On Two-dimensional Hamiltonian Transport Equations with $\mathbb{L}^p_{loc}$ coefficients

M. Hauray

Cérémade
Université Paris IX-Dauphine
Place du Maréchal Lattre de Tassigny
75775 Paris Cédex 16, France
E-mail : hauray@clipper.ens.fr

Abstract. *We consider two-dimensional autonomous divergence free vector-fields in $\mathbb{L}^2_{loc}$. Under a condition on direction of the flow and on the set of critical points, we prove the existence and uniqueness of a stable a.e. flow and of renormalized solutions of the associated transport equation.*

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1 Introduction

We consider the following transport equation,

$$\frac{\partial u}{\partial t}(t,x) + b(x) \cdot \nabla_x u(t,x) = 0 \tag{1}$$

with initial conditions

$$u(0,x) = u^o(x) \tag{2}$$

where $t \in \mathbb{R}$, $x \in \Omega$, $u^o : \Omega \to \mathbb{R}$, $b : \Omega \to \mathbb{R}^2$ satisfies $div\, b = 0$ and $u : \mathbb{R} \times \Omega \to \mathbb{R}$. The domain Ω is the torus Π^2, or $\mathbb{R}^2$ but in that case we must assume that b satisfies some natural growth conditions, or a bounded open regular subset of $\mathbb{R}^2$ and b is then required to be tangent to the surface $\partial\Omega$. We assume that $u^0 \in \mathbb{L}^p$ for some $p \in [1, \infty]$.

As is well known, this transport equation is in some sense equivalent to the ODE

$$\dot{X}(t) = b(X(t)) \tag{3}$$

Let us begin with some definitions and a proposition in which we always assume that b belongs at least to $\mathbb{L}^1_{loc}$.

Definition 1. *Given an initial condition in $\mathbb{L}^\infty$, a solution of (1)-(2) is a function in $\mathbb{L}^\infty([0,\infty) \times \Omega)$ satisfying for all $\phi \in \mathcal{C}^\infty_c([0,\infty) \times \Omega)$*

$$\int_{[0,\infty)\times\Omega} u\left(\frac{\partial \phi}{\partial t} + b \cdot \nabla_x \phi\right) = -\int_\Omega u^o \phi(0,\cdot) \tag{4}$$

Definition 2. *We shall call renormalized solution a function u in $\mathbb{L}^1_{loc}([0,\infty) \times \Omega)$ such that $\beta(u)$ is a solution of (1) with initial value $\beta(u^o)$, for all $\beta \in \mathcal{C}^1_b(\mathbb{R})$, the set of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ with bounded continuous derivative.*

Remark In this definition, we do not ask u to be a solution because if u only belongs to $\mathbb{L}^1_{loc}$, we cannot give a sense to the product uv. This is one of the reasons why we introduce this definition. But, this is of course an extension of the notion of solution. If $u \in \mathbb{L}^\infty$ is a renormalized solution, it may be shown using good β that u is a solution.

We will give the next definition only for the case where $\Omega = \Pi^2$ or a bounded open subset of $\mathbb{R}^2$. We refer to [1] for the adaptation to the case of $\mathbb{R}^2$ in order to simplify the presentation.

Definition 3. *A flow defined almost everywhere (or a.e. flow) solving (3) is a function X from $\mathbb{R} \times \Omega$ to Ω satisfying*

i. $X \in \mathcal{C}(\mathbb{R}, \mathbb{L}^1)^2$

ii. $\int_\Omega \phi(X(t,x))\, dx = \int_\Omega \phi(x)\, dx \quad \forall \phi \in \mathcal{C}^\infty, \quad \forall t \in \mathbb{R}$ *(preservation of the Lebesgue's measure)*

iii. $X(s+t,x) = X(t, X(s,x))$ *a.e. in* x, $\forall s,t \in \mathbb{R}$

iv. (3) is satisfied in the sense of distributions.

These properties implies that for almost all x, $\forall t \in \mathbb{R}, \quad X(t,x) = x + \int_0^t b(X(s,x))\, ds.$

Moreover, the useful following result is stated in [2].

Proposition 1. *The two following statements are equivalent*

i. For all initial condition $u^o \in \mathbb{L}^1$, *there exists a unique stable renormalized solution of (1).*

ii. There exists a unique stable a.e. flow solution of (3).

Moreover the following condition (R) implies these two equivalent statements

(R) *Every solution of (1) belonging to* $\mathbb{L}^\infty(\mathbb{R} \times \Omega)$ *is a renormalized solution.*

This method of resolution of ODE's and associated transport equations was introduced by R.J. DiPerna and P.L. Lions in [1]. In this article, they show that if $b \in W^{1,1}_{loc}$, the problem (1)-(2) has a unique renormalized solution u. In fact, even if it is not stated in these terms in their article, we can adapt the method used in it to prove Proposition 1 and the fact that (R) is true when $b \in W^{1,1}_{loc}$. In our paper we will show that (R) holds provided that $b \in \mathbb{L}^2_{loc}$ and that the following condition (P_x) on the local direction of b is true for a sufficiently large set of points x

$$(P_x) \qquad \exists \xi \in \mathbb{R}^2,\ \alpha > 0,\ \epsilon > 0 \quad \text{such that for almost all} \quad y \in B(x,\epsilon) \quad b(y) \cdot \xi \geq \alpha$$

This is a local condition and the quantities ξ, α, ϵ depend on x.

We will also show that (R) still holds in the case of a physical Hamiltonian $H(x,y) = y^2/2 + V(x)$ with $V' \in \mathbb{L}^1_{loc}$.

This paper is a extension of L. Desvillettes and F. Bouchut [3], in which similar results are shown when b is continuous. The authors use the fact that since we have an Hamiltonian, we can integrate the ODE to obtain a one dimensionnal problem, that we are able to solve. We will adapt this method with less regularity on b.

2 Main result

Since we are in dimension two and that $div(b) = 0$, there exists a scalar function H (the hamiltonian) such that $\nabla H^\perp = b$. If b belongs to $\mathbb{L}^p$, then H is in $W^{1,p}$.

Theorem 1. *Let* Ω' *be an open subset of* Ω. *Assume that* $b \in \mathbb{L}^2_{loc}(\Omega')$ *and* (P_x) *holds for every* $x \in \Omega'$, *Then the condition (R) holds in* Ω'.

Remarks

i. Two is the critical exponent. It corresponds to the critical case $W^{1,1}$ in [1] since in two dimension we have the Sobolev embedding from $W^{1,1}$ to $\mathbb{L}^2$. In the fourth paragraph, we shall describe a flow which is in $\mathbb{L}^p$ for all $p < 2$, which satisfy the condition (P_x) everywhere but for which uniqueness is false.

ii. This theorem does not extend the result in [1] in this particular case because a vector-fields in $W^{1,1}$ does not necessary satisfy the condition (P_x). We can construct divergence free vector-fields in $W^{1,1}$ which does not satisfy the condition (P_x) at any point x.

iii. Our method allow to prove the existence and the uniqueness directly (i.e. without using (R)), but it raises many difficulties concerning localisation and the addition of critical points.

iv. Here we state a result for a subset of Ω. Of course, a particular case of interest is the case when $\Omega' = \Omega$, where we may then use proposition 1 to obtain the existence and the uniqueness of an a.e. flow and of the solution of the transport equation. But the case $\Omega' \subsetneq \Omega$ will be useful when we will shall take into account some points where (P_x) is not true.

Proof. We shall prove this result in several steps. First, we shall state and prove some results about a change of variables. Then, we shall justify its application in formula (1), and obtain a new transport equation. Finally, we reduces this problem to a one dimensionnal one, that we are able to solve.

Step 1. A change of variable.
It is sufficient to show the result stated in Theorem 1 locally. Then, we shall work in a bounded neighbourhood U of x_0, in which we assume that $b \cdot \xi > \alpha$ a.e. as in (P_x). We define Φ on U by

$$\Phi(x) = ((x - x_0) \cdot \xi, H(x))$$

We wish to use Φ as a change of variable. For this, we use the following lemma

Lemma 1. *Assume that $H \in W^{1,p}(U)$ for $p \geq 2$, then there exist a bounded connected open set V containing $(0,0)$ and $\Phi^{-1} \in W^{1,p}(V)$ such that*

$$\text{for almost all } x \in U,\ \Phi(x) \in V \text{ and } \Phi^{-1} \circ \Phi(x) = x,$$
$$\text{for almost all } y \in V,\ \Phi^{-1}(y) \in U \text{ and } \Phi \circ \Phi^{-1}(y) = y,$$

Φ and Φ^{-1} leave invariant zero-measure sets.
Moreover, we have for $f \in \mathbb{L}^\infty(V)$ the following formula:

$$\int_U f \circ \Phi(x) |D\Phi(x)|\, dx = \int_V f(y)\, dy \tag{5}$$

Proof of the lemma. Without loss of generality, we may assume that $x_0 = 0$, $\xi = (-1, 0)$, $U = (-\eta, \eta) \times (-\eta, \eta)$. According to [Zie] we can assume, since H is $W^{1,p}$, that H is absolutely continuous on almost all lines parallel to the coordinate axes and that it is true in particular for the lines $\{y = \pm\eta\}$. Then we define a open set V by

$$V = \{(y_1, y_2) \in \mathbb{R}^2 \mid H(y_1, -\eta) < y_2 < H(y_1, \eta)\}$$

To show that V is connected we have to show that $H(x_1, -\eta) < H(x_1, \eta)$ for all $x_1 \in (-\eta, \eta)$. But we have $|b_1(x)| > \alpha$ for almost all $x \in U$ then $H(x_1, \eta) - H(x_1, -\eta) > 2\eta\alpha$ for almost all $x_1 \in (-\eta, \eta)$, then for all those x_1 by continuity.

Φ preserve the first coordinate, and for almost all $x_1 \in (-\eta, \eta)$, $H(x_1, \cdot)$ is a strictly increasing homeomorphism from $(-\eta, \eta)$ to $(H(x_1, -\eta), H(x_1, \eta))$. Hence we can define a suitable mesurable Φ^{-1}.

Now, we can prove the equation (5) using Fubini's theorem. First we consider the case when f is continuous. Then, we have

$$\int_U f \circ \Phi(x) |D\Phi(x)|\, dx = \int_{-\eta}^{\eta} \left(\int_{-\eta}^{\eta} f(x_1, H(x_1, x_2)) |b_1(x_1, x_2)|\, dx_2 \right) dx_1$$

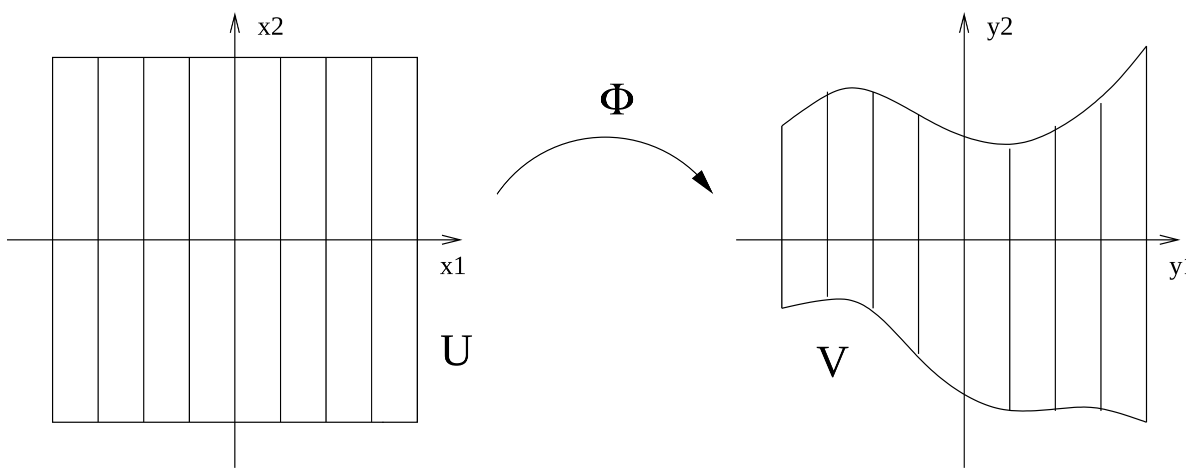

Figure 1: The Φ map

Next, if F is $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ and ϕ is in $W^{1,p}([a,b])$, then $F\circ\phi$ is in $W^{1,p}([a,b])$ and $(F\circ\phi)'=(F'\circ\phi)\phi'$. We now use this fact with F a primitive of f. Therefore, we can write

$$\int_{-\eta}^{\eta} f(x_1,H(x_1,x_2))|b_1|(x_1,x_2)\,dx_2=\int_{H(x_1,-\eta)}^{H(x_1,\eta)} f(x_1,y)\,dy$$

And if we use Fubini's theorem again, we obtain the result.

Now, we prove (5) for an arbitrary function in $\mathbb{L}^\infty$. If O is an open subset of V, we choose a sequence of f_n continuous such that $f_n\to\chi_O$ everywhere when n goes to ∞. By increasing convergence, the result is still true for χ_O. We have it for the caracteristic function of an open set. If we use the fact that $|b_1|\ge\alpha$, we obtain the inequality

$$\lambda(\Phi^{-1}(O))\le\frac{1}{\alpha}\lambda(O)$$

where λ is the Lebesgue measure. Next, if E is a zero measure subset of V, we obtain (using the above inequality with open set of small measure containing E) that $\Phi^{-1}(E)$ is also a zero-measure set.

Now, if we approximate a $\mathbb{L}^\infty$-function f by a sequence of continuous functions f_n converging to f a.e., then the sequence $f_n\circ\Phi$ converges to $f\circ\Phi$ a.e. and with the dominated convergence theorem, we obtain the result for f.

The formula (5) may be rewritten as follows

$$\int_U f(x)|D\Phi(x)|\,dx=\int_V f\circ\Phi^{-1}(y)\,dy$$

By approximation, it is always true provided the left hand side is meaningful, as it is the case, for instance when f belongs to $\mathbb{L}^q(U)$, with q the conjugate exposant of p ($p^{-1}+q^{-1}=1$). And if $f\in\mathbb{L}^a(U)$, then $f\circ\Phi^{-1}$belongs to $\mathbb{L}^b(V)$ with $b=a/q$.

To show that Φ^{-1} belongs to $W^{1,p}(V)$, and that $D(\Phi^{-1})=(D\Phi)^{-1}\circ\Phi^{-1}$, the most difficult case is to show that

$$\frac{\partial\Phi_2^{-1}}{\partial x_1}=-(\frac{b_2}{|b_1|})\circ\Phi^{-1} \tag{6}$$

First, since $b_2\in\mathbb{L}^p(U)$ and $|b_1|>\alpha$, we can use the change of variables to deduce

$$\int_V|\frac{b_2}{b_1}|^p\circ\Phi^{-1}=\int_U b_2^p b_1^{1-p}$$

Hence, the right handside of (6) belongs to $\mathbb{L}^p$.

Then, let ϕ be in $\mathcal{C}_o^\infty(V)$, we have

$$\begin{aligned}\int_V\Phi_2^{-1}(y)\frac{\partial\phi}{\partial x_1}\,dy &= \int_U x_2\frac{\partial\phi}{\partial x_1}\circ\Phi(x)\,|b_1(x)|\,dx\\ &= \int_U x_2\Big(\frac{\partial(\phi\circ\Phi)}{\partial x_1}(x)|b_1(x)|-\frac{\partial(\phi\circ\Phi)}{\partial x_2}(x)b_2(x)\Big)\,dx\\ &= \int_U\phi\circ\Phi(x)b_2(x)\,dx\\ &= \int_V\phi(y)\frac{b_2}{|b_1|}\circ\Phi^{-1}(x)\,dx\end{aligned}$$

and this is the expected result. To obtain the second line from the first, we write

$$\partial_{x_1}(\phi\circ\Phi)=\partial_{x_1}\phi\circ\Phi-b_2\partial_{x_2}\phi\circ\Phi$$

$$\partial_{x_2}(\phi\circ\Phi)=b_1\partial_{x_2}\phi\circ\Phi$$

And when $p\ge2$, these two quantities belong to $\mathbb{L}^2$ and we may multiply the first by b_1, the second by $-b_2$ and add them to obtain the desired identity. To obtain the third line from the second, we use an integration by parts and the fact that $div\,b=0$. □

Step 2. Equivalence with a new transport equation.

We now wish to apply the change of variables in the formula (we recall that we assume that $\xi = (-1, 0)$)

$$\int_{[0,\infty)\times U} u(\partial_t \phi + b.\nabla\phi) = -\int_U u^o \phi^o \tag{7}$$

Since u belongs to $\mathbb{L}^\infty(U)$, this expression make sense for ϕ in $W_0^{1,q}([0,\infty)\times U)$ (here and below q is always the conjugate exponent of p). But, we want to apply (7) with $\phi(t,x) = \psi(t,\Phi(y))$, where $\psi \in \mathcal{C}_o^\infty([0,\infty)\times V)$. In this case ϕ will belong to $W^{1,p}([0,\infty)\times U)$ and will also have a compact support because of the form of Φ. In addition, since $p \geq 2$, we may write

$$\begin{aligned} b\cdot\nabla\phi &= b_1(\partial_{x_1}\psi\circ\Phi - b_2\partial_{x_2}\psi\circ\Phi) + b_2\, b_1\partial_{x_2}\psi\circ\Phi \\ &= b_1\partial_{x_1}\psi\circ\Phi \end{aligned}$$

and we obtain, denoting by $v(t,y) = u(t,\Phi^{-1}(y))$ and $J = |b_1|\circ\Phi^{-1}$

$$\int_{[0,\infty)\times V} v(\frac{1}{J(y)}\partial_t\psi(y) + \partial_{x_1}\psi(y)) = -\int_V \frac{v^o\psi^o}{J}$$

for all ψ in $\mathcal{C}_o^\infty([0,\infty)\times V)$. In other words, v is solution in V (in the sense of the distributions) of

$$\partial_t(\frac{v}{J}) + \partial_{x_1} v = 0 \tag{8}$$

with the initial condition $(v/J)(0,\cdot) = v^o/J$.

Conversely, if $v \in \mathbb{L}^\infty([0,\infty)\times V)$ is a solution of (8), we may test it against functions ψ in $W_0^{1,1}([0,\infty)\times V)$, and if ϕ is in $\mathcal{C}_o^\infty([0,\infty)\times U)$ then $\phi\circ\Phi^{-1}$ is in $W_0^{1,1}([0,\infty)\times V)$. Thus we may follow the above argument backwards, and we obtain that (8) is equivalent to (1).

Step 3. Solution of the one dimensionnal problem.

In view of the precedent steps, it is sufficient for us to show that (R) hold for the equation (8). But, in this equation there is no derivative with respect to y_2. Therefore, it is equivalent to say that for almost all y_2, $\partial_t(v/J)+\partial_{x_1}v = 0$ on the set $\mathbb{R}\times V_{y_2}$ with the good initial conditions (here $V_{y_2} = \{y \in \mathbb{R}\,|(y,y_2)\in V\}$). This would be obvious if V were of the form $(a,b)\times(c,d)$, but we can always see V as a countable union of such rectangular sets. And since an open subset of $\mathbb{R}$ is a countable union of open intervals, we just have to show that the property (R) is true for the equation (8) on an interval $I = (a,b)$ of $\mathbb{R}$, with $J \geq \alpha$ a.e. on I.

Let F be a primitive of $1/J$. F is continuous, strictly increasing on (a,b) onto $(F(a),F(b))$, and its inverse F^{-1} belongs to $W^{1,1}(F(a),F(b))$. Again, we may performe the change of variables $y \mapsto z = F(y)$ and we obtain that the equation (8) on I is equivalent to

$$\partial_t w + \partial_z w = 0 \qquad \text{on} \quad [0,\infty)\times(F(a),F(b)) \tag{9}$$

where $w(t,z) = v(t,F^{-1}(z))$. For this equation the property (R) is true. In fact we have a flow $X(t,x) = F^{-1}(F(x)+t)$ for (8), but we need to be careful because we are not exactly on the whole line and so this quantity is not defined for all t. □

3 Critical points

In the preceding result, we assumed that the condition (P_x) was true for all x. We want here to take into account possible critical points. However, since we only assume that $b \in \mathbb{L}^1_{loc}$, we cannot define critical points (of the Hamiltonian) as points where b vanishes (the usual notion when the flow is continuous). In some sense, critical points mean for us all those points where (P_x) is not true. In fact, this yields a "larger" set of critical points.

3.1 Isolated critical points

Our first result is the following

Corollary 1. *If b satisfies (P_x) everywhere in Ω except on a set of isolated points, then the (R) hypothesis holds.*

Proof. Without loss of generality we may assume that $\Omega = \mathbb{R}^2$, that (P_x) holds everywhere except at the origin $(0,0)$ and that $b \in \mathbb{L}^2$. We take $\psi \in \mathcal{C}_o^\infty(\mathbb{R})$ so that $\psi \equiv 1$ on a neighbourhood of $(0,0)$ and vanishes outside the ball B_1 of radius 1. We define $\psi_\epsilon = \psi(\frac{\cdot}{\epsilon})$.

Let $\phi \in \mathcal{C}_o^\infty([0,\infty)\times\mathbb{R}^2)$, $\beta \in \mathcal{C}^1(\mathbb{R})$ and u be a solution of the transport equation on $\mathbb{R}^2$, then $(1-\psi_\epsilon)\phi \in \mathcal{C}_o^\infty(\mathbb{R}^2\backslash\{(0,0)\}$, and since u is a renormalized solution on $\mathbb{R}^2\backslash\{(0,0)\}$, we may write

$$\langle \partial_t\beta(u) + div(b\beta(u)), (1-\psi_\epsilon)\phi\rangle = \int_{\mathbb{R}^2} \beta(u^o)(1-\psi_\epsilon)\phi^o \tag{10}$$

i.e.
$$\int_{[0,\infty)\times\mathbb{R}^2} \beta(u)(1-\psi_\epsilon)(\partial_t\phi + b\cdot\nabla\phi) - \int_{[0,\infty)\times\mathbb{R}^2} \beta(u)\phi b\cdot\nabla\psi_\epsilon = -\int_{\mathbb{R}^2} \beta(u^o)(1-\psi_\epsilon)\phi^o \tag{11}$$

When ϵ goes to 0, the first integral converges to $\langle \partial_t\beta(u) + div(b\beta(u)), \phi\rangle$, the second one converges to 0 since

$$\begin{aligned} \left|\int_{\mathbb{R}^2} \beta(u)\phi\nabla\psi_\epsilon\right| &\leq C\|b\|_{\mathbb{L}^2(B_\epsilon)}\|\nabla\psi_\epsilon\|_{\mathbb{L}^2} \\ &\leq C\|\nabla\psi\|_{\mathbb{L}^2}\|b\|_{\mathbb{L}^2(B_\epsilon)} \end{aligned}$$

and the right hand side converges to $-\int_{\mathbb{R}^2}\beta(u^o)\phi^o$.

We conclude that

$$\langle \partial_t\beta(u) + div(b\beta(u)), \phi\rangle = \int_{\mathbb{R}^2} \beta(u^o)\phi^o$$

for all $\phi \in \mathcal{C}_o^\infty([0,\infty)\times\mathbb{R}^2)$. Hence u is a renormalized solution. □

3.2 A result with more regularity on H

The above result, of course, does not allow for many critical points. But we can allow much more with stronger conditions on H. First, points where there exists a neighbourhood on which b vanishes, are obviously easy to handle. We shall call O the set of all these points, and P the set of the points where (P_x) is true. We denote $Z = (O\cup P)^c$ (this complementary is taken in Ω). It is closed, since O and P are open . Then we have the following corolary.

Corollary 2. *Assume that H is continuous, Z is a set of zero-measure in $\mathbb{R}^2$ and $H(Z)$ is a set of zero-measure in $\mathbb{R}$. Then (R) still holds.*

Remarks

i. These conditions where introduced by L. Desvillettes and F. Bouchut in [3] in the case when b is continuous. Here, we only rewrite their proof in a less regular case.

ii. If $p > 2$, according to Sobolev embeddings, H is automatically continuous.

iii. We do not know if $H(Z)$ has zero-measure since we cannot apply Sard's lemma.

Proof. Let u be a solution of the transport equation (1) in Ω, $\beta \in \mathcal{C}^1(\mathbb{R})$, ϕ a $\mathcal{C}^\infty$-test function, and K_o a compact set containing the support of $\phi(t,\cdot)$ for all t. We define $Z_o = Z\cap K_o$ and $K = H(Z_o)$. Then K is a zero-measure compact set. Then, we can find functions $\chi_n \in \mathcal{C}_o^\infty(\mathbb{R})$ such that, $0 \leq \chi_n \leq 1$, $\chi_n \equiv 1$ on a neighbourhood of K and $\chi_n \to \chi_K$, the caracteristic function of K, when n goes to ∞. We set $\Psi_n = \chi_n \circ H$. Ψ_n is continuous, belongs to $W^{1,p}(\mathbb{R}^2)$ and $\Psi_n \equiv 1$ on a neighbourhood of Z_o.

By theorem 1, $\beta(u)$ is a solution of (1) in P, and is also a solution in O, because on this set u is independent of the time. Since this two sets are open, $\beta(u)$ is a solution in $P\cup O$. $(1-\Psi_n)\phi$ is continuous and belongs to $W^{1,p}([0,\infty) \times K_o)$ and has its support in $[0,\infty) \times (K_o \backslash Z_o)$. Hence, since $(K_o \backslash Z_o) \subset P \cup O$ we can use it as a test fonction. We have

$$\langle \partial_t \beta(u) + div(b\beta(u)), (1-\Psi_n)\phi \rangle = \int_\Omega \beta(u^o)(1-\Psi_n)\phi^o$$

i.e. $$\int_{[0,\infty)\times\Omega} \beta(u)(1-\Psi_n)(\partial_t \phi + b\cdot\nabla\phi) - \int_{[0,\infty)\times\Omega} \beta(u) b\cdot\nabla\Psi_n = -\int_\Omega \beta(u^o)(1-\Psi_n)\phi^o \tag{12}$$

The second integral vanishes because $\nabla\Psi_n = (\Phi_n' \circ H)\nabla H$ and $b = \nabla H^\perp$.
The first integral converges by dominated convergence to

$$\int_{[0,\infty)\times H^{-1}(K)^c} \beta(u)(\partial_t\phi + b\cdot\nabla\phi)$$

while the left hand side goes to $-\int_{H^{-1}(K)^c} \beta(u^o)\phi^o$.

Then, to prove that (4) holds, we just have to show that

$$\int_{[0,\infty)\times H^{-1}(K)} \beta(u)(\partial_t\phi + b\cdot\nabla\phi) = -\int_{H^{-1}(K)} \beta(u^o)\phi^o \tag{13}$$

But $H \in W^{1,p}(\mathbb{R}^2)$ and K is a zero-measure set, and this is a classical result that in that case $\nabla H = 0$ a.e. on $H^{-1}(K)$ (see for instance [4]). Then, $b = \nabla H^\perp = 0$ a.e. on this set, and

$$\int_{[0,\infty)\times H^{-1}(K)} \beta(u)(\partial_t\phi + b\cdot\nabla\phi) = \int_{[0,\infty)\times H^{-1}(K)} \beta(u)\partial_t\phi$$

Moreover, $H^{-1}(K) \cap P$ is a set of zero-measure because on P, $\nabla H \neq 0$ a.e.. Hence the following quantity will not change if we integrate only on $H^{-1}(K) \cap (O \cup Z)$ or on $H^{-1}(K) \cap O$ since Z has zero-measure. But we already know that u is independent of the time on this set, then we can integrate in time to obtain the equality (13). □

As a conclusion to this section we just wanted to say that we do not know what happens when the condition (P_x) is not true on a sufficiently large set. Of course, we can construct divergence free vector-fields which do not satisfy (P_x) at every point, but it seems difficult to work with such flows because their definition is complex.

4 One example

We observe in this section that the example introduced by R. DiPerna and P.L. Lions in [1] provides an example of an divergence free vector fields b such that $b \in \mathbb{L}^\infty_{loc}(\mathbb{R}^2 \backslash (0,0))$, b is in $\mathbb{L}^p$ in a neighbourhood of the origin for all $p < 2$ but not for $p = 2$, b satisfies the condition (P_x) everywhere, but there exist several solutions to the transport equations and several a.e. flows solving the associated ODE.

4.1 Definition of the vector-field

We define the hamiltonian H as follows (see fig 2)

$$H(x) = \begin{cases} -\frac{x_1}{|x_2|} & \text{if } |x_1| \le |x_2| \\ -(x_1 - |x_2| + 1) & \text{if } x_1 > |x_2| \\ -(x_1 + |x_2| - 1) & \text{if } x_1 < -|x_2| \end{cases}$$

Then, b is given by

$$b_1(x) = -\frac{\partial H}{\partial x_2} = -sign(x_2)\left(\frac{x_1}{|x_2|^2} 1_{|x_1|\le|x_2|} + sign(x_1) 1_{|x_1|>|x_2|}\right)$$

$$b_2(x) = \frac{\partial H}{\partial x_1} = -\left(\frac{1}{|x_2|} 1_{|x_1|\le|x_2|} + 1_{|x_1|>|x_2|}\right)$$

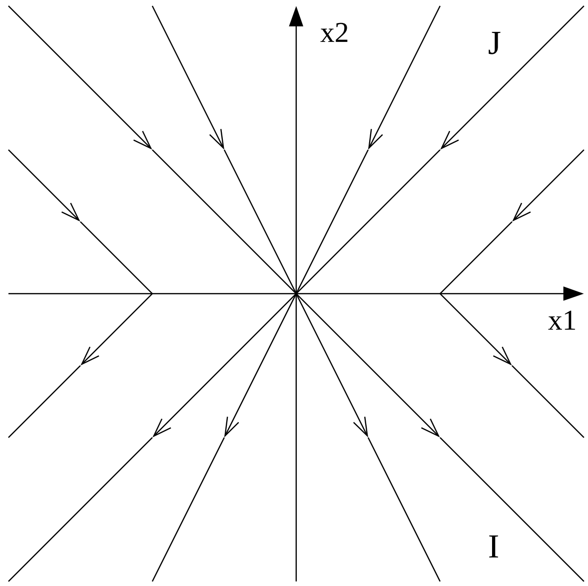

Figure 2: Flow lines of the example

4.2 Form of the solutions

First, we construct an a.e. flow X solution of the associated ODE. Since it would be symmetric in relation to the x_2-axis, we only defined it for $x_1 \geq 0$. We also define it just for $t \geq 0$.

In the case when $0 \leq x_2 \leq x_1$, we set

$$X(t,x) = (x_1 - t, x_2 - t) \qquad \text{for} \quad t \leq x_2$$
$$X(t,x) = (x_1 - 2x_2 + t, x_2 - t) \qquad \text{for} \quad t \geq x_2$$

while for $0 \leq -x_2 \leq x_1$, we set

$$X(t,x) = (x_1 + t, x_2 - t)$$

In the case when $0 \leq x_1 < x_2$, we set

$$X(t,x) = \sqrt{1 - \frac{2t}{(x_2)^2}}\,(x_1, x_2) \qquad \text{for} \quad t \leq \frac{(x_2)^2}{2}$$
$$X(t,x) = \sqrt{\frac{2t}{(x_2)^2} - 1}\,(x_1, -x_2) \qquad \text{for} \quad t \geq \frac{(x_2)^2}{2}$$

And if $0 \leq x_1 < -x_2$,

$$X(t,x) = \sqrt{1 + \frac{2t}{(x_2)^2}}\,(x_1, x_2)$$

In the sequels, we denote $I = \{x \in \mathbb{R}^2 | 0 < x_1 < -x_2\}$ and $J = \{x \in \mathbb{R}^2 | 0 < x_1 < x_2\}$. For an initial condition u^o, some tedious computation easily shows that the solutions of the transport equation (we use the fact that $u(t, X(t,x))$ is independent of t as long as X(t,x) does not reach the origin, and then we use the change of variable $(t,x) \to (t, X(t,x))$ on all the space, paying attention to what happens at the origin). They are of the form

$$u(t,x) = \begin{cases} u^o(X(-t,x)) & \text{if} \quad x \notin I \quad \text{or} \quad x \in I \quad \text{and} \quad t \leq \frac{(x_2)^2}{2} \\ \tilde{u}(X(-t,x)) & \text{if} \quad x \in I \quad \text{and} \quad t \geq \frac{(x_2)^2}{2} \end{cases} \tag{14}$$

where $\tilde{u}$ is any function defined on J satisfying the condition

$$\forall x_2 > 0, \quad \int_{-x_2}^{x_2} \tilde{u}(x_1, x_2)\, dx_1 = \int_{-x_2}^{x_2} u^o(x_1, x_2)\, dx_1 \tag{15}$$

Indeed, we use here the flow X for simplicity but these solutions are not defined according to this flow when a trajectory pass through the origin. We will try to explain what happens at the origin. For $x_2 > 0$, if the quantity u represent a density of mass, all the mass on the segment $\{(x, x_2) | x \in (-x_2, x_2)\}$ reaches the origin at the time $(x_2)^2/2$. After this time it continues to move in I always on segments parallel to the x_1-axis, but it can be redistributed on them in any way provided the total mass on this segment in conserved. This is what means the condition (15).

The renormalized solutions are always of this form, but the condition (15) should be replaced by

$$\forall x_2 > 0, \forall \beta \in \mathcal{C}^1(\mathbb{R}) \quad \int_{-x_2}^{x_2} \beta(\tilde{u})(x_1, x_2)\, dx_1 = \int_{-x_2}^{x_2} \beta(u^o)(x_1, x_2)\, dx_1 \tag{16}$$

This condition (16) is equivalent to the fact that for all $x_2 \in \mathbb{R}$, we have a measure-preserving transformation Φ from $(-x_2, x_2)$ into itself such that $\tilde{u} = u^o \circ \Phi$. We refer to [Roy] for this point.

Moreover, we can also find all the flows solutions of the associated ODE. Choosing a mesurable measure-preserving transformation Ψ from $(-1, 1)$ into itself, we defined a flow X_Ψ by

$$X_\Psi(t, x) = \begin{cases} X(t, x) & \text{if} \quad x \notin J \quad \text{or} \quad t \leq \frac{(x_2)^2}{2} \\ \Psi(x) X(t, x) & \text{if} \quad x \in J \quad \text{and} \quad t \geq \frac{(x_2)^2}{2} \end{cases}$$

To see that this defined a a.e. flow, we use the property stated in the definition of an a.e. flow and the fact that an a.e. flow is measure preserving. Let us try to illustrate this definition. A particle with an initial position x^o in J moving according to X_Ψ behaves as follow. It moves on the half-line $\{x | x_1/x_2 = \lambda, \quad x_2 > 0\}$ (with $\lambda = x_1^o/x_2^o$) until it reaches the origin. Then it continues to move in I but on the half-line $\{x | x_1/x_2 = \Psi(\lambda), \quad x_2 < 0\}$. Indeed, Ψ may be seen as a mapping between the upper half-lines and the lower ones.

We can thus see that in this case, we have renormalized solutions that are not defined according to an a.e. flow. Indeed, for a renormalized solution, we can choose different mappings between the upper half-lines and the lower ones for each x_2 (in other words $\tilde{u} = u^o(\Psi_{x_2}(x_1/x_2)x_2, x_1)$ where Ψ_{x_2} is measure-preserving transformation from $(-1, 1)$ into itself depending on x_2), while for a solution defined according to an a.e. flow, this correspondance will be independant of x_2 .

4.3 Remark about the uniqueness of the solution

First we remark that the flow X is a specific one. It is the only one for which the hamiltonian remains constant on all the trajectories. Moreover, we observe that the solution $\overline{u}$ defined according to X is specific among all the others. This is the only one which satisfies also the above family of equations (17), which says that the hamiltonian remains constant on the trajectories.

$$\forall f \in \mathcal{C}(\mathbb{R}, \mathbb{R}) \qquad \partial_t(f(H)u) + \operatorname{div}(f(H)bu) = 0 \tag{17}$$

Indeed, we do the same computation that leads to (15) with these equations and we obtain the following conditions

$$\forall x_2 > 0, \quad \forall f \in \mathcal{C}(\mathbb{R}, \mathbb{R}), \quad \int_{-x_2}^{x_2} \tilde{u}(x_1, x_2) f(x_1)\, dx_1 = \int_{-x_2}^{x_2} u^o(x_1, x_2) f(x_1)\, dx_1 \tag{18}$$

This implies that $\tilde{u} = u^o$ and then that $u = \overline{u}$.

Hence, adding the conditions (17) in the definition of a solution, we are able to define it uniquely. Moreover, if we try to solve this problem by approximation, choosing a sequence of divergence free vector-field b_m converging to b in all $\mathbb{L}^p_{loc}$, for $p < 2$, (this implies that H_m converge to H up to a constant in all $W^{1,p}_{loc}$, for $p < 2$), we obtain a sequence of solutions u_m, that satisfy all the equations (17) with the initial condition u^o. This sequence u_m is weakly compact in $\mathbb{L}^\infty$. Extracting a converging subsequence, we see that the equations (17) are always true at the limit and then the sequence u_m converge to $\overline{u}$. This solution is therefore the only that we can construct by approximation.

5 The case of a particule moving on a line

We consider here a classical Hamiltonian

$$H(x,y) = y^2/2 + V(x) \quad \text{or} \quad b(x,y) = (y, -V'(x))$$

with V a potential in $W^{1,p}_{loc}(\mathbb{R})$. Then V' belongs to $\mathbb{L}^p_{loc}(\mathbb{R})$. In this case, b satisfies the (P_x) assumption in $\mathbb{R}^2\backslash\{y=0\}$. The set of criticals points Z has then zero-measure. We can apply the preceding results, if H satisfies $m(H(Z)) = 0$. When V' is continuous, this is true because we can apply the Sard Lemma. But this is false for a general $V' \in \mathbb{L}^1_{loc}$. If V' oscillates very quickly, Z may even be the whole line. And then $H(Z)$ is an interval because H is continuous. However, we will show that the result is always true in this case. Moreover, we can only assume that V' belongs to $\mathbb{L}^1_{loc}(\mathbb{R})$, since the other composant of b is in $\mathbb{L}^\infty_{loc}(\mathbb{R})$.

The transport equation we are considering has the form

$$\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial x} - V'(x)\frac{\partial u}{\partial y} = 0 \tag{19}$$

Here we can solve the differential equation $x' = y$, $y' = -a(x)$ directly if we use the fact that the Hamiltonian is constant on a trajectory and integrate the system. But this flow is not regular, and we do not know how to work directly with it in order to solve the transport equation.

Theorem 2. *For a flow $b(x,y) = (y, -V'(x))$ with $V' \in \mathbb{L}^1_{loc}(\mathbb{R})$ and $1/\sqrt{max(1, -V(x))}$ not integrable at $\pm\infty$, the transport equation has an unique renormalized solution.*

Remarks

i. The condition of integrability on V is there to insure that a point does not reach $\pm\infty$ in a finite time. It could be replaced by a stronger condition like $V(x) \geq -C(1+x^2)$.

ii. This result can be adapted to the case of two particles moving on a line according to a interaction potential in $W^{1,1}_{loc}$. In order to do so this we just have to use a change of variable which follows the classical way of reducing this two-body problem to a one-body problem.

Proof. We can use our previous theorem in the neighboorhood of a point with $y \neq 0$. This will give us "the result" on two half-planes, but we need to "glue" together the information available on this two half-planes. Then we need to work differently, and we shall follow the same sketch of proof as in our first theorem.

Step 1. A change of variables.
First we define

$$\Phi_+(x,y) = (x, y^2/2 + V(x)) \qquad \text{from} \quad \mathbb{R}\times(0,\infty) \quad \text{to} \quad B$$
$$\Phi_-(x,y) = (x, y^2/2 + V(x)) \qquad \text{from} \quad \mathbb{R}\times(-\infty,0) \quad \text{to} \quad B$$

where $B = \{(x,E) \in \mathbb{R}^2 | V(x) < E\}$. Then Φ_+ and Ψ_- are continuous and belong to $W^{1,1}_{loc}$ with

$$D\Phi_\pm = \begin{pmatrix} 1 & 0 \\ V'(x) & y \end{pmatrix}$$

and the same for Φ_-.

These transformations are one-to-one and onto and $\Phi^{-1}_\pm(x,E) = (x, \pm\sqrt{2(E - V(x))})$.

$$D\Phi^{-1}_+ = \begin{pmatrix} 1 & 0 \\ -\frac{V'(x)}{\sqrt{2(E-V(x))}} & \frac{1}{\sqrt{2(E-V(x))}} \end{pmatrix},$$

with a similar formula for Φ^{-1}_-.

The following change of variables is true

$$\int_{y>0} f(x,y)\,dxdy = \int_B \frac{f\circ\Phi_+^{-1}(x,E)}{\sqrt{2(E-V(x))}}\,dxdE$$

for f in $\mathbb{L}^\infty$ and even in $\mathbb{L}^1$.

Before going further, we state some properties about the set $W^{1,1}(B)$. We define $\mathcal{C}^\infty(\overline{B})$ (resp.$\mathcal{C}_o^\infty(\overline{B})$) the space of restriction to B of $\mathcal{C}^\infty$-functions on $\mathbb{R}^2$ (resp. such functions with compact support). We recall that $\partial B = \{(x,V(x))|x\in\mathbb{R}\}$.

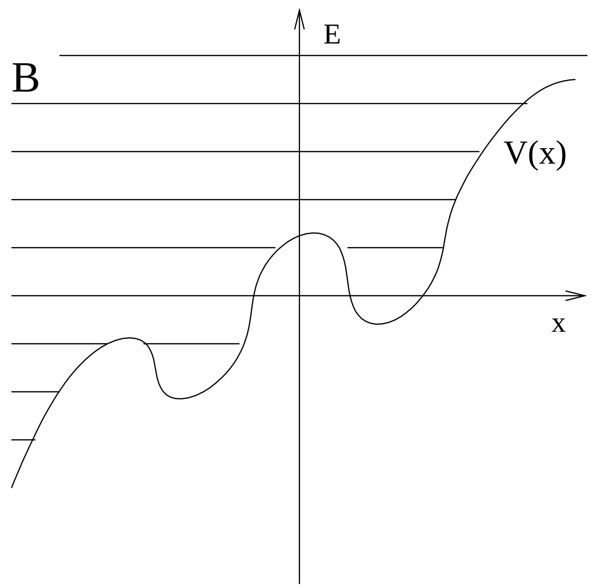

Figure 3: The domain B

Proposition 2.

i. $\mathcal{C}_o^\infty(\overline{B})$ *is dense in* $W^{1,1}(B)$.

ii. The trace of a function in $W^{1,1}(B)$ *has a sense in* $\mathbb{L}^1$. *More precisely, there exist a continuous application* $Tr: W^{1,1}(B)\to\mathbb{L}^1(\mathbb{R})$ *such that* $Tr(\phi)=\phi(\cdot,V(\cdot))$ *if* $\phi\in\mathcal{C}_o^\infty(\overline{B})$.

iii. $\overline{\mathcal{C}_o^\infty(B)} = Ker(Tr)$, *the kernel of the trace, also denoted by* $W_o^{1,1}(B)$.

iv. The same results are true for $\mathbb{R}\times B$ *and* $[0,\infty)\times B$ *as well as locally.*

Proof of the proposition. We refer to theorem 3.18 in [Ada] for a complete proof. But we may adapt the proof of this theorem to this simpler case. For a $f\in W^{1,1}(B)$, we define for $\epsilon>0$

$$f_\epsilon = \rho_\epsilon * (f(\cdot+2\epsilon e)\chi_{\{E>V(x)-2\epsilon\}})$$

where $e=(0,1)$ and $\rho_\epsilon(x,E)=\rho(x/\epsilon,E/\epsilon)$ with $\rho\in\mathcal{C}_o^\infty(\mathbb{R}^2)$ satisfying $\int\rho=1$. Then, the functions f_ϵ belong to $\mathcal{C}_o^\infty(\overline{B})$ and converges to f in $W^{1,1}(B)$ as $\epsilon\to 0$.

For the second point, we choose $f\in\mathcal{C}_o^\infty(\overline{B})$. Then

$$f(x,V(x)) = -\int_{V(x)}^\infty \frac{\partial f}{\partial E}(x,E)\,dE$$

taking the absolute value and integrating in x leads to

$$\int_{\mathbb{R}} |f(\cdot,V(\cdot))| \le \|\nabla f\|_{\mathbb{L}^1(B)}$$

Then, the trace is a contraction from $\mathcal{C}_o^\infty(\overline{B})$ with the $W^{1,1}$-norm into $\mathbb{L}^1(\mathbb{R})$, and since $\mathcal{C}_o^\infty(\overline{B})$ is dense in $W^{1,1}(B)$, we may extend this application to $W^{1,1}(B)$.

For the third point, we take $f \in Ker(Tr)$ and extend it by zero outside B. We obtain a $\tilde{f}$ in $W^{1,1}(\mathbb{R}^2)$. Then if we translate $\tilde{f}$ in the direction of $e = (1,0)$ and smooth it by convolution, we can construct $\mathcal{C}^\infty$-approximations of f with support in B. $\square$

Step 2. Equivalence with a simpler transport equation.

Now, let u be a solution of the transport equation. We may write

$$\int_{[0,\infty)\times\mathbb{R}^2} u(\partial_t\phi + y\partial_x\phi - V'(x)\partial_y\phi)\,dxdy = -\int_{\mathbb{R}^2} u^o\phi^o \tag{20}$$

for all $\phi \in W^{1,1}([0,\infty)\times\mathbb{R}^2)$ with compact support (in the sense of distributions) satisfying moreover $\partial_y\phi \in \mathbb{L}^\infty([0,\infty)\times\mathbb{R}^2)$.

Let Ψ_+ and Ψ_- be in $\mathcal{C}_o^\infty([0,\infty)\times\overline{B})$, and Ψ_+ and Ψ_- satisfy the compatibility condition $\Psi_{+|[0,\infty)\times\partial B} = \Psi_{-|[0,\infty)\times\partial B}$. We define ϕ from $[0,\infty)\times\mathbb{R}^2$ to $\mathbb{R}$ with

$$\phi(t,x,y) = \begin{cases} \Psi_+(t,\Phi_+(x,y)) & \text{if} \quad y>0 \\ \Psi_-(t,\Phi_-(x,y)) & \text{if} \quad y<0 \end{cases}$$

Then, ϕ belongs to $W^{1,1}([0,\infty)\times\mathbb{R}^2)$, has a compact support, and ϕ, $\partial_t\phi$, $\partial_y\phi$ are in $\mathbb{L}^\infty$. Moreover,

$$\begin{gathered} \partial_x\phi = (\partial_x\Psi_+)\circ\Phi_+ + E(x)(\partial_y\Psi_+)\circ\Phi_+ \quad \text{for } y>0 \\ \partial_y\phi = y(\partial_y\Psi_+)\circ\Phi_+ \\ \partial_t\phi = (\partial_t\Psi_+)\circ\Phi_+ \end{gathered}$$

Then we have $\quad \partial_t\phi + y\partial_x\phi - E(x)\partial_y\phi = (\partial_t\Psi_+)\circ\Phi_+ + y(\partial_x\Psi_+)\circ\Phi_+$ for all $y>0$.

We write $v_\pm = u\circ\Phi_\pm^{-1}$, defined on $[0,\infty)\times B$. Then, (20) may be written as follows.

$$\begin{aligned} \int_{[0,\infty)\times\{y>0\}} &[v_+(\partial_t\Psi_+ + \sqrt{2(E-V(x))}\partial_x\Psi_+)]\circ\Phi_+ \\ &+ \int_{[0,\infty)\times\{y<0\}} [v_-(\partial_t\Psi_- - \sqrt{2(E-V(x))}\partial_x\Psi_-)]\circ\Phi_- \\ &= \int_{y>0}(v_+^0\Psi_+^0)\circ\Phi_+ + \int_{y<0}(v_-^0\Psi_-^0)\circ\Phi_- \end{aligned} \tag{21}$$

We can apply the change of variables, and we obtain

$$\begin{aligned} \int_{[0,\infty)\times B} v_+\left(\frac{\partial_t\Psi_+}{\sqrt{2(E-V(x))}} + \partial_x\Psi_+\right) + \int_{[0,\infty)\times B} v_-\left(\frac{\partial_t\Psi_-}{\sqrt{2(E-V(x))}} - \partial_x\Psi_-\right) \\ = \int_B \frac{v_+^0\Psi_+^0 + v_-^0\Psi_-^0}{\sqrt{2(E-V(x))}} \end{aligned} \tag{22}$$

It is difficult to work with Φ_+ and Φ_- because of the compatibility condition. But we may make the particular choice $\Phi_+ = \Phi_-$ (below we will omit the indices $\pm$). Then (22) becomes

$$\int_{[0,\infty)\times B}(v_+ + v_-)\frac{\partial_t\Psi}{\sqrt{2(E-V(x))}} + (v_+ - v_-)\partial_x\Psi = \int_B \frac{(v_+^0 + v_-^0)}{\sqrt{2(E-V(x))}}\Psi^0 \tag{23}$$

Now, we choose $\Psi_+ = -\Psi_-$ and $\Psi_{|[0,\infty)\times\partial B} = 0$ (we omit the indices $\pm$). In this case, (22) becomes

$$\int_{[0,\infty)\times B}(v_+ - v_-)\frac{\partial_t\Psi}{\sqrt{2(E-V(x))}} + (v_+ + v_-)\partial_x\Psi = \int_B \frac{(v_+^0 - v_-^0)}{\sqrt{2(E-V(x))}}\Psi^0 \tag{24}$$

Then, (20) implies (23) for all Ψ in $\mathcal{C}_o^\infty([0,\infty)\times\overline{B})$, and (24) for all $\Psi \in \mathcal{C}_o^\infty([0,\infty)\times\overline{B})$ with $\Psi_{|[0,\infty)\times\partial B} = 0$, or equivalently for all $\Psi \in \mathcal{C}_o^\infty([0,\infty)\times B)$ since $\mathcal{C}_o^\infty([0,\infty)\times B)$ is dense in $W_o^{1,1}([0,\infty)\times$

B). And conversly, these two statements are equivalent with (22) for all Ψ_+ and Ψ_- in $\mathcal{C}_o^\infty([0,\infty)\times\overline{B})$ having the same trace on the boundary.

Thus, we have to solve

$$\partial_t\Big(\frac{v_+ + v_-}{\sqrt{2(E-V(x))}}\Big) + \partial_x(v_+ - v_-) = 0 \qquad \text{on} \quad \mathcal{D}'([0,\infty)\times\overline{B}) \tag{25}$$

$$\partial_t\Big(\frac{v_+ - v_-}{\sqrt{2(E-V(x))}}\Big) + \partial_x(v_+ + v_-) = 0 \qquad \text{on} \quad \mathcal{D}'([0,\infty)\times B) \tag{26}$$

with the convenient initial conditions. In (25), $\mathcal{D}'([0,\infty)\times\overline{B})$ means that we allow test functions in $\mathcal{C}^\infty([0,\infty)\times\overline{B})$.

We can do the same arguments backwards. Therefore, solving (25)-(26) is equivalent to solve (19)

Step 3. Reduction to one dimension.

These two equations do not contain any derivative in E. As in the proof of the first result, we want to reduce them to equations in one dimension of space. For the second equation (26), we can make the same argument and we obtain that this equation holds on B_E, for almost all E in $\mathbb{R}$. (with $B_E=\{x\in\mathbb{R}|(x,E)\in B\}$).

For the first equation (25) we can still apply the argument. We shall be more precise since it is a little bit more involved. We choose a test function ϕ of the form $\phi_1\phi_2$ with ϕ_1 depending only on (t,x) and ϕ_2 depending on E. We obtain

$$\int_{[0,\infty)\times B}\left((v_+ + v_-)\frac{\partial_t\phi_1}{\sqrt{2(E-V(x))}} + (v_+ - v_-)\partial_x\phi_1\right)\phi_2 = \int_B \frac{(v_+^0 + v_-^0)}{\sqrt{2(E-V(x))}}\phi_1^o\phi_2 \tag{27}$$

Since the linear combinaisons of functions of the form $\phi_1\phi_2$ are dense in $\mathcal{C}_o^\infty([0,\infty))$ with the $W^{1,1}$-norm, (27) for all $\mathcal{C}_o^\infty$ ϕ_1 and ϕ_2 is equivalent with (23) for all $\mathcal{C}_o^\infty$ Ψ. Moreover, since $W^{1,1}([0,\infty)\times\mathbb{R})$ is separable, it is sufficient (and necessary) to write (27) for ϕ_1 choosen among a countable subset F_1 of $\mathcal{C}_o^\infty$-functions.

Now, using Fubini's theorem (27) may be rewritten

$$\int_{\mathbb{R}}\left(\int_{[0,\infty)\times B_E}(v_+ + v_-)\frac{\partial_t\phi_1}{\sqrt{2(E-V(x))}} + (v_+ - v_-)\partial_x\phi_1\, dtdx\right)\phi_2\, dE$$
$$= \int_{\mathbb{R}}\left(\int_{B_E}\frac{(v_+^0 + v_-^0)}{\sqrt{2(E-V(x))}}\phi_1^o\, dx\right)\phi_2\, dE \tag{28}$$

for a fixed ϕ_1. Since it is satisfied for all $\mathcal{C}_o^\infty$-ϕ_2, we obtain that

$$\int_{[0,\infty)\times B_E}(v_+ + v_-)\frac{\partial_t\phi}{\sqrt{2(E-V(x))}} + (v_+ - v_-)\partial_x\phi = \int_{B_E}\frac{(v_+^0 + v_-^0)}{\sqrt{2(E-V(x))}}\phi^o \tag{29}$$

for all $E\in\mathbb{R}\backslash N$ where N is a zero-measure set depending on ϕ_1. Now, if we write this equation for all $\phi_1\in F_1$, we obtain that (25) is satisfied, but this time in $[0,\infty)\times B_E$ for allmost all $E\in\mathbb{R}$. And we can do the argument backwards to show that this is equivalent to the initial problem. Finally, we just have to solve (25)-(26) on B_E instead of B.

Step 4. Solution of the one dimensionnal problem.

B_E is a countable union of disjoint open intervals. We denote $B_E=\cup_n(a_n,b_n)$, where a_n, b_n are disjoints reals. But, since we shall also work on $\overline{B_E}$ we want that these open intervals are not to "close" to each other. For instance, if there exist n, m such that $b_n=a_m$ and if $1/\sqrt{2(E-V(x))}$ is integrable on a neighboorhood of b_n, a particle reaching b_n from the left may continue to go further right or may change direction and go backwards. This will give rise to distinct solutions of the transport equation. But, we shall show that for almost all E, we have some "free zone" around each (a_n,b_n). More precisely, for almost all E there exists an $\epsilon_n > 0$ such that $B_E \cap (a_n - \epsilon_n, b_n + \epsilon_n) = (a_n, b_n)$. If we admit this point, we see that we just have to solve (25)-(26) on an interval of the type (a', b'), where a' belongs to $[-\infty, +\infty)$ and b' to $(-\infty, +\infty]$. Before going further, we prove the

Lemma 2. *For almost all E, if we write $B_E = \cup_n (a_n, b_n)$ then, for each n, there exists some $\epsilon_n > 0$ such that $B_E \cap (a_n - \epsilon_n, b_n + \epsilon_n) = (a_n, b_n)$*

Proof of the lemma. First we recall that since V belongs to $W^{1,1}_{loc}$, the image by V of a zero-measure set is a zero-measure set. Then, we state a result similar to the Sard's lemma for V. Let Z be the set were V' vanishes. We claim that $V(Z)$ has zero-measure. Of course , Z is defined up to a zero-measure set, but this is irrelevant for our claim in view of the fact recalled above. In order to prove our claim, we choose a sequence of open sets O_n such that $Z \subset O_n$ and $\lambda(O_n \backslash Z)$ goes to 0 as n goes to ∞. Here and below λ denotes the Lesbegue measure on $\mathbb{R}$ or $\mathbb{R}^2$. We may write $O_n = \cup_m I_{n,m}$ where the $I_{n,m}$ are disjoint intervals of $\mathbb{R}$. Then,

$$\begin{aligned} \lambda(V(O_n)) &= \lambda(V(\cup_m I_{n,m})) \leq \sum_m \lambda(V(I_{n,m})) \\ &\leq \sum_m \int_{I_{n,m}} |V'| = \int_{O_n} |V'| \\ &\leq \int_{O_n \backslash Z} |V'| \end{aligned}$$

and the last quantity goes to 0 as $n \to \infty$ since $\lambda(O_n \backslash Z)$ goes to 0 as $n \to \infty$ and our claim is shown.

Next, we denote by Z_1 the set such that Z_1^c is the set of Lebesgue points of V' (i.e. the set of points such that $1/(2\epsilon) \int_{x-\epsilon}^{x+\epsilon} |V'(y) - V'(x)|\, dy$ goes to zero as $\epsilon \to 0$). Then, $\lambda(Z_1) = 0$. According to what we proved above, we know that $\lambda(V(Z \cup Z_1)) = 0$. Now, if we choose $E \in V(Z \cup Z_1)^c$,and write $B_E = \cup_n (a_n, b_n)$ as above, we know that a_n and b_n are Lebesgue's point of V' with $V'(a_n) \neq 0$ and $V'(b_n) \neq 0$. Then necessarily, $V'(b_n) > 0$ and V is strictly increasing in a neighboorhood of b_n because it is a Lebesgue's point. Since we may make the same argument near a_n, we have then shown the existence of ϵ_n as stated in the lemma. □

To solve (25)-(26) on (a', b') we use the change of variable $x \mapsto z = F(x)$ where F is a primitive of $1/\sqrt{2(E - V(x))}$ from (a', b') to (a, b). We can because this quantity is locally integrable on almost all lines (this result is easily seen using Fubini's theorem). Then, we obtain the two following equations

$$\partial_t (w_+ + w_-) + \partial_z (w_+ - w_-) = 0 \qquad \text{on} \quad [0, \infty) \times [a, b] \tag{30}$$

$$\partial_t (w_+ - w_-) + \partial_z (w_+ + w_-) = 0 \qquad \text{on} \quad [0, \infty) \times (a, b) \tag{31}$$

with appropriate initial conditions. And as before, in (30) we use test functions in $\mathcal{C}^\infty_o([0, \infty) \times [a, b])$ (in others words the tests functions do not necessarily vanish on $\{z = a\}$ and $\{z = b\}$ when a and b are finite).

Here, if $a' = -\infty$ or $b' = +\infty$ we need the assumption of non-integrability on V. If it is not verified, a (or b) will be finite, and we cannot use test functions which do not vanish on $\{z = a\}$ (or $\{z = b\}$) in (25). And we shall not have the uniqueness of solutions of the equivalent problem (as will become clearerr below).

Adding and substracting the two equations in $\mathcal{D}'([0, \infty) \times (a, b))$ yields

$$\partial_t w_+ + \partial_z w_+ = 0 \qquad \text{in} \quad \mathcal{D}'([0, \infty) \times (a, b))$$

$$\partial_t w_- - \partial_z w_- = 0 \qquad \text{in} \quad \mathcal{D}'([0, \infty) \times (a, b))$$

Hence, the solutions are of the form $w_+(t, z) = \Phi_+(z - t)$ and $w_-(t, z) = \Phi_-(z + t)$ with Φ_+ and Φ_- belonging to $\mathbb{L}^\infty(\mathbb{R})$. but we have not used yet the fact that (30) is true on [a,b]. This tells us formally that $w_+(t, a) = w_-(t, a)$ when $a \neq -\infty$ and $w_+(t, b) = w_-(t, b)$ when $b \neq +\infty$. This can be justified. Indeed, let us assume that $b \neq +\infty$ and let we choose some $\phi \in \mathcal{C}^\infty_o((0, \infty))$, an $\epsilon \in (0, b - a)$ and

$\chi_\epsilon \in \mathcal{C}^\infty(\mathbb{R})$ increasing such that $\chi_\epsilon(z) = 0$ for $z < b - \epsilon$ and some $\chi_\epsilon(z) = 1$ for $z > b$. We use $\phi\chi_\epsilon$ as a test function in (30). We then obtain

$$\int_{[0,\infty)\times(b-\epsilon,b)} (\Phi_+(z-t) + \Phi_-(z+t))\partial_t\phi(t)\chi_\epsilon(z)\,dtdz + \int_{[0,\infty)\times(b-\epsilon,b)} (\Phi_+(z-t) - \Phi_-(z+t))\phi(t)\partial_z\chi_\epsilon(z)\,dtdz = 0$$

When $\epsilon \to 0$, the first integral goes to 0. The second integral goes to $\int_{[0,\infty)}(\Phi_+(b-t) - \Phi_-(b+t))\phi(t)\,dt$. Since it holds for all $\phi \in \mathcal{C}_o^\infty((0,\infty))$, we obtain that $\Phi_+(b-t) = \Phi_-(b+t)$. We can prove similary that $\Phi_+(a-t) = \Phi_-(t+a)$ if $a \neq -\infty$.

Now, we shall assume that a and b are both finite (the other cases are similar and simpler) and we define $l = b - a$. Without using the boundary conditions, the initial conditions on w_+ and w_- impose the value of Φ_+ and Φ_- on the interval (a,b). Of course, it should be understood in sense of functions defined almost everywhere, but here it does not raise any difficulty and we will omit to specify it afterwards. Using the boundary condition $\Phi_+(b-t) = \Phi_-(b+t)$, we see that Φ_+ and Φ_- are determined in $(b, b+l)$. And the condition $\Phi_+(a-t) = \Phi_-(t+a)$ determines Φ_+ and Φ_- in $(a-l, a)$. If we continue to use this symmetry argument further, we see that Φ_+ and Φ_- are uniquely determined in $\mathbb{R}$, provided we know them in (a,b) (we remark here that it is not the case if one of the boundary counditions is missing, as it is the case when $a = -\infty$ or $b = +\infty$ and the assumption of non-integrability on V is not satisfied).Then, for every intial condition (on w_+ and w_-) in $\mathbb{L}^\infty$, there exists a unique solution to the system (30)-(31). And in view of the form of those solutions, we see that they are renormalized ones. This concludes the proof.

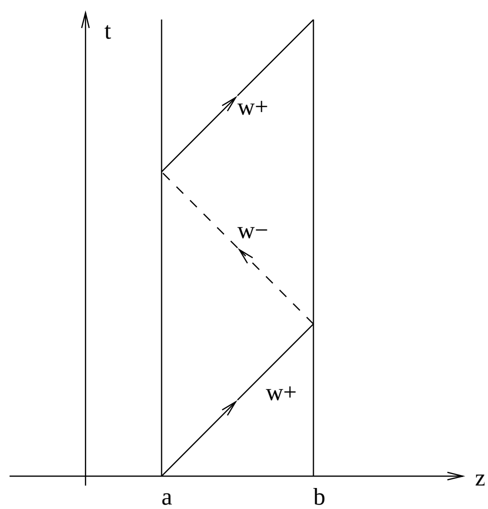

Figure 4: behaviour of w_+ and w_-

□

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