2)$ of affine transformations preserves $q_{1,1}$ and its time-orientation, and equals in fact the group $\mathrm{Isom}^0(\mathbb{R}^{1,1})$ of orientation and time-orientation preserving isometries of $\mathbb{R}^{1,1}$. In particular, $\mathrm{Isom}^0(\mathbb{R}^{1,1})$ acts transitively on $\mathbb{R}^{1,1}$ with stabilizer $\mathrm{SO}^0(1,1)$ at $\mathsf{0} = (0,0)$, which induces a $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,2)$-equivariant identification of $\mathbb{R}^{1,1}$ with the homogeneous space $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,2)/\mathrm{SO}^0(1,1)$.

2.1.3. *The de-Sitter space.* We now introduce the Lorentzian homogeneous space of non-zero constant curvature. We will denote by $[S]$ the projection of $S \subset \mathbb{R}^{n+1} \setminus \{0\}$ in the projective space $\mathbb{R}\mathbf{P}^n$, by (e_i) the standard basis of $\mathbb{R}^n$, and use the identification

$$\varphi_0 \colon \begin{cases} t \in \mathbb{R} & \mapsto \hat{t} \coloneqq [t:1] \in \mathbb{R}\mathbf{P}^1 \setminus [e_1] \\ \infty & \mapsto \hat{\infty} \coloneqq [e_1] \end{cases} \tag{2.2}$$

between $\mathbb{R} \cup \{\infty\}$ and $\mathbb{R}\mathbf{P}^1$. Since any pair of distinct points of $\mathbb{R}\mathbf{P}^1$ is contained in the image U of the map $\varphi \coloneqq g \circ \varphi_0|_{\mathbb{R}} \colon \mathbb{R} \to U$ for some $g \in \mathrm{PSL}_2(\mathbb{R})$, the set

$$\mathbf{dS}^2 \coloneqq (\mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^1) \setminus \Delta \text{ with } \Delta \coloneqq \left\{ (p,p) \;\middle|\; p \in \mathbb{R}\mathbf{P}^1 \right\}$$

is covered by the domains of maps of the form

$$\phi \colon (p,q) \in (U \times U) \setminus \Delta \mapsto (\varphi^{-1}(p), \varphi^{-1}(q)) \in \mathbb{R}^2 \setminus \{\text{diagonal}\} \tag{2.3}$$

which we will call *affine charts of* $\mathbf{dS}^2$. The transition map between any two such affine charts is by construction of the form $(x,y) \in I^2 \setminus \{\text{diagonal}\} \mapsto (g(x), g(y)) \in \mathbb{R}^2$, with $I \subset \mathbb{R}$ some interval, and g abusively denoting the homography

$$g(t) \coloneqq \frac{at+b}{ct+d} \text{ associated to } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}), \tag{2.4}$$

characterized by the relation $g\left(\hat{t}\right) = \widehat{g(t)}$. A direct computation shows that the Lorentzian metric

$$\mu^0_{\mathbf{dS}^2} \coloneqq \frac{1}{|x-y|^2} dxdy$$

on $\mathbb{R}^2 \setminus \{\text{diagonal}\}$ is preserved by the transition maps $g \times g$ (2.4) between affine charts of $\mathbf{dS}^2$, which allows the following.

Definition 2.1. $\boldsymbol{\mu}$ is defined as the Lorentzian metric of $\mathbf{dS}^2$ equaling $\phi^* \mu^0_{\mathbf{dS}^2}$ on the domain of any affine chart ϕ of the form (2.3). The Lorentzian surface $(\mathbf{dS}^2, \boldsymbol{\mu})$ will be called the *de-Sitter space.*

We endow $\mathbb{R}\mathbf{P}^1$ with the $\mathrm{PSL}_2(\mathbb{R})$-invariant orientation induced by the standard one of $\mathbb{R}$ through the identification (2.2), and $\mathbf{dS}^2 \subset \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^1$ with the orientation induced by the one of $\mathbb{R}\mathbf{P}^1$. We also endow $\mathbf{dS}^2$ with the time-orientation (respectively space-orientation) for which the set of future timelike (resp. spacelike) vectors is the top left quadrant $\{(u,v) \mid u < 0, v > 0\}$ (resp. top right quadrant $\{(u,v) \mid u > 0, v > 0\}$), in a tangent space endowed with the coordinates coming from an affine chart (2.3).

By construction, $\boldsymbol{\mu}$ is invariant by the diagonal action $g(x,y) \coloneqq (g(x), g(y))$ of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbf{dS}^2$. This action is moreover transitive and the stabilizer of $\mathsf{o} \coloneqq ([e_1],[e_2]) \in \mathbf{dS}^2$ is the diagonal group

$$A \coloneqq \left\{ a^t \;\middle|\; t \in \mathbb{R} \right\},$$

hence $\mathbf{dS}^2$ is identified with $\mathrm{PSL}_2(\mathbb{R})/A$ in a $\mathrm{PSL}_2(\mathbb{R})$-equivariant way. Note that the projection $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ induces an isomorphism from $\mathrm{SO}^0(1,1)$ defined in (2.1) with A.

We now give another (more usual) description of the de-Sitter space. The quadratic form $q_{1,2}$ of the Minkowki space $\mathbb{R}^{1,2}$ equips (by restriction to its tangent bundle) the quadric

$$\mathrm{dS}^2 := \left\{x \in \mathbb{R}^3 \,\middle|\, q_{1,2}(x) = 1\right\}$$

with a Lorentzian metric μ_{dS^2} of sectional curvature constant equal to 1 (see for instance [O'N83, Proposition 4.29]), and the Lorentzian surface $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$ is the two-dimensional *hyperboloid model of the de-Sitter space*. Observe that endowing dS^2 with the restriction of the quadratic form $q_{2,1} := -q_{1,2}$ defines a Lorentzian metric of constant curvature equal to -1. In other words, the de-Sitter and anti-de-Sitter spaces are anti-isometric in dimension 2 and have thus the same geometry.

Lemma 2.2. *(1)* $\mathrm{PSL}_2(\mathbb{R})$ *is the subgroup of isometries of* $(\mathbf{dS}^2, \boldsymbol{\mu})$ *preserving both its orientation and time-orientation.*

(2) $(\mathbf{dS}^2, \boldsymbol{\mu})$ *is isometric to* $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$ *up to a multiplicative constant. For the sake of clarity, we normalize henceforth* $(\mathbf{dS}^2, \boldsymbol{\mu})$ *to have constant curvature* 1.

Proof. (1) This claim follows from the facts that $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathbf{dS}^2$, that the stabilizer of points in $\mathrm{PSL}_2(\mathbb{R})$ realize all linear isometries (*i.e.* that $a \in A \mapsto \mathrm{D}_{\mathbf{o}}a \in \mathrm{O}(\mathrm{T}_{\mathbf{o}}\mathbf{dS}^2, \boldsymbol{\mu}_{\mathbf{o}})$ is surjective), and that the one-jet determines pseudo-Riemannian isometries (a local isometry defined on a connected open subset, fixing a point x and of trivial differential at x, is the identity).
(2) One checks that the stabilizer in $\mathrm{SO}^0(1,2)$ of a point of dS^2 is a one-parameter hyperbolic subgroup, which gives an identification between dS^2 and $\mathrm{PSL}_2(\mathbb{R})/A$, equivariant with respect to some isomorphism between $\mathrm{SO}^0(1,2)$ and $\mathrm{PSL}_2(\mathbb{R})$. This yields two $\mathrm{PSL}_2(\mathbb{R})$-invariant Lorentzian metrics on $\mathrm{PSL}_2(\mathbb{R})/A$, respectively coming from the identifications with $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$ and $(\mathbf{dS}^2, \boldsymbol{\mu})$. But up to multiplication by a constant, $\mathfrak{sl}_2/\mathfrak{a}$ admits a unique Lorentzian quadratic form which is invariant by the adjoint action of A, and $\mathrm{PSL}_2(\mathbb{R})/A$ admits therefore a unique $\mathrm{PSL}_2(\mathbb{R})$-invariant Lorentzian metric up to multiplication by a constant. □

Remark 2.3. We emphasize that $\mathcal{C} := \mathbf{P}^+(q_{1,2}^{-1}(0)) = \left\{l \subset \mathbb{R}^{1,2} \,\middle|\, \text{null half-line}\right\}$ can be naturally interpreted as the *conformal boundary* of dS^2, and that this interpretation yields a natural identification of dS^2 with $\mathbf{dS}^2$ where each $\mathbb{R}\mathbf{P}^1$ appears as a connected component of $\mathcal{C}$. We refer to the proof of Proposition C.2 for more details on this construction.

2.1.4. *Lorentzian* $(\mathbf{G}, \mathbf{X})$*-surfaces.* We will be interested in this paper in the Lorentzian surfaces locally modelled on one of the two formerly introduced homogeneous spaces. Denoting henceforth by $(\mathbf{G}, \mathbf{X})$ one of the pairs $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,2), \mathbb{R}^{1,1})$ or $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$, we will use in this text the convenient language of $(\mathbf{G}, \mathbf{X})$-structures that we now introduce.

Definition 2.4. A $(\mathbf{G}, \mathbf{X})$-*atlas* on an oriented topological surface S is an atlas of orientation-preserving $\mathcal{C}^0$-charts $\varphi_i \colon U_i \to \mathbf{X}$ from connected open subsets $U_i \subset S$ to $\mathbf{X}$, whose transition maps $\varphi_j \circ \varphi_i^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ equal on every connected component of their domain the restriction of an element of $\mathbf{G}$ (henceforth, we will assume that any two domains of any atlas have a connected intersection). A $(\mathbf{G}, \mathbf{X})$-*structure* is a maximal $(\mathbf{G}, \mathbf{X})$-atlas, and a $(\mathbf{G}, \mathbf{X})$-*surface* is an oriented surface endowed with a $(\mathbf{G}, \mathbf{X})$-structure. A $(\mathbf{G}, \mathbf{X})$-*morphism* between two $(\mathbf{G}, \mathbf{X})$-surfaces is a map which reads in any connected $(\mathbf{G}, \mathbf{X})$-chart as the restriction of an element of $\mathbf{G}$.

Convention 2.5. All along this paper, $\mathbf{X}$ will be considered solely with the action of the group $\mathbf{G}$. In order to make the text lighter, we thus drop henceforth $\mathbf{G}$ from our notations, and talk simply of $\mathbf{X}$-chart, $\mathbf{X}$-structure, $\mathbf{X}$-surface and $\mathbf{X}$-morphism.

For any $\mathbf{X}$-structure on a surface S, each covering $\pi \colon S' \to S$ of S is induced with the unique $\mathbf{X}$-structure for which π is a $\mathbf{X}$-morphism. In particular, $\pi_1(S)$ acts on the universal cover $\tilde{S}$ by $\mathbf{X}$-morphisms of its $\mathbf{X}$-structure. Moreover for any $\mathbf{X}$-morphism f from a connected open subset $U \subset \tilde{S}$ to $\mathbf{X}$, there exists a unique extension

$$\delta \colon \tilde{S} \to \mathbf{X} \tag{2.5}$$

of f to a $\mathbf{X}$-morphism defined on $\tilde{S}$, and such a map is called a *developing map* of S. For any developing map δ, there exists furthermore a group morphism

$$\rho\colon \pi_1(S) \to \mathbf{G} \tag{2.6}$$

with respect to which δ is equivariant, entirely determined by δ and called the *holonomy morphism* associated to δ. Such a pair (δ, ρ) associated to the $\mathbf{X}$-structure of S is moreover unique up to the action

$$g \cdot (\delta, \rho) \coloneqq (g \circ \delta, g\rho g^{-1})$$

of $\mathbf{G}$. Reciprocally any $\mathbf{G}$-orbit of such local diffeomorphisms (2.5) equivariant for some morphism (2.6) defines a unique compatible $\mathbf{X}$-structure on S. We refer the reader to [Thu97, CEG87] for more details on $(\mathbf{G}, \mathbf{X})$-structures.

The core idea of $\mathbf{X}$-surfaces is that any $\mathbf{G}$-invariant geometric object on $\mathbf{X}$ gives rise to a corresponding object on any $\mathbf{X}$-surface. Let $\varepsilon_{\mathbf{X}}$ denote the constant sectional curvature of $\mathbf{X}$.

Proposition-Definition 2.6. *On any orientable surface S, $\mathbf{X}$-structures are in equivalence with time-oriented Lorentzian metrics of constant curvature $\varepsilon_{\mathbf{X}}$ in the following way.*

1. *For any $\mathbf{X}$-structure on S, there exists a unique Lorentzian metric for which $(\mathbf{G}, \mathbf{X})$-charts are local isometries. The latter metric is time-oriented and has constant curvature $\varepsilon_{\mathbf{X}}$.*
2. *Conversely, any time-oriented Lorentzian metric of constant curvature $\varepsilon_{\mathbf{X}}$ on S is induced by a unique $\mathbf{X}$-structure.*
3. *Moreover under this correspondence, the $\mathbf{X}$-morphisms between $\mathbf{X}$-surfaces are exactly their orientation-preserving isometries between connected open subsets.*

We will denote henceforth by the same letter μ a $\mathbf{X}$-structure on an orientable surface S and its induced Lorentzian metric.

Proof of Proposition 2.6. (1) Since $\mathbf{G}$ preserves the time-orientation of $\mathbf{X}$, the Lorentzian metric induced by a $\mathbf{X}$-structure is time-oriented, and of constant curvature $\varepsilon_{\mathbf{X}}$.
(2) Let μ be a time-oriented Lorentzian metric on S of constant sectional curvature $\varepsilon_{\mathbf{X}}$. Then it is locally isometric to $\mathbf{X}$ according to [O'N83, Corollary 8.15], and there exists thus an atlas of local isometric charts of S to $\mathbf{X}$ preserving both orientation and time-orientation. We claim that the transition maps of such an atlas and between two such atlases are restrictions of elements of $\mathbf{G}$, which will prove the claim. This is essentially due to the analog of the *Liouville theorem* for $(\mathbf{G}, \mathbf{X})$, claiming that any orientation and time-orientation preserving local isometry between two connected open subsets of $\mathbf{X}$, is the restriction of an element of $\mathbf{G}$. This last claim is easily obtained from the proof of Lemma 2.2.(2).
(3) Liouville theorem proves in particular the last claim. ☐

2.1.5. *Lightlike α and β-foliations of $\mathbf{X}$-surfaces.* We now describe the lightlike foliations of our models.

Definition 2.7. We will call α and β-*foliation* and denote by $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ the foliations of $\mathbf{dS}^2$ (respectively $\mathbb{R}^{1,1}$) whose leaves are the respective fibers of the second and first projections of $\mathbf{dS}^2 \subset \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^1$ to $\mathbb{R}\mathbf{P}^1$ (resp. the horizontal and vertical affine lines of $\mathbb{R}^{1,1}$). We call and denote in the same way the lightlike foliations induced by the latter on any $\mathbf{dS}^2$-surface (resp. $\mathbb{R}^{1,1}$-surface).

In other words, the α-leaves (resp. β-leaves) of $\mathbf{dS}^2$ read as horizontal (resp. vertical) lines in any affine chart (2.3) (hence the denomination to match the one for $\mathbb{R}^{1,1}$). Observe that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbf{dS}^2$ (respectively of $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,2)$ on $\mathbb{R}^{1,1}$) preserve both the α and the β-foliation, which induce thus indeed foliations on any $\mathbf{dS}^2$-surface (resp. $\mathbb{R}^{1,1}$-surface).

We endow the lightlike leaves of $\mathbf{dS}^2$ with the $\mathrm{PSL}_2(\mathbb{R})$-invariant orientation induced by the one of $\mathbb{R}\mathbf{P}^1$, and the lightlike leaves $\mathbb{R} \times \{b\}$ and $\{a\} \times \mathbb{R}$ of $\mathbb{R}^{1,1}$ with the $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,2)$-invariant one induced by $\mathbb{R}$. This further induces an orientation on the lightlike foliations of any $\mathbf{X}$-surface, compatible with its orientation, time-orientation and space-orientation as illustrated by Figure 2.1 below. The lightlike leaves of $\mathbf{dS}^2$ and $\mathbb{R}^{1,1}$ are embeddings of $\mathbb{R}$, and we denote by $\mathcal{F}_\alpha^{+*}(p)$ and $\mathcal{F}_\alpha^{-*}(p)$ the *half α-leaves*, *i.e.* the two connected components of $\mathcal{F}_\alpha(p) \setminus \{p\}$ emanating respectively in the positive and negative directions, by $\mathcal{F}^+_\alpha(p)$ and $\mathcal{F}^-_\alpha(p)$ their closures, and accordingly for $\mathcal{F}^\pm_\beta(p)$.

2.1.6. *Cyclic order, intervals of a circle and rectangles of* $\mathbf{dS}^2$. The circles $\mathbb{R}\mathbf{P}^1$ and $\mathbf{S}^1$ inherit from their orientation a $\mathrm{PSL}_2(\mathbb{R})$-invariant *cyclic ordering*, *i.e.* a partition of triplets $(x_1, x_2, x_3) \in (\mathbb{R}\mathbf{P}^1)^3$ (respectively $(\mathbf{S}^1)^3$) between *positive* and *negative* ones which is invariant by cyclic permutations, exchanged by transpositions and defined in the following way. Any n-tuple ($n \geq 3$) of two-by-two distinct points of $\mathbb{R}\mathbf{P}^1$ has an ordering $(x_1, \dots, x_n)$, unique up to the n cyclic permutations $(1, \dots, n)^k$ for $1 \leq k \leq n$, such that for any $1 \leq i \leq n-1$, the positively oriented injective path of $\mathbb{R}\mathbf{P}^1$ from x_i to x_{i+1} does not meet any of the x_j for $j \notin \{i, i+1\}$. In this case $(x_1, \dots, x_n)$ is said to be *positively cyclically ordered*, and two n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are said to have *the same cyclic order* if there exists a permutation σ such that $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ and $(y_{\sigma(1)}, \dots, y_{\sigma(n)})$ are both positive. For any $x, y \in \mathbb{R}\mathbf{P}^1$, we denote

$$[x\,;y] \coloneqq \{x, y\} \cup \left\{ z \in \mathbb{R}\mathbf{P}^1 \;\middle|\; (x, z, y) \text{ is positively cyclically ordered} \right\} \subset \mathbb{R}\mathbf{P}^1$$

with $[x\,;y] = \{x\}$ if $x = y$, and adopt the same notation for any oriented topological circle. For any $p = (x_p, y_p), q = (x_q, y_q) \in \mathbf{dS}^2$ such that $q \in \mathcal{F}^+_\alpha(p)$ – respectively $q \in \mathcal{F}^+_\beta(p)$ – we denote

$$[p\,;q]_\alpha \coloneqq [x_p\,;x_q] \times \{y_p\}, [p\,;q]_\beta \coloneqq \{x_p\} \times [y_p\,;y_q],$$

with obvious corresponding notations in $\mathbb{R}^{1,1}$ and for (half-)open intervals. More generally in any $\mathbf{X}$-surface, $[p\,;q]_{\alpha/\beta}$ denotes the segment of the oriented leaf $\mathcal{F}_{\alpha/\beta}(p)$ from p to q.

Definition 2.8. For any four distinct points $(A, B, C, D) \in \mathbf{dS}^2$ such that $(x_A, y_A) = A = \mathcal{F}^-_\alpha(B) \cap \mathcal{F}^-_\beta(D)$ and $(x_C, y_C) = C = \mathcal{F}^+_\beta(B) \cap \mathcal{F}^+_\alpha(D)$,

$$\mathcal{R}_{ABCD} = \mathcal{R}_{(x_A, x_C, y_A, y_C)} \coloneqq [x_A\,;x_C] \times [y_A\,;y_C]$$

will be called a *rectangle of* $\mathbf{dS}^2$ *with lightlike boundary*.

Note that by convention, the rectangles that we consider are non-degenerated (*i.e.* have distinct edges), and that we name the vertices of a rectangle $\mathcal{R}_{ABCD}$ of $\mathbf{dS}^2$ in the positive cyclic order by starting with its "bottom-left" vertex A. The area of an orientable surface S for the area form induced by a Lorentzian metric μ (which, by definition, gives volume 1 to an orthogonal basis of norms $(1, -1)$ for μ), will be denoted by $\mathcal{A}_\mu(S)$.

Lemma 2.9. *Two rectangles of* $\mathbf{dS}^2$ *with lightlike boundaries are in the same orbit under* $\mathrm{PSL}_2(\mathbb{R})$ *if, and only if they have the same area.*

Proof. For any rectangle $\mathcal{R}_{(x_A, x_C, y_A, y_C)}$, (y_A, y_C, x_A) is a positively cyclically ordered triplet of $\mathbb{R}\mathbf{P}^1$, and we can thus assume without lost of generality that $\mathcal{R}_{(x_A, x_C, y_A, y_C)} = \mathcal{R}_{(\hat{1}, \hat{t}, \hat{\infty}, \hat{0})}$. Since $t \in\,]1\,;+\infty[\, \mapsto \mathcal{A}_{\boldsymbol{\mu}}(\mathcal{R}_{(\hat{1}, \hat{t}, \hat{\infty}, \hat{0})}) \in \mathbb{R}^*_+$ is bijective, two rectangles have the same area if, and only if the 4-tuples defining them have the same cross-ratio, which happens if and only if they are in the same orbit under $\mathrm{PSL}_2(\mathbb{R})$. □

2.2. **The local model of standard singularities.** We define in this subsection the local singularities that will be considered in this text (which appeared in [BBS11, §3.3]), and prove some of their fundamental properties.

$(\mathbf{G}, \mathbf{X})$ denotes one of the pairs $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,2), \mathbb{R}^{1,1})$ or $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$, $\boldsymbol{\mu}$ the Lorentzian metric of $\mathbf{X}$, and $g_{\boldsymbol{\mu}}$ its associated bilinear form. We also fix a base-point $\mathsf{o} \in \mathbf{X}$, respectively equal to $(0,0)$ or $([e_1], [e_2])$, and denote by $A = \{a^t\}_{t \in \mathbb{R}}$ its stabilizer in $\mathbf{G}$.

Convention 2.10. Henceforth, we will use the unique parametrization of $A = \{a^t\}_{t \in \mathbb{R}}$ satisfying the following for any non-zero future spacelike vector $u \in \mathrm{T}_\mathsf{o}\mathbf{X}$.

(1) With u_t the unique point of $\mathbb{R}^+\mathrm{D}_\mathsf{o} a^t(u)$ belonging to the unit circle C of a fixed Euclidean quadratic form on $\mathrm{T}_\mathsf{o}\mathbf{X}$, $t \mapsto u_t$ is a positively oriented curve on C (endowed with the orientation induced from the one of $\mathbf{X}$).

(2) Moreover denoting by cosh the hyperbolic cosine function, for any $t \in \mathbb{R}$ we have:

$$\frac{g_{\boldsymbol{\mu}}(u, a^t(u))}{\boldsymbol{\mu}(u)} = \cosh(t).$$

This convention will be crucial for the correspondence (2.7) between angles and areas given below by Gauß-Bonnet formula. Apart from this formula, the convention does not matter. We emphasize that for $\mathbf{X} = \mathbb{R}^{1,1}$, the parametrization $A = \{a^t\}_{t \in \mathbb{R}}$ is simply the usual one given by (2.1).

2.2.1. *Standard singularities as identification spaces.* We denote by $\mathbf{X}_*$ the surface with boundary and one conical point obtained from $\mathbf{X}$ by cutting it along $\mathcal{F}^{+*}_\alpha(\mathsf{o})$. The interior of $\mathbf{X}_*$ is identified with $\mathbf{X} \setminus \mathcal{F}^+_\alpha(\mathsf{o})$, its conical point o' with o, and its two boundary components are "upper" and "lower" embeddings $\iota_\pm \colon \mathcal{F}^+_\alpha(\mathsf{o}) \to \mathbf{X}_*$ of $\mathcal{F}^+_\alpha(\mathsf{o})$ with $\iota_\pm(\mathsf{o}) = \mathsf{o}'$. Furthermore $\mathbf{X}_*$ is endowed with an action of the diagonal subgroup A for which the embeddings $\iota_\pm$ are equivariant.

For $\theta \in \mathbb{R}$, we introduce the equivalence relation generated by the relations $\iota_+(x) \sim_\theta \iota_-(a^\theta(x))$ for any $x \in \mathcal{F}^{+*}_\alpha(\mathsf{o})$, and we denote by

$$\pi_\theta \colon \mathbf{X}_* \to \mathbf{X}_\theta = \mathbf{X}_* / \sim_\theta$$

the canonical projection onto the topological quotient of $\mathbf{X}_*$ by $\sim_\theta$. This identification space is illustrated in Figure 2.1.

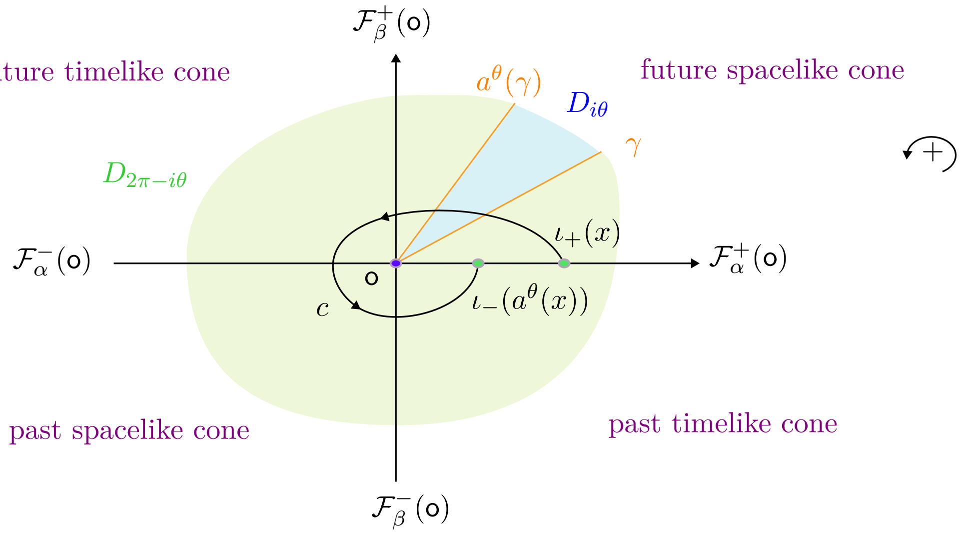

FIGURE 2.1. Standard singularity, quadrants and orientations.

We define $\mathsf{o}_\theta := \pi_\theta(\mathsf{o}')$ and endow $\mathbf{X}_\theta \setminus \{\mathsf{o}_\theta\}$ with its *standard* $\mathbf{X}$*-structure* defined by the following atlas.

(1) For any open set $U \subset \mathbf{X} \setminus \mathcal{F}^+_\alpha(\mathsf{o})$, we consider the chart $\varphi_{\pi_\theta(U)} \colon \pi_\theta(U) \to U$ satisfying $\varphi_{\pi_\theta(U)} \circ \pi_\theta|_U = \mathrm{id}|_U$.

(2) Let $U \subset \mathbf{X} \setminus \{\mathsf{o}\}$ be an open set such that $U \setminus \mathcal{F}^+_\alpha(\mathsf{o})$ has two respectively up and down connected components U_+ and U_-, and $a^\theta(U) \cap U = \varnothing$. Then we consider the open set $V = \pi_\theta(U_+ \cup \iota_+(U \cap \mathcal{F}^+_\alpha(\mathsf{o})) \cup a_\theta(U_-))$ of $\mathbf{X}_\theta$, and the chart $\varphi_V \colon V \to U$ satisfying:
   - $\varphi_V \circ \pi_\theta = \mathrm{id}$ in restriction to $U_+ \cup \iota_+(U \cap \mathcal{F}^+_\alpha(\mathsf{o}))$,
   - and $\varphi_V \circ \pi_\theta = a^{-\theta}$ in restriction to $a^\theta(U_-)$.

Definition 2.11. The *standard* $\mathbf{X}$*-cone of angle* θ is the oriented topological surface $\mathbf{X}_\theta$ endowed with its marked point o_θ, its standard $\mathbf{X}$-structure on $\mathbf{X}_\theta \setminus \{\mathsf{o}_\theta\}$ and its associated Lorentzian metric denoted by $\boldsymbol{\mu}_\theta$.

Note that our definition makes sense for $\theta = 0$, and that in this case $\mathbf{X}_0 = \mathbf{X}$.

Remark 2.12. The standard cones that we have introduced do not exhaust the natural geometric singularities, and we refer to Remark 3.6 for a discussion of other kind of examples. However these singularities are the *dynamically natural* ones: they are essentially the only ones at which the lightlike foliations extend to two continuous foliations, in a sense made more precise in Lemma 2.14. The existence of these continuous foliations is our main motivation for considering this specific type of singularities, and is the subject of the next paragraph.

2.2.2. *Lightlike foliations at a standard singularity.* To investigate the behaviour of the lightlike foliations at the singularity, we consider a continuous chart of $\mathbf{X}_\theta$ at o_θ defined as follows. Let $\exp_{\mathsf{o}}\colon \mathrm{T}_{\mathsf{o}}\mathbf{X} \to \mathbf{X}$ denote the exponential chart of $\mathbf{X}$ at o, and $d_\nu \subset \mathrm{T}_{\mathsf{o}}\mathbf{X}$ be the half-open line making a positive euclidean angle $\nu \in [0\,;2\pi[$ with d_0, where $\exp_{\mathsf{o}}(d_0) \subset \mathcal{F}^+_\alpha(\mathsf{o})$. Note that $a^\theta \circ \exp_{\mathsf{o}} = \exp_{\mathsf{o}} \circ \mathrm{D}_{\mathsf{o}} a^\theta$, hence with $\theta' \in \mathbb{R}$ characterized by $\mathrm{D}_{\mathsf{o}} a^\theta(X) = e^{-2\theta'}X$ for $X \in \mathrm{T}_{\mathsf{o}}\mathcal{F}_\alpha(\mathsf{o})$, we have $\iota_+(\exp_{\mathsf{o}}(X)) \sim_\theta \iota_-(\exp_{\mathsf{o}}(e^{-2\theta'}X))$. With D an open disk centered at $\mathsf{0}$ in $\mathrm{T}_{\mathsf{o}}\mathbf{X}$, we consider the open neighbourhood

$$U \coloneqq \iota_+ \circ \exp_{\mathsf{o}}(d_0 \cap D) \cup \bigcup_{\nu \in]0;2\pi[} \exp_{\mathsf{o}}(e^{-\frac{\nu}{\pi}\theta'}(d_\nu \cap D))$$

of o' in $\mathbf{X}_*$, so that $V = \pi_\theta(U)$ is an open neighbourhood of o_θ in $\mathbf{X}_\theta$. We define then a map $\psi_\theta\colon V \to D$, for any $\nu \in [0\,;2\pi[$ and $X \in e^{-\frac{\nu}{\pi}\theta'}(d_\nu \cap D)$, by

$$\psi_\theta \circ \pi_\theta(\exp_{\mathsf{o}}(X)) = e^{\frac{\nu}{\pi}\theta'}X.$$

In the above equation for $p \in \mathcal{F}^+_\alpha(\mathsf{o})$, we abusively denoted $\iota_+(p)$ simply by p. It is easily checked that ψ_θ is a homeomorphism from V to D.

Proposition 2.13. *The lightlike foliations of $\mathbf{X}_\theta \setminus \{\mathsf{o}_\theta\}$ extend uniquely to two topological one-dimensional foliations on $\mathbf{X}_\theta$, that we call the* lightlike foliations *of $\mathbf{X}_\theta$ and continue to denote by $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$. Moreover for any small enough open neighbourhoods I and J of o_θ in $\mathcal{F}_\alpha(\mathsf{o}_\theta)$ and $\mathcal{F}_\beta(\mathsf{o}_\theta)$,*

$$\Phi\colon (x,y) \in I \times J \mapsto \mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$$

is a homeomorphism onto its image, restricting outside of o_θ to a $\mathcal{C}^\infty$-diffeomorphism onto its image. The continuous α and β-foliations are thus transverse *in the sense that Φ defines a simultaneous $\mathcal{C}^0$ foliated chart.*

Proof. Since $\psi_\theta(\pi_\theta(\iota_+(\mathcal{F}^{+*}_\alpha(\mathsf{o})) \cup \mathcal{F}^{-*}_\alpha(\mathsf{o}))) = \mathbb{R}\cdot d_0 \setminus \{\mathsf{0}\}$ and $\psi_\theta(\pi_\theta(\mathcal{F}^{+*}_\beta(\mathsf{o}) \cup \mathcal{F}^{-*}_\beta(\mathsf{o}))) = \mathbb{R}\cdot d_\beta \setminus \{\mathsf{0}\}$ where $\exp_{\mathsf{o}}(\mathbb{R}\cdot d_\beta) = \mathcal{F}_\beta(\mathsf{o})$, the only possible definition of the α and β-leaves of o_θ for it to define a foliation with continuous leaves, is: $\mathcal{F}_\alpha(\mathsf{o}_\theta) = \pi_\theta \circ \iota_+(\mathcal{F}^+_\alpha(\mathsf{o})) \cup \pi_\theta(\mathcal{F}^{-*}_\alpha(\mathsf{o}))$ and $\mathcal{F}_\beta(\mathsf{o}_\theta) = \{\mathsf{o}_\theta\} \cup \pi_\theta(\mathcal{F}^{+*}_\beta(\mathsf{o}) \cup \mathcal{F}^{-*}_\beta(\mathsf{o}))$. This makes $\mathcal{F}_\alpha(\mathsf{o}_\theta)$ and $\mathcal{F}_\beta(\mathsf{o}_\theta)$ two topological 1-manifolds. Now for any small enough open neighbourhoods I and J of o_θ in $\mathcal{F}_\alpha(\mathsf{o}_\theta)$ and $\mathcal{F}_\beta(\mathsf{o}_\theta)$, and any $(x,y) \in I \times J$: $\mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$ is a single point which we denote by $[x,y]$. Moreover for $x, x' \in \mathcal{F}_\alpha(\mathsf{o}_\theta)$, $x \neq x'$ implies $\mathcal{F}_\beta(x) \cap \mathcal{F}_\beta(x') = \varnothing$, and similarly for $y \neq y' \in \mathcal{F}_\beta(\mathsf{o}_\theta)$. Therefore $\Phi\colon (x,y) \in I \times J \mapsto [x,y]$ is an injective map from $I \times J$ to the topological surface $\mathbf{X}_\theta$, which is clearly continuous, and $\Phi(\mathsf{o}_\theta, \mathsf{o}_\theta) = \mathsf{o}_\theta$. By Brouwer's invariance of domain theorem, Φ is thus a homeomorphism onto its image U, which is an open neighbourhood of o_θ. Observe moreover that Φ is a $\mathcal{C}^\infty$-diffeomorphism onto its image on restriction to any small enough open subset of $\mathbf{X}_\theta \setminus \{\mathsf{o}_\theta\}$, since it is so in $\mathbf{X}$. Furthermore $\Phi(\{x\} \times J)$ contains an open neighbourhood of x in $\mathcal{F}_\beta(x)$, and $\Phi(I \times \{y\})$ an open neighbourhood of y in $\mathcal{F}_\alpha(y)$. The restriction of Φ to suitable subsets defines thus a simultaneous continuous foliated chart for the α and β-foliations, which concludes the proof. ☐

2.2.3. *Characterization of standard singularities and their angles by developing maps and holonomy morphisms.* We now characterize the singularity o_θ of $\mathbf{X}_\theta$ among the $\mathbf{X}$-structures of a punctured disk. Let us call *slit neighbourhood of* $\mathbf{X}$ an open set of the form $U' = U \setminus \mathcal{F}^+_\alpha(p)$ for U an open neighbourhood of a point $p \in \mathbf{X}$.

Lemma 2.14. *Let D be an oriented topological disk, $x \in D$, and $D^* \coloneqq D \setminus \{x\}$ be endowed with a $\mathbf{X}$-structure. Let R denote the* positive generator *of $\pi_1(D^*)$,* i.e. *the homotopy class of a positively oriented closed loop around x generating $\pi_1(D^*)$. Then the following properties (1) and (2) are equivalent.*

(1) There exists $\theta \in \mathbb{R}$, and a homeomorphism φ from an open neighbourhood U of x to an open neighbourhood of o_θ in $\mathbf{X}_\theta$, such that: $\varphi(x) = \mathsf{o}_\theta$, and φ is a $\mathbf{X}$-morphism in restriction to $U^* = U \setminus \{x\}$.

(2) (a) The lightlike foliations of D^* extend uniquely to two continuous 1-dimensional foliations of D;

(b) and there exists an open disk $U \subset D$ containing x, and a $\mathbf{X}$-isomorphism ψ from $U' = U \setminus \mathcal{F}^+_\alpha(x)$ to a slit neighbourhood of o.

Furthermore property (1) for $\theta \in \mathbb{R}$ is equivalent to (2).(a) and (2).(b) together with:

(2).(c) $\rho(R) = a^\theta$, with ρ the holonomy morphism associated to the developing map extending the lift of a $\mathbf{X}$-morphism ψ like in (2).(b).

In particular, there exists at most one $\theta \in \mathbb{R}$ for which the equivalent properties (1) and (2) can be satisfied for θ.

Definition 2.15. Let $D^* := D \setminus \{x\}$ be an oriented topological punctured disk endowed with a $\mathbf{X}$-structure. We will say that x is a *standard singularity of angle θ* of D if the equivalent properties (1) and (2).(a)-(c) of Lemma 2.14 are satisfied at x for $\theta \in \mathbb{R}$. A developing map of D^* extending a lift of φ like in (1) (equivalently of ψ like in (2).(b)) and its holonomy morphism are said *compatible at x*.

Remark 2.16. The holonomy of a positively oriented loop around a singularity is well defined only *up to conjugacy*, and for $\theta \in \mathbb{R}$ and $g \in \mathrm{PSL}_2(\mathbb{R})$: $a^\theta = ga^{-\theta}g^{-1}$ if, and only if g is an anti-diagonal matrix. Hence if the angle of singularities were to be simply defined as the latter holonomy conjugacy class, then it would be well-defined only *up to sign*. For this reason one has to consider specific developing maps around a standard singularity x to define the sign of its angle: the *compatible* ones as introduced in Definition 2.15. Let $\pi \colon E \to D^* = D \setminus \{x\}$ be the universal covering of a singular $\mathbf{X}$-disk with a single singularity at x, and $F \subset E$ be a closed fundamental domain of π, such that $\pi|_{\mathrm{Int}\, F}$ is injective, $\pi(F) = D^*$ and ∂F is a copy of two lifts I^d and $I^u = R(I^d)$ of $\mathcal{F}^{+*}_\alpha(x)$. Then a developing map $\delta \colon E \to \mathbf{X}$ is compatible at x if, and only if $\delta(\mathrm{Int}\, F)$ is a slit neighbourhood of o. We will see in Lemma 2.20 and Remark 2.21 another intrinsic characterization of the angle of a singularity.

Lemma 2.14 implies directly the following results.

Corollary 2.17. *Let $D^* := D \setminus \{x\}$ be an oriented punctured disk endowed with a $\mathbf{X}$-structure. If x is a standard singularity of angle* 0*, equivalently a standard singularity of trivial holonomy, then the $\mathbf{X}$-structure of D^* uniquely extends to D. In other words, x is actually a regular point.*

Corollary 2.18. *Let x be a standard singularity of a $\mathbf{X}$-structure on an oriented punctured disk $D^* := D \setminus \{x\}$, $\rho \colon \pi_1(D^*) \to \mathbf{G}$ be a compatible holonomy map at x, and c be a positively oriented loop of D^* whose homotopy class $[c]$ generates $\pi_1(D^*)$. Then x is of angle $\theta \in \mathbb{R}$ if, and only if $\rho([c]) = a^\theta$.*

The interpretation of the angle θ of a standard singularity x as the holonomy of a positive closed loop c around it is illustrated in Figure 2.1.

Proof of Lemma 2.14. **(1) for $\theta \Rightarrow$ (2).(a),(b)&(c).** The unique continuous extension of the lightlike foliations follows from Proposition 2.13. The restriction of the map φ of (1) to a slit neighbourhood U' of x is a $\mathbf{X}$-isomorphism to a slit neighbourhood of o_θ which is canonically identified with a slit neighbourhood of o by the projection map π_θ, giving us the desired map ψ. Now let O be an open subset of the universal cover of D^* projecting homeomorphically to U', and δ be the developing map extending a lift of ψ to O. Then δ satisfies $\delta \circ R = a^\theta \circ \delta$ (on the non-empty open subset where this equality is well-defined) by the very definition of $\mathbf{X}_\theta$, which shows that $\rho(R) = a^\theta$ and concludes the proof of this implication.

(2).(a)&(b) $\Rightarrow$ (1) for some θ. Let $\pi \colon E \to U^* = U \setminus \{x\}$ be the universal covering map of U^*, and $O \subset E$ be an open set such that $\pi|_O$ is a diffeomorphism onto $U' = U \setminus \mathcal{F}^+_\alpha(x)$. The existence of ψ shows that the restriction of the developing map $\delta \colon E \to \mathbf{X}$ to O is an isometry onto $V' = V \setminus \mathcal{F}^+_\alpha(\mathsf{o})$, with V an open neighbourhood of o. The lightlike leaf spaces of V' have the following description:

– the leaf space $\mathcal{L}_\beta$ of the β-foliation of V' is homeomorphic to the non-Hausdorff topological 1-manifold $(L^+ \cup L^-)/\sim$, with $L^\pm$ two copies of $\mathbb{R}$ and $p^- \sim p^+$ for $p \in \mathbb{R}_{<0}$, the special points $0^\pm$ corresponding to the special leaves $J_\beta^\pm \coloneqq \mathcal{F}_\beta^\pm(\mathsf{o}) \cap V'$;
– the leaf space of the α-foliation of V' has one specific point $J_\alpha^- \coloneqq \mathcal{F}_\alpha^-(\mathsf{o}) \cap V'$, which is the only α-leaf intersecting none of the leaves $p^\pm \in \mathcal{L}_\beta$ for $p \geq 0$.

Since the lightlike foliations of D^* extend by assumption to continuous foliations of D, we can choose U to be a small enough neighbourhood of x for it to be a trivialization domain of both lightlike foliations of D. The same above description holds then for the lightlike leaf spaces of U' than for the ones of V'. Let us denote by $I_\beta^\pm$, respectively I_α^- the lifts of $\mathcal{F}_\beta^\pm(x) \cap U$, resp. $\mathcal{F}_\alpha^-(x) \cap U$ in O, and by $I_\alpha^{d/u}$ the "down and up" lifts of $\mathcal{F}_\alpha^+(x) \cap U$, so that $\partial O = I_\alpha^d \cup I_\alpha^u$ and $R(I_\alpha^d) = I_\alpha^u$. Then since δ is a simultaneous equivalence between the lightlike foliations, the descriptions of the leaf spaces impose $\delta(I_\beta^\pm) = J_\beta^\pm$, $\delta(I_\alpha^-) = J_\alpha^-$ and $\delta(I_\alpha^{d/u}) =]\mathsf{o}\,;p^{d/u}[_\alpha$ with $p^{d/u} \in \mathcal{F}_\alpha^{+*}(\mathsf{o})$. With ρ the holonomy morphism associated to δ we have thus $\rho(R)(]\mathsf{o}\,;p^d[_\alpha) =]\mathsf{o}\,;p^u[_\alpha$, which shows that $\rho(R)$ fixes o, *i.e.* $\rho(R) = a^\theta$ for some θ, and thus $\delta \circ R = a^\theta \circ \delta$.

We now define a map $\varphi \colon U \to \mathbf{X}_\theta$ by:
– $\varphi(x) = \mathsf{o}_\theta$;
– $\varphi \circ \pi = \pi_\theta \circ \delta$ on O;
– $\varphi \circ \pi = \pi_\theta \circ \iota_+ \circ \delta$ on I_α^d;

and show that φ satisfies the properties of (1). Let W be an open neighbourhood of $p \in I_\alpha^d$ so that $\pi|_W$ is a diffeomorphism onto $\pi(W)$, and $W \setminus I_\alpha^d$ has two connected components $W^\pm$, with $W^+ \subset O$ and $R(W^-) \subset O$. Since $\delta \circ R = a^\theta \circ \delta$, we have $\varphi \circ \pi = \pi_\theta \circ a^\theta \circ \delta$ on W^-, $\varphi \circ \pi = \pi_\theta \circ \iota_+ \circ \delta$ on $I_\alpha^d \cap W$ and $\varphi \circ \pi = \pi_\theta \circ \delta$ on W^+, which shows that φ is a $\mathbf{X}$-morphism to $\mathbf{X}_\theta$ on the neighbourhood of $\pi(p)$.

It thus only remains to show that φ is continuous at x. Our former description shows that $\varphi(\mathcal{F}_{\alpha/\beta}(x) \cap U) = \mathcal{F}_{\alpha/\beta}(\mathsf{o}_\theta)$, and thus that φ induces two maps $\phi_{\alpha/\beta}$ between the respective leaf spaces of the α, resp. β-foliations of U and $\varphi(U) \subset \mathbf{X}_\theta$. These foliations being continuous and transverse, it moreover suffices to show that the maps $\phi_{\alpha/\beta}$ induced by φ between the leaf spaces are continuous at $\mathcal{F}_{\alpha/\beta}(x) \cap U$, to conclude that φ is continuous at x. But our former description of the leaf spaces of the slit neighbourhoods U' and V' showed that $\delta(I_\alpha^-) = J_\alpha^-$, and thus for any sequence L_n of α-leaves contained in U' and converging to $\mathcal{F}^\alpha(x) \cap U$, $\varphi(L_n)$ converges to $\mathcal{F}_\alpha^-(\mathsf{o}_\theta)$, which shows the continuity of ϕ_α at $\mathcal{F}^\alpha(x) \cap U$. In the same way, the fact that $\delta(I_\beta^\pm) = J_\beta^\pm$ shows that ϕ_β is continuous at $\mathcal{F}^\beta(x) \cap U$, which concludes the proof of the second implication.

Unicity of θ. If θ_1 and θ_2 both satisfy the equivalent properties (1) and (2), then the holonomy morphism of a developing map extending the lift of a $\mathbf{X}$-isomorphism like in (b) should satisfy $a^{\theta_1} = \rho(R) = a^{\theta_2}$ according to (c) (note that (b) is indeed independent of θ). Hence $\theta_1 = \theta_2$, which concludes the proof of the Lemma. □

2.2.4. *Standard singularities as quotients.* Let D be an open disk around o in $\mathbf{X}$, and E be the universal cover of $D^* \coloneqq D \setminus \{\mathsf{o}\}$. Since a^θ fixes o, it induces an isometry of D^* which lifts to a unique isometry $\widetilde{a^\theta}$ of E fixing each lift of the punctured lightlike leaves of o. On the other hand, E admits also a preferred isometry R which is the positive generator of its covering automorphism group.

Lemma 2.19. *$\widetilde{a^\theta} \circ R$ acts properly discontinuously on E, and $E/\langle \widetilde{a^\theta} \circ R \rangle$ is $\mathbf{X}$-isomorphic to $\mathbf{X}_\theta \setminus \{\mathsf{o}_\theta\}$. More precisely, there is a natural embedding of $E/\langle \widetilde{a^\theta} \circ R \rangle$ as the complement of a point o_θ in a topological disk $\bar{E}$, for which o_θ is a standard singularity of angle θ of $\bar{E}$.*

Proof. Any lift $\tilde{\mathcal{F}}_\alpha$ of $\mathcal{F}_\alpha^{+*}(\mathsf{o})$ is an embedding of $\mathbb{R}$ separating $E \simeq \mathbb{R}^2$ in two connected components, and since $\langle R \rangle \simeq \mathbb{Z}$ acts properly discontinuously on E, the images of $\tilde{\mathcal{F}}_\alpha$ by $\langle R \rangle$ are pairwise disjoint and form a discrete set. The complement of $\langle R \rangle \cdot \tilde{\mathcal{F}}_\alpha$ in E is a disjoint union of topological disks, the boundary of each of them being the disjoint union of an upper and a lower translate of $\tilde{\mathcal{F}}_\alpha$, and the closure of any of these connected components is a fundamental domain for the action of $\langle R \rangle$ on E. The important observation is now that by definition, $\langle \widetilde{a^\theta} \rangle$

preserves the interior and the boundary of any of these fundamental domains and acts properly on it, which shows that $\widetilde{a^\theta} \circ R$ acts indeed properly discontinuously on E.

We add to $E/\langle\widetilde{a^\theta} \circ R\rangle$ a point o_θ, with a neighbourhood basis composed of images of sets of the form $U \cup \{o_\theta\}$, for all the $\widetilde{a^\theta} \circ R$-invariant open sets $U \subset E$ projecting to punctured neighbourhoods of o in D. This defines a topological disk $\bar{E}$, in which the lightlike foliations of $E/\langle\widetilde{a^\theta} \circ R\rangle = \bar{E} \setminus \{o_\theta\}$ extend to two continuous transverse foliations. The complement of $\tilde{\mathcal{F}}_\alpha = \mathcal{F}_\alpha^{+*}(o_\theta)$ in $\bar{E}$ is $\mathbf{X}$-isomorphic to the interior of one of the previously described fundamental domains, themselves isomorphic to the slit neighbourhood $D \setminus \mathcal{F}_\alpha^{+*}(\mathsf{o})$ in $\mathbf{X}$. The result now follows from Lemma 2.14. □

2.2.5. *Standard singularities as angle defaults.* Let D be a small disk around o in $\mathbf{X}$, γ be a half-open future-oriented spacelike geodesic starting from o, $\theta > 0$ and $\gamma_\theta := a^\theta(\gamma)$. Then $D \setminus (\gamma \cup \gamma_\theta)$ has two connected components. One of them is contained in the future spacelike quadrant of o and its closure is denoted by $D_{i\theta}$. The other one contains the three other quadrants and its closure is denoted by $D_{2\pi - i\theta}$. We denote by $\bar{D}_{2\pi - i\theta}$ the quotient of $D_{2\pi - i\theta}$ by the relation $\gamma \ni x \sim a^\theta(x) \in \gamma_\theta$ on its boundary (in particular $\mathsf{o} \sim \mathsf{o}$). As we did in paragraph 2.2.1, we consider the surface D_* obtained from D by cutting it open along $\gamma \setminus \{\mathsf{o}\}$, with two upper and lower boundary components $\iota_\pm \colon \gamma \to D_*$. We can now form the quotient $\bar{D}_{2\pi + i\theta}$ of $D_* \cup D_{i\theta}$ by the relation: $\iota_-(x) \sim x \in \gamma$ and $\iota_+(x) \sim a^\theta(x) \in \gamma_\theta$ for $x \in \gamma$. Both topological disks $\bar{D}_{2\pi \pm i\theta}$ have a marked point o_θ, image of o, and bear a natural $\mathbf{X}$-structure on $\bar{D}_{2\pi \pm i\theta} \setminus \{o_\theta\}$ which is defined as in paragraph 2.2.1. These constructions are illustrated in Figure 2.1.

Lemma 2.20. *o_θ is a standard singularity of angle $-\theta$ (respectively θ) of $\bar{D}_{2\pi - i\theta} \setminus \{o_\theta\}$ (resp. of $\bar{D}_{2\pi + i\theta} \setminus \{o_\theta\}$). The obvious analogous statement can be given for any two half-geodesics of the same signature and orientation. In particular, any lightlike half-leaf can be used to define a standard singularity.*

Proof. The first important observation is that both $D_{2\pi - i\theta}$ and D_* contain three quadrants of D at o, and thus that the lightlike foliations of $\bar{D}_{2\pi \pm i\theta} \setminus \{o_\theta\}$ extend to two transverse continuous foliations of $\bar{D}_{2\pi \pm i\theta}$. Let E be the universal cover of $D \setminus \{\mathsf{o}\}$, $\widetilde{a^\theta}$ the lift of a^θ fixing each lift of the punctured lightlike leaves of o and R the positive generator of the automorphism group of E. It is then easy to check that $\bar{D}_{2\pi - i\theta} \setminus \{o_\theta\}$ is isometric to the quotient of E by $\langle\widetilde{a^{-\theta}} \circ R\rangle$, and $\bar{D}_{2\pi + i\theta} \setminus \{o_\theta\}$ to the quotient of E by $\langle\widetilde{a^\theta} \circ R\rangle$. The claim is now a consequence of Lemma 2.19. □

Remark 2.21. Lemma 2.20 provides us with the Lorentzian counterpart of the usual interpretation of Riemannian singularities as *angles defaults*. Indeed, we will see in the proof of Proposition 2.32 that for a natural notion of *Lorentzian angle* (for which angles are *complex* numbers), $D_{i\theta}$ is a sector of angle $i\theta$ (oriented from γ to $a^\theta(\gamma)$), and $D_{2\pi - i\theta}$ a sector of angle $2\pi - i\theta$ (oriented from $a^\theta(\gamma)$ to γ). Hence a standard singularity x has angle $\nu \in \mathbb{R}$ if, and only if the total angle around x is $2\pi + i\nu$. This gives in particular a new intrinsic characterization of the angle of a standard singularity (and especially of its sign).

Our main interest being in this text for the extension of the lightlike foliations at the singularities as topological foliations, it seems to us that the use of lightlike geodesics to define a standard singularity is clearer at first sight. However the point of view of definite geodesics will be useful for some aspects. We emphasize that contrarily to the Riemannian case, the *same* (lightlike) geodesic ray can be used in the Lorentzian setting to define a singularity of non-zero cone angle.

2.3. **Singular X-surfaces.** We use in this subsection the local model of singularities described in paragraph 2.2, to define singular $\mathbf{X}$-surfaces and to prove some of their fundamental properties.

Definition 2.22. A *singular* $\mathbf{X}$*-structure* (Σ, μ) on an oriented topological surface S is the data:

1. of a set $\Sigma \subset S$ of *singular points* in S;
2. and of a $\mathbf{X}$*-structure* μ on $S^* := S \setminus \Sigma$ for which any $x \in \Sigma$ is a *standard singularity*, *i.e.* for which there exists $\theta_x \in \mathbb{R}$ (the *angle* at x) and a homeomorphism φ from an open neighbourhood $U \subset S$ of x to an open neighbourhood V of o_{θ_x} in $\mathbf{X}_{\theta_x}$, such that:

(a) $U \cap \Sigma = \{x\}$,
(b) $\varphi(x) = \mathsf{o}_{\theta_x}$,
(c) and φ is a $\mathbf{X}$-morphism in restriction to $U \setminus \{x\}$.

Such a map φ is called a *singular* $\mathbf{X}$*-chart at* x.

A *singular* $\mathbf{X}$*-surface* (S, Σ) is an oriented topological surface S endowed with a singular $\mathbf{X}$-structure of singular set Σ. $S^* = S \setminus \Sigma$ will always be endowed with the $\mathcal{C}^\infty$ structure defined by its $\mathbf{X}$-structure, and S with a $\mathcal{C}^\infty$ structure extending the one of S^* (see for instance [Hat]). The points of S which are not singular are called *regular*, and S itself is said *regular* if it does not have any singular point (*i.e.* if it is a $\mathbf{X}$-surface). If we want to specify them, we will denote by Θ the (ordered) set of angles of the (ordered) singularities Σ.

A *singular* $\mathbf{X}$*-atlas* (φ_i, U_i) on S is an atlas of $\mathcal{C}^0$-charts $\varphi_i \colon U_i \to V_i$ from connected open subsets U_i of S to either $\mathbf{X}$ (*regular charts*) or some $\mathbf{X}_{\theta_i}$ (*singular charts*), such that:

(1) any two distinct singular chart domains are disjoint;
(2) regular charts cover $S \setminus \Sigma$, with $\Sigma = \{\varphi^{-1}(\mathsf{o}_{\theta_i}) \mid \varphi \text{ singular chart to } \mathbf{X}_{\theta_i}\}$ the set of *singularities* of the atlas;
(3) and the transition map between any two charts is a $\mathbf{X}$-morphism (which makes sense since $U_i \cap U_j \cap \Sigma = \varnothing$ for any two distinct chart domains U_i, U_j).

An *isometry* between two singular $\mathbf{X}$-surfaces $(S_i, \Sigma_i, \mu_i)_{i=1,2}$ is a homeomorphism $f \colon S_1 \to S_2$ such that:

(1) $f(\Sigma_1) = \Sigma_2$;
(2) and f is a $\mathbf{X}$-morphism in restriction to $S_1 \setminus \Sigma_1$.

The *area* of a singular $\mathbf{X}$-surface (S, Σ, μ) is the area of $S \setminus \Sigma$ for μ.

Remark 2.23. Let us say that a time-oriented Lorentzian metric μ of constant sectional curvature $\varepsilon_{\mathbf{X}}$ defined on the complement of a discrete subset Σ of an orientable surface S is *singular*, if it is induced by a singular $\mathbf{X}$-structure. Then according to Proposition 2.6, time-oriented singular Lorentzian metrics of constant sectional curvature $\varepsilon_{\mathbf{X}}$ are equivalent to singular $\mathbf{X}$-structures.

2.3.1. *First properties of singular* $\mathbf{X}$*-surfaces.* We prove now some elementary but fundamental properties of singular $\mathbf{X}$-surfaces.

Lemma 2.24. *Let* (S, Σ) *be a singular* $\mathbf{X}$*-surface.*

(1) Σ *is discrete, hence finite if* S *is closed.*
(2) For any singularity $x \in \Sigma$ *of angle* θ_x*,* $\rho \colon \pi_1(S \setminus \Sigma) \to \mathbf{G}$ *a holonomy representation of* S^* *compatible at* x *(see Definition 2.15), and* $[\gamma] \in \pi_1(S \setminus \Sigma)$ *the homotopy class of a positively oriented loop around* x *homotopic to* x *in* S*:* $\rho([\gamma]) = a^{\theta_x}$*. In particular,* $\rho([\gamma])$ *is conjugated to* a^{θ_x}*.*
(3) If S *is closed, then the area of* (S, Σ) *is finite.*

Proof. (1) Any singular $\mathbf{X}$-chart contains indeed a unique singularity.
(2) Since x is a standard singularity of angle θ_x, this is a direct consequence of Lemma 2.14.
(3) For any compact measurable subset $K \subset S \setminus \Sigma$, $\mathcal{A}_{\mu_S}(K)$ is finite, and the claim follows thus from the fact that for any compact neighbourhood K of o_θ in $\mathbf{X}_\theta$, the area of $K \setminus \{\mathsf{o}_\theta\}$ equals the one of K and is thus finite. □

We emphasize that the second claim of Lemma 2.24 shows that the singularities and their angles are characterized by μ_S, and are geometrical invariants in the following sense.

Corollary 2.25. *Let* $f \colon S_1 \to S_2$ *be an isometry between two singular* $\mathbf{X}$*-surfaces. Then for any singular point* x *of* S_1*,* $x \in \Sigma_1$ *and* $f(x) \in \Sigma_2$ *have the same angle:* $\theta_x = \theta_{f(x)}$*.*

Proof. Let $[\gamma] \in \pi_1(S_1 \setminus \Sigma_1)$ be the homotopy class of a positively oriented loop homotopic to x, and $\rho \colon \pi_1(S_1 \setminus \Sigma_1) \to \mathbf{G}$ be a compatible holonomy representation of S_1 at x. Then $[f(\gamma)] \in \pi_1(S_2 \setminus \Sigma_2)$ and the morphism $\rho \circ f_*^{-1} \colon \pi_1(S_2 \setminus \Sigma_2) \to \mathbf{G}$ induced by f has the same properties with respect to $f(x)$, hence $a^{\theta_x} = \rho([\gamma]) = \rho \circ f_*^{-1}([f \circ \gamma]) = a^{\theta_{f(x)}}$, *i.e.* $\theta_x = \theta_{f(x)}$. □

Observe that for any $u \in \mathbb{R}$, a^u preserves the equivalence relation $\sim_\theta$ used to define $\mathbf{X}_\theta$. It induces thus a map on $\mathbf{X}_\theta$ preserving o_θ that we denote by $\bar{a}^u$, characterized by $\bar{a}^u \circ \pi_\theta = \pi_\theta \circ a^u$.

Proposition 2.26. *Let φ be a singular $\mathbf{X}$-chart of $\mathbf{X}_\theta$ at o_θ, or equivalently a homeomorphism between two neighbourhoods of o_θ and fixing o_θ which is an isometry on its complement. Then φ is the restriction of some $\bar{a}^u$.*

Proof. First according to Corollary 2.25, a singular $\mathbf{X}$-chart of $\mathbf{X}_\theta$ at o_θ is indeed a local isometry of $\mathbf{X}_\theta$ fixing o_θ. Denoting $U^* \coloneqq U \setminus \{\mathsf{o}_\theta\}$ we can assume without lost of generality that $\mathcal{F}_\beta(\mathsf{o}_\theta) \cap U^*$ is the union of two down and up connected components $I_- =]x\,;\mathsf{o}_\theta[_\beta$ and $I_+ =]\mathsf{o}_\theta\,;y[_\beta$. The first natural but important observation is that φ preserves both ends of $\mathcal{F}^*_\beta(\mathsf{o}_\theta)$ in the sense that $\varphi(I_-) =]x'\,;\mathsf{o}_\theta[_\beta$ and $\varphi(I_+) =]\mathsf{o}_\theta\,;y'[_\beta$ for some x' and y'. Likewise both ends of $\mathcal{F}^*_\alpha(\mathsf{o}_\theta)$ are preserved, the proof being identical. Indeed $\varphi(I_-)$ and $\varphi(I_+)$ are intervals of β-leaves since $\varphi|_{U^*}$ is a $\mathbf{X}$-morphism, containing furthermore o_θ in their closure since $\varphi(\mathsf{o}_\theta) = \mathsf{o}_\theta$. Hence the only alternative to the above claim is that $\varphi(I_-) =]\mathsf{o}_\theta\,;x'[_\beta$ and $\varphi(I_+) =]y'\,;\mathsf{o}_\theta[_\beta$ for some x' and y'. But since $\varphi(\mathsf{o}_\theta) = \mathsf{o}_\theta$, φ would then reverse the canonical orientation defined on β-leaves by the $\mathbf{X}$-structure of U^* (see paragraph 2.1.5), which contradicts the fact that $\varphi|_{U^*}$ is a $\mathbf{X}$-morphism.

With $V = \varphi(U)$, let $\mathcal{U}, \mathcal{V}$ be open neighbourhoods of o in $\mathbf{X}$, so that with $U' \coloneqq \mathcal{U} \setminus \mathcal{F}^+_\alpha(\mathsf{o})$: $U = \pi_\theta(U' \cup \iota_-(U \cap \mathcal{F}^+_\alpha(\mathsf{o})) \cup \iota_+(U \cap \mathcal{F}^+_\alpha(\mathsf{o})))$, and likewise for V and $V' =\coloneqq \mathcal{V} \setminus \mathcal{F}^+_\alpha(\mathsf{o})$. Then the restriction of π_θ to U' and V' is a $\mathbf{X}$-morphism, and $\pi_\theta|^{-1}_{V'} \circ \varphi \circ \pi_\theta|_{U'}$ is thus the restriction of an element $g \in \mathbf{G}$. But our previous claim shows that g is simultaneously in the stabilizer of $\mathcal{F}_\alpha(\mathsf{o})$ and $\mathcal{F}_\beta(\mathsf{o})$ whose intersection is $\mathrm{Stab}(\mathsf{o}) = A$. In other words there exists $u \in \mathbb{R}$ so that $\varphi = a^u$ on U^* and thus on U, which concludes the proof. ☐

For any $\mathbf{X}$-surface (S, Σ), the union of a $\mathbf{X}$-atlas of $S \setminus \Sigma$ with a (small enough) singular $\mathbf{X}$-chart at each singularity defines a singular $\mathbf{X}$-atlas of S. Conversely, any singular $\mathbf{X}$-atlas of S defines of course on S a singular $\mathbf{X}$-structure with the same singularities. The following result follows directly from Proposition 2.26.

Corollary 2.27. *Let S be an oriented topological surface. Then the transition maps between any two singular $\mathbf{X}$-atlases defining the same singular $\mathbf{X}$-structure on S are:*

- *either restrictions of some a^u between two singular charts at the same singularity,*
- *or $\mathbf{X}$-morphisms outside of singularities.*

Two singular $\mathbf{X}$-atlases whose transition maps are of this form are said equivalent*, and singular $\mathbf{X}$-structures are in correspondence with equivalence classes of singular $\mathbf{X}$-atlases.*

Consequently, any $\mathbf{G}$-invariant object or notion on $\mathbf{X}$ which projects well to $\mathbf{X}_\theta$ through π_θ will make sense on any singular $\mathbf{X}$-structure. The main application of this vague remark will be the Definition 4.6 given below of geodesics in singular $\mathbf{X}$-surfaces.

2.3.2. *First-return maps, suspensions and regularity of the lightlike foliations.* If T is a homeomorphism of the circle $\mathbf{S}^1$, the vertical foliation of $\mathbf{S}^1 \times [0\,;1]$ of leaves $\{p\} \times [0\,;1]$ induces on the quotient $M_T \coloneqq \mathbf{S}^1 \times [0\,;1]/\{(1,p) \sim (0, T(p))\}$, homeomorphic to a torus, a foliation $\mathcal{F}_T$ called the *suspension of* T. We will be interested in this text with lightlike foliations of singular $\mathbf{X}$-structures which are suspensions of circle homeomorphisms, and it happens that the dynamics of a circle homeomorphism T, hence of its suspension, is highly dependent of the regularity of T. Indeed, circle homeomorphisms can in general have pathological behaviours by admitting *exceptional minimal sets* (see [HH86, Chapter I §5]), but the seminal work of Herman [Her79] showed that regular enough circle homeomorphisms behave nicely. In this paragraph we give the main technical properties of the lightlike foliations of a singular $\mathbf{X}$-surface, and show in particular that if they are suspensions of a circle homeomorphism T, then T is a $\mathcal{C}^2$ diffeomorphism with breaks.

Definition 2.28. A homeomorphism $f\colon I = [a\,;b] \to J$ between two intervals of $\mathbb{R}$ is an *orientation-preserving $\mathcal{C}^k$-diffeomorphism with breaks* ($1 \leq k \leq \infty$) if there exists a finite number of points $a = x_0 < \cdots < x_N = b$ in I such that for any $1 \leq i \leq N$:

(1) $f|_{]x_{i-1};x_i[}$ is an orientation-preserving $\mathcal{C}^k$-diffeomorphism onto its image,
(2) for any $1 \leq l \leq k$, the l^{th} derivative of f has finite limites from above at x_{i-1} and from below at x_i,
(3) $f'_+(x_{i-1}) \coloneqq \lim_{t \to x^+_{i-1}} f'(t)$ and $f'_-(x_i) \coloneqq \lim_{t \to x^-_i} f'(t)$ are > 0.

If $f'_+(x_{i-1}) \neq f'_-(x_i)$, then x_i is a *break point* of f. A homeomorphism of $\mathbf{S}^1$ is a $\mathcal{C}^k$*-diffeomorphism with breaks* if it is a $\mathcal{C}^k$-diffeomorphism with breaks in restriction to any interval of $\mathbf{S}^1$.

The following naive observation will be useful to us.

Lemma 2.29. *Let two consecutive intervals $[a\,;b]$ and $[b\,;c]$ of $\mathbb{R}$ be endowed with $\mathcal{C}^\infty$-structures $\mathcal{C}^0$-compatible with the topology of $\mathbb{R}$, and $\varphi\colon [a\,;c] \to I \subset \mathbb{R}$ be a homeomorphism. Then for any $1 \leq k \leq \infty$, the following are equivalent.*

(1) φ restricts on $[a\,;b]$ and $[b\,;c]$ to $\mathcal{C}^k$-diffeomorphisms with breaks, and $\lim\limits_{t\to b^\pm} \varphi'(t) > 0$.

(2) In a $\mathcal{C}^\infty$-structure of $[a\,;c]$ which is $\mathcal{C}^\infty$-compatible with the structures of both of its subintervals, φ is a $\mathcal{C}^k$-diffeomorphism with breaks.

Let $\mathcal{F}$ be an oriented topological one-dimensional foliation on a surface S, I and J be two *transversals* of $\mathcal{F}$, *i.e.* one-dimensional topological submanifolds transverse to $\mathcal{F}$ in a foliation chart, and $x \in I$ be such that $\mathcal{F}(x) \cap J \neq \varnothing$. Then by transversality, $\mathcal{F}(x)$ has a *first intersection point* denoted by $H(x)$ with J (with respect to the orientation of $\mathcal{F}$), and there exists an open neighbourhood I' of x in I such that $H(y) \in J$ is well-defined for any $y \in I'$. The map $H\colon I' \to J$ obtained in this way is a homeomorphism onto its image (which is an open neighbourhood of $H(x)$), and is called the *holonomy of $\mathcal{F}$ from I to J*. We refer to [CLN85, §IV.1] for more details on the notion of holonomy of foliations. A *section* of $\mathcal{F}$ is a simple closed curve γ in S transverse to $\mathcal{F}$ and intersecting all of its leaves. In this case, if the holonomy of $\mathcal{F}$ from γ to itself is well-defined, it will be called the *first-return map* of $\mathcal{F}$ on γ and be denoted by $P^\gamma_{\mathcal{F}}$ (in reference to Poincaré). We recall that a homeomorphism (respectively a foliation) of a manifold M is said *minimal* if all of its orbits (resp. leaves) are dense in M.

Lemma 2.30. *Let (S, Σ) be a singular $\mathbf{X}$-surface.*

(1) The lightlike foliations of $S \setminus \Sigma$ extend uniquely to two one-dimensional continuous foliations on S, still denoted by $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$.

(2) There exists at any point of S a simultaneous $\mathcal{C}^0$ foliation chart for $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ (in the sense of Proposition 2.13).

Let $\mathcal{F}$ be one of the lightlike foliations of S.

(3) Let $T_1, T_2 \subset S$ be two small $\mathcal{C}^\infty$ transversals of $\mathcal{F}$ such that $T_1 \cap \Sigma = \{x\}$ and $T_2 \subset S \setminus \Sigma$ intersects $\mathcal{F}(x)$, and $H\colon T_1 \to T_2$ be the holonomy of $\mathcal{F}$ from T_1 to T_2. Then H is a $\mathcal{C}^\infty$-diffeomorphism with breaks.

(4) If S is homeomorphic to $\mathbf{T}^2$ and $\mathcal{F}$ is $\mathcal{C}^0$-conjugated to the suspension of an orientation-preserving homeomorphism H of $\mathbf{S}^1$, then H is $\mathcal{C}^0$-conjugated to a $\mathcal{C}^\infty$-diffeomorphism with breaks of $\mathbf{S}^1$, and has no exceptional minimal set. If H has moreover an irrational rotation number $\rho \in \mathbf{S}^1$, then H is $\mathcal{C}^0$-conjugated to the rotation $R_\rho\colon x \in \mathbf{S}^1 \mapsto x + \rho \in \mathbf{S}^1$ and is thus minimal. In particular $\mathcal{F}$ is then $\mathcal{C}^0$-equivalent to the corresponding linear foliation of $\mathbf{T}^2$ and is thus minimal.

The notion of rotation number is introduced in Proposition-Definition 3.18. We will prove below in Proposition C.2 a "geometric version" of claims (3) and (4) of the above Lemma, showing that the holonomy is not only $\mathcal{C}^\infty$ with breaks but more precisely *piecewise projective* when the transverse curves are geodesics of the surface. The latter fact will moreover be clearly illustrated by the examples of $\mathbf{dS}^2$-tori $\mathcal{T}_{\theta,x}$ and $\mathcal{T}_{\theta,x,y}$ constructed in Propositions 3.12 and 3.17. These structures are indeed precisely defined for the first-return maps of their lightlike foliations to be induced by *homographic interval exchange maps*, and as such, they are in particular naturally piecewise projective (see paragraph 3.4.1 and Lemmas 3.38 and 3.39 for more details).

Proof of Lemma 2.30. (1) follows directly from Proposition 2.13, using singular $\mathbf{X}$-charts at the singularities.
(2) follows from Proposition 2.13 at the singularities and from the $\mathbf{X}$-charts at regular points. Indeed the affine charts (2.3) are simultaneous foliated charts of the lightlike foliations of $\mathbf{X}$.
(3) Without lost of generality, we can assume that $S = \mathbf{X}_\theta$, $x = \mathsf{o}_\theta$, $\mathcal{F} = \mathcal{F}_\alpha$, and that $T_1 = \mathcal{F}_\beta(\mathsf{o}_\theta)$ and $T_2 = \mathcal{F}_\beta(p)$ with $p \in \mathcal{F}^+_\alpha(\mathsf{o}_\theta)$. These reductions being done, and since the $\mathcal{C}^\infty$-structure of

S is by definition compatible with the $\mathbf{X}$-structure of $S \setminus \Sigma$, it only remains to check according to Lemma 2.29 that the restriction of H to the closure of each component of $\mathcal{F}_\beta(\mathsf{o}_\theta) \setminus \{\mathsf{o}_\theta\}$ is a $\mathcal{C}^\infty$-diffeomorphism with breaks, with a positive limit of the derivative at o_θ from below and above. We do it for $\mathcal{F}^+_\beta(\mathsf{o}_\theta)$, the case of the other component being analogous. According to Proposition 2.13, for I and J small open neighbourhoods of o_θ in $\mathcal{F}_\alpha(\mathsf{o}_\theta)$ and $\mathcal{F}_\beta(\mathsf{o}_\theta)$, the map $(x,y) \in I \times J \mapsto \mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$ defines outside of o_θ a smooth diffeomorphism onto a punctured open neighbourhood of o_θ in $\mathbf{X}_\theta$. But the holonomy H reads in this chart as the identity of the vertical factor, and extends thus on the closure I^+ of the upper component to a $\mathcal{C}^\infty$-diffeomorphism whose derivative has a positive limit at o_θ.
(4) Since $\Sigma \cap S$ is finite and $\mathcal{F}$ is by assumption a suspension, there exists a $\mathcal{C}^\infty$ section $T \subset S \setminus \Sigma$ of $\mathcal{F}$. The first-return map $H \colon T \to T$ of $\mathcal{F}$ on T is then well-defined, and is according to (3) a $\mathcal{C}^2$-diffeomorphisms with breaks as a composition of such homeomorphisms. The two last claims follow then from Denjoy Theorem [Den32] (see also [Her79, Théorème VI.5.5 p.76]): if an orientation-preserving homeomorphism T of $\mathbf{S}^1$ is a $\mathcal{C}^2$-diffeomorphism with breaks, then it has no exceptional minimal set. If T has moreover irrational rotation number ρ, then it is $\mathcal{C}^0$-conjugated to the rotation R_ρ.[1] □

Corollary 2.31. *Any closed connected orientable surface which bears a singular $\mathbf{X}$-structure, is homeomorphic to a torus.*

Proof. According to [HH86, Theorem 2.4.6], any closed connected orientable surface bearing a topological foliation is indeed homeomorphic to a torus. □

This corollary shows the necessity of introducing *branched covers* of the standard singularities to obtain singular $\mathbf{X}$-structures on higher-genus surfaces.

2.3.3. *Gauß-Bonnet formula.* The standard Riemannian Gauß-Bonnet formula has a natural counterpart for singular constant curvature Lorentzian surfaces, which imposes a relation between the singularities and the area of a singular $\mathbf{X}$-torus. We recall that $\varepsilon_{\mathbf{X}}$ denotes the constant sectional curvature of $\mathbf{X}$: $\varepsilon_{\mathbb{R}^{1,1}} = 0$ and $\varepsilon_{\mathbf{dS}^2} = 1$.

Proposition 2.32 (Gauß-Bonnet formula)**.** *Let S be a closed and connected orientable surface endowed with a singular $\mathbf{X}$-structure of area $\mathcal{A}(S) \in \mathbb{R}^*_+$, having $n \in \mathbb{N}^*$ singularities of angles $(\theta_1, \dots, \theta_n) \in \mathbb{R}^n$. Then:*

$$\varepsilon_{\mathbf{X}}.\mathcal{A}(S) = \sum_{i=1}^{n} \theta_i. \tag{2.7}$$

In particular, we have the following consequences.

1. *A closed singular $\mathbb{R}^{1,1}$-surface S cannot have a single singularity. More precisely:*
   (a) *either S is regular,* i.e. *is a flat Lorentzian torus;*
   (b) *or S has exactly two singularities of opposite signs;*
   (c) *or else S has at least three singularities.*
2. *The area of a closed singular $\mathbf{dS}^2$-surface is entirely determined by the angles at its singularities.*
3. *If a closed singular $\mathbf{dS}^2$-surface S has a single singularity x, then x has a positive angle equal to the area $\mathcal{A}(S) \in \mathbb{R}^*_+$ of S.*

Proof. Let us denote by Σ the singular set of S, and by $S^* = S \setminus \Sigma$ the $\mathbf{X}$-surface associated to S. A general topological fact ensures that S admits a finite triangulation *subordinate* to any given covering, *i.e.* each of which triangle is contained in an open set of the chosen covering. Let us choose a singular $\mathbf{X}$-atlas of S, each of which chart domain is a normal convex neighbourhood of any of its points. Around a singular point of S, we use a natural generalization in the singular setting of the usual notion of normal convex neighbourhood, introduced in Proposition 4.8 below. This allows us to consider a triangulation $\mathcal{T}$ of S, whose set of vertices, edges and faces (namely triangles) are respectively denoted by $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$, and such that:

[1] Note that this theorem of Denjoy holds more generally for the so-called *class P homeomorphisms*, of which $\mathcal{C}^2$-diffeomorphisms with breaks are specific examples.

(1) Σ is contained in the vertex set $\mathcal{V}$;
(2) the interior of any edge $e \in \mathcal{E}$ is a geodesic interval of timelike or spacelike signature.

Formula (2.7) will follow from a Lorentzian counterpart of the Gauß-Bonnet formula, proved in [Dza84, p.225] for compact subsets of regular Lorentzian surfaces whose boundary are piecewise smooth timelike or spacelike curves, and taking into account the angles between consecutive smooth segments at the breaking points (see also [Ave63, Che63] for analogous formula in any signatures and dimensions and with intrisic proofs, but in the boundaryless setting). This formula needs thus the definition of *angles* between tangent vectors of Lorentzian manifolds, which is done in [Dza84, §3 p.217]. For any non-zero future-oriented spacelike vector $u \in \mathrm{T}_{\mathsf{o}}\mathbf{X}$, the angle between u and $\mathrm{D}_{\mathsf{o}}a^t(u)$ in Dzan's convention is simply given by

$$\varnothing(u, \mathrm{D}_{\mathsf{o}}a^t(u)) = i.t. \tag{2.8}$$

This relation follows from our Convention 2.10 on the parametrization of the stabilizer $\{a^t\}_t$ of o. We draw the attention of the reader on the fact (surprising at first sight) that Lorentzian angles have *complex* values (for instance pure imaginary in (2.8)). One can then define the angle axiomatically by stating that it is additive in the usual sense (see [Dza84, Definition 7 p.220]), and that $\varnothing(u, v) = \frac{\pi}{2}$ if $g_{\boldsymbol{\mu}}(u, v) = 0$ for u and v two non-zero tangent vectors. Let T^i be a vertex of a triangle $T \in \mathcal{F}$, and (e^i_-, e^i_+) be the two edges of T incident to T^i, each of them being oriented from T^i to its other extremity, and such that a positively oriented path from $\mathrm{Int}(e^i_-)$ to $\mathrm{Int}(e^i_+)$ remains in $\mathrm{Int}(T)$. Then with $u^i_\pm$ a vector at T^i tangent to $e^i_\pm$ and compatible with its orientation, the *interior* and *exterior angles* at T^i are naturally defined by

$$\alpha(T^i) = \varnothing(u^i_-, u^i_+) \text{ and } \lambda(T^i) \coloneqq \pi - \alpha(T^i). \tag{2.9}$$

If T^i is a singular point then the tangent vectors $u^i_\pm$ are well-defined in any singular chart at T^i, and the angle $\varnothing(u^i_-, u^i_+)$ being invariant by isometry, it will not depend on the chosen singular chart according to Proposition 2.26. Therefore, the definitions (2.9) of the angles still make sense at a singular vertex. Denoting by (T^1, T^2, T^3) the vertices of a triangle $T \in \mathcal{F}$, the Gauß-Bonnet formula proved in [Dza84, p.225] becomes then:

$$i\varepsilon_{\mathbf{X}}\mathcal{A}(T) + \sum_{i=1}^{3} \lambda(T^i) = 2\pi \tag{2.10}$$

with $\mathcal{A}(T)$ the area of T. To translate Dzan's formula into the equation (2.10) for our geodesic triangle T, the following remarks are in order about the successive terms of the left-hand-side of the Gauß-Bonnet formula in [Dza84, p.225]:

(1) the area element dS appearing in the formula is purely imagery, equal to idS_0 with dS_0 the standard area element of S (see [Dza84, (55) p.224]);
(2) the edges of our triangle T being geodesic, the integral of the geodesic curvature k_g vanishes;
(3) the "directed sectorial measure of the exterior angle λ_i" at T^i, equals our exterior angle $\lambda(T^i)$ defined in (2.9).

For any $v \in \mathcal{V}$, we denote by $\mathcal{F}_v$ the set of triangles containing v as a vertex, and for $T \in \mathcal{F}_v$, by T^{i_v} the (unique) vertex of T equal to v. The remark preceding [Dza84, Definition 3 p.218] and the additivity of the Lorentzian angle imply then that the total angle at any regular vertex $v \in \mathcal{V}$ is 2π, *i.e.* that:

$$\sum_{T \in \mathcal{F}_v} \alpha(T^{i_v}) = 2\pi.$$

Thanks to the interpretation of standard singularities as angle defaults in Remark 2.21, this relation becomes:

$$\sum_{T \in \mathcal{F}_v} \alpha(T^{i_v}) = 2\pi + i\theta_v. \tag{2.11}$$

at a singular point $v \in \mathcal{V}$ of angle θ_v.

We are finally ready to sum the formula (2.10) on the faces of our triangulation. To this end, we denote by V, E, F and F_v the respective cardinals of the sets $\mathcal{V}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{F}_v$ for any $v \in \mathcal{V}$. We first translate (2.11) into:

$$\sum_{T \in \mathcal{F}_v} \lambda(T^{i_v}) = \pi(F_v - 2) - i\theta_v,$$

which gives

$$\begin{aligned}\sum_{T \in \mathcal{F}} \sum_{i=1}^{3} \lambda(T^i) &= \sum_{v \in \mathcal{V}} \sum_{T \in \mathcal{F}_v} \lambda(T^{i_v}) \\ &= \sum_{v \in \mathcal{V}} (\pi(F_v - 2) - i\theta_v) \\ &= \pi(3F - 2V) - i \sum_{v \in \mathcal{V}} \theta_v \end{aligned} \tag{2.12}$$

by summing on the vertices. In the last equality, we used the obvious relation $\sum_{v \in \mathcal{V}} F_v = \sum_{T \in \mathcal{F}} 3 = 3F$. Using (2.12), we obtain from (2.10):

$$\begin{aligned} \sum_{T \in \mathcal{F}} i\varepsilon_{\mathbf{X}} \mathcal{A}(T) + \sum_{T \in \mathcal{F}} \sum_{i=1}^{3} \lambda(T^i) &= \sum_{T \in \mathcal{F}} 2\pi \\ \Leftrightarrow i\varepsilon_{\mathbf{X}} \mathcal{A}(S) + \pi(3F - 2V) - i \sum_{v \in \mathcal{V}} \theta_v &= 2\pi F \\ \Leftrightarrow i\varepsilon_{\mathbf{X}} \mathcal{A}(S) + \pi(F - 2V) - i \sum_{v \in \mathcal{V}} \theta_v &= 0. \end{aligned} \tag{2.13}$$

Since $\mathcal{T}$ is a triangulation, each of its edge belongs to exactly two of its faces, which translates as $\sum_{e \in \mathcal{E}} 2 = \sum_{T \in \mathcal{F}} 3$ and thus $E = \frac{3F}{2}$. Hence $\pi(F - 2V) = 2\pi(-F + E - V) = -2\pi\chi(S)$ with $\chi(S)$ the Euler characteristic of S, and (2.13) becomes thus:

$$i\left(\varepsilon_{\mathbf{X}} \mathcal{A}(S) - \sum_{v \in \mathcal{V}} \theta_v\right) = 2\pi\chi(S). \tag{2.14}$$

But S is homeomorphic to a torus according to Corollary 2.31, hence $\chi(S) = 0$, and (2.14) yields the expected formula (2.7) which concludes the proof of the Proposition. □

3. Constructions of singular $\mathbf{dS}^2$-tori

In this section, we present the constructions of $\mathbf{dS}^2$-tori with one singularity yielding the existence results from Theorem B, C and D. More precisely, we will prove the following.

Theorem 3.1. *Let $\theta \in \mathbb{R}_+^*$, $c_\alpha \neq c_\beta \in \pi_1(\mathbf{T}^2)$ be two distinct primitive elements, and $A_\alpha \neq A_\beta \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$ be two distinct irrational rays. Then there exists on $\mathbf{T}^2$ a singular $\mathbf{dS}^2$-structure having a unique singularity of angle θ at $\mathsf{0} = [0,0]$, whose lightlike foliations are suspensions of circle homeomorphisms, and satisfy moreover any of the following properties.*

1. *$\mathcal{F}_\alpha(\mathsf{0})$ and $\mathcal{F}_\beta(\mathsf{0})$ are closed leaves of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$, and $([\mathcal{F}_\alpha(\mathsf{0})], [\mathcal{F}_\beta(\mathsf{0})]) = (c_\alpha, c_\beta)$. We can moreover assume that either $\mathcal{F}_\alpha(\mathsf{0})$ or $\mathcal{F}_\beta(\mathsf{0})$ is the unique closed leaf of its foliation, and that both of them are such if (c_α, c_β) is a basis of $\pi_1(\mathbf{T}^2)$.*
2. *$([\mathcal{F}_\alpha(\mathsf{0})], A^+(\mathcal{F}_\beta)) = (c_\alpha, A_\beta)$ (in particular, $\mathcal{F}_\beta$ is minimal), and $\mathcal{F}_\alpha(\mathsf{0})$ is the unique closed leaf of $\mathcal{F}_\alpha$.*
3. *$(A^+(\mathcal{F}_\alpha), A^+(\mathcal{F}_\beta)) = (A_\alpha, A_\beta)$ (in particular, $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are both minimal).*

We recall that according to Proposition 2.32, the positive angles are the only one which can be realized by a single singularity of a $\mathbf{dS}^2$-torus, hence the necessary condition $\theta \in \mathbb{R}_+^*$ which is not a restriction. The proof of Theorem 3.1 will be concluded in paragraph 3.8.

$A^+(\mathcal{F}) \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$ denotes the *oriented projective asymptotic cycle* of the oriented foliation $\mathcal{F}$, which will be introduced in paragraph 3.6. An element $a \in \pi_1(\mathbf{T}^2)$ is *primitive* if it cannot be written as $a = b^k$ with $b \in \pi_1(\mathbf{T}^2)$ and $k \geq 2$ – equivalently if a is represented by simple closed curves of $\mathbf{T}^2$. We denote by $[\gamma]$ the homotopy class of a curve γ in $\pi_1(\mathbf{T}^2)$. A half-line

$l \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$ is *rational* if $l = \mathbb{R}a$ with $a \in \pi_1(\mathbf{T}^2) \equiv \mathrm{H}_1(\mathbf{T}^2, \mathbb{Z}) \subset \mathrm{H}_1(\mathbf{T}^2, \mathbb{R})$, and *irrational* otherwise.

We fix for this whole section a positive angle $\theta \in \mathbb{R}_+^*$, and recall that according to the Gauß-Bonnet formula (2.7) in Proposition 2.32, a singular $\mathbf{dS}^2$-torus having a single singularity x has area θ, if, and only if x has angle θ. We also identify in the whole section $\mathbb{R}\mathbf{P}^1$ with $\mathbb{R} \cup \{\infty\}$ and elements of $\mathrm{PSL}_2(\mathbb{R})$ with their associated homography of $\mathbb{R} \cup \{\infty\}$, as defined in (2.2) and (2.4).

3.1. **Gluings of polygons in $\mathbf{dS}^2$.** Let us denote by $y_\theta := 1 - e^{-\theta} \in \,]0\,;1[$ the unique number such that $\mathcal{A}_{\boldsymbol{\mu}}(\mathcal{R}_{(1,\infty,0,y_\theta)}) = \theta$. According to Lemma 2.9, $\mathcal{R}_\theta := \mathcal{R}_{(1,\infty,0,y_\theta)}$ is, up to the action of $\mathrm{PSL}_2(\mathbb{R})$, the unique rectangle with lightlike edges and area θ in $\mathbf{dS}^2$. Our goal is to define a quotient of $\mathcal{R}_\theta$ with a single singularity, which will *a posteriori* necessarily have angle θ by Gauß-Bonnet formula (2.7). A first easy way to do this is to consider the unique elements $g = g_\theta$ and h_θ of $\mathrm{PSL}_2(\mathbb{R})$ such that

$$g(1, 0, y_\theta) = (\infty, 0, y_\theta) \text{ and } h_\theta(1, \infty, 0) = (1, \infty, y_\theta), \tag{3.1}$$

and to form the quotient of $\mathcal{R}_\theta$ by gluing its edges through g and h_θ (see Figure 3.1). The gluing being made by isometries, the $\mathbf{dS}^2$-torus obtained in this way will have, as sought, a unique singularity at the class of the vertices. However by such a construction, both lightlike leaves of the singularity will always be closed. To obtain a structure with a minimal lightlike foliation, it is thus necessary to consider another type of gluing.

3.1.1. *Suspension of homographic interval exchange transformations.* Inspired from the constructions of translation surfaces as "suspensions" of (classical) interval exchange transformations, a natural idea to obtain minimal lightlike foliations is to keep gluing the β-edges of $\mathcal{R}_\theta$ through g, but to glue its two α-edges through a *homographic interval exchange transformation* (*HIET*) with two components of the closed α-leaf. Such a map is a bijection of an interval I of $\mathbb{R}\mathbf{P}^1$ exchanging the components of two partitions of I called *top* and *bottom* partitions, and which is homographic on each component of the top partition (*i.e.* equals the restriction of an element of $\mathrm{PSL}_2(\mathbb{R})$). The notion of HIET is both a natural generalization of the ones of (classical) IET and affine IET, and a restriction of the notion of *generalized interval exchange transformation* (*GIET*). We refer the reader to the excellent [Yoca, Yocb] for more informations on theses notions (which will however not be needed in this text).

For any $x, x' \in \,]1\,;\infty[$, we introduce the following subintervals of $I = [1\,;\infty[$:

$$I_1^t = [1\,;x'[, I_2^t = [x'\,;\infty[, I_1^b = [1\,;x[, I_2^b = [x\,;\infty[, \tag{3.2}$$

delimiting a *top* partition $I = I_1^t \sqcup I_2^t$ and a *bottom* partition $I = I_1^b \sqcup I_2^b$ of I. By three-transitivity of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{R}\mathbf{P}^1$, there exists a unique pair h_1, h_2 of elements of $\mathrm{PSL}_2(\mathbb{R})$ such that $h_1(0) = h_2(0) = y_\theta$, $h_1(I_1^t) = I_2^b$ and $h_2(I_2^t) = I_1^b$, and we define a HIET $E \colon I \to I$ by:

$$E|_{I_1^t} = h_1|_{I_1^t}, E|_{I_2^t} = h_2|_{I_2^t}. \tag{3.3}$$

We now "suspend" this HIET E, obtaining the quotient $\mathcal{T}_{\theta,E}$ of the rectangle $\mathcal{R}_\theta$ by the following edges identifications:

$$\begin{cases} [1\,;\infty[\times \{0\} \ni (p, 0) \sim (E(p), y_\theta) \in [1\,;\infty[\times \{y_\theta\}, \\ \{1\} \times [0\,;y_\theta] \ni (1, p) \sim (\infty, g(p)) \in \{\infty\} \times [0\,;y_\theta]. \end{cases}$$

These gluings, illustrated in Figure 3.1, give us a first family of examples of singular $\mathbf{dS}^2$-tori. Vertices of $\mathcal{R}_\theta$ of the same color indicate points identified in the quotient $\mathcal{T}_{\theta,E}$. To prevent any confusion, we emphasize that the denominations of top and bottom partitions are the usual ones in the literature of GIET's which is the reason why we used them, but that they do *not* correspond to their positions in the Figure 3.1: the top partition corresponds to the lower interval and the bottom one to the upper interval.

Proposition 3.2. *For any $\theta \in \mathbb{R}_+^*$ and $x, x' \in \,]1\,;\infty[$, $\mathcal{T}_{\theta,E}$ is homeomorphic to $\mathbf{T}^2$ and the $\mathbf{dS}^2$-structure of the interior of $\mathcal{R}_\theta$ extends to a unique singular $\mathbf{dS}^2$-structure on $\mathcal{T}_{\theta,E}$. The latter has area θ, the α-leaf of $[\infty, 0]$ is closed, its unique (potentially) singular points are $[\infty, 0]$ and $[x', 0]$, and the holonomies of small positively oriented loops around them are:*

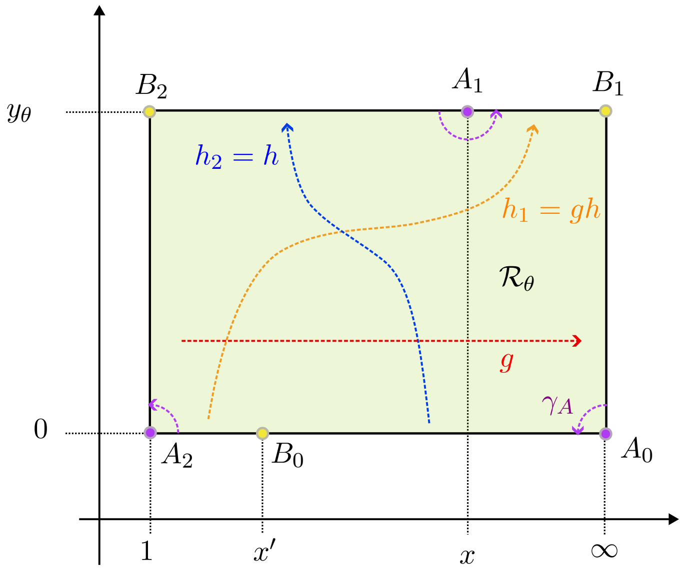

FIGURE 3.1. $\mathbf{dS}^2$-torus with one singularity and a closed α-lightlike leaf.

(1) holonomy around $(\infty, 0) = h_2^{-1} h_1 g^{-1}$,
(2) holonomy around $(x', 0) = h_1^{-1} g h_2$.

Proof. Let us denote by $\pi \colon \mathcal{R}_\theta \to \mathcal{T}_{\theta,E}$ the canonical projection, and $[a, b] = \pi(a, b)$ for $(a, b) \in \mathcal{R}_\theta$. We first observe that the gluing of the edges are well-defined for the quotient to be topologically a torus, as a Euler characteristic computation directly shows. The edges being moreover identified by elements of $\mathrm{PSL}_2(\mathbb{R})$, the $\mathbf{dS}^2$-structure of $\pi(\mathrm{Int}(\mathcal{R}_\theta))$ for which $\pi|_{\mathrm{Int}(\mathcal{R}_\theta)}$ is a $\mathbf{dS}^2$-morphism extends to a $\mathbf{dS}^2$-structure of area θ on the complement of the vertices, *i.e.* on $\mathcal{T}_{\theta,E} \setminus \{[\infty, 0], [x', 0]\}$. Lastly, observe that the lightlike foliations of $\pi(\mathrm{Int}(\mathcal{R}_\theta))$ clearly extend to two transverse continuous foliations of $\mathcal{T}_{\theta,E}$.

The top and bottom partitions (3.2) of $[1\,; \infty[$ define associated partitions of the α and β boundary parts of $\mathcal{R}_\theta$, that we will call *edges*, and their extremities will be called *vertices*. Let us detail in the specific case of $A = [\infty, 0] \in \mathcal{T}_{\theta,E}$ a general "recipe" to compute the holonomy around any vertex P of $\mathcal{T}_{\theta,E}$, illustrated in Figure 3.1. First of all, note that each vertex P is associated with a positively cyclically ordered periodic orbit $(P_0, P_1, \dots, P_d)$, which has length 2 for A. A small positively oriented closed loop γ_P around P defines indeed a cyclic ordering on the (finite) equivalence class of P for $\sim$, describing in which order the points are met in $\mathcal{R}_\theta$ when following γ_P. For instance in the case of A if we start with $A_0 = (\infty, 0)$, then we successively meet $A_1 = (x, y_\theta)$, $A_2 = (1, 0)$ and finally come back to A_0. Moreover at each step P_i, $i \geq 1$ of this periodic orbit, γ_P meets in $\mathcal{T}_{\theta,E}$ an interval of a lightlike half-leaf emenating from P which corresponds both to a top edge $e^t_{P_i}$ and to a bottom edge $e^b_{P_i}$ of $\mathcal{R}_\theta$, having respectively P_{i-1} and P_i as one of their extremities. These are for instance $e^t_{A_1} = [x'\,; \infty] \times \{0\}$ (A_0 as right extremity) and $e^b_{A_1} = [1\,; x] \times \{y_\theta\}$ (A_1 as right extremity) for $P_i = A_1$. These edges are then identified in the quotient by some $f_{P_i} \in \mathrm{PSL}_2(\mathbb{R})$, characterized by $f_{P_i}(e^b_{P_i}) = e^t_{P_i}$ (for instance $f_{A_1} = h_2^{-1}$ in our example $P_i = A_1$). Lastly, each point P_i of the periodic orbit $(P_0, P_1, \dots, P_d)$ contributes for a certain sequence Q_{P_i} of *quadrants* around P, ordered as they are met by γ_P. For instance for A, $Q_{A_0} =$ future timelike, $Q_{A_1} =$ (past spacelike, past timelike) and $Q_{A_2} =$ future spacelike. We will say that *the identification of the quadrants around P is standard*, if the sequence $(Q_{P_0}, \dots, Q_{P_d})$ equals the *standard sequence*: (future timelike, past spacelike, past timelike, future spacelike), up to cyclic permutations.

Fact 3.3. *Let assume that the identification of the quadrants around a vertex P is standard. Then P is a standard singularity of $\mathcal{T}_{\theta,E}$. Moreover with ρ the holonomy morphism associated to the developing map extending the section $s \colon \pi(\mathrm{Int}(\mathcal{R}_\theta)) \to \mathrm{Int}(\mathcal{R}_\theta)$ of π, we have:*

$$\rho(\gamma_P) = f_{P_1} f_{P_2} \dots f_{P_d} f_{P_0} \in \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{R})}(P_0). \tag{3.4}$$

Proof. For the sake of clarity, we write the proof in the specific case of A, but it is formally identical in any situation. We define $\varphi_0 = s$ as a $\mathbf{dS}^2$-chart on $\pi(U_0)$, with U_0 a small neighbourhood of A_0 in $\mathcal{R}_\theta$. Now let U_1 be a small neighbourhood of A_1 in $\mathcal{R}_\theta$, and φ_1 be a $\mathbf{dS}^2$-chart defined on a neighbourhood of $\overline{\pi(U_1)}$ in $\mathcal{T}_{\theta,E} \setminus \{[\infty,0],[x',0]\}$, and agreeing with φ_0 on a neighbourhood of $[\infty,0]$ in $\pi(]1\,;\infty] \times \{0\})$. Then $\varphi_1 = g_{A_1} \circ s$ on $\pi(U_1)$ with $g_{A_1} \in \mathrm{PSL}_2(\mathbb{R})$ agreeing with $f_{A_1} = h_2^{-1}$ on a neighbourhood of A_1 in $[1\,;x] \times \{y_\theta\}$. The naive but important observation is now that if $g, g' \in \mathrm{PSL}_2(\mathbb{R})$ have the same action on a non-empty open lightlike interval, then $g = g'$. Indeed, it is sufficient to check this for $g, g' \in \mathrm{Stab}(\mathsf{o})$, for which this claim simply follows from the fact that a non-trivial element of $\mathrm{Stab}(\mathsf{o})$ has a non-trivial action on any non-empty open lightlike interval of extremity o. This shows that $g_{A_1} = f_{A_1}$, *i.e.* that $\varphi_1 = f_{A_1} \circ s$ on $\pi(U_1)$.

Continuing in the same way, we conclude that if U_2 is a neighbourhood of A_2 in $\mathcal{R}_\theta$, and φ_2 a $\mathbf{dS}^2$-chart defined on a neighbourhood of $\overline{\pi(U_2)}$ and agreeing with φ_1 on the suited α-interval, then $\varphi_2 = f_{A_1} \circ f_{A_2} \circ s$ on $\pi(U_2)$. To understand this relatively counter-intuitive order in the compositions, observe first that $f_{A_2} \circ s|_{\pi(U_2)}$ and $s|_{\pi(U_1)}$ glue together to define a $\mathbf{dS}^2$-chart on a punctured neighbourhood of $[1,0]$ in $\pi([1\,;x'] \times \{0\})$, hence that $f_{A_1} \circ f_{A_2} \circ s$ and $f_{A_1} \circ s = \varphi_1$ agree on the intersection of their domains.

In the end $\varphi_3 = f_{A_1} \circ f_{A_2} \circ f_{A_0} \circ \varphi_0$, and the maps φ_i for $i = 0, \dots, 3$ agree on the intersection of their domains. They glue thus together to give a $\mathbf{dS}^2$-isomorphism ψ from a slit neighbourhood $U' = U \setminus \mathcal{F}_\alpha([\infty,0])$ of $[\infty,0]$ to a slit neighbourhood of $(\infty,0) = \mathsf{o}$ in $\mathbf{dS}^2$. This map satisfies the hypotheses of Lemma 2.14.(2), and we conclude thus that $[\infty,0] = A$ is a standard singularity of the $\mathbf{dS}^2$-structure of $\mathcal{T}_{\theta,E} \setminus \{[1,0],[x',0]\}$, and that $\rho(\gamma_A) = f_{A_1} \circ f_{A_2} \circ f_{A_0} \in \mathrm{Stab}(\mathsf{o})$. □

Fact 3.3 shows our claim for the vertices $[\infty,0]$ and $[x',0]$, and concludes thus the proof of the proposition. □

3.1.2. *Further remarks on identification spaces of polygons.* To clarify our exposition, avoid unnecessary notations and rather emphasize the main ideas, we chose to focus on the constructions of singular $\mathbf{dS}^2$-tori that will be developed in the sequel of the text in the case of one singularity. However, the same formal proof than the one of Fact 3.3 offers a general way of constructing singular $\mathbf{X}$-tori, and proves the following result. We refer to the proof of Proposition 3.2 for the definition of a *standard identification of quadrants around a vertex*, and of the related notions appearing in the statement below. We will call *polygon* a compact connected subset of $\mathbf{X}$, homeomorphic to a closed disk and whose boundary is a finite union of geodesic segments. We also denote by $(\mathbf{G},\mathbf{X})$ the pair $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ or $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1), \mathbb{R}^{1,1})$.

Proposition 3.4. *Let $\mathcal{P}$ be a polygon of $\mathbf{X}$, whose boundary is lightlike and endowed with:*

(1) a decomposition into an even number of edges which are segments of lightlike leaves,
(2) and pairwise identifications between these edges by elements of $\mathbf{G}$.

Assume that the identification of the quadrants around each vertex is standard. Then the quotient of $\mathcal{P}$ by the edges identifications is a torus endowed with a unique singular $\mathbf{X}$-structure compatible with the one of $\mathcal{P}$. This singular $\mathbf{X}$-torus has the same area than $\mathcal{P}$, and the holonomies at the vertices are given by the formula (3.4).

Remark 3.5. Proposition 3.4 proves in particular the existence of singular $\mathbb{R}^{1,1}$-tori or *singular flat tori*, and offers a way to construct a large family of them. The investigation of singular flat tori will be considered in a future work.

Remark 3.6. Proposition 3.4 could be stated more generally: the quotient of any connected polygon of $\mathbf{X}$ whose boundary is lightlike and endowed with an even partition into edges, by any pairwise identifications of the edges by elements of $\mathbf{G}$, is endowed with a natural $\mathbf{X}$-structure on the complement of the vertices. But these vertices are *not* standard singularities as studied in this text when the identification of quadrants around them is not standard. For instance, non-standard singularities do not see four lightlike half-leaves emanating from them, and in particular the lightlike foliations do not extend to topological foliations at non-standard singularities. This should however not exclude the attention for such examples, particularly interesting ones arising for instance when the lightlike foliations have themselves standard singularities at the singularities of the metric (for instance when they are the stable and unstable foliations of a pseudo-Anosov map). The study of this very interesting class of examples will be the content of a future work.

Lastly, Lemma 2.20 shows that standard singularities do not need to be constructed from lightlike geodesics, and that definite geodesics work just as well. A natural analog to Proposition 3.4 can therefore be stated and proved in the same way for any polygon of $\mathbf{X}$ having a geodesic boundary endowed with a partition into an even number of edges and pairwise identifications between them by elements of $\mathbf{G}$.

In what follows, all the graphs are assumed to be finite.

Definition 3.7. A graph C embedded in a singular $\mathbf{X}$-surface S is said *lightlike*, if any vertex of C has degree at least 2, and any edge of C is a connected subset of a lightlike geodesic. S is *rectangular* if there exists a lightlike graph C, said *rectangular*, embedded in S and such that:

(1) any singularity of S is a vertex of C,
(2) $S \setminus C$ is a topological disk,
(3) and the oriented boundary of the surface $S \setminus\!\!\setminus C$ obtained from cutting S along C is a lightlike rectangle, namely the successive union of a positive α-segment, a positive β-segment, a negative α-segment and a negative β-segment.[2]

Proposition 3.8. *Let S be a rectangular singular $\mathbf{X}$-torus. Then S is isometric to the quotient of a lightlike rectangle of $\mathbf{X}$, endowed with a decomposition of its boundary into an even number of edges, by pairwise identifications of its edges by elements of $\mathbf{G}$ as given by Proposition 3.4.*

Proof. Let $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be endowed with a rectangular singular $\mathbf{X}$-structure, and $\bar{C} \subset \mathbf{T}^2$ be a rectangular graph as in Definition 3.7. We endow $\mathbb{R}^2$ with the $\mathbb{Z}^2$-invariant singular $\mathbf{X}$-structure for which the universal covering $\pi\colon \mathbb{R}^2 \to \mathbf{T}^2$ is a local isometry, and denote by $\tilde{C} = \pi^{-1}(\bar{C})$ the lift of $\bar{C}$. This is an embedded graph in $\mathbb{R}^2$ satisfying properties (2) and (3) of Definition 3.7 for $S = \mathbb{R}^2$, and such that each connected component of $\mathbb{R}^2 \setminus \tilde{C}$ is a topological disk. We denote by E the closure of one of these connected components, and by C the subgraph of $\tilde{C}$ which is the boundary of E. Then E is a fundamental domain for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$, and $\mathbf{T}^2$ is thus isometric to the quotient of E by the identifications of the edges of C by suitable elements of $\mathbb{Z}^2$. Note that any edge of $\bar{C}$ has two lifts in C, hence C has an even number of edges.

(a) Injectivity of the developing map on a fundamental domain. Since the singularities $\bar{\Sigma}$ of $\mathbf{T}^2$ are by assumption contained in $\bar{C}$, the singularities $\tilde{\Sigma} = \pi^{-1}(\bar{\Sigma})$ of $\mathbb{R}^2$ are contained in $\tilde{C}$, and with $\Sigma = \tilde{\Sigma} \cap C$, we have $\pi(\Sigma) = \bar{\Sigma}$. In particular $E^* := E \setminus \Sigma$ is contained in $\mathbb{R}^2 \setminus \tilde{\Sigma}$, and with U a simply connected open neighbourhood of E^* contained in $\mathbb{R}^2 \setminus \tilde{\Sigma}$, there exists a $\mathbf{X}$-morphism

$$\delta\colon U \to \mathbf{X},$$

which is the developing map of the $\mathbf{X}$-structure of U. Note that U is a topological disk, as is any connected and simply connected open subset of the plane.

Fact 3.9. *δ extends to a continuous map D from a neighbourhood $\mathcal{U}$ of E to $\mathbf{X}$. There exists moreover a lightlike rectangle E_0 in $\mathbf{X}$, a decomposition of the boundary of E_0 into a graph C_0 whose edges are segments of lightlike leaves, and a subset Σ_0 of the vertices of C_0, such that:*

(1) $D(E) \subset E_0$,
(2) $D(\Sigma) = \Sigma_0$ and D is a graph morphism from C to C_0,
(3) D is injective in restriction to C.

Proof. By assumption, any vertex of $\tilde{C}$ has degree at least 2, and since any edge is a segment of lightlike leave, the vertices also have degree at most 4 inside $\tilde{C}$ (in the maximal case, segments of the four lightlike half-leaves emanate from a vertex). But C being the boundary of E hence a topological circle, any vertex of C has of course degree exactly 2 inside C. Now we endow the circle $C = \partial E$ with the orientation induced by the one of E, fix $v \in \Sigma$ a singular vertex of C, and

[2]Equivalently, the graph $C = \partial E$ embedded in the universal cover of S appearing in the proof of Proposition 3.8 is a lightlike rectangle of $\mathbf{dS}^2$; or equivalently: C has two edges, one of them being a closed α-lightlike leaf and the other one a segment of β-lightlike leaf (up to interverting the α and β closed leaf).

denote by e_-, e_+ the two (closed) edges of C of extremity v ($e_- \neq e_+$ since v has degree 2), e_+ being met after e_- in the positive orientation of C. Up to a cyclic permutation of the quadrants, the three following situations are the only one that can arise.

(1) e_- is a segment of the α-leaf of v denoted by $[x_-\,;v]_\alpha$, going from x_- to v for the positive orientation of C. Similarly, e_+ is a segment of the β-leaf of v of the form $[v\,;x_+]_\beta$. Moreover, v admits an open neighbourhood $Q_v \subset E^* \cup \{v\}$ in E which is a small future timelike quadrant, and such that $Q_v \cap \Sigma = \{v\}$.
(2) e_- is an α-segment $[x_-\,;v]_\alpha$, e_+ an α-segment $[v\,;x_+]_\alpha$, and v admits an open neighbourhood $Q_v \subset E^* \cup \{v\}$ in E which is the union of a small future timelike quadrant and of a small future spacelike quadrant.
(3) e_- is an α-segment $[x_-\,;v]_\alpha$, e_+ a β-segment $[x_+\,;v]_\beta$, and v admits an open neighbourhood $Q_v \subset E^* \cup \{v\}$ in E which is the union of a small future timelike quadrant, a small future spacelike quadrant and a small past timelike quadrant.

Note that the segments $e_\pm$ are endowed with two orientations, respectively induced by the one of $C = \partial E$ and by the lightlike foliations. These two orientations coincide for $[x_-\,;v]_\alpha$ in the three above cases and for $[v\,;x_+]_\beta$ and $[v\,;x_+]_\alpha$ in cases (1) and (2), but they are opposite for $[x_+\,;v]_\beta$ in case (3).

Since v is a standard singularity, denoting by $Q_{\mathsf{o}} \subset \mathbf{X}$ the union of quadrants at o corresponding to Q_v, $Q_v^* := Q_v \setminus \{v\}$ is isometric to $Q_{\mathsf{o}}^* := Q_{\mathsf{o}} \setminus \{\mathsf{o}\}$. Namely, there exists an isometry φ from a neighbourhood $V \subset U$ of Q_v^* in $\mathbb{R}^2$ to a neighbourhood V_0 of Q_{o}^* in $\mathbf{X}$, such that $\varphi(Q_v^*) = Q_0^*$ (see Lemma 2.14). Since $\delta|_V$ is another $\mathbf{X}$-morphism from V to $\mathbf{X}$, there exists moreover $g \in \mathbf{G}$ such that $\delta|_V = g \circ \varphi$. Hence $\delta(Q_v^*) = g(Q_0^*) = Q_{v_0}^*$, with Q_{v_0} the union of quadrants at $v_0 := g(\mathsf{o})$ corresponding to Q_v. In particular, this shows that $\delta|_V$ extends to an injective continuous map D_v from a neighbourhood $W \subset \mathbb{R}^2$ of Q_v to a neighbourhood $W_0 \subset \mathbf{X}$ of Q_{v_0}, sending v to v_0.

We can now glue together these maps D_v, to define a map D from a neighbourhood $\mathcal{U}$ of E to $\mathbf{X}$. Since δ is a local diffeomorphism, it is injective in restriction to any open edge of C, and D is thus injective in restriction to any closed edge since the lightlike leaves of $\mathbf{X}$ are embeddings of $\mathbb{R}$. By construction, $C_0 := D(C)$ is a lightlike rectangular closed loop in $\mathbf{X}$, and we define a decomposition of C_0 by stating that D is a graph morphism (which makes sense since D is injective in restriction to any edge). A simple but important observation is now that any lightlike rectangular closed loop in $\mathbf{X}$ is simple, *i.e.* without any self-intersection. Since E is moreover always on the same side of C by definition of its orientation (namely on the left), $D(E)$ is always on the same side of C_0, hence $D(E)$ is contained in the (unique) lightlike rectangle E_0 of $\mathbf{X}$ bounded by C_0.

We know at this stage that $D|_C$ is a continuous map from the topological circle $C = \partial E$ to the topological circle C_0, which is locally injective hence a local homeomorphism. But since the oriented graph C contains only one positively travelled α-segment, $D|_C$ cannot have degree > 1. Therefore $D|_C$ is injective, which concludes the proof of the fact. □

Now since the continuous map $D|_E\colon E \to E_0$ is locally injective and injective in restriction to ∂E, $D|_E$ is injective according to [MO63, Theorem 1 p.75] (see also Definition 3 p.74 therein). And since δ is a local diffeomorphism, D is actually injective in restriction to a small enough neighbourhood $\mathcal{U} \subset \mathbb{R}^2$ of E, and is thus a homeomorphism from $\mathcal{U}$ to a neighbourhood $\mathcal{U}_0$ of E_0 in $\mathbf{X}$ according to Brouwer's invariance of domain theorem. In particular, $D(E)$ is a compact subset of E_0 of boundary ∂E_0, *i.e.* $D(E) = E_0$.

(b) Edges identifications. Recall that $C = \partial E$ has an even number of edges denoted by $\{(e_i^t, e_i^b)\}_i$, and that $\mathbf{T}^2$ is isometric to the quotient $\mathcal{E}$ of E by the identification of each e_i^t with the corresponding e_i^b through a translation T_{u_i} (where $u_i \in \mathbb{Z}^2$ and $T_{u_i}(e_i^t) = e_i^b$). Since integral translations are isometries of $\mathbb{R}^2$, there exists moreover unique elements $g_i \in \mathbf{G}$ such that

$$\delta \circ T_{u_i} = g_i \circ \delta$$

in restriction to a connected neighbourhood of e_i^t. Since D is a graph morphism according to Fact 3.9, we can define a decomposition of C_0 associated to the one of C by $f_i^t = D(e_i^t)$ and $f_i^b = D(e_i^b)$. We have then $g_i(f_i^t) = f_i^b$, and we can thus form the quotient $\mathcal{E}_0$ of $\mathcal{E}$ by these edges identifications, given by Proposition 3.4. By construction, D induces then an isometry from $\mathcal{E} \simeq \mathbf{T}^2$ to $\mathcal{E}_0$, which concludes the proof of the Proposition. $\square$

Lemma 3.10. *Let S be a closed singular $\mathbf{X}$-torus with a unique singularity x, and such that one of the lightlike leaves of x is closed and for the other lightlike foliation $\mathcal{F}$: either $\mathcal{F}$ is minimal or $\mathcal{F}(x)$ is closed. Then S is rectangular.*

Proof. To fix the ideas, we can assume without any lost of generality that $\mathcal{F}_\alpha(x)$ is closed. We then begin the construction of the rectangular graph C of Definition 3.7 with the vertex x and the edge $\mathcal{F}_\alpha(x)$. Let us denote by y the first intersection point of $\mathcal{F}_\beta(x)$ with $\mathcal{F}_\alpha(x)$ (for the positive orientation of $\mathcal{F}_\beta(x)$), and by e the positive β-segment from x to y. Note that y exists since by assumption, either $\mathcal{F}_\beta(x)$ is closed or $\mathcal{F}_\beta$ is minimal, hence $\mathcal{F}_\beta(x)$ eventually comes back in its future to $\mathcal{F}_\alpha(x)$. We define then $C = \mathcal{F}_\alpha(x) \cup e$, with set of vertices $V = \{x, y\}$, and edges given by the connected components of $C \setminus V$. Since $S \setminus \mathcal{F}_\alpha(x)$ is a cylinder, $S \setminus C$ is indeed a topological disk, while the other properties of Definition 3.7 are clearly satisfied. $\square$

Henceforth, we come back to the homogeneous model space $(\mathbf{G}, \mathbf{X}) = (\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$, and investigate thoroughly two families of $\mathbf{dS}^2$-tori with a single singularity.

3.2. **A one-parameter family of $\mathbf{dS}^2$-tori with one singularity having a closed leaf.**

We now apply Proposition 3.2 to obtain a first one-parameter family of $\mathbf{dS}^2$-tori.

3.2.1. *Definition of the one-parameter family.* For any $x \in \,]1\,;\infty]$ and $x' \in [1\,;\infty[$, let $h = h_{(x,x')}$ be the unique element of $\mathrm{PSL}_2(\mathbb{R})$ such that

$$h(x', \infty, 0) = (1, x, y_\theta), \tag{3.5}$$

i.e. $h = h_2$ in the notations of Proposition 3.2. Proposition 3.2 and Corollary 2.17 indicate us that $[x', 0] \in \mathcal{T}_{\theta,E}$ is regular if, and only if $h_1 = gh_2 = gh$, or equivalently if:

$$gh(1, x', 0) = (x, \infty, y_\theta). \tag{3.6}$$

Since $gh(x', 0) = (\infty, y_\theta)$ is automatically satisfied according to the equations (3.5) and (3.1), the regularity of $[x', 0] \in \mathcal{T}_{\theta,E}$ is eventually equivalent to $gh(1) = x$.

Lemma 3.11. *$gh(1) = x$ if, and only if $x' = \frac{x}{x-1}$. Moreover, g and h are hyperbolic.*

Proof. The last claim follows from a direct observation of the dynamics of g and h on $\mathbb{R}\mathbf{P}^1$. With $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the definition of g reads: $c + d = 0$, $b = 0$, $ay_\theta + b = y_\theta(cy_\theta + d)$, *i.e.* $y_\theta(cy_\theta - c - a) = 0$ and thus $a = c(y_\theta - 1)$. Hence $g = (1 - y_\theta)^{-1/2}\left(\begin{smallmatrix} -(1-y_\theta) & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and $g(t) = (y_\theta - 1)\frac{t}{t-1}$. Now if $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the definition of h reads: $ax' + b = cx' + d$, $a = cx$, $b = dy_\theta$, hence $d = \frac{cx'(x-1)}{(1-y_\theta)}$ and thus

$$h(t) = \frac{x(1 - y_\theta)t + x'(x-1)y_\theta}{(1 - y_\theta)t + x'(x-1)}.$$

A direct computation shows that $x - gh(1) = ((1 + e^\theta(-1 + x))(x(-1 + x') - x'))/(e^\theta(-1 + x)(-1 + x'))$. Since $x > 1 > 1 - e^{-\theta}$, this quantity vanishes if, and only if $x(-1 + x') - x' = 0$ *i.e.* $x' = x/(x-1)$, which concludes the proof. $\square$

We now fix $x \in [1\,;\infty]$ and denote:

1. $x' = x'_x := \frac{x}{x-1} \in [1\,;\infty]$ (with $x'_\infty = 1$ and $x'_1 = \infty$),
2. and $h = h_x := h_{(x,x'_x)}$ if $x > 1$, extended by $h_1 := g^{-1}h_\infty$ for $x = 1$.

The equations (3.5) and (3.6) show that $\lim\limits_{x\to 1} gh_x = h_\infty$, hence that $\lim\limits_{x\to 1} h_x = \lim\limits_{x\to 1} g^{-1}(gh_x) = h_1$, so that the maps

$$x \in [1\,;\infty] \mapsto h_x \in \mathrm{PSL}_2(\mathbb{R}) \text{ and } x \in [1\,;\infty] \mapsto gh_x \in \mathrm{PSL}_2(\mathbb{R})$$

are continuous. Using the top and bottom partitions of $I = [1\,;\infty[$ defined in (3.2), we consider the HIET $E = E_x\colon I \to I$ defined by

$$E_x|_{I_1^t} = gh_x|_{I_1^t}\colon I_1^t \to I_2^b \text{ and } E_x|_{I_2^t} = h_x|_{I_2^t}\colon I_2^t \to I_1^b, \tag{3.7}$$

and denote by $\mathcal{T}_{\theta,x} \coloneqq \mathcal{T}_{\theta,E_x}$ the suspension of E_x defined in Proposition 3.2 and illustrated in Figure 3.1. Note that $E_1 = E_\infty$ is simply the restriction of h_∞ to I, so that $\mathcal{T}_{\theta,1} = \mathcal{T}_{\theta,\infty}$. The following result summarizes the construction, and is a reformulation of Proposition 3.2 in the case $x' = \frac{x}{x-1}$.

Proposition 3.12. *For any $\theta \in \mathbb{R}_+^*$ and $x \in [1\,;\infty]$, $\mathcal{T}_{\theta,x}$ is homeomorphic to $\mathbf{T}^2$ and the $\mathbf{dS}^2$-structure of the interior of $\mathcal{R}_\theta$ extends to a unique singular $\mathbf{dS}^2$-structure on $\mathcal{T}_{\theta,x}$. The latter has area θ, and its unique singular point $[1,0] = [\infty,0]$ has a closed α-leaf and angle θ.*

Remark 3.13. Of course, one can realize the symmetric construction to obtain a quotient of $\mathcal{R}_\theta$ with this time the β-leaf of $[\infty,0]$ being closed. This is done by gluing the α-edges of $\mathcal{R}_\theta$ by the restriction of h_θ defined in (3.1), and its β-edges by a HIET with two components of $J = \{1\} \times [0\,;y_\theta]$ with top and bottom partitions

$$J_1^t = [0\,;y'[,\ J_2^t = [y'\,;y_\theta[,\ J_1^b = [0\,;y[,\ J_2^b = [y\,;y_\theta[.$$

These $\mathbf{dS}^2$-tori of area θ, with one singularity at $[\infty,0]$ whose β-leaf is closed, will be denoted by $\mathcal{T}_{\theta,y,*}$.

Corollary 3.14. *Let S be a closed singular $\mathbf{dS}^2$-surface with a single singularity x, such that one of the lightlike leaves of x is closed and for the other lightlike foliation $\mathcal{F}$: either $\mathcal{F}$ is minimal or $\mathcal{F}(x)$ is closed. Then S is isometric to a torus $\mathcal{T}_{\theta,x}$ given by Proposition 3.12, or to a torus $\mathcal{T}_{\theta,y,*}$ described in Remark 3.13.*

Proof. According to Lemma 3.10 and Proposition 3.8, such a closed singular $\mathbf{dS}^2$-surface S is the quotient of a lightlike rectangle $\mathcal{R} \subset \mathbf{dS}^2$, endowed with a decomposition of $\partial\mathcal{R}$ into an even number of edges, by pairwise identifications of its edges by elements of $\mathbf{G}$ as described in Proposition 3.4. Since S has moreover a unique singularity, the HIET's gluing the α and β-edges of $\partial\mathcal{R}$ have at most two components, *i.e.* are of the form described in (3.3), and S is thus isometric to a singular $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,E}$ as described in Proposition 3.2, up to interverting the α and β closed leaves. But we saw in Lemma 3.11 that $\mathcal{T}_{\theta,E}$ has a unique singularity if, and only if $x' = \frac{x}{x-1}$, and S is thus isometric to a singular $\mathbf{dS}^2$-torus $\mathcal{T}_{\theta,x}$ or $\mathcal{T}_{\theta,y,*}$, which concludes the proof. ☐

3.2.2. *Investigation of the holonomy.* Let γ denote the positively oriented closed α-lightlike leaf $[1\,;\infty] \times \{0\}$ in $\mathcal{T}_{\theta,x}$. Let η_1 be the β-lightlike positively oriented geodesic segment $\{1\} \times [0\,;y_\theta]$ going from $[1,0]$ to $[x',0]$, η_2 be the α-lightlike negatively oriented geodesic segment $[1\,;x'] \times \{0\}$ going from $[x',0]$ to $[1,0]$, and $\eta = \eta_1\eta_2$ be their concatenation, a piecewise geodesic closed loop. Then with γ' and η' slight deformations of these closed loops avoiding $\mathtt{0}$, the homotopy classes (a,b) of (γ',η') in $\mathcal{T}_{\theta,x}^* \coloneqq \mathcal{T}_{\theta,x} \setminus \{[\infty,0]\}$ freely generate the rank-two free group $\pi_1(\mathcal{T}_{\theta,x}^*) = \langle a,b\rangle$, and $K \coloneqq aba^{-1}b^{-1}$ is the homotopy class of a small positively oriented closed loop around $[\infty,0]$ in $\mathcal{T}_{\theta,x}^*$.

With $\rho = \rho_{\theta,x}\colon \pi_1(\mathcal{T}_{\theta,x}^*) \to \mathrm{PSL}_2(\mathbb{R})$ the holonomy representation of $\mathcal{T}_{\theta,x}^*$, we have $\rho(a) = g$, $\rho(b) = h$ and thus

$$\rho(K) = ghg^{-1}h^{-1},$$

which is coherent with Proposition 3.2. A direct computation using the description of g and h in Lemma 3.11 shows moreover that

$$\mathrm{tr}(gh_x) = \frac{-\sqrt{x}(2-y_\theta)}{\sqrt{(x-y_\theta)}} \text{ and } \mathrm{tr}(gh_xg^{-1}h_x^{-1}) = \frac{y_\theta^2 - 2y_\theta + 2}{1-y_\theta},$$

and in particular that for any $\theta \in \mathbb{R}_+^*$:

(1) $\mathrm{tr}(\rho_{\theta,x}(K)) > 2$;
(2) $\mathrm{tr}(gh_x) < 0$, and the function $x \in\,]1\,;\infty[\, \mapsto \mathrm{tr}(gh_x) + 2$ takes any sign, *i.e.* gh_x can be hyperbolic, elliptic or parabolic depending on the value of x.

We emphasize that, while the traces of g and h are not well-defined, any lifts of g and h to $\mathrm{SL}_2(\mathbb{R})$ give the same $\mathrm{tr}(ghg^{-1}h^{-1})$ (the signs vanishing in the commutator). This trace is thus a well-defined quantity associated to $\mathcal{T}_{\theta,x}$.

A particularly important class of $\mathbf{dS}^2$-surfaces are the *Kleinian* (or uniformizable) ones, of the form $S = \Gamma\backslash\Omega$ with Ω an open subset of $\mathbf{dS}^2$ and $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ a discrete subgroup acting properly discontinuously on Ω. In this case, the holonomy morphism ρ_S of S has image Γ, and is thus in particular discrete (though non necessarily faithful if Ω is not simply connected). It is relatively easy to check that g and h_∞ satisfy a ping-pong configuration on $\mathbf{dS}^2$, and that $\mathcal{T}^*_{\theta,\infty}$ is therefore a Kleinian punctured torus. However, the following claim shows that this is far to be the case for any x, and that the behaviour of $\rho_{\theta,x}$ and of the family of $\mathbf{dS}^2$-structures $\mathcal{T}^*_{\theta,x}$ is very diverse.

Lemma 3.15. *Let $x \in \,]1\,;\infty]$ be such that $-2 < \mathrm{tr}(gh_x) < 0$.*

(1) Then gh_x is elliptic, and ρ is not both discrete and faithful.
(2) There exists $x \in \,]1\,;\infty]$ such that $\mathrm{tr}(gh) \notin 2\cos(2\pi\mathbb{Q})$, and then $\mathcal{T}^*_{\theta,x}$ is not Kleinian.

Proof. (1) The condition on $\mathrm{tr}(gh_x)$ is a classical characterization of elliptic elements of $\mathrm{PSL}_2(\mathbb{R})$. Now if ρ was by contradiction both discrete and faithful, then the subgroup $\langle gh\rangle$ generated by gh would be both contained in the compact one-parameter elliptic subgroup of $\mathrm{PSL}_2(\mathbb{R})$ containing gh and in the discrete subgroup $\rho(\pi_1(\mathcal{T}^*_{\theta,x}))$, and would thus be finite. In particular gh would have finite order, contradicting the fact that ρ is injective. This contradiction concludes the proof of the first claim.
(2) If $\mathcal{T}^*_{\theta,x}$ is Kleinian, then $\rho(\pi_1(\mathcal{T}^*_{\theta,x}))$ hence $\langle gh\rangle$ is discrete. Since gh_x is elliptic, this forces it to have finite order, therefore $\mathrm{tr}(gh_x) = 2\cos(\nu)$ for some angle ν such that $k\nu = 2n\pi$ for some $(k,n) \in \mathbb{N}^* \times \mathbb{Z}$, and thus $\mathrm{tr}(gh) \in 2\cos(2\pi\mathbb{Q})$. By continuity of $x \mapsto \mathrm{tr}(gh_x)$, there exists x such that $\mathrm{tr}(gh) \notin 2\cos(2\pi\mathbb{Q})$, and then $\mathcal{T}^*_{\theta,x}$ is not Kleinian. □

Since $\pi_1(\mathcal{T}^*_{\theta,x})$ is free, $\rho\colon \pi_1(\mathcal{T}^*_{\theta,x}) \to \mathrm{PSL}_2(\mathbb{R})$ lifts to a representation into $\mathrm{SL}_2(\mathbb{R})$, and singular $\mathbf{dS}^2$-tori give thus a new geometric interpretation to the representations ρ of a rank-two free group $\langle a,b\rangle$ into $\mathrm{SL}_2(\mathbb{R})$, for which $\rho(a)$ and $\rho(b)$ are hyperbolic and $\mathrm{tr}(\rho(aba^{-1}b^{-1})) > 2$. We refer to the seminal work [Gol03] where such representations were thoroughly studied.

3.3. **A two-parameter family of $\mathbf{dS}^2$-tori with one singularity.** Our goal being to eventually construct singular $\mathbf{dS}^2$-tori with one singularity both of whose lightlike foliations are minimal, we should first make sure that both leaves of the singularity are non-closed. To this end we fix $0 < y \leq y_\theta$ and $1 < x \leq \infty$, and we apply the Proposition 3.4 to the "L-shaped polygon"

$$\mathcal{L}_{\theta,x,y} := \mathcal{R}_{(1,\infty,0,y_+)} \setminus \,]x\,;\infty] \times \,]y\,;y_+] \subset \mathbf{dS}^2$$

of area θ illustrated in Figure 3.2. The point

$$y_+ = y_{+(x,y)} := \frac{-x + e^\theta(x-y)}{-1 + e^\theta(x-y)} \in [y_\theta\,;1[$$

is fixed by (x,y), and is the unique one so that $\mathcal{A}_{\boldsymbol{\mu}}(\mathcal{L}_{\theta,x,y}) = \theta$. We emphasize that, conversely to lightlike rectangles, the orbit space of L-shaped polygons of area θ under the action of $\mathrm{PSL}_2(\mathbb{R})$ is not trivial but two-dimensional, and is parametrized by (x,y).

3.3.1. *A pair of HIETs.* As we previously did for the rectangle $\mathcal{R}_\theta$, we want to glue the edges of $\mathcal{L}_{\theta,x,y}$ through HIETs of the intervals $I = [1\,;\infty[$ and $J = [0\,;y_+[$ exchanging the two components of their top and bottom partitions defined by

$$\begin{cases} I_1^t = [1\,;x'[, I_2^t = [x'\,;\infty[,\ I_1^b = [1\,;x[, I_2^b = [x\,;\infty[, \\ J_1^t = [0\,;y'[, J_2^t = [y'\,;y_+[,\ J_1^b = [0\,;y[, J_2^b = [y\,;y_+[, \end{cases}$$

for $x' \in [1\,;\infty]$ and $y' \in [0\,;y_+]$. We denote by $h_1 = h_{1(x,x',y)}$ and $h_2 = h_{2(x,x',y)}$ the unique elements of $\mathrm{PSL}_2(\mathbb{R})$ realizing the gluing of the α-edges of $\mathcal{L}_{\theta,x,y}$ according to these partitions, characterized by

$$h_1(I_1^t \times \{0\}) = I_2^b \times \{y\} \text{ and } h_2(I_2^t \times \{0\}) = I_1^b \times \{y_+\}$$

or equivalently by

$$h_1(1,x',0) = (x,\infty,y) \text{ and } h_2(x',\infty,0) = (1,x,y_+). \tag{3.8}$$

We denote in the same way by (g_1, g_2) the elements of $\mathrm{PSL}_2(\mathbb{R})$ realizing the gluing of the β-edges and illustrated in Figure 3.2.

We can then form the quotient of $\mathcal{L}_{\theta,x,y}$ by these gluings as described in Proposition 3.4, and compute the holonomy around the vertices of $\mathcal{L}_{\theta,x,y}$. Formula (3.4) indicate us that $C = [1, y']$ and $B = [x', 0]$ are regular points in the quotient if, and only if

$$g_1 = h_2 h_1 h_2^{-1} \text{ and } g_2 = h_1 h_2^{-1}.$$

These two relations impose two equations on (x, y, x', y'), given by the following Lemma which follows from direct computations similar to the ones detailed in Lemma 3.11.

Lemma 3.16. *(1) $h_1 h_2^{-1}$ and h_2 are hyperbolic.*

(2) $h_2 h_1 h_2^{-1}(0) = y$ if, and only if $x' = \frac{x}{e^\theta(y-1)+x}$ ($= 1$ if $x = \infty$).

(3) $\frac{x}{e^\theta(y-1)+x} > 1$ if, and only if $y > 1 - e^{-\theta}x$.

(4) If $x' = \frac{x}{e^\theta(y-1)+x}$ and $y > 1 - e^{-\theta}x$, then $h_2 h_1^{-1}(0) = \frac{x+e^\theta x(y-1)}{1+e^\theta x(y-1)+y(x-1)} \in [0\,;y_+[$.

We thus fix henceforth $x \in\,]1\,;\infty]$ and $y \in\,]1 - e^{-\theta}x\,;y_\theta[$, and define

$$\text{(3.9)} \qquad \begin{cases} x' = x'_{(x,y)} := \frac{x}{e^\theta(y-1)+x}, \\ h_1 = h_{1(x,y)} := h_{1\left(x,x'_{(x,y)},y\right)}, h_2 = h_{2(x,y)} := h_{2\left(x,x'_{(x,y)},y\right)}, \\ y' := h_2 h_1^{-1}(0) \\ g_1 := h_2 h_1 h_2^{-1}, g_2 := h_1 h_2^{-1}. \end{cases}$$

Then according to Lemma 3.16.(3) and (4): $x' \in [1\,;\infty]$ and $y' \in [0\,;y_+[$. Moreover according to Lemma 3.16.(2) and the definition of h_1 and h_2 in (3.8) we have

$$\text{(3.10)} \qquad g_1(1, 0, y') = (x, y, y_+) \text{ and } g_2(1, y', y_+) = (\infty, 0, y).$$

This allows us to define a pair $E = E_{x,y} \colon I \to I$ and $F = F_{x,y} \colon J \to J$ of HIET with two components by

$$\text{(3.11)} \qquad \begin{cases} E_{x,y}|_{I_1^t} = h_{1(x,y)}|_{I_1^t} \colon I_1^t \to I_2^b \text{ and } E_{x,y}|_{I_2^t} = h_{2(x,y)}|_{I_2^t} \colon I_2^t \to I_1^b, \\ F_{x,y}|_{J_1^t} = g_{1(x,y)}|_{J_1^t} \colon J_1^t \to J_2^b \text{ and } F_{x,y}|_{J_2^t} = g_{2(x,y)}|_{J_2^t} \colon J_2^t \to J_1^b. \end{cases}$$

3.3.2. *Gluing of the L-shaped polygon.* We can now form the quotient $\mathcal{T}_{\theta,x,y}$ of $\mathcal{L}_{\theta,x,y}$ by the following edges identifications, given by E and F and illustrated in Figure 3.2:

$$\begin{cases} [1\,;x'[\, \times \{0\} \ni (p, 0) \sim (h_1(p), y) \in [x\,;\infty[\, \times \{y\}, [x'\,;\infty[\, \times \{0\} \ni (p, 0) \sim (h_2(p), y_+) \in [1\,;x[\, \times \{y_+\}, \\ \{1\} \times [0\,;y'[\, \ni (1, p) \sim (x, g_1(p)) \in \{x\} \times [y\,;y_+[, \{1\} \times [y'\,;y_+[\, \ni (1, p) \sim (\infty, g_2(p)) \in \{\infty\} \times [0\,;y[. \end{cases}$$

The following result summarizes this construction, and follows from Proposition 3.4.

Proposition 3.17. *For any $\theta \in \mathbb{R}_+^*$ and (x, y) in*

$$\text{(3.12)} \qquad \mathcal{D} := \left\{(x, y) \in [1\,;\infty] \times\,]0\,;y_\theta] \;\middle|\; y > 1 - e^{-\theta}x\right\} \cup (\{\infty\} \times [0\,;y_\theta]) \cup ([1\,;\infty] \times \{y_\theta\}),$$

$\mathcal{T}_{\theta,x,y}$ is homeomorphic to $\mathbf{T}^2$ and the $\mathbf{dS}^2$-structure of the interior of $\mathcal{L}_{\theta,x,y}$ extends to a unique singular $\mathbf{dS}^2$-structure on $\mathcal{T}_{\theta,x,y}$. The latter has area θ, $[1, 0]$ is its unique singular point and it has angle θ.

3.3.3. *At the boundary of the domain.* Let us investigate what happens at the boundary of the domain $\mathcal{D}$ where our parameters (x, y) take their values.

If $x = \infty$ and $y \in [0\,;y_\theta]$: Then $y_+ = y_\theta$ hence $\mathcal{L}_{\theta,\infty,y} = \mathcal{R}_\theta$, $x' = 1$, $E := h_2|_I$, and $\mathcal{T}_{\theta,\infty,y}$ is an example of the form $\mathcal{T}_{\theta,y,*}$ described in Remark 3.13.

If $x \in [1\,;\infty]$ and $y = y_\theta$: Then $y_+ = y = y_\theta$ hence $\mathcal{L}_{\theta,x,y_\theta} = \mathcal{R}_\theta$, $y' = 0$, $F := g_2|_J$, and $\mathcal{T}_{\theta,x,y_\theta}$ is simply the quotient $\mathcal{T}_{\theta,x}$ constructed in paragraph 3.2.1.

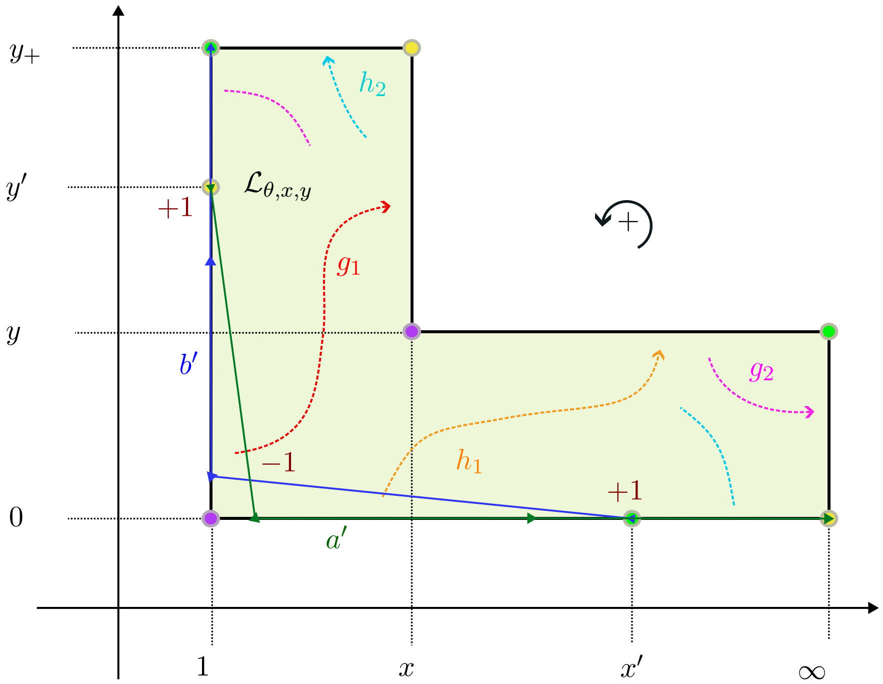

FIGURE 3.2. $\mathbf{dS}^2$-torus with one singularity and two minimal foliations.

If $x \in [e^\theta\,;\infty[$ and $y = 0$: Then $y' = y_+ \in \,]0\,;1[$ and $\mathcal{L}_{\theta,\infty,y}$ is degenerated, hence $\mathcal{T}_{\theta,x,0}$ is not defined, but $F \coloneqq g_1|_J$ is well-defined. According to (3.9), $x'_{x,0} = \frac{x}{x-e^\theta} \in [1\,;\infty]$, hence $E_{x,0}$ is well-defined except for $x = e^\theta$, where $x' = \infty$ and $I_2^t = \varnothing$. However, we can still define then $E^{-1}_{e^\theta,0} \colon I \to I$ by

$$E^{-1}_{e^\theta,0}|_{I_2^b} = h_1^{-1}|_{I_2^b} \text{ and } E^{-1}_{e^\theta,0}|_{I_1^b} \equiv \infty, \tag{3.13}$$

so that $E^{-1}_{x,y}$ converges to $E^{-1}_{x_0,0}$ when $(x,y) \in \mathcal{D}$ converges to $(x_0,0) \in [e^\theta\,;\infty[\times \{0\}$.

If $x \in \,]1\,;e^\theta]$ and $y = 1 - e^{-\theta}x$: Then $x' = \infty$, hence E and $\mathcal{T}_{\theta,x,1-e^{-\theta}x}$ are not defined. However F is well-defined, and we can moreover define $E^{-1} \colon I \to I$ as in (3.13), with the same continuity property.

3.4. **Rotation numbers.** Our goal is to prescribe the dynamics of the lightlike foliations of the $\mathbf{dS}^2$-tori that we constructed. They will be essentially characterized by those of the HIET's that we suspended to construct our examples, and in the end by the dynamics of circle homeomorphisms induced by these HIET's. Therefore, we introduce now the basic invariant of circle homeomorphisms, namely the rotation number.

3.4.1. *From HIET to circle homeomorphisms and rotation numbers.* We see the circle as the additive group $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$, denote by $\pi \colon \mathbb{R} \to \mathbf{S}^1$ the canonical projection when we need it, and also use the notation $[x] \coloneqq \pi(x) \in \mathbf{S}^1$ for $x \in \mathbb{R}$. We endow $\mathbf{S}^1$ with the orientation induced by the one of $\mathbb{R}$, for which a continuous map $f \colon I \to \mathbf{S}^1$, I being an interval of $\mathbb{R}$, is *non-decreasing* if for any lift $F \colon I \to \mathbb{R}$ of f, F is non-decreasing. In the same way a continuous map $f \colon \mathbf{S}^1 \to \mathbf{S}^1$ is non-decreasing if any lift $F \colon \mathbb{R} \to \mathbb{R}$ of f is so. We adopt the natural analogous definitions for non-increasing, and strictly increasing or decreasing maps.

Any HIET E of an interval $I = [a\,;b[\,\subset \mathbb{R}\mathbf{P}^1$ with one or two components naturally induces a bijection E of the quotient $\mathbf{S}^1_I \coloneqq [a\,;b]/\{a \sim b\}$, defined by

$$\forall p \in I, \mathsf{E}([p]) = [E(p)].$$

$\mathbf{S}^1_I$ is homeomorphic to the circle $\mathbf{S}^1$ and bears a natural orientation induced by the one of I, and it is moreover easily checked that E is an *orientation-preserving homeomorphism* of $\mathbf{S}^1_I$ (since the HIET E exchanges at most two components).

If $f\in\mathrm{Homeo}^+(\mathbf{S}^1)$ is an orientation-preserving homeomorphism of the circle, then any lift $F\colon\mathbb{R}\to\mathbb{R}$ of f is a strictly increasing homeomorphism of $\mathbb{R}$ commuting with every integer translation $T_n\colon x\in\mathbb{R}\mapsto x+n\in\mathbb{R}$ ($n\in\mathbb{Z}$). Following [Her79] and the literature, we denote by $\mathrm{D}(\mathbf{S}^1)$ the subgroup of all such homeomophisms of $\mathbb{R}$, *i.e.* of all the lifts of elements of $\mathrm{Homeo}^+(\mathbf{S}^1)$ to $\mathbb{R}$ ($\mathrm{D}(\mathbf{S}^1)$ is precisely the centralizer in $\mathrm{Homeo}^+(\mathbb{R})$ of the translation T_1). Denoting by $\pi(F)\in\mathrm{Homeo}^+(\mathbf{S}^1)$ the map $\pi(F)([x])=[F(x)]$, we have a short exact sequence

$$0\to\{T_n\mid n\in\mathbb{Z}\}\to\mathrm{D}(\mathbf{S}^1)\xrightarrow{\pi}\mathrm{Homeo}^+(\mathbf{S}^1)\to 0.$$

The *translation number* of $F\in\mathrm{D}(\mathbf{S}^1)$ is the asymptotic average amount by which F translates the points of $\mathbb{R}$. We refer to [Her79, II.2 p.20] and [dFG22, §2.1] for a proof of the following classical results.

Proposition-Definition 3.18. *Let $f,g\in\mathrm{Homeo}^+(\mathbf{S}^1)$ and $F\in\mathrm{D}(\mathbf{S}^1)$ be any lift of f.*

(1) The limit

$$\tau(F)=\lim_{n\to\pm\infty}\frac{F^n(x)-x}{n} \tag{3.14}$$

exists for any $x\in\mathbb{R}$, is independent of x, and is uniform on $\mathbb{R}$. It is called the translation number of *F.*

(2) If $G=F+d$ is another lift of f ($d\in\mathbb{Z}$), then $\tau(G)=\tau(F)+d$, and

$$\rho(f)=[\tau(F)]\in\mathbf{S}^1$$

is called the rotation number *of f.*

(3) The maps $F\in\mathrm{D}(\mathbf{S}^1)\to\tau(F)\in\mathbb{R}$ and $f\in\mathrm{Homeo}^+(\mathbf{S}^1)\to\rho(f)\in\mathbf{S}^1$ are continuous for the compact-open topology.

(4) Moreover ρ is a conjugacy invariant: $\rho(g\circ f\circ g^{-1})=\rho(f)$.

The following simple observation will be useful to us all along this text.

Lemma 3.19. *Let C be an oriented topological circle and $f\in\mathrm{Homeo}^+(C)$. Then for any orientation-preserving homeomorphisms $\varphi_1,\varphi_2\colon C\to\mathbf{S}^1$: $\rho(\varphi_1\circ f\circ\varphi_1^{-1})=\rho(\varphi_2\circ f\circ\varphi_2^{-1})$. This common number will still be called the* rotation number of *f and be denoted by $\rho(f)\in\mathbf{S}^1$.*

Proof. Since $\varphi_2\circ f\circ\varphi_2^{-1}=\varphi\circ(\varphi_1\circ f\circ\varphi_1^{-1})\circ\varphi^{-1}$ with $\varphi=\varphi_2\circ\varphi_1^{-1}\in\mathrm{Homeo}^+(\mathbf{S}^1)$, the claim follows from Proposition 3.18.(4). ☐

Lemma 3.20. *In the $\mathbf{dS}^2$-tori $\mathcal{T}_{\theta,x}$ constructed in Proposition 3.12, $\mathsf{E}_x^{-1}\in\mathrm{Homeo}^+\left(\mathbf{S}^1_{[1;\infty]}\right)$ is the first-return map of the β-foliation on the closed α-leaf $[1\,;\infty]\times\{y_\theta\}$. Moreover if E has irrational rotation number $\rho\in\mathbf{S}^1$, then it is $\mathcal{C}^0$-conjugated to the rotation $R_\rho\colon x\in\mathbf{S}^1\mapsto x+\rho\in\mathbf{S}^1$.*

Proof. The first claim follows directly from the construction of $\mathcal{T}_{\theta,x}$. Since $\mathcal{F}_\beta$ is the suspension of E, the second claim is a direct consequence of Lemma 2.30.(4). ☐

3.4.2. *Rotation numbers as cyclic ordering of the orbits.* For $\theta\in\mathbb{R}$, we will say that a sequence $(p_n)_{n\in\mathbb{Z}}$ in $\mathbf{S}^1$ is *of cyclic order* $[\theta]\in\mathbf{S}^1$ if it is cyclically ordered as an orbit of $R_{[\theta]}$, namely if for any $(n_1,n_2,n_3)\in\mathbb{Z}^3$: the three points $(p_{n_1},p_{n_2},p_{n_3})\in(\mathbf{S}^1)^3$ are two-by-two distinct and positively cyclically ordered if, and only if $(R_{[\theta]}^{n_1}([0]),R_{[\theta]}^{n_2}([0]),R_{[\theta]}^{n_3}([0]))=([n_1\theta],[n_2\theta],[n_3\theta])$ are such. We will henceforth assume every rational $\frac{p}{q}\in\mathbb{Q}$ to be *written in reduced form*, *i.e.* such that:

– either $\frac{p}{q}=0$ and then $(p,q)=(0,1)$;
– or $p\in\mathbb{Z}^*$, $q\in\mathbb{N}^*$ and p,q are coprimes.

We refer to [dFG22, §1.1] and [dMvS93, I.1] for a proof of the following classical results.

Proposition 3.21. *Let $T\in\mathrm{Homeo}^+(\mathbf{S}^1)$.*

(1) $\rho(T)=[\frac{p}{q}]\in[\mathbb{Q}]$ if, and only if there exists a periodic orbit of T of cyclic order $[\frac{p}{q}]$. Moreover if this is the case, then any periodic orbit of T is of this form, and has thus in particular minimal period q. In particular, $\rho(T)=[0]$ if, and only if T has a fixed point.

(2) $\rho(T)=\theta\in[\mathbb{R}\setminus\mathbb{Q}]$ if, and only if any orbit of T is of cyclic order θ.

3.5. **Realization of rotation numbers.** We now come back to the HIETs that we suspended in paragraphs 3.2 and 3.3, and show existence results for their rotation numbers.

3.5.1. *Rotation number for a single HIET.* We will use the notations of the paragraph 3.2.1. For any $x \in [1\,;\infty]$, we consider the orientation-preserving homeomorphism E_x of $\mathbf{S}^1_I := [1\,;\infty]/\{1 \sim \infty\}$ induced by the HIET E_x of $I = [1\,;\infty[$ defined in (3.7).

Note that when x converges to 1, x'_x converges to ∞ and gh_x to $h_\infty = gh_1$, since

$$gh_x(1, x'_x, 0) = (x, \infty, y_\theta).$$

Hence E_x converges to $\mathsf{E}_1 = \mathsf{E}_\infty$ for the compact-open topology of $\mathrm{Homeo}^+(\mathbf{S}^1_I)$ when $x \to 1$, and the map

$$\mathsf{E}\colon [x] \in \mathbf{S}^1_I \mapsto \mathsf{E}_x \in \mathrm{Homeo}^+(\mathbf{S}^1_I) \tag{3.15}$$

is therefore continuous.

Let $\{g^t\}_{t\in\mathbb{R}} \subset \mathrm{PSL}_2(\mathbb{R})$ denote the one-parameter hyperbolic subgroup containing g, parametrized so that $g = g^1$ (with g defined by (3.1)).

Lemma 3.22. *Let $x_1 \leq x_2 \in [1\,;\infty]$.*

(1) $h_{x_1}^{-1} g h_{x_1} g^{-1} = h_{x_2}^{-1} g h_{x_2} g^{-1}$.

(2) There exists a unique $\tau \in [0\,;1]$ such that $x_2 = g^\tau(x_1)$, and $h_{x_2} = g^\tau h_{x_1}$.

(3) Moreover $E_{x_2} = S_\tau \circ E_{x_1}$, with S_τ the HIET defined by

$$\begin{cases} \forall p \in [1\,; E_{x_1}(x'_2)[, S_\tau(p) = g^\tau(p) \in [g^\tau(1)\,;\infty[, \\ \forall p \in [E_{x_1}(x'_2)\,;\infty[, S_\tau(p) = g^{\tau-1}(p) \in [1\,; g^\tau(1)[. \end{cases}$$

Proof. (1) According to Proposition 3.2, the holonomy around $[\infty, 0]$ in $\mathcal{T}_{\theta,x_i}$ is equal to $h_{x_i}^{-1} g h_{x_i} g^{-1}$ (for a developing map compatible at $[\infty, 0]$, see Lemma 2.14), hence $h_{x_1}^{-1} g h_{x_1} g^{-1} = a^\theta = h_{x_2}^{-1} g h_{x_2} g^{-1}$. Note that this extends to the case $x_1 = 1$ since by definition of h_1 we have $h_1^{-1} g h_1 g^{-1} = (h_\infty^{-1} g) g (g^{-1} h_\infty) g^{-1} = h_\infty^{-1} g h_\infty g^{-1}$.
(2) According to (1), $hgh^{-1} = g$ with $h = h_{x_2} h_{x_1}^{-1}$. Hence h is in the centralizer of $g = g^1$ in $\mathrm{PSL}_2(\mathbb{R})$, which is equal to $\{g^t\}_t$. Now if $h_{x_2} = g^\tau h_{x_1}$ we obtain directly from (3.5) that $x_2 = g^\tau(x_1)$. Moreover $g^1(1) = \infty$ according to (3.1), and thus $\tau \in [0\,;1]$ since $x_1, x_2 \in [1\,;\infty]$.
(3) Indeed for any $p \in [1\,; x_1[$, $E_{x_1}^{-1}(p) = H_1^{-1}(p) \in [x'_1\,;\infty[$, and $x'_2 < x'_1$ hence $E_{x_2} \circ E_{x_1}^{-1}(p) = H_2 H_1^{-1}(p) = g^\tau(p) \in [g^\tau(1)\,; x_2[$. Note that $gH_1(x'_2) \in]x_1\,;\infty]$, so that for $p \in [x_1\,; gH_1(x'_2)[$, $E_{x_1}^{-1}(p) = H_1^{-1} g^{-1}(p) \in [1\,; x'_2[$ and $E_{x_2} \circ E_{x_1}^{-1}(p) = gH_2 H_1^{-1} g^{-1}(p) = g^\tau(p) \in [x_2\,;\infty[$. Lastly for $p \in [gH_1(x'_2)\,;\infty[$, $E_{x_1}^{-1}(p) = H_1^{-1} g^{-1}(p) \in [x'_2\,; x'_1[$, and thus $E_{x_2} \circ E_{x_1}^{-1}(p) = g^\tau H_1 H_1^{-1} g^{-1}(p) = g^{\tau-1}(p) \in [x_2\,;\infty[$. □

Proposition 3.23. *The map $[x] \in \mathbf{S}^1_I \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1$ is continuous, non-decreasing, and has degree one (in particular, it is surjective). Moreover it is strictly increasing at any x for which $\rho(\mathsf{E}_x) \in [\mathbb{R} \setminus \mathbb{Q}]$. In particular for any $u \in [\mathbb{R} \setminus \mathbb{Q}]$, there exists a unique $[x] \in \mathbf{S}^1_I$ such that $\rho(\mathsf{E}_x) = u$. Lastly, for any $r \in [\mathbb{Q}]$ there exists $x \in [1\,;\infty]$ such that the orbit of $[1, 0]$ under E_x is periodic and of cyclic order r.*

Proof. The continuity of $x \in [1\,;\infty] \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1$ follows from the continuity of E (see (3.15)) and of the rotation number itself (see for instance [Her79, Proposition 2.7]), for the compact-open topology of $\mathrm{Homeo}^+(\mathbf{S}^1_I)$. Note that both E_1 and E_∞ have $[1] \in \mathbf{S}^1_I$ as a fixed point, and thus that $\rho(\mathsf{E}_1) = \rho(\mathsf{E}_\infty) = [0] \in \mathbf{S}^1$. On the other hand it is easily checked that for any $x \in]1\,;\infty[$, E_x does not have any fixed point and thus that $\rho(\mathsf{E}_x) \neq [0]$. In particular, $x \mapsto \rho(\mathsf{E}_x)$ is not constant. According to Lemma 3.22.(3), we have moreover $\mathsf{E}_{g^\tau(1)} = S_\tau \circ \mathsf{E}_1$ with $\tau \in [0\,;1] \mapsto S_\tau \in \mathrm{Homeo}^+(\mathbf{S}^1_I)$ a continuous map such that $\tau \in [0\,;1] \mapsto S_\tau(p) \in \mathbf{S}^1_I$ is strictly increasing for any $p \in \mathbf{S}^1_I$. According to Lemma B.1, $x \in [1\,;\infty] \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1_I$ is thus non-decreasing. But since it is also not constant and attains the same value $[0]$ at 1 and ∞, it is actually surjective according to the Intermediate value theorem. The value $[0]$ being attained only at the point $[1] = [\infty]$ of $\mathbf{S}^1_I$, the map $[x] \in \mathbf{S}^1_I \mapsto \rho(\mathsf{E}_x) \in \mathbf{S}^1$ has moreover degree one. It is also strictly increasing at any x for which $\rho(\mathsf{E}_x) \in [\mathbb{R} \setminus \mathbb{Q}]$ according to Lemma B.1, which forbids any element of $[\mathbb{R} \setminus \mathbb{Q}]$ to have more than one pre-image in $\mathbf{S}^1_I$ since the map also has degree one. By surjectivity, there exists $[x] \in \mathbf{S}^1_I$ such that $\rho(\mathsf{E}_x)$ is irrational, and since E_x is a $\mathcal{C}^\infty$-diffeomorphism with breaks it is then minimal according to Denjoy theorem (see also Lemma 2.30.(4)). The existence of periodic orbits of any rational cyclic order under the maps E_x for $[1,0]$ follows then from Lemma B.1.(5), which concludes the proof of the Proposition. □

3.5.2. *Rotation numbers for a pair of HIET.* We now want to realize rotation numbers for the pair (E,F) of HIETs introduced in paragraph 3.3.1. For any $(x,y) \in \mathcal{D}$ (defined in (3.12)), we consider the orientation-preserving homeomorphisms $\mathsf{E}_{x,y}$ of $\mathbf{S}^1_I := [1\,;\infty]/\{1 \sim \infty\}$ and $\mathsf{F}_{x,y}$ of $\mathbf{S}^1_J := [0\,;y_+]/\{0 \sim y_+\}$ induced by the HIETs $E_{x,y}$ and $F_{x,y}$ defined in (3.11). Note that with the definitions introduced in paragraph 3.3.3 for $(x,y) \in \mathrm{Cl}(\mathcal{D})$, $\mathsf{F}_{x,y}$ is a well-defined orientation-preserving homeomorphism of $\mathbf{S}^1_J$. On the other hand for $x \in \,]1\,;e^\theta]$ and $y = 1 - e^{-\theta}x$, $\mathsf{E}^{-1}_{x,y}$ is a well-defined orientation-preserving *endomorphism* of $\mathbf{S}^1_I$, *i.e.* by definition a continuous, degree-one and orientation-preserving self-map of $\mathbf{S}^1_I$. Equivalently, f is an orientation-preserving endomorphism of $\mathbf{S}^1$ if it admits a lift F to $\mathbb{R}$ which is a continuous, non-decreasing self-map of $\mathbb{R}$ commuting with integer translations. According to [PJM82, Appendix Lemma 3] and [NPT83, Chapter III Proposition 3.3], the Proposition-Definition 3.18 defining the rotation number extends to endomorphisms of $\mathbf{S}^1$, and the rotation number $\rho(\mathsf{E}^{-1}_{x,y})$ is thus well-defined. Lastly, the maps

$$\mathsf{E}^{-1}\colon (x,y) \in \mathrm{Cl}(\mathcal{D}) \mapsto \mathsf{E}^{-1}_{x,y} \in \mathrm{End}^+(\mathbf{S}^1_I) \text{ and } \mathsf{F}\colon (x,y) \in \mathrm{Cl}(\mathcal{D}) \mapsto \mathsf{F}_{x,y} \in \mathrm{Homeo}^+(\mathbf{S}^1_I)$$

are continuous. The author want to thank Florestan Martin-Baillon, who helped him to obtain a more elegant proof for this result than in a first version.

Proposition 3.24. *The map $(x,y) \in \mathcal{D} \mapsto (\rho(\mathsf{E}_{x,y}), \rho(\mathsf{F}_{x,y})) \in (\mathbf{S}^1)^2$ is continuous and surjective.*

Proof. Since the maps E^{-1} and F are continuous, and such is the rotation number as well according to [NPT83, Chapter III Proposition 3.3], the map

$$R\colon (x,y) \in \mathrm{Cl}(\mathcal{D}) \mapsto (\rho(\mathsf{E}^{-1}_{x,y}), \rho(\mathsf{F}_{x,y})) \in (\mathbf{S}^1)^2$$

is continuous. We recall that $\rho(T^{-1}) = \rho(T)^{-1}$ for any $T \in \mathrm{Homeo}^+(\mathbf{S}^1)$ (see for instance [dFG22, §2.1]). We begin by investigating what happens for the rotation numbers on the boundary of $\mathcal{D}$, as we did in paragraph 3.3.3.

If $x = \infty$ and $y \in [0\,;y_\theta]$: Then $\rho(\mathsf{E}^{-1}_{\infty,y}) = [0]$ since $[1]$ is a fixed point of $\mathsf{E}^{-1}_{\infty,y}$, and

$$y \in \mathbf{S}^1_J \mapsto \rho(\mathsf{F}_{\infty,y}) \in \mathbf{S}^1$$

is a continuous degree-one map as we proved in Proposition 3.23.

If $x \in [1\,;\infty]$ and $y = y_\theta$: Then $\rho(\mathsf{F}_{\infty,y}) = [0]$ since $[0]$ is a fixed point of F_{x,y_θ}, and

$$x \in \mathbf{S}^1_I \mapsto \rho(\mathsf{E}^{-1}_{x,y_\theta})) \in \mathbf{S}^1$$

is a continuous degree-one map as we proved in Proposition 3.23.

If $x \in [e^\theta\,;\infty[$ and $y = 0$: Then $\rho(\mathsf{F}_{\infty,y}) = [0]$ since $[0]$ is a fixed point of F_{x,y_θ}. On the other hand $x \in [e^\theta\,;\infty] \mapsto x'_{x,0} = \frac{x}{x-e^\theta} \in [1\,;\infty]$ is surjective (see (3.9)), $\rho(\mathsf{E}^{-1}_{e^\theta,0}) = \rho(\mathsf{E}^{-1}_{\infty,0}) = [0]$ since $[\infty]$ is a fixed point of both, and $\rho(\mathsf{E}^{-1}_{x,0}) \neq [0]$ for any $x \in \,]e^\theta\,;\infty[$ since $\mathsf{E}^{-1}_{x,0}$ has no fixed points. Therefore, the same argument than in Proposition 3.23 shows that

$$[x] \in [e^\theta\,;\infty]/\{e^\theta \sim \infty\} \mapsto \rho(\mathsf{E}^{-1}_{x,0}) \in \mathbf{S}^1$$

is a continuous monotonous map with value $[0]$ only at $x = e^\theta$ and $x = \infty$, hence a degree-one map.

If $x \in \,]1\,;e^\theta]$ and $y = 1 - e^{-\theta}x$: Then $x' = \infty$, hence $[1] = [\infty]$ is a fixed point of $\mathsf{E}^{-1}_{\infty,y}$, and therefore $\rho(\mathsf{E}^{-1}_{x,1-e^{-\theta}x}) = [0]$. On the other hand $x \in [1\,;e^\theta] \mapsto y(x) := 1 - e^{-\theta}x \in [0\,;y_\theta]$ is surjective, with $y = y_+ = y_\theta$ for $x = 1$, and $(y = 0, y' = y_+)$ for $x = e^\theta$. The same argument than in Proposition 3.23 shows thus that

$$[x] \in [1\,;e^\theta]/\{1 \sim e^\theta\} \mapsto \rho(\mathsf{F}_{\infty,y}) \in \mathbf{S}^1$$

is a continuous monotonous map with value $[0]$ only at $x = 1$ and $x = e^\theta$, hence a degree-one map.

We conclude from this description that there exists two continuous monotonous and surjective maps $f_h\colon [e^\theta\,;\infty]\times\{0\}\to[1\,;\infty]\times\{y_\theta\}$ and $f_v\colon \left\{(x,1-e^{-\theta}x)\,\middle|\,x\in[1\,;e^\theta]\right\}\to\{\infty\}\times[0\,;y_\theta]$ between the horizontal and vertical edges of $\partial\mathcal{D}$, such that $R\circ f_h=R$ on $[e^\theta\,;\infty]\times\{0\}$ and $R\circ f_v=R$ on $\left\{(x,1-e^{-\theta}x)\,\middle|\,x\in[1\,;e^\theta]\right\}$. R induces therefore a continuous map

$$\bar{R}\colon \mathcal{T}\to(\mathbf{S}^1)^2$$

such that $\bar{R}\circ\pi=R$, with $\pi\colon \mathrm{Cl}(\mathcal{D})\to\mathcal{T}$ the quotient of $\mathrm{Cl}(\mathcal{D})$ by the identifications $p\sim f_h(p)$ and $p\sim f_v(p)$ of its edges under f_h and f_v. Note that $\mathcal{T}$ is homeomorphic to a two-torus.

Assume now by contradiction that the restriction of R to $\mathcal{D}$ misses a point in the torus $(\mathbf{S}^1)^2$. Since our previous description of $R|_{\partial\mathcal{D}}$ shows that $\mathbf{S}^1\times[0]\cup[0]\times\mathbf{S}^1\subset R((\{\infty\}\times[0\,;y_\theta])\cup([1\,;\infty]\times\{y_\theta\}))\subset R(\mathcal{D})$, we have thus $\bar{R}(\mathcal{T})\subset(\mathbf{S}^1)^2\setminus\{p\}$ for some $p\in(\mathbf{S}^1)^2\setminus(\mathbf{S}^1\times[0]\cup[0]\times\mathbf{S}^1)$. Since $(\mathbf{S}^1)^2\setminus\{p\}$ retracts to a bouquet of two circles, its fundamental group is a free group F_2 in two generators represented by the loops $\mathbf{S}^1\times[0]$ and $[0]\times\mathbf{S}^1$, and $\bar{R}$ induces moreover in homotopy a morphism $\bar{R}_*$ from $\pi_1(\mathcal{T})\simeq\mathbb{Z}^2$ to $\pi_1((\mathbf{S}^1)^2\setminus\{p\})=F_2$. The image of this morphism is then an abelian subgroup of F_2. Moreover $\bar{R}_*$ sends the horizontal and vertical generators of $\pi_1(\mathcal{T})$, given by the projections of the horizontal and vertical edges of $\partial\mathcal{D}$, to the respective generators $\mathbf{S}^1\times[0]$ and $[0]\times\mathbf{S}^1$ of the free group $F_2=\pi_1((\mathbf{S}^1)^2\setminus\{p\})$. Since the latter elements do not commute, this contradicts the fact that $\bar{R}_*(\pi_1(\mathcal{T}))$ is abelian and concludes the proof of the Proposition. □

3.6. **Projective asymptotic cycles and class A structures.** Our goal is to prove the existence of singular $\mathbf{dS}^2$-tori whose lightlike foliations are prescribed in terms of an invariant which is in a sense a global version of the rotation number of the first-return map: the *projective asymptotic cycle*. The notion of *asymptotic cycle* was introduced by Schwartzman in [Sch57]. It associates to any suitable orbit O of a topological flow on a closed manifold M, an element of the first homology group of M which is in a sense the "best approximation of O by a closed loop in homology". This notion has a natural projective counterpart for the leaves of an oriented topological one-dimensional foliation $\mathcal{F}$, that we now quickly describe, referring to [Sch57, Yan85] for more details.

We consider an auxiliary smooth Riemannian metric μ on M, the induced metric and its induced distance $d_{\mathcal{F}}$ on the leaves of $\mathcal{F}$. For $x\in M$ and $T\in\mathbb{R}$ we denote by $\gamma_{T,x}$ the closed curve on M obtained by: first following $\mathcal{F}(x)$ from x in the positive direction until the unique point $y\in\mathcal{F}(x)$ such that $d_{\mathcal{F}}(x,y)=T$, and then closing the curve by following the minimal geodesic of μ from y to x. Following [Sch57, Yan85], we then define the *oriented projective asymptotic cycle* of $\mathcal{F}$ at x as the half-line

$$A^+_{\mathcal{F}}(x)\coloneqq\mathbb{R}^+\left(\lim_{T\to+\infty}\frac{1}{T}[\gamma_{T,p}]\right)\in\mathbf{P}^+(\mathrm{H}_1(M,\mathbb{R})) \tag{3.16}$$

in the first homology group of M, if this limit exists and is non-zero. Note that the orientation of $A^+_{\mathcal{F}}(x)$ obviously depends of the orientation of the foliation $\mathcal{F}$, and is reversed when the orientation of $\mathcal{F}$ is. We also denote by $A_{\mathcal{F}}(x)=\mathbb{R}A^+_{\mathcal{F}}(x)$ the unoriented *projective asymptotic cycle*. This line (if it exists) is by definition constant on leaves, does not depend on the auxiliary Riemannian metric, and is moreover natural with respect to any homeomorphism f:

$$A^+_{f_*\mathcal{F}}(f(x))=f_*(A^+_{\mathcal{F}}(x)). \tag{3.17}$$

In particular, any homeomorphism isotopic to the identity acts trivially on projective asymptotic cycles. For these properties of aymptotic cycles, we refer to [Sch57, Theorem p.275] proving the equivalence between the geometric interpretation (3.16) and the equivariant definition.

In the case of foliations on the torus, asymptotic cycles are described by the following result which is a reformulation of [Yan85, Theorem 6.1 and Theorem 6.2]. We identify henceforth $\mathrm{H}_1(\mathbf{T}^2,\mathbb{R})$ with $\mathbb{R}^2$ through the isomorphism induced by the covering map $\mathbb{R}^2\to\mathbf{T}^2=\mathbb{R}^2/\mathbb{Z}^2$, and we say that a line in $\mathrm{H}_1(\mathbf{T}^2,\mathbb{R})$ is *rational* if it passes through a point of the lattice $\mathrm{H}_1(\mathbf{T}^2,\mathbb{Z})=\mathbb{Z}^2$.

Proposition 3.25 ([Yan85])**.** *Let $\mathcal{F}$ be an oriented topological one-dimensional foliation of $\mathbf{T}^2$, which is the suspension of a $\mathcal{C}^\infty$ circle diffeomorphism with breaks.*

(1) *$A^+_{\mathcal{F}}(x)$ exists at any $x \in \mathbf{T}^2$. It is moreover constant on $\mathbf{T}^2$ and will be denoted by $A^+(\mathcal{F})$ (respectively $A(\mathcal{F}) = \mathbb{R}A^+(\mathcal{F})$ for the unoriented asymptotic cycle).*
(2) *If $\mathcal{F}$ has a closed leaf F, then $A^+(\mathcal{F})$ is equal to the homology class $[F]$ of F, and is in particular* rational.
(3) *If $\mathcal{F}$ is the linear oriented foliation induced by a half-line $l \subset \mathbb{R}^2$, then $A^+(\mathcal{F}) = l$.*

We will later apply the notion of projective asymptotic cycle to the lightlike foliations of singular $\mathbf{dS}^2$-structures which are suspensions of circle homeomorphisms. According to Lemma 2.30, these foliations are topologically equivalent to suspensions of $\mathcal{C}^\infty$-diffeomorphisms with breaks and have thus no exceptional minimal set. It will be useful to have in mind a rough classification of such suspensions, that we summarize in the following statement. Those results are well-known, and are for instance proved in [HH86, §4]. We recall that a foliation (respectively a homeomorphism) is said *minimal* if all its leaves (resp. orbits) are dense.

Proposition 3.26. *Let $\mathcal{F}$ be a topological foliation of $\mathbf{T}^2$. Then:*
(1) *either $\mathcal{F}$ has at least one Reeb component, and in this case $\mathcal{F}$ is not minimal;*
(2) *or $\mathcal{F}$ is a suspension.*

Assume now that $\mathcal{F}$ is the suspension of a $\mathcal{C}^\infty$ circle diffeomorphism T with breaks. Then one of the two following exclusive situations arise.
(1) *Either T has rational rotation number, and then $\mathcal{F}$ has closed leaves, all of which are freely homotopic, and every non-closed leaf is past- and future-asymptotic to one of these closed leaves.*
(2) *Or T has irrational rotation number ρ, and then $\mathcal{F}$ is minimal and topologically equivalent to a linear foliation of slope ρ.*

The following result is classical, and we recall its statement for the convenience of the reader.

Lemma 3.27. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two oriented topological foliations of $\mathbf{T}^2$ having the same oriented projective asymptotic cycles, and γ_1, γ_2 be freely homotopic oriented sections of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then the first-return maps on γ_1 and γ_2 have the same rotation number:*

$$\rho(P^{\gamma_1}_{\mathcal{F}_1}) = \rho(P^{\gamma_2}_{\mathcal{F}_2}).$$

The next result state that conversely, the rotation number of the first-return map is locally equivalent to the oriented asymptotic cycle. While well-known by experts of the area, we give a short proof of this fact for the convenience of the reader.

Lemma 3.28. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two oriented topological foliations of $\mathbf{T}^2$, and γ_1, γ_1 be two freely homotopic oriented sections of $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\rho(P^{\gamma_1}_{\mathcal{F}_1}) = \rho(P^{\gamma_2}_{\mathcal{F}_2})$. Then there exists a Dehn twist D of $\mathbf{T}^2$ around γ_2, such that $A^+(\mathcal{F}_1) = A^+(D_*\mathcal{F}_2)$.*

Moreover if $A^+(\mathcal{F}_1) = A^+(\mathcal{F}_2)$, then for any oriented foliations $\mathcal{F}'_1$ and $\mathcal{F}'_2$ respectively sufficiently close to $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\rho(P^{\gamma_1}_{\mathcal{F}'_1}) = \rho(P^{\gamma_2}_{\mathcal{F}'_2}) \Rightarrow A^+(\mathcal{F}'_1) = A^+(\mathcal{F}'_2).$$

Proof. Let γ be a simple closed curve of $\mathbf{T}^2$ of homotopy class b, and $a \in \pi_1(\mathbf{T}^2)$ be any simple closed curve completing b into a basis (a, b) of $\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}) \equiv \mathbb{R}^2$. Then for any suspension $\mathcal{G}$ of $\mathbf{T}^2$ having γ as a section, it is easily checked by lifting $\mathcal{G}$ to $\mathbb{R}^2$ that $\rho(P^\gamma_{\mathcal{G}}) = [u]$ if, and only if there exists $n \in \mathbb{Z}$ such that $A^+(\mathcal{G}) = \mathbb{R}^+[a + (u + n)b]$. Since these rays of $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$ are in the same orbit under the action of Dehn twists around b, $\rho(P^{\gamma_1}_{\mathcal{F}_1}) = \rho(P^{\gamma_2}_{\mathcal{F}_2})$ implies thus the existence of a Dehn twist D around γ_2 such that $A^+(\mathcal{F}_1) = D_*(A^+(\mathcal{F}_2)) = A^+(D_*\mathcal{F}_2)$, the latter equality being due to the naturality (3.17) of the asymptotic cycle with respect to homeomorphisms. This shows the first claim.

In the other hand if $A^+(\mathcal{F}_1) = A^+(\mathcal{F}_2) =: l$, then there exists a neighbourhood U of l in $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$ containing at most one of the half-lines $\{\mathbb{R}^+[a + (u + n)b] \mid n \in \mathbb{Z}\}$ for any $[u] \in \mathbf{S}^1$. Since the oriented asymptotic cycle vary continuously with the foliation, for any oriented foliations $\mathcal{F}'_1$ and $\mathcal{F}'_2$ respectively sufficiently close to $\mathcal{F}_1$ and $\mathcal{F}_2$, $A^+(\mathcal{F}'_1)$ and $A^+(\mathcal{F}'_2)$ are contained in U, and therefore $\rho(P^{\gamma_1}_{\mathcal{F}'_1}) = \rho(P^{\gamma_2}_{\mathcal{F}'_2})$ implies $A^+(\mathcal{F}'_1) = A^+(\mathcal{F}'_2)$ which concludes the proof of the Lemma. ∎

We will say, following [Suh13], that a singular $\mathbf{X}$-surface S is *class A* if the projective asymptotic cycles of its α and β lightlike foliations are distinct: $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$; and that it is *class B* otherwise. All of the structures studied in this text are class A (see Lemma A.8 for more details) and both of their lightlike foliations are moreover suspensions.

3.7. **Parameter families in the deformation space.** We now want to deduce, from the singular $\mathbf{dS}^2$-tori constructed in Propositions 3.12 and 3.17, parameter families of singular $\mathbf{dS}^2$-structures on a *fixed* torus $\mathbf{T}^2$. To achieve this process sometimes described as a *marking*, we first have to introduce a suited deformation space to work in.

3.7.1. *Deformation space of singular* $\mathbf{dS}^2$*-structures.* For any oriented surface S and any set $\Theta = \{\theta_i\}_i$ of angles $\theta_i \in \mathbb{R}$, we denote by $\mathcal{S}(S,\Theta)$ the set of singular $\mathbf{dS}^2$-structures on S whose singular points angles are given by Θ. We will endow $\mathcal{S}(S,\Theta)$ with the usual topology on $(\mathbf{G},\mathbf{X})$-structures, defined as follows (see [CEG87, §1.5] for more details).

Let (S,Σ,μ) be a singular $\mathbf{dS}^2$-surface of singular $\mathbf{dS}^2$-atlas $(\varphi_i\colon U_i \to X_i)_i$, where $X_i = \mathbf{dS}^2$ if φ_i is a regular chart, and $X_i = \mathbf{dS}^2_{\theta_i}$ at a singular point x_i of angle θ_i. Let $(U_i')_i$ be a shrinking of $(U_i)_i$, *i.e.* an open covering of $\mathbf{T}^2$ such that $\overline{U_i'} \subset U_i$ for each i, and assume moreover that for any singular chart $\varphi_i\colon U_i \to X_i$, U_i' contains the unique singular point x_i of U_i. Note that the $\overline{U_i'}$ for singular charts are pairwise disjoint, since the associated U_i are such and $\overline{U_i'} \subset U_i$. Lastly, let $\mathcal{V}_i$ be for any i an open neighbourhood of $\varphi_i|_{U_i'}$ in the compact-open topology of $\mathrm{C}(U_i', X_i)$, small enough so that for any singular chart φ_i of angle θ_i, $\mathsf{o}_{\theta_i} \in \psi(U_i')$ for any $\psi \in \mathcal{V}_i$.

Definition 3.29. The set $\mathcal{S}(S,\Theta)$ of singular $\mathbf{dS}^2$-structures of angles Θ on an oriented surface S is endowed with the topology for which the sets of the form

$$\left\{\mu' \in \mathcal{S}(S,\Theta) \text{ defined by a singular } \mathbf{dS}^2\text{-atlas } \psi_i\colon U_i' \to X_i \;\middle|\; \psi_i \in \mathcal{V}_i\right\}$$

form a sub-basis of the topology, for any initial singular $\mathbf{dS}^2$-structure $(\Sigma,\mu) \in \mathcal{S}(S,\Theta)$ on S, and any choice of shrinking $(U_i')_i$ and of compact-open neighbourhoods $\mathcal{V}_i$ as above. We denote by $\mathcal{S}(S,\Sigma,\Theta) \subset \mathcal{S}(S,\Theta)$ the subspace of singular $\mathbf{dS}^2$-structures on S of (ordered) singular set Σ with (ordered) angles Θ.

Let $\mu \in \mathcal{S}(S,\Sigma,\Theta)$ be a singular $\mathbf{dS}^2$-structure of singular $\mathbf{dS}^2$-atlas (φ_i, U_i). If f is an orientation-preserving homeomorphism of S acting as the identity on Σ, then the singular $\mathbf{dS}^2$-structure $f^*\mu \in \mathcal{S}(S,\Sigma,\Theta)$ is defined by the singular $\mathbf{dS}^2$-atlas $(\varphi_i \circ f, f^{-1}(U_i))$, so that f is an isometry from $(S, f^*\mu)$ to (S,μ). This defines a right action of the subgroup $\mathrm{Homeo}^+(S,\Sigma)$ of orientation-preserving homeomorphisms of S acting as the identity on Σ, on each $\mathcal{S}(S,\Sigma,\Theta)$.

The *deformation space* of singular $\mathbf{dS}^2$-structures on S with singular set Σ of angles Θ, denoted by $\mathsf{Def}_\Theta(S,\Sigma)$, is defined as the quotient of $\mathcal{S}(S,\Sigma,\Theta)$ by the subgroup $\mathrm{Homeo}^0(S,\Sigma) \subset \mathrm{Homeo}^+(S,\Sigma)$ of homeomorphisms of S isotopic to the identity relative to Σ.

We recall that a $f \in \mathrm{Homeo}^+(S,\Sigma)$ is said *isotopic to the identity relative to* Σ, if there exists a continuous family $t \in [0;1] \mapsto f_t \in \mathrm{Homeo}^+(S,\Sigma)$ such that $f_0 = f$ and $f_1 = \mathrm{id}_S$. The quotient $\mathrm{PMod}(S,\Sigma)$ of $\mathrm{Homeo}^+(S,\Sigma)$ by $\mathrm{Homeo}^0(S,\Sigma)$ is called the *pure mapping class group* of (S,Σ) and acts naturally on $\mathsf{Def}_\Theta(S,\Sigma)$. The quotient of this action is the *moduli space* of $\mathbf{dS}^2$-structures on S with singular set Σ of angles Θ.

Remark 3.30. The projective asymptotic cycles of the lightlike foliations of a point $[\mu] \in \mathrm{Def}_\Theta(\mathbf{T}^2,\Sigma)$ in the deformation space is well-defined, since homeomorphisms isotopic to the identity act trivially on projective asymptotic cycles according to (3.17). In particular, the notion of class A and B structures is invariant by the action of $\mathrm{Homeo}^0(S,\Sigma)$, and makes thus sense in $\mathrm{Def}_\Theta(\mathbf{T}^2,\Sigma)$.

Lemma 3.31. *The subset* $\mathrm{Def}_\Theta(\mathbf{T}^2,\Sigma)^A$ *of class A (respectively class B) singular* $\mathbf{dS}^2$*-structures on* $\mathbf{T}^2$ *is a union of connected components of* $\mathrm{Def}_\Theta(\mathbf{T}^2,\Sigma)$*.*

Proof. The condition $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$ of class A structures being clearly open, the set of class A structures is open. In the other hand according to Lemma A.8, if a structure μ is class B then its lightlike α and β foliations respectively have closed leaves F_α and F_β, such that F_α is freely homotopic to $\pm[F_\beta]$. This prevents any of the lightlike foliations to have only closed leaves. Assume indeed by contradiction that $\mathcal{F}_\alpha$ has only closed leaves. Then all of them are freely homotopic to F_α and thus to $\pm[F_\beta]$, which prevents F_β to be transverse to $\mathcal{F}_\alpha$. Hence $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ have both closed and non-closed leaves, and these foliations are thus stable in the sense that any small deformation of them still contains a closed leaf, which is furthermore freely homotopic to the original closed leaf $F_{\alpha/\beta}$. In particular any small deformation of μ remains class B, which shows that the subset of class B structures is open. Since class A and B structures form a partition of all singular $\mathbf{dS}^2$-structures in $\mathcal{S}_\Theta(\mathbf{T}^2,\Sigma)$, this shows in the end that the set of class A (respectively class B) structures is both open and closed, *i.e.* is a union of connected components of $\mathcal{S}_\Theta(\mathbf{T}^2,\Sigma)$. □

3.7.2. *Definition of the markings.* We denote $\mathcal{T}^*_{\theta,x}=\mathcal{T}_{\theta,x}\setminus\{[1,0]\}$ and $\mathcal{T}^*_{\theta,x,y}=\mathcal{T}_{\theta,x,y}\setminus\{[1,0]\}$. Taking the homotopy classes of the loops (γ',η') defined in paragraph 3.2.2 in $\pi_1(\mathcal{T}_{\theta,x})\simeq\mathbb{Z}^2$ and not anymore in $\pi_1(\mathcal{T}^*_{\theta,x})$, we obtain for any x a fixed generating set

$$m_x \coloneqq (a,b)_x$$

of $\pi_1(\mathcal{T}_{\theta,x})$ (denoted in the same way by a slight abuse of notations). It is indeed easy to check that b is freely homotopic to a closed path b' intersecting a at a single point.

In the same way with γ_1 the positively oriented α-lightlike segment of $\mathcal{T}_{\theta,x,y}$ from $[1,0]$ to $[\infty,0]$ and γ_2 the negatively oriented β-lightlike segment from $[\infty,0]=[1,y']$ to $[1,0]$, we denote by a the homotopy class in $\pi_1(\mathcal{T}^*_{\theta,x,y})$ of a small deformation γ' of $\gamma_1\gamma_2$ avoiding $[1,0]$. Lastly with η_1 the positively oriented β-lightlike segment from $[1,0]$ to $[1,y_+]$ and η_2 the negatively oriented α-lightlike segment from $[1,y_+]=[x',0]$ to $[1,0]$, we denote by b the homotopy class in $\pi_1(\mathcal{T}^*_{\theta,x,y})$ of a small deformation η' of $\eta_1\eta_2$ avoiding $[1,0]$. With a slight abuse of notations, we still denote by (a,b) the homotopy classes of these curves in $\pi_1(\mathcal{T}_{\theta,x,y})$, and obtain in this way for any (x,y) a fixed basis

$$m_{x,y} \coloneqq (a,b)_{x,y} \tag{3.18}$$

of $\pi_1(\mathcal{T}_{\theta,x,y})$. To see that a and b indeed generate $\pi_1(\mathcal{T}_{\theta,x,y})$, one easily check that these homotopy classes contain two transverse closed curves a' and b' indicated in Figure 3.2 which have algebraic intersection number 1 (the signs of their three intersection points being indicated in brown).

We lastly denote by $\mathsf{0}=[0,0]$ the origin of $\mathbf{T}^2=\mathbb{R}^2/\mathbb{Z}^2$, fix a basis

$$m^0=(a^0,b^0)$$

of $\pi_1(\mathbf{T}^2\setminus\mathsf{0})$ and denote in the same way the induced basis of $\pi_1(\mathbf{T}^2)$.

Lemma 3.32. *Up to pre-composition by homeomorphisms of $\mathbf{T}^2$ isotopic to the identity relative to $\mathsf{0}$, there exists:*

(1) for any fixed $x\in[1\,;\infty]$, a unique homeomorphism $M_x\colon\mathbf{T}^2\to\mathcal{T}_{\theta,x}$ such that $M_x(\mathsf{0})=[1,0]$ and whose action in homotopy sends m^0 to m_x;

(2) for any fixed $(x,y)\in\mathcal{D}$, a unique homeomorphism $M_{x,y}\colon\mathbf{T}^2\to\mathcal{T}_{\theta,x,y}$ such that $M_{x,y}(\mathsf{0})=[1,0]$ and whose action in homotopy sends m^0 to $m_{x,y}$.

For any fixed $x\in[1\,;\infty]$ (respectively $(x,y)\in\mathcal{D}$), all such homeomorphisms M_x (resp. $M_{x,y}$) define thus a unique point $[M_x^*\mathcal{T}_{\theta,x}]$ (resp. $[M_{x,y}^*\mathcal{T}_{\theta,x,y}]$) in $\mathsf{Def}_\theta(\mathbf{T}^2,\mathsf{0})$ which will be denoted by

$$\mu_{\theta,x}\ (\textit{resp. } \mu_{\theta,x,y}).$$

Proof. The existence being clear, we only have to prove that a homeomorphism of $\mathbf{T}^2$ fixing $\mathsf{0}$ and acting trivially in homotopy, is isotopic to the identity relative to $\mathsf{0}$. This fact is well-known but we outline here the proof for sake of completeness. First, for a homeomorphism f of $\mathbf{T}^2$ fixing $\mathsf{0}$ and with h the restriction of f to $\mathbf{T}^2\setminus\{\mathsf{0}\}$, f is isotopic to $\mathrm{id}_{\mathbf{T}^2}$ relative to $\mathsf{0}$ if and only h is isotopic to $\mathrm{id}_{\mathbf{T}^2\setminus\{\mathsf{0}\}}$ (see for instance [BCLR20, Proposition 1.6]). Then, h is isotopic to $\mathrm{id}_{\mathbf{T}^2\setminus\{\mathsf{0}\}}$ if, and only if it is homotopic to $\mathrm{id}_{\mathbf{T}^2\setminus\{\mathsf{0}\}}$, due to a result of Epstein in [Eps66] (see also [BCLR20, Theorem 2]). Lastly, h is homotopic to $\mathrm{id}_{\mathbf{T}^2\setminus\{\mathsf{0}\}}$ if, and only if it acts trivially on $\pi_1(\mathbf{T}^2\setminus\{\mathsf{0}\})$ (see [BCLR20, Theorem 2 and §2.2]). But if f acts trivially on $\pi_1(\mathbf{T}^2)$, then h acts trivially on $\pi_1(\mathbf{T}^2\setminus\{\mathsf{0}\})$, which concludes the proof. □

We summarize the constructions of this paragraph in the following result.

Proposition 3.33. *The maps*

$$x \in [1\,;\infty] \mapsto \mu_{\theta,x} \in \mathsf{Def}_\theta(\mathbf{T}^2, 0) \text{ and } (x,y) \in \mathcal{D} \mapsto \mu_{\theta,x,y} \in \mathsf{Def}_\theta(\mathbf{T}^2, 0) \tag{3.19}$$

are continuous.

Proof. This follows from the continuity of the HIETs proved in paragraphs 3.5.1 and 3.5.2. □

Remark 3.34. We emphasize that $\mu_{\theta,1} \neq \mu_{\theta,\infty}$. Indeed $\mathcal{T}_{\theta,1} = \mathcal{T}_{\theta,\infty}$, but $m_1 = (a_1, b_1) = (a_\infty, -a_\infty + b_\infty) \neq m_\infty$. Hence with Φ the element of the pure mapping class group of $(\mathbf{T}^2, 0)$ induced by the matrix

$$\phi = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

we have $\mu_{\theta,1} = \Phi^*(\mu_{\theta,\infty})$. In other words, $\mu_{\theta,x}$ does not define a closed loop, but a path in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$.

3.7.3. *Ehresman-Thurston principle.* We now describe the local topology of the deformation space through the celebrated *Ehresman-Thurston principle.* We emphasize that, although we will prove more below, we will only use in this text the weakest imaginable topological property of the deformation space, namely the fact that $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ is *Hausdorff* (this will only be used in the proof of Theorem A in paragraph 4.5).

We saw in paragraph 2.1.4 that the holonomy morphism

$$\rho\colon \pi_1(\mathbf{T}^2 \setminus \Sigma) \to \mathrm{PSL}_2(\mathbb{R}) \tag{3.20}$$

of a singular $\mathbf{dS}^2$-structure μ of singular set Σ on $\mathbf{T}^2$ is well defined up to conjugacy by $\mathrm{PSL}_2(\mathbb{R})$. If μ moreover has angles $\Theta = (\theta_p)_{p\in\Sigma}$, then

$$\forall p \in \Sigma, \rho(\delta_p) \text{ is conjugated to } a^{\theta_p} \tag{3.21}$$

with δ_p a small positively oriented loop around p (see Lemma 2.24). The subspace $\mathrm{Hom}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R}))$ of morphisms (3.20) satisfying the condition (3.21) is obviously invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$ by conjugation, and we denote by

$$\mathcal{X}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R})) = \mathrm{PSL}_2(\mathbb{R}) \backslash \mathrm{Hom}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R})) \tag{3.22}$$

its quotient by the latter action, called a *relative character variety.* A singular $\mathbf{dS}^2$-structure (Σ, μ) on $\mathbf{T}^2$ is finally associated to a point $\mathrm{hol}(\mu)$ in the relative character variety (3.22). The main utility of quotienting by homeomorphisms *isotopic to the identity* in the Definition 3.29 of the deformation space $\mathrm{Def}_\Theta(\mathbf{T}^2, \Sigma)$, is that distinct representants of a point $[\mu] \in \mathrm{Def}_\Theta(\mathbf{T}^2, \Sigma)$ will have the same holonomy morphism (up to conjugacy by $\mathrm{PSL}_2(\mathbb{R})$). We obtain thus a well-defined map

$$[\mathrm{hol}]\colon [\mu] \in \mathrm{Def}_\Theta(\mathbf{T}^2, \Sigma) \mapsto \mathrm{hol}(\mu) \in \mathcal{X}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R})), \tag{3.23}$$

continuous for the quotient topology induced on $\mathcal{X}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R}))$ by the compact-open topology on $\mathrm{Hom}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R}))$ (which is simply a product topology since $\pi_1(\mathbf{T}^2 \setminus \Sigma)$ is a finitely generated free group). While this topology is not Hausdorff on the whole relative character variety[3], the holonomy morphisms of the singular $\mathbf{dS}^2$-structures appearing in the present text belong to the open subspace $\mathcal{X}^{irr}_\theta(\mathbf{T}^2 \setminus \{0\}, \mathrm{PSL}_2(\mathbb{R})) \subset \mathcal{X}_\theta(\mathbf{T}^2 \setminus \{0\}, \mathrm{PSL}_2(\mathbb{R}))$ of *irreducible* representations, which is not only Hausdorff but actually a topological surface. We refer to [Gol09, §2.3 & 3.4], [Gol03] and [Gol84, §1] for more details on character varieties and their topology, including the definition of irreducible representations and a proof of the latter claim.

The importance of holonomy morphisms lies in the following crucial statement, due to Ehresman and popularized by Thurston.

Theorem 3.35 (Ehresman-Thurston)**.** *The map* $[\mathrm{hol}]$ *defined in* (3.23) *is a local homeomorphism in restriction to* $[\mathrm{hol}]^{-1}(\mathcal{X}^{irr}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R})))$. *In particular,* $[\mathrm{hol}]^{-1}(\mathcal{X}^{irr}_\theta(\mathbf{T}^2 \setminus \{0\}, \mathrm{PSL}_2(\mathbb{R})))$ *is a topological surface in* $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$.

[3] This is the reason why the character variety is frequently not defined as a simple quotient by conjugation, but as an object coming from algebraic geometry and known as a GIT quotient, that we will not use in this text.

We did not find a suitable reference proving this statement for structures with singularities, and give thus a proof which relies on [CEG87, §1.7] for the case of regular structures on compact manifolds with boundary. The proof given below came out from a very enlightening correspondence on this subject with Nicolas Tholozan, that the author wants to thank. Other useful references for this result are [BG04, §2] for manifolds with boundary, and [Gol22, §7.2], [Gol88, p.177] for closed manifolds (the original reference of Thurston being [Thu22, Chapter 5]).

Proof of Theorem 3.35. We already know that [hol] is continuous. We now fix a singular $\mathbf{dS}^2$-structure μ_0 of holonomy ρ_0 and an open neighbourhood $\mathcal{U} \subset \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$ of μ_0, and prove the existence of an open neighbourhood $U \subset \mathrm{Hom}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R}))$ of ρ_0 such that any $\rho \in U$ is the holonomy of a structure $\mu \in \mathcal{U}$. For any singularity $p \in \Sigma$ we choose a trivial neighbourhood $D_p \ni p$, *i.e.* admitting a singular $\mathbf{dS}^2$-chart $\varphi_p \colon D_p \to \mathbf{dS}^2_{\theta_p}$, we fix a closed disk $\Delta_p \subset D_p$ containing p in its interior, denote $S \coloneqq \mathbf{T}^2 \setminus \cup_{p\in\Sigma}\Delta_p$ and identify $\pi_1(S)$ with $\pi_1(\mathbf{T}^2 \setminus \Sigma)$. There exists then a neighbourhood $\mathcal{U}'$ of $\mu_0|_S$ in the space of $\mathbf{dS}^2$-structures on the compact surface with boundary $\mathrm{Cl}(S)$, such that $\mathcal{U}' \cap \mathrm{hol}^{-1}(\mathrm{Hom}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R}))) = \mathcal{U}|_S \coloneqq \{\mu|_S \mid \mu \in \mathcal{U}\}$. According to [CEG87, Theorem 1.7.1], there exists thus a neighbourhood U of ρ_0 in $\mathrm{Hom}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R}))$ realized by holonomies of structures $\mu^S \in \mathcal{U}|_S$. Extending these structures μ^S to structures $\mu \in \mathcal{U}$ proves our claim.

We now consider two close enough structures $\mu_1, \mu_2 \in \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$ having the same holonomy ρ, and show the existence of a diffeomorphism f of $\mathbf{T}^2$ isotopic to the identity relative to Σ and such that $\mu_2 = f^*\mu_1$. Using notations analogous to the ones of the previous paragraph, we choose the disks D_p's small enough for μ_1 and μ_2 to be trivial on D_p. Since $\mu_1|_S$ and $\mu_2|_S$ are as close as we want and have the same holonomy morphism, there exists according to the proof of [CEG87, Lemma 1.7.4] a diffeomorphism ϕ of S isotopic to the identity and close to the identity such that $\mu_2 = \phi^*\mu_1$. We can then extend ϕ to a diffeomorphism Φ of $\mathbf{T}^2$ isotopic to the identity relative to Σ, and introduce $\mu_2' \coloneqq \Phi^*\mu_1 \in \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$ which satisfies by construction $\mu_2'|_S = \mu_2|_S$. The problem is in this way reduced to the following local situation at the neighbourhood of singularities. Let ν_1, ν_2 be two close enough trivial $\mathbf{dS}^2$-structures on an oriented open disk D, having a unique singularity of the same angle θ at the same point $p \in D$ and coinciding on an annulus $A \coloneqq D \setminus \Delta$. The ν_i, $i = 1, 2$ admit then singular $\mathbf{dS}^2$-charts $\varphi_i \colon D \to \mathbf{dS}^2_\theta$ which coincide on A, and $\nu_2 = \psi^*\nu_1$ with $\psi \coloneqq \varphi_1^{-1} \circ \varphi_2$ an orientation-preserving diffeomorphism of D of support Δ and fixing p, hence isotopic to the identity as any such diffeomorphism. This observation concludes the proof of our claim by considering such a diffeomorphism ψ_p at each singularity, extending $\cup_{p\in\Sigma}\psi_p$ to a diffeomorphism Ψ of $\mathbf{T}^2$ of support $\cup_{p\in\Sigma}\Delta_p$ and isotopic to the identity relative to Σ, and by considering $f = \Phi \circ \Psi$.

Let us denote by $\mathcal{D}(\mathbf{T}^2, \Sigma, \Theta)$ the space of developing maps $\delta \colon \widetilde{\mathbf{T}^2 \setminus \Sigma} \to \mathbf{dS}^2$ of structures in $\mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$, endowed with a natural right action of $\mathrm{Homeo}^0(\mathbf{T}^2, \Sigma)$ by pre-composition and with a natural left action of $\mathrm{PSL}_2(\mathbb{R})$ by post-composition. Then with $\mathcal{S}'(\mathbf{T}^2, \Sigma, \Theta)$ the quotient of $\mathcal{D}(\mathbf{T}^2, \Sigma, \Theta)$ by $\mathrm{Homeo}^0(\mathbf{T}^2, \Sigma)$, $\mathrm{Def}_\Theta(\mathbf{T}^2, \Sigma)$ is identified with the quotient of $\mathcal{S}'(\mathbf{T}^2, \Sigma, \Theta)$ by $\mathrm{PSL}_2(\mathbb{R})$. The holonomy map hol is moreover well-defined and $\mathrm{PSL}_2(\mathbb{R})$-equivariant from $\mathcal{S}'(\mathbf{T}^2, \Sigma, \Theta)$ to $\mathrm{Hom}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R}))$, and the two previous paragraphs show that hol is open and locally injective. Now since $\mathrm{PSL}_2(\mathbb{R})$ acts properly on the open set of irreducible representations, it is easy to check that the map [hol] induced between the quotients by the $\mathrm{PSL}_2(\mathbb{R})$-actions remains open and locally injective in restriction to $[\mathrm{hol}]^{-1}(\mathcal{X}^{irr}_\Theta(\mathbf{T}^2 \setminus \Sigma, \mathrm{PSL}_2(\mathbb{R})))$. It is therefore a local homeomorphism on restriction to this open subset. We refer to [Gol88, pp.178–179] for more details on these arguments.

The fact that $\mathcal{X}^{irr}_\theta(\mathbf{T}^2 \setminus \{0\}, \mathrm{PSL}_2(\mathbb{R}))$ is a topological surface is for instance proved in [Gol09, §2.3 & 3.4], [Gol03] and [Gol84, §1], and concludes the proof of the Theorem. $\square$

Remark 3.36. The map (3.23) is in general *not* locally injective for (G, X)-structures with singularities, as the examples of isomonodromic deformations of branched projective structures show (see for instance [CDF14]). The reason why it is locally injective in our case is morally because "the angles at singularities are less than 2π", meaning that we forbidedd in the present text branching points. More precisely, it is because of the very definition of our singularities, at which local charts of the structure extend to local homeomorphisms.

Remark 3.37. Another natural geometrical proof of Theorem 3.35 would have been to express $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ as a space of polygons in the model space with prescribed identifications of their edges. We refer the reader to [FM11, §10.4.2] for a very nice presentation of the latter type of arguments in the classical case of hyperbolic structures on surfaces, easily adaptable to our case.

In the case of one singularity that we are interested with in this text, we can actually specify a local parametrization of the deformation space and show that the map $(x, y) \in \mathcal{D} \mapsto \mu_{\theta,x,y} \in \mathsf{Def}_\theta(\mathbf{T}^2, 0)$ introduced in (3.19) is a local homeomorphism (it is however *not* globally injective). This follows from Ehresman-Thurston principle by proving that the map $(x, y) \in \mathcal{D} \mapsto (\mathrm{tr}(g_2(x, y)), \mathrm{tr}(h_2(x, y)))$ is itself locally injective. It is moreover relatively easy to show that any singular $\mathbf{dS}^2$-structure on $\mathbf{T}^2$ with a single singularity and whose lightlike foliations are minimal is isometric to a structure $\mu_{\theta,x,y}$.

3.8. **Conclusion of the proof of Theorem 3.1.** We can now use the structures constructed in Propositions 3.12 and 3.17 to conclude the proof of the existence Theorem 3.1.

Let (a', b') be two closed curves of $\mathcal{T}_{\theta,x,y}$, belonging to the homotopy classes (a, b) defined in (3.18) and respectively transverse to the β and the α-foliation. To fix the ideas, we define

(1) for $t \in [1\,; x]$, a'_t as the closed loop obtained by following positively $[t\,; \infty] \times \{0\}$ and then the affine segment of $\mathcal{L}_{\theta,x,y} \subset \mathbb{R}^2$ from $(1, y')$ to $(t, 0)$;
(2) for $t \in [0\,; y]$, b'_t as the closed loop obtained by following positively $\{1\} \times [t\,; y_+]$ and then the segment of $\mathcal{L}_{\theta,x,y}$ from $(x', 0)$ to $(1, t)$.

Then $t \mapsto a'_t$ and $t \mapsto b'_t$ are homotopies, respectively beginning at $a'_1 = a$ and $b'_0 = b$ and illustrated in Figure 3.2. We moreover fix an identification from a to a', given by the identity on common points of the loops, and by following β-leaves positively until the first meeting point elsewhere. Accordingly, we fix an identification from b to b' given by following α-leaves positively. Through these identifications, the homeomorphisms E and F of $\mathbf{S}^1_I = [1\,; \infty]/\{1 \sim \infty\}$ and $\mathbf{S}^1_J = [0\,; y_+]/\{0 \sim y_+\}$ introduced in paragraph 3.5.2 induce homeomorphisms E' and F' of a' and b', which are by definition respectively conjugated to E and F. Let

$$P^{a'}_\beta : a' \to a' \text{ and } P^{b'}_\alpha : b' \to b'$$

denote the respective first-return maps of $\mathcal{F}_\beta$ on a' and of $\mathcal{F}_\alpha$ on b' in $\mathcal{T}_{\theta,x,y}$.

Lemma 3.38. *$P^{a'}_\beta = \mathsf{E}'^{-1}$ and $P^{b'}_\alpha = \mathsf{F}'^{-1}$, and these maps are respectively conjugated to E^{-1} and F^{-1}.*

Proof. This follows from the definition of $\mathcal{T}_{\theta,x,y}$. □

With obvious corresponding notations, we will use in any torus $\mathcal{T}_{\theta,x}$ a closed loop b' homotopic to b and transverse to $\mathcal{F}_\alpha$, denote by $\bar{g}'$ the homeomorphism of b' induced by g, and by $P^{b'}_\alpha$, P^a_β the first-return maps of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ on b' and a. We obtain then:

Lemma 3.39. *$P^a_\beta = \mathsf{E}^{-1}$, and $P^{b'}_\alpha = \bar{g}'^{-1}$ and is conjugated to $\bar{g}^{-1}$.*

Conclusion of the proof of Theorem 3.1. We will repeatedly use the Propositions 3.25 and 3.26 to translate the dynamics of a torus foliation in terms of its projective asymptotic cycle. We will only consider the non-oriented projective asymptotic cycles, since the latter yield all the expected oriented projective asymptotic cycles by composing with orientation-reversing maps.

(1) It is clear from the dynamics of g and h_1 that $\mathcal{F}_\alpha([1, 0])$ (respectively $\mathcal{F}_\beta([1, 0])$) is the unique closed α-leaf (resp. β-leaf) of the torus $\mathcal{T}_{\theta,1}$, and by acting with the pure mapping class group of $(\mathcal{T}_{\theta,1}, [1, 0])$, one obtains any basis of $\pi_1(\mathcal{T}_{\theta,1})$. On the other hand, Proposition 3.23 and Lemma 3.39 show that any periodic cyclic order for the orbit of $[1, 0]$ under the first-return map E^{-1}_x of $\mathcal{F}_\beta$ on a_x is reached. Since Dehn twists around a_x belong to the pure mapping class group of $(\mathcal{T}_{\theta,x}, [1, 0])$ and fix a_x, we can act by such Dehn twists to obtain points $[\mu] \in \mathsf{Def}_\theta(\mathbf{T}^2, 0)$ so that $[\mathcal{F}^\mu_\alpha(0)] = [1, 0]$ and $[\mathcal{F}^\mu_\beta(0)]$ is any primitive element of $\pi_1(\mathbf{T}^2)$ distinct from $(1, 0)$. We lastly observe that $\mathcal{F}_\alpha([1, 0])$ remains the unique closed α-leaf of $\mathcal{T}_{\theta,x}$ by such operations, and that the same can be achieved for $\mathcal{F}_\beta$ according to Remark 3.13. This concludes the proof of the first claim.

(2) The first-return map of the β-foliation is given by the map E^{-1} according to Lemma 3.39. Proposition 3.23 shows thus in particular that the map $x \in [0\,;\infty] \mapsto A(\mathcal{F}_\beta^{\mu_{\theta,x}}) \in \mathbf{P}(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R})) \equiv \mathbb{R}\mathbf{P}^1$ is continuous, monotonous and non-constant. Therefore $A(\mathcal{F}_\beta^{\mu_{\theta,x}})$ reaches an interval $I \subset \mathbf{P}(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ of non-empty interior, hence containing irrational lines. On the other hand, Remark 3.34 shows that $\mu_{\theta,\infty} = \Phi_*(\mu_{\theta,1})$ with Φ a Dehn twist around $(1,0)$ fixing $A(\mathcal{F}_\alpha^{\mu_{\theta,x}}) = [\mathcal{F}_\alpha^{\mu_{\theta,x}}](\mathsf{0}) = [1,0]$. It is now easily checked that the translates of I by the iterates of Φ cover $\mathbf{P}(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R})) \setminus \{[1,0]\}$, which shows the first claim of (2). The fact that $\mathcal{F}_\alpha(\mathsf{0})$ is the unique closed leaf of $\mathcal{F}_\alpha$ follows again from the fact that $[0]=[y_\theta]$ is the unique periodic point of g.
(3) The first-return maps of the α and β foliations are conjugated to the maps F^{-1} and E^{-1} according to Lemma 3.38. Proposition 3.24 shows thus that $(A(\mathcal{F}_\alpha^{\mu_{\theta,x,y}}), A(\mathcal{F}_\beta^{\mu_{\theta,x,y}}))$ reaches a subset $K \subset \mathbf{P}(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))^2 \setminus \{\text{diagonal}\}$ of non-empty interior, hence containing pairs of irrational lines. As in (2), the claim follows then from the fact that the translates of K by the action of the pure mapping group of $(\mathbf{T}^2,\mathsf{0})$ cover the pairs of distinct irrational lines in $\mathbf{P}(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))^2$. □

4. Rigidity of singular $\mathbf{dS}^2$-tori

We conclude in this section the proofs of the rigidity Theorems A, C and D.

4.1. **Conclusion of the proof of Theorem C.** The existence part was proved in Theorem 3.1. Let μ_1,μ_2 be two singular $\mathbf{dS}^2$-structures on $\mathbf{T}^2$ with a unique singularity of angle θ at $\mathsf{0}$, and whose lightlike leaves at $\mathsf{0}$ are closed and homotopic:

$$([\mathcal{F}_\alpha^{\mu_1}(\mathsf{0})],[\mathcal{F}_\beta^{\mu_1}(\mathsf{0})]) = ([\mathcal{F}_\alpha^{\mu_2}(\mathsf{0})],[\mathcal{F}_\beta^{\mu_2}(\mathsf{0})]). \tag{4.1}$$

According to Lemma 3.10 and to the proof of Proposition 3.8, there exists then homotopic isometries respectively sending μ_1 and μ_2 to structures μ_{θ,x_1} and μ_{θ,x_2} with $x_1,x_2 \in [1\,;\infty[$. In particular, $\mathcal{F}_\beta^{\mu_{\theta,x_1}}(\mathsf{0})$ and $\mathcal{F}_\beta^{\mu_{\theta,x_2}}(\mathsf{0})$ are closed and homotopic. There only remains to prove now that $x_1=x_2$, which will conclude the proof of Theorem C. This will indeed show that $\mu_2=\varphi^*\mu_1$ for some homeomorphism of $\mathbf{T}^2$ fixing $\mathsf{0}$, but (4.1) will then imply that φ acts trivially in homotopy, *i.e.* is isotopic to identity relative to $\mathsf{0}$ (see the proof of Lemma 3.32 for more details), and thus $[\mu_1]=[\mu_2]$ in $\mathsf{Def}_\theta(\mathbf{T}^2,\mathsf{0})$.

From now on we implicitly identify $(\mathbf{T}^2,\mu_{\theta,x_i})$ and $\mathcal{T}_{\theta,x_i}$ as explained in Lemma 3.32, to simplify the notations. The first return map of $\mathcal{F}_\beta^{\mu_{\theta,x_i}}$ on $\mathcal{F}_\alpha^{\mu_{\theta,x_i}}(\mathsf{0})$ being $\mathsf{E}_{x_i}^{-1}$ according to Lemma 3.39, we can translate the fact that $\mathcal{F}_\beta^{\mu_{\theta,x_1}}(\mathsf{0})$ and $\mathcal{F}_\beta^{\mu_{\theta,x_2}}(\mathsf{0})$ are closed and homotopic in terms of orbits of the E_{x_i}'s: $[1] \in \overline{[1\,;\infty]} \coloneqq [1\,;\infty]/\{1\sim\infty\}$ under E_{x_1} and E_{x_2} are periodic, say of minimal period $q\in\mathbb{N}^*$, and of the same cyclic order on the circle $\overline{[1\,;\infty]}$. We can moreover assume without any lost of generality that $x_1,x_2 \in\,]1\,;\infty[$ and that $q\geq 2$, since E_x has no fixed points unless $x=1$. For $p\in\overline{[1\,;\infty]}$, let us denote:

(1) $l(p)=a$ if $p\in[1\,;x_i'[$, equivalently if $\mathsf{E}_{x_i}(p) = gh_{x_i}(p)$;
(2) and $l(p)=b$ if $p\in[x_i'\,;\infty[$, equivalently if $\mathsf{E}_{x_i}(p)=h_{x_i}(p)$.

Then with $l_1=l([1])$ and $l_{k+1}=l(l_k([1]))$, the word $w=l_q\dots l_1$ in the letters a and b is the *coding* of the periodic orbit of $[1]$ under E_{x_i}, and is equivalent to its cyclic ordering. In other words, the respective codings of $[1]$ under E_{x_1} and E_{x_2} are equal to a common word $w=l_q\dots l_1$, characterized by

$$\mathsf{E}_{x_i}^k([1]) = w_k(gh,h)([1]) \tag{4.2}$$

for any $1\leq k\leq q$, where $w_k=l_k\dots l_1$ and $v(A,B)\in\mathrm{PSL}_2(\mathbb{R})$ is obtained for any $A,B\in\mathrm{PSL}_2(\mathbb{R})$ from a word v in the letters a and b by replacing a by A and b by B.

According to Lemma 3.22 there exists $T\in[0\,;1]$ such that $x_2=g^T(x_1)$ and $h_{x_2}=g^Th_{x_1}$, and we thus only have to show that $T=0$. From now on we denote $h\coloneqq h_{x_1}$ to simplify notations, and work in $\mathbb{R}\cup\{\infty\}$ identified with $\mathbb{R}\mathbf{P}^1$ (in the same $\mathrm{PSL}_2(\mathbb{R})$-equivariant way (2.2) than usually). The equalities (4.2) translate then as:

$$\begin{cases} w(gh,h)(1) = w(g^{T+1}h,g^Th)(1) = 1 \\ \forall k\in\{1,\dots,q-1\} : w_k(gh,h)(1) \text{ and } w_k(g^{T+1}h,g^Th)(1)\in\,]1\,;\infty[. \end{cases} \tag{4.3}$$

Fact 4.1. *For any $k \in \{1, \dots, q\}$, the map $s \in [0\,;T] \mapsto w_k(g^{s+1}h, g^s h)(1)$ is strictly increasing and has values in $[1\,;\infty[$.*

Fact 4.1 concludes the proof of our claim, and thus of Theorem C. In the map $s \in [0\,;T] \mapsto w_q(g^{s+1}h, g^s h)(1) = w(g^{s+1}h, g^s h)(1)$ is in particular strictly increasing, but has according to (4.3) the same value 1 at $s = 0$ and $s = T$ which implies $T = 0$.

Proof of Fact 4.1. We prove the claim by recurrence on k.

Case $k = 1$. Then $w_1 = l_1 = a$ and since $gh(1) \in]1\,;\infty[$, $s \in \mathbb{R} \mapsto w_1(g^{s+1}h, g^s h)(1) = g^{s+1}h(1)$ is strictly increasing in $\mathbb{R} \cup \{\infty\}$. Since $g^{T+1}h(1) \in]1\,;\infty[$ as well according to (4.3), we have thus $g^{s+1}h(1) \in]1\,;\infty[$ for any $s \in [0\,;T]$ by the intermediate values Theorem.

From $k \in \{1, \dots, q-1\}$ to $k+1$. Then $w_{k+1}(g^{s+1}h, g^s h)(1) = l_{k+1}(g, \mathrm{id})g^s h(\alpha(s))$ for $s \in [0\,;T]$, with $\alpha \colon s \in [0\,;1] \mapsto w_k(g^{s+1}h, g^s h)(1)$ a strictly increasing map having values in $[1\,;\infty[$ by recurrence. Since h is orientation-preserving, $s \in [0\,;T] \mapsto h \circ \alpha(s)$ is strictly increasing as well. The dynamics of h show moreover that its attractive and repulsive fixed points respectively satisfy $h_+ \in]y_\theta\,;1[$ and $h_- \in]\infty\,;0[$, and the attractive and repulsive fixed points of g are on the other hand 0 and y_θ. We have thus $h \circ \alpha([0\,;T]) \subset]h_+\,;\infty[\subset [y_\theta\,;0]$, and denoting $G(s,p) \coloneqq g^s(p)$ for any $(s,p) \in \mathbb{R} \times]y_\theta\,;0[$ we have: $\frac{\partial G}{\partial s}(s,p) > 0$ due to the dynamics of g, and $\frac{\partial G}{\partial p}(s,p) > 0$ due to the fact that g^s is orientation-preserving. Therefore:

$$\frac{d}{ds} g^s h(\alpha(s)) = \frac{d}{ds} G(s, h(\alpha(s))) = \frac{\partial G}{\partial s}(s, h(\alpha(s))) + \frac{d}{ds} h(\alpha(s)) \frac{\partial G}{\partial p}(s, h(\alpha(s)))$$

is strictly positive for any $s \in [0\,;T]$ as a sum of strictly positive terms. Therefore $s \in [0\,;T] \mapsto w_{k+1}(g^{s+1}h, g^s h)(1) = l_{k+1}(g, \mathrm{id})g^s h(\alpha(s))$ is strictly increasing, since g is orientation-preserving. Since $w_{k+1}(gh, h)(1)$ and $w_{k+1}(g^{T+1}h, g^T h)(1)$ are moreover in $[1\,;\infty[$ according to (4.3), we have $w_{k+1}(g^{s+1}h, g^s h)(1) \in [1\,;\infty[$ for any $s \in [0\,;T]$, which concludes the proof of the Fact. □

4.2. **Conclusion of the proof of Theorem D.** The existence part is given by Theorem 3.1. Let μ_1, μ_2 be two singular $\mathbf{dS}^2$-structures on $\mathbf{T}^2$ with a unique singularity of angle θ at $\mathsf{0}$, whose α-leaves at $\mathsf{0}$ are closed and such that:

$$([\mathcal{F}_\alpha^{\mu_1}(\mathsf{0})], A^+(\mathcal{F}_\beta^{\mu_1})) = ([\mathcal{F}_\alpha^{\mu_2}(\mathsf{0})], A^+(\mathcal{F}_\beta^{\mu_2})) \tag{4.4}$$

with $A^+(\mathcal{F}_\beta^{\mu_i})$ an irrational half-line. Then as in the beginning of paragraph 4.1, there exists according to Lemma 3.10 and to the proof of Proposition 3.8 homotopic isometries respectively sending μ_1 and μ_2 to structures μ_{θ,x_1} and μ_{θ,x_2} with $x_1, x_2 \in [1\,;\infty[$. The leaves $\mathcal{F}_\alpha^{\mu_1}(\mathsf{0})$ and $\mathcal{F}_\alpha^{\mu_2}(\mathsf{0})$ are then closed and homotopic, and $A^+(\mathcal{F}_\beta^{\mu_{\theta,x_1}}) = A^+(\mathcal{F}_\beta^{\mu_{\theta,x_2}})$. There only remains now for us to prove that $x_1 = x_2$, which will conclude the proof of Theorem D. Indeed this will show that $\mu_2 = \varphi^*\mu_1$ for some homeomorphism of $\mathbf{T}^2$ fixing $\mathsf{0}$, and (4.4) will then imply that φ acts trivially in homotopy and is thus isotopic to identity relative to $\mathsf{0}$, showing finally that $[\mu_1] = [\mu_2]$ in $\mathsf{Def}_\theta(\mathbf{T}^2, \mathsf{0})$.

Let us denote by $\gamma_i \colon [0\,;1] \to \mathbf{T}^2$ the unique future affine parametrization of the closed leaf $\mathcal{F}_\alpha^{\mu_{\theta,x_i}}(\mathsf{0})$ such that $\gamma_i(0) = \gamma_i(1) = \mathsf{0}$ and $\gamma_i|_{]0;1[}$ is injective, and by P_i the first-return map of $\mathcal{F}_\beta^{\mu_{\theta,x_i}}$ on $\mathcal{F}_\alpha^{\mu_{\theta,x_i}}(\mathsf{0})$ (well-defined since $\mathcal{F}_\beta^{\mu_{\theta,x_i}}$ is minimal by assumption). Then since $\mathcal{F}_\alpha^{\mu_{\theta,x_1}}(\mathsf{0})$ and $\mathcal{F}_\alpha^{\mu_{\theta,x_2}}(\mathsf{0})$ are homotopic, the equality of the oriented projective asymptotic cycles of the β-foliations implies that $\rho(P_1) = \rho(P_2)$ according to Lemma 3.27. Since the cycle $A^+(\mathcal{F}_\beta^{\mu_{\theta,x_1}}) = A^+(\mathcal{F}_\beta^{\mu_{\theta,x_2}})$ is irrational, the rotation number $\rho(P_1) = \rho(P_2)$ is irrational as well, and this equality implies therefore that $x_1 = x_2$ according to Proposition 3.23. This concludes the proof of our claim and thus of Theorem D.

4.3. **Geodesics and affine circles.** Denoting by $(\mathbf{G}, \mathbf{X})$ the pair $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ or $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1), \mathbb{R}^{1,1})$, we define in this subsection the natural notion of geodesics in a singular $\mathbf{X}$-surface.

4.3.1. *Geodesics of* $\mathbf{X}$. On an oriented topological one-dimensional manifold, we will call:

(1) *affine structure* an $(\mathrm{Aff}^+(\mathbb{R}), \mathbb{R})$-structure, with $\mathrm{Aff}^+(\mathbb{R}) \simeq \mathbb{R}_+^* \ltimes \mathbb{R}$ the group of (orientation-preserving) affine transformations $\lambda \operatorname{id} + u \colon x \mapsto \lambda x + u$ of $\mathbb{R}$ (with $\lambda \in \mathbb{R}_+^*$ and $u \in \mathbb{R}$);

(2) and *translation structure* a $(\mathbb{R}, \mathbb{R})$-structure (which induces obviously an affine structure);

the charts of both structures being assumed to be orientation-preserving homeomorphisms. An *affine automorphism* is of course a $(\mathrm{Aff}^+(\mathbb{R}), \mathbb{R})$-morphism of affine structures. As for any affine connection, the geodesic of $\mathbf{X}$ have a natural affine structure given by parametrizations satisfying the geodesic equation, and its definite geodesics even have a natural translation structure given by constant speed parametrizations. For $\mathbf{X} = \mathbb{R}^{1,1}$, the affinely parametrized geodesics are simply the affinely parametrized affine segments.

Lemma 4.2. *Let γ be a geodesic of $\mathbf{X}$.*

(1) The stabilizer of γ in $\mathbf{G}$ acts transitively on γ. It is moreover:
  - *(a) a one-parameter group if γ is timelike, which is hyperbolic for $\mathbf{X} = \mathbf{dS}^2$;*
  - *(b) a one-parameter group if γ is spacelike, which is elliptic for $\mathbf{X} = \mathbf{dS}^2$*
  - *(c) and a two-dimensional group if γ is lightlike, which is parabolic (*i.e. *conjugated to a triangular subgroup) for $\mathbf{X} = \mathbf{dS}^2$.*

(2) There exists for any $x \in \gamma$ a one-parameter subgroup (g^t) stabilizing γ and acting freely at x, and $t \in \mathbb{R} \mapsto g^t(x) \in \gamma$ is then an affine parametrization of an open subset of γ.

(3) Let $\varphi \colon I \to J$ be an affine transformation between two non-empty open intervals of γ, which is a translation if γ is definite. Then there exists a unique $g \in \mathbf{G}$ such that $g|_I = \varphi$.

Proof. (1) For $\mathbf{X} = \mathbf{dS}^2$ we can work with the hyperboloid model dS^2 thanks to Lemma 2.2. The stabilizer of a plane $P \subset \mathbb{R}^{1,2}$ is also the one of its orthogonal for $q_{1,2}$, which is respectively spacelike, timelike and lightlike in the three above cases. Straightforward computations show then that these stabilizers are of the announced form and act transitively (observe that $\mathrm{Stab}_{\mathrm{SO}^0(1,2)}(\gamma)$ preserves each connected component of $P \cap \mathrm{dS}^2$).
(2) This fact follows easily from the identification of $\mathbf{X}$ with the homogeneous space $\mathbf{G}/A$.
(3) The action of $\mathrm{Stab}_{\mathbf{G}}(\gamma)$ defines a subgroup of affine transformations of γ, which is according to (1) a one-dimensional subgroup of translations in the definite case, and a two-dimensional subgroup in the lightlike case. This observation shows that the announced affine transformations of γ are indeed induced by elements of $\mathbf{G}$, which proves the existence.

For $x = (p, q) \in \mathbf{dS}^2$, let denote $x^{opp} \coloneqq (q, p) \in \mathbf{dS}^2$.

Fact 4.3. *Let $x \neq y \in \mathbf{X}$ such that $y \neq x^{opp}$ if $\mathbf{X} = \mathbf{dS}^2$, and $g_1, g_2 \in \mathbf{G}$ such that: $g_1(x) = g_2(x)$ and $g_1(y) = g_2(y)$. Then $g_1 = g_2$.*

Proof. This claim follows from the straightforward observation that with $A = \mathrm{Stab}_{\mathbf{G}}(\mathsf{o})$ and $x \neq o$, $x \neq o^{opp}$ if $\mathbf{X} = \mathbf{dS}^2$: $a \in A \mapsto a(x)$ is injective. □

Fact 4.3 shows the uniqueness, which concludes the proof of the Lemma. □

4.3.2. *Geodesics in singular* $\mathbf{X}$*-surfaces.* We observe now that any affinely parametrized geodesic segment $\gamma \colon I \to \mathbf{X}$ passing through o avoids a quadrant, and that we can therefore assume without lost of generality that $\gamma(I) \cap \mathcal{F}_\alpha^{+*}(\mathsf{o}) = \varnothing$. Using the projection $\pi_\theta \colon \mathbf{X}_* \to \mathbf{X}_\theta$ introduced in paragraph 2.2.1 for the standard $\mathbf{X}_\theta$-cone, $\pi_\theta \circ \gamma \colon I \to \mathbf{X}_\theta$ is thus well-defined and will be called an *affinely parametrized geodesic* of $\mathbf{X}_\theta$. The orientations of time and space induce a natural notion of *future* timelike and spacelike geodesic in any $\mathbf{X}$-surface (the one whose derivative is future-pointing), and this notion persists in $\mathbf{X}_\theta$ by saying that a geodesic is *future* timelike or spacelike if it is the projection of such a geodesic of $\mathbf{X}$. For lightlike geodesics namely lightlike leaves, the *future* orientation is by definition the positive orientation of the foliation.

Lemma 4.4. *(1) Singular $\mathbf{X}$-charts of $\mathbf{X}_\theta$ at o_θ send future affinely parametrized geodesics of $\mathbf{X}_\theta$ to future affinely parametrized geodesics of the same signature.*

(2) Let γ be a parametrized curve of a singular $\mathbf{X}$-surface S passing through a singularity x. Then γ is mapped to a future affinely parametrized geodesic of $\mathbf{X}_\theta$ by a singular $\mathbf{X}$-chart of S at x, if and only if it is mapped to a future affinely parametrized geodesic of the same signature in any singular $\mathbf{X}$-chart at x.

Proof. (1) According to Proposition 2.26, the singular $\mathbf{X}$-charts of $\mathbf{X}_\theta$ at o_θ are the maps $\bar{a}$ induced by elements $a \in \mathrm{Stab}(\mathsf{o})$ and characterized by $\bar{a} \circ \pi_\theta = \pi_\theta \circ a$. Since a preserve the affine structure of any geodesic of $\mathbf{X}$ and its future orientation, $\bar{a} \circ \pi_\theta \circ \gamma = \pi_\theta \circ a \circ \gamma$ is thus a future affine parametrization for any future affinely parametrized geodesic γ of $\mathbf{X}_\theta$, which proves the claim.
(2) This is a direct consequence of (1). □

Remark 4.5. The fundamental consequence of Lemma 4.4 is that, contrarily to the Riemannian case, the notion of *straight* geodesic segment through a singular point always makes sense in a singular Lorentzian surface. In other words, every future geodesic segment I^- converging to a singular point x has a preferred associated geodesic segment I^+ arising from x of the same signature: the one for which $I^- \cup I^+$ is a geodesic through x in a singular chart. The affine structure of such a geodesic is moreover well-defined. This new manifestation of the higher rigidity of singular Lorentzian surfaces compared to their Riemannian couterparts allows the following definition.

Definition 4.6. A *geodesic* γ of a singular $\mathbf{X}$-surface (S, Σ) is a curve of S which maps in any regular (respectively singular) chart of the singular $\mathbf{X}$-atlas to a geodesic segment of $\mathbf{X}$ (resp. of $\mathbf{X}_\theta$), and the *affine structure* of γ is given by the parametrizations mapping in the singular $\mathbf{X}$-atlas to affine parametrizations of geodesics in $\mathbf{X}$ or $\mathbf{X}_\theta$. The *signature* (timelike, spacelike or lightlike) of γ is the one of $\gamma \cap (S \setminus \Sigma)$, and the *translation structure* of a definite geodesic γ is given by the affine parametrizations which have constant speed in $S \setminus \Sigma$.

Remark 4.7. Note that the signature of any of the connected components of $\gamma \cap (S \setminus \Sigma)$ is the same, hence the signature of γ is indeed well-defined. There is also of course a natural notion of *piecewise geodesic* in a singular $\mathbf{X}$-surface (S, Σ): a curve γ such that any connected component of $\gamma \setminus \Sigma$ is a geodesic of $S \setminus \Sigma$. However this notion will not be used in this text.

Proposition 4.8. *Let (S, Σ) be a singular $\mathbf{X}$-surface.*

(1) Geodesics of S are one-dimensional $\mathcal{C}^0$-submanifolds, and are $\mathcal{C}^\infty$ in $S \setminus \Sigma$.
(2) Any geodesic of S is contained in a unique maximal geodesic.
(3) Any point $x \in S$ admits a connected open neighbourhood U homeomorphic to a disk, and such that:
   (a) U is the domain of a chart of the singular $\mathbf{X}$-atlas centered at x;
   (b) U is the domain of a simultaneous foliated $\mathcal{C}^0$-chart of the lightlike foliations;
   (c) $U \setminus (\mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x))$ has four connected components, called the (open) quadrants of U at x;
   (d) for any two points $y \neq z \in U$, there exists a unique geodesic segment $[y\,;z]_U \subset U$ of endpoints y and z, and $[y\,;z]_U$ is moreover disjoint from (at least) one of the open quadrants at x.

   Such an U will be called a normal convex neighbourhood of x. *Moreover quadrants are themselves* convex, i.e. *if y, z are in a same open quadrant Q of U at x, then $[y\,;z]_U \subset Q$.*

A quadrant of U will be said *future timelike* and denoted by U^+ (respectively *past timelike* U^-) according to the notations of Figure 2.1, namely if it is crossed by a future timelike geodesic segment starting at x of the same signature. Obvious similar denominations are used for spacelike and causal quadrants.

Proof of Proposition 4.8. (1) This is immediate from the definition.
(2) This claim is of course true in $\mathbf{X}$ and thus on $S \setminus \Sigma$, and we only have to prove it at a singular point $x \in \Sigma$. Namely for two geodesics γ_1, γ_2 such that $\gamma_1 \cap \gamma_2$ contains a geodesic interval I having x as one of its endpoints, we want to prove that $\gamma_1 \cap \gamma_2$ contains a geodesic interval J containing x in its interior. With $\varphi \colon U \to \mathbf{X}_\theta$ a singular $\mathbf{X}$-chart at x, $\varphi(I)$ is contained in the projection in $\mathbf{X}_\theta$ of a maximal geodesic C of $\mathbf{X}$. In the same way, $\varphi(\gamma_1)$ (resp. $\varphi(\gamma_2)$) is contained in the projection of a maximal geodesic of $\mathbf{X}$ which intersects C on the open interval $\pi_\theta^{-1}(\varphi(I))$. But C is the only such geodesic of $\mathbf{X}$, and $\varphi(\gamma_1 \cap \gamma_2)$ contains thus some neighbourhood of x in $\pi_\theta(C)$, which proves our claim.

(3) This claim is easily proved in $\mathbf{X}$, and thus on $S \setminus \Sigma$ by using a standard normal convex neighbourhood. On the other hand (2) proves it on the neighbourhood of a singular point. □

Geodesics are natural in the following sense.

Lemma 4.9. *Let $f\colon S_1 \to S_2$ be an isometry of singular $\mathbf{X}$-surfaces. Then for any affine parametrization of a geodesic γ of S_1, $f \circ \gamma$ is an affine parametrization of a geodesic of S_2 of the same signature than γ.*

Proof. If $(\varphi_i\colon U_i \to V_i)_i$ is a singular $\mathbf{X}$-atlas of S_1, then $(\varphi_i \circ f^{-1}\colon f^{-1}(U_i) \to V_i)_i$ is a singular $\mathbf{X}$-atlas of S_2. Hence γ and $f \circ \gamma$ read as the same path in these singular $\mathbf{X}$-atlases, which directly implies the claim. □

4.3.3. *Affine structures of closed geodesics.* The easiest example of affine circle is given by the natural translation structure of $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$. For any $\mu \in \mathbb{R}_+^*$, $\mathbb{R}_+^*/\langle \mu \operatorname{id}\rangle$ gives in the other hand an example of affine circle which is not induced by a translation structure. Those two types of affine circles are in fact the only ones.

Lemma 4.10. *An affine circle C is either isomorphic to $\mathbb{R}/\mathbb{Z}$, or to $\mathbb{R}_+^*/\langle \mu \operatorname{id}\rangle$ for some $\mu \in \mathbb{R}_+^*$. Moreover:*

- *the affine automorphisms of $\mathbb{R}/\mathbb{Z}$ are the translations;*
- *the affine automorphisms of $\mathbb{R}_+^*/\langle \mu \operatorname{id}\rangle$ are induced by homotheties $\lambda \operatorname{id}$, $\lambda \in \mathbb{R}_+^*$.*

In both cases $\operatorname{ev}_x\colon \varphi \in \operatorname{Aff}^+(C) \mapsto \varphi(x) \in C$ is a homeomorphism for any $x \in C$, and we endow the circle $\operatorname{Aff}^+(C)$ with the orientation induced by C through any of the identifications ev_x.

Proof. With E the universal cover of C and γ a generator of its covering automorphism group, an affine structure on C is determined by a pair (δ, g), with $g = \lambda \operatorname{id} + u \in \operatorname{Aff}^+(\mathbb{R})$ and $\delta\colon E \to \mathbb{R}$ an orientation-preserving local homeomorphism such that $\delta \circ \gamma = g \circ \delta$. In particular δ is globally injective, and g has thus no fix point on the g-invariant interval $I = \delta(E)$. Up to the action of $\operatorname{Aff}^+(\mathbb{R})$, we can assume that I is either $\mathbb{R}$ or $\mathbb{R}_+^*$. In the first case $\lambda \neq 1$ would imply that $g = \lambda \operatorname{id} + u$ has a fixed point on $\mathbb{R}$, hence $\lambda = 1$ and g is a translation. The latter can moreover be assumed to be $\operatorname{id} + 1$ up to conjugation by $\operatorname{Aff}^+(\mathbb{R})$, proving that C is isomorphic to $\mathbb{R}/\mathbb{Z}$. In the second case, the fact that $g = \lambda \operatorname{id} + u$ preserves $\mathbb{R}_+^*$ shows that $u = 0$, hence that C is isomorphic to some $\mathbb{R}_+^*/\langle \mu \operatorname{id}\rangle$, which proves the first claim.

The second claim of the Lemma follows from the fact that affine automorphisms of C are induced by the affine automorphisms of $\delta(E)$ that normalize the holonomy group $\langle g \rangle$.

The last claim follows then from a direct observation. □

Closed definite geodesics in singular $\mathbf{X}$-surfaces have a translation structure as we have seen in Definition 4.6, and are thus isomorphic to $\mathbb{R}/\mathbb{Z}$. In the other hand, it is easy to check that the closed lightlike geodesics passing through the singular point of the singular $\mathbf{dS}^2$-tori $\mathcal{T}_{\theta,x}$ introduced in Proposition 3.12 are isomorphic to some affine circle $\mathbb{R}_+^*/\langle \mu \operatorname{id}\rangle$.

4.4. **Surgeries of singular constant curvature Lorentzian surfaces.** In this subsection we introduce a useful notion of surgery for singular $\mathbf{X}$-surfaces, $(\mathbf{G}, \mathbf{X})$ denoting as before the pair $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ or $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1,1), \mathbb{R}^{1,1})$. If it is well-defined, then we denote by

$$P^{\gamma}_{\alpha/\beta}\colon \gamma \to \gamma$$

the first-return map of the lightlike foliation $\mathcal{F}_{\alpha/\beta}$ on a simple closed geodesic γ. It is characterized by the fact that for any $x \in \gamma$, $P^{\gamma}_{\alpha/\beta}(x)$ is the first intersection point of $\mathcal{F}_{\alpha/\beta}(x)$ with γ starting from x (for the orientation of $\mathcal{F}_{\alpha/\beta}$). Our goal is to prove the following result.

Proposition 4.11. *Let γ be a simple closed geodesic in a singular $\mathbf{X}$-surface (S, Σ, μ) of ordered angle set Θ. Then for any $T \in \operatorname{Aff}^+(\gamma)$, there exists a singular $\mathbf{X}$-structure μ_T on S called a* surgery of μ around γ with respect to T, *such that $\mu_{\operatorname{id}_\gamma} = \mu$, and:*

1. *the ordered singular set of μ_T is Σ, and its ordered angle set Θ;*
2. *for any injective, continuous and orientation-preserving map $u \in [0\,;1] \mapsto \mu_{T_u} \in \operatorname{Aff}^+(\gamma)$ starting at $T_0 = \operatorname{id}_\gamma$: $u \in [0\,;1] \mapsto \mu_{T_u} \in \mathcal{S}(S, \Sigma, \Theta)$ is continuous;*

(3) γ remains a (simple closed) geodesic of μ_T of the same signature and with the same affine structure;

(4) if the first-return map $P^{\gamma}_{\alpha/\beta,\mu}\colon \gamma \to \gamma$ of a lightlike foliation of μ is well-defined on γ, then the first-return map of this foliation for μ_T is also well-defined on γ and is equal to $P^{\gamma}_{\alpha/\beta,\mu_T} = P^{\gamma}_{\alpha/\beta,\mu} \circ T$.

Moreover, μ_T can be chosen to coincide with μ outside of a tubular neighbourhood of γ as small as one wants.

We emphasize that this surgery construction is by no mean canonical, which does however not prevent it to be very useful. We will moreover observe during the proof that the surgeries μ_T are actually well-defined up to isotopy (see footnote [4] at the bottom of page 49), but we will not use this fact in the present paper.

4.4.1. *A one-parameter family of isometries.* We recall that if two $\mathbf{X}$-morphisms $f_1\colon U_1 \to V_1$ and $f_2\colon U_2 \to V_2$ coincide on a non-empty connected open subset $U \subset U_1 \cap U_2$, then f_1 and f_2 coincide on the connected component of $U_1 \cap U_2$ containing U. This is well-known and due to the analyticity of the action of $\mathbf{G}$ on $\mathbf{X}$. We fix henceforth an injective, continuous and orientation-preserving map $t \in [0\,;1] \mapsto S_t \in \mathrm{Aff}^+(\gamma)$ starting at $S_0 = \mathrm{id}$.

Lemma 4.12. *Let A, B be two small open tubular neighbourhoods of γ such that $\mathrm{Cl}(A) \subset B$ and $B \cap \Sigma = \gamma \cap \Sigma$. Then there exists $\varepsilon > 0$ and a continuous family $\Phi\colon t \in [0\,;\varepsilon] \mapsto \Phi_t \in \mathrm{C}(A^+ \cup \gamma, B^+ \cup \gamma)$ of continuous maps Φ_t defined on $A^+ \cup \gamma$ and with values in $B^+ \cup \gamma$, which are homeomorphisms onto their images, $\mathbf{X}$-morphisms in restriction to A^+, and such that $\Phi_t|_{\gamma} = S_t$ for any $t \in [0\,;\varepsilon]$.*

Observe that γ has indeed *two-sided* tubular neighbourhoods since S is orientable. We denoted in the above statement by A^+ and A^- the up and down connected components of $A \setminus \gamma$ (with respect to the orientation of γ and to the one of S, meaning that A^+ is on the left when γ is travelled positively) and likewise for $B^{\pm}$. While the Φ_t are $\mathbf{X}$-morphisms in restriction to A^+, note that they are in general *not* isometries of the singular $\mathbf{dS}^2$-surface S since they move its singular points if $\gamma \cap \Sigma \neq \varnothing$ and $S_t \neq \mathrm{id}_{\gamma}$.

Proof of Lemma 4.12. We first prove the existence of the Φ_t away from the singularities. Let U be a topological disk of closure contained in $A \setminus \Sigma$ and such that $U \cap \gamma \neq \varnothing$ is connected. Lemma 4.2.(3) shows then the existence of $\varepsilon > 0$ and of a unique continuous family $t \in [0\,;\varepsilon] \mapsto \phi_t$ of $\mathbf{X}$-morphisms defined on U and with values in B, such that $\phi_t|_{\gamma\cap U} = S_t|_{\gamma\cap U}$ for any $t \in [0\,;\varepsilon]$. Indeed if ϕ_t^1 and ϕ_t^2 are two such morphisms, then in any $\mathbf{X}$-chart $\varphi\colon V \to \mathbf{X}$ from an open set $V \subset U$, $\varphi \circ \phi_t^i \circ \varphi^{-1}$ is the restriction of a $g_t^i \in \mathbf{G}$ such that $g_t^i|_{\varphi(\gamma\cap U)} = \varphi \circ S_t \circ \varphi^{-1}|_{\varphi(\gamma\cap U)}$, hence $g_t^1 = g_t^2$ according to Lemma 4.2.(3) and thus $\phi_t^1 = \phi_t^2$. The uniqueness on any small enough connected open subset of U gives the existence on U by gluing these $\mathbf{X}$-morphisms ϕ_t^i together.

We now handle the singularities. Since Σ is discrete and γ compact, $\gamma \cap \Sigma$ is finite and $\gamma \setminus \Sigma$ is thus a finite union of intervals. We can assume without any lost of generality that γ is not an α-leaf, the arguments being formally analogous if γ is an α-leaf by replacing slit neighbourhoods of the form $V \setminus \mathcal{F}^+_{\alpha}(x)$ by slit neighbourhoods of the form $V \setminus \mathcal{F}^+_{\beta}(x)$ (which is authorized by Lemma 2.20). Let $x \in \gamma \cap \Sigma$ be a singular point of angle θ, $V \subset A$ be a normal convex neighbourhood of x, and let denote by I^+ the positive half-leaf of $\mathcal{F}_{\alpha}(x) \cap V$ starting from x for the orientation of $\mathcal{F}_{\alpha}$. Let U_1, U_2 be two topological disks of closures contained in $V \setminus I^+$. In particular, $\mathrm{Cl}(U_i)$ admits thus for $i = 1$ and 2 a neighbourhood contained in $S \setminus \Sigma$. We assume also that $U_1 \cap U_2 \neq \varnothing$, $U_1 \cap \gamma \neq \varnothing$ and $U_2 \cap \gamma \neq \varnothing$, and that those three intersections are connected. Possibly exchanging U_1 and U_2, we can moreover assume that with γ_i ($i = 1$ or 2) the connected component of $\gamma \setminus \Sigma$ containing $U_i \cap \gamma$: x is the future (respectively past) endpoint of γ_1 (resp. γ_2). Observe that possibly $\gamma_1 = \gamma_2$ if $\gamma \cap \Sigma$ is a point, in which case x is both a future and past endpoint. We showed in the previous paragraph the existence for $i = 1$ and 2 and for $\varepsilon > 0$ small enough, of unique continuous families $t \in [0\,;\varepsilon] \mapsto \phi_i^t$ of $\mathbf{X}$-morphisms defined on U_i and with values in $B \setminus \Sigma$, such that $\phi_t^i|_{\gamma\cap U_i} = S_t|_{\gamma\cap U_i}$ for $i = 1$ and 2 and any $t \in [0\,;\varepsilon]$. Let now $\varphi\colon V \setminus I^+ \to \mathbf{X}$ be a $\mathbf{X}$-chart of S such that $\varphi(V \setminus I^+) = V_0 \setminus \mathcal{F}^+_{\alpha}(\mathsf{o})$ with V_0 a connected neighbourhood of o, and such that $\pi_\theta \circ \varphi$ extends to a singular $\mathbf{X}$-chart at x with values in $\mathbf{X}_\theta$. This exists by definition of a standard singularity, see Lemma 2.14 for more details. Then $\varphi \circ \phi_t^i \circ \varphi^{-1}|_{\varphi(U_i)}$ is the restriction of a $g_i^t \in \mathbf{G}$ such that

$$g_t^i|_{\varphi(\gamma\cap U_i)} = \varphi \circ S_t \circ \varphi^{-1}|_{\varphi(\gamma\cap U_i)} \tag{4.5}$$

for $i = 1$ and 2 and any $t \in [0\,;\varepsilon]$. On the other hand by Definition 4.6 of a geodesic and since $\pi_\theta \circ \varphi$ extends to a singular $\mathbf{X}$-chart at x, $\varphi(\gamma_1 \cap V)$ and $\varphi(\gamma_2 \cap V)$ are two intervals of a common geodesic $L \subset \mathbf{X}$ through o, of future (resp. past) endpoint o. Therefore (4.5) implies $g_t^1 = g_t^2$ according to Lemma 4.2.(3), and thus $\phi_t^1 = \phi_t^2$ on $U_1 \cap U_2$, which shows that Φ_t is well-defined on $U_1 \cup U_2$. But if A is chosen small enough, such topological disks of the form U_i together with the open sets U of the first paragraph cover $\mathrm{Cl}(A^+) \setminus \Sigma$, which yields the existence of Φ_t on A^+ by gluing the ϕ_t^i together.

It is finally easy to observe on restriction to domains $U \subset A^+$ of the singular $\mathbf{X}$-atlas of S, that $\Phi_t|_U$ extends on $\mathrm{Cl}(U)$ to a continuous embedding that we still denote by Φ_t, and such that $\Phi_t|_{\gamma\cap U} = S_t|_{\gamma\cap U}$. This concludes the proof of the Lemma. □

Remark 4.13. It could at first sight seem surprising that we experienced no issue of holonomy while constructing in Lemma 4.12 the $\mathbf{X}$-morphisms Φ_t on the (non simply connected) annulus A^+. This is however due to the fact that the topology of A^+ is carried by γ, that $\mathbf{X}$-morphisms are entirely determined by their action on geodesics according to Lemma 4.2.(3), and that the affine automorphisms S_t used to define Φ_t are by definition *globally* defined on γ.

4.4.2. *Proof of Proposition 4.11.* We fix henceforth an injective, continuous and orientation-preserving map $t \in [0\,;1] \mapsto S_t \in \mathrm{Aff}^+(\gamma)$ starting at $S_0 = \mathrm{id}$. We observe first that it is sufficient to construct the surgery $\mu_t := \mu_{S_t}$ for any $t \in [0\,;\varepsilon]$ with some $\varepsilon > 0$ depending only on γ, since one only has to apply later the same construction to μ_ε and to compose with further surgeries. There exists a small open tubular neighbourhood $A \subset S$ of γ, whose two boundary components are transverse to whichever lightlike foliation γ is transverse to, *i.e.* to both lightlike foliations if γ is definite, and to $\mathcal{F}_\beta$ (respectively $\mathcal{F}_\alpha$) if γ is an α (resp. β) leaf. We moreover assume that $\mathrm{Cl}(A) \cap \Sigma = \gamma \cap \Sigma$, and that $A \setminus \gamma$ has two up and down connected components $A^\pm$. There exists also a closed curve $\sigma \subset A^+$ freely homotopic to γ within A^+ and that we orient compatibly with γ, which is transverse to whichever lightlike foliation γ is transverse to, and such that A^+ is itself a tubular neighbourhood of σ. In particular, $A^+ \setminus \sigma$ is the union of an upper boundary component A_1^+, and of a lower one A_2^+. If A is chosen small enough, there exists moreover an open tubular neighbourhood B of $\mathrm{Cl}(A)$ such that $\mathrm{Cl}(B) \cap \Sigma = \gamma \cap \Sigma$, and $\varepsilon > 0$ such that the continuous map $\Phi\colon t \mapsto \Phi_t$ satisfying

$$\Phi_t^{-1}|_\gamma = S_t$$

given by Lemma 4.12 is defined for any $t \in [0\,;\varepsilon]$. We recall that each Φ_t is defined on $A^+ \cup \gamma$, has values in $B^+ \cup \gamma$ (with the obvious similar notations for $B^\pm$ than the one we defined for $A^\pm$), and is a $\mathbf{X}$-morphism on restriction to A^+. We can moreover choose A small enough, so that for any $t \in [0\,;\varepsilon]$ we have:

$$\max_{x\in A^+} d_S(x, \Phi_t(x)) \leq 2\max_{x\in\gamma} L([x\,;S_t(x)]_\gamma), \tag{4.6}$$

with $L([a\,;b]_\gamma)$ the length of the intervals $[a\,;b]_\gamma$ of the oriented curve γ for a fixed Riemannian metric on S, and d_S the distance induced by this metric on S. Then there exists a continuous map

$$F\colon t \in [0\,;\varepsilon] \mapsto F_t \in \mathrm{C}(A, A \cup B^+),$$

such that $F_0 = \mathrm{id}_A$ and for any $t \in [0\,;\varepsilon]$:

(1) F_t is an orientation-preserving homeomorphism onto its image,
(2) F_t equals the identity on $A^- \cup \gamma$,
(3) and F_t equals Φ_t on $A_1^+ \cup \sigma$.

We can moreover assume that

$$\max_{x\in A} d_S(x, F_t(x)) \leq \max_{x\in A_1^+} d_S(x, F_t(x)). \tag{4.7}$$

Of course there exists many such maps F that give different surgeries, but we fix one.[4] We define then a singular $\mathbf{X}$-structure μ_t^* on A by $\mu_t^* = F_t^*\mu$. Observe that:

(1) since $F_t|_\gamma = \mathrm{id}|_\gamma$ and $A \cap \Sigma = \gamma \cap \Sigma$, the singular points of μ_t^* and their angles coincide with the one of $\mu|_A$, and the singular set of μ_t^* is thus equal to $\gamma \cap \Sigma$;
(2) since $F_t|_{A^-} = \mathrm{id}|_{A^-}$, $\mu_t^*|_{A^-} = \mu|_{A^-}$;
(3) since $F_t|_{A_1^+} = \Phi_t|_{A_1^+}$ is a $\mathbf{X}$-morphism of μ, the $\mathbf{X}$-atlas of $\mu|_{A_1^+}$ is compatible with its pullback by F_t, in other words $\mu_t^*|_{A_1^+} = \mu|_{A_1^+}$.

Since $(S\setminus \mathrm{Cl}(A_2^+)) \cap A = A^- \sqcup A_1^+$, the singular $\mathbf{X}$-structures μ_t^* of A and $\mu|_{S\setminus \mathrm{Cl}(A_2^+)}$ of $S\setminus \mathrm{Cl}(A_2^+)$ are compatible, *i.e.* the union of their singular $\mathbf{X}$-atlases defines a singular $\mathbf{X}$-structure μ_t on $S = (S \setminus \mathrm{Cl}(A_2^+)) \cup A$. By construction, the singular points of μ_t and their angles coincide with the ones of μ, and $t \mapsto \mu_t$ is moreover continuous since $t \mapsto F_t$ is so. Furthermore for any small enough chart $\varphi\colon U \to \mathbf{X}'$ of the singular $\mathbf{X}$-structure of S at a point $x \in \gamma$ ($\mathbf{X}' = \mathbf{X}$ if x is regular, and $\mathbf{X}' = \mathbf{X}_\theta$ if x is singular of angle θ), $U \subset F_t(A)$ and $F_t^{-1}(U)$ contains $x = F_t(x)$ since $F_t|_\gamma = \mathrm{id}_\gamma$. Moreover $\varphi \circ \gamma|_{\gamma^{-1}(U)}$ is a geodesic segment of $\mathbf{X}'$ according to Definition 4.6 of a geodesic, and $\varphi \circ F_t \circ \gamma|_{\gamma^{-1}(F_t^{-1}(U))}$ as well since $F_t|_\gamma = \mathrm{id}_\gamma$. This shows that γ remains a geodesic of μ_t with the same affine structure since $\varphi \circ F_t$ is a chart of the singular $\mathbf{X}$-atlas of μ_t.

We now prove the claim concerning the first-return maps of the lightlike foliations, and it is sufficient to do so for $\mathcal{F}_\alpha^\mu$ up to interverting the two foliations (note that if γ is lightlike, then γ is a closed β-leaf). We assume thus that the first-return map P_μ of $\mathcal{F}_\alpha^\mu$ on γ is well-defined, and denote by $P_\mu^1(x)$ the first intersection point of $\mathcal{F}_\alpha^\mu(x)$ with σ for $x \in \gamma$. Note that our orientation conventions impose to the interval of $\mathcal{F}_\alpha^\mu(x)$ from x to $P_\mu^1(x)$ to be contained in $\mathrm{Cl}(A_2^+)$. By definition of $\mu_t|_A = \mu_t^* = F_t^*\mu$, F_t is an isometry from $\mu_t|_A$ to μ, and therefore $\mathcal{F}_\alpha^{\mu_t}(x) \cap A = F_t^{-1}(\mathcal{F}_\alpha^\mu(x)) \cap A$ for any $x \in \gamma$. Since $F_t|_{A_1^+} = \Phi_t$ and $\Phi_t^{-1}|_\gamma = S_t$ by definition, this shows that $\mathcal{F}_\alpha^{\mu_t}(x) \cap \sigma = \Phi_t^{-1}(\mathcal{F}_\alpha^\mu(x)) \cap \sigma = \mathcal{F}_\alpha^\mu(S_t(x)) \cap \sigma$, using the fact that Φ_t is an isometry in the last equality. This shows that $\mathcal{F}_\alpha^{\mu_t}(x)$ intersects σ, and that its first intersection point satisfies:

$$P_{\mu_t}^1(x) = P_\mu^1 \circ S_t(x) \tag{4.8}$$

for any $x \in \gamma$. By definition we have $\mu_t = \mu$ in restriction to $S \setminus \mathrm{Cl}(A_2^+)$, and moerover due to our orientation conventions, $\mathcal{F}_\alpha^{\mu_t}(y)$ travelled positively leaves $\mathrm{Cl}(A_2^+)$ as long as it does not intersect γ, for any $y \in \sigma$. Therefore $\mathcal{F}_\alpha^{\mu_t}(y)$ travelled positively coincide with $\mathcal{F}_\alpha^\mu(y)$ as long as it does not intersect γ. The first intersection point $P_{\mu_t}^2(y)$ of $\mathcal{F}_\alpha^{\mu_t}(y)$ with γ exists thus, and is equal to the one of $\mathcal{F}_\alpha^\mu(y)$ with γ denoted by $P_\mu^2(y)$. This shows that the first-return map P_{μ_t} of $\mathcal{F}_\alpha^{\mu_t}$ on γ is well-defined, and since $P_{\mu/\mu_t} = P_{\mu/\mu_t}^2 \circ P_{\mu/\mu_t}^1$, we have moreover

$$P_{\mu_t} = P_\mu \circ S_t$$

according to (4.8), which concludes the proof of Proposition 4.11.

4.4.3. *Bounding the size of a surgery.* The topology of the space $\mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$ of singular $\mathbf{X}$-structures on $\mathbf{T}^2$ with singular points Σ of angles Θ was introduced in Definition 3.29, and we use the notations of this definition. We endow this space with a distance d defined as follows. Let $(\varphi_i\colon U_i \to X_i)_i$ be a finite singular $\mathbf{dS}^2$-atlas of $\mu \in \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$ (where $X_i = \mathbf{dS}^2$ if φ_i is a regular chart and $X_i = \mathbf{dS}^2_{\theta_i}$ at a singular point of angle θ_i) and $\mathcal{U}' = (U_i')_i$ be a shrinking of $(U_i)_i$ as in Definition 3.29. Then with d_i a fixed distance on X_i and $d_i^\infty(f,g) = \max_{x \in U_i} d_i(f(x), g(x))$ the associated uniform distance on continuous maps from U_i' to X_i, for any $\mu' \in \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$ defined by a singular $\mathbf{dS}^2$-atlas $\mathcal{A}' = (\psi_i\colon U_i' \to X_i)_i$, we define:

$$d(\mu', \mu) = \min\left(1, \sup\left\{\max_i d_i^\infty(\varphi_i|_{U_i'}, \psi_i) \;\middle|\; \mathcal{A}' \text{ atlas for } \mu' \text{ defined on } \mathcal{U}'\right\}\right). \tag{4.9}$$

[4] One can actually show that any two such maps are homotopic, and yield therefore isotopic surgeries.

We fix a Riemannian metric on $\mathbf{T}^2$, endow $\mathbf{T}^2$ with the induced distance $d_{\mathbf{T}^2}$, and denote by $L([x\,;y]_\gamma)$ the length of intervals $[x\,;y]_\gamma$ of any oriented piecewise smooth curve $\gamma\subset\mathbf{T}^2$ for this metric.

Lemma 4.14. *Let $\mu\in\mathcal{S}(\mathbf{T}^2,\Sigma,\Theta)$, and γ be a simple closed timelike geodesic of μ. Then there exists a constant $C>0$, such that for any surgery ν of μ around γ given by Proposition 4.11 and having a closed α-leaf $\mathcal{F}$, for any affine transformation $T\in\mathrm{Aff}^+(\mathcal{F})$, and for any surgery ν' of ν around $\mathcal{F}$ with respect to T given by Proposition 4.11:*

$$d(\nu,\nu')\leq C\max_{x\in\mathcal{F}}L([x\,;T(x)]_{\mathcal{F}}).$$

Proof. By construction ν' coincides with ν outside of A (we refer henceforth to the notations introduced in the proof of Proposition 4.11 paragraph 4.4.2 for the surgery ν' of ν). With F the homeomorphism used to define $\nu'|_A=F^*\nu|_A$ on A, we thus want to prove that $d(F^*\nu|_A,\nu|_A)\leq C\max_{x\in\mathcal{F}}L([x\,;T(x)]_{\mathcal{F}})$ for some constant $C>0$. It is sufficient to prove this claim for any small enough surgery ν' of ν, since the inequality follows then for further surgeries by triangular inequality. With $(\varphi_i\colon U_i\to X_i)_i$ a finite singular $\mathbf{dS}^2$-atlas of ν and $(U_i')_i$ a shrinking of $(U_i)_i$ as above, we can thus assume that $F(U_i')\subset U_i$. By finiteness of the atlas and continuity of the φ_i's, there exists furthermore a constant $C>0$ such that $d_i^\infty(\varphi_i\circ F|_{U_i'},\varphi_i|_{U_i'})\leq Cd^\infty_{\mathbf{T}^2}(F|_{U_i'},\mathrm{id}|_{U_i'})$ for any i and F, and therefore

$$d(F^*\nu|_A,\nu|_A)\leq Cd^\infty_{\mathbf{T}^2}(\mathrm{id}_{\mathbf{T}^2},F). \tag{4.10}$$

Moreover F satisfies $d^\infty_{\mathbf{T}^2}(\mathrm{id}_{\mathbf{T}^2},F)\leq\max_{x\in A_1^+}d_S(x,F(x))$ according to (4.7). Let Φ be the isometry used to define $F|_{\sigma\cup A_1^+}=\Phi|_{\sigma\cup A_1^+}$. We have then:

$$\begin{aligned}\max_{x\in A_1^+}d_S(x,F(x))&=\max_{x\in A_1^+}d_S(x,\Phi(x))\\&\leq 2\max_{x\in\mathcal{F}}L([x\,;T(x)]_{\mathcal{F}}),\end{aligned}$$

the last inequality being due to (4.6). In the end, the latter inequality implies together with (4.10) that $d(F^*\nu|_A,\nu|_A)\leq 2C\max_{x\in\mathcal{F}}L([x\,;T(x)]_{\mathcal{F}})$, which concludes the proof of the Lemma. ☐

4.5. **Conclusion of the proof of Theorem A.** Let S_1 and S_2 be two closed singular $\mathbf{dS}^2$-surfaces having a unique singularity of the same angle $\theta\in\mathbb{R}_+^*$, and with minimal and topologically equivalent lightlike bifoliations. Without lost of generality we can assume that $S_1=S_2=\mathbf{T}^2$, and that the oriented lightlike α-foliations (respectively β-foliations) of μ_1 and μ_2 coincide (this is possible since our definition of singular $\mathbf{X}$-structures authorizes $\mathcal{C}^0$-charts, hence singular $\mathbf{X}$-structures can be pulled back by homeomorphisms). According to Theorem A.1, μ_1 and μ_2 admit then freely homotopic simple closed timelike geodesics γ_1 and γ_2 (since they are class A according to Lemma A.8). Up to translations of $\mathbf{T}^2$ we can moreover assume that 0 is the unique singularity of both μ_1 and μ_2, which does not change the existence of freely homotopic simple closed timelike geodesics γ_1 and γ_2, nor the equality

$$A^+(\mathcal{F}^{\mu_1}_{\alpha/\beta})=A^+(\mathcal{F}^{\mu_2}_{\alpha/\beta})$$

of the oriented projective asymptotic cycles of the lightlike foliations. Our goal is to show the following approximation result.

Proposition 4.15. *Let μ_1,μ_2 be two singular $\mathbf{dS}^2$-structures on $\mathbf{T}^2$:*

- *having 0 as unique singularity of the same angle θ;*
- *admitting freely homotopic simple closed timelike geodesics γ_1 and γ_2;*
- *and whose lightlike bi-foliations are minimal, and have the same asymptotic cycles denoted by $A^+_{\alpha/\beta}\coloneqq A^+(\mathcal{F}^{\mu_1}_{\alpha/\beta})=A^+(\mathcal{F}^{\mu_2}_{\alpha/\beta})$.*

Then there exists sequences $\nu_{1,n},\nu_{2,n}$ of singular $\mathbf{dS}^2$-structures in $\mathcal{S}(\mathbf{T}^2,0,\theta)$ respectively converging to μ_1 and μ_2, and such that for any n:

(1) $\mathcal{F}^{\nu_{1,n}}_\alpha(0)$ and $\mathcal{F}^{\nu_{2,n}}_\alpha(0)$ are closed and freely homotopic;

(2) and $A^+(\mathcal{F}_\beta^{\nu_{1,n}}) = A^+(\mathcal{F}_\beta^{\nu_{2,n}}) = A_\beta^+$.

We first show how to conclude the proof of Theorem A with the help of Proposition 4.15. Since the α-leaves $\mathcal{F}_\alpha^{\nu_{1,n}}(0)$ and $\mathcal{F}_\alpha^{\nu_{2,n}}(0)$ are closed and freely homotopic in the one hand, and the β-foliations are minimal with identical irrational oriented projective asymptotic cycles $A^+(\mathcal{F}_\beta^{\nu_{1,n}}) = A^+(\mathcal{F}_\beta^{\nu_{2,n}})$ in the other hand, Theorem D shows that $[\nu_{1,n}] = [\nu_{2,n}]$ in the deformation space $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$. The same sequence $[\nu_{1,n}] = [\nu_{2,n}]$ converges thus both to $[\mu_1]$ and to $[\mu_2]$ in $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$, and since $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ is Hausdorff in the neighbourhood of $[\mu_1]$ and $[\mu_2]$ according to Theorem 3.35, this shows that $[\mu_1] = [\mu_2]$ and concludes the proof of Theorem A. □

Proof of Proposition 4.15. We denote by x_i the first intersection point of $\mathcal{F}_\alpha^{\mu_i}(0)$ with γ_i. Since $\mathcal{F}_\alpha^{\mu_i}$ and $\mathcal{F}_\beta^{\mu_i}$ are both assumed minimal, the first-return maps $P_{\alpha/\beta,\mu_i}^{\gamma_i} \colon \gamma_i \to \gamma_i$ are well-defined, and moreover have the same rotation numbers

$$\rho(P_{\alpha/\beta,\mu_1}^{\gamma_1}) = \rho(P_{\alpha/\beta,\mu_2}^{\gamma_2})$$

according to Lemma 3.27, since γ_1 and γ_2 are freely homotopic. According to Lemmas B.1.(5) and 4.10, there exists thus a sequence $r_n \in \mathbf{S}^1$ of rationals converging to $\rho(P_{\alpha,\mu_1}^{\gamma_1}) = \rho(P_{\alpha,\mu_2}^{\gamma_2}) \in [\mathbb{R} \setminus \mathbb{Q}]$ and sequences $T_{i,n} \in \mathrm{Aff}^+(\gamma_i)$ of affine transformations of γ_i converging uniformly to id_{γ_i}, such that for $i = 1$ and 2 and for any n: the orbit of x_i for $P_{\alpha,\mu_i}^{\gamma_i} \circ T_{i,n}$ is periodic and of rational cyclic order r_n. Proposition 4.11 yields then a surgery $\mu_{i,n} = (\mu_i)_{T_{i,n}}$ of μ_i around the geodesic γ_i with respect to $T_{i,n}$ such that:

(1) $\mu_{i,n}$ has a unique singularity of angle θ at 0;
(2) γ_i remains a timelike simple closed geodesic of $\mu_{i,n}$;
(3) the first-return map of $\mathcal{F}_{\alpha/\beta}^{\mu_{i,n}}$ on γ_i is well-defined and equals the circle homeomorphism

$$P_{\alpha/\beta,\mu_{i,n}}^{\gamma_i} = P_{\alpha/\beta,\mu_i}^{\gamma_i} \circ T_{i,n}. \tag{4.11}$$

Possibly exchanging the direction of the surgeries and passing to a subsequence, we can moreover assume that $T_{i,n}$ converges uniformly *and monotonically to* id_{γ_i} *from above*, *i.e.* that for any $x \in \gamma_i$, $(T_{i,n}(x))_n$ is decreasing for the cyclic order of γ_i and converges uniformly to x. Therefore:

$$\lim \mu_{i,n} = \mu_i \tag{4.12}$$

according to Proposition 4.11.(2). Hence $\mathcal{F}_{\alpha/\beta}^{\mu_{i,n}}$ converges to $\mathcal{F}_{\alpha/\beta}^{\mu_i}$, and in particular $A^+(\mathcal{F}_{\alpha/\beta}^{\mu_{i,n}})$ converges to $A^+(\mathcal{F}_{\alpha/\beta}^{\mu_i})$. Moreover according to (4.11) and by construction of $T_{i,n}$, the respective orbits of x_1 and x_2 for $P_{\alpha,\mu_{1,n}}^{\gamma_1}$ and $P_{\alpha,\mu_{2,n}}^{\gamma_2}$ are periodic and of the same rational cyclic order r_n, hence $\rho(P_{\alpha,\mu_{1,n}}^{\gamma_1}) = \rho(P_{\alpha,\mu_{2,n}}^{\gamma_2}) = r_n$ according to Proposition 3.21. In particular, the α-lightlike leaves $\sigma_{1,n} \coloneqq \mathcal{F}_\alpha^{\mu_{1,n}}(0)$ and $\sigma_{2,n} \coloneqq \mathcal{F}_\alpha^{\mu_{2,n}}(0)$ are thus closed. For any large enough n, Lemma 3.28 shows moreover that $\rho(P_{\alpha,\mu_{1,n}}^{\gamma_1}) = \rho(P_{\alpha,\mu_{2,n}}^{\gamma_2})$ implies

$$A^+(\mathcal{F}_\alpha^{\mu_{1,n}}) = A^+(\mathcal{F}_\alpha^{\mu_{2,n}}),$$

since γ_1 and γ_2 are freely homotopic and $\mathcal{F}_\alpha^{\mu_{1,n}}, \mathcal{F}_\alpha^{\mu_{2,n}}$ close enough. In particular the closed α-lightlike leaves $\sigma_{1,n}$ and $\sigma_{2,n}$ are thus freely homotopic, since $A^+(\mathcal{F}_\alpha^{\mu_{i,n}}) = [\sigma_{i,n}]$ according to Proposition 3.25.

We now perform on $\mu_{i,n}$ a second surgery around $\sigma_{i,n}$, allowing us to keep the closed α-leaves $\sigma_{i,n}$ unchanged while modifying the asymptotic cycle of the β-foliation until recovering the original one of $\mathcal{F}_\beta^{\mu_i}$.

Lemma 4.16. *Let μ be a singular $\mathbf{dS}^2$-structure on $\mathbf{T}^2$, with 0 as unique singular point of angle θ, and whose lightlike foliations are minimal. Let γ be a simple closed timelike geodesic of μ, and $T_n \in \mathrm{Aff}^+(\gamma)$ be a sequence converging uniformly and monotonically to id_γ from above, and such that $\sigma_n \coloneqq \mathcal{F}_\alpha^{\mu_n}(0)$ is closed for any n, with $\mu_n \coloneqq \mu_{T_n}$ the surgery of μ around γ with respect to T_n given by Proposition 4.11. Then there exists a sequence $S_n \in \mathrm{Aff}^+(\sigma_n)$ such that:*

(1) S_n converges uniformly and monotonically to the identity from above, in the sense that:

$$\lim \max_{x \in \sigma_n} L([x\,; S_n(x)]_{\sigma_n}) = 0 \tag{4.13}$$

with $L([a\,;b]_{\sigma_n})$ the length of intervals $[a\,;b]_{\sigma_n}$ of the oriented curve σ_n for a fixed Riemannian metric on $\mathbf{T}^2$;

(2) $A^+(\mathcal{F}_\beta^{\nu_n}) = A^+(\mathcal{F}_\beta^{\mu})$, with $\nu_n := (\mu_n)_{S_n}$ the surgery of μ_n around σ_n with respect to S_n given by Proposition 4.11.

Let us temporarily admit this statement and conclude thanks to it the proof of Proposition 4.15. Denoting by $S_{i,n} \in \mathrm{Aff}^+(\sigma_{i,n})$ the affine transformations given by Lemma 4.16 and by $\nu_{i,n}$ the surgery $(\mu_{i,n})_{S_{i,n}}$, the limit (4.13) shows that $\lim d(\nu_{i,n}, \mu_{i,n}) = 0$ according to Lemma 4.14, with d the distance on $\mathcal{S}(\mathbf{T}^2, 0, \theta)$ defined in (4.9). We finally conclude that $\nu_{i,n}$ converges to μ_i in $\mathcal{S}(\mathbf{T}^2, 0, \theta)$, since $\mu_{i,n}$ does so according to (4.12). Since the closed α-leaf $\sigma_{i,n}$ is unchanged during the surgery given by Proposition 4.11, the α-leaves $\mathcal{F}_\alpha^{\nu_{1,n}}(0) = \sigma_{1,n}$ and $\mathcal{F}_\alpha^{\nu_{2,n}}(0) = \sigma_{2,n}$ remain closed and homotopic. Moreover $A^+(\mathcal{F}_\beta^{\nu_{1,n}}) = A^+(\mathcal{F}_\beta^{\nu_{2,n}}) = A_\beta^+$ by assumption on the $S_{i,n}$, which concludes the proof of Proposition 4.15. □

The last step in the proof of Theorem A is thus the:

Proof of Lemma 4.16. Note that our assumption on T_n implies that μ_n converges to μ according to Proposition 4.11.(2), hence that $\mathcal{F}_{\alpha/\beta}^{\mu_n}$ converges to $\mathcal{F}_{\alpha/\beta}^{\mu}$. We first check that the surgery around σ_n indeed allows us to modify the asymptotic cycle of the β-foliation, since:

Fact 4.17. *Possibly passing to a subsequence, σ_n is a section of $\mathcal{F}_\beta^{\mu_n}$, and the first-return map $P_{\beta,\mu_n}^{\sigma_n}$ is thus well-defined.*

Proof. The surface with boundary A_n obtained from cutting $\mathbf{T}^2$ along σ_n is an annulus whose boundary components are two copies of σ_n, and the foliation $\mathcal{F}_\beta^{\mu_n}$ induces a foliation $\mathcal{F}_n$ of A_n transverse to its boundary. By "spiraling" $\mathcal{F}_n$ around ∂A_n, one obtains a foliation $\mathcal{F}'_n$ of A_n tangent to its boundary, and it is known that such a foliation $\mathcal{F}'_n$ is obtained by gluing together a finite number of Reeb components and suspensions (see for instance [HH86, Remark 4.2.1 and Theorem 4.2.15] for more details). As a consequence, either σ_n intersects every leaf of $\mathcal{F}_\beta^{\mu_n}$, or else $\mathcal{F}'_n$ and thus $\mathcal{F}_n$ admit a closed leaf F_n^0 in the interior of A_n, corresponding to a closed leaf F_n of $\mathcal{F}_\beta^{\mu_n}$. In the latter case F_n^0 is freely homotopic to the boundary of A_n within A_n, and F_n is thus freely homotopic to σ_n in $\mathbf{T}^2$. In particular according to Proposition 3.25, $\mathcal{F}_\alpha^{\mu_n}$ and $\mathcal{F}_\beta^{\mu_n}$ have then the same projective asymptotic cycle given by the homotopy class of σ_n. But since $A(\mathcal{F}_{\alpha/\beta}^{\mu_n})$ converges to $A(\mathcal{F}_{\alpha/\beta}^{\mu})$, and $A(\mathcal{F}_\alpha^{\mu}) \neq A(\mathcal{F}_\beta^{\mu})$ according to Lemma A.8, there exists $N \in \mathbb{N}$ such that for any $n \geq N$: $A(\mathcal{F}_\alpha^{\mu_n}) \neq A(\mathcal{F}_\beta^{\mu_n})$. This prevents thus $\mathcal{F}'_n$ hence $\mathcal{F}_n$ to have a closed leaf F_n^0 in the interior of A_n, and implies in turn that for any $n \geq N$: σ_n intersects every leaf of $\mathcal{F}_\beta^{\mu_n}$, concluding the proof of the Fact. □

We fix henceforth a Riemannian metric on $\mathbf{T}^2$, and denote by d_{σ_n} the induced distance on σ_n and by $d_{\sigma_n}^\infty$ the uniform distance induced on continuous maps of σ_n. Note that the first-return map $P_{\beta,\mu}^{\sigma_n}$ is well-defined since $\mathcal{F}_\beta^{\mu}$ is minimal, and that $\lim\ d_{\sigma_n}^\infty(P_{\beta,\mu_n}^{\sigma_n}, P_{\beta,\mu}^{\sigma_n}) = 0$ since $\mathcal{F}_\beta^{\mu_n}$ converges to $\mathcal{F}_\beta^{\mu}$. More precisely, our choices of orientation show that $P_{\beta,\mu_n}^{\sigma_n}(x)$ converges to $P_{\beta,\mu}^{\sigma_n}(x)$ from below on the oriented α-leaf σ_n, in the sense that:

$$\lim \max_{x\in\sigma_n} L([P_{\beta,\mu_n}^{\sigma_n}(x)\,; P_{\beta,\mu}^{\sigma_n}(x)]_{\sigma_n}) = 0 \tag{4.14}$$

with $L([a\,;b]_{\sigma_n})$ the length of intervals of σ_n for the Riemannian metric of $\mathbf{T}^2$.

For $S \in \mathrm{Aff}^+(\sigma_n)$ let us denote by $(\mu_n)_S$ the surgery of μ_n around the closed α-leaf σ_n with respect to S given by Proposition 4.11, such that:

(1) $(\mu_n)_S$ has a unique singularity of angle θ at 0;
(2) $\mathcal{F}_\alpha^{(\mu_n)_S}(0) = \mathcal{F}_\alpha^{\mu_n}(0) = \sigma_n$;
(3) the first-return map of $\mathcal{F}_\beta^{(\mu_n)_S}$ on σ_n is well-defined and equal to the circle homeomorphism $P_{\beta,(\mu_n)_S}^{\sigma_n} = P_{\beta,\mu_n}^{\sigma_n} \circ S$.

Note that while γ is not anymore a geodesic of $(\mu_n)_S$, it remains however a section of $\mathcal{F}_\beta^{(\mu_n)_S}$ since it is a section of $\mathcal{F}_\beta^{\mu_n}$, and the first-return map $P_{\beta,(\mu_n)_S}^{\gamma}$ is therefore well-defined. Let

$t \in [0\,;1] \mapsto S_t \in \mathrm{Aff}^+(\sigma_n)$ be a continuous, orientation-preserving and surjective map, injective on restriction to $[0\,;1[$ and such that $S_0 = S_1 = \mathrm{id}_{\sigma_n}$. According to Lemma 4.10,

$$t \in [0\,;1[\mapsto P^{\sigma_n}_{\beta,(\mu_n)_{S_t}}(x) = P^{\sigma_n}_{\beta,\mu_n} \circ S_t(x) \in \sigma_n \tag{4.15}$$

is then a bijective, continuous and increasing map for any $x \in \sigma_n$, and $t \in [0\,;1[\mapsto (\mu_n)_{S_t}$ is moreover continuous according to Proposition 4.11. This shows that

$$t \in [0\,;1[\mapsto P^{\gamma}_{\beta,(\mu_n)_{S_t}} \in \mathrm{Homeo}^+(\gamma) \tag{4.16}$$

is continuous, and that

$$t \in [0\,;1[\mapsto P^{\gamma}_{\beta,(\mu_n)_{S_t}}(x) \in \gamma \tag{4.17}$$

is surjective and strictly decreasing for any $x \in \gamma$, since the holonomy of the β-foliation of μ_n induces homeomorphisms from small intervals of $\mathcal{F}^{\mu_n}_{\alpha}$ to small intervals of γ. We emphasize that our orientation conventions induce a reversal of the direction of the perturbation, wether it is observed on the first-return map on σ_n in (4.15) or on the first-return map on γ in (4.17). To say it roughly: “turning positively on σ_n implies turning negatively on γ”.

Due to this change of orientation, the continuous maps $t \in [0\,;1[\mapsto \rho(P^{\gamma}_{\beta,(\mu_n)_{S_t}}) \in \mathbf{S}^1$ and $t \in [0\,;1[\mapsto A^+(\mathcal{F}^{(\mu_n)_{S_t}}_{\beta}) \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ are non-increasing according to Lemma B.1 (the topological circle $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ being endowed with the natural orientation induced by $\mathbf{T}^2$). On the other hand, $A^+(\mathcal{F}^{\mu_n}_{\beta})$ is decreasing to the irrational line $A^+(\mathcal{F}^{\mu}_{\beta})$ since T_n is assumed to converge to id_γ from above. In conclusion for any large enough n, $A^+(\mathcal{F}^{(\mu_n)_{S_t}}_{\beta})$ is slightly above $A^+(\mathcal{F}^{\mu}_{\beta})$ at $t=0$ and is decreasing with t. The distance of $A^+(\mathcal{F}^{(\mu_n)_{S_t}}_{\beta})$ to $A^+(\mathcal{F}^{\mu}_{\beta})$ on the circle $\mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2,\mathbb{R}))$ is thus non-increasing as long as $A^+(\mathcal{F}^{(\mu_n)_{S_t}}_{\beta})$ does not meet $A^+(\mathcal{F}^{\mu}_{\beta})$ (which may *a priori* not happen, but we show now that it does happen). Since the images of $A^+(\mathcal{F}^{\mu}_{\beta})$ by Dehn twists around γ do not accumulate on $A^+(\mathcal{F}^{\mu}_{\beta})$, this shows in particular that for any large enough n and as long as $A^+(\mathcal{F}^{(\mu_n)_{S_t}}_{\beta})$ does not meet $A^+(\mathcal{F}^{\mu}_{\beta})$:

$$A^+(\mathcal{F}^{(\mu_n)_{S_t}}_{\beta}) = D_*(A^+(\mathcal{F}^{\mu}_{\beta})) \Rightarrow A^+(\mathcal{F}^{(\mu_n)_{S_t}}_{\beta}) = A^+(\mathcal{F}^{\mu}_{\beta}), \tag{4.18}$$

for any Dehn twist D around γ.

Now since $t \in [0\,;1[\mapsto P^{\gamma}_{\beta,(\mu_n)_{S_t}}(x) \in \gamma$ is surjective for any $x \in \gamma$, the map $t \in [0\,;1[\mapsto \rho(P^{\gamma}_{\beta,(\mu_n)_{S_t}}) \in \mathbf{S}^1$ is surjective according to Lemma B.1.(3), and there exists thus a smallest time $t_n \in [0\,;1]$ such that

$$\rho(P^{\gamma}_{\beta,\nu_n}) = \rho(P^{\gamma}_{\beta,(\mu_n)_{S_{t_n}}}) = \rho(P^{\gamma}_{\beta,\mu}), \tag{4.19}$$

denoting $S_n := S_{t_n} \in \mathrm{Aff}^+(\sigma_n)$ and $\nu_n := (\mu_n)_{S_n}$. This implies that $A^+(\mathcal{F}^{\nu_n}_{\beta}) = D_*(A^+(\mathcal{F}^{\mu}_{\beta}))$ for some Dehn twist D around γ according to Lemma 3.28, and thus that

$$A^+(\mathcal{F}^{\nu_n}_{\beta}) = A^+(\mathcal{F}^{\mu}_{\beta}) \tag{4.20}$$

according to the observation (4.18). Note that for any n, denoting $F_n(t) = \rho(P^{\gamma}_{\beta,(\mu_n)_{S_t}})$ we have:

$$F_n([0\,;t_n]) = [\rho(P^{\gamma}_{\beta,\mu})\,;\rho(P^{\gamma}_{\beta,\mu_n})] \tag{4.21}$$

since t_n is the smallest time where the equality (4.19) is satisfied.

We have thus reach in (4.20) our goal of modifying the β-foliation until recovering the original asymptotic cycle of $\mathcal{F}^{\mu}_{\beta}$. To conclude the proof of Lemma 4.16, it remains now to control the size of the surgery ν_n around σ_n, by proving the limit (4.13) that we recall for the convenience of the reader:

$$\lim \max_{x\in\sigma_n} L([x\,;S_n(x)]_{\sigma_n}) = 0. \tag{4.22}$$

Denoting $P^{\sigma_n}_{\beta,\mu_n} = P^{\sigma_n}_{\beta,\mu} \circ U_n$ so that $P^{\sigma_n}_{\beta,\nu_n} = P^{\sigma_n}_{\beta,\mu} \circ U_n \circ S_n$, it is important to note at this point that U_n is *not* an affine transformation of σ_n. Indeed, even though $P^{\gamma}_{\beta,\mu_n} = P^{\gamma}_{\beta,\mu} \circ T_n$ with T_n an affine transformation of γ, the computation of U_n involves the holonomy of $\mathcal{F}^{\mu}_{\beta}$ between γ and segments of leaves of $\mathcal{F}^{\mu}_{\alpha}$, which is not affine but only *projective* according to Proposition C.2. Therefore, while U_n converges to the identity since $\mathcal{F}^{\mu_n}_{\beta}$ converges to $\mathcal{F}^{\mu}_{\beta}$, we are now comparing maps U_n and S_n of σ_n which are *not* in the same one-parameter group of $\mathrm{Homeo}^+(\sigma_n)$, and this is what makes the proof of (4.22) slightly more technical than one could expect.

We proceed by contradiction and assume thus that the limit (4.22) does not hold. Passing to a subsequence, there exists then $\varepsilon_1 > 0$ such that

$$\max_{x\in\sigma_n} L([x\,;S_n(x)]_{\sigma_n}) \geq \varepsilon_1 \tag{4.23}$$

for any n. Since $\mathcal{F}^{\mu_n}_{\beta}$ converges to $\mathcal{F}^{\mu}_{\beta}$, there exists furthermore an $\varepsilon_2 > 0$ such that

$$L([P^{\sigma_n}_{\beta,\mu_n}(x)\,;P^{\sigma_n}_{\beta,\mu_n}(y)]_{\sigma_n}) \geq \varepsilon_2 L([x\,;y]_{\sigma_n}) \tag{4.24}$$

for any n and $x, y \in \sigma_n$. Now since $P^{\sigma_n}_{\beta,\nu_n} = P^{\sigma_n}_{\beta,\mu_n} \circ S_n$, (4.24) and (4.23) imply together that

$$\max_{x\in\sigma_n} L([P^{\sigma_n}_{\beta,\mu_n}(x)\,;P^{\sigma_n}_{\beta,\nu_n}(x)]_{\sigma_n}) \geq \varepsilon \tag{4.25}$$

for any n, with $\varepsilon \coloneqq \varepsilon_1\varepsilon_2 > 0$. Since $P^{\sigma_n}_{\beta,\mu_n}$ converges to $P^{\sigma_n}_{\beta,\mu}$ from below according to (4.14), we would like to infer from (4.25) that for any large enough n, $P^{\sigma_n}_{\beta,\nu_n}$ pushes every point x above $P^{\sigma_n}_{\beta,\mu}(x)$ by a distance bounded from below. This would show that $\rho(P^{\sigma_n}_{\beta,\nu_n}) \neq \rho(P^{\sigma_n}_{\beta,\mu})$ according to Lemma B.1.(4), contradicting (4.20) according to Lemma 3.27, and concluding thus the proof. The only possible phenomenon preventing us to apply this argument straightforwardly this way, and forcing us to be more cautious, is that some points x may be moved by $P^{\sigma_n}_{\beta,\nu_n}$ above $P^{\sigma_n}_{\beta,\mu}(x)$ while some other may move between $P^{\sigma_n}_{\beta,\mu_n}(x)$ and $P^{\sigma_n}_{\beta,\mu}(x)$. But since all of them are in any case pushed above $P^{\sigma_n}_{\beta,\mu_n}(x)$ which itself uniformly approaches $P^{\sigma_n}_{\beta,\mu}(x)$ from below, the uniform lower bound (4.25) will allow us to apply the same argument on the limit, and to conclude by continuity of the rotation number. We now implement this strategy as follows.

By compactness of σ_n, there exists $x_n \in \sigma_n$ such that

$$L([P^{\sigma_n}_{\beta,\mu_n}(x_n)\,;P^{\sigma_n}_{\beta,\nu_n}(x_n)]_{\sigma_n}) = \max_{x\in\sigma_n} L([P^{\sigma_n}_{\beta,\mu_n}(x)\,;P^{\sigma_n}_{\beta,\nu_n}(x)]_{\sigma_n}) \geq \varepsilon \tag{4.26}$$

for any n, and passing to a subsequence we can moreover assume by compactness of $\mathbf{T}^2$ that x_n converges to some $x_\infty \in \mathbf{T}^2$. Let us denote by $y_\infty \in \gamma$ the first intersection point of $\mathcal{F}^{\mu}_{\beta}(x_\infty)$ with γ. Since $\mathcal{F}^{\mu_n}_{\alpha/\beta}$ converges to $\mathcal{F}^{\mu}_{\alpha/\beta}$, there exists then an open neighbourhood U of x_∞, an open interval J around y_∞ in γ and a constant $C > 0$, such that for any n and any interval $I \subset U$ of the foliation $\mathcal{F}^{\mu_n}_{\alpha}$, the holonomy of $\mathcal{F}^{\mu_n}_{\beta}$ defines a homeomorphism H_n from I to an interval contained in J, and satisfies

$$L([H_n(x)\,;H_n(y)]_{\gamma}) \geq C L([x\,;y]_{\mathcal{F}^{\mu_n}_{\alpha}}) \tag{4.27}$$

for any $x, y \in I$. Since $P^{\gamma}_{\beta,\mu_n} \circ H_n(x) = H_n \circ P^{\sigma_n}_{\beta,\mu_n}(x)$ and $P^{\gamma}_{\beta,\nu_n} \circ H_n(x) = H_n \circ P^{\sigma_n}_{\beta,\nu_n}(x)$ for any $x \in \sigma_n$ for which this equality makes sense, (4.26) and (4.27) imply together the existence of $\eta > 0$ such that

$$L([P^{\gamma}_{\beta,\nu_n}(y_\infty)\,;P^{\gamma}_{\beta,\mu_n}(y_\infty)]_{\gamma}) \geq \eta \tag{4.28}$$

for any n. The reversal of the orientation of intervals between (4.26) and (4.28) is due as previously to our orientation conventions. We recall indeed that $P^{\gamma}_{\beta,\mu_n}(x)$ converges to $P^{\gamma}_{\beta,\mu}(x)$ from above, while $t \in [0\,;1[\mapsto P^{\gamma}_{\beta,(\mu_n)_{S_t}}(x) \in \gamma$ is strictly decreasing (see (4.17)).

Since $\mathcal{F}^{\mu_n}_{\beta}$ converges to $\mathcal{F}^{\mu}_{\beta}$, P^{γ}_{β,μ_n} converges to $P^{\gamma}_{\beta,\mu}$. On the other hand by passing to a subsequence, we may assume according to Arzelà-Ascoli theorem that P^{γ}_{β,ν_n} converges to some continuous map $P_\infty\colon \gamma \to \gamma$. Note that while P_∞ is not necessarily a homeomorphism, it remains an orientation-preserving *endomorphism* of γ, *i.e.* by definition a continuous, degree-one and orientation-preserving self-map of γ. According to [PJM82, Appendix Lemma 3] and [NPT83, Chapter III Proposition 3.3], the Proposition-Definition 3.18 defining the rotation number extends to endomorphisms of γ, and the rotation number remains moreover continuous on $\mathrm{End}^+(\gamma)$. The equality (4.19) yields thus

$$\rho(P_\infty) = \rho(P^\gamma_{\beta,\mu}) \tag{4.29}$$

at the limit, while the uniform bound (4.28) becomes:

$$L([P_\infty(y_\infty)\,;P^\gamma_{\beta,\mu}(y_\infty)]_\gamma) \geq \eta. \tag{4.30}$$

For any n, $G_n\colon s\in[0\,;1]\mapsto P^\gamma_{\beta,(\mu_n)_{S_{stn}}}\in\mathrm{Homeo}^+(\gamma)$ is according to (4.16) a continuous one-parameter family from $(G_n)_0 = P^\gamma_{\beta,\mu_n}$ to $(G_n)_1 = P^\gamma_{\beta,\nu_n}$, and $s\in[0\,;1]\mapsto G_s(y)$ is moreover non-increasing for any $y\in\gamma$ according to (4.17). Possibly passing to a subsequence, these continuous maps G_n uniformly converge to a continuous map $G\colon[0\,;1]\mapsto G_t\in\mathrm{Homeo}^+(\gamma)$ such that $G_0 = P^\gamma_{\beta,\mu}$, $G_1 = P_\infty$ and $t\mapsto G_t(y)$ is non-increasing for any $y\in\gamma$. Moreover (4.21) shows that $t\in[0\,;1]\mapsto\rho(G_t)\in\mathbf{S}^1$ is not surjective, while (4.30) shows that $G_1(y_\infty)\neq G_0(y_\infty)$. The proof of Lemma B.1.(4) holds now without any modification for $G_t\in\mathrm{End}^+(\gamma)$ and shows thus that $\rho(P_\infty)\neq\rho(P^\gamma_{\beta,\mu})$, which contradicts (4.29). This contradiction eventually shows that the limit (4.22) holds, and concludes the proof of the Lemma. □

Appendix A. Simple closed definite geodesics in singular constant curvature Lorentzian surfaces

The main goal of this appendix is to prove the existence of simple closed definite geodesics in any closed constant curvature singular Lorentzian surface. This result is well-known for regular Lorentzian surfaces, see for instance [Tip79, Gal86, Suh13], and we check here that the classical proof remains valid in our singular setting. This appendix being entirely independent from the rest of the paper, the reader may choose to use Theorem A.1 below as a "black-box" in a first reading and to come back to its proof later on.

We will work in this section in the general setting of singular $\mathbf{X}$-surfaces, $(\mathbf{G},\mathbf{X})$ denoting as usual the pair $(\mathrm{PSL}_2(\mathbb{R}),\mathbf{dS}^2)$ or $(\mathbb{R}^{1,1}\rtimes\mathrm{SO}^0(1,1),\mathbb{R}^{1,1})$. *Geodesics* of singular $\mathbf{X}$-surfaces were defined in Definition 4.6. The goal of the section is to prove the following existence result, which will be a direct consequence of the Proposition A.9 and the Theorem A.16 proved below.

Theorem A.1. *Let μ_1 and μ_2 be two class A singular $\mathbf{X}$-structures on a closed surface S, having identical oriented lightlike bi-foliations. Then μ_1 and μ_2 admit freely homotopic simple closed timelike geodesics, which are not null-homotopic. The obvious analogous statement holds for the spacelike signature.*

While it is *a priori* not clear that the usual tools and results of Lorentzian geometry can be used in our singular setting, the goal of this appendix is precisely to show that this toolbox persists in the setting of singular $\mathbf{X}$-surfaces – which is likely to have an independent interest in the future for their further study. Notions and results of this section are well-known in the classical setting of regular Lorentzian manifolds, and their proofs are mainly adapted from [Min19] or [BEE96]. We essentially follow the proof of [Tip79] to show Theorem A.1, with slight adaptations more suited to our setting. The main idea is to prove the existence of a simple closed timelike curve which *maximizes* the Lorentzian length, which is the extremal property of Lorentzian timelike geodesics in contrast with Riemannian ones.

A.1. **Timelike curves and causality notions.** The following definition is identical to the classical one, to the exception of condition (1) handling the singular points.

Definition A.2. In a singular $\mathbf{X}$-surface (S,Σ), a *timelike* (respectively *causal*, *spacelike*) curve is a continuous curve $\sigma\colon[a\,;b]\to S$ such that:

1. for any $t_0\in[a\,;b]$ for which $\gamma(t_0)\in\Sigma$, there exists $\varepsilon>0$ and a normal convex neighbourhood U of $\gamma(t_0)$ such that $\gamma|_{]t_0-\varepsilon;t_0[}\subset U^-$ and $\gamma|_{]t_0;t_0+\varepsilon[}\subset U^+$, with U^- and U^+ the past and future timelike (resp. spacelike, causal) quadrants in U;
2. σ is locally Lipschitz;
3. $\sigma'(t)$ is almost everywhere non-zero, future-directed and timelike (resp. causal, spacelike).

We emphasize that timelike, causal and spacelike curves are in particular always assumed to be relatively compact and future-oriented, unless explicitly stated otherwise. They are moreover not *trivial* (*i.e.* reduced to a point), and $\sigma^{-1}(\Sigma)$ is discrete according to (1), hence finite. S will always be endowed with an auxiliary $\mathcal{C}^\infty$ Riemannian metric h and its induced distance d, with respect to which the Lipschitz conditions are considered. Note that σ is compact and locally Lipschitz, hence Lipschitz. A locally Lipschitz function being almost everywhere differentiable according to Rademacher's Theorem, $\sigma'(t)$ is almost everywhere defined which gives sense to the condition (3). *Past* timelike, causal and spacelike curves are defined as future-oriented curves of the same signature travelled in the opposite direction.

Definition A.3. In a singular $\mathbf{X}$-surface S, we denote for $x \in S$ by:

(1) $I^+(x)$ (respectively $I^-(x)$) the set of points that can be reached from x by a timelike (resp. past timelike) curve;
(2) $J^+(x)$ (respectively $J^-(x)$) the set of points that can be reached from x by a causal (resp. past causal) curve.

We will denote $I^+_S(x)$ and likewise for the other notions, to specify that the curves are assumed to be contained in S. An open set U of a singular $\mathbf{X}$-surface S is *causally convex* if there exists no causal curve of S which intersects U in a disconnected set. S is said *strongly causal* if any point of S admits arbitrarily small causally convex open neighbourhoods. In particular S is then *causal*, *i.e.* admits no closed causal curves. S is *globally hyperbolic* if it is strongly causal, and if for any $p, q \in S$, $J^+(p) \cap J^-(q)$ is relatively compact.

Observe that for any convex normal neighbourhood U of x of future and past timelike quadrant U^+ and U^-, $I^\pm_U(x) = U^\pm$. This is classical in the regular Lorentzian setting (see for instance [Min19, Theorem 2.9 p.29]) and follows from our definition of timelike and causal curves at a singular point (see Proposition 4.8 concerning normal convex neighbourhoods of singular points). Observe moreover that a $\mathbf{X}$-structure on $\mathbb{R}^2$ has no closed lightlike leaves, as a consequence of the classical Poincaré-Hopf theorem for topological foliations proved for instance in [HH86, Theorem 2.4.6]. The two following results are well-known for regular Lorentzian metrics on $\mathbb{R}^2$, and the proofs respectively given in [BEE96, Proposition 3.42 and Corollary 3.44] persist in our singular setting. We repeat below a quick version of these proofs, and refer to the above reference for more details.

Lemma A.4. *In a singular $\mathbf{X}$-surface homeomorphic to $\mathbb{R}^2$, a timelike curve intersects a given lightlike leaf at most once.*

Proof. We henceforth endow $\mathbb{R}^2$ with a singular $\mathbf{X}$-structure, and assume by contradiction the existence of a timelike curve intersecting a given lightlike leaf at least twice. It is well-known that timelike curves of regular Lorentzian manifolds are locally injective, and on the other hand it follows readily from our definition of a timelike curve σ in a singular $\mathbf{X}$-surface that it is also locally injective at a singularity (since $\sigma^{-1}(\Sigma)$ is discrete). Hence timelike curves are locally injective (see also Fact A.13 below for an alternative proof), and it is thus easy to reduce the proof to the case of an injective timelike curve $\sigma\colon [a\,;b] \to \mathbb{R}^2$ such that $\sigma(a)$ and $\sigma(b)$ belong to a leaf F of the, say α-foliation, and such that $\sigma|_{]a;b[}$ does not intersect F. Traversing σ from $\sigma(a)$ to $\sigma(b)$ followed by the interval of F from $\sigma(b)$ to $\sigma(a)$ defines a Jordan curve in $\mathbb{R}^2$, bounding a unique compact subset $E \subset \mathbb{R}^2$. Moreover since σ is timelike, $\sigma(t_1)$ admits for any $t_1 \in]a\,;b[$ a punctured neighbourhood in $\mathcal{F}^-_\alpha(\sigma(t_1))$ which is contained in $\mathrm{Int}(E)$, and the first point of $\mathcal{F}^-_\alpha(\sigma(t_1))$ contained in ∂E is then necessarily of the form $\sigma(t_1')$ for some $t_1' \in]t_1\,;b[$ (the existence of t_1' follows from the existence of foliated charts and from the compactness of E). From there we construct recursively $t_{n+1} = \frac{t_n+t_n'}{2}$, and using the same notations we obtain sequences $t_n, t_n' \in]a\,;b[$ converging to a same point $t_0 \in]a\,;b[$ and such that $t_n < t_{n+1} < t'_{n+1} < t'_n$. Hence for n big enough, $\sigma([t_n\,;t_n'])$ is contained in a normal convex neighbourhood U of $\sigma(t_n)$, and $\sigma(t_n') \in \mathcal{F}_\alpha(\sigma(t_n)) \cap I^+_U(\sigma(t_n))$. But we have seen that $I^+_U(\sigma(t_n))$ is the open future timelike quadrant U^+, which does not contain any point of $\mathcal{F}_\alpha(\sigma(t_n))$. This contradiction concludes the proof. □

Lemma A.4 implies in particular that for any α-lightlike (respectively β-lightlike) leaf F of a singular $\mathbf{X}$-structure on $\mathbb{R}^2$ and for any $x \in F$, there exists a transversal T to the α-foliation (resp. β-foliation) intersecting F only at x. It suffices indeed to take for T a timelike curve through x. This means by definition that the lightlike leaves of a singular $\mathbf{X}$-structure on $\mathbb{R}^2$ are *proper*.

Corollary A.5. *In a singular $\mathbf{X}$-surface homeomorphic to $\mathbb{R}^2$, two distinct lightlike leaves intersect at most once.*

Proof. Assume by contradiction that two distinct lightlike leaves intersect at least two times. Then these are necessarily leaves F_α and F_β of distinct lightlike foliations, and there exists $x, y \in F_\alpha \cap F_\beta$ such that the open intervals $]x\,;y[_{\alpha/\beta}$ of $F_{\alpha/\beta}$ from x to y are disjoint. To fix the ideas we furthermore assume that these intervals are positively oriented, which can be achieved by inversing the orientations. The curve J formed by following $[x\,;y]_\alpha$ from x to y and then $[x\,;y]_\beta$ from y to x is then a Jordan curve of the $\mathbf{X}$-surface $S \simeq \mathbb{R}^2$, bounding a unique compact domain E. With γ a timelike curve starting from y, γ enters E and cannot leave it, or it would intersect $\partial E = F_\alpha \cup F_\beta$ and contradict Lemma A.4. We can obviously extend γ at its endpoint to a larger curve, and we obtain in this way timelike curves of arbitrarily large arclength with respect to a fixed Riemannian metric, and contained in E. Since E is compact, this contradicts the Fact A.13 which will be independently proved below, and concludes the proof. $\square$

Corollary A.6. *Any singular $\mathbf{X}$-surface homeomorphic to $\mathbb{R}^2$ is strongly causal.*

Proof. Assume by contradiction that a singular $\mathbf{X}$-structure on $\mathbb{R}^2$ is not strongly causal. Then there exists a point $x \in \mathbb{R}^2$, a normal convex neighbourhood U of x, and a causal curve starting from x, leaving U and returning to it. It is easy to deform this curve to a timelike curve σ with the same properties. We can moreover choose the boundary of U to be the union of lightlike segments, and denote by I one of these segments which is first met by σ when it leaves U. We can then clearly extend σ if necessary, for it to be a timelike curve intersecting I twice. This contradicts Lemma A.4 and concludes the proof. $\square$

Corollary A.7. *A singular $\mathbf{X}$-surface of universal cover homeomorphic to $\mathbb{R}^2$ does not admit any null-homotopic closed causal curve.*

Proof. Indeed, such a null-homotopic closed causal curve would lift to a closed causal curve of a singular $\mathbf{X}$-structure on $\mathbb{R}^2$, contradicting Corollary A.6. $\square$

We recall that for $S \simeq \mathbf{T}^2$ a closed singular $\mathbf{X}$-surface, a line l in $\mathrm{H}_1(S, \mathbb{R}) \simeq \mathbb{R}^2$ is said *rational* if it passes through $\mathrm{H}_1(S, \mathbb{Z}^2) \simeq \mathbb{Z}^2$ and *irrational* otherwise, and that S is *class A* if the projective asymptotic cycles of its α and β lightlike foliations are distinct: $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$, and is *class B* otherwise.

Lemma A.8. *A closed singular $\mathbf{X}$-surface S is class B if, and only if both of its lightlike foliations have closed leaves which are freely homotopic up to orientation, and is class A otherwise. In particular, if one of the lightlike foliations has irrational projective asymptotic cycle, then S is class A.*

Proof. If the lightlike foliations have closed leaves which are not freely homotopic up to orientation, then since two primitive element $c_\alpha \neq \pm c_\beta$ of $\pi_1(S)$ are not proportional in $\mathrm{H}_1(\mathbf{T}^2, \mathbb{R})$, the projective asymptotic cycles are distinct according to Lemma 3.25 and S is thus class A. If only one of the lightlike foliations has a closed leaf, then it has a rational projective asymptotic cycle while the other lightlike foliation has an irrational cycle, hence $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$.

If none of the lightlike foliations have closed leaves, then none of them has a Reeb component, hence both of them is a suspension of a homeomorphism with irrational rotation number according to Proposition 3.26. The latter is a $\mathcal{C}^\infty$ diffeomorphism with breaks and is thus minimal according to Lemma 2.30.(4). By definition (3.16) of the asymptotic cycle, and because any line of $\mathrm{H}_1(S, \mathbb{R})$ is the limit of a sequence of rational lines, there exists a smooth simple closed curve a representing $A(\mathcal{F}_\alpha)$ and as close as one wants to a segment of a leave of $\mathcal{F}_\alpha$. In particular we can assume a to be transverse to $\mathcal{F}_\beta$. Moreover a meets all the leaves of $\mathcal{F}_\beta$ since the latter is minimal, and $\mathcal{F}_\beta$ is therefore the suspension of a homeomorphism of a. There exists thus a simple closed curve b

representing $A(\mathcal{F}_\beta)$, whose class generates $\mathrm{H}_1(S,\mathbb{R})$ together with $[a]$. In particular $\mathbb{R}[a]\neq\mathbb{R}[b]$, which shows that $A(\mathcal{F}_\alpha)\neq A(\mathcal{F}_\beta)$ and concludes the proof of the Lemma. ☐

Proposition A.9. *Let μ_1 and μ_2 be two class A singular $\mathbf{X}$-structures on a closed surface S having identical oriented lightlike bi-foliations. Then for any $x\in S$ we have the following.*

(1) μ_1 and μ_2 admit freely homotopic simple closed timelike curves passing through x which are not null-homotopic.

(2) For any simple closed timelike curve a of μ_1 (respectively μ_2), there exists a simple closed spacelike curve b intersecting a in a single point.

(3) There exists two simple closed timelike and spacelike curves (a_1,b_1) of μ_1 and (a_2,b_2) of μ_2, such that a_1 is freely homotopic to a_2, b_1 freely homotopic to b_2, and $([a_1],[b_1])=([a_2],[b_2])$ is a basis of $\pi_1(\mathbf{T}^2)$.

Proof. The oriented projective asymptotic cycles of the lightlike foliations of a class A singular $\mathbf{X}$-surface (S,μ) delimit an open *timelike cone*

$$\mathcal{C}_\mu=\mathrm{Int}(\mathrm{conv}(A^+(\mathcal{F}_\beta)\cup(-A^+(\mathcal{F}_\alpha))))\subset\mathrm{H}_1(S,\mathbb{R}) \tag{A.1}$$

in the homology, and likewise an open *spacelike cone* $\mathcal{C}_\mu^{\mathrm{space}}=\mathrm{Int}(\mathrm{conv}(A^+(\mathcal{F}_\alpha)\cup A^+(\mathcal{F}_\beta)))$.

(1) Since S is homeomorphic to a torus we let S be equal to $\mathbf{T}^2$ to fix the ideas, identify the action of $\pi_1(\mathbf{T}^2)$ on the universal cover $\pi\colon\mathbb{R}^2\to\mathbf{T}^2$ with the translation action of $\mathbb{Z}^2$, and endow $\mathbb{R}^2$ with the induced singular $\mathbf{X}$-structures $\tilde{\mu}_1\coloneqq\pi^*\mu_1$ and $\tilde{\mu}_2\coloneqq\pi^*\mu_2$ and with a $\mathbb{Z}^2$-invariant auxiliary complete Riemannian metric. With $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ the common lightlike foliations of $\tilde{\mu}_1$ and $\tilde{\mu}_2$, the half-leaves $\mathcal{F}_\beta^+(p)$ and $\mathcal{F}_\alpha^-(p)$ are for any $p\in\mathbb{R}^2$ proper embeddings of $\mathbb{R}^+$. They intersect furthermore only at p according to Corollary A.5, and delimit thus a closed subset $C_p\subset\mathbb{R}^2$ of boundary $\mathcal{F}_\alpha^-(p)\cup\mathcal{F}_\beta^+(p)$ containing all the timelike curves emanating from p. On the other hand there exists a constant $K>0$ such that for any $p\in\mathbb{R}^2$, $\mathcal{F}_\alpha(p)$ and $\mathcal{F}_\beta(p)$ are respectively contained in the K-neighbourhoods of the affine lines $p+A(\mathcal{F}_\alpha)$ and $p+A(\mathcal{F}_\beta)$. This property follows from the equivalence between asymptotic cycles and winding numbers described in [Sch57, p. 278], which is also very well explained in [Suh13, §3.1]. In particular, there exists an affine sub-cone $\mathcal{C}'$ of non-empty interior of the timelike cone $\mathcal{C}_{\mu_1}=\mathcal{C}_{\mu_2}$ in homology defined in (A.1), such that $x+\mathcal{C}'\subset\mathrm{Int}(C_x)$ for any $x\in\mathbb{R}^2$. We fix henceforth $x\in\mathbb{R}^2$ and $c\in\mathcal{C}'$, and we have then $x+c\in\mathrm{Int}(C_x)$, and in particular $x+c\notin\mathcal{F}_\alpha(x)\cup\mathcal{F}_\beta(x)$. Moreover the half-leaves $\mathcal{F}_\beta^-(x+c)$ and $\mathcal{F}_\alpha^-(x)$ intersect, at a unique point y according to Corollary A.5, and $y\notin\{x,x+c\}$ since $x+c\notin\mathcal{F}_\alpha(x)\cup\mathcal{F}_\beta(x)$.

Let $\tilde{\nu}$ denote the curve from x to $x+c$ defined in $\mathbb{R}^2$ by following $\mathcal{F}_\alpha^-(x)$ from x to y and then $\mathcal{F}_\beta^+(y)$ from y to $x+c$. Then by construction, $\tilde{\nu}$ is a piecewise lightlike and a causal curve of $\tilde{\mu}_1$ and $\tilde{\mu}_2$, and it is furthermore contained in the closure of the cone $C_x\subset\mathbb{R}^2$. In particular, $\tilde{\nu}$ is *not* entirely contained in a lightlike leaf $\mathcal{F}_\alpha(x)$ or $\mathcal{F}_\beta(x+c)$ since $y\notin\{x,x+c\}$. Let ν denote the projection of $\tilde{\nu}$ to $\mathbf{T}^2$, which a piecewise lightlike and causal closed curve of μ_1 and μ_2 passing through $\bar{x}\coloneqq\pi(x)$. Since the causal curve ν is not entirely contained in a single lightlike leaf, it can be slightly deformed to a closed timelike curve σ of μ_1 and μ_2, passing through $\bar{x}$ and homotopic to ν. Note that the condition of being timelike depends only on the lightlike bi-foliation, and that ν can therefore indeed be deformed to a curve σ which is timelike both for μ_1 and for μ_2.

Let $t=\sup\left\{s\in[0\,;1]\;\middle|\;\sigma|_{[0;s[}\text{ is injective}\right\}$ (note that $t>0$ since timelike curves are locally injective) so that $\sigma(t)$ is the first self-intersection point of σ with itself, and let $u\in[0\,;t[$ denote the unique time for which $\sigma(t)=\sigma(u)$. If $u=0$, *i.e.* $\sigma(t)=\sigma(u)=\sigma(0)$, then we define $\gamma\coloneqq\sigma|_{[0;t]}$. If $u\neq0$, then we define σ_1 as the curve constituted by $\sigma|_{[0;u]}$ followed by $\sigma|_{[t;1]}$, and repeat the process on σ_1. Using for instance Fact A.13 to be proved below, there exists $\varepsilon>0$ such that for any $s\in[0\,;1]$, $\sigma|_{]s-\varepsilon;s+\varepsilon[}$ is injective. Therefore this process finishes in a finite number of steps by compactness of σ, and yields a simple closed subcurve γ of σ passing through $\bar{x}\in\mathbf{T}^2$. This simple closed timelike curve γ of μ_1 and μ_2 passing through $\bar{x}$ cannot be null-homotopic according to Corollary A.7, which concludes the proof of the claim.

(2) Let $\mathcal{C}'$ be the sub-cone of the future spacelike cone $\mathcal{C}_S^{\mathrm{space}}$ in homology introduced previously,

such that $p + \mathcal{C}' \subset \mathrm{Int}(C_p^{\mathrm{space}})$ for any $p \in \mathbb{R}^2$ with $C_p^{\mathrm{space}} \subset \mathbb{R}^2$ the closed subset of boundary $\mathcal{F}_\alpha^+(p) \cup \mathcal{F}_\beta^+(p)$ in the future of p. Then there exists a free homotopy class $c \in \pi_1(S)$ contained in $\mathcal{C}'$ and of algebraic intersection number $\hat{\imath}(c, [a]) = 1$ with $[a]$. The proof of the first claim of the Proposition yields moreover a closed timelike curve σ through $x = a(0)$ in the free homotopy class c. Since σ and a intersect only transversally and with a positive sign according to our orientations conventions (see Figure 2.1), $\hat{\imath}([\sigma], [a]) = 1$ implies moreover that σ and a intersect actually only at x. With γ the simple closed subcurve of σ through x constructed in the first part of the proof, a and γ intersect again only at $x = a(0) = \sigma(0)$, and have in particular algebraic intersection number $\hat{\imath}([\gamma], [a]) = 1$, which concludes the proof of the claim.
(3) This last claim is a direct consequence of the two first ones. □

A.2. **Lorentzian length, time-separation and maximizing causal curves.** We define the *Lorentzian length* of a causal curve $\gamma\colon [0\,;l] \to S$ in a singular $\mathbf{X}$-surface (S, Σ) by

$$L(\gamma) \coloneqq \int_0^l \sqrt{-\mu_S(\gamma'(t))}dt \in [0\,;+\infty].$$

Causal curves being almost everywhere differentiable (see paragraph A.1 for more details), this quantity is well-defined and moreover independent of the (locally Lipschitz) parametrization of γ thanks to the change of variable formula. An important remark to keep in mind for this whole paragraph is that singular points do not play any role in the length of a causal curve γ in S. Indeed since $\gamma^{-1}(\Sigma)$ is finite, γ is the concatenation of a finite number n of *regular pieces*, namely the connected components γ_i of $\gamma \cap S^*$ with $S^* \coloneqq S \setminus \Sigma$, and we have

$$L(\gamma) = \sum_{i=1}^n L(\gamma_i), \tag{A.2}$$

the lengths appearing in the right-hand finite sum being computed in the regular Lorentzian surface S^*. The Lorentzian length allows us to define on $S \times S$ the *time-separation function* by

$$\tau_S(x, y) \coloneqq \sup_\sigma L_S(\sigma) \in [0\,;+\infty], \tag{A.3}$$

the sup being taken on all future causal curves in S going from x to y if such a curve exists (*i.e.* if $y \in J^+(x)$), and by $\tau_S(x, y) = 0$ otherwise. To avoid any confusion, we emphasize that, on the contrary to τ_S, the Lorentzian length $L(\gamma)$ computed in any open subset $U \subset S$ of course agrees with the one computed in S, which is why we do not bother to specify S in the notation $L(\gamma)$.

Lemma A.10. *Let $y \in J^+(x)$ and $z \in J^+(y)$, then $\tau_S(x, z) \geq \tau_S(x, y) + \tau_S(y, z)$.*

Proof. The same exact proof than in the regular setting (see for instance [Min19, Theorem 2.32]) works in our case, and we repeat it here for the reader to get a grasp of the Lorentzian specificities. If $\tau(x, y)$ or $\tau(y, z)$ is infinite, then using concatenations of causal curves from x to y and from y to z, one easily constructs causal curves of arbitrarily large lengths going from x to z, which proves the inequality (with equality). Assume now that $\tau(x, y)$ and $\tau(y, z)$ are both finite, let $\varepsilon > 0$ and γ, σ be causal curves respectively going from x to y and from y to z such that $L(\gamma) \geq \tau_S(x, y) - \varepsilon$ and $L(\sigma) \geq \tau_S(y, z) - \varepsilon$. Then the causal curve ν equal to the concatenation of γ and σ goes from x to z, hence $\tau_S(x, z) \geq L(\nu) = L(\gamma) + L(\sigma) \geq \tau_S(x, y) + \tau_S(y, z) - 2\varepsilon$ by the definition of τ_S, which proves the claim by letting ε converge to 0. □

It is important to keep in mind that all the usual inequalities, suprema and infima encountered in Riemannian geometry when dealing with lengths of curves and geodesics are exchanged in Lorentzian geometry (for causal curves), as the reverse triangle inequality of Lemma A.10 already showed. The best way to understand this phenomenon (confusing at first sight), is for the reader to explicitly check in the case of the Minkoswki plane $\mathbb{R}^{1,1}$ that timelike geodesics realize the maximal length of a causal curve between two points. We now generalize this observation in the following classical result.

Proposition A.11. *In a singular $\mathbf{X}$-surface S, a future causal curve $\gamma\colon I \to S$ is geodesic up to reparametrization if, and and only if it is* locally maximizing, *namely if for any $t \in I$ there exists* *a connected neighbourhood $I_t = [a_t\,; b_t]$ of t in I and a connected open neighbourhood U_t of $\gamma(t)$ in S, such that $\gamma(I_t) \subset U_t$ and*

$$L(\gamma|_{I_t}) = \tau_{U_t}(\gamma(a_t), \gamma(b_t)).$$

If $I = [a\,; b]$ and $L(\gamma) = \tau_S(\gamma(a), \gamma(b))$ then we say that γ is maximizing. *In this case γ is in particular locally maximizing, and is thus a geodesic (of timelike signature if $L(\gamma) > 0$).*

Proof. We first prove that a maximizing causal curve $\gamma\colon [a\,; b] \to S$ is locally maximizing. For any $a < t < b$ we have:

$$L(\gamma|_{[a;t]}) + L(\gamma|_{[t;b]}) = L(\gamma) = \tau_S(\gamma(a), \gamma(b)) \geq \tau_S(\gamma(a), \gamma(t)) + \tau_S(\gamma(t), \gamma(b)) \tag{A.4}$$

according to the reverse triangular inequality (Lemma A.10). Since on the other hand $L(\gamma|_{[a;t]}) \leq \tau_S(\gamma(a), \gamma(t))$ and $L(\gamma|_{[t;b]}) \leq \tau_S(\gamma(t), \gamma(b))$ by definition of τ_S, both of the latter inequalities have to be equalities to match (A.4). Applying twice this argument to $a_t \in [a\,; b]$ and then $b_t \in [a_t\,; b]$ we obtain $L(\gamma|_{[a_t;b_t]}) = \tau_S(\gamma(a_t), \gamma(b_t)) \geq \tau_{U_t}(\gamma(a_t), \gamma(b_t))$, the latter inequality following from the definition of τ as a supremum. On the other hand $L(\gamma|_{[a_t;b_t]}) \leq \tau_{U_t}(\gamma(a_t), \gamma(b_t))$ by definition of τ_{U_t}, hence $L(\gamma|_{[a_t;b_t]}) = \tau_{U_t}(\gamma(a_t), \gamma(b_t))$, *i.e.* γ is locally maximizing.

The first claim of the Proposition is classical for causal curves of regular Lorentzian manifolds, and is for instance proved in [Min19, Theorem 2.9 and 2.20]. We now treat the case of singularities. Let U be a normal convex neighbourhood of o_θ in $\mathbf{X}_\theta$, and $\sigma \subset U$ be a causal curve from x to y which is maximizing in U. Since σ avoids one of the four lightlike half-leaves of o_θ, we can assume without lost of generality that σ avoids $\mathcal{F}^{+*}_\alpha(\mathsf{o}_\theta)$, hence that $\sigma = \pi_\theta \circ \tilde{\sigma}$ with $\tilde{\sigma}$ a causal curve. Since σ is maximizing in U, $\tilde{\sigma}$ is maximizing as well and is thus a geodesic according to the regular case of the Proposition. Therefore σ is a geodesic. Using small enough normal convex neighbourhoods, this observation shows that a locally maximizing causal curve is geodesic at the singular points, and concludes the proof of the Proposition. $\square$

The following result is well-known in the classical setting of regular Lorentzian manifolds, where it is a particular case of the *Limit curve theorems*. We give here the main arguments of its proof to make it clear that it persists in our singular setting, refering for instance to [Min19, §2.11 and Theorem 2.53] for more details.

Lemma A.12. *Let γ_n be a sequence of causal curves in a globally hyperbolic singular $\mathbf{X}$-surface S joining two points x and y. The (γ_n) have then uniformly bounded arclength with respect to a fixed Riemannian metric h on S. Let σ_n denote the reparametrization of γ_n by h-arclength. Then there exists a causal curve σ from x to y and a subsequence σ_{n_k} of σ_n converging to σ in the $\mathcal{C}^0$-topology. Moreover* $\limsup L(\sigma_{n_k}) \leq L(\sigma) < +\infty$.

Proof. The first important and classical fact is:

Fact A.13. *For any relatively compact normal convex neighbourhood U of a $\mathbf{X}$-surface S (not necessarily globally hyperbolic), causal curves contained in U are equi-Lipschitz, of uniformly bounded Riemannian length, and leave U in a uniform bounded time. Namely for any Riemannian metric h on U, there exists a constant $K > 0$ and a time-function f such that for any causal curve γ in U:*

1. *γ may be reparametrized by f to be K-Lipschitz;*
2. *with this reparametrization, γ leaves U in a time bounded by K;*
3. *and the h-arclength of γ is bounded by K.*

Proof. We explain the main ideas leading to these properties for a causal curve γ contained in a relatively compact normal convex neighbourhood U of $p \in S^*$, and refer to [BEE96, p.75] and [Min19, Theorem 1.35, Remark 1.36 and Theorem 2.12] for more details. Denoting by g the Lorentzian metric of S^*, let $x = (x_1, x_2)$ be coordinates on U such that $g_p(\partial x_1, \partial x_1) = -1$, $g_p(\partial x_2, \partial x_2) = 1$ and $g_p(\partial x_1, \partial x_2) = 0$. Then there exists $\varepsilon > 0$ such that, possibly shrinking U further around p, the timelike cones of the Lorentzian metric $-(1+\varepsilon)dx_1^2 + dx_2^2$ of U strictly contain the causal cones of g (this is indeed true at p by assumption, hence on a neighbourhood of p by continuity of g). Introducing the Riemannian metric $h = dx_1^2 + dx_2^2$ on U and $K_0 := \sqrt{2+\varepsilon} > 0$, this inclusion translates as $\|u\|_h < K_0 dx_1(u)$ for any g-causal vector u, hence as

$$\int_0^t \|\gamma'(t)\|_h < K_0(x_1(\gamma(t)) - x_1(\gamma(0))) \tag{A.5}$$

for any causal curve $\gamma \subset U$ by integration. This last inequality shows that the h-arclengths of causal curves contained in U for h is uniformly bounded, that x_1 is strictly increasing over them, hence that they leave U in a uniformly bounded time when reparametrized by x_1, and that they are moreover equi-Lipschitz for this reparametrization. Note that for any function f sufficiently close to x_1, the causal curves in U retain these uniform properties when reparametrized by f (possibly changing the constants).

To conclude the proof we only have to argue that these properties persist on the neighbourhood of a singular point p. We first consider normal convex neighbourhoods U^- and U^+ contained in S^*, respectively avoiding the future and past timelike quadrants at p, and such that $U := U^- \cup U^+ \cup \{p\}$ is a neighbourhood of p. We next choose coordinates (x_1, x_2) on U so that x_1 is sufficiently close to the respective functions $x_1^{\pm}$ of the previous discussion on the neighbourhoods $U^{\pm}$, for the uniform properties to be satisfied. Property (1) of Definition A.2 implies then that x_1 is strictly increasing on any causal curve γ in U, hence that γ leaves U in uniformly bounded time. When reparametrized by x_1, the causal curves of U are moreover clearly equi-Lipschitz and of uniformly bounded length for a fixed Riemannian metric, since the inequality (A.5) does not take into account the singular point p. This concludes the proof of the Fact. □

We now come back to the proof of the Lemma and fix a Riemannian metric h on S. Since S is strongly causal and $J^+(x) \cap J^-(y)$ relatively compact by global hyperbolicity, we can cover $J^+(x) \cap J^-(y)$ by a finite number of normal convex neighbourhoods U_i which are causally convex. Since the causal curves γ_n join x to y, they are contained in $J^+(x) \cap J^-(y)$. We reparametrize then each γ_n in U_i thanks to the Fact A.13, obtaining in this way an equi-Lipshitz family. Since each of the γ_n meets a given U_i at most once by causal convexity, since the h-arclengths of the $\gamma_n|_{U_i}$ are uniformly bounded for any i according to Fact A.13, and since the covering $(U_i)_i$ is finite, the h-arclength of the γ_n is in the end uniformly bounded.

In particular, the sequence of causal curves $\sigma_n \colon [0\,; a_n] \to S$ obtained by reparametrizing the γ_n by h-arclength remains equi-Lipschitz (because the changes of parametrizations are themselves equi-Lipschitz by boundedness of the arclengths). The sequence (a_n) being bounded, we can moreover assume by passing to a subsequence that it converges to some $a \in]0\,; +\infty[$. We now extend the σ_n to *future inextendible* causal curves $\nu_n \colon \mathbb{R}^+ \to S$, *i.e.* such that $\nu_n(t)$ has no limit when $t \to +\infty$. One easily proves using Fact A.13 that the h-arclength of the ν_n is infinite, and we can therefore reparametrize them by h-arclength on $[a_n\,; \infty]$, obtaining in this way an equi-Lipschitz family $\eta_n \colon \mathbb{R}^+ \to S$ of causal curves.

For any $m \in \mathbb{N}$, we can now apply Arzela-Ascoli Theorem to $(\eta_n|_{[0;m]})_n$. This shows that a subsequence of $(\eta_n|_{[0;m]})_n$ uniformly converges to a continuous curve η_∞^m in S, which is Lipschitz as a uniform limit of equi-Lipschitz curves. By a diagonal argument, we conclude to the existence of a subsequence $(\eta_{n_k})_k$ and of a continuous curve $\eta_\infty \colon \mathbb{R}^+ \to S$ obtained as the union of the η_∞^m, such that $(\eta_{n_k}|_I)_k$ uniformly converges to $\eta_\infty|_I$ for any compact interval $I \subset \mathbb{R}^+$. It is moreover easy to show that η_∞ is a causal curve as a uniform limit of such curves (see for instance [Min19, top of p.46]). With σ the restriction of η_∞ to $[0\,; a]$, the subsequence $(\sigma_{n_k})_k$ uniformly converges to σ, which proves the second claim.

Lastly the proof that $\limsup L(\gamma_{n_k}) \leq L(\sigma)$ given in [Min19, Theorem 2.41] works without any variation in our singular setting, using the decomposition (A.2) of the length into the ones of its regular pieces. This concludes the proof of the Lemma. □

A.3. **Conclusion of the proof of Theorem A.1.** Let S be a closed singular $\mathbf{X}$-surface of class A, $\bar{b}$ be a simple closed spacelike curve in S, and $\pi_C \colon C \to S$ be the $\mathbb{Z}$-covering of S for which $\pi_{C*}(\pi_1(C))$ is generated by $[\bar{b}]$, endowed with the singular $\mathbf{X}$-structure induced by S. Note that S is homeomorphic to $\mathbf{T}^2$, and C to a cylinder $\mathbf{S}^1 \times \mathbb{R}$.

Lemma A.14. *C is a globally hyperbolic singular $\mathbf{X}$-surface.*

Proof. Let T denote the automorphism of the universal cover $\Pi\colon \tilde{S} \to C$ of C induced by $\bar{b}$. Then Π induces a homeomorphism from the quotient $\tilde{S}/\langle T\rangle$ to C. Since $\bar{b}$ lifts to spacelike curves in $\tilde{S}$ and $\tilde{S}$ is a class A surface, each point $x \in C$ admits a connected open neighbourhood U such that $\Pi^{-1}(U)$ is the disjoint union of open sets $U_i \subset \tilde{S}$, so that any two distinct $U_i \neq U_j$ are not causally related, *i.e.* there exists no causal curve joining a point of U_i to a point of U_j. Since $\tilde{S}$ is moreover strongly causal according to Corollary A.6 (because $\tilde{S}$ is homeomorphic to $\mathbb{R}^2$), we can choose U arbitrarily small and so that each U_i is causally convex in $\tilde{S}$. Now let $\gamma\colon [0\,;1] \to C$ be a causal curve joining $p = \gamma(0) \in U$ to $q = \gamma(1) \in U$. Then γ lifts in $\tilde{S}$ to a causal curve $\tilde{\gamma}$ from p' in some U_i to q' in some U_j. By definition of the U_k's this implies that $U_i = U_j$ since $\tilde{\gamma}$ is causal, hence that $\tilde{\gamma} \subset U_i$ since U_i is causally convex. In the end $\gamma \subset U$, which shows the strong causality of C since $x \in C$ is arbitrary and U can be chosen arbitrarily small.

Let R denote the generator of the automorphism group of $\pi_C\colon C \to S$, which is positive in the sense that it is induced by the action on $\tilde{S}$ of a homotopy class $[\bar{a}]$ of closed curves of S of algebraic intersection number $\hat{i}([\bar{b}],[\bar{a}]) = 1$ with $\bar{b}$. We denote in the same way the action of R on $\tilde{S}$ and on C. Then for $x, y \in C$, there exists a lift $b \subset C$ of $\bar{b}$ and $k \in \mathbb{N}^*$, such that x and y are contained in the interior of the unique connected compact annulus $E \subset C$ bounded by b and $R^k(b)$ (this is due to the compactness of S). For $\gamma\colon [0\,;1] \to C$ a causal curve from x to y, we show now that γ is contained in E. This will prove the relative compactness of $J^+(x) \cap J^-(y)$, and conclude the proof of the Lemma. Since $x, y \in \mathrm{Int}(E)$ by assumption, there exists $\varepsilon > 0$ such that $\gamma([0\,;\varepsilon[) \subset \mathrm{Int}(E)$ and $\gamma(]1-\varepsilon\,;1]) \subset \mathrm{Int}(E)$. Furthermore by connectedness, γ cannot leave $\mathrm{Int}(E)$ before meeting b or $R^k(b)$. But if γ meets b (resp. $R^k(b)$), then it meets b twice and in two opposite directions since $C \setminus E$ has two connected components having respectively b and $R^k(b)$ as unique boundary components. Since b and $R^k(b)$ are spacelike, this contradicts the fact that γ is causal and future-oriented. Hence $\gamma \subset E$ as we claimed previously, which concludes the proof. □

Let $\bar{a}$ be a closed timelike curve of S intersecting $\bar{b}$ at a point $\bar{x} = \bar{a}(0) = \bar{b}(0)$, and of algebraic intersection number $\hat{i}([\bar{b}],[\bar{a}]) = 1$ with $\bar{b}$. In particular $([\bar{a}],[\bar{b}])$ is a basis of $\pi_1(S) \simeq \mathbb{Z}^2$. We fix a lift $x_1 \in \pi_C^{-1}(\bar{x})$ of $\bar{x}$ in C, and denote by $a\colon [0\,;1] \to C$ and $b_1\colon [0\,;1] \to C$ the lifts of $\bar{a}$ and $\bar{b}$ starting from $x_1 = a(0) = b_1(0)$. By definition of C we have $b_1(1) = x_1$, *i.e.* b_1 is a simple closed curve in C. On the other hand a is a simple segment but is not closed, and $x_2 \coloneqq a(1) = R(x_1)$ with R the positive generator of the covering automorphism group of π_C induced by $[\bar{a}]$. We denote by $b_2\colon [0\,;1] \to C$ the lift of $\bar{b}$ starting from x_2, so that $b_2 = R \circ b_1$. For $p \in b_1$ we denote by $\mathcal{S}_p$ the set of causal curves of C from p to $R(p)$ which are *causally freely homotopic to* a, *i.e.* freely homotopic to a through causal curves. The following result is a version of the classical Avez-Seifert theorem (see for instance [Min19, Theorem 4.123]), suitably adapted to our setting.

Proposition A.15. *The function*

$$F\colon p \in b_1 \mapsto \sup_{\sigma \in \mathcal{S}_p} L(\sigma) \in [0\,;\infty[\tag{A.6}$$

has finite values, is continuous, and moreover for any $p \in b_1$ there exists $\sigma \in \mathcal{S}_p$ such that $L(\sigma) = F(p)$.

Proof. We fix on C a complete Riemannian metric and endow C with its induced distance. Let $p \in b_1$ and $\sigma_n \in \mathcal{S}_p$ be a sequence of causal curves such that $\lim L(\sigma_n) = F(p)$. Since C is globally hyperbolic according to Lemma A.14, there exists according to Lemma A.12 a subsequence σ_{n_k} converging to a causal curve σ from p to $R(p)$. For any normal convex neighbourhood U, there exists $\varepsilon_U > 0$ and $V \subset U$ such that for any causal curve $\gamma \subset V$, all the causal curves ε_U-close to γ are contained in U and causally homotopic to γ. Since $J^+(p) \cap J^-(R(p))$ is compact by global hyperbolicity and contains any curve of $\mathcal{S}_p$, we can cover $J^+(p) \cap J^-(R(p))$ by a finite number of normal convex neighbourhoods V as before, and we conclude to the existence of $\varepsilon > 0$ such that for any $\gamma \in \mathcal{S}_p$, any causal curve ε-close to γ is causally homotopic to γ. Hence for any large enough k, σ is causally homotopic to $\sigma_{n_k} \in \mathcal{S}_p$, and therefore $\sigma \in \mathcal{S}_p$. Hence $L(\sigma) \leq F(p)$ by definition of F, and since $F(p) = \lim L(\sigma_{n_k}) \leq L(\sigma)$ according to Lemma A.12, this shows that $F(p) = L(\sigma) < +\infty$ and proves the first and third claims.

The proof that F is lower semi-continuous is a straightforward adaptation of [Min19, Theorem 2.32], to which we refer for more details. Let $p \in b_1$, $\varepsilon > 0$ be such that $0 < 3\varepsilon < F(p)$ and $\gamma \in \mathcal{S}_p$ so that $L(\gamma) > F(p) - \varepsilon > 0$. We slightly modify γ for it to be timelike and still satisfy the latter inequality. We choose then $p' \in \gamma$ close enough to p so that $L(\gamma|_{[p;p']}) < \varepsilon$, and $q' \in \gamma$ close enough to $R(p)$ so that $L(\gamma|_{[q';R(p)]}) < \varepsilon$, hence $L(\gamma|_{[p';q']}) > F(p) - 3\varepsilon > 0$. If p' and q' are close enough to p and $R(p)$, then the respective past and future timelike quadrants U and V of normal convex neighbourhoods of p' and q' are neighbourhoods of p and $R(p)$, $I \coloneqq U \cap b_1$ is a neighbourhood of p in b_1, and $R(I)$ is a neighbourhood of $R(p)$ in b_2. We recall that $[a\,; b]_U \subset U$ denotes the unique geodesic contained in U going from $a \in U$ to $b \in J^+(a) \cap U$. For any $x \in I$, let γ_x denote the causal curve going from x to $R(x)$ formed by first following the geodesic $[x\,; p']_U \subset U$, then $\gamma|_{[p';q']}$ and finally $[q'\,; R(x)]_V \subset V$. This curve γ_x is freely causally homotopic to $\gamma \in \mathcal{S}_p$, hence $\gamma_x \in \mathcal{S}_p$ and $F(x) \geq L(\gamma_x) \geq L(\gamma|_{[p';q']}) > F(p) - 3\varepsilon$. This proves the lower semi-continuiuty of F.

Assume now by contradiction that F is not upper semi-continuous, *i.e.* that there exists $p_n \to p$ in b_1 and $\varepsilon > 0$ such that $F(p_n) \geq F(p) + 2\varepsilon$ for any n. Then with $\gamma_n \in \mathcal{S}_{p_n}$ such that $L(\gamma_n) \geq F(p_n) - \varepsilon$, since p_n converges to p and $R(p_n)$ to $R(p)$, Lemma A.12 shows the existence of a causal curve γ from p to $R(p)$ to which a subsequence $(\gamma_{n_k})_k$ converges. Indeed with $p' \in I^-(p)$ and $q' \in I^+(R(p))$ sufficiently close to p and $R(p)$, there exists for any large enough n timelike geodesics γ_n^- and γ_n^+ respectively from p' to p_n and from $R(p_n)$ to q', contained in normal convex neighbourhoods of p' and q'. We can now directly apply Lemma A.12 to the sequence of causal curves formed by following γ_n^-, γ_n and then γ_n^+, and restrict the obtained limit curve to its segment γ from p to $R(p)$. According to Lemma A.12 and by assumption on $L(\gamma_n)$ and $F(p_n)$, we have then $L(\gamma) \geq \limsup L(\gamma_{n_k}) \geq \limsup F(p_{n_k}) - \varepsilon \geq F(p) + \varepsilon$. But the argument of the first paragraph of this proof shows that $\gamma \in \mathcal{S}_p$, and this last inequality contradicts thus the definition of $F(p)$. This concludes the proof of the upper semi-continuity, hence the one of the Lemma. ☐

We can finally conclude the proof of Theorem A.1 thanks to the following result.

Theorem A.16. *Let S be a closed singular $\mathbf{X}$-surface of class A. Then any simple closed timelike (resp. spacelike) curve in S admits a freely homotopic simple closed timelike (resp. spacelike) geodesic.*

Proof. We prove the claim for a simple closed timelike curve $\bar{a}$, and the proof follows then in the spacelike case by replacing the metric of S with its opposite. According to Proposition A.9, there exists a simple closed spacelike curve $\bar{b}$ intersecting $\bar{a}$ at a single point $\bar{x} = \bar{a}(0) = \bar{b}(0)$. We use the notations introduced before Proposition A.15 for the $\mathbb{Z}$-covering $\pi_C \colon C \to S$ of S such that $\pi_{C*}(\pi_1(C)) = \langle[\bar{b}]\rangle$, for the lifts a, b_i and x_i $(i = 1, 2)$ of $\bar{a}$, $\bar{b}$ and $\bar{x}$, and for the covering automorphism R induced by the action of $[\bar{a}]$. With this setup, we want to find a simple timelike geodesic segment $\gamma \colon [0\,; l] \to C$ freely homotopic to a, such that $\gamma(0) \in b_1$ and $\gamma(l) = R(\gamma(0)) \in b_2$. According to Proposition A.15, the function F defined in (A.6) is continuous and finite on the compact set b_1, and reaches thus its maximum at a point $p_0 \in b_1$. There exists moreover according to the same Proposition a causal curve $\gamma \in \mathcal{S}_{p_0}$ such that

$$L(\gamma) = F(p_0) = \sup_{p \in b_1} \sup_{\sigma \in \mathcal{S}_p} L(\sigma). \tag{A.7}$$

In particular, note that $L(\gamma) \geq L(a) = L(\bar{a}) > 0$.

We now prove that $\gamma \colon [0\,; 1] \to C$ is locally maximizing. Indeed let $t \in [0\,; 1]$, U be a normal convex neighbourhood of $\gamma(t)$ and $I = [a\,; b]$ be a connected neighbourhood of t in $[0\,; 1]$ such that $\gamma(I) \subset U$. Then the unique geodesic segment $[\gamma(a)\,; \gamma(b)]_U$ of U from $\gamma(a)$ to $\gamma(b)$ is (future) timelike, and homotopic to $\gamma|_I$ through causal curves while fixing the extremities. In other words the curve ν obtained by concatenating $\gamma|_{[0;a]}$, $[\gamma(a)\,; \gamma(b)]_U$ and $\gamma|_{[b;1]}$ is in $\mathcal{S}_{p_0}$, and thus $L(\nu) \leq L(\gamma)$ according to (A.7). But on the other hand $L([\gamma(a)\,; \gamma(b)]_U) = \tau_U(\gamma(a), \gamma(b))$ since $[\gamma(a)\,; \gamma(b)]_U$ is maximizing in U, and thus $\tau_U(\gamma(a), \gamma(b)) \geq L(\gamma|_{[a;b]})$ by definition, hence $L(\nu) \geq L(\gamma)$. The latter inequality is therefore an equality, which imposes $\tau_U(\gamma(a), \gamma(b)) = L(\gamma|_{[a;b]})$. This proves that γ is locally maximizing, hence that it is a timelike geodesic up to reparametrization according to Proposition A.11.

Let us reparametrize $\gamma\colon [0\,;l] \to C$ to be geodesic. Since C is strongly causal according to Lemma A.14, it contains in particular no closed timelike curve, and γ is thus injective. Furthermore, $\gamma(]0\,;l[)$ is contained in the interior of the unique compact connected annulus E of C bounded by b_1 and b_2 (as we have already seen in the second part of the proof of Lemma A.14), and in particular $\gamma(]0\,;l[)$ is thus disjoint from $b_1 \cup b_2$. Since $\pi_C\colon C \to S$ is injective in restriction to $\mathrm{Int}(E)$ and $\pi_C(\gamma(0)) = \pi_C(\gamma(l))$, this proves that $\bar{\gamma} = \pi_C \circ \gamma\colon [0\,;l] \to S$ is a simple closed timelike curve of S, freely homotopic to a (since γ is freely homotopic to a). We already know that γ is geodesic in restriction to $]0\,;l[$. In a small normal convex neighbourhood U of $x := \bar{\gamma}(0)$, $\bar{\gamma}$ is thus the union of two future timelike geodesic segments I_- and I_+ of extremity x, and respectively contained in the past timelike and future timelike quadrants at x.

Assume by contradiction that $\bar{\gamma}$ is not a geodesic of S, *i.e.* that $I_\pm$ are not parts of the same geodesic segment of U. Then according to Proposition A.11, there exists two points $x_\pm \in I_\pm$ distinct from x such that the unique geodesic segment $[x_-\,;x_+]_U$ from x_- to x_+ in U is future timelike and longer than the segment $\bar{\gamma}|_{[x_-;x_+]}$ of $\bar{\gamma}$ going from x_- to x_+:

$$L([x_-\,;x_+]_U) > L(\bar{\gamma}|_{[x_-;x_+]}). \tag{A.8}$$

With $\bar{\gamma}_*$ the segment of $\bar{\gamma}$ from x_+ to x_-, the curve $\bar{\nu}$ formed by following $\bar{\gamma}_*$ and then $[x_-\,;x_+]_U$ is thus future timelike, and satisfies $L(\bar{\nu}) > L(\bar{\gamma})$ according to (A.8). We denote by ν its lift in C starting from the lift $q \in b_1$ of the (unique) intersection point of $[x_-\,;x_+]_U$ with $\bar{b}$. Observe that, if $x_\pm$ are chosen sufficiently close to x then ν is causally freely homotopic to a, *i.e.* $\nu \in \mathcal{S}_q$. Since $L(\nu) > L(\gamma)$ this contradicts the characterization of γ in (A.7) as the maximizer of $L(\sigma)$ for $p \in b_1$ and $\sigma \in \mathcal{S}_p$. This shows that $\bar{\gamma}$ is a geodesic and concludes the proof of the Theorem. $\square$

Corollary A.17. *Let $c \in \mathrm{Def}_\Theta(\mathbf{T}^2, \Sigma)$ be an isotopy class of class A singular $\mathbf{X}$-structures on a closed surface S. Then there exists a basis (A, B) of $\pi_1(\mathbf{T}^2)$, such that any $\mu \in c$ admits simple closed timelike and spacelike geodesics a and b respectively freely homotopic to A and to B. In particular a and b intersect at a single point.*

Proof. Theorem A.16 and Proposition A.9 yield a pair of simple closed timelike and spacelike geodesics defining a basis of $\pi_1(\mathbf{T}^2)$. In the other hand, the action of $\mathrm{Homeo}^0(S, \Sigma)$ sends such a pair on a freely homotopic one, which proves the claim. $\square$

Appendix B. Some classical results on the rotation number

The claims (1) and (2) of Lemma B.1 below are classical, and Selim Ghazouani indicated us that the claims (3) and (4) are also well-known to specialists of one-dimensional dynamics (related results can for istance be found in [Gha, Chapter 3 and 4]). However we did not find a reference proving these specific results, and we give thus a proof here for sake of completeness.

Lemma B.1. *Let $f \in \mathrm{Homeo}^+(\mathbf{S}^1)$, and $t \in [0\,;1] \mapsto g_t \in \mathrm{Homeo}^+(\mathbf{S}^1)$ be a continuous map such that:*

- $g_0 = \mathrm{id}_{\mathbf{S}^1}$,
- *and $t \in [0\,;1] \mapsto g_t(x) \in \mathbf{S}^1$ is non-decreasing for any $x \in \mathbf{S}^1$.*

Then with $f_t := g_t \circ f$, the map $t \in [0\,;1] \mapsto \rho(f_t) \in \mathbf{S}^1$ is:

(1) continuous;
(2) and non-decreasing.

Moreover:

(3) Assume that $g_1 = \mathrm{id}_{\mathbf{S}^1}$, and that there exists $x_0 \in \mathbf{S}^1$ such that $t \in [0\,;1] \mapsto g_t(x_0) \in \mathbf{S}^1$ is surjective. Then $t \in [0\,;1] \mapsto \rho(f_t) \in \mathbf{S}^1$ is surjective.
(4) Assume that f is minimal, and that there exists $x_0 \in \mathbf{S}^1$ such that $t \in [0\,;1] \mapsto g_t(x_0) \in \mathbf{S}^1$ is not constant. Then $t \in [0\,;1] \mapsto \rho(f_t)$ is not constant at 0. More precisely for any $\varepsilon > 0$ such that $t \in [0\,;\varepsilon] \mapsto \rho(f_t) \in \mathbf{S}^1$ is not surjective and $f_\varepsilon(x_0) \neq f(x_0)$: $\rho(f_\varepsilon) \neq \rho(f)$.
(5) Assume that f is minimal, and that $t \in [0\,;1] \mapsto g_t(x) \in \mathbf{S}^1$ is strictly increasing for any $x \in \mathbf{S}^1$. Then for any $\varepsilon > 0$, there exists $\eta > 0$ such that for any rational $r \in [\rho(f)\,;\rho(f)+\eta] \subset \mathbf{S}^1$ and any $x \in \mathbf{S}^1$, there exists $t \in [0\,;\varepsilon]$ such that the orbit of x under f_t is periodic and of cyclic order r. In particular $\rho(f_t) = r$.

The obvious analogous statements hold for non-increasing maps, and for a family $t \mapsto f \circ g_t$ of deformations.

Proof. The obvious analogous claims for non-increasing maps $t \mapsto g_t(x)$ follow from the non-decreasing case by interverting orientations. The same claims follow then for the family of deformations $t \mapsto f \circ g_t$ by taking the inverse of $f \circ g_t$, since $\rho(f^{-1}) = -\rho(f)$ for any circle homeomorphism.
(1) The continuity follows readily from the ones of the rotation number (see Proposition 3.18) and of $t \mapsto g_t$.
(2) The assumptions on (g_t) ensure the existence of a family of lifts $G_t \in \mathrm{D}(\mathbf{S}^1)$ of g_t such that for any $x \in \mathbb{R}$: $t \mapsto G_t(x)$ is non-decreasing. Let F be a lift of f, and $s \leq t \in [0\,;1]$. Then $G_s \circ F(0) \leq G_t \circ F(0)$ and if we assume that $(G_s \circ F)^n(0) \leq (G_t \circ F)^n(0)$ for some $n \in \mathbb{N}$, then since F and the G_u are strictly increasing and $x \mapsto G_u(x)$ is non-decreasing for any $x \in \mathbb{R}$ we obtain: $(G_s \circ F)^{n+1}(0) \leq G_t(F \circ (G_s \circ F)^n)(0) \leq (G_t \circ F)^{n+1}(0)$. In the end $(G_s \circ F)^n(0) \leq (G_t \circ F)^n(0)$ for any $n \in \mathbb{N}$, which shows that $\tau(G_s \circ F) \leq \tau(G_t \circ F)$ according to (3.14). Hence $u \in [0\,;1] \mapsto \tau(G_u \circ F) \in \mathbb{R}$ is non-decreasing. Since the latter is a lift of the map $u \in [0\,;1] \mapsto \rho(g_u \circ f) \in \mathbf{S}^1$, this proves our claim.
(3) Assume that $F\colon t \in [0\,;1] \mapsto \rho(f_t)$ is not constant. Then there exists $t_0 \in \,]0\,;1]$ such that $F(t_0) \in \mathbf{S}^1 \setminus \{\rho(f)\}$, and since F is continuous and non-decreasing according to (1) and (2), and in the other hand $F(1) = \rho(f)$ by assumption on $g_1 = \mathrm{id}_{\mathbf{S}^1}$, we obtain $\mathbf{S}^1 = [\rho(f)\,; F(t_0)] \cup [F(t_0)\,; \rho(f)] \subset F([0\,;1])$, which proves the claim. It remains now to argue that $F\colon t \in [0\,;1] \mapsto \rho(f_t)$ is not constant, from the existence of $x_0 \in \mathbf{S}^1$ such that $t \in [0\,;1] \mapsto g_t(x_0) \in \mathbf{S}^1$ is surjective. If $x_0 = f(x_0) = f_0(x_0)$, then there exists by surjectivity some $t \in [0\,;1]$ such that $f_t(x_0) \neq x_0$, proving that $\rho(f_t) \neq [0] = \rho(f)$ and thus that F is not constant. If $x_0 \neq f(x_0)$, there exists some $t \in [0\,;1]$ such that $f_t(x_0) = x_0$, proving that $\rho(f_t) = [0] \neq \rho(f)$ and thus again that F is not constant, which concludes the proof of the claim.
(4) For any interval I of $\mathbf{S}^1$, we will denote by $L(I)$ the length of I for a fixed Riemannian metric on $\mathbf{S}^1$ of total length $L(\mathbf{S}^1) = 1$. Let $\varepsilon > 0$ be such that $t \in [0\,;\varepsilon] \mapsto \rho(f_t) \in \mathbf{S}^1$ is not surjective and $f_\varepsilon(x_0) \neq f(x_0)$. Since $(t,x) \mapsto f_t(x)$ is continuous, there exists then a neighbourhood $I \coloneqq [x_0^-\,; x_0^+]$ of x_0 in $\mathbf{S}^1$ and a constant $\alpha > 0$, such that for any $x \in I$:

$$L([f(x)\,; f_\varepsilon(x)]) \geq \alpha. \tag{B.1}$$

Since f is moreover minimal, there exists a strictly increasing sequence $n_k \in \mathbb{N}^*$ such that $f^{n_K}(x_0) \in [x_0^-\,; x_0[$ is strictly increasing and converges to x_0. In particular $\lim f^{n_K+1}(x_0) = f(x_0)$, and there exists thus a smallest $K \in \mathbb{N}$ so that

$$L([f^{n_K+1}(x_0)\,; f(x_0)]) < \alpha. \tag{B.2}$$

Since $f^{n_K}(x_0) \in [x_0^-\,; x_0[$ by construction of the n_k's, we have $L([f^{n_K+1}(x_0)\,; f_\varepsilon \circ f^{n_K-1}(f(x_0))]) \geq \alpha$ according to (B.1), hence $L([f^{n_K+1}(x_0)\,; f_\varepsilon^{n_K}(f(x_0))]) \geq \alpha$ since $t \mapsto f_t(x)$ is non-decreasing for any $x \in \mathbf{S}^1$. Therefore $f(x_0) \in [f_0^{n_K}(f(x_0))\,; f_\varepsilon^{n_K}(f(x_0))]$ according to (B.2), and since $t \in [0\,;\varepsilon] \mapsto f_t^{n_K}(f(x_0))$ is continuous, there exists thus $t_0 \in \,]0\,;\varepsilon]$ such that $f_{t_0}^{n_K}(f(x_0)) = f(x_0)$. But $f(x_0)$ is then a periodic point of f_{t_0}, and $\rho(f_{t_0})$ is thus rational and in particular distinct from $\rho(f)$. The continuous and non-decreasing map $t \in [0\,;\varepsilon] \mapsto \rho(f_t) \in \mathbf{S}^1$ is thus not constant, and since it is also not surjective by assumption, this shows that $\rho(f_\varepsilon) \neq \rho(f)$ which concludes the proof of the claim.
(5) Since f is minimal, $F\colon t \mapsto \rho(f_t)$ is not constant on a neighbourhood of 0 according to (3), and there exists thus by continuity of F some $\eta > 0$ such that $[\rho(f)\,; \rho(f)+\eta] \subset [\rho(f)\,; \rho(f_\varepsilon)]$. Then for any rational $r \in [\rho(f)\,; \rho(f)+\eta]$, there exists because of the continuity and the monotonicity of F some $t_1 \leq t_2 \in \,]0\,;\varepsilon]$ and some small $\varepsilon' > 0$ such that:

- $F(t) \in [\rho(f)\,; r[$ for any $t \in [0\,; t_1[$,
- $F([t_1\,; t_2]) = \{r\}$,
- $F(t) \in \,]r\,; \rho(f)+\eta]$ for any $t \in \,]t_2\,; t_2+\varepsilon']$.

Let $x \in \mathbf{S}^1$, and assume that x is not periodic for $f_{t_1} = g_{t_1} \circ f$. We first assume that $r \neq [0]$, which implies $q \geq 2$ with $r = [\frac{p}{q}]$ in reduced form. Denoting $(x_1, \ldots, x_n) \sim r$ if $(x_1, \ldots, x_n)$ has the same cyclic order than $([0], r, 2r, \dots, (q-1)r)$, we have for any $\theta \in \mathbf{S}^1$:

$$\left\{([0], \theta, \dots, (q-1)\theta) \sim r \text{ and } q\theta \in \mathrm{Cl}(I_\theta^-)\right\} \Leftrightarrow \theta \in \left]r - \frac{1}{q}\,; r\right],$$

$$\left\{([0], \theta, \dots, (q-1)\theta) \sim r \text{ and } q\theta \in \mathrm{Cl}(I_\theta^+)\right\} \Leftrightarrow \theta \in \left[r\,; r + \frac{1}{q}\right[,$$

with I_θ^- (respectively I_θ^+) the connected component of $\mathbf{S}^1 \setminus \{0, \theta, \dots, (q-1)\theta\}$ having $[0]$ as right extremity (respectively as left extremity). It is well-known that using the interpretation of the rotation number in terms of cyclic ordering of the orbits given by Proposition 3.21, the above equivalences adapt for any $T \in \mathrm{Homeo}^+(\mathbf{S}^1)$ to give the following:

$$\left\{(x, T(x), \dots, T^{q-1}(x)) \sim r \text{ and } T^q(x) \in \mathrm{Cl}(I_T^-)\right\} \Leftrightarrow \rho(T) \in \left]r - \frac{1}{q}\,; r\right], \tag{B.3a}$$

$$\left\{(x, T(x), \dots, T^{q-1}(x)) \sim r \text{ and } T^q(x) \in \mathrm{Cl}(I_T^+)\right\} \Leftrightarrow \rho(T) \in \left[r\,; r + \frac{1}{q}\right[, \tag{B.3b}$$

with I_T^- (respectively I_T^+) the connected component of $\mathbf{S}^1 \setminus \{x, T(x), \dots, T^{q-1}(x)\}$ having x as right extremity (respectively as left extremity). Now $f_{t_1}^q(x) \neq x$ since we assumed x to be non-periodic, and since moreover $\rho(f_{t_1}) = r$, $f_{t_1}^q(x)$ is actually either in $I_{f_{t_1}}^-$ or in $I_{f_{t_1}}^+$ according to (B.3a) and (B.3b). If $f_{t_1}^q(x) \in I_{f_{t_1}}^+$, then $f_t^q(x) \in I_{f_t}^+$ for any $t \in [0\,; t_1[$ sufficiently close to t_1 by continuity of $t \mapsto f_t^q(x)$, which implies $\rho(f_t) \in [r\,; r + \frac{1}{q}[$ for any such t according to (B.3b) and contradicts the definition of t_1. Therefore $f_{t_1}^q(x) \in I_{f_{t_1}}^-$, and since $t \mapsto f_t^q(x)$ is continuous and increasing, with moreover $\rho(f_t) = r$ for any $t \in [t_1\,; t_2]$: either $f_t^q(x) = x$ for some $t \in\,]t_1\,; t_2]$, or $f_{t_2}^q(x)$ remains in $I_{f_{t_2}}^-$. In the latter case, $f_t^q(x) \in I_{f_t}^-$ for any $t \in\,]t_2\,; t_2 + \varepsilon']$ sufficiently close to t_2, which implies $\rho(f_t) \in\,]r - \frac{1}{q}\,; r]$ for such a t according to (B.3a) and contradicts the definition of t_2. In conclusion, $f_t^q(x) = x$ for some $t \in\,]t_1\,; t_2]$.

We assume now that $\rho(f_{t_1}) = r = [0]$. According to the interpretation of the rotation number in terms of cyclic ordering of the orbits given by Proposition 3.21 and equations (B.3a)- (B.3b), this is equivalent to say that the sequence $(f_{t_1}^n(x))_{n\in\mathbb{N}}$ is monotonically cyclically ordered. More precisely, the cyclic monotonicity of $(f_t^n(x))_{n\in\mathbb{N}}$ forces $\rho(f_t)$ to be rational according to Proposition 3.21 and to be zero by equations (B.3a)- (B.3b), and reciprocally if $(f_t^n(x))_{n\in\mathbb{N}}$ is not cyclically monotonous, then equations (B.3a)- (B.3b) implies that $\rho(f_t) \neq [0]$. Assume by contradiction that $(f_{t_1}^n(x))_{n\in\mathbb{N}}$ is positively cyclically ordered, hence strictly since $f_{t_1}(x) \neq x$ by assumption. Then since $t \mapsto f_t^n(x)$ is increasing for any n, the sequence $(f_t^n(x))_{n\in\mathbb{N}}$ is strictly positively cyclically ordered for any $t \in [0\,; t_1[$ close enough to t_1. But this implies $\rho(f_t) = [0]$ for such a t as we have seen previously, which contradicts the definition of t_1. Therefore $(f_{t_1}^n(x))_{n\in\mathbb{N}}$ is negatively cyclically ordered, and thus using again that $t \mapsto f_t^n(x)$ is increasing for any n: either $f_t(x) = x$ for some $t \in\,]t_1\,; t_2]$, or $(f_{t_2}^n(x))_{n\in\mathbb{N}}$ remains strictly negatively cyclically ordered. But in the latter case $(f_t^n(x))_{n\in\mathbb{N}}$ is strictly negatively cyclically ordered for any $t \in [t_2\,; t_2 + \varepsilon'[$ close enough to t_2, which implies $\rho(f_t) = [0]$ for such a t and contradicts the definition of t_2. In conclusion $f_t(x) = x$ for some $t \in\,]t_1\,; t_2]$, which concludes the proof. $\square$

Appendix C. Holonomies of lightlike foliations are piecewise Möbius

This appendix is entirely independent from the rest of the paper, and is not used anywhere in the text. We prove here that the holonomies of lightlike foliations in a singular $\mathbf{X}$-surface are piecewise Möbius maps.

A *projective structure* on a topological one-dimensional manifold is a $(\mathrm{PSL}_2(\mathbb{R}), \mathbb{R}\mathbf{P}^1)$-structure consisting of orientation preserving charts, and we call *projective* the $(\mathrm{PSL}_2(\mathbb{R}), \mathbb{R}\mathbf{P}^1)$-morphisms between two projective curves. We endow $\mathbb{R}$ with its standard projective structure for which $x \in \mathbb{R} \mapsto [x : 1] \in \mathbb{R}\mathbf{P}^1$ is a global chart, so that projective morphisms between intervals of $\mathbb{R}$ are precisely the (restrictions of) homographies. We recall that geodesics of singular $\mathbf{dS}^2$-surfaces have well-defined affine structures (see Definition 4.6), and observe that these affine structures define in particular a projective structure on geodesics (through the embedding $\mathbb{R} \hookrightarrow \mathbb{R}\mathbf{P}^1$, equivariant for the natural embedding $\mathrm{Aff}^+(\mathbb{R}) \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$).

Definition C.1. A homeomorphism $F\colon I \to J$ between two projective 1-dimensional manifolds is *piecewise projective* if there exists a finite number of points $x_1, \dots, x_N$ in I, called the *singular points* of F, such that F is projective in restriction to any connected component C of $I \setminus \{x_1, \dots, x_N\}$.

Proposition C.2. *Let $H\colon I \to J$ be the holonomy of a lightlike foliation between two connected subsets I and J of geodesics in a singular $\mathbf{X}$-surface ($I = J$ being allowed). Then H is piecewise projective.*

Proof. **Case of $\mathbb{R}^{1,1}$.** In this case, the leaves of the α and β foliations are the affine lines respectively parallel to the vector lines $\mathbb{R}e_1$ and $\mathbb{R}e_2$. On the other hand the affinely parametrized geodesics are the affinely parametrized segments, and the holonomy between them is thus a dilation, *i.e.* an affine and in particular projective transformation.

Case of $\mathbf{dS}^2$. If the surface is $\mathbf{dS}^2$, the claim follows from a series of naive but fundamental observations. Thanks to Proposition 2.6 we can work with the hyperboloid model dS^2 of the de-Sitter space, that we will see here as the set $\{l \in \mathbf{P}^+(\mathbb{R}^{1,2}) \mid \text{spacelike}\}$ of spacelike half-lines of $\mathbb{R}^{1,2}$. Note that the tangent spaces of dS^2 are identified with Lorentzian planes, and its geodesics with connected components of $P \cap \mathrm{dS}^2$ with P a plane of the same signature than the geodesic. We identified here P with the set of half-lines that it contains, a slight abuse of notations that we will frequently repeat below for any homothety-invariant subset of $\mathbb{R}^{1,2}$, in the hope to simplify the reading.

We now describe an affine parametrization of geodesics of dS^2 by the ($\mathrm{SO}^0(1,2)$-invariant) positive copy $\mathcal{C} \coloneqq \{l \in \mathbf{P}^+(\mathbb{R}^{1,2}) \mid \text{lightlike and positive}\}$ of its conformal boundary. The latter is equipped with the *projective structure* for which $t \in \mathbb{R} \mapsto g^t(l) \in \mathcal{C}$ is a $(\mathrm{PSL}_2(\mathbb{R}), \mathbb{R}\mathbf{P}^1)$-chart for any one-parameter subgroup $\{g^t\}_{t\in\mathbb{R}} \subset \mathrm{SO}^0(1,2)$ and $l \in \mathcal{C}$ (it is easily checked that this defines indeed a projective structure on $\mathcal{C}$). We define two $\mathrm{SO}^0(1,2)$-equivariant natural projections

$$\pi_{\alpha/\beta}\colon l \in \mathrm{dS}^2 \mapsto l_{\alpha/\beta} \in \mathcal{C}$$

whose fibers are the α and β lightlike foliations of dS^2. Any $l \in \mathrm{dS}^2$ is contained in exactly two null planes $N^l_{\alpha/\beta}$ defining two lightlike geodesics $n^l_{\alpha/\beta}$ containing l (the connected components of $N^l_{\alpha/\beta} \cap \mathrm{dS}^2$ containing l), and we name them in such a way that with $l_{\alpha/\beta} = N^l_{\alpha/\beta} \cap \mathcal{C}$, the positive orientation of n^l_α (respectively n^l_β) goes from l to l_α (resp. l_β). We emphasize that $\pi_\alpha(l) \neq \pi_\beta(l)$, $l = n^l_\alpha \cap n^l_\beta$ for any $l \in \mathrm{dS}^2$, and that

$$l \in \mathrm{dS}^2 \mapsto (\pi_\alpha(l), \pi_\beta(l)) \in \mathcal{C}^2 \setminus \{\text{diagonal}\} \equiv \mathbf{dS}^2$$

is a $\mathrm{SO}^0(1,2)$-equivariant bijection which naturally identifies dS^2 with $\mathbf{dS}^2$ once $\mathcal{C}$ is projectively identified with $\mathbb{R}\mathbf{P}^1$ (compare with Remark 2.3).

For any plane $S \subset \mathbb{R}^{1,2}$ and for any geodesic $s \subset \mathrm{dS}^2$ defined by S (*i.e.* a connected component of $S \cap \mathrm{dS}^2$) which is not α-lightlike, we claim that the map $\pi_\alpha|_s\colon s \to \mathcal{C}$ is projective for the affine structure of s and the projective structure of $\mathcal{C}$ (the same proof showing that $\pi_\beta|_s$ is projective if s is not β-lightlike). Indeed the stabilizer of S in $\mathrm{SO}^0(1,2)$ contains a one-parameter subgroup (g^t) acting transitively on s, and $t \in \mathbb{R} \mapsto g^t(x) \in s$ is an affine parametrization of s for any $x \in s$. The equivariance $\pi_\alpha(g^t(x)) = g^t(\pi_\alpha(x))$ of π_α concludes then the proof of the claim by definition of the projective structure of $\mathcal{C}$. Observe moreover that, unless s is α-lightlike (in which case $\pi_\alpha|_s$ is by definition constant), $\pi_\alpha|_s$ is injective and defines thus a projective isomorphism onto its image (which equals $\mathcal{C}$ if s is spacelike and an open proper subset in the other cases).

But for any two geodesics s_1, s_2 of dS^2, the holonomy H of $\mathcal{F}_\alpha$ from s_1 to s_2 satisfies by definition the invariance $\pi_\alpha|_{s_2} \circ H = \pi_\alpha|_{s_1}$ on the open subset where this equality is well-defined, showing that H is a projective isomorphism since the $\pi_\alpha|_{s_i}$ are such.

General case. Let (S, Σ) be a singular $\mathbf{X}$-surface. Without lost of generality, we can assume that H is the holonomy of the α foliation between relatively compact connected subsets I and J of geodesics of S. Since Σ is discrete and $\mathcal{F}_\alpha$ continuous, the set I_Σ of points $p \in I$ such that $[p\,; H(p)]_\alpha \cap \Sigma \neq \varnothing$ is discrete in I, hence finite (we denote by $[p\,; H(p)]_\alpha$ the interval of the oriented leaf $\mathcal{F}_\alpha(p)$ from p to $H(p)$). Let C be a connected component of $I \setminus I_\Sigma$. Then for any

$x \in C$, we can cover $[x\,;H(x)]_\alpha$ by a finite chain of compatible regular $\mathbf{X}$-charts. This expresses $H|_C$ as a finite composition of holonomies H_i between geodesics which are, for any i, contained in the domain of a given regular $\mathbf{X}$-chart. We proved previously that each H_i is projective, and $H|_C$ is thus projective as a composition of such maps. This shows that H is piecewise projective and concludes the proof. □

Appendix D. Singular constant curvature Lorentzian surfaces as Lorentzian length spaces

We show in this appendix, entirely independent from the rest of the text, that globally hyperbolic singular $\mathbf{X}$-surfaces give examples of the *Lorentzian length spaces* introduced in [KS18].

The latter are natural Lorentzian counterparts of the usual metric length spaces (for which [BH99] is a classical reference), and give a synthetic approach to Lorentzian geometry by forgetting the metric itself and rather looking at its main geometrical byproducts. Existing examples included for now (beyond smooth Lorentzian metrics) the Lorentzian metrics with low regularity, the cone structures [KS18, §5], the so-called "generalized cones" [AGKS21] and some gluing constructions [BR24]. To the best of our knowledge and understanding, the singular constant curvature Lorentzian surfaces as we introduce them here were not considered yet in the literature as examples of Lorentzian length spaces. It seems to us that they provide natural examples, as the constant curvature Riemannian metrics with conical singularities give important examples of metric length spaces.

We will quickly describe the relation with Lorentzian length spaces without entering into too much details, most of the technical work beeing done in the Appendix A. Until the end of this section, S denotes a singular $\mathbf{X}$-surface endowed with the distance d_S induced by a fixed complete Riemannian metric.

The structure of a *causal space* on a set X is defined in [KS18, Definition 2.1] by a causal relation $\leq$ (formally a reflexive and transitive relation) and a chronological relation $\ll$ (formally a transitive relation contained in $\leq$) on X. We endow of course our singular $\mathbf{X}$-surface S with the chronological and causal relations defined by the timelike and causal futures (see Definition A.3), namely by definition:

(1) $x \leq y$ if, and only if $y \in J^+(x)$;
(2) $x \ll y$ if, and only if $y \in I^+(x)$.

On a metrizable causal space $(X, d, \leq, \ll)$, a *time-separation* function is then defined as a map $\tau\colon X \times X \to [0\,;+\infty]$ such that $x \not\leq y$ implies $\tau(x,y) = 0$, $\tau(x,y) > 0$ if and only if $x \ll y$, τ satisfies the reverse triangular inequality

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \tag{D.1}$$

for any $x \leq y \leq z$, and τ is lower semi-continuous. The two first conditions are by definition satisfied by the time-separation function τ_S of S defined in (A.3), which also satisfies the reverse triangular inequality (D.1) according to Lemma A.10. Lastly, the lower semi-continuity of τ_S is proved in the same way than the second part of the proof of Proposition A.15, which does not rely on global hyperbolicity (see also [Min19, Theorem 2.32]). $(S, d_S, \leq, \ll, \tau_S)$ is then a *Lorentzian pre-length space* as defined in [KS18, Definition 2.8], and it is moreover automatically causally path connected as defined in [KS18, Definition 2.18, Definition 3.1].

We assume from now on that S is globally hyperbolic in the sense of Definition A.3. In this case the Lorentzian pre-length space $(S, d_S, \leq, \ll, \tau_S)$ satisfies some additional nice properties. Lemma A.12 first shows that S is *causally closed* in the sense that if $p_n \leq q_n$ respectively converge to p and q, then $p \leq q$. It is moreover easy to show that the restriction of τ_S to a normal convex neighbourhood of S (see Proposition 4.8) gives a localizing neighbourhood as defined in [KS18, Definition 3.16], hence that $(S, d_S, \leq, \ll, \tau_S)$ is *strongly localizable*.

The last step to Lorentzian length spaces mimics the definition of usual metric length spaces. The τ_S-length of a causal curve $\gamma\colon [a\,;b] \to S$ is defined in [KS18, Definition 2.24] as

$$L_{\tau_S}(\gamma) = \inf \left\{ \sum_{i=0}^{N} \tau_S(\gamma(t_i), \gamma(t_{i+1})) \,\middle|\, N \in \mathbb{N}, a = t_0 < t_1 < \cdots < t_N = b \right\}.$$

Note that our usual notion of causal curve coincides with the one of [KS18, Definition 2.18] according to [KS18, Lemma 2.21]. Using [KS18, Proposition 2.32] and the decomposition (A.2) of the usual Lorentzian length $L(\gamma)$ into the ones of its regular pieces, one easily shows that $L(\gamma) = L_{\tau_S}(\gamma)$. This last equality shows the following.

Proposition D.1. *Any globally hyperbolic singular* **X***-surface* S *has a natural structure of a* regular Lorentzian length space $(S, d_S, \leq, \ll, \tau_S)$ *as defined in* [KS18, Definition 3.22].

We recall that according to Proposition A.9, any class A closed singular **X**-surface admits a simple closed spacelike curve, and that $\mathbb{Z}$-coverings with respect to such curves give according to Lemma A.14 examples of globally hyperbolic singular **X**-surfaces. Such coverings are regular Lorentzian length spaces according to Proposition D.1.

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