N-particles approximation of the Vlasov equations with singular potential

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Abstract. We prove the convergence in any time interval of a point-particle approximation of the Vlasov equation by particles initially equally separated for a force in $1/|x|^\alpha$, with $\alpha \leq 1$. We introduce discrete versions of the L^∞ norm and time averages of the force field. The core of the proof is to show hat these quantities are bounded and that consequently the minimal distance between particles in the phase space is bounded from below.
Key words. Derivation of kinetic equations. Particle methods. Vlasov equations.

1 Introduction

We are interested here by the validity of the modeling of a continuous media by a kinetic equation, with a density of presence in space and velocity. In other words, do the trajectories of many interacting particles follow the evolution given by the continuous media if their number is sufficiently large? This is a very general question and this paper claims to give a (partial) answer only for the mean field approach.

Let us be more precise. We study the evolution of N particles, centered at $(X_1, \ldots, X_n)$ in $\mathbb{R}^d$ with velocities $(V_1, \ldots, V_n)$ and interacting with a central force $F(x)$. The positions and velocities satisfy the following system of ODEs

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = E(X_i) = \displaystyle\sum_{j \neq i} \frac{\alpha_i\, \alpha_j}{m_i}\, F(X_i - X_j), \end{cases} \tag{1.1}$$

where the initial conditions $(X_1^0, V_1^0, \ldots, X_n^0, V_n^0)$ are given. The prime example for (1.1) consists in charged particles with charges α_i and masses m_i, in which case $F(x) = -x/|x|^3$ in dimension three.

To easily derive from (1.1) a kinetic equation (at least formally), it is very convenient to assume that the particles are identical which means $\alpha_i = \alpha_j$. Moreover we will rescale system (1.1) in time and space to work with quantities of order one, which means that we may assume that

$$\frac{\alpha_i\, \alpha_j}{m_i} = \frac{1}{N}, \quad \forall i, j. \tag{1.2}$$

We now write the Vlasov equation modelling the evolution of a density f of particles interacting with a radial force in $F(x)$. This is a kinetic equation in the sense that the density depends on the position and on the velocity (and of course on the time)

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + E(x) \cdot \nabla_v f = 0, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d, \\ &E(x) = \int_{\mathbb{R}^d} \rho(t, y)\, F(x - y)\, dy, \\ &\rho(t, x) = \int_v f(t, x, v)\, dv. \end{aligned} \tag{1.3}$$

Here ρ is the spatial density and the initial density f^0 is given.

When the number N of particles is large, it is obviously easier to study (or solve numerically) (1.3) than (1.1). Therefore it is a crucial point to determine whether (1.3) can be seen as a limit of (1.1).

Remark that if $(X_1, \ldots, X_N, V_1, \ldots, V_n)$ is a solution of (1.1), then the measure

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^{n} \delta(x - X_i(t)) \otimes \delta(v - V_i(t))$$

is a solution of the Vlasov equation in the sense of distributions. And the question is whether a weak limit f of μ_N solves (1.3) or not. If F is C^1 with compact support, then it is indeed the case (it is proved in the book by Spohn [23] for example). The purpose of this paper is to justify this limit if

$$|F(x)| \leq \frac{C}{|x|^{\alpha}}, \quad |\nabla F(x)| \leq \frac{C}{|x|^{1+\alpha}} \quad |\nabla^2 F(x)| \leq \frac{C}{|x|^{2+\alpha}}, \quad \forall x \neq 0, \quad (1.4)$$

for $\alpha < 1$, which is the first rigorous proof of the limit in a case where F is not necessarily bounded.

Before being more precise concerning our result, let us explain what is the meaning of (1.1) in view of the singularity in F. Here we assume either that we restrict ourselves to the initial configurations for which there are no collisions between particles over a time interval $[0,\ T]$ with a fixed T, independent of N. Or we assume that F is regular or regularized but that the norm $\|F\|_{W^{1,\infty}}$ may depend on N; This procedure is well presented in [1] and it is the usual one in numerical simulations (see [24] and [25]). In both cases, we have classical solutions to (1.1) but the only bound we may use is (1.4).

Other possible approaches would consist in justifying that the set of initial configurations $X_1(0), \ldots, X_N(0), V_1(0), \ldots, V_N(0)$ for which there is at least one collision, is negligible or that it is possible to define a solution (unique or not) to the dynamics even with collisions.

Finally notice that the condition $\alpha < 1$ is not unphysical. Indeed if F derives from a potential, $\alpha = 1$ is the critical exponent for which repulsive and attractive forces seem very different. In other words, this is the point where the behavior of the force when two particles are very close takes all its importance.

1.1 Important quantities

The derivation of the limit requires a control on many quantities. Although some of them are important only at the discrete level, many were already used to get the existence of strong solutions to the Vlasov-Poisson equation (we refer to [10], [11] and [18], [20] as being the closest from our method).

The first two are quite natural and are bounds on the size of the support of the initial data in space and velocity,

$$R(T) = \sup_{t\in[0,T],\, i=1,\ldots N} |X_i(t)|, \quad K(T) = \sup_{t\in[0,T],\, i=1,\ldots N} |V_i(t)|. \quad (1.5)$$

Of course R is trivially controlled by K since

$$R(T) \leq R(0) + T\,K(T). \tag{1.6}$$

Now a very important and new parameter is the discrete scale of the problem denoted ε. This quantity represents roughly the minimal distance between two particles or the minimal time interval which the discrete dynamics can see. We fix this parameter from the beginning and somehow the main part of our work is to show that it is indeed correct, so take

$$\varepsilon = \frac{R(0)}{N^{1/2d}}. \tag{1.7}$$

At the initial time, we will choose our approximation so that the minimal distance between two particles will be of order ε.
The force term cannot be bounded at every time for the discrete dynamics (a quantity like $F \star \rho_N$ is not bounded even in the case of free transport), but we can expect that its average on a short interval of time will be bounded. So we denote

$$\overline{E}(T) = \sup_{t\in[0,\,T-\varepsilon],i=1,\ldots,N} \left\{ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |E(X_i(s))|\,ds \right\}, \tag{1.8}$$

with for $T < \varepsilon$

$$\overline{E}(T) = \sup_{i=1,\ldots,N} \left\{ \frac{1}{\varepsilon} \int_0^{T} |E(X_i(s))|\,ds \right\}, \tag{1.9}$$

thus obtaining a unique and consistant definition for all $T > 0$. Moreover we denote by E^0 the supremum over all i of $|E(X_i(0))|$.
This definition comes from the following intuition. The force is big when two particles are close together. But if their speeds are different, they will not stay close for a long time. So we can expect the interaction force between these two particles to be integrable in time even if they "collide". There just remains the case of two close particles with almost the same speed. To estimate the force created by them, we need an estimate on their number. One way of obtaining it is to have a bound on

$$m(T) = \sup_{t\in[0,T],i\neq j} \frac{\varepsilon}{|X_i(t) - X_j(t)| + |V_i(t) - V_j(t)|}. \tag{1.10}$$

The control on m requires the use of a discretized derivative of E, more precisely, we define for any exponent $\beta \in \,]1,\ d-\alpha[$, which also satisfies $\beta < 2d - 3\alpha$ ($\beta = 1$ would be enough for short time estimates)

$$\Delta\overline{E}(T) = \sup_{t\in[0,\,T-\varepsilon]}\ \sup_{i,j=1,\dots,N,} \left\{ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \frac{|E(X_i(s)) - E(X_j(s))|}{\varepsilon^\beta + |X_i(s) - X_j(s)|}\, ds \right\}, \qquad (1.11)$$

with as for $\overline{E}$, when $T < \varepsilon$

$$\Delta\overline{E}(T) = \sup_{i,j=1,\dots,N} \left\{ \frac{1}{\varepsilon} \int_0^{T} \frac{|E(X_i(s)) - E(X_j(s))|}{\varepsilon^\beta + |X_i(s) - X_j(s)|}\, ds \right\}. \qquad (1.12)$$

Now, we introduce what we called the discrete infinite norm of the distribution of the particle μ_N. This quantity is the supremum over all the boxes of size ε of the total mass they contain divided by the size of the box. That is, for a measure μ we denote

$$\|\mu\|_{\infty,\varepsilon} = \frac{1}{(2\varepsilon)^{2d}} \sup_{(x,v)\in\mathbb{R}^{2d}} \left\{ \mu(B_\infty((x,v),\varepsilon)) \right\}. \qquad (1.13)$$

where $B_\infty((x,v),\varepsilon)$ is the ball of radius ε centered at (x,v) for the infinite norm. Note that we may bound $\|\mu_N(T,\cdot)\|_{\infty,\varepsilon}$ by

$$\|\mu_N(T,\cdot)\|_{\infty,\varepsilon} \le (4\, m(T))^{2d}. \qquad (1.14)$$

We may also introduce discrete L^∞ norm at other scales by defining in general

$$\|\mu\|_{\infty,\eta} = \frac{1}{(2\eta)^{2d}} \sup_{(x,v)\in\mathbb{R}^{2d}} \left\{ \mu(B_\infty((x,v),\eta)) \right\}. \qquad (1.15)$$

The quantities R, K, m will always be assumed to be bounded at the initial time $T = 0$ uniformly in N.

1.2 Main results

The main point in the derivation of the Vlasov equation is to obtain a control on the previous quantities. We first do it for a short time as given by

Theorem 1.1. *If $\alpha < 1$, there exists a time T and a constant c depending only on $R(0)$, $K(0)$, $m(0)$ but not on N such that for some $\alpha < \alpha' < 3$*

$$R(T) \le 2\,(1 + R(0)), \quad K(T) \le 2\,(1 + K(0)), \quad m(T) \le 2\, m(0),$$
$$\overline{E}(T) \le c\,(m(0))^{2\alpha'}\,(K(0))^{\alpha'}\,(R(0))^{\alpha'-\alpha}, \quad \sup_{t\le T} \|\mu_N(t,\cdot)\|_{\infty,\varepsilon} \le (8\, m(0))^{2d}.$$

Remark
The constant 2, which appears in the bounds, is of course only a matter of convenience. This means that another theorem could be written with 3 instead of 2 for instance; The time T would then be larger. However increasing this value is not really helpful because the kind of estimates which we use for this theorem blow up in finite time, no matter how large the constant in the bounds is.

This theorem can, in fact, be extended on any time interval

Theorem 1.2. *For any time $T > 0$, there exists a function $\tilde{N}$ of $R(0)$, $K(0)$, $m(0)$ and T and a constant $C(R(0), K(0), m(0), T)$ such that if $N \geq \tilde{N}$ then*

$$R(T),\ K(T),\ m(T),\ \overline{E}(T) \leq C(R(0), K(0), m(0), T).$$

From this last theorem, it is easy to deduce the main result of this paper, which reads

Theorem 1.3. *Consider a time T and sequence $\mu_N(t)$ corresponding to solutions to* (1.1) *such that $R(0)$, $K(0)$ and $m(0)$ are bounded uniformly in N. Then any weak limit f of $\mu_N(t)$ in $L^\infty([0,\ T],\ M^1(\mathbb{R}^{2d}))$ belongs to $L^\infty([0,\ T],\ L^1 \cap L^\infty(\mathbb{R}^{2d}))$, has compact support and is a solution to* (1.3).

Of course the main limitation of our results is the condition $\alpha < 1$ and the main open question is to know what happens when $\alpha \geq 1$. However this condition is not only technical and new ideas will be needed to prove something for $\alpha \geq 1$. It would also be interesting to extend our result to more complicated forces like the ones found in the formal derivation of [14]. The second important limitation is that $m(0)$ be uniformly bounded. The two applications of Theorem 1.3 concern the numerical simulation of kinetic equations and a justification of the model through the derivation of the equation in statistical mechanics. Concerning numerical simulation, the approximations of the initial data which are usually chosen imply a bound on $m(0)$. For statistical mechanics, determining the initial data is more of a problem. A natural way would be to take identically distributed particles; In that case, the average distance in the phase space between one particle and the closest one, is of the order of $\varepsilon \sim N^{-1/2d}$. However the probability that the minimal distance between any particles be always at least ε decreases exponentially fast with N, making the assumption on $m(0)$ much more restrictive.

Finally the two conditions of compact support in space and velocity are very usual, for instance to prove the existence of "strong solutions" to Vlasov equations. In the case $\alpha < 1$ which we consider here, getting strong solutions is rather easy which explains why passing from Theorem 1.1 to Theorem 1.2 "only" requires the proof of the almost preservation of discrete L^∞ bounds. For the sake of completeness, we recall the proof of existence of strong solutions in an appendix at the end of the paper.
The derivation of kinetic equations is an important question both for numerical and theoretical aspects. The first results for Vlasov equations are due to Neunzert and Wick [16], Dobrushin [6] and Braun and Hepp [4]. We also refer to works of Batt [1], Spohn [23], Victory and Allen [24] and Wollmann [25]. Another interesting case concerns Boltzmann equation, for which we refer to the book by Cercignani, Illner and Pulvirenti [5] and the paper by Illner and Pulvirenti [12].
On the other hand, the derivation of hydrodynamic equations is somewhat different and some results are already known (although not since a very long time) even in cases with singularity. In particular and that is more or less the hydrodynamic equivalent of our result, the convergence of the point vortex method for $2-D$ Euler equations was obtained by Goodman, Hou and Lowengrub [9] (see also the works by Schochet [21] and [22]). The main part of the proof for hydrodynamic systems consists in controlling the minimal distance between two particles in the physical space (as it is also clear in [13]). The situation for kinetic equations is different: First of all, such a control is impossible to obtain. And then, having it is not necessary as the two particles could still be far away in the phase space. On the other hand, for a hydrodynamic system, the velocity of a particle only depends on its position in the physical space and therefore two particles with the same position, at a given time, still have the same position at any latter time. As a consequence preventing collisions is really a necessity for a hydrodynamic system; This more or less implies that the proofs are simpler but more demanding for hydrodynamic systems and that a more complex approach is required for kinetic equations.
Our method of proof makes full use of the method of characteristics developed for the Vlasov-Poisson equation in dimension two and three. This method was introduced by Horst in [10] and [11] with the aim of obtaining strong solutions in large time and was, eventually and successfully, used to do that in [18] and [20]. These results were extended to the periodic case by Batt and Rein in [3]. At about the same time strong solutions were obtained by

Lions and Perthame in [15] with a different method (see also [7] for a slightly simpler proof and [17] for an application to the asymptotic behavior of the equation). Their method controls the moments, *i.e.* quantities of the kind $\int |v|^k f\,dv$ with f the solution, and is therefore closer to the notion of weak solutions. It was then applied to the Vlasov-Poisson-Fokker-Planck equation by Bouchut in [2]. Still for the Vlasov-Poisson-Fokker-Planck equation, L^∞ bounds were obtained by Pulvirenti and Simeoni in [19], this time with the method of characteristics. The proof is interesting because it also shows the need to integrate in time to control the oscillations of the force. For a given problem, choosing between the method of characteristics and the control of the moments is obviously not easy and could simply be a matter of "taste". The reason why we opted for the characteristics is that it seems more appropriate for a discrete setting. Finally we refer to the book by Glassey [8] for a general discussion of the existence theory for kinetic equations.
In the rest of the paper, C will denote a generic constant, depending maybe on $R(0)$, $K(0)$, or $m(0)$ but not on N or any other quantity. We first prove Theorem 1.1, then we show a preservation of discrete L^∞ norms which proves Theorem 1.2. In the last section we explain how to deduce Theorem 1.3, the appendix being devoted to the proof of existence of strong solutions to (1.3).

2 Proof of Theorem 1.1

The first steps are to estimate all quantities in terms of themselves. Then if this is done correctly it is possible to deduce bounds for them on a short interval of time.

2.1 Estimate on $\overline{E}$

In this section we will prove a usefull estimate on $\overline{E}$. As explain above, we will decomposate the force that a particle see in the force created by the distants particles, at an order larger than ε, the close particles but with a different speed, again at order ε, and the particles with almost the same position and speed at order ε. So we have three terms to estimate. As we will often have to estimate terms of the same type in the rest of the article, we will in a first lemma prove estimate for all this terms, and unite it in the second lemma.

Lemma 2.1. *We choose an δ in $(0,d)$ and a particles i and assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\,K(t_0)\,\Delta\overline{E}(t_0)}.$$

We defined three subsets of $\{1,\dots,N\}\backslash\{i\}$, G_i, B_i and U_i by

$$G_i = \{j||X_i(t)-X_j(t)| \geq 2K(t)\varepsilon\}$$

$$B_i = \{j||X_i(t)-X_j(t)| \leq 2K(t)\varepsilon \quad \textit{and} \quad |V_i(t)-V_j(t)| \geq 2\overline{E}(t)\varepsilon\}$$
$$U_i = \{j||X_i(t)-X_j(t)| \leq 2K(t)\varepsilon \quad \textit{and} \quad |V_i(t)-V_j(t)| \leq 2\overline{E}(t)\varepsilon\}$$

Then, for any δ' satisfying $\delta \leq \delta' \leq d$, we have the following estimates

i. $$\frac{1}{N}\sum_{j\in G_i}\int_{t-\varepsilon}^{t}\frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds \leq \|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\,K^{\delta'}\,R^{\delta'-\delta}$$

If we assume moreover that δ and δ' satisfy $\delta < \delta' < 1$, we have the following estimates

ii. $$\frac{1}{N}\sum_{j\in B_i}\int_{t-\varepsilon}^{t}\frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds \leq \varepsilon^{d-\delta}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\delta}$$

iii. $$\frac{1}{N}\sum_{j\in U_i}\int_{t-\varepsilon}^{t}\frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds \leq \varepsilon^{2d-3\delta}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\delta}\overline{E}^d$$

Proof. The first estimate. For the first point, we denote

$$I_1 = \frac{1}{N}\sum_{j\in G_i}\int_{t-\varepsilon}^{t}\frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds$$

and divided again G_i in

$$G_{i,k} = \left\{ i \,\middle|\, 3\,\varepsilon\,K(t_0)\,2^{k-1} < |X_i(t_1)-X_1(t_1)| \leq 3\,\varepsilon\,K(t_0)\,2^k \right\}. \tag{2.1}$$

Remark that after $k_0 = (\ln(R(t)/4\,\varepsilon\,K(t_0)))/\ln 2$, the set $G_{i,k}$ is empty.

Approximate stability of the $G_{i,k}$. Given their definition, the $G_{i,k}$ enjoy the following property: For any $i \in G_{i,k}$ with $k > 1$, we have for any $t \in [t_1,\ t_0]$

$$|X_1(t) - X_i(t)| \geq \varepsilon\,K(t_0)\,2^{k-1}.$$

Indeed, we of course know that

$$\left|\frac{d}{dt}(X_i(t)-X_1(t))\right| = |V_i(t)-V_1(t)| \leq 2\,K(t_0),$$

and then

$$\begin{aligned}|X_j(t)-X_i(t)| &\geq |X_j(t_1)-X_i(t_1)| - 2\,(t_0-t_1)\,K(t_0)\\ &\geq 3\,\varepsilon\,K(t_0)\,2^{k-1} - 2\,\varepsilon\,K(t_0),\end{aligned}$$

with the corresponding result since $k \geq 1$. Of course the same argument also shows that if $i \in B_i$ then for any $t \in [t_1,\ t_0]$,

$$|X_j(t)-X_i(t)| \leq 5\,\varepsilon\,K(t_0).$$

This prove also show that B_i is approximately stable.

Sommation over the $G_{i,k}$ Using the result from the previous step, we deduce that for any $j \in G_{i,k}$ with $k \geq 1$,

$$\frac{1}{|X_i(t)-X_1(t)|^\delta} \leq \frac{C\,2^{-\delta k}}{\varepsilon^\delta\,(K(t_0))^\delta}.$$

On the other hand, we have of course $|G_{i,_k}| \leq N$. Moreover the set of points (x,v) in the phase space with $3\varepsilon K(t_0)\,2^{k-1} < |x - X(t_1)| < 3\varepsilon K(t_0)\,2^k$, can be covered by $K^d \times \varepsilon^{-2d} \times (3K(t_0)\,2^k)^d$ balls of radius ε in the phase space. According to the definition of the discrete L^∞ norm (1.13), this implies that $|G_{i,k}| \leq C\,\varepsilon^{-d}\,K^{2d}\,2^{d\,k} \times \|\mu_N\|_{\infty,\varepsilon}$.
Consequently for any $\delta' < d$, since $\varepsilon^{2d} = C/N$, interpolating between these two values, we get

$$|G_{i,k}| \leq C\,N\,(K(t_0))^{2\delta'}\,\varepsilon^{\delta'}\,2^{\delta' k} \times \|\mu_N(t_0,.)\|_{\infty,\varepsilon}^{\delta'/d}.$$

Now we can use this two bounds to compute I_1.

$$\begin{aligned}I_1 &\leq \sum_{k=1}^{k_0}\sum_{j\in G_{i,k}}\int_{t-\varepsilon}^{t}\frac{1}{N|X_j(s)-X_i(s)|^\alpha}\,ds\\ &\leq \sum_{k=1}^{k_0}|G_{i,k}| \times N^{-1}\,(K(t_0))^{-\delta}\,\varepsilon^{-\delta}\,2^{-\delta\,k}\\ &\leq C\,\|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\,K^{2\delta'-\delta}\,\varepsilon^{\delta'-\delta}\sum_{k=1}^{k_0}2^{(\delta'-\delta)k}.\end{aligned}$$

Eventually for any $\delta < \delta' < 1$, we deduce that

$$I_1 \le C \,\|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\, K^{2\delta'-\delta}\, \varepsilon^{\delta'-\delta}\, 2^{(\delta'-\delta)k_0} \le C\,\|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\, r^{\delta'-\delta}\, K^{\delta'}, \tag{2.2}$$

all the values being taken at t. This gives the point $i.$ in Lemma 2.1.

The second estimate. We denote

$$I_2 = \frac{1}{N}\sum_{j\in B_i}\int_{t-\varepsilon}^{t} \frac{1}{|X_i(s)-X_j(s)|^{\delta}}\, ds$$

and decompose again the set B_i in

$$B_{i,l} = \left\{ j \in B_i \mid 3\,\varepsilon\,\overline{E}(t_0)\,2^{l-1} < |V_1(t_1) - V_j(t_1)| \le 3\,\varepsilon\,\overline{E}(t_0)\,2^{l} \right\}, \tag{2.3}$$

for $l \ge 1$. Remark that the set $B_{i,l}$ is empty if $l > l_0 = \ln(K(t_0)/(\varepsilon\,\overline{E}(t_0)))/\ln 2$. As before we decompose I_2 in

$$I_2 = \sum_{l=1}^{l_0}\sum_{j\in Q_l}\frac{1}{\varepsilon}\int_{t_1}^{t_0}\frac{dt}{N\,|X_j(t)-X_1(t)|^{\delta}}, \tag{2.4}$$

The idea behind this new decomposition is that although the particles in $B_{i,l}$ with $l \ge 1$ are close to X_i, their speed is different from V_i. So even if they come very close to X_i they will stay close only for a very short time. Since the singularity of the potential is not too high, we will be able to bound the force.

Approximate stability of the $B_{i,l}$. Just as for the $G_{i,k}$, we may prove that for any time t in $[t_1,\ t_0]$ and any $j \in B_{i,l}$ with $l \ge 1$

$$|V_j(t) - V_i(t)| > \varepsilon\,\overline{E}(t_0)\,2^{l-1}.$$

This is again due to the fact that

$$|V_j(t) - V_j(t_1)| \le \int_{t_1}^{t_0} |E(X_j(s))|\, ds \le \varepsilon\,\overline{E}(t_0),$$

so that in fact the result is even more precise in the sense that the relative velocity $V_j(t) - V_i(t)$ remains close to $V_j(t_1) - V_i(t_1)$ up to exactly $\varepsilon\,\overline{E}(t_0)$. We also remind that B_i was approximately stable and so that $\forall l,\ \forall j \in B_{i,l}$ and $\forall t \in [t_1,\ t_0]$

$$|X_j(t) - X_i(t)| \le 5\,\varepsilon\,K(t_0).$$

Therefore all the particles which now concern us are in a spatial box of size $C\,\varepsilon\,K(t_0)$.

Control of I_2. Together with the next one, this is the only step which uses the condition $\delta < 1$. Given this previous point, for any $j \in Q_l$ with $l > 0$ and any $t \in [t_1,\ t_2]$, we have, denoting by t_m the time in the interval $[t_1,\ t_0]$ where $|X_j(t) - X_i(t)|$ is minimal

$$|X_i(t) - X_j(t)| \geq \left| |X_i(t_m) - X_j(t_m)| - \frac{1}{2}(t - t_m)|V_1(t_m) - V_j(t_m)| \right|.$$

Then,

$$\begin{aligned} \frac{1}{\varepsilon}\int_{t_1}^{t_0} \frac{1}{|X_1(t) - X_j(t)|^\delta}\,dt &\leq \frac{C}{\varepsilon}\,|V_i(t_m) - V_j(t_m)|^{-\delta}\varepsilon^{1-\delta} \\ &\leq C\,\varepsilon^{-2\delta}\,(\overline{E}(t_0))^{-\delta}\,2^{-\delta l}. \end{aligned}$$

Summing up on l, we obtain

$$|I_2| \leq C\sum_{l=1}^{l_0} |B_{i,l}|\frac{1}{N}\,\varepsilon^{-2\delta}\,(\overline{E}(t_0))^{-\delta}\,2^{-\delta l}.$$

We bound $|B_{i,l}|$ by $|B_{i,l}| \leq C\,\|\mu\|_{\infty,\varepsilon}\,(K(t_0)\,\varepsilon)^d(2^l\,\overline{E}(t_0)\,\varepsilon)^d$ using again the definition of the discrete L^∞ norm and recalling that $Q_l \subset C_0$. It gives us the inequality

$$\begin{aligned} I_2 &\leq C\,(K(t_0))^d\,(\overline{E}(t_0))^{d-\delta}\,\varepsilon^{2d-2\delta}\,\|\mu_N\|_{\infty,\varepsilon} \times \sum_{l=2}^{l_0} 2^{(d-\delta)l} \\ &\leq C\,(K(t_0))^d\,(\overline{E}(t_0))^{d-\delta}\,\|\mu_N\|_{\infty,\varepsilon}\,\varepsilon^{2d-2\delta}\,\left(\frac{K(t_0)}{\overline{E}(t_0)\,\varepsilon}\right)^{d-\delta} \\ &\leq C\,(K(t_0))^{2d-\delta}\,\|\mu_N\|_{\infty,\varepsilon}\,\varepsilon^{d-\delta}, \end{aligned}$$

which is the point ii. of the Lemma 2.1.

The point iii. We denote

$$I_3 = \frac{1}{N}\sum_{j\in U_i}\int_{t-\varepsilon}^{t} \frac{1}{|X_i(s) - X_j(s)|^\delta}\,ds$$

We will also uses the condition $\delta < 1$ for this step and it is the only one where m is needed. The first point to note is that for any $j \in U_i$ and any $t \in [t_1,\ t_0]$, as $U_i \subset B_i$ we have that

$$|X_j(t) - X_i(t)| \leq 5\,\varepsilon\,K(t_0).$$

Consequently, by the definition (1.11) of $\Delta\overline{E}$

$$|V_j(t) - V_i(t) - V_j(t_1) - V_i(t_1)| \leq 5\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0).$$

It is thus logical to decompose (again) U_i in $U_i' \cup U_i''$ and I_3 in the corresponding $I_3' + I_3''$ with

$$U_i' = \left\{ j \in Q_0 \,\middle|\, |V_j(t_1) - V_i(t_1)| \geq 6\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0) \right\},$$

U_i'' the remaining part of U_i and I_3', I_3'' the sums on the corresponding indices. Then for any $j \in U_i'$, the same computation as in the fifth step, shows that

$$\frac{1}{\varepsilon}\int_{t_1}^{t_0} \frac{dt}{N\,|X_j(t) - X_1(t)|^\delta} \leq C\,\varepsilon^{2d-3\delta}\,(K(t_0))^{-\delta}\,(\Delta\overline{E}(t_0))^{-\delta}.$$

The cardinal of U_i' is bounded by the one of U_i and using as always the discrete L^∞ bound

$$|U_i'| \leq C\,(K(t_0))^d\,(\overline{E}(t_0))^d\,\|\mu_N\|_{\infty,\varepsilon}.$$

Eventually that gives

$$\begin{aligned}
I_3' &\leq |U_i'| \times \sup_{j\in U_i'} \frac{1}{\varepsilon}\int_{t_1}^{t_0} \frac{dt}{N\,|X_j(t) - X_1(t)|^\delta} \\
&\leq C\,\varepsilon^{2d-3\delta}\,(K(t_0))^{d-\delta}\,(\overline{E}(t_0))^d\,\|\mu_N(t_0,.)\|_{\infty,\varepsilon} \times (\Delta\overline{E}(t_0))^{-\delta} \\
&\leq C\,\varepsilon^{2d-3\delta}\,(K(t_0))^{d-\delta}\,(\overline{E}(t_0))^d\,\|\mu_N(t_0,.)\|_{\infty,\varepsilon},
\end{aligned}$$

as $\Delta\overline{E}(t_0) \geq \Delta\overline{E}(0)$ and this last quantity is bounded easily in terms of $m(0)$, $K(0)$ and $R(0)$.
Let us conclude the proof with the bound on I_3''. Of course if $j \in U_i''$ then for any $t \in [t_1,\ t_0]$,

$$\begin{aligned}
|V_j(t) - V_i(t)| &\leq |V_j(t_1) - V_i(t_1)| + |V_j(t) - V_i(t) - V_j(t_1) + V_i(t_1)| \\
&\leq (6+5)\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0).
\end{aligned}$$

Now we use the definition (1.10) of m and the assumption in the lemma to deduce that

$$|X_j(t) - X_i(t)| \geq \frac{\varepsilon}{m(t_0)} - |V_j(t) - V_i(t)| \geq \varepsilon^2 \, K(t_0) \, \Delta\overline{E}(t_0).$$

We bound $|U_i''|$ by $|U_i|$ which is the best we can do since the discrete L^∞ norm cannot see the scales smaller than ε and we obtain

$$I_3'' \leq C \, \varepsilon^{2d-2\delta} \, (K(t_0))^{d-\delta} \, (\overline{E}(t_0))^d \, \|\mu_N(t_0,.)\|_{\infty,\varepsilon},$$

which is dominated by the bound which we have just obtained on I_3'. This give the point iii. □

We will now just state a corrolary that will be usefull in the last section.

Corollary 2.2. *We choose an δ in $(0,d)$ and a particle i and a real $r > 0$ and assume that*

$$m(t_0) \leq \frac{1}{12 \, \varepsilon \, K(t_0) \, \Delta\overline{E}(t_0)},$$

We defined the subset G_i^r of $\{1,\dots,N\}\backslash\{i\}$,

$$G_i^r = \{j | 2K(t)\varepsilon \leq |X_i(t) - X_j(t)| \leq r\}$$

Then, for any δ' satisfying $\delta \leq \delta' \leq d$, we have the following estimate $\displaystyle\frac{1}{N} \sum_{j \in G_i^r} \int_{t-\varepsilon}^{t} \frac{1}{|X_i(s) - X_j(s)|^\delta} \, ds \leq \|\mu_N\|_{\infty,\varepsilon}^{\delta'/d} \, K^{\delta'} \, r^{\delta'-\delta}$

Proof. We only have to replace $R(t)$ by r in the proof of the point i. of the preceding lemma 2.1. □

Now we can use this lemma to get an estimate on $\overline{E}$.

Lemma 2.3. *For any α' with $\alpha < \alpha' < 1$, and any $t_0 > 0$, if*

$$m(t_0) \leq \frac{1}{12 \, \varepsilon \, K(t_0) \, \Delta\overline{E}(t_0)},$$

then there exist a constant $C(\alpha')$ so that

$$\begin{aligned} \overline{E}(t_0) \leq C \, (&\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d} \, K^{\alpha'} \, R^{\alpha'-\alpha} + \varepsilon^{d-\alpha} \, \|\mu_N\|_{\infty,\varepsilon} \, K^{2d-\alpha} \\ &+ \varepsilon^{2d-3\alpha} \, \|\mu_N\|_{\infty,\varepsilon} \, K^{d-\alpha} \overline{E}^d), \end{aligned}$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

Of course if any of the above quantity is infinite then the result is obvious. This lemma could appear stupid since we control $\overline{E}(t_0)$ by itself (and with a power larger than 1 in addition). But the point is that except for the first term, the other two are very small because of the ε in front of them so that they almost do not count.

Proof. We choose a particles i and apply the preceding lemma 2.1. We separate the remaining particles in the three set G_i,B_i, and U_i. Combining the three estimates in which we use $\delta = \alpha$ and $\delta' = \alpha'$, we obtain

$$\int_{t_0-\varepsilon}^{t^0} |E(X_i(s))|\, ds \leq C\,(\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\, K^{\alpha'}\, R^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} \\ + \varepsilon^{2d-3\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha}\overline{E}^d).$$

since this is independant of the particle we choose, we get the estimate on $\overline{E}(t_0)$. □

2.2 Estimate on $\Delta\overline{E}$

We may show the following with the same remarks as for Lemma 2.3,

Lemma 2.4. *For any α' with $\alpha < \alpha' < 1$, and for any t_0, if*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\, K(t_0)\, \Delta\overline{E}(t_0)},$$

then there exist a constant $C(\alpha')$

$$\Delta\overline{E}(t_0) \leq C\,(\|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d}\, K^{1+\alpha'}\, R^{\alpha'-\alpha} + \varepsilon^{d-\alpha-\beta}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} \\ + \varepsilon^{2d-3\alpha-\beta}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha}\overline{E}^d),$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

Proof. We choose a time t, two particles i and j and introduce the sets G_i, G_j, B_i, B_j, U_i and U_j. We decomposed the term in sums on these sets:

$$\Delta I = \frac{1}{\varepsilon} \int_{t_1}^{t_0} \frac{|E(X_i(t)) - E(X_j(t))|}{\varepsilon^\beta + |X_i(t) - X_j(t)|}\, dt.$$

$$\Delta I \leq \frac{1}{N} \sum_{k \in G_i \cap G_j} \int_{t_1}^{t_0} \frac{|F(X_i(t) - X_k(t)) - F(X_j(t) - F(X_k(t)|}{\varepsilon^\beta + |X_i(t) - X_j(t)|}\, dt$$
$$+ \frac{1}{\varepsilon^\beta} \sum_{k \in B_i \cup U_i} |F(X_i(t) - X_k(t))|$$
$$+ \frac{1}{\varepsilon^\beta} \sum_{k \in B_j \cup U_j} |F(X_j(t) - X_k(t))|$$
$$+ \frac{1}{\varepsilon^\beta} \sum_{k \in B_i \cup U_i} |F(X_j(t) - X_k(t))|$$
$$+ \frac{1}{\varepsilon^\beta} \sum_{k \in B_j \cup U_j} |F(X_i(t) - X_k(t))|. \quad (2.5)$$

We denote the term of the right hand side, keeping the order

$$\Delta I \leq \Delta I_1 + \Delta I_2 + \Delta I_3 + \Delta I_4 + \Delta I_5$$

Both the term ΔI_2 and ΔI_3 can be bounded by $C\varepsilon^{d-\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{2d-\alpha} + C\varepsilon^{2d-3\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{d-\alpha}\overline{E}^d$ using point ii. and iii. of the lemma (2.1).
The term ΔI_4 and ΔI_5 are of the same form (just exchange the indices i and j). So we will give a bound for ΔI_4 which will be valid for ΔI_5. For this, we decompose again ΔI_4 in the sum on the index in $C' = (B_i \cup U_i) \cap (B_j \cup U_j)$ denoted $\Delta I_4'$ and the sum on the rest $C'' = (B_i \cup U_i) \backslash (B_j \cup U_j)$ denoted $\Delta I_4''$. The first one is bounded by the sum on $B_j \cup U_j$) which is bounded by $C\varepsilon^{d-\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{2d-\alpha} + C\varepsilon^{2d-3\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{d-\alpha}\overline{E}^d$ according to points ii. and iii. of the lemma (2.1). For the second term $\Delta I_4'$, if $k \in C''$, then $|X_k(t) - X_j(t)| \geq 2K(t)\varepsilon$. Since, $B_i \cup U_i$ and C'' can be cover by ε-balls of total volume $4(K(t)\varepsilon)^d$, we can bound $\Delta I_4''$ by $CK(t)^{d-\alpha}\varepsilon^{d-\alpha-\beta}$ a term that will be bounded by the one bounding ΔI_3 if K is greater than one.
Now for ΔI_1, we observe that for $i \notin B$ and for any t

$$|F(X_1(t) - X_i(t)) - F(X_2(t) - X_i(t))| \leq C|X_1(t) - X_2(t)| \times \left(\frac{1}{N\,|X_1(t) - X_i(t)|^{\alpha+1}} + \frac{1}{N\,|X_1(t) - X_i(t)|^{\alpha+1}} \right),$$

since it is always possible to find a regular path $x_t(s)$ of length less than $2\,|X_1(t) - X_2(t)|$ such that $x_t(0) = X_1(t)$, $x_t(1) = X_2(t)$ and $|x_t(s) - X_i(t)|$ is always larger than the minimum between $|X_1(t) - X_i(t)|$ and $|X_2(t) - X_i(t)|$.

The only problem if we always choose the direct line between X_1 and X_2 arises when X_i is almost on this line, because $F(x-X_i)$ has a singularity at X_i. So,

$$\Delta I_1 \leq C \sum k \in B_i \cap B_j \left(\frac{1}{N\,|X_k(t)-X_i(t)|^{\alpha+1}} + \frac{1}{N\,|X_k(t)-X_i(t)|^{\alpha+1}} \right)$$

This two sums can be bounded thanks to the point i. of the lemma (2.1) with $\delta = \alpha + 1$ by

$$\Delta I_1 \leq C \|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d}\, K^{1+\alpha'}\, R^{\alpha'-\alpha}$$

putting all the bound together we get the result of the lemma. $\square$

2.3 Control on m and K

We prove the

Lemma 2.5. *Assume that for a given $t > 0$*

$$m(t) \leq \frac{1}{\varepsilon^{\beta-1}},$$

then we also have that

$$m(t) \leq m(0) \times e^{Ct + C\varepsilon\,\Delta\overline{E}(t) + C\int_0^t \Delta\overline{E}(s)\,ds},$$

and we may eliminate the $\varepsilon\,\Delta\overline{E}(t)$ term if $t > \varepsilon$.

Note that we still need an assumption on m but it is a bit different (and somewhat "harder" to satisfy) than the corresponding one for Lemmas 2.3 and 2.4. And note also that by definition $m(t)$ is a non decreasing quantity therefore if $m(t) \geq \varepsilon^{1-\beta}$ then it is true for all $s < t$.

Proof. We consider any two indices $i \neq j$. Then we write

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}\right) &= \frac{\varepsilon}{(|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|)^2} \\
&\times \left(\frac{X_i-X_j}{|X_i-X_j|}\cdot(V_i-V_j) + \frac{V_i-V_j}{|V_i-V_j|}\cdot(E(X_i)-E(X_j))\right) \\
&\leq \frac{\varepsilon\,(|V_i(s)-V_j(s)|+|E(X_i(s))-E(X_j(s))|)}{(|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|)^2}.
\end{aligned}$$

Since $m(t) \le \varepsilon^{1-\beta}$, the same is true of $m(s)$ and at least one of the quantities $|X_i(s) - X_j(s)|$ and $|V_i(s) - V_j(s)|$ is larger than $\varepsilon^\beta/2$, therefore

$$\frac{d}{ds}\left(\frac{\varepsilon}{|X_i(s)-X_j(s)| + |V_i(s)-V_j(s)|}\right) \le \frac{C\,\varepsilon}{|X_i(s)-X_j(s)| + |V_i(s)-V_j(s)|} \times \left(1 + \frac{|E(X_i(s)) - E_(X_j(s))|}{\varepsilon^\beta + |X_i(s) - X_j(s)|}\right).$$

But by the definition of $\Delta\overline{E}$, see (1.11), we know that for $t > \varepsilon$

$$\int_\varepsilon^t \frac{|E(X_i(s)) - E_(X_j(s))|}{\varepsilon^\beta + |X_i(s) - X_j(s)|}\,ds \le \int_0^t \Delta\overline{E}(s)\,ds,$$

and of course for $t < \varepsilon$

$$\int_0^t \frac{|E(X_i(s)) - E_(X_j(s))|}{\varepsilon^\beta + |X_i(s) - X_j(s)|}\,ds \le \varepsilon\Delta\overline{E}(t).$$

Hence, integrating in time, we find

$$\frac{\varepsilon}{|X_i(s)-X_j(s)| + |V_i(s)-V_j(s)|} \le \frac{\varepsilon}{|X_i(0)-X_j(0)| + |V_i(0)-V_j(0)|} \times e^{Ct + C\varepsilon\,\Delta\overline{E}(t) + C\int_0^t \Delta\overline{E}(s)\,ds},$$

wich after taking the supremum in i and j is precisely the lemma. $\square$

As for K, using the equation that $\dot{V}_i(t) = E_i(X_i(t))$, we may prove by the same method which we do not repeat, the result

Lemma 2.6. *We have that for any t*

$$K(t) \le K(0) + Ct + C\varepsilon\,\overline{E}(t) + C\int_0^t \overline{E}(s)\,ds.$$

2.4 Conclusion on the proof of Theorem 1.1

Here (but only in this subsection) for a question of clarity, we keep the notation C for the constants appearing in Lemmas 2.3, 2.4, 2.5 and 2.6 and we denote by $\tilde{C}$ any other constant depending only on $R(0)$, $K(0)$ and $m(0)$.

We assume that on a time interval $[0,\ T]$, we have (for a given α') for a constant k

$$\begin{aligned} &m(t) \le k\, m(0), \quad \overline{E}(t) \le kC\, k^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha}, \\ &K(t) \le k\,(1+K(0)), \quad R(0) \le k\,(1+R(0)), \quad \forall\, t\in [0,\ T], \end{aligned} \tag{2.6}$$

which we may always do since all these quantities are continuous in time (although they may a priori increase very fast). The constant k is chosen to be equal to 2, however we keep the notation k in order to let the reader keep more easily track of this constant.
Then we show that if T is too small we have in fact the same inequalities but with a $3k/4$ constant instead of k. By contradiction this of course shows that we can bound T from below in terms of only $R(0)$, $K(0)$ and $m(0)$ and it proves Theorem 1.1 with $c = C\times k^{8\alpha'-\alpha+1}$.

First of all, we note that since $m(t)\le k\, m(0)$, we may apply Lemmas 2.3, 2.4, and 2.5. Furthermore we immediately know from (1.14) that

$$\|\mu_N(t,.)\|_{\infty,\varepsilon} \le (k^3\, m(0))^{2d}.$$

Let us start with Lemma 2.3, using the assumption (2.6) we deduce that for any $t\in[0,\ T]$,

$$\overline{E}(t) \le C\, k^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha} + \tilde{C}\,\varepsilon^{d-a} + \tilde{C}\,\varepsilon^{2d-3\alpha}.$$

For ε small enough this proves that

$$\overline{E}(t) \le \frac{3\,k\,C}{4}\, k^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha},$$

which is the first point.

Next applying Lemma 2.4, we deduce that for any $t\in[0,\ T]$

$$\Delta\overline{E}(t) \le \tilde{C}.$$

From Lemma 2.5, we obtain that

$$m(t) \le m(0)\times e^{\tilde{C}T},$$

so if T is such that $\tilde{C}\,T < \ln(3\,k/4)$ then we get

$$m(t) \le \frac{3\,k}{4}\, m(0).$$

Lemma 2.6 implies that for $t \in [0,\ T]$

$$K(t) \leq K(0) + \tilde{C}\, T,$$

so that again for T small enough

$$K(t) \leq \frac{3\,k}{4}\,(1 + K(0)).$$

Eventually thanks to relation (1.6), we know that for $t \in [0,\ T]$

$$R(t) \leq R(0) + T\, K(t) \leq R(0) + \tilde{C}\, T,$$

hence the corresponding estimate for R provided $\tilde{C}\, T \leq 3\,k/4$.
In conclusion we have shown that if (2.6) holds and if T is smaller than a given time depending only on $R(0)$, $K(0)$ and $m(0)$ then the same inequalities are true with $3/2$ instead of $k = 2$. By the continuity of R, K, m and $\overline{E}$ this has for consequence that (2.6) is indeed valid at least on this time interval thus proving Theorem 1.1.

3 Preservation of $\|\mu_N\|_{\infty,\eta}$

From the form of the estimate on m in Lemma 2.5, it is clear that with this estimate we will never get a result for a long time. Indeed, even assuming that we have bounded before K and R, we would have the equivalent of $\dot{m} \leq m \times \Delta\overline{E} \leq C\, m \times m^{2+2\alpha'}$.
On the other hand this suggests the possibility that we did not use enough the structure of the equation since, in the limit, the L^∞ norm is conserved. And this preservation is very useful in the proof of the existence and uniqueness of the solution of the Vlasov equation, see for instance [15] or the appendix. But, how to obtain the analog of this in the discrete case? At this time, we just have a bound on $\|\mu_N\|_{\infty,\varepsilon}$ on a small time, and the bound is too huge to allow us to prove convergence results for long time. Of course, this norm is not preserved at all because we are looking at the scheme at the scale of the discretization. And in our calculation we do not use the fact that the flow is divergence free, a property that is the key for the preservation of the L^∞ norm.
So what else can we do? One of the solutions is to look at a scale $\eta > \varepsilon$, with ε/η going to zero as ε goes to zero. At this scale, we have many more

particles in a cell and we will be able to obtain the asymptotical preservation of this norm. This will be very useful because it will allow us to sharpen our estimate on E and ΔE. And with this we will obtain long time convergence results.

3.1 Sketch of the proof

Now, we will try to give roughly the idea of the proof in dimension 1 before beginning the genuine calculations. We choose a time t and a box S_t in the phase space of size ε centered at (X_t, V_t). The field $(v, E(t,x))$ is divergence free, so it preserves the volume; Heuristically speaking because this field is not regular. This will be the first problem we will have to resolve. If it is solved, we can deform the set S_t backwards in time according to the flow. We obtain at time 0 the set S_0, which is of the same volume than S_t. Our question is: "How many particles contains S_0?". Remember that we only control the norm $L_{\infty,\varepsilon}$ of μ_N^0. So we need to recover the set S_0 by balls of size ε. In order to obtain a not too huge number of balls, we need a control on the shape of S_0. By instance, if S_0 is the set $\{(x,v) | |x| \leq \varepsilon^2, |v| \leq (\eta/\varepsilon)^2\}$, then we need $(\eta/\varepsilon)^{2d} \times (1/\varepsilon)^d$ balls to recover it. It will give us

$$\|\mu_N(t)\|_{\infty,\eta} \geq \frac{1}{\eta^{2d}} \mu_N^0(S_0) \geq \frac{1}{\varepsilon^d} \|\mu_N^0\|_{\infty,\varepsilon},$$

which is a very bad estimate.

For the control of the shape, we will move backwards with steps of size ε in time. So first, we look at $S_{t-\varepsilon}$. Assume that a particle is in S_t at time t. Since

$$\begin{aligned} X_i(t) &\equiv X_i(t-\varepsilon) + \varepsilon V(t-\varepsilon), \\ V_i(t) &\equiv V_i(t-\varepsilon) + \varepsilon E(t-\varepsilon, X_i(t - s\varepsilon)), \end{aligned}$$

if we assume that the field E is Lipschitz, we obtain approximatively that

$$\begin{aligned} &|X_i(t-\varepsilon) - X_t - \varepsilon V_i(t-\varepsilon)| \leq \eta, \\ &\big|V_i(t-\varepsilon) - (V_t - \varepsilon E(t-\varepsilon, X_t - \varepsilon V_t)) \\ &\quad - \nabla E(t-\varepsilon, X_t - \varepsilon V_t)) \cdot (X_i(t-\varepsilon) - X_t - \varepsilon V_t)| \leq \eta. \end{aligned}$$

We denote $X_{t-\varepsilon} = X_t - \varepsilon V_t$ and $V_{t-\varepsilon} = V_t - \varepsilon E(t-\varepsilon, X_t - \varepsilon V_t)$, the approximate positions of the center of the balls at time $t - \varepsilon$. This two equations may be rewritten

$$|X_i(t-\varepsilon) - X_{t-\varepsilon} - \varepsilon(V_i(t-\varepsilon) - V_{t-\varepsilon})| \leq \eta,$$

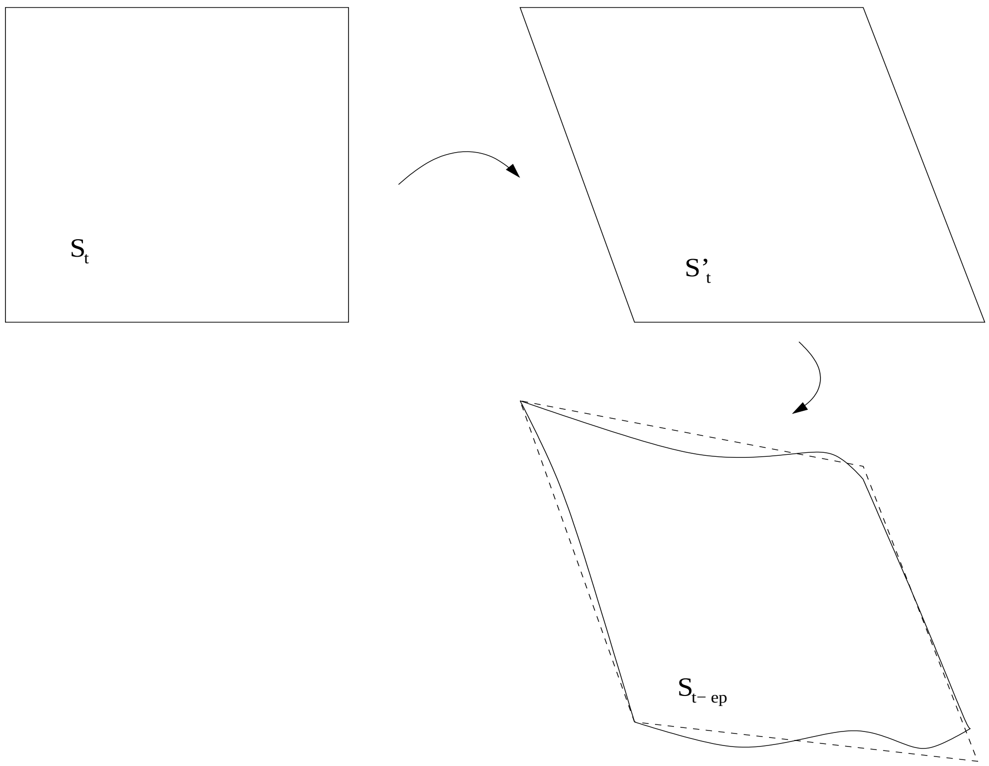

Figure 1: Evolution of S_t.

$$|V_i(t-\varepsilon) - V_{t-\varepsilon} - \varepsilon\nabla E(t-\varepsilon, X_t - \varepsilon V_t)) \cdot (X_i(t-\varepsilon) - X_{t-\varepsilon})| \leq \eta.$$

So the particles are at time $t-\varepsilon$ in the set

$$S_{t-\varepsilon} = \left\{ (x,v) \left| \begin{array}{c} |X_i(t-\varepsilon) - X_{t-\varepsilon} - \varepsilon(V_i(t-\varepsilon) - V_{t-\varepsilon})| \leq \eta \\ \big| V_i(t-\varepsilon) - V_{t-\varepsilon} \\ \quad -\varepsilon\nabla E(t-\varepsilon, X_t - \varepsilon V_t)) \cdot (X_i(t-\varepsilon) - X_{t-\varepsilon}) \big| \leq \eta \end{array} \right. \right\}.$$

If $d=1$, this set is a paralellogram (see the Figure 1), and for commodity we will still call it paralellogram in higher dimension.
If we define the matrix $M_{t-\varepsilon}$ of dimension $2d \times 2d$ by

$$M_{t-\varepsilon} = \left(\begin{array}{cc} I & \varepsilon I \\ \nabla E(t-\varepsilon, X_t - \varepsilon V_t)) & I \end{array} \right),$$

$$\text{then,} \qquad S_{t-\varepsilon} = \left\{ (x,v) \left| \left\| M_{t-\varepsilon} \cdot \left(\begin{array}{c} x - X_{t-\varepsilon} \\ v - V_{t-\varepsilon} \end{array} \right) \right\| \leq \rho \right. \right\}.$$

This definition involving the matrix M will considerably simplify our work.

Definition 3.1. *We call parallelogram a subset S of $\mathbb{R}^{2d}$ defined as above:*

$$S = \left\{ (x,v) \left| \left\| M \cdot \begin{pmatrix} x - X \\ v - V \end{pmatrix} \right\| \leq \rho \right. \right\},$$

where (X,V) in $\mathbb{R}^{2d}$ is the center of the paralellogram, ρ in $\mathbb{R}$ is the size, M is a matrix in $\mathcal{M}(\mathbb{R}^{2d})$. The norm used is defined by $\|(x,v)\| = \max(|x|,|v|)$. We will always decompose the matrix M in four block

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Because we need to control the deformation of a paralellogram, we introduce the following definition

Definition 3.2. *A paralellogram S will be called not too stretched if the corresponding matrix M satisfies $|\det(M) - 1| \leq 1/2$ and*

$$\|A - Id\|, \|B\|, \|C\|, \|D - Id\| \leq \frac{1}{3}.$$

3.2 The notion of ε-volume

Now, we need to control the number of ε-balls needed to cover a paralellogram S. For this, we introduce the following definition:

Definition 3.3. *The ε-volume, denoted $Vol_\varepsilon(S)$, of a subset S of $\mathbb{R}^{2d}$ is the volume of the minimal number of balls of size ε needed to recover S times $(2\varepsilon)^{2d}$ (the volume of a ball).*

Notice that the ε-volume can be very different from the volume. For instance the set

$$T = \{(x,v) |\ |x| \leq \varepsilon^2,\ |v| \leq 1\}$$

has volume of order ε^{2d}, but ε-volume ε^d. We can also see that, up to a constant, the ε-volume is the volume of the set $S_{+\varepsilon/2} = \{(x,v) | d((x,v),S) \leq \varepsilon\} = S + B(0,\varepsilon/2)$.
This notion is usefull to compute the number of particles in a set at time 0. At this time, we only control the number of particles by balls of size ε.

Then, the best estimate we can obtain on the total mass $\mu_N(S)$ of particles in a set S is

$$\mu_N(S) \leq Vol_\varepsilon(S) \times \|\mu_N\|_{\infty,\varepsilon}.$$

We need this estimation of the ε-volume of the set S_0. Roughly speaking, we know its volume and want to show that its ε-volume is closed to its volume. Thanks to the following lemma, we will be able to proof this if S_0 is a not too stretched paralellogram.

Lemma 3.1. *Let S be a not too stretched paralellogram, then we have the following inequality:*

$$Vol_\varepsilon(S) \leq Vol(S) \times (1 + \frac{2\varepsilon}{\rho})^{2d}.$$

Proof of the lemma. We define, for all positive integer k

$$S^+_{k\varepsilon} = \left\{ (x,v) | \left\| M \cdot \begin{pmatrix} x - X \\ v - V \end{pmatrix} \right\| \leq \rho + k\varepsilon \right\} ,$$

and $P = \varepsilon\mathbb{Z} \cap S^+_{2\varepsilon}$. Here, ρ, M, (X,V) stands for the size, the matrix and the center of the paralellogram as in the definition. We look at the set $P_+\varepsilon$ consisting of the union of all the balls of size ε centered at points of P, that is $P_\varepsilon = P + B(0,\varepsilon)$. We will show that this set is included in $S^+_{4\varepsilon}$. For this, we choose $(x,v) \in P_\varepsilon$ and a couple (m,n) in $\mathbb{Z}^2$ such that $\|(x - \varepsilon m, v - \varepsilon n)\| \leq \varepsilon/2$. Then,

$$\begin{aligned} \left\| M \cdot \begin{pmatrix} x \\ v \end{pmatrix} \right\| &\leq \left\| M \cdot \begin{pmatrix} x - \varepsilon m \\ v - \varepsilon n \end{pmatrix} \right\| + \left\| M \cdot \begin{pmatrix} \varepsilon m \\ \varepsilon n \end{pmatrix} \right\| \\ &\leq \|M\| \frac{\varepsilon}{2} + \rho + \varepsilon \\ &\leq \eta + 2\varepsilon. \end{aligned}$$

In the last line, we use $\|M\| \leq 2$. This inequalitiy is implied by the condition in the definition of a not too streched paralellogram. Therefore we have the inclusion $P_\varepsilon \subset S^+_{2\varepsilon}$.

Moreover, if we choose a point $(x,v) \in S$, we can find a point $(\varepsilon m, \varepsilon n)$of $\varepsilon\mathbb{Z}^{2d}$ such that $\|(x - \varepsilon m, v - \varepsilon n)\| \leq \varepsilon/2$. As above, we have

$$\left\| M \cdot \begin{pmatrix} \varepsilon m \\ \varepsilon n \end{pmatrix} \right\| \leq \eta + 2\varepsilon.$$

Thus, $\varepsilon(m,n) \in P$. That proves that $S \subset P_{+\varepsilon}$. So, we have the inclusions

$$S \subset P_{+\varepsilon} \subset S^+_{2\varepsilon}.$$

The first is the recovering we want. The second gives us an estimate on the cardinal of P. Comparing the volume of P_ε and $S^+_{2\varepsilon}$ we obtain

$$(\varepsilon)^{2d}|P| \leq Vol(S^+_{2\varepsilon}) = det(M)^{-1}(\rho + 2\varepsilon)^{2d}.$$

Since $Vol(S) = det(M)^{-1}\rho^{2d}$ we obtain

$$Vol_\varepsilon(S) \leq Vol(S) \times (1 + \frac{2\varepsilon}{\rho})^{2d}.$$

$\square$

3.3 Asymptotic preservation of $\|\mu\|_{\infty,\eta}$ for small time

Now, given a box S_t, our goal is to find a not too stretched paralellogram S_0 which contains at time 0 all the particles that are in S_t at time t. For this, we will go from t to $t - \varepsilon$ using the following lemma:

Lemma 3.2. *Assume as before that*

$$m(t) \leq \frac{1}{12\varepsilon\, K(t) \Delta \overline{E}(t)}.$$

Then, for any $1 < \beta < d - 1$, there exists a constant K_1 depending on t, R, K, $\overline{E}$, $\|\mu\|_{\infty,\varepsilon}$ such that for all not too stretched paralellogram S_t, of center (X_t, V_t), matrix M_t (decomposed in A_t,B_t,C_t,D_t) and size ρ_t, there exists a paralellogram $S_{t-\varepsilon}$ of center $(X_{t-\varepsilon}, V_{t-\varepsilon})$ and so on, satisfying the following conditions

i. $\|A_{t-\varepsilon} - A_t\|, \|B_{t-\varepsilon} - B_t\|, \|C_{t-\varepsilon} - C_t\|, \|D_{t-\varepsilon} - D_t\| \leq K_1\varepsilon$

ii. $|det(M_{t-\varepsilon}) - \det(M)| \leq K_1\varepsilon^2$

iii. $\rho_{t-\varepsilon} \leq \rho_t + K_1\varepsilon(\rho_t^\beta + \varepsilon)$

and that contains at time $t - \varepsilon$ all the particles that are in S_t at time t.

Remarks

- We always use the heavy expression " contains at time t' all the particles that are in S at time t" because here we can not speak of the reverse image by the flow. There is not a flow that all the particles follow because a particle do not see the force-field it creates.

- What is important here is that $\mu_N(t, S_t) \leq \mu_N(t-\varepsilon, S_{t-\varepsilon})$.

Proof. We want to rewrite our inequalities involving $X_j(t)$, $V_j(t)$, X_t and V_t in inequalities involving $X_j(t-\varepsilon)$, $V_j(t-\varepsilon)$, $X_{t-\varepsilon}$ and $V_{t-\varepsilon}$ (and we have to choose the last position and speed). Of course, the center of the paralellogram will approximately move according to the flow created by all the particles. We write approximately because particles close from the center will induce pertubation in his trajectory (these perturbation are however negligible). The best way to do this is to regularise the flow at order ε. So, we introduce

$$E_\varepsilon(t,x) = \sum_{i=1}^{n} F * \xi_{B(0,\varepsilon)}(x - X_i(t))$$

Remark that the kernel $F_\varepsilon = F * \xi_{B(0,\varepsilon)}$ satisfy the same assumptions that the kernel ∇F, it means

$$F_\varepsilon, |x||\nabla F_\varepsilon|, |x|^2|\nabla^2 F_\varepsilon| \leq C|x|^{-\alpha}$$

At this point, we define the center $(X_{t-\varepsilon}, V_{t-\varepsilon})$ of the paralellogram $S_{t-\varepsilon}$. It will be the center (X_t, V_t) moved backward to the time $t-\varepsilon$ according to E_ε. Moreover, all the estimates on $\overline{E}$, $\Delta\overline{E}$ can be applied to this virtual particle. More precisely, the two first point of the lemma (2.1) are true even for a virtual particle because for this two estimation we do not use the minimal distance between particles m. The last one is more easy to obtain because the approximate kernel F_ε is bounded by ε. We wanted an estimate of $|X_j(t-\varepsilon) - X_{t-\varepsilon}|$ and $|V_j(t-\varepsilon) - V_{t-\varepsilon}|$. We will begin with the second and integrate it.

Step 1: Estimation of $|V_j(t-\varepsilon) - V_{t-\varepsilon}|$.

For the particle j, we have

$$\begin{aligned}
V_j(t-\varepsilon) - V_{t-\varepsilon} &= V_j(t) - V_t - \varepsilon \int_0^1 (E(X_j(t-s\varepsilon)) - E_\varepsilon(X_{t-s\varepsilon}))\, ds \\
&= V_j(t) - V_t - \varepsilon \int_0^1 (E(X_j(t-s\varepsilon)) - E_\varepsilon(X_j(t-s\varepsilon)))\, ds \\
&\qquad +\varepsilon \int_0^1 (E_\varepsilon(X_j(t-s\varepsilon)) - E_\varepsilon(X_{t-s\varepsilon}))\, ds \\
&= \varepsilon(J_1 \quad + \quad J_2).
\end{aligned}$$

We need to bound the first term J_1. The approximation error is

$$\begin{aligned}
J_1 &= \int_0^1 E(X_j(t-s\varepsilon)) - E_\varepsilon(X_j(t-s\varepsilon))\, ds \\
&= \frac{1}{N} \sum_{k \neq j} \int_0^1 \big(F(X_j(t-s\varepsilon) - X_k(t-s\varepsilon)) - F_\varepsilon(X_j(t-s\varepsilon) - X_k(t-s\varepsilon)) \big)\, ds.
\end{aligned}$$

We can bound this term using the two bounds $|F(x) - F_\varepsilon(x)| \leq C\varepsilon/|x|^{\alpha+1}$ and $|F(x) - F_\varepsilon(x)| \leq C/|x|^\alpha$. We write, recalling the notation $G_j = \{k||X_k(t) - X_j(t)| \geq 2K(t)\varepsilon\}$ of the lemma (2.1)

$$\begin{aligned}
J_1 \leq \frac{C\varepsilon}{N} \sum_{k \in G_j} \int_{t-\varepsilon}^t \frac{1}{|X_k(s) - X_j(s)|^{1+\alpha}}\, ds \\
+ \frac{C}{N} \sum_{k \notin G_j} \int_{t-\varepsilon}^t \frac{1}{|X_k(s) - X_j(s)|^{\alpha}}\, ds \quad (3.1)
\end{aligned}$$

We choose an α' so that $\alpha < \alpha' < 1$. Using the point i. of the lemma (2.1) with $\delta = 1 + \alpha$ and $\delta' = 1 + \alpha'$, we can bound the first term of the right hand side by $C\varepsilon^2 \|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d} K^{1+\alpha'} R^{\alpha'-\alpha}$. Using the point ii. and iii., we can bound the second term by $\varepsilon^{d-\alpha} \|\mu_N\|_{\infty,\varepsilon} K^{2d-\alpha} + \varepsilon^{2d-3\alpha} \|\mu_N\|_{\infty,\varepsilon} K^{d-\alpha} \overline{E}^d K^d)$. So, without forgetting is dependance, we can write

$$J_1 \leq K_2(\varepsilon + \varepsilon^{d-\alpha}).$$

The term J_2 contains only term using the approximate field. In that case the estimates are simpler because we do not need to integrate it over a small interval of time. We state them in the following lemma

Lemma 3.3. *We assume that the force K_ε satifyies*

$$K_\varepsilon(x) \leq \frac{C}{(\varepsilon + |x|)^\delta}.$$

We choose a particle i and define two set

$$G_{\varepsilon,i} = \{j||X_j(t) - X_i(t)| \geq \varepsilon\}$$

$$B_{\varepsilon,i} = \{j||X_j(t) - X_i(t)| < \varepsilon\}$$

then for any δ' satisfying $\delta < \delta' < d$, there exist a numerical constant C such that the two following inequality are true.

$$i.) \qquad \left| \frac{1}{N} \sum_{j \in G_i} K_\varepsilon(X_i(t) - X_j(t) \right| \leq C \|\mu_N\|_{\infty,\varepsilon}^{\delta'/d} R^{\delta'-\delta}$$

$$ii.) \qquad \left| \frac{1}{N} \sum_{j \in B_i} K_\varepsilon(X_i(t) - X_j(t) \right| \leq \|\mu_N\|_{\infty,\varepsilon} K^d \varepsilon^{d-\delta}$$

Proof. The point i. This is exactly the estimate i. of the lemma (2.1) with $K = 1$, so we will not write it again.
the point ii. The ε-volume of the set $B_{\varepsilon,i}$ is $(K\varepsilon)^d$. So the mass in it is less than $\|\mu_N\|_{\infty,\varepsilon}(K\varepsilon)^d$. Moreover, K_ε is bounded by C/ε^δ. So we get

$$\frac{1}{N} \sum_{j \in B_i} K_\varepsilon(X_i(t) - X_j(t)| \leq \|\mu_N\|_{\infty,\varepsilon} K^d \varepsilon^{d-\delta}$$

□

Now, we want to approximate J_2 by $\int_{t-\varepsilon}^t \nabla E_\varepsilon(X_s) ds \cdot (X_j(t-\varepsilon) - X_{t-\varepsilon})$. First, we can replace $X_j(s) - X_s$ by $X_j(t-\varepsilon) - X_{t-\varepsilon}$ in the expression of J_2, because for all $s \in (t - \varepsilon, t)$

$$|(X_j(t-\varepsilon) - X_{t-\varepsilon}) - (X_j(s) - X_s)| \leq 2K\varepsilon,$$

and then,

$$\left| \int_{t-\varepsilon}^t \nabla E_\varepsilon(X_s) ds \cdot (X_j(t-\varepsilon) - X_{t-\varepsilon}) - \int_{t-\varepsilon}^t \nabla E_\varepsilon(X_s) \cdot (X_j(s) - X_s)\, ds \right| \leq 2K\Delta\overline{E}\varepsilon^2.$$

We write $\Delta\overline{E}$ even for the approximate field because it is bounded by the bound of lemma (2.4) like the true field, eventually addind a numerical factor in front of the bound.
Now, the main term is

$$J_2' = \int_{t-\varepsilon}^{t} E_\varepsilon(X_j(s)) - E_\varepsilon(X_s) - \nabla E_\varepsilon(X_s)\, ds \cdot (X_j(s) - X_s)\, ds.$$

This is a sum of terms of the form

$$\int_{t-\varepsilon}^{t} \left(F_\varepsilon(X_j(s) - X_i(s)) - F_\varepsilon(X_s - X_i(s)) - \nabla F_\varepsilon(X_s - X_i(s)) \cdot (X_j(s) - X_s)\right)\, ds.$$

So, for each i, j and s, we choose a path $I(s,\cdot)$ between $X_j(s)$ and X_s so that its length is less than $4|X_j(s) - X_s|$ and so that $|I(s,u) - X_i(s)|$ always stays between in the interval between $|X_j(s) - X_i(s)|$ and $|X_s - X_i(s)|$. We can rewrite the previous term as

$$\int_{t-\varepsilon}^{t} \int_0^1 \nabla F_\varepsilon(I(s,u) - X_i(s)) - \nabla F_\varepsilon(X_s - X_i(s)) \cdot (X_j(s) - X_s)\, du\, ds.$$

The integrand may be bounded in two ways. First by

$$\frac{C|X_j(s) - X_s|}{(\varepsilon + \min(|I(s,u) - X_i(t)|, |X_s - X_i(t)|))^{1+\alpha}},$$

if we bound it by the sum of the two terms and also by

$$C\frac{|I(s,u) - X_s|^2}{(\varepsilon + \min(|I(s,u) - X_i(s)|, |X_s - X_i(s)|))^{2+\alpha}},$$

if we use the derivative. We need a majoration by a term with a small power of $|I(s,u) - X_s|$ on the top, and an exponent sufficiently small below. For this, we pick a γ in $(0,1)$ and bound the integrand by the first bound at the power $1-\gamma$ and the second at the power γ. So, we bound the term by

$$J_2' \le \sum_{i \neq j} \int_{t-\varepsilon}^{t} \int_0^1 \frac{|X_j(s) - X_s|^{1+\gamma}}{\varepsilon + \min(|X_j(s) - X_i(s)|, |X_s - X_i(s)|)^(1 + \alpha + \gamma)}\, ds.$$

First, as $|X_j(s) - X_s| \leq |X_j(t-\varepsilon) - X_{t-\varepsilon}| + K\varepsilon$ for all $s \in [t-\varepsilon]$, we can write

$$J_2' \leq C\frac{K\,\varepsilon}{N}\sum_{i\neq j}\int_{t-\varepsilon}^{t}\int_0^1 \frac{1}{\varepsilon + \min(|X_j(s)-X_i(s)|, |X_s - X_i(s)|)^{1+\alpha+\gamma}}$$
$$+\left(\frac{1}{N}\sum_{i\neq j}\int_{t-\varepsilon}^{t}\int_0^1 \frac{}{\varepsilon + \min(|X_j(s)-X_i(s)|, |X_s - X_i(s)|)^{1+\alpha+\gamma}}\right)|X_j(t-\varepsilon) - X_{t-\varepsilon}|^{1+\gamma} \tag{3.2}$$

Now we can use the estimates of the lemma (3.3) with $\delta = 1+\alpha+\gamma$ to bound J_2'. We obtain

$$J_2' \leq C(K\varepsilon + |X_j(t-\varepsilon) - X_{t-\varepsilon}|^{1+\gamma})\|\mu_N\|_{\infty,\varepsilon}^{\delta'/d} R^{\delta'-\delta}$$
$$+ C|X_j(t-\varepsilon) - X_{t-\varepsilon}|^{1+\gamma}\|\mu_N\|_{\infty,\varepsilon} K^d \varepsilon^{d-\delta}. \tag{3.3}$$

This gives us a nice bound if $\gamma < d-1-\alpha$. In this case, defining $\beta = 1+\gamma$, we may rewrite it as

$$J_2' \leq \tilde{K}_2 \varepsilon(|X_j(t) - X_t|^\beta + \varepsilon)$$

without forgetting the dependance of K_2. Now, putting everything together and denoting $\widetilde{\nabla E_\varepsilon} = (1/\varepsilon)\int_{t-\varepsilon}^{t} \nabla E_\varepsilon(X_s)\,ds$, we have:

$$|(V_j(t-\varepsilon) - V_{t-\varepsilon})) - (V_j(t) - V_t) - \varepsilon\widetilde{\nabla E_\varepsilon}\cdot(X_j(t-\varepsilon) - X_{t-\varepsilon})|$$
$$\leq K_1\varepsilon(|X_j(t) - X_t|^\beta + \varepsilon + \varepsilon^{d-1-\alpha}).$$

This is the estimation we will use.
Step 2: Estimation of $|X_j(t-\varepsilon) - X_{t-\varepsilon}|$
The bound on the position is easier to obtain. We have

$$(X_j(t-\varepsilon) - X_{t-\varepsilon}) = (X_j(t) - X_t) - \varepsilon(V_j(t-\varepsilon) - V_{t-\varepsilon}) - \varepsilon^2 R_\varepsilon,$$

with

$$R_\varepsilon = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}(s-t+\varepsilon)(E(X_j(s) - E(X_s))\,ds.$$

Here, a bound on R_ε will be sufficient. And we have $R_\varepsilon \leq 2\overline{E}$ which gives

$$|(X_j(t-\varepsilon) - X_{t-\varepsilon}) - (X_j(t) - X_t) + \varepsilon(V_j(t-\varepsilon) - V_{t-\varepsilon})| \leq K_3\varepsilon^2.$$

Step 3: The new paralellogram
Consequently if we apply $A\times$ 3.3 $-B\times$ 3.3 and use the fact that $|A_t\cdot(X_j(t)-X_t)-B_t\cdot(V_j(t)-V_t)|\leq\rho$, we obtain

$$|(A_t+\varepsilon B_t\widetilde{\nabla E_\varepsilon})\cdot(X_j(t_\varepsilon)-X_{t-\varepsilon})-(B_t\varepsilon A_t)\cdot(V_j(t-\varepsilon)-V_{t-\varepsilon})| \\ \leq\rho_t+C(\|A_t\|+\|B\|)\,\varepsilon\,(\rho_t^\beta+\varepsilon).$$

So, if we denote $A_{t-\varepsilon}=A_t+\varepsilon B_t\widetilde{\nabla E_\varepsilon}$, $B_{t-\varepsilon}=B_t\varepsilon A_t$ and $\rho_{t-\varepsilon}=\rho_t+C(\|A_t\|+\|B\|)\varepsilon(\rho_t^{1+\gamma}+\varepsilon+\varepsilon^{d-1-\alpha})$, we obtain that

$$|A_{t-\varepsilon}\cdot(X_j(t-\varepsilon)-X_{t-\varepsilon})-B_{t-\varepsilon}\cdot(V_j(t-\varepsilon)-V_{t-\varepsilon})|\leq\rho_{t-\varepsilon}.$$

We can do the same for the second line of the matrix. If we denote $C_{t-\varepsilon}=C_t+\varepsilon D_t\widetilde{\nabla E_\varepsilon}$ and $D_{t-\varepsilon}=D_t\varepsilon C_t$, and

$$M_{t-\varepsilon}=\begin{pmatrix}A_{t-\varepsilon} & B_{t-\varepsilon}\\ C_{t-\varepsilon} & D_{t-\varepsilon}\end{pmatrix}$$

we obtain that $(X_j(t-\varepsilon,V_j(t-\varepsilon))\in S_{t-\varepsilon}$, the paralellogram of center $(X_{t-\varepsilon},V_{t-\varepsilon})$, matrix $M_{t-\varepsilon}$ and size $\rho_{t-\varepsilon}$.
It remain to prove the estimates on $M_{t-\varepsilon}$. For this, remark that, $M_{t-\varepsilon}=M_tJ_t$ with

$$J_t=\begin{pmatrix}I & \varepsilon I\\ \varepsilon\tilde{\nabla}E_\varepsilon & I\end{pmatrix}=I+\varepsilon N_t.$$

Then, $\det M_{t-\varepsilon}=\det(M_t)\det(I+\varepsilon N_t)$. And $|det(I+\varepsilon N_t)-1-\varepsilon tr(N_t)|\leq C\|N_t\|^2\varepsilon^2$. Moreover, $tr(N_t)=0$. Remark that this is here that we use the fact that our field in the phase space is divergence free. And we obtain

$$|\det(M_{t-\varepsilon}-\det(M_t)|\leq C\det(M_t)\varepsilon^2,$$

where C is of the form $K(\overline{E}+\Delta\overline{E})$. And of course, $\|A_{t-\varepsilon}-A_t\|\leq K\Delta\overline{E}\varepsilon$, $\|B_{t-\varepsilon}-B_t\|\leq K\varepsilon$ and so on. This is all we needed to prove. □

Now, we need to go from a time t to time 0, by backward jumps in time of size ε. At each step we obtain a new paralellogram. We can go on till this paralellogram is too stretched. This will happens in a time of order $1/\Delta\overline{E}$, because of the inequality $\|A_{t-\varepsilon}-A_t\|\leq\Delta\overline{E}\varepsilon$. We would be able to conclude if we had a bound on ρ_0, the size of the paralellogram obtain at time 0. The following lemma provides it.

Lemma 3.4. *Assume that $t' = t - M\varepsilon$, that S_t' is obtain from S_t by iteration of the lemma 3.2 and that $3^\beta K_1 (t - t')(\rho_t^\beta + \varepsilon) \le \rho_t$. Then, the folowing inequality holds*

$$\rho_{t'} \le \rho_t + 3^\beta K_1 (t - t') (\rho_t^\beta + \varepsilon).$$

Proof. We recall that $\rho_{t-\varepsilon} = \rho_t + K_1 \varepsilon (\rho_t^\beta + \varepsilon)$. From these formulas, we expect that $\rho_{t-n\varepsilon} \approx \rho_t + K_1 n\varepsilon(\rho_t^\beta + \varepsilon)$.
To prove this rigourously, we define $\alpha_n = (\rho_{t-n\varepsilon} - \rho_t - 3^\beta K_1 n\varepsilon (\rho_t^\beta + \varepsilon))_+$. We have

$$\alpha_{n+1} - \alpha_n \le K_1 \varepsilon (\rho_t + 3^\beta nK_1 \varepsilon \rho_t^\beta + 3^\beta nK_1 \varepsilon^2 + \alpha_n)^\beta - 3^\beta K_1 \varepsilon \rho_t^\beta - (3^\beta - 1) K_1 \varepsilon^2.$$

Provided $3^\beta K_1 \varepsilon n (\rho_t^{\beta-1} + \varepsilon/\rho_t) \le 1$ and $\alpha_n \le \rho_t$, we have that

$$\alpha_{n+1} - \alpha_n \quad \le \quad 3^\beta K_1 \varepsilon \rho_t^\beta - 3^\beta K_1 \le 0$$

Therefore α_n remains equal to 0 which gives the corresponding result for $\rho_{t'}$. □

Now that we control the growth of ρ_t, we are able to prove the following theorem

Theorem 3.1. *There exists a numerical constant K_2 such that if*

$$m(t) \le \frac{1}{12 \varepsilon K(t) \Delta \overline{E}(t)},$$

$t \le 1/(2\Delta\overline{E})$ and ε is small enough, then the following inequality holds:

$$\|\mu_N(t)\|_{\infty,\eta} \le \|\mu\|_{\infty,\varepsilon} + K_2 \left(\eta^\beta + \frac{\varepsilon}{\eta} \right).$$

Proof. We start at time t from a box $S_t = \{(x,v) \mid \|x - X_t, v - V_t\| \le \eta\}$. It means that $\rho_t = \eta$ and

$$M_t = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

We define backward all the $S_{t-n\varepsilon}$ till S_0. If t is not a multiple of ε, we use a last step less than ε, but all our estimates are still true for a step like this. As $\|A_{t-(n+1)\varepsilon} - A_{t-n\varepsilon}\| \le 2\Delta\overline{E}$ and as $A_t = I$, we have $\|A_s - I\| \le 1/2$. And the same estimates hold for B, C, and D. That means that all our parallelograms are always not too streched. We may then apply the previous lemma to get the corresponding estimate on ρ_0. Using the definition of the discrete L^∞ norm at ε, we control the number of particles in S_0 which is also the numer of particles in S_t. This last number is the bound on the discrete L^∞ norm at η and at time t. □

3.4 New estimates on $\overline{E}$ and $\Delta\overline{E}$

The almost preservation of the $\|\mu_N\|_{\infty,\eta}$ norms will enable us to prove a new estimate on $\overline{E}$, namely

Lemma 3.5. *For any α' with $\alpha < \alpha' < d$, assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\,K(t_0)\,\Delta\overline{E}(t_0)},$$

then the following inequality holds

$$\begin{aligned}\overline{E}(t_0) \leq C\,(&\|\mu_N\|_{\infty,\eta}^{\alpha'/d}\,K^{\alpha'}\,R^{\alpha'-\alpha} + \|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,K^{2\alpha'-\alpha}\,\eta^{\alpha'-\alpha}\\ &+ \varepsilon^{d-\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\alpha} + \varepsilon^{2d-3\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\alpha}\overline{E}^d),\end{aligned}$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

The only non-negligable term in this estimate is sub-linear if α' is chosen sufficiently close to α.

Proof. The idea is very similar to Lemma 2.1. We do the same separation of the position space in dyadic cells, but we begin with cells $\tilde{C}_k$ satisfying

$$\tilde{C}_k = \left\{ i \,\middle|\, 3\,\eta\,K(t_0)\,2^{k-1} < |X_i(t_1) - X_1(t_1)| \leq 3\,\eta\,K(t_0)\,2^k \right\},$$

with k between 0 and $k_0 = \ln(R/(3\eta\,K))/\ln 2$.
For $\tilde{C}_0$, we apply estimate (2.2) with $r = 3\eta\,K(t_0)$ which gives

$$I_{\tilde{C}_0} \leq \|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,K^{2\alpha'-\alpha}\,\eta^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\alpha} + \varepsilon^{2d-3\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\alpha}\overline{E}^d).$$

Next, notice that $\tilde{C}_k$ can be covered by at most $C\,(K(t_0))^d\,2^{kd} \times \eta^{-d}\,(K(t_0))^d$ balls of radius η and therefore by the definition of $\|\mu_N\|_{\infty,\eta}$, we have that

$$|\tilde{C}_k| \leq C\,N\,2^{kd}\,(K(t_0))^{2d}\,\eta^d\,\|\mu_N\|_{\infty,\eta}.$$

On the other hand $|\tilde{C}_k| \leq N$ so for any $\alpha' < d$

$$|\tilde{C}_k| \leq C\,N\,\eta^{\alpha'}\,2^{k\alpha'}\,K^{2\alpha'}\,\|\mu_N\|_{\infty,\eta}^{\alpha'/d}.$$

Of course the $\tilde{C}_k$ are also approximatly stable in the sense that if $i \in C_k$ then $|X_i(t) - X_1(t)| \geq \eta\, K(t_0)\, 2^{k-1}$ for any $t \in [t_1,\ t_0]$. Therefore

$$\begin{aligned}
I_1 &= \sum_{k=1}^{k_0} \sum_{i \in \tilde{C}_k} \frac{1}{\varepsilon} \int_{t_1}^{t_0} \frac{1}{N\,|X_1(t) - X_i(t)|^\alpha}\, dt \\
&\leq C \sum_{k=1}^{k_0} |\tilde{C}_k|\ N^{-1}\, \eta^{-\alpha}\, K^{-\alpha}\, 2^{-k\alpha} \\
&\leq C\, \eta^{\alpha'-\alpha}\, K^{2\alpha'-\alpha} \sum_{k=1}^{k_0} 2^{k(\alpha'-\alpha)} \\
&\leq C\, \eta^{\alpha'-\alpha}\, K^{2\alpha'-\alpha}\, 2^{k_0(\alpha'-\alpha)},
\end{aligned}$$

provided that $\alpha' > \alpha$. Therefore

$$I_1 \leq C\, R^{\alpha'-\alpha}\, K^{\alpha'}\, \|\mu_N\|_{\infty,\eta}^{\alpha'/d}.$$

Summing I_1 with $I_{\tilde{C}_0}$ proves the lemma. ☐

Of course we can perform the same changes for the estimates on $\Delta\overline{E}$ to get

Lemma 3.6. *For any α' with $\alpha < \alpha' < 3$, assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\, K(t_0)\, \Delta\overline{E}(t_0)},$$

then the following inequality holds

$$\begin{aligned}
\Delta\overline{E}(t_0) \leq C\,(&\|\mu_N\|_{\infty,\eta}^{(1+\alpha')/d}\, K^{1+\alpha'}\, R^{\alpha'-\alpha} + \|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d}\, K^{1+2\alpha'-\alpha}\, \eta^{\alpha'-\alpha} \\
&+ \varepsilon^{d-\alpha-\beta}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} + \varepsilon^{2d-3\alpha-\beta}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha}\overline{E}^d),
\end{aligned}$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

3.5 Proof of Theorem 1.2

Let us fix any time $T > 0$. The aim is to show that we have bounds for R, K, $\overline{E}$ and m, uniform in N on $[0,\ T]$.
Next we choose $\eta_0 = \varepsilon^{1/2}$ for instance and $\eta' = \varepsilon^{1/4}$.

Since for any N the quantities R, K, $\overline{E}$ and m are continuous in time, we may define $T_N < T$ as the first time t (if it exists) such that one of the following inequality at least is not true for some integer M to be chosen after

$$\begin{aligned}
&T' = T(R(t), K(t), \overline{E}(t), \sup_{s\leq t} \|\mu_N\|_{\infty,\varepsilon}) \geq \frac{T}{M},\\
&m(t) \leq \frac{1}{12\,\varepsilon\, K(t)\, \Delta\overline{E}(t)}, \quad C(R(t), K(t), \overline{E}(t), \sup_{s\leq t} \|\mu_N\|_{\infty,\varepsilon}) \leq \varepsilon^{-1/8M},\\
&\varepsilon^{d-\alpha}(m(t))^{-2d}\,(K(t))^{2d-\alpha} \leq \varepsilon^{\beta}, \quad \varepsilon^{2d-3\alpha}(m(t))^{-2d}\,(\overline{E}(t))^{d}\,(K(t))^{d-\alpha} \leq \varepsilon^{\beta}.
\end{aligned} \tag{3.4}$$

The quantity T' and C are the time and constant defined in Theorem 3.1. Therefore on $[0,\ T_N]$ all inequalities (3.4) are true and we may apply both Theorem 3.1 and Lemma 3.5.
We define $t_i = i\,T'$ and $\eta_i = \eta_0 \times r^i$ with $r = \varepsilon^{-1/4M}$ so that $\eta_M = \eta'$. We are going to apply M times Theorem 3.1, once on every interval $[t_{i-1},\ t_i]$ (of size less than T') and with $\eta = \eta_i$ and ε replaced by η_{i-1}. That gives

$$\sup_{t\in[t_{i-1},t_i]} \|\mu_N(t)\|_{\infty,\eta_i} \leq \|\mu_N\|_{\infty,\eta_{i-1}} + C(\overline{E}(t_i), \Delta\overline{E}(t_i))\,(\eta_i^{\gamma} + \varepsilon^{1/4M}),$$

and consequently thanks to (3.4)

$$\sup_{t\leq T_N} \|\mu_N(t)\|_{\infty,\eta'} \leq \|\mu_N\|_{\infty,\varepsilon} + C(\overline{E}(T_N), \Delta\overline{E}(T_N))\,M\,\varepsilon^{1/4M} \leq 2\,\|\mu_N^0\|_{\infty,\varepsilon}, \tag{3.5}$$

independently of N (and T_N). Now we apply Lemma 3.5 at time T_N and because of (3.4), we obtain

$$\begin{aligned}
\overline{E}(T_N) &\leq C\,\|\mu_N(T_N)\|_{\infty,\eta}^{\alpha'/d}\,(K(T_N))^{\alpha'}\,(R(T_N))^{\alpha'-\alpha}\\
&\leq C\,(K(T_N))^{\alpha'}\,(R(T_N))^{\alpha'-\alpha},
\end{aligned} \tag{3.6}$$

using (3.5). As $T_N > \varepsilon$, Lemma 2.6 implies that

$$K(T_N) \leq K(0) + C\int_0^{T_N} \overline{E}(t)\,dt \leq K(0) + C\,T_N\,\overline{E}(T_N).$$

From this inequality, we immediately deduce that

$$R(T_N) \leq R(0) + T_N\,K(0) + C\,T_N^2\,\overline{E}(T_N) \leq C\,T + C\,T^2\,\overline{E}(T_N).$$

Inserting these last two inequalities in (3.6), we find

$$\overline{E}(T_N) \leq C\,T + C\,T^2\,(\overline{E}(T_N))^{2\alpha'-\alpha}.$$

Since $2\alpha' - \alpha < 1$, there exists a constant $C(T)$ depending only on T and the initial distribution such that

$$\overline{E}(T_N) \leq C(T), \quad K(T_N) \leq C(T), \quad R(T_N) \leq C(T). \tag{3.7}$$

We are almost ready to conclude, we only need to apply once Lemma 3.6 and by (3.4), (3.5) and (3.7)

$$\Delta\overline{E}(T_N) \leq C(T). \tag{3.8}$$

Inserting (3.8) in Lemma 2.5, we eventually get

$$m(T_N) \leq C(T). \tag{3.9}$$

Together (3.7), (3.8) and (3.9) imply that all the inequalities of (3.4) are true with a factor $1/2$ at time T_N, provided N and M are large enough. Therefore (3.4) is still true on at least a short time interval after T_N and that means that necessarily $T_N = T$. The consequence is that (3.7), (3.8) and (3.9) are true on any time interval $[0,\ T]$ which is exactly Theorem 1.2.
Finally note that we have implicitly used the short time result when we said that $T_N > \varepsilon$.

4 Convergence of the density in the approximation

The existence of the bounds on R, K, $\overline{E}$, $\Delta\overline{E}$ and $\|\mu_N\|_{\infty,\eta}$ implies the weak convergence of the distribution μ_N to a weak solution of the Vlasov equation and Theorem 1.3 is only a consequence of Theorem 1.2 and the following proposition

Proposition 4.1. *Let μ_N be the distributions associated with the solutions to (1.1). We assume that the initial conditions μ_N^0 converges weakly in $M^1(\mathbb{R}^{2d})$ to some $f_0 \in L^1 \cap L^\infty(\mathbb{R}^{2d})$. We choose a time $T > 0$. Assume furthermore that there exists a constant $C(T)$ independent of N such that*

$$\sup_{\varepsilon>0}(R(T),\ K(T),\ \overline{E}(T),\ \Delta\overline{E}(T),\ \|\mu_{\infty,\eta}\|) < +\infty\ ,$$

where η depends on ε and N and goes to zero when ε goes to zero. Then, $\mu_N(t)$ converges weakly to $f(t)$, a solution to the Vlasov equation with initial conditions f^0.

Proof. We recall that the distribution of the particles μ_N satisfies the Vlasov equation in the sense of distribution provided the force field is correctly written. Moreover, the sequence μ_N is bounded in $C([0,T], M^1(\mathbb{R}^{3d}))$. Up to an extraction, we may assume that μ_N converges weakly to some $f \in L^\infty([0,T], M^1(\mathbb{R}^{2d}))$. In addition, the fact that $\|\mu_N\|_{\infty,\eta}$ is bounded implies that $f \in L^\infty$. To see this, we choose a regular test function Φ with compact support. We have

$$\langle \mu_N, \Phi \rangle = \frac{1}{N} \sum_{i=1}^N \Phi(X_i(t), V_i(t)).$$

Now, we define $\rho_\eta(x,v) = \chi_C(x/\eta, v/\eta)$ where χ_C is the characteristic function of the set $C = \{(x,v) | \|(x,v)\| \le 1\}$ and we write

$$\begin{aligned}\langle \mu_N, \Phi \rangle = &\frac{1}{N} \sum_{i=1}^N \Phi * \rho_\eta(X_i(t), V_i(t)) \\ &+ \frac{1}{N} \sum_{i=1}^N (\Phi(X_i(t), V_i(t)) - \Phi * \rho_\eta(X_i(t), V_i(t))).\end{aligned}$$

The first term is $\int \phi * \rho_\eta(x,v)\, d\mu_N(x,v) = \int \phi(\mu_N * \rho_\eta)\, dxdv$. So it is bounded by $\|\phi\|_1 \times \|\mu_N * \rho_\eta\|_\infty$. But $\|\mu_N * \rho_\eta\|_\infty$ is exactly $\|\mu_N\|_{\infty,\eta}$. The second term is easily bounded by $\eta \|\nabla \Phi\|_\infty$. Putting all together, we obtain that

$$\langle \mu_N, \Phi \rangle \le \|\mu_N\|_{\infty,\eta} \|\Phi\|_1 + \eta \|\nabla \Phi\|_\infty.$$

At the limit,

$$\langle f, \Phi \rangle \le \liminf_{N \to \infty} \|\mu_N\|_{\infty,\eta} \|\Phi\|_1,$$

which means that $f \in L^\infty$ and that $\|f\|_\infty \le \liminf_{N\to\infty} \|\mu_N\|_{\infty,\eta}$.

The passage to the limit in the linear part of the equation does not raise any difficulty. For the term in $F \cdot \nabla_v f$, we need a strong convergence in the force. We denote by F_∞ the force induced by f and by F_N the force induced by μ_N

$$\begin{aligned} F_\infty(x) &= \int \frac{x-y}{|x-y|^{1+\alpha}}\, dydw, \\ F_N(x) &= \frac{1}{N} \sum_{i=1}^N \frac{x - X_i(t)}{|x - X_i(t)|^{1+\alpha}}. \end{aligned}$$

We have

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} F_N(X_i(t)) - F_\infty(X_i(t))\,dt &= I_1 + I_2 + I_3 \\
&= \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|y-X_i(s)|\geq r} \frac{y-X_i(s)}{|y-X_i(s)|^{\alpha+1}}\,d(\mu_N - f)(y)\,ds \\
&\quad + \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|y-X_i(t)|\leq r} \frac{y-X_i(s)}{|y-X_i(s)|^{\alpha+1}}\,d\mu_N(y)\,ds \\
&\quad - \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|y-X_i(t)|\leq r} \frac{y-X_i(s)}{|y-X_i(s)|^{\alpha+1}}\,df(y)\,ds,
\end{aligned}
$$

for all $r > 0$. The first term I_1 in the right hand side always goes to zero because μ_N converges weakly to f. The second term is dominated by $\|f\|_\infty \int_{B(0,R)} dy/|y|^\alpha$, a quantity which is less than $C\|f\|_\infty r^{d-\alpha}$. The last one is the field created by the close particles in the discrete case. To estimate it, we use estimate (2.2), which gives

$$
\begin{aligned}
I_3 \leq C\,(\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,K^{\alpha'}\,r^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\alpha} \\
+ \varepsilon^{2d-3\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\alpha}\overline{E}^d\,K^d) \leq C\,r^{\alpha'-\alpha}.
\end{aligned}
$$

And these bounds are independent of N or i.
Then, letting ε going to 0 and then r, we find that

$$
\sup_{i,t}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |F_N(X_i(s)) - F_\infty(X_i(s))|\,ds \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.1}
$$

With this strong convergence, we are able to prove the convergence of the term $F_N \cdot \nabla_v \mu_N$ towards $F_\infty \cdot \nabla_v f$ in the sense of distributions. We choose a smooth test function ϕ with compact support and compute

$$
\begin{aligned}
J = \int_0^T \Big(\int_{x,v} F_\infty(t,x)\cdot\nabla_v\phi(t,x,v) f(t,x,v)\,dxdv \\
- \sum_{i=1}^N F_N(t, X_i(t), V_i(t))\cdot\nabla_v\phi(t, X_i(t), V_i(t))\Big)\,dt.
\end{aligned} \tag{4.2}
$$

We separate J in $J_1 + J_2$, with

$$
J_1 = \int_0^T \int_{x,v} F_\infty(t,x)\cdot\nabla_v\phi(t,x,v) d(f-\mu_N)(.,x,v)\,dt,
$$

and

$$J_2 = \int_0^T \Big(\sum_{i=1}^N F_\infty(t, X_i(t), V_i(t)) - F_N(t, X_i(t), V_i(t)) \cdot \nabla_v \phi(t, X_i(t), V_i(t))\Big).\, dt$$

Because of the continuity of F_∞, J_1 vanishes as ε goes to zero. To show that J_2 vanishes as well, we decompose it in $M = [T/\varepsilon] + 1$ integrals on M intervals of time with length ε. The last interval is of length less than ε, but that does not create any difficulty and we do as if it were of length ε. We obtain,

$$\begin{aligned} J_2 = \sum_{k=1}^M \int_{k\varepsilon}^{(k+1)\varepsilon} &\Big(\sum_{i=1}^N \big(F_\infty(t, X_i(t), V_i(t)) - F_N(t, X_i(t), V_i(t))\big) \\ &\cdot \nabla_v \phi(t, X_i(t), V_i(t))\Big)\, dt \qquad (4.3) \\ \le C \sum_{k=1}^M \int_{k\varepsilon}^{(k+1)\varepsilon} &\Big(\sum_{i=1}^N \big|F_\infty(t, X_i(t), V_i(t)) - F_N(t, X_i(t), V_i(t))\big|\Big)\, dt. \end{aligned}$$

This sum may be bounded by

$$CT \sup_{i,t} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |F_N(X_i(s)) - F_\infty(X_i(s))|\, ds,$$

a quantity which goes to zero according to (4.1). Thus, J goes to zero when ε goes to zero and the proof is done. □

Appendix : Existence of strong solutions to Equation (1.3)

We mean by strong solution on a time interval $[0,\ T]$, a function $f \in L^\infty([0,\ T] \times \mathbb{R}^{2d})$ with compact support in space and velocity and which satisfies (1.3) in the sense of distributions.

Obtaining such solutions for any time was a major issue for the Vlasov-Poisson system (finally solved in [15], [20] and [18]) because from strong solutions it is easy to get uniqueness or classical solutions. However if the potential is not as singular (and it is the case here), the issue of strong solutions is relatively simple

Theorem 4.1. *Assume that* (1.4) *with* $\alpha < 1$. *Let* $f^0 \in L^\infty(\mathbb{R}^{2d})$ *with compact support and* $T > 0$. *Then there exists* $f \in L^\infty([0,\ T] \times \mathbb{R}^{2d})$ *with compact support, satisfying* (1.3) *in the sense of distribution.*

Corollary 4.1. *This solution is unique.*

The proof of the corollary is immediate as the theorem implies that $E(t,x) = F \star_x f$ is lipschitz thanks to (1.4).
The core of the proof of the theorem is the following estimate

Lemma 4.2. *Let* $f \in L^\infty([0,\ T],\ \mathbb{R}^{2d})$ *with compact support be a solution to* (1.3) *in the sense of distribution with* (1.4) *and* $\alpha < d-1$. *Then if we denote by* $R(t)$ *and* $K(t)$ *the size of the supports of* f *in space and velocity, they satisfy for a numerical constant* C

$$
\begin{aligned}
&R(t) \leq R(0) + \int_0^t K(s)\, ds,\\
&K(t) \leq K(0) + C\, \|f(t=0,.,.)\|_{L^\infty}^{\alpha/d}\, \|f(t=0,.,.)\|_{L^1}^{1-\alpha/d} \times \int_0^t (R(s))^\alpha\, ds.
\end{aligned}
$$

Proof of the lemma. Given the estimate on f, ρ also belongs to L^∞ with the bound

$$\|\rho(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq C\,(K(t))^d\, \|f(t,.,.)\|_{L^\infty(\mathbb{R}^{2d})}.$$

As we have (1.4) with $\alpha < d-1$, $E = F \star_x \rho$ is lipschitz. Therefore the solution to (1.3) is unique and is given by the characteristics. Namely, we define X and V the unique solutions to

$$
\begin{aligned}
&\partial_t X(t,s,x,v) = V(t,s,x,v), \quad \partial_t V(t,s,x,v) = E(t, X(t,s,x,v)),\\
&X(s,s,x,v) = x, \quad V(s,s,x,v) = v.
\end{aligned}
$$

The solution f is now given by

$$f(t,x,v) = f(0, X(0,t,x,v), V(0,t,x,v)),$$

with the consequence that

$$R(t) \leq R(0) + \int_0^t K(s)\, ds, \quad K(t) \leq K(0) + \int_0^t \|E(s,.)\|_{L^\infty}\, ds.$$

Then
$$\|E\|_{L^\infty} \leq \|\rho\|_{L^1}^{1-\alpha/d} \, \|\rho\|_{L^\infty}^{\alpha/d},$$
and it is enough to notice that the L^1 and L^∞ norms of f are preserved in this case. □

From Lemma 4.2, one may obtain very easily Theorem 4.1 with a standard approximation procedure. The only thing to check is that the estimates on the support are independent of the parameter of the approximation and this is ensured by Lemma 4.2 in the case $\alpha < 1$ thanks to Gronwall Lemma.

References

[1] J. Batt, *N-Particle approximation to the nonlinear Vlasov-Poisson system*, Preprint.

[2] F. Bouchut, *Smoothing effect for the non-linear Vlasov-Poisson-Fokker-Planck system*, J. Diff. Eq., 122, pp 225-238, 1995.

[3] J. Batt, G. Rein, *Global classical solutions of the periodic Vlasov-Poisson system in three dimensions*, C.R. Acad. Sci. Paris, 313, serie 1, pp. 411-416, 1991.

[4] W. Braun, K. Hepp,*The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles*, Comm. Math. Phys., 56, no. 2, pp. 101–113, 1977.

[5] C. Cercignani, R. Illner and M. Pulvirenti, The mathematical theory of dilute gases, Applied Mathematical Sciences, 106, *Springer-Verlag New-York*, 1994.

[6] R. L. Dobrušin, *Vlasov equations*, Funktsional. Anal. i Prilozhen., 13, pp 48–58, 1979.

[7] I. Gasser, P.E. Jabin, B. Perthame, *Regularity and propagation of moments in some nonlinear Vlasov systems*, Proc. Roy. Soc. Edinburgh Sect. A, 130, pp 1259–1273, 2000.

[8] R.T. Glassey, *The Cauchy problem in kinetic theory*, Philadelphia PA, SIAM, 1996.

[9] J. Goodman, T.Y. Hou and J. Lowengrub, *Convergence of the point vortex method for the 2-D Euler equations*, Comm. Pure Appl. Math., 43, pp 415–430, 1990.

[10] E. Horst, *On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I*, Math. Meth. Appl. Sci., 3, pp 229-248, 1981.

[11] E. Horst, *On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation II*, Math. Meth. Appl. Sci., 4, pp 19-32, 1982.

[12] R. Illner and M. Pulvirenti, *Global validity of the Boltzmann equation for two and three-dimensional rare gas in vacuum*, Comm. Math. Phys., 121, pp 143–146, 1989.

[13] P.E. Jabin, F. Otto, *Identification of the dilute regime in particle sedimentation*, to appear in Comm. Math. Phys.

[14] P.E. Jabin and B. Perthame, Notes on mathematical problems on the dynamics of dispersed particles interacting through a fluid, *Modelling in applied sciences*, 111–147, Model. Simul. Sci. Eng. Technol., *Birkhauser Boston*, 2000.

[15] P.L. Lions, B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson System*, Invent. Math., 105, pp. 415-430, 1991.

[16] H. Neunzert, J. Wick, Theoretische und numerische Ergebnisse zur nichtlinearen Vlasov-Gleichung. *Numerische Lsung nichtlinearer partieller Differential- und Integrodifferentialgleichungen (Tagung, Math. Forschungsinst., Oberwolfach, 1971)*, pp. 159–185. Lecture Notes in Math., Vol. 267, Springer, Berlin, 1972.

[17] B. Perthame, *Time decay, propagation of low moments and dispersive effects for kinetic equations*, Comm. P.D.E., 21, pp 659-686, 1996.

[18] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data*, J. Diff. Eq., 95, pp 281-303, 1992.

[19] M. Pulvirenti, C. Simeoni, *L^∞-estimates for the Vlasov-Poisson-Fokker-Planck equation*, Math. Methods Appl. Sci., 23, pp 923–935, 2000.

[20] J. Schaeffer, *Global existence of smooth solutions to the Vlasov-Poisoon system in three dimensions*, Comm. P.D.E., 16, pp 1313-1335, 1991.

[21] S. Schochet, *The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation*, Comm. P.D.E., 20, pp 1077-1104, 1995.

[22] S. Schochet, *The point-vortex method for periodic weak solutions of the 2-D Euler equations*, Comm. P. A. M., 49, pp 911-965, 1996.

[23] H. Spohn, Large scale dynamics of interacting particles, *Springer-Verlag Berlin*, 1991.

[24] H.D. Victory, jr., E.J. Allen, *The convergence theory of particle-in-cell methods for multidimensional Vlasov-Poisson systems*, SIAM J. Numer. Anal., 28, pp 1207–1241, 1991.

[25] S. Wollman, *On the approximation of the Vlasov-Poisson system by particles methods*, SIAM J. Numer. Anal., 37, pp 1369–1398, 2000.