GEOMETRIC SURGERIES OF THREE-DIMENSIONAL FLAG STRUCTURES AND NON-UNIFORMIZABLE EXAMPLES

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Abstract. In this paper, we introduce a notion of geometric surgery for flag structures, which are geometric structures locally modelled on the three-dimensional flag space under the action of $\mathrm{PGL}_3(\mathbb{R})$. Using such surgeries we provide examples of flag structures, of both uniformizable and non-uniformizable type.

1. Introduction

A fundamental problem in the study of any locally homogeneous geometric structure is simply to construct examples of such structures, and a basic way to produce new examples is to combine formerly known ones. With this goal in mind, an easy way to topologically combine two manifolds is to form their connected sum, which raises then the natural question wether the connected sum of two geometric manifolds can be endowed with a geometric structure combining the ones of the two pieces. In the case of flat conformal Riemannian manifolds for instance, locally modelled on the round sphere $\mathbf{S}^n$ with the conformal action of $\mathrm{PO}(1, n+1)$, or in the one of spherical CR-manifolds modelled on $\partial\mathbf{H}^n_{\mathbb{C}}$ with the CR action of $\mathrm{PU}(1, n)$, previous works established such *geometric connected sums* (see [Kul78] for the former and [BS76, Fal92] for the latter).

However, both of these structures share the important common feature that they are modelled on *rank one* simple Lie groups. Our goal in this paper is to introduce such a notion of *geometric surgery* for three-dimensional *flag structures*, which are one of the simplest higher-rank geometries and are modelled on the three-dimensional flag space under the action of $\mathrm{PGL}_3(\mathbb{R})$. Flag structure surgeries will involve gluings along genus two surfaces. We then initiate the study of these surgeries and use them to produce new examples of flag structures.

1.1. **Flag structures in dimension three.** We will be interested in this paper in a three-dimensional homogeneous space of the Lie group $\mathrm{PGL}_3(\mathbb{R})$. Denoting by $\mathbb{R}\mathbf{P}^2_*$ the space of projective lines of $\mathbb{R}\mathbf{P}^2$, the *flag space* is the set

$$\mathbf{X} = \{(p, D) \mid p \in D\} \subset \mathbb{R}\mathbf{P}^2 \times \mathbb{R}\mathbf{P}^2_* \tag{1.1}$$

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of pointed projective lines of $\mathbb{R}\mathbf{P}^2$. This is a closed orientable three-manifold, endowed with a transitive diagonal (projective) action of $\mathrm{PGL}_3(\mathbb{R})$. We will study in this paper three-manifolds that are locally modelled on $\mathbf{X}$. We will also call *flag structure* a $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-structure (see Definition 2.1), and *flag manifold* a three-manifold endowed with a flag structure.

Flag structures can also be defined as *flat path structures*, a path structure on a three-manifold being a pair (E^α, E^β) of transverse rank one distributions in the tangent bundle whose sum $E^\alpha \oplus E^\beta$ is a contact distribution (we will give in subsection 2.1 below more details about this interpretation). Although any orientable closed three-manifold can be given a path structure, it is a basic and open problem to decide which ones bear a flag structure. For instance, while a flag structure was constructed by the first author and Thebaldi on a (finite-volume and complete) non-compact hyperbolic three-manifold in [FT15], it is not known whether a flag structure exists or not on a *closed* hyperbolic three-manifold.

1.2. **A rank-two surgery.** In the rank-one flat conformal or CR-spherical geometries, it is the existence of North-South dynamics for the action of any generic element which ultimately allows one to endow the connected sum of two geometric manifolds with a compatible structure. On the other hand, in the dynamics of the rank two group $\mathrm{PGL}_3(\mathbb{R})$ on the flag space $\mathbf{X}$, the attracting and repelling points of North-South dynamics are replaced by geometric one-dimensional objects that we call *$\alpha - \beta$ bouquet of circles*, described as follows.

The flag space $\mathbf{X}$ enjoys two natural $\mathrm{PGL}_3(\mathbb{R})$-equivariant circle-bundle projections π_α and π_β, which are the respective first and second coordinate projections onto $\mathbb{R}\mathbf{P}^2$ and $\mathbb{R}\mathbf{P}^2_*$, and whose fibers define two $\mathrm{PGL}_3(\mathbb{R})$-invariant transverse foliations of $\mathbf{X}$ by circles, respectively called α and *β-circles* $\mathcal{C}_\alpha(x)$ and $\mathcal{C}_\beta(x)$. These foliations being $\mathrm{PGL}_3(\mathbb{R})$-invariant they define on any flag manifold M two transverse one-dimensional foliations $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$, and through any point x in $\mathbf{X}$ passes thus a *$\alpha - \beta$ bouquet* $B_{\alpha\beta}(x) = \mathcal{C}_\alpha(x) \cup \mathcal{C}_\beta(x)$ of two circles, which are the attracting sets of loxodromic elements of $\mathrm{PGL}_3(\mathbb{R})$ acting on $\mathbf{X}$ as we will see in subsection 2.2.

In order to obtain a surgery we need our manifolds to contain full $\alpha - \beta$ bouquet of circles. We define then a *flag surgery along $\alpha - \beta$ bouquets of circles* properly introduced in Definition 3.1. The gluing being made along the boundary of a tubular neighbourhood of the attracting set, and a neighbourhood of a bouquet of two circles being a genus two handlebody, the flag surgery is a geometric realization of the gluing of two three-manifolds along two genus two handlebodies (see subsection 3.1 for more details). Note that a similar phenomenon already appeared in [Fra04] where the case of conformal Lorentzian geometry was investigated – another rank two geometry. In this case, the geometric surgeries were however not defined by gluing along genus two handlebodies, but along solid tori. The following result providing flag surgeries is proved in section 3.3.

Theorem A. *Let M and N be two flag manifolds, and $B_M \subset M$, $B_N \subset N$ be two $\alpha - \beta$ bouquets of circles admitting open neighbourhoods flag isomorphic to open subsets of $\mathbf{X}$. There exists then a flag surgery of M and N along B_M and B_N.*

It was established in [Kul78], that there exits a conformal structure on a connected sum of conformal structures. This fact was systematically used in [KP86] to construct examples enjoying different geometrical properties. The authors show for instance that connected sums of Kleinian Möbius structures are Kleinian, but they also use these geometric connected sums to obtain examples of non-Kleinian structures. Inspired by [KP86], one of our main motivations for Theorem A is to provide new examples of flag structures both Kleinian and non-Kleinian.

1.3. **Combination of Kleinian flag manifolds by surgery.** For any discrete subgroup $\Gamma \subset \mathrm{PGL}_3(\mathbb{R})$ acting properly discontinuously on an open subset $\Omega \subset \mathbf{X}$, the quotient $\Gamma\backslash\Omega$ bears a canonical flag structure and such flag structures are called *Kleinian*. Kleinian examples will be produced by surgery through the following result proved in subsection 3.4 (see Theorem 3.7), following an argument initially proved in [KP86, §5.6] for conformal structures.

Theorem B. *A flag surgery of Kleinian flag manifolds is a Kleinian flag structure.*

Actually, most of the known Kleinian flag structures arise from *Anosov representations*, forming an important and deeply studied class of examples. Such representations $\rho\colon \Gamma \to \mathrm{PGL}_3(\mathbb{R})$ of hyperbolic groups Γ into $\mathrm{PGL}_3(\mathbb{R})$ admit a $\rho(\Gamma)$-invariant open subset $\Omega_\rho \subset \mathbf{X}$ with a proper, discontinuous and cocompact action of $\rho(\Gamma)$, and yield thus a closed Kleinian flag manifold $\rho(\Gamma)\backslash\Omega_\rho$. We refer to [Bar01, Bar10] for the first examples of flag Anosov representations (having actually appeared before the introduction of Anosov representations themselves) and to [GW12, KLP18] for the general theory, providing domains of discontinuity for Anosov subgroups in a more general setting. In the recent preprint [NR24], three-dimensional flag structures with a Hitchin holonomy are studied in link with some specific distinguished foliations.

A rich class of examples are provided by *Schottky flag manifolds*, obtained from free Anosov subgroups of $\mathrm{PGL}_3(\mathbb{R})$ which are Schottky in a natural sense defined in [MM22a, §1.3.1]. Building up on the latter, Theorem B yields the following examples of Kleinian flag manifolds (see Corollary 3.10).

Corollary C. *Let M and N be two Schottky flag manifolds, and B_M, B_N be two $\alpha-\beta$ bouquets of circles contained in the respective fundamental domains of M and N. There exists then a flag surgery of M and N along B_M and B_N, which is a Kleinian flag manifold.*

Results generalizing the Klein-Maskit combination theorem were proved for Anosov subgroups in [DKL19, DK23], to which Theorem B gives a concrete geometric interpretation in the case where the considered Kleinian flag manifolds are quotients of Anosov subgroups. However, not all Kleinian flag manifolds arise from Anosov subgroups, and Theorem B gives thus a more general "combination" result for Kleinian flag manifolds. A first necessary condition for a Kleinian example to arise from an Anosov subgroup is indeed that the holonomy group should be word-hyperbolic, which excludes, for instance, the following examples.

The action of the Heisenberg group $\mathrm{Heis}(3) \subset \mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$ admits an open orbit O (and actually only one). The action of the lattice $\mathrm{Heis}_{\mathbb{Z}}(3)$ on O is thus properly discontinuous and cocompact. This yields a Kleinian flag manifold $\mathrm{Heis}_{\mathbb{Z}}(3)\backslash O$ whose holonomy is nilpotent, preventing it to arise from an Anosov representation (see [MM22b, §4.2.3] for more details on these examples). We note, however, that the flag surgeries cannot be applied to these flag manifolds, since they do not contain any $\alpha-\beta$ bouquet of circles. The α and β-leaves of these nil-manifolds examples are indeed the stable and unstable leaves of partially hyperbolic diffeomorphisms (see [MM22b, §1.1]), and as such none of them is closed. The authors do not know of any example of Kleinian flag manifold which does not arise up to finite index from an Anosov subgroup and which contains an embedded $\alpha-\beta$ bouquet of circles.

1.4. **Flag structures beyond Kleinian examples.** To the best of our knowledge all of the flag structures described in the literature are Kleinian (most of them arising furthermore from *Anosov representations* as we explained previously), to the exception of the non-compact flag structure constructed in [FT15] on the complement of a knot. A general way to produce examples of closed flag structures (more generally of (G,X)-structures) is the *Ehresman-Thurston* principle, asserting that the set of morphisms from $\pi_1(M)$ to $\mathrm{PGL}_3(\mathbb{R})$ that are holonomy morphisms of a flag structure on a closed manifold M is open. In a very specific situation, suggested to the authors by Charles Frances and described in subsection 4.1, this approach yields indeed non-Kleinian deformations. These are however topologically constrained to $\Sigma_2\times\mathbf{S}^1$ with Σ_2 a genus two closed, connected and orientable surface.

The initial motivation of this paper and of the surgery introduced therein was precisely to widen the realm of known flag structures by producing other *non-Kleinian* closed flag manifolds. In the following result proved in subsections 4.2 and 4.3, we use flag structures surgeries to provide a general recipe to construct non-Kleinian flag manifolds. We denote by Σ_3 the genus three closed, orientable and connected surface, and we say that a flag structure is *virtually Kleinian* if it has a Kleinian covering.

Theorem D. *There exists a virtually Kleinian flag structure on $M=\Sigma_3\times\mathbf{S}^1$ such that, for any $\alpha-\beta$ bouquet $B_M\subset M$ of two circles, and for any Kleinian flag manifold N satisfying*

(1) N contains an $\alpha-\beta$ bouquet B_N of two circles admitting a neighbourhood which embeds in $\mathbf{X}$,
(2) the holonomy group of N contains a loxodromic element,

we have the following. For any flag surgery S of M and N along B_M and B_N, the developing map δ of S is surjective onto $\mathbf{X}$. In particular δ is not a covering map, and S is not virtually Kleinian.

Proposition 3.8 shows that Theorem D can may be applied when N is a Schottky flag manifold. This construction is inspired by the one of [KP86] for conformal geometry and of [FG94] for CR structures.

1.5. **Further questions.** The flag surgery introduced here raises a number of questions that the authors hope to consider in the future. The first question concerns the possible *deformations* of flag surgeries for fixed initial flag manifolds and $\alpha-\beta$ bouquets inside those. This question was investigated in [Ize96] for conformal structures, where non-trivial deformations in the Teichmüller space of conformal structures were described.

Another question is the higher-dimensional generalization of the procedure described here. There are several higher-dimensional analogs to flag structures, among which there are those modelled on complete flags but also structures modelled on partial flags. The spaces $\mathbf{X}_{2n+1}$ of pointed projective hyperplanes of $\mathbb{R}\mathbf{P}^{n+1}$ under the action of $\mathrm{PGL}_{n+2}(\mathbb{R})$ are particularly interesting to the authors. These are indeed the flat *Lagrangian-contact structures*, which are the natural higher-dimensional analog to path structures.

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2. Flag structures and Schottky flag manifolds

In this section, we summarize the geometric and dynamical information that we will need about the flag space and the action of $\mathrm{PGL}_3(\mathbb{R})$, and present the examples described in [MM22a] on which our constructions will be based. We refer to the latter paper for more details and for the proofs of the results claimed in this section.

2.1. **Flag structures.** The flag space $\mathbf{X}$ is an orientable three-dimensional closed manifold, with universal cover $\mathbf{S}^3$ and fundamental group of cardinality 8. The stabilizer of the standard flag $o \coloneqq ([e_1], [e_1, e_2])$ for the transitive action of $\mathrm{PGL}_3(\mathbb{R})$ is the subgroup $\mathbf{P}_{min} \subset \mathrm{PGL}_3(\mathbb{R})$ of upper-triangular matrices, minimal parabolic subgroup of $\mathrm{PGL}_3(\mathbb{R})$ (where (e_i) denotes in all of this paper the standard basis of $\mathbb{R}^n$). The action of $\mathrm{PGL}_3(\mathbb{R})$ induces thus an equivariant identification of $\mathbf{X}$ with the homogeneous space $\mathrm{PGL}_3(\mathbb{R})/\mathbf{P}_{min}$.

A structure locally modelled on $\mathbf{X}$ is formalized in the following way.

Definition 2.1. A $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-*atlas* on a three-manifold M is an atlas of connected charts of M with values in $\mathbf{X}$, whose transition functions are restrictions of elements of $\mathrm{PGL}_3(\mathbb{R})$. A $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-*structure*, or *flag structure* on M is a maximal $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-atlas on M, and a *flag manifold* a three-manifold endowed with a flag structure. A $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-*morphism*, or *flag morphism* between two flag structures is a map which reads in any connected $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-charts as the restriction of an element of $\mathrm{PGL}_3(\mathbb{R})$.

We recall (see for instance [Thu97, CEG87] for more details) that for any $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-structure on M, there exists:

(1) a local diffeomorphism $\delta\colon \tilde{M} \to \mathbf{X}$ which is a $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-morphism, called the *developing map*,
(2) and a *holonomy morphism* $\rho\colon \pi_1(M) \to \mathrm{PGL}_3(\mathbb{R})$ for which δ is ρ-equivariant.

Moreover, if the flag structure is fixed, then such a pair (δ, ρ) is unique up to the action $g \cdot (\delta, \rho) = (g \circ \delta, g\rho g^{-1})$ of $\mathrm{PGL}_3(\mathbb{R})$ and, reciprocally, any such pair defines a unique compatible flag structure.

Definition 2.2. The α- and β-circles of $x = (p, D) \in \mathbf{X}$ are denoted by

$$\mathcal{C}_\alpha(x) = \{(p, D') \mid D' \ni p\} = \mathcal{C}_\alpha(p) \text{ and } \mathcal{C}_\beta(x) = \{(p', D) \mid p' \in D\} = \mathcal{C}_\beta(D). \tag{2.1}$$

The α- and β-circles of $\mathbf{X}$, being $\mathrm{PGL}_3(\mathbb{R})$-equivariant, induce on any flag manifold M a pair $\mathcal{L}_M = (\mathcal{F}_\alpha, \mathcal{F}_\beta)$ of transverse one-dimensional α- and β-foliations, which we call the *flat path structure* induced by the flag structure of M.

The reason for this terminology is that the pair of α- and β-foliations induced on a three-manifold by a flag structure are specific instances of a *path structure*, which is a pair (E^α, E^β) of smooth transverse line fields on a three-manifold whose sum $E^\alpha \oplus E^\beta$ is a contact structure (see [IL] or [MM22a, §1.2] for more details). A path structure (E^α, E^β) is indeed equivalent to its integral foliations $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$, and it can be checked that the α- and β-foliations of a flag structure define a path structure, *i.e.* that $\mathrm{T}\mathcal{C}_\alpha \oplus \mathrm{T}\mathcal{C}_\beta$ is a contact distribution in $\mathbf{X}$. Path structures can be interpreted as *Cartan geometries* modelled on the flag space $\mathbf{X}$ which endows them with a notion of *curvature*, and the path structures induced by flag structures are precisely the ones whose curvature vanishes and are therefore called *flat* (see [SC97] or [MM22b, §2] for more details). A flag structure and its associated flat path structure turns out to be equivalent in the following sense.

Proposition 2.3. *Let M and N be two flag manifolds, of associated flat path structures $\mathcal{L}_M$ and $\mathcal{L}_N$. Then a diffeomorphism $f\colon M \to N$ is a flag isomorphism if, and only if it is a path structure isomorphism.*

A diffeomorphism $f\colon M \to N$ is a path structure isomorphism if for any $x \in M$: $\mathrm{D}_x f(E^\alpha_M(x)) = E^\alpha_N(f(x))$ and $\mathrm{D}_x f(E^\beta_M(x)) = E^\beta_N(f(x))$.

Proof of Proposition 2.3. The direct implication is straightforward, and the reverse one follows from a path structure analog of the "Liouville theorem". Let $f\colon M \to N$ be a path structure isomorphism from $\mathcal{L}_M$ to $\mathcal{L}_N$. Then in any two $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-charts φ and ψ from connected open subsets $U \subset M$ and $f(U) \subset N$, $F = \psi \circ f \circ \varphi^{-1}$ is a diffeomorphism between the connected open subsets $\varphi(U)$ and $\psi \circ f(U)$ of $\mathbf{X}$, preserving the flat path structure of $\mathbf{X}$ defined by α and β-circles. According to the "Liouville theorem" [MM22b, Theorem 2.9], F is thus the restriction of an element $g \in \mathrm{PGL}_3(\mathbb{R})$. Hence f is indeed a $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-isomorphism which concludes the proof. □

2.2. **Dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$.** It is known that the action of the conformal group $\mathrm{PO}(1, n+1)$ on the round sphere $\mathbf{S}^n$ exhibits a dynamics called "North-South". For any sequence (g_n) going to infinity, there exists a subsequence of (g_n) which has a repelling point p_- and an attracting point p_+, and the restriction of (g_n) to $\mathbf{S}^n \setminus \{p_-\}$ converges moreover to the constant map p_+. This dynamics is due to the fact that $\mathrm{PO}(1, n+1)$ has rank one as opposed to $\mathrm{PGL}_3(\mathbb{R})$ which has *rank two*, meaning that the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{sl}_3$ has dimension two. The dimension two leaves space in a Weyl chamber of $\mathfrak{a}$ for three different ways to go to infinity, namely along one of the walls, or in the interior of the chamber. In the latter case however, a kind of North-South dynamics persists with the attracting and repelling points being respectively replaced by an attracting and a repelling *$\alpha - \beta$ bouquet of circles.*

Definition 2.4. For any $x \in \mathbf{X}$, the *$\alpha - \beta$ bouquet* of x is the union

$$B_{\alpha\beta}(x) = \mathcal{C}_\alpha(x) \cup \mathcal{C}_\beta(x)$$

of the α and β-circles of x.

Let us say that $g \in \mathrm{PGL}_3(\mathbb{R})$ is *loxodromic* if any representative of g has three real eigenvalues of pairwise distinct absolute values $a > b > c > 0$, whose corresponding eigenlines are denoted by p_+, $p_\pm$ and p_-. Then $x_- = (p_-, [p_-, p_\pm])$ and $x_+ = (p_+, [p_+, p_\pm])$ are the *repelling and attracting fixed points of* g in $\mathbf{X}$, and we call their $\alpha - \beta$ bouquet of circles

$$B^-_{\alpha\beta}(g) = \mathcal{C}_\alpha(x_-) \cup \mathcal{C}_\beta(x_-) \text{ and } B^+_{\alpha\beta}(g) = \mathcal{C}_\alpha(x_+) \cup \mathcal{C}_\beta(x_+) \tag{2.2}$$

the *repelling and attracting bouquet of circles of* g. Note in particular that $B^+_{\alpha\beta}(g^{-1}) = B^-_{\alpha\beta}(g)$ (see for instance [MM22a, Remark 2.14]).

Example 2.5. Let $g = \mathrm{Diag}(a, b, c)$ be a diagonal matrix with $a > b > c > 0$. Then $x_- = ([e_1], [e_1, e_2])$, $x_+ = ([e_3], [e_2, e_3])$, and thus:

$$B^-_{\alpha\beta}(g) = \mathcal{C}_\alpha[e_1] \cup \mathcal{C}_\beta[e_1, e_2] \text{ and } B^+_{\alpha\beta}(g) = \mathcal{C}_\alpha[e_3] \cup \mathcal{C}_\beta[e_2, e_3].$$

The set of compact subsets of $\mathbf{X}$ is endowed with the topology defined by the *Hausdorff distance* (induced by any Riemannian metric on $\mathbf{X}$, see for instance [MM22a, §2.1] for more details), for which it is a compact metrizable space. We have then the following result, which is a direct reformulation of [MM22a, Lemma 2.2, Lemma 2.21 and Example 2.22].

Lemma 2.6. *Let $g \in \mathrm{PGL}_3(\mathbb{R})$ be a loxodromic element and $K \subset \mathbf{X} \setminus B^-_{\alpha\beta}(g)$ be a compact subset. Then any accumulation point of the sequence $(g^n(K))$ is contained in $B^+_{\alpha\beta}(g)$.*

2.3. **Schottky flag manifolds.** Let $\{g^t\}_{t\in\mathbb{R}} \subset \mathrm{PGL}_3(\mathbb{R})$ be a 1-parameter subgroup of loxodromic elements, of respective repelling and attracting points $x_-, x_+ \in \mathbf{X}$, and repelling and attracting bouquet of circles $B^- = B_{\alpha\beta}(x_-)$, $B^+ = B_{\alpha\beta}(x_+)$. Let $\Omega = \mathbf{X} \setminus (B^- \cup B^+)$, $g = g^1$ and Γ be the cyclic group $\langle g \rangle \simeq \langle \mathbb{Z} \rangle$. Then according to [MM22a, Lemma 3.1], there exists a compact connected neighbourhood H of B^-, which is homeomorphic to a genus two handlebody and such that:

(1) $\Sigma \coloneqq \partial H$ is a genus two closed surface, transverse to the orbits of $\{g^t\}$;
(2) and

$$(x, t) \in \Sigma \times \mathbb{R} \mapsto g^t(x) \in \Omega \tag{2.3}$$

is a diffeomorphism.

We recall that a genus two handlebody is a compact, orientable and connected three-manifold with boundary, which is obtained from a three-ball by attaching two 1-handles. Equivalently, it is obtained from a solid torus by attaching a single 1-handle.

The diffeomorphism (2.3) induces a diffeomorphism from $\Sigma \times \mathbf{S}^1$ to the quotient $M_0 \coloneqq \Gamma \backslash \Omega$ (where $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$), as well as an identification between the fundamental group of M_0 and $\pi_1(\Sigma) \times \mathbb{Z}$ ($\pi_1(\Omega)$ being identified with $\pi_1(\Sigma)$ since Σ is a deformation retract of Ω). Furthermore with $H^- \coloneqq H$, $H^+ \coloneqq \mathbf{X} \setminus \mathrm{Int}(g(H^-))$ and

$$D \coloneqq \mathbf{X} \setminus (H^- \cup H^+), \tag{2.4}$$

D is a fundamental domain for the action of Γ on Ω, and is naturally identified to $\Sigma \times (0, 1)$ by the diffeomorphism (2.3).

Since M_0 is the quotient of Ω by the free and proper action of Γ, it inherits a Kleinian flag structure from the one of $\Omega \subset \mathbf{X}$, and we now describe the developing map and holonomy morphism of this flag structure. With $\pi_\Omega \colon E \to \Omega$ the universal covering map of Ω and $\pi_\Gamma \colon \Omega \to M_0$ the canonical projection, $\pi_\Gamma \circ \pi_\Omega \colon E \to M_0$ is the universal covering map of M_0. Hence $\pi_1(M_0) \equiv \pi_1(\Sigma) \times \mathbb{Z}$ acts on E, π_Ω is $\pi_1(\Sigma) \equiv \pi_1(\Omega)$-invariant, and is thus equivariant for the action of $\pi_1(M_0) \equiv \pi_1(\Sigma) \times \mathbb{Z}$ with respect to the morphism

$$\rho_{M_0} \colon (\lambda, n) \in \pi_1(\Sigma) \times \mathbb{Z} \mapsto g^n. \tag{2.5}$$

Hence (π_Ω, ρ_{M_0}) is a pair of developing map and holonomy morphism of M_0, and we denote $\delta_{M_0} = \pi_\Omega$.

In [MM22a, §1.3.1 Theorem D], the second author describes Kleinian flag manifolds $\Gamma \backslash \Omega(\Gamma)$ obtained from *Schottky subgroups* of $\mathrm{PGL}_3(\mathbb{R})$, that we will call *Schottky flag manifolds*. These examples are the analog of the previous construction for non-cyclic free subgroups of $\mathrm{PGL}_3(\mathbb{R})$ (their existence also follows from [GW12]).

2.4. **An obstruction to be Kleinian.** An important and basic question concerning locally homogeneous structures is to find examples of *non-Kleinian* structures, and to this end it is necessary to find an obstruction for a structure to be Kleinian. In this subsection we establish such a useful criterion for flag structures, relying on dynamical information on the holonomy group of the structure. This is inspired from the criterion initially developed in [KP86, p.20-22] for conformal Riemannian structures.

Lemma 2.7. *Let $\delta(\tilde{M}) \subset \mathbf{X}$ be the developing map image and $\Gamma \subset \mathrm{PGL}_3(\mathbb{R})$ be the holonomy group of a flag structure on a closed three-dimensional manifold M.*

(1) Assume that there exists a loxodromic element $g \in \Gamma$ such that $B^-_{\alpha\beta}(g) \subset \delta(\tilde{M})$. Then $\mathbf{X} \setminus B^+_{\alpha\beta}(g) \subset \delta(\tilde{M})$.

(2) Assume that Γ contains a loxodromic element g such that $\delta(\tilde{M})$ contains the attracting and repelling bouquets $B^-_{\alpha\beta}(g)$ and $B^+_{\alpha\beta}(g)$ of g. Then $\delta(\tilde{M}) = \mathbf{X}$.

(3) If $\delta(\tilde{M}) = \mathbf{X}$ and $\pi_1(M)$ is infinite, then δ is not a covering map. In particular, M is not virtually Kleinian.

Proof. 1. Since $B^-_{\alpha\beta}(g) \subset \delta(\tilde{M})$, there exists a compact neighbourhood P of $B^-_{\alpha\beta}(g)$ which is contained in $\delta(\tilde{M})$. Then with $K = \mathbf{X} \setminus \mathrm{Int}(P)$, any accumulation point of $g^n(K)$ is contained in $B^+_{\alpha\beta}(g)$ according to Lemma 2.6. In particular for any open neighbourhood O of $B^+_{\alpha\beta}(g)$, there exists n such that $g^n(K) \subset O$, *i.e.* such that $\mathbf{X} \setminus O \subset g^n(\mathrm{Int}(P))$. But $\delta(\tilde{M})$ is Γ-invariant, hence $g^n(\mathrm{Int}(P)) \subset \delta(\tilde{M})$ and thus $\mathbf{X} \setminus O \subset \delta(\tilde{M})$. Finally $\mathbf{X} \setminus B^+_{\alpha\beta}(g) = \cup_O \mathbf{X} \setminus O \subset \delta(\tilde{M})$, the union being taken on neighbourhoods O of $B^+_{\alpha\beta}(g)$, which concludes the proof of the claim.

2. According to the first claim $\delta(\tilde{M})$ contains $\mathbf{X} \setminus B^+_{\alpha\beta}(g)$, and it contains $B^+_{\alpha\beta}(g)$ by assumption, hence $\delta(\tilde{M}) = \mathbf{X}$.

3. Assume by contradiction that δ is a covering map onto $\mathbf{X}$. Then since $\pi_1(\mathbf{X})$ is finite, δ is a finite degree covering and $\tilde{M}$ is thus compact. This contradicts the fact that $\pi_1(M)$ is infinite, and δ is thus not a covering map. Assume now by contradiction that M' is a covering of M which is Kleinian. Then the developing map of M' remains $\delta \colon \tilde{M} \to \mathbf{X}$,

and is a covering map since M' is Kleinian. This contradicts what we have proved and concludes the proof. □

3. Flag surgeries and Kleinian examples

3.1. **Definition and main result.** Let M, N be two three-manifolds, $H_M \subset M$ and $H_N \subset N$ be compact submanifolds of dimension three with boundary of M and N, and $f\colon \partial H_M \to \partial H_N$ be a diffeomorphism between their boundaries (which are closed two-dimensional submanifolds in M). Then the *surgery of M and N along f*

$$M\#_f N := (M \setminus \mathrm{Int}(H_M)) \sqcup (N \setminus \mathrm{Int}(H_N))/\{\forall x \in \partial H_M : x \sim f(x)\} \tag{3.1}$$

contains natural embeddings of $M \setminus \mathrm{Int}(H_M)$ and $N \setminus \mathrm{Int}(H_N)$ denoted by j_M and j_N, and has a unique natural smooth structure extending the ones of $j_M(M \setminus \mathrm{Int}(H_M))$ and $j_N(N \setminus \mathrm{Int}(H_N))$. Moreover, the homeomorphism type of $M\#_f N$ only depends on the homotopy type of f. For instance, if H_M and H_N are balls then $M\#_f N = M\#N$ is simply the connected sum of M and N.

If M and N are two flat conformal Riemannian manifolds, then it is possible to endow their connected sum $M\#N$ with a compatible flat conformal Riemannian structure, as shown in [Kul78]. This construction relies on the existence of specific conformal automorphisms: for any open set $U \subset \mathbf{S}^n$, there exists an *inversion* reversing the two boundary spheres of $B_2 \setminus B_1$, with $B_1 \subset B_2$ two balls contained in U. An important feature of the construction of the geometric connected sum in [Kul78] is that the spheres ∂B_1 and ∂B_2 which are exchanged by the inversion, can be chosen as small as one wants.

For flag structures however, such geometric inversions with respect to spheres do not exist anymore. Indeed, because of the rank two dynamics described in subsection 2.2, the best that we can hope for is an involution exchanging two $\alpha-\beta$ bouquet of circles. These involutions will exchange the boundaries of two nested neighbourhoods of a bouquet of two circles, which are genus two handlebodies.

Our goal in this section is to construct a compatible flag structure on the surgery of two flag manifolds along genus two handlebodies. In a flag manifold M, we call $\alpha - \beta$ *bouquet of circles* a bouquet of two circles which is the union $B = \mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x)$ of two closed α- and β-leaves intersecting at a single point. For the following definition see Figure 3.1.

Definition 3.1. Let M and N be two flag manifolds, and $B_M \subset M$, $B_N \subset N$ be two $\alpha - \beta$ bouquet of circles. We will say that a flag manifold S is *a flag surgery of M and N along B_M and B_N*, if there exists:

(1) genus two handlebodies $K_M \subset M$, respectively $K_N \subset N$, containing B_M, resp. B_N in their interiors, and flag structure embeddings $j_M\colon M \setminus K_M \to S$ of image M' and $j_N\colon N \setminus K_N \to S$ of image N', such that $S = M' \cup N'$;
(2) and open subsets $U_M \supset K_M$ and $U_N \supset K_N$ with flag structure embeddings into $\mathbf{X}$, such that $M' \cap N' \subset V = j_M(U_M \setminus K_M) \cap j_N(U_N \setminus K_M)$, with $\partial M'$ and $\partial N'$ isotopic within V.

This definition directly implies that a flag surgery S of M and N along B_M and B_N is diffeomorphic to a surgery of M and N along some diffeomorphism $f\colon \partial H_M \to \partial H_N$, with $H_M \subset U_M$ a genus two handlebody containing B_M in its interior and of boundary

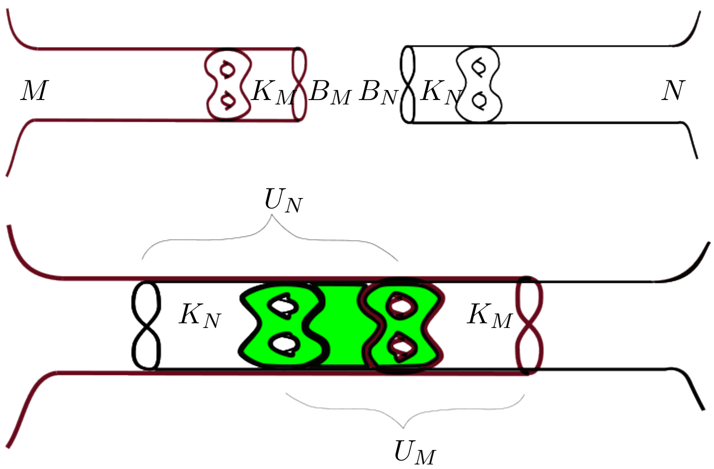

FIGURE 3.1. The surgery S along B_M and B_N. We glue the two manifolds M and N through embeddings $j_M : M \setminus K_M \to S$ and $j_N : N \setminus K_N \to S$. The regions K_N and K_M are deleted and the green region $j_M(U_M \setminus K_M) \cap j_N(U_N \setminus K_N)$ contains the identifications defined by the surgery on a tubular neighborhood of a genus two surface.

isotopic to ∂K_M and likewise for H_N. Definition 3.1 fulfills thus our initial goal, namely to geometrically realize topological surgeries along genus two handlebodies, its condition (1) translating the compatibility of the surgery with the structures of the two pieces.

Remark 3.2. If the condition (2) of Definition 3.1 asking for a flag embedding of a neighbourhood of the $\alpha-\beta$ bouquets is absent in the conformal case, it is simply because any open subset of $\mathbb{R}^n$ contains a conformally embedded euclidean ball, and thus so does any conformally flat manifold. This allows one to form conformal connected sums of any two conformally flat manifolds, while it is not clear that any $\alpha - \beta$ bouquet of circles in a flag manifold admits a neighbourhood embedding in $\mathbf{X}$.

The necessary condition (2) of Definition 3.1 imposed on the $\alpha-\beta$ bouquets is sufficient to obtain the existence of the surgery, given by Theorem A in subsection 1.2 and proved in subsection 3.3 below.

3.2. **Anti-flag involutions of the flag space.** Denoting by $V^\perp$ the orthogonal of a vector subspace $V \subset \mathbb{R}^3$ for the standard euclidean quadratic form, the *standard involution*

$$\kappa\colon (p, D) \in \mathbf{X} \mapsto (D^\perp, p^\perp) \in \mathbf{X} \tag{3.2}$$

of the flag space will be used to glue flag structures, replacing in this context the inversion of the round sphere used for conformal structures. The standard inversion is easily seen to be equivariant for the involutive morphism

$$\Theta\colon g \in \mathrm{PGL}_3(\mathbb{R}) \to {}^t g^{-1} \in \mathrm{PGL}_3(\mathbb{R}), \tag{3.3}$$

where ${}^t g$ denotes the transpose of the matrix g. It also exchanges the $\alpha-$ and $\beta-$ foliations, namely for any $x \in \mathbf{X}$:

$$\kappa(\mathcal{C}_\alpha(x)) = \mathcal{C}_\beta(\kappa(x)) \text{ and } \kappa(\mathcal{C}_\beta(x)) = \mathcal{C}_\alpha(\kappa(x)). \tag{3.4}$$

Observe in particular that κ is not an element of the automorphism group $\mathrm{PGL}_3(\mathbb{R})$ of the flat path structure $(\mathcal{C}_\alpha, \mathcal{C}_\beta)$ of $\mathbf{X}$. The group of automorphisms of the *unordered* pair $\{\mathcal{C}_\alpha, \mathcal{C}_\beta\}$ is actually generated by $\mathrm{PGL}_3(\mathbb{R})$ and κ, and its two connected components are $\mathrm{PGL}_3(\mathbb{R})$ and the left coset of *anti-flag morphisms*, *i.e.* diffeomorphisms of $\mathbf{X}$ exchanging the α and β-foliations, which are of the form $\kappa \circ g$ for $g \in \mathrm{PGL}_3(\mathbb{R})$. Note moreover that κ is not a canonical involution of $\mathbf{X}$, its choice being equivalent to the one of an euclidean quadratic form on $\mathbb{R}^3$. The set $g \circ \kappa \circ g^{-1}$ of conjugates of κ by elements of $\mathrm{PGL}_3(\mathbb{R})$ is however canonical (a change of quadratic form being indeed equivalent to conjugating κ by some g). These conjugates are precisely the involutive anti-flag morphisms of $\mathbf{X}$, that we will call more simply *anti-flag involutions*.

Remark 3.3. Note that for any $x \in \mathbf{X}$, $\kappa(B_{\alpha\beta}(x))$ is disjoint from $B_{\alpha\beta}(x)$.

Remark 3.4. The definition of κ can be generalized to the higher dimensional flag spaces (see more details in [FW17]), and the procedure of flag surgery could be applied to higher dimensional structures. In this paper we restrict ourselves to the three-dimensional case, and the higher-dimensional *Lagrangian-contact structures* or *complete flag structures* will be considered in a future work.

Lemma 3.5. *Let $B \subset \mathbf{X}$ be an $\alpha - \beta$ bouquet of circles and $H_2 \subset \mathbf{X}$ be a genus two handlebody which is a neighbourhood of B. Then there exists a genus two handlebody H_1 which is a neighbourhood of B, as close as one wants from B, and $g \in \mathrm{PGL}_3(\mathbb{R})$ such that with $\varphi = g^{-1} \circ \kappa \circ g$:*

(1) $H_1 \subset \mathrm{Int}(H_2)$,
(2) $\varphi(H_1) = \mathbf{X} \setminus \mathrm{Int}(H_2)$,
(3) $\varphi(\partial H_1) = \partial H_2$,
(4) $\varphi(H_2 \setminus \mathrm{Int}(H_1)) = H_2 \setminus \mathrm{Int}(H_1)$.

Proof. See Figure 3.2. Let $B = \mathcal{C}_\alpha(x) \cup \mathcal{C}_\beta(x)$ and g be a loxodromic element in the stabilizer of x which has B as repelling bouquet (see subsection 2.2 for more details). We will show below that with $\varphi_n = g^{-n} \circ \kappa \circ g^n$, any accumulation point of $H_1^n \coloneqq \mathbf{X} \setminus \varphi_n^{-1}(\mathrm{Int}(H_2))$ is contained in B. For any neighbourhood O of B, there exists thus n such that $H_1^n \subset O \cap \mathrm{Int}(H_2)$, and the claims (1), (2) and (3) follow then directly for $H_1 = H_1^n$ and $\varphi = \varphi_n$. Moreover $\mathrm{Int}(H_1) = \mathbf{X} \setminus \varphi^{-1}(H_2)$, hence $\varphi(H_2 \setminus \mathrm{Int}(H_1)) = \varphi(H_2 \cap \varphi^{-1}(H_2)) = \varphi(H_2) \cap H_2 = H_2 \setminus \mathrm{Int}(H_1)$ since φ is involutive. Note that H_1 can be chosen in O, *i.e.* as close from B as one wants.
For $n_k \to \infty$ such that $\mathbf{X} \setminus g^{-n_k} \circ \kappa \circ g^{n_k}(\mathrm{Int}(H_2))$ converges to K_∞, it only remains to show that $K_\infty \subset B$. Since the compact subset $\mathbf{X} \setminus \mathrm{Int}(H_2)$ is disjoint from $B = B^-_{\alpha\beta}(g)$, any accumulation point of $g^n(\mathbf{X} \setminus \mathrm{Int}(H_2))$ is contained in $B^+_{\alpha\beta}(g)$ according to Lemma 2.6, *i.e.* any accumulation point of $\mathbf{X} \setminus \kappa \circ g^n(\mathrm{Int}(H_2))$ is contained in $\kappa(B^+_{\alpha\beta}(g))$, which

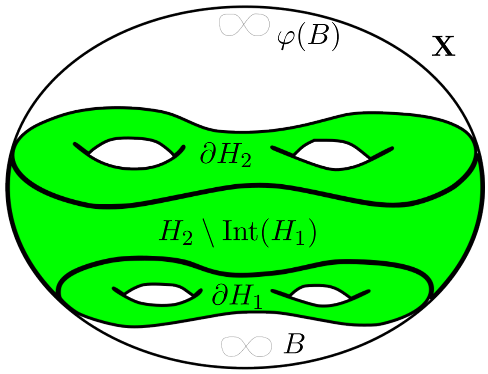

FIGURE 3.2. The green region is defined as the intersection of the handlebody H_2 and the complement of the handlebody H_1. It is invariant under the map $\varphi = g^{-1} \circ \kappa \circ g$. The $\alpha - \beta$ bouquet is contained in the handlebody H_1.

is disjoint from $B^+_{\alpha\beta}(g) = B^-_{\alpha\beta}(g^{-1})$. Taking a subsequence, we can thus assume that $\mathbf{X} \setminus \kappa \circ g^{n_k}(\mathrm{Int}(H_2)) \subset P$ with $P \subset \mathbf{X} \setminus B^-_{\alpha\beta}(g^{-1})$ a compact subset. But according to Lemma 2.6 again, any accumulation point of $g^{-n}(P)$ is then contained in $B^+_{\alpha\beta}(g^{-1}) = B^-_{\alpha\beta}(g) = B$, hence $K_\infty = \lim g^{-n_k}(\mathbf{X} \setminus \kappa \circ g^{n_k}(\mathrm{Int}(H_2)))$ is contained in B, which concludes the proof of the Lemma. $\square$

3.3. **Proof of Theorem A.** Possibly taking smaller neighbourhoods and composing with the action of some element of $\mathrm{PGL}_3(\mathbb{R})$, there exists an $\alpha - \beta$ bouquet of circles $B = \mathcal{C}_\alpha(x) \cup \mathcal{C}_\beta(x) \subset \mathbf{X}$, connected neighbourhoods $U \subset \mathbf{X}$ of B, $U_M \subset M$ of B_M, $U_N \subset N$ of B_N, and flag isomorphisms $\phi_M \colon U_M \to U$ and $\phi_N \colon U_N \to \kappa(U)$. Hence $\phi'_N := \kappa \circ \phi_N$ is a diffeomorphism from U_N to U. According to Lemma 3.5, there exists two genus two handlebodies H_1 and $H_2 \subset U$ which are neighbourhoods of B satisfying $H_1 \subset \mathrm{Int}(H_2)$, and $g \in \mathrm{PGL}_3(\mathbb{R})$ such that $\varphi = g^{-1} \circ \kappa \circ g$ preserves $H_2 \setminus \mathrm{Int}(H_1)$ and exchanges ∂H_1 and ∂H_2. We now introduce the manifold

$$S := (M \setminus \phi_M^{-1}(H_1)) \sqcup (N \setminus \phi_N'^{-1}(H_1)) / \{\forall x \in \mathrm{Int}(H_2) \setminus H_1 : \phi_M^{-1}(x) \sim \phi_N'^{-1}(\varphi(x))\}, \tag{3.5}$$

together with the natural embeddings j_M and j_N of $M \setminus \phi_M^{-1}(H_1)$ and $N \setminus \phi_N'^{-1}(H_1)$ in S. According to the equivariance of κ (see (3.3)), $\phi_N'^{-1} \circ \varphi \circ \phi_M$ equals $\phi_N^{-1} \circ {}^t g g \circ \phi_M$ and is thus a flag morphism in restriction to $\phi_M^{-1}(\mathrm{Int}(H_2) \setminus H_1)$. In other words, the equivalence relation $\sim$ defining S preserves the flag structures of the open subsets $\phi_M^{-1}(\mathrm{Int}(H_2) \setminus H_1)$ and $\phi_N'^{-1}(\mathrm{Int}(H_2) \setminus H_1)$ of M and N.

Therefore, the union of the $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-atlases of $j_M(M \setminus \phi_M^{-1}(H_1))$ and $j_N(N \setminus \phi_N^{-1}(H_1))$ defines a $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-atlas on S, which induces by definition the *canonical flag structure of* S. Conversely, any flag structure on S for which j_M and j_N are $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-morphisms has to coincide with this specific flag structure. S is a surgery of M and N along B_M and B_N, which concludes the proof of Theorem A.

Remark 3.6. Let us emphasize that in the procedure described previously, the choice of open subsets used to form the surgery along a given pair of $\alpha-\beta$ bouquets is non-unique. The investigation of these distinct flag structures will be the subject of a subsequent work.

3.4. **Kleinian flag manifolds by surgery.** Using the surgery procedure introduced in the previous subsection we can prove, using the same argument as in [KP86, §5.6], the following result (Theorem B of the introduction) yielding Kleinian flag structures. We point out a related work in [DKL19, DK23] where combination results generalizing the Klein-Maskit theorem were proved for Anosov subgroups, and to which Theorem 3.7 below gives a concrete geometric interpretation when the considered Kleinian flag manifolds are quotients of Anosov subgroups.

Theorem 3.7. *A flag surgery of Kleinian flag manifolds is a Kleinian flag structure.*

Proof. Let Γ_1 and Γ_2 be the holonomy groups of two Kleinian flag manifolds $M_1 = \Gamma_1\backslash\Omega_1$ and $M_2 = \Gamma_2\backslash\Omega_2$ (which we can assume to be connected without loss of generality), Ω_1 and Ω_2 being connected open subsets of $\mathbf{X}$ where Γ_1 and Γ_2 act freely and properly discontinuously. We will say that an open connected set $D \subset \Omega$ is a *fundamental domain* for the action of Γ on Ω if it coincides with the interior of its closure, is disjoint from its translates by any non-trivial element of Γ, $\Omega \subset \Gamma \cdot \bar{D}$ and for any compact subset $K \subset \Omega$: $\left\{\gamma \in \Gamma \,\middle|\, \gamma\bar{D} \cap K \neq \varnothing\right\}$ is finite. The actions of each Γ_i on Ω_i admit a fundamental domain D_i (see, for instance, [Kap23, Theorem 25]).

For $i = 1, 2$, let $U_i \subset \Omega_i$ be the neighbourhood of an $\alpha-\beta$ bouquet B_i which embedds in M_i, *i.e.* such that $\overline{U_i} \subset D_i$, and so that S is the flag surgery along U_1 and U_2 in the following sense. Using notation as in Definition 3.1, we assume that $\partial j_{M_1}(\mathcal{U}_1) = \partial j_{M_2}(\mathcal{U}_2)$ and that $S = j_{M_1}(M_1 \setminus \mathcal{U}_1) \cup j_{M_2}(M_2 \setminus \mathcal{U}_2)$, with $\mathcal{U}_i$ the projection of U_i in M_i (in other words as in Definition 3.1, $\mathcal{U}_i$ is an open set contained in U_{M_i}, containing K_{M_i} and of boundary isotopic to ∂K_{M_i}). We can assume, without loss of generality, the U_i to coincide with the interior of their closure, and that there exists $g \in \mathrm{PGL}_3(\mathbb{R})$ such that $U_2 = g(U_1)$, and anti-flag involutions φ_1 and $\varphi_2 = g\varphi_1 g^{-1}$ such that $\varphi_i(\partial U_i) = \partial U_i$ and $\varphi_i(\mathbf{X} \setminus \overline{U_i}) = U_i$ (see subsection 3.3 for more details). We emphasize that φ_1 and φ_2 are conjugate because any two anti-flag involutions are conjugate (see subsection 3.2).

For $i = 1, 2$, Γ_i acts freely and properly discontinuously on

$$\Omega_i' \coloneqq \Omega_i \setminus \bigcup_{\gamma \in \Gamma_i} (\gamma\overline{U_i}) \subset \mathbf{X} \tag{3.6}$$

with fundamental domain $D_i \setminus \overline{U_i}$, and we will construct recursively an open subset O of $\mathbf{X}$ as a tree-like gluing of Ω_1' and Ω_2' thanks to the involutions φ_i. For the first step, we glue to $O_1 \coloneqq \Omega_1'$ a copy of Ω_2' by attaching them through $g^{-1}\varphi_2 = \varphi_1 g^{-1}$ on the components ∂U_1 and ∂U_2 of their boundaries. Namely, since $\Omega_2'^{*} \coloneqq g^{-1}\varphi_2(\Omega_2') \subset U_1$ and $g^{-1}\varphi_2(\partial U_2) = \partial U_1$, $O_2^{\mathrm{id}} = \Omega_1' \cup \overline{\Omega_2'^{*}}$ is an open subset of Ω_1. For any $\gamma \in \Gamma_1$, we can attach on the same way the copy $\Omega_2^{\gamma} = \gamma(\Omega_2'^{*}) \subset \gamma(U_1)$ of Ω_2' at the boundary component $\partial(\gamma U_1)$ of Ω_1' to get an open set O_2^{γ}, obtaining eventually the open subset $O_2 \coloneqq \cup_{\gamma \in \Gamma_1} O_2^{\gamma}$ where each "hole" $\gamma\overline{U_1}$ has been "filled in" with the corresponding copy Ω_2^{γ} of Ω_2'. Note that $g^{-1}\varphi_2(U_2) = \mathbf{X} \setminus \overline{U_1}$ and thus with $\Omega_2^{*} \coloneqq g^{-1}\varphi_2(\Omega_2)$, $\Omega_1' \subset \Omega_2^{*}$. Therefore $\overline{O_1} \subset O_2$, $\overline{O_2} \subset \Omega_1 \cap \Omega_2^{*}$ and $O_2 \setminus \overline{O_1} = \cup_{\gamma \in \Gamma_1} \Omega_2^{\gamma}$ with $\Omega_2^{\gamma} \subset \gamma(U_1)$.

In the second step for any $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2 \setminus \{\mathrm{id}\}$, let $U_2(\gamma_1, \gamma_2) = \gamma_1 g^{-1}\varphi_2\gamma_2(U_2)$ be the copy of U_2 contained in $\gamma_1(U_1) \setminus \Omega_2^{\gamma_1}$ and corresponding to $\gamma_2(U_2)$. As we previously did for the copies of Ω'_2 glued to Ω'_1, we can now glue to each created boundary component $\partial U_2(\gamma_1, \gamma_2)$ of O_2 the suitable copy $\Omega_1^{\gamma_1,\gamma_2}$ of Ω'_1 on ∂U_1 through φ_1. More precisely, as before $\Omega_1'^{*} := g\varphi_1(\Omega'_1) \subset U_2$ and $\Omega'_2 \cup \overline{\Omega_1'^{*}} \subset \Omega_2$ is open, hence with $\Omega_1^{\gamma_1,\gamma_2} := (\gamma_1 g^{-1}\varphi_2\gamma_2)(\Omega_1'^{*})$, $O_3^{\gamma_1,\gamma_2} := O_2 \cup \Omega_1^{\gamma_1,\gamma_2} \subset \Omega_1 \cap \Omega_2^*$ is open. This leads to an open subset O_3 such that $\overline{O_2} \subset O_3$, $\overline{O_3} \subset \Omega_1 \cap \Omega_2^*$ and $O_3 \setminus \overline{O_2} = \cup \Omega_1^{\gamma_1\gamma_2}$, the union being taken on all $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2 \setminus \{\mathrm{id}\}$. We can then continue this procedure recursively to obtain an increasing sequence O_n of open sets, and eventually define $O = \cup_n O_n$ which is an open subset of $\Omega_1 \cap \Omega_2^*$.

We now introduce the group $\Gamma'_2 = (g^{-1}\varphi_2)\Gamma_2(g^{-1}\varphi_2)^{-1}$, and emphasize that $\Gamma'_2 \subset \mathrm{PGL}_3(\mathbb{R})$ since $\varphi_2 = g\varphi_1 g^{-1}$ is involutive and equivariant. Since $(g^{-1}\varphi_2)^{-1} = \varphi_2 g = g\varphi_1$, O is by construction Γ_1- and Γ'_2-invariant. Let us denote by $O^i := \cup_{k\equiv i[2]} O_k \setminus \overline{O_{k-1}}$ the "Ω'_i-part" of O (where $O_0 = \varnothing$). Then Γ_1 (resp. Γ'_2) preserves O^1 (resp. O^2). Since Γ_1 (respectively Γ'_2) acts freely and properly discontinuously on Ω_1 (resp. on Ω_2^*) and $O \subset \Omega_1 \cap \Omega_2^*$, both groups act freely and properly discontinuously on O. For the same reason, Γ_1 (respectively Γ'_2) acts freely and properly discontinuously on O^1 (resp. on O^2), and $\Gamma_1 \backslash O^1$ (resp. $\Gamma_2 \backslash O^2$) is moreover canonically identified with $M_1 \setminus \overline{\mathcal{U}_1}$ (resp. $M_2 \setminus \overline{\mathcal{U}_2}$) with $\mathcal{U}_i$ the projection of U_i in M_i.

A standard ping-pong-like argument shows now that the group Γ generated by Γ_1 and Γ'_2 is a free product of Γ_1 and Γ'_2, and acts freely and properly discontinuously on O. Indeed using the notation introduced previously, for any $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$: $\Omega_2^{\gamma_1} = \gamma_1(\Omega_2'^{*})$ and $\Omega_1^{\gamma_1,\gamma_2} = \gamma_1\gamma_2^*(\Omega'_1)$ with $\gamma_2^* := (g^{-1}\varphi_2)\gamma_2(g^{-1}\varphi_2)^{-1}$ (observe that $\Omega_1^{\gamma_1,\mathrm{id}} = \Omega'_1$). More generally, continuing the recursive construction described above with analog notations, for any reduced word $w = g_1 g_2 \dots g_n$ whose letters are alternatively in Γ_1 and Γ_2, and denoting by $\bar{w}$ the image of w in Γ obtained by replacing any $\gamma \in \Gamma_2$ by γ^*, we obtain $\Omega_2^w = \bar{w}(\Omega_2'^{*})$ and $\Omega_1^w = \bar{w}(\Omega'_1)$. The key-remark is now that by the very construction of O, for any two distinct reduced words w and w' the subsets Ω_1^w, $\Omega_1^{w'}$, Ω_2^w and $\Omega_2^{w'}$ are pairwise disjoint. This fact allows us to conclude, by the same argument than the usual ping-pong lemma, that the map $w \mapsto \bar{w} \in \Gamma$ sending a reduced word to its image induces an isomorphism between the free product $\Gamma_1 \star \Gamma_2$ and Γ, and that the action of Γ on O is free and properly discontinuous.

In the end $M := \Gamma \backslash O$ is a Kleinian flag manifold containing a copy M'_i of $M_i \setminus \overline{\mathcal{U}_i}$ and such that $M = \overline{M'_1} \cup \overline{M'_2}$, *i.e.* M is flag isomorphic to S which concludes the proof. ☐

Note that the use of anti-flag involutions is crucial in the construction of the open domain of discontinuity O in the previous proof.

We now construct using Theorem A a large family of examples of Kleinian flag manifolds. We use in the statement below the notation from [MM22a].

Proposition 3.8. *Let $\Gamma \subset \mathrm{PGL}_3(\mathbb{R})$ be a Schottky subgroup of rank d of separating handlebodies $\{H_i^-, H_i^+\}_{i=1}^d$. Let $M = \Gamma \backslash \Omega(\Gamma)$ be the associated Schottky flag manifold. Then for any $\alpha - \beta$ bouquet of circles B contained in the fundamental domain $\mathbf{X} \setminus \cup_{i=1}^d (H_i^- \cup H_i^+)$, B admits a neighbourhood which embedds in M. Theorem A can thus be applied to glue M along the resulting $\alpha - \beta$ bouquet of circles B_M in M.*

Proof. This follows from the fact that $D = \mathbf{X} \setminus \cup_i (H_i^- \cup H_i^+)$ is a fundamental domain for the action of Γ on $\Omega(\Gamma)$ according to [MM22a, Theorem D]. The restriction of the canonical projection $\Omega(\Gamma) \to M$ to $D \supset B$ is thus a flag embedding, which proves the claim. □

Remark 3.9. Observe that not every α (respectively β) leaf of a Schottky flag manifold is closed. For instance, the examples of [MM22a, §4.4] are isomorphic up to a finite index to Schottky flag manifolds, and are compactifications of $\mathrm{T}^1\Sigma$ with Σ a non-compact hyperbolic surface. Here $\mathrm{T}^1\Sigma$ is endowed with a natural flag structure whose α and β leaves are the stable and unstable horocycle of the geodesic flow (see [MM22b, §1 and Lemma 4.3]). They are thus not closed, and [MM22a, Proposition 4.9] ensures that some of them remain unclosed in the compactification.

Corollary 3.10. *Let M and N be two Schottky flag manifolds, and B_M, B_N be two $\alpha - \beta$ bouquets of circles contained in the respective fundamental domains of M and N. Then there exists a flag surgery of M and N along B_M and B_N, which is a Kleinian flag manifold.*

Proof. This is a direct consequence of Proposition 3.8 and Theorem 3.7. □

Remark 3.11. Comparing with [MM22a, Theorem C], we note that the flag surgery of two flag manifolds M and N along two $\alpha - \beta$ bouquets, is related to the procedure of removing two bouquets B_1 and B_2 from the *same* flag manifold and gluing together neighbourhoods of these bouquets. Comparing with topological surgeries, the former procedure is a flag structure realization of an amalgated product, while the latter one is analog to an HNN-extension. In particular, one can obtain Schottky flag manifolds by surgery of several of the examples M_0 (see subsection 2.3).

4. New non-Kleinian flag structures

The goal of this section is to construct new examples of non-Kleinian flag manifolds. We describe in subsection 4.1 a first very specific family of such examples, obtained as deformations of cyclic Schottky flag manifolds on $\Sigma_2 \times \mathbf{S}^1$. We use then surgeries to give a general recipe producing non-Kleinian examples, and conclude in subsection 4.3 the proof of Theorem D.

4.1. First example: deformations of the flag structure on M.

The first examples of non-Kleinian flag structures arise as deformations of the examples of subsection 2.3 which have infinite cyclic holonomy. These examples were suggested to us by C. Frances and are analogous to the construction of affine structures on the torus with non-discrete holonomy ([Gun67, §6 p. 79]). Consider a loxodromic element $g \in \mathrm{PGL}_3(\mathbb{R})$ as in subsection 2.3. We let again $x_-, x_+ \in \mathbf{X}$ be the repelling and attracting points and $B^- = B_{\alpha\beta}(x_-)$, $B^+ = B_{\alpha\beta}(x_+)$ the associated $\alpha - \beta$ bouquet of circles. Define $\Omega = \mathbf{X} \setminus (B^- \cup B^+)$ and $\Gamma \coloneqq \langle g \rangle$. Then $M_0 = \Gamma \backslash \Omega$ (see subsection 2.3) is a Kleinian flag manifold diffeomorphic to $\Sigma_2 \times \mathbf{S}^1$.

Proposition 4.1. *There exists on M_0 distinct pairwise non-isomorphic flag structures, which are continuous deformations of the Kleinian structure of $\Gamma\backslash\Omega$, but which are not virtually Kleinian.*

Proof. The fundamental group of the quotient manifold M_0 is $\pi_1(M_0) = \pi_1(\Sigma_2) \times \mathbb{Z}$. The holonomy map $\rho\colon \pi_1(M_0) \to \mathrm{PGL}_3(\mathbb{R})$ is defined on generators a_1, b_1, a_2, b_2, z, where a_1, b_1, a_2, b_2 are standard generators of $\pi_1(\Sigma_2)$ subject to the single relation

$$[a_1, b_1][a_2, b_2] = \mathrm{id},$$

and z is the positive generator of $\mathbb{Z}$, by $\rho(a_i) = \rho(b_i) = id$, $1 \leq i \leq 2$, and $\rho(z) = g$. Let $x = (s_1, s_2, t_1, t_2) \in \mathbb{R}^4$ be any choice of four real numbers such that $\mathbb{Z}s_1 + \mathbb{Z}s_2 + \mathbb{Z}t_1 + \mathbb{Z}t_2$ is dense in $\mathbb{R}$. Note that this choice can be made with $\|x\|_1 = |s_1| + |s_2| + |t_1| + |t_2|$ as small as one wants. Since the one-parameter group $\{g^t\}_{t\in\mathbb{R}}$ is abelian, the relations

- $\rho_x(a_1) = g^{s_1}$ and $\rho_x(a_2) = g^{s_2}$,
- $\rho_x(b_1) = g^{t_1}$ and $\rho_x(b_2) = g^{t_2}$,
- $\rho_x(z) = g$,

define a unique morphism $\rho_x\colon \pi_1(M_0) \to \mathrm{PGL}_3(\mathbb{R})$. According to the Ehresmann-Thurston principle (see [Thu97] and [CEG87]), for any small enough $\|x\|_1$, ρ_x is close enough to ρ to be the holonomy morphism of a flag structure $\mathcal{L}_x$ on M_0 which is close to the original Kleinian one. In particular, for $\|x\|_1$ small enough the resulting flag structure is thus a continuous deformation of the original Kleinian structure of $\Gamma\backslash\Omega$ (since the moduli space of (G, X)-structures is locally arwise-connected for any homogeneous space X under a connected Lie group G, see for instance [Gol88, Deformation theorem p.178]). Note furthermore that there exists choices of x with $\|x\|_1$ as small as one wants, and whose associated sets of traces $\left\{\mathrm{tr}(g^{js_1+ks_2+lt_1+mt_2+n}) \;\middle|\; (j,k,l,m,n) \in \mathbb{Z}^5\right\}$ are pairwise distinct. These sets, being conjugacy invariants of the representation ρ_x in $\mathrm{PGL}_3(\mathbb{R})$, are invariants of the associated flag structures $\mathcal{L}_x$, which are thus pairwise non-isomorphic. Lastly, the holonomy group of any finite-index covering of the flag manifold $(M_0, \mathcal{L}_x)$ is a finite-index subgroup of $\mathrm{Im}(\rho_x)$, and is thus non-discrete since $\mathrm{Im}(\rho_x)$ is dense by assumption. In particular, $\mathcal{L}_x$ has no finite-index Kleinian covering. □

4.2. **A suitable covering.** Recall that $\Omega = \mathbf{X} \setminus (B^- \cup B^+)$ denotes the image of the developing map δ_{M_0} as in subsection 2.3. The quotient manifold is $M_0 = \Gamma_0\backslash\Omega$, where $\rho_0(\pi_1(M_0)) = \Gamma_0 \coloneqq \langle g\rangle$ is the holonomy group, and B^- and B^+ are the repelling and attracting bouquets of circles. We also denote by $E \simeq \tilde{\Omega}$ the universal cover of M_0, and by $\delta_{M_0}\colon E \to \Omega = \delta_{M_0}(E)$ the developing map of M_0 which is just the universal covering map of Ω.

Let B be a $\alpha - \beta$ bouquet of circles disjoint from B^- and B^+. Then, possibly replacing g by some big enough power and H by a smaller neighbourhood, we can assume that B is contained in the fundamental domain D. There exists then an open neighbourhood $\mathcal{U}_0 \subset \mathbf{X}$ of B contained in D, and with $\pi_\Gamma\colon \Omega \to M_0$ the canonical projection, $\pi_\Gamma|_{\mathcal{U}_0}\colon \mathcal{U}_0 \to U_0 \coloneqq \pi_\Omega(\mathcal{U}_0)$ is thus an embedding of $\mathcal{U}_0$ in M_0 of image U_0, neighbourhood of the $\alpha - \beta$ bouquet of circles $B_{M_0} = \pi_\Gamma(B)$.

The goal of this subsection is to show the following:

Lemma 4.2. *There exists a Galoisian order 2 covering $F\colon M \to M_0$ such that with $\pi_M\colon E \to M$ the universal covering map of M, there exists a connected component U of $F^{-1}(U_0)$ such that:*

(1) $F|_U\colon U \to U_0$ is a diffeomorphism;
(2) $\delta_M(E \setminus \pi_M^{-1}(U)) = \delta_M(E) = \Omega$.

We will construct this covering from a suitable covering Σ_3 of Σ by a genus 3 surface, and to this end we first have to homotope the bouquet of circles B to Σ.

4.2.1. *Homotopy of B to Σ.* In (2.4), we described a fundamental domain D for the action of a loxodromic element g on the flag space $\mathbf{X}$. This fundamental domain is naturally identified with a product $\Sigma \times (0,1)$, with Σ a genus two surface which is the boundary of a tubular neighbourhood of the repelling bouquet B^- of g. The quotient space $M_0 = \Gamma_0 \backslash \Omega$ is a flag manifold homeomorphic to $\Sigma \times \mathbf{S}^1$ with Σ a genus 2 closed connected and orientable surface.

An explicit tubular neighbourhood of a bouquet can be constructed as follows. Consider the two $\mathrm{PGL}_3(\mathbb{R})$-equivariant fiber-bundle projections of $\mathbf{X}$, π_α and π_β, which are the first and second coordinate projections onto $\mathbb{R}\mathbf{P}^2$ and $\mathbb{R}\mathbf{P}^2_*$. The $\alpha - \beta$ *bouquet* passing through x, $B_{\alpha\beta}(x) = \mathcal{C}_\alpha(x) \cup \mathcal{C}_\beta(x)$, is the union of the two fibers $\mathcal{C}_\alpha(x) = \pi_\alpha^{-1}(\pi_\alpha(x))$ and $\mathcal{C}_\beta(x) = \pi_\beta^{-1}(\pi_\beta(x))$.

A convenient description of a tubular neighbourhood of $B_{\alpha\beta}(x)$ is given as the union of fibered neighbourhoods of each of the circles of the bouquet. Let U_α and U_β be two neighbourhoods of $\pi_\alpha(x)$ and $\pi_\beta(x)$ respectively, which we suppose to be homeomorphic to discs. The fibrations are trivial over each neighbourhood and, therefore, $\pi_\alpha^{-1}(U_\alpha)$ and $\pi_\beta^{-1}(U_\beta)$ are tubular neighbourhoods of the two circles forming the bouquet. Each neighbourhood is a fibered full torus.

Lemma 4.3. *Let $B_{\alpha\beta}(x)$ be the bouquet of circles associated to the flag x. Then, for any neighbourhoods U_α and U_β in $\mathbb{R}\mathbf{P}^2$ and $\mathbb{R}\mathbf{P}^2_*$ homeomorphic to discs as above, $\pi_\alpha^{-1}(U_\alpha) \cup \pi_\beta^{-1}(U_\beta)$ is a tubular neighbourhood of $B_{\alpha\beta}(x)$.*

Proof. Clearly, $\pi_\alpha^{-1}(U_\alpha)$ and $\pi_\beta^{-1}(U_\beta)$ are tubular neighbourhoods of the two circles forming the bouquet. We note that the intersection of these two neighbourhoods is homeomorphic to a ball in a special case. Consider an affine chart containing U_α, which we can assume to be a disc centered at the origin. The flag x can be identified to the pair consisting of the origin and the x-axis. The neighbourhood U_β can be chosen to be formed of parallel lines of slopes ranging, for small ϵ, from $-\epsilon$ to ϵ, and intersecting the disc U_α. The intersection $\pi_\alpha^{-1}(U_\alpha) \cap \pi_\beta^{-1}(U_\beta)$ is then a cylinder parametrized by $U_\alpha \times (-\epsilon, \epsilon)$. □

From this lemma we conclude that any two bouquets $B_{\alpha\beta}(x_i)$, $i = 1, 2$, with $\pi_\alpha(x_i) \in U_\alpha$ and $\pi_\beta(x_i) \in U_\beta$ are homotopic. This implies that one can choose a bouquet $B \subset \Sigma \times (0,1)$ contained in the fundamental domain and it will be homotopic to a bouquet contained in the surface Σ.

The goal now is to construct a double cover of the quotient space $M_0 = \Gamma_0 \backslash \Omega = \Sigma_2 \times \mathbf{S}^1$ such that the bouquet B lifts to two bouquets, each of them homeomorphic to the original one by the covering map. It is sufficient to construct a double cover of the surface with the same property as B can be deformed to the surface Σ.

4.2.2. *Suitable covering of Σ.* Let Σ be a genus two closed connected and orientable surface and a_1, b_1, a_2, b_2 be standard generators of $\pi_1(\Sigma)$. The bouquet B can be identified, through a homotopy, to the element $a_1 \cup a_2$ in Σ. We need now the following :

Lemma 4.4. *Let Σ be a genus two surface and $a_1, b_1, a_2, b_2 \subset \Sigma$ be standard generators of $\pi_1(\Sigma)$. Then, there exists a Galoisian double cover by a genus three surface $\pi : \Sigma_3 \to \Sigma$, such that the inverse image of $a_1 a_2$ has two connected components. In other words, the covering exact sequence*

$$\{e\} \to \pi_1(\Sigma_3) \to \pi_1(\Sigma) \to \mathbb{Z}/2\mathbb{Z} \to \{0\}$$

satisfies $a_1, a_2 \in \ker \phi$, where $\phi : \pi_1(\Sigma) \to \mathbb{Z}/2\mathbb{Z}$ is the quotient map.

Proof. This lemma is straightforward once we choose $\pi_1(\Sigma_3)$ to be the kernel of the homomorphism $\pi_1(\Sigma) \to \mathbb{Z}/2\mathbb{Z}$ given by $a_1 \to 0, a_2 \to 0, b_1 \to 1, b_2 \to 1$. □

4.2.3. *Proof of Lemma 4.2.* We can now prove Lemma 4.2. Consider the double cover $\pi : \Sigma_3 \to \Sigma$ of Lemma 4.4. Recall that $M_0 \simeq \Sigma \times \mathbf{S}^1$ and define the double cover $M = \Sigma_3 \times \mathbf{S}^1$

$$F : M \to M_0$$

induced by the cover π. Let U_0 be a tubular neighbourhood of the bouquet B (which, by homotopy, we may consider to be the union of the two generators a_1 and a_2 in Σ). By lemma 4.4, $F^{-1}(U_0)$ has two connected components U and U', each of them homeomorphic to U_0 through F.

Consider now the universal covering $\pi_M \colon E \to M$ and the developing map $\delta_M : E \to \Omega$. The latter coincides with $\delta_{M_0} \colon E \to \Omega$ since M is a cover of M_0, and satisfies thus $\delta_M(\gamma x) = \rho_{M_0}(\gamma)\delta_M(x)$ for any $\gamma \in \pi_1(M_0)$ and $x \in E$. Since ρ_{M_0} restricted to $\pi_1(\Sigma)$ is trivial according to (2.5), we have thus

$$\delta_M(\gamma x) = \delta_M(x) \tag{4.1}$$

for all $\gamma \in \pi_1(\Sigma)$. Choose one of the connected components $U \subset F^{-1}(U_0)$. Observe that π_M restricts to a covering map $E \setminus \pi_M^{-1}(U) \to M \setminus U$ and the developing map defined on the universal cover of $M \setminus U$ descends to a map defined on $E \setminus \pi_M^{-1}(U)$. Moreover if $x \in \pi_M^{-1}(U)$, then since the covering $F : M \to M_0$ is Galois there exists an element $\gamma \in \pi_1(\Sigma)$ such that $\gamma x \in \pi_M^{-1}(U') \subset E \setminus \pi_M^{-1}(U)$, hence $\delta(x) = \delta(\gamma x) \in \delta_M(E \setminus \pi_M^{-1}(U))$ according to (4.1). We obtain thus that $\delta_M(E \setminus \pi_M^{-1}(U)) = \delta_M(E) = \Omega$, which concludes the proof of the lemma.

4.3. **Conclusion of the proof of Theorem D.** Let N be a Kleinian flag manifold such that

(1) there exists $B_N \subset N$ an $\alpha - \beta$ bouquet of two circles admitting a neighbourhood U_N flag isomorphic to the neighbourhood of an $\alpha - \beta$ bouquet of circles in $\mathbf{X}$;
(2) denoting $\Gamma_N \subset \mathrm{PGL}_3(\mathbb{R})$ the holonomy group of N, there exists a loxodromic element $h \in \Gamma_N$.

We now use the open set $U_M \coloneqq U \subset M$ given by Lemma 4.2, which is a neighbourhood of the $\alpha - \beta$ bouquet of circles B_M, and the neighbourhood $U_N \subset N$, to form the flag surgery S of M and N along U_M and U_N (see the proof of Theorem A in subsection 3.3). Let $\delta_S \colon \tilde{S} \to \mathbf{X}$ and $\rho_S \colon \pi_1(S) \to \mathrm{PGL}_3(\mathbb{R})$ be the developing map and holonomy morphism of S, and $\Gamma_S = \rho_S(\pi_1(S))$ its holonomy group. Recall that E is the universal cover of M.

Lemma 4.5. *Possibly composing δ_S with an element of* $\mathrm{PGL}_3(\mathbb{R})$*, we have* $\delta_M(E \setminus \pi_M^{-1}(U)) \subset \delta_S(\tilde{S})$*.*

Proof. According to the Definition 3.1 of a flag surgery, there exists a flag embedding $j\colon M \setminus U \to S$. More precisely, the proof of Theorem 3.7 shows that one can obtain a cover Y of S by taking the tree formed by the components $E \setminus \pi_M^{-1}(U_M)$ and $\tilde{N} \setminus \pi_N^{-1}(U_N)$ glued accordingly. Note that contrary to the case of Theorem 3.7, Y is not embedded in $\mathbf{X}$, however the developing map $\delta\colon \tilde{S} \to \mathbf{X}$ descends to a flag morphism $\bar{\delta}_S$ from Y to $\mathbf{X}$. In particular, we can choose for δ_S the unique developing map of S such that $\bar{\delta}_S|_{E\setminus\pi_M^{-1}(U_M)}$ coincides with $\delta_M|_{E\setminus\pi_M^{-1}(U_M)}$, so that $\delta_M(E \setminus \pi_M^{-1}(\bar{U})) \subset \delta_S(\tilde{S})$ as claimed. □

According to Lemmas 4.2 and 4.5, $\delta_S(\tilde{S})$ contains Ω. We denote by $\Omega_N \subset \mathbf{X}$ the domain of discontinuity of the holonomy Γ_N.

Now, following the proof of Theorem 3.7 we construct a tree-like manifold Y which is a cover of S by attaching E (using involutions as described in subsection 3.1) along each component $\partial\gamma\pi_M^{-1}U_M$, with $\gamma \in \Gamma$, to $\Omega_N \setminus \bigcup_{\gamma\in\Gamma_N}(\gamma U'_N)$ (here $U'_N \subset \Omega_N$ is a lift of U_N). We repeat the attaching procedure recursively as in Theorem 3.7. Remark that in this case, contrary to the surgery of two Kleinian structures, the manifold Y is not embedded into $\mathbf{X}$. But the developing map defined on $\tilde{S}$, descends to a map defined on Y. The holonomy group contains the group Γ and a subgroup isomorphic to Γ_N as in the proof of Theorem 3.7. This can be checked by observing that the developing map defined on Y is equivariant with respect to the action of these groups.

By construction, the neighbourhood $\pi_M^{-1}U_M \subset \mathbf{X}$ contains a bouquet in the limit set of the holonomy of the surgery. In fact, this is the case for each $\gamma U'_N$ with $\gamma \in \Gamma$. Indeed, $\mathbf{X} \setminus U'_N$ contains a bouquet in the limit set of the holonomy of N and by the appropriate inversion used in the surgery, it will be translated to inside the neighborhood $\gamma\pi_M^{-1}U_M$. The conclusion follows from Lemma 2.7.

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