WASSERSTEIN DISTANCES FOR VORTICES APPROXIMATION OF EULER-TYPE EQUATIONS

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Abstract

We establish the convergence of a vortex system towards equations similar to the 2D Euler equation in vorticity formulation. The only but important difference is that we use singular kernel of the type $x^\perp/|x|^{\alpha+1}$, with $\alpha < 1$, instead of the Biot-Savard kernel $x^\perp/|x|^2$. This article follows a previous work of Jabin and the author about the particles approximation of Vlasov equation in [13]. Here we study a different mean-field equation, simplify the proofs and weaken non physical initial conditions. The simplification are due to the introduction of the infinite Wasserstein distance. The results are obtained for $L^1 \cap L^\infty$ vorticities without any sign assumption, in the periodic setting, on the whole space and on the half space (with Neumann boundary conditions). A vortex-blob result is also given, that is valid for short times in the true vortex case.

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1 Introduction

1.1 The fluid and discrete dynamics

The well known Euler equation

$$\begin{cases} \partial_t u + u \cdot \nabla u = 0 \\ \text{div } u = 0 \end{cases}, \tag{1}$$

is long-past studied. There remains many open problems in dimension three (See [16] for more details), but the 2D case is better understood, mainly because of the existence in that case of a scalar vorticity $\omega = \text{rot}(u)$ with the help of which the Euler equation may be rewritten (on the domain Ω):

$$\begin{cases} \partial_t \omega + u(t,x) \cdot \nabla \omega = 0 \\ u(t,\cdot) = \int K(x,y)\omega(y)\,dy \\ \omega(0,\cdot) = \omega^0 \\ u.n = 0 \quad \text{on } \partial\Omega \end{cases}, \tag{2}$$

where Ω may be $\mathbb{R}^2$, an open subset of $\mathbb{R}^2$, or $\mathbb{T}^2$ (the torus of dimension 2). ω^0 the intial condition is given and the kernel K is the gradient of the Green kernel G on Ω, rotated by a angle $\pi/2$: $K(x,y) = \nabla G^\perp(x,y)$. For smooth solutions, that vorticity formulation is completely equivalent to the original equation (1), but in the low regularity setting, the equivalence is not so clear (See [6] or [22] for more details). In the sequel, we interest us only to solutions of the Euler equation written in vorticity formulation (2) (written VF in the rest of the article).

What is the value of K? In the case $\Omega = \mathbb{R}^2$, the kernel K_{BS} is given by the Biot-Savard law:

$$K_{BS}(x,y) = K_{BS}(x-y) = \frac{(x-y)^\perp}{2\pi|x-y|^2}.$$

In the case $\Omega = \mathbb{T}^2$, the kernel K_{per} is given by a periodisation of K_{BS} (writing $K_{per}(x,y) = K_{per}(x-y)$ with some misuse of notations):

$$K_{per}(z) = \frac{1}{2\pi} \sum_{(m,n)\in\mathbb{Z}^2} \frac{(z-(m,n))^\perp}{(z_1-m)^2+(z_2-n)^2} \ . \tag{3}$$

It is proven in [23] that this sum over $\mathbb{Z}^2$ converges (to see it, associate (m,n) to the points obtained by rotation by angle $\pi/2$, π and $-\pi/2$). Precisely, we may decompose the kernel in:

$$K_{per}(z) = \frac{z^\perp}{2\pi|z|^2} + f(z), \tag{4}$$

where $|z|$ denote the norm on $\mathbb{T}^d$, and f is C^∞. In the bounded domain case, $K_\Omega(x,y) = (2\pi|x-y|^2)^{-1}(x-y)^\perp + \gamma(x,y)$ where γ is regular as long as x and y are far away from the border. In the half-space case γ is explicit (see Section 3.3.4). In all these cases, we must precise the conventionnal value $K_{per}(0) = 0$ (no self-interaction) because we shall also deals with non regular vorticities.

The existence and uniqueness of strong solutions for (2) with bounded vorticity was obtained by Yudovich [26] in the sixties and improved more recently by the same author for vorticities in all L^p with very slowly increasing L^p-norms (as $\ln(\ln(p))$ for instance) [27]. The existence of weak solutions in the case of L^1 vorticity was shown by Di Perna and Majda [7]. This result was extended to positive H^{-1} vorticity by Delort in [6] (see also the simplified proof of Schochet [22]). This case is of great importance because it includes vortex sheets. Unfortunately, the uniqueness is lost in the low regularity setting (L^p and less). Di Perna and Madja have explained that concentration-cancellation phenomens can occur and have given explicit examples of solutions with vorticity lose. We may also mentioned that for the vortex sheets, which evolution is given by Birkhoff-Rott integro-differential equation (See [24]), a singularity appears in a finite time after which what happens is not well known.

The 2D Euler equation in VF is a mean-field equation, which means that the speed field depends on the vorticity over the whole space. But, contrarily to the Vlasov equation, which is also a a mean field equation but in kinetic setting, there is no underlying system of "physical" particles that it describes statistically. However, physicists have introduced an abstract microscopic system which may be described statistically by the Euler equation in VF: the system of vortices. It consists in writing the Euler equation for a vorticity sum of Dirac masses. First references to that system may be found in works of Birkhoff and we cite also a more recent but important paper by Onsäger which deals with the statistical properties of system of vortices [20]. Since the seventies, this system has been widely studied an used for numerical simulation. For a system of N vortices with positions $(X_i)_{1\le i\le N}$ and vorticities ω_i/N (in the sequel, we will assume $\omega_i \in [-C,C]$ to keep the total absolute vorticity bounded by some constant C), the dynamic of the vortices is described by the system of ODE written below:

$$\forall i \le N \qquad \left\{ \begin{array}{l} \dot X_i(t) = \frac{1}{N}\sum_{i\neq j} \omega_j K(X_i(t) - K_j(t)) \\ X_i(0) = X_i^0 \end{array} \right. . \tag{5}$$

As we already mentioned before, the empirical distribution $\omega_N(t)$ of the vortices:

$$\omega_N(t) = \frac{1}{N}\sum_{i=1}^N \omega_i \delta_{(X_i(t),V_i(t))} \ , \tag{6}$$

satisfies the Euler equation in the sense of distributions.

1.2 The question of convergence of vortex systems towards Euler equation in VF

Since $\omega_N(t)$ and $\omega(t)$ satisfy the same equation, we may ask what happens when the number of vortices became large and that their initial positions are choosen so that $\omega_N^0 \rightharpoonup \omega^0$ weakly? Does we get $\omega_N(t) \rightharpoonup \omega(t)$ for small time or for every time?

This question about the convergence of the empirical distribution of vortices towards solutions of the Euler equation has two mathematical interests. Theoritically, it will be a justification of the consistency of the vortices approximation introduced by physicists. Numerically, it may provide methods of calculation with small or even without truncature parameters.

Positive answers to that question have been already given. First, the well-known vortex-blob methods is known to converge towards solutions with bounded vorticity of the Euler equation in VF. Roughly speaking, it consists in the use of small rigid blobs of vorticity rather than true Dirac. The kernel is then regularized by convolution: $K_\varepsilon = K * \phi_\varepsilon$, where ϕ_ε is some approximation of the identity and $\varepsilon = \varepsilon(N) \to 0$ when N increases. See for instance the introduction of [14] for more details and references. We only mention that usually (with the noteworthy exceptions of the two works cited below) , it is requested that $\varepsilon(N) = N^{-q/d}$ with $q \in (0,1)$. A condition that ensures that $\varepsilon(N)$ is larger than the average distance $N^{-1/d}$ between a particle and its closest neighbour, and that implies that the regularisation is effectively seen by all the particles.

Moreover, in the case of the true vortex system (without regularization), two methods, one of Goodman, Hou and Lowengrub (see [11]) and the second by Schochet [23] provide positive answers to that question without the help of any regularisation. We will discuss these two methods in the section 4. There exists also a work of Liu and Xin [17], [18]) that we will not discuss here (it is similar to results of Schochet).

In this article, we will provide a third result of convergence of vortices system towards Euler equation in VF, valid for a kernel K (even if in dimension $d \geq 3$ equation (2) is no more related to the Euler equation (1) we will provide results in that setting) satisfying a C_α condition with $\alpha < d-1$:

$$(C_\alpha) \qquad \operatorname{div} K = 0\,, \quad \forall x, \quad |K(x)|,\ |x|\,|\nabla K(x)| \leq \frac{C}{|x|^\alpha} \tag{7}$$

Unfortunately, this result cannot be applied with the Biot-Savard periodic kernel in 2D that satisfy only a C_{d-1} conditions. This is of course not satisfactory, but we think that this result has nevertheless some interests. We will discuss them in the section 4.3. For such kernels, the convergence will hold for all time provided that another condition over the minimal distance between vortices at time 0 is fulfilled. We will provide estimates of convergence in terms of infinite Wasserstein distance. The precise initial conditions are introduced in the statement of the theorem and their relevance are discussed in the section 2.

The article is organized as follows: the next section is devoted to the statement of our results, and the next one to the proofs. In the last section, we discuss the already known methods and the interests of our new method. At the end, some appendices contain definitions and proofs of some results used in the proof or in the discussion.

2 Main results

We begin by two important remarks. First, in the sequel $(smthg)_1 \leq C(smthg)_2$ will mean that: "There exists a positive real C, depending only on α and not on N and the initial conditions, such that $(smthg)_1 \leq C(smthg)_2$." We shall use the same notation C for many different numerical constants depending only on α. Moreover, we will not provide result in dimension 1, so that in the sequel we will always implicitly assume that $d \geq 2$ (exception made of the Appendix (A.A.2)).

Before stating precisely our result, we need to introduce a condition and two quantities depending of the empirical distribution $\omega_N(t)$ (defined in (6)). The condition is a compatibility condition that will allow us to construct a transport map between two signed measures ω_1 and ω_2:

$$\omega_1^+(\Omega) = \omega_2^+(\Omega)\,, \quad \text{and} \quad \omega_1^-(\Omega) = \omega_2^-(\Omega)\,, \tag{8}$$

where $\omega^{\pm}$ denote respectively the positive and negative part of the signed measure ω. When ω_N and ω satisfy this condition, we may define:

$$\eta(t) = W_\infty(\mu_N(t), f(t)) \qquad\qquad \eta_m(t) = \inf_{i\neq j}(|X_i(t) - X_j(t)|)\,, \tag{9}$$

where W_∞ stands for the infinite Wassertein distance. That distance is usually defined for probability measure, but it may be also defined for signed measure with bounded absolute mass satisfying the compatibility condition (8) (See Appendix (A.A.1) and Definition 1). η_m is the distance between the two closest particles. The initial value of all these quantities will be denoted with a 0 subscript, i.e. $\mu_N(0) = \mu_N^0$, $\eta(0) = \eta^0$ and $\eta_m(0) = \eta_m^0$. With the notations $a^+ = \max(0, a)$ and $a^- = \max(0, -a)$, our results are the following:

Theorem 1. *Assume that the kernel K satisfy a (C_α) condition (See (7)) with $0 < \alpha < d-1$, that the vorticity $\omega \in L_t^\infty(L^\infty \cap L^1(\Omega))$ (where Ω may be the torus $\mathbb{T}^d$, or the whole space $\mathbb{R}^d$), and that for each N, the N initial positions and vorticities satisfy $\sum \omega_i^+ = \int \omega^+$, $\sum \omega_i^- = \int \omega^-$ and:*

$$\lim_{N\to\infty} \frac{(\eta^0)^d}{(\eta_m^0)^{(1+\alpha)}} = 0\,. \tag{10}$$

Then, there exists a numerical constant C such that for all $T > 0$, and for sufficiently large N:

$$\forall t \in [0, T], \quad \eta(t) \le \eta^0 e^{C\|\omega^0\|_\Omega t}, \tag{11}$$

where $\|\omega^0\|_\Omega = \|\omega^0\|_\infty$ if $\Omega = \mathbb{T}^d$ and $\|\omega^0\|_\Omega = \|\omega^0\|_\infty + \|\omega^0\|_1$ if $\Omega = \mathbb{R}^d$.

Remark 1. *The constant C in the previous theorem is independant of the time T. However, we cannot state the previous for all time T, because the rank N, from which it is true, depends on T. We need more particles to stay close from the smooth profile for a long time.*

Remark 2. *This theorem and the following are also valid in the half-space case, provided that the definition of η_m is modified in*

$$\eta_m = \min\big(\inf_{i\neq j}(|X_i - X_j|), 2\inf_i d(X_i, \partial\Omega)\big)\,. \tag{12}$$

We explain in section 3.3.4 the symetrization technics needed to obtain the results in that settings.

Remark 3. *The condition that the positive and negative parts of the discrete and continuous vorticity have the same total weight is technical. We need it to use Wasserstein distance between these two vorticity distributions. If this condition is not exactly satisfied, we may introduced a new continuous vorticity distribution ω' such that ω_N and ω' fulfill the compatibility condition (8) and use it as a pivot in our estimates. The distance between ω and ω' may be estimated using theorem 4 of section 3.1 and adding a small L^∞ perturbation in it. However, as it will be rather technical, we do not emphasize on that and will only provide results valid for the exact compatibility condition.*

The hypothesis on the initial distribution of vortices are thus that: $(\eta^0)^d(\eta_m^0)^{-(1+\alpha)}$ converges to zero. Is it a relevant assumption? In other words, if the positions of the vortices are independant random variables in $\mathbb{T}^d$, with law ω^0 (which is assume to be a probability measure for simplicity), does the limit goes to zero with probability one? In the Appendix (A.A.3), we show that in that case η_m is roughly speaking of order $N^{-2/d}$. In Appendix (A.A.2), we give heuristics arguments that shows that W_∞ may be of order smaller than $c\ln(N)/N^{1/d}$. If we assume it, η_m^0 is roughly speaking of order of $(\eta^0)^2$, and the condition became $(\eta^0)^{d-2(1+\alpha)} \to 0$. This is satisfied if

$$2(1+\alpha) < d\,. \tag{13}$$

That cannot be achieved in 2D. But this is true in 3D, if $\alpha < 1/2$. In 2D however, this result shows that we do not need a precise control on the distance η_m, which may be shorter than η^0 or the grid size in numerical simulations. This of course, except in the limit case α very close to $d-1$ (the most interesting one), in which the condition becomes "η and η_m are of the same order".

The previous theorem deals with the singular kernel, but if we wish to use vortex-blob methods (mentioned in the introduction, see [14] for more details), we can replace η_m by the parameter of regularisation η_b. We obtain the following result:

Theorem 2 (Vortex-blob result)**.** *Assume that our kernel satisfy*

$$|K(x)|, (|x|+\eta_b)|\nabla K(x)| \le \frac{C}{(|x|+\eta_b)^\alpha},$$

for some $\alpha < d-1$, and that the vorticity $\omega \in L^\infty_t(L^\infty \cap L^1(\Omega))$. Assume also that for each N, the N vortices satisfy $\sum \omega_i^\pm = \int \omega^\pm$ and:

$$\lim_{N\to\infty} \frac{(\eta^0)^d}{\eta_b^{(1+\alpha)}} = 0\,. \tag{14}$$

Then, there exist a numerical constant C depending only on α such that for all $T>0$, if N is sufficiently large:

$$\forall t \in [0,T], \quad \eta(t) \le \eta^0 e^{C\|\omega^0\|_\infty t}, \tag{15}$$

All the remark done for the Theorem 1 are still valid there. We also mention that our technic may provide a short time result for true-vortex blobs:

Theorem 3 (true Vortex-blob short time result)**.** *Assume that our kernel satisfy*

$$|K(x)|, (|x|+\eta_b)|\nabla K(x)| \le \frac{C}{(|x|+\eta_b)^{d-1}}\,,$$

and that the vorticity $\omega \in L^\infty_t(L^\infty \cap L^1)$. Assume also that for each N, ω_N and ω satisfy the compatibility condition (8) *and that there exists a $\gamma \in (0,1)$ such that:*

$$\lim_{N\to\infty} \frac{(\eta^0)^\gamma}{\eta_b} = 0\,. \tag{16}$$

Then, there exists a time $T^\star = C(1-\gamma)$ such that:

$$\forall t \in [0,T^\star], \quad \eta(t) \to 0 \text{ when } N \to +\infty, \tag{17}$$

Unfortunately, a regularization is needed in the true vortex case. The reason of that is explained at the end of the Section 3.2. Remark also that in the case of an approximation of a profil ω^0 by a sum of Dirac masses ω_N^0 disposed on a grid, the distance $W_\infty(\omega_N^0, \omega^0)$ is of order of the grid width h. And the condition of our theorem 3 (and of theorem 2 if α is close to $d-1$) is nothing more than the assumption $\lim_{N\to\infty} h^\gamma/\eta_b = 0$. We retrieve a basic condition for vortex blob methods, which are known to converge (in the general case, without the strong assumptions on the initial positions of vortices made in [11]) if the blob parameter is larger than the width of the grid. A condition that implies that the regularisation is effectively seen by all the particles.

We end this section by stating a important Proposition, only to emphasize that for the results staten above we need a control on the minimal distance between vortices. Physically it is not very relevant, but what may be interesting is that we are able to control the minimal distance at a scale smaller than η^0:

Proposition 1. *Under the same assumptions made in Theorem 1 and in the periodic setting, the quantities η and η_m, defined in* (9)*, satisfy the following system of differential inequalities:*

$$\begin{cases} \dfrac{d\eta}{dt} \le C\|\omega^0\|_\infty \eta(1+\eta^{d-1}\eta_m^{-\alpha}), \\ \dfrac{d\eta_m}{dt} \ge -C\|\omega^0\|_\infty \eta_m(1+\eta^d \eta_m^{-(1+\alpha)}). \end{cases} \tag{18}$$

3 Proof of the theorem

This section is devoted to the proof of the Theorem 1, 2 and 3, in the periodic, half-space and whole space setting. We begin by proving a result of stability of solutions with bounded vorticity in the periodic setting. We will not use it, but its proof will be a good introduction to the technics used in the rest of the section. After that, we perform in details the proof of Theorem 1 in the periodic setting. In the last subsection 3.3, we explain how to adapt the proof for Theorems 1, 2 and 3 and for the whole space and half space case.

3.1 Stability of bounded solution of the Euler equation

Here, we adapt for the Euler equation a theorem of G. Loeper [19] which give a stability estimate in terms of Wasserstein distance for solutions with bounded densities of the Vlasov-Poisson equation. That estimate implies existence, uniqueness and stability (for this distance) of L^∞-solutions of the Euler equation in VF with a given initial vorticity.

Theorem 4 (adapted from G. Loeper)**.** *Let $p \in [1, +\infty]$, and ω_1 and ω_2 be two $L^\infty(\mathbb{R} \times \mathbb{T}^d)$ solutions of the Euler equation in VF (2) on the Torus. We assume that initially these two measures satisfy the compatibility conditions (8). Then, that condition is satisfied for all time and the infinite norm is preserved. Moreover, if the kernel is the true Biot-Savard kernel, the following inequality bound the growth of the Wasserstein distance between the two solution $W_p(t) := W_p(\omega_1(t), \omega_2(t))$:*

$$\frac{d}{dt} W_p(t) \leq C \max\{\|\omega_1^0\|_\infty, \|\omega_2^0\|_\infty\} W_p(t) \max(1, -\ln(W_p(t))) \ . \tag{19}$$

If the kernel satifies a (C_α) condition with $\alpha < d-1$, we have the linear growth estimate:

$$\frac{d}{dt} W_p(t) \leq C \max\{\|\omega_1^0\|_\infty, \|\omega_2^0\|_\infty\} W_p(t) \ . \tag{20}$$

Remark 4. *In the whole space setting, the theorem is still true provided we assume that the two solutions belongs to $L^\infty_t(L^\infty \cap L^1)$, and replace all the occurrences of $\|\omega_i^0\|_\infty$ by $\|\omega_i^0\|_\infty + \|\omega_i^0\|_1$ for $i = 1, 2$. This may be seen with the arguments used in the section 3.3.3.*

Remark 5. *In the following proof, we will use existing results about the existence and uniqueness of bounded solution of the Euler equation in VF. We choose this presentation to go faster on this non central point. But, everything may be recovered in our proof using the standard procedure: We may replace K by a smooth approximation K_ε, then obtain our bound for the solutions of the approximated equation, and then pass to the limit and obtain existence, uniqueness and stability for the original equation. This may be an elegant way to retrieve previously known results.*

We shall prove that theorem only for $p = +\infty$. The other cases are already treated in [19] and may be handle similarly.

Proof. We begin by the proof in the case where K satisfy a (C_α) condition with $\alpha < d - 1$. As ω_1 and ω_2 satisfies the compatibility condition (8), we may choose a optimal transport map T^0 between ω_2^0 and ω_1^0 : $\omega_1^0 = T^0_\# \omega_2^0$ (See Appendix (A.A.1) for precise definition). We also introduce the flows Φ^1 and Φ^2 : $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ generated by the two vorticities, defined as solutions of:

$$\begin{cases} \frac{d}{dt}(\Phi^i(t,s,x)) = u^i(t,x) \ , \quad \forall s,t \in \mathbb{R} \\ \Phi^i(s,s,x) = x \ , \quad \forall s \in \mathbb{R} \end{cases} , \tag{21}$$

for $i = 1, 2$, where $u_i = K * \omega_i$ is the speed field created by the vorticity distribution ω_i. In fact, $\Phi^i(t,s,x)$ is the position at time t of the particles moving with speed u_i that was at x at time s. When the vorticity is $L^\infty \cap L^1$, a standard estimate shows that the gradient of the speed field u_i is Lipschitz if K satisfy a (C_α) condition ($\alpha < d - 1$), or has a modulus of continuity bounded by $-x\ln(x)$ if K is the Biot-Savard kernel or its periodisation. That ensures the existence and uniqueness of solutions of (21). The solutions ω_i of the Euler equation in VF are then given by the theory of caracteristics. Because K and then the u_i are divergence free, Φ preserves the Lebesgue measure, and we have:

$$\text{For } i = 1,2 \qquad \omega_i(t) = \Phi^i(t,0,\cdot)_\# \omega_i^0 = \omega_i^0(\Phi^i(0,t,\cdot)) \,, \tag{22}$$

If we translate T^0 along the two flows, we obtain a mapping T^t between $\omega_1(t)$ and $\omega_2(t)$ defined by:

$$T^t = \Phi^1(t,0,\cdot) \circ T^0 \circ \Phi^2(0,t,\cdot). \tag{23}$$

In fact,

$$\begin{aligned} T^t_{\#}\omega_2(t) &= [\Phi^1(t,0,\cdot)\circ T^0 \circ \Phi^2(0,t,\cdot)]_{\#}\omega_2(t) && (24)\\ &= \Phi^1(t,0,\cdot)\circ T^0_{\#}\omega_2^0 = \Phi^1(t,0,\cdot)_{\#}\omega_1^0 = \omega_1(t). && (25)\end{aligned}$$

As T^t is a transport map, we have the inequality:

$$W_\infty(t) := W_\infty(\omega_1(t),\omega_2(t)) \le \|T^t - Id\|_\infty .$$

Using the change of variable by $x \mapsto \Phi^2(0,t,x)$, we get:

$$\|T^t - Id\|_\infty = \|\Phi^1(t,0,\cdot)\circ T^0 - \Phi^2(t,0,\cdot)\|_\infty .$$

The expression inside the right hand side is derivable with respect to the time and at $t=0$:

$$\frac{d}{dt}(\Phi^1(t,0,x)\circ T^0 - \Phi^2(t,0,x))\Big|_{t=0} = u_1^0(T^0\,x) - u_2^0(x)$$

Putting all toghether, we get:

$$\frac{d}{dt}\|\Phi^1(0,t,\cdot)\circ T^0 - \Phi^2(0,t,\cdot)\|_\infty\Big|_{t=0} \le \|u_1^0\circ T^0 - u_2^0\|_\infty . \tag{26}$$

To get the estimate of the theorem, it remains to bound the term $\|u_1^0\circ T^0 - u_2^0\|_\infty$ by $C\,W_\infty(\omega_1^0,\omega_2^0)$. For this, observe that:

$$\begin{aligned} u_1(T^0\,x) - u_2(x) &= \int K(T^0\,x - y)\omega_1(y)\,dy - \int K(x-y)\omega_2(y)\,dy\ ,\\ &= \int (K(T^0\,x - T^0\,y) - K(x-y))\omega_2(y)\,dy\,, && (27)\end{aligned}$$

where we have used the fact that T^0 is a transport map between ω_2 and ω_1 to get the last line. To estimate the term under the integral, we use the following inequality true for a kernel K satisfying the C_α condition ($\alpha < d-1$):

$$|K(x) - K(y)| \le \frac{2|x-y|}{\min(|x|,|y|)^{\alpha+1}}. \tag{28}$$

This estimate is obtained by integration of ∇K on a path S of length less than $2|x-y|$ such that $S \subset \mathbb{R}^d - B(0,\min(|x|,|y|))$ (choose by instance the union of a segment and an arc). We use it in (27):

$$\begin{aligned} \|u_1\circ T^0 - u_2\|_\infty &\le \sup_x \int |K(T^0\,x - T^0\,y) - K(x-y)|\omega_2(y)\,dy && (29)\\ &\le 2\sup_x \int \frac{|T^0\,x - x| + |T^0\,y - y|}{\min(|T^0\,x - T^0\,y|,|x-y|)^{\alpha+1}}\omega_2(y)\,dy\\ &\le 4\|T^0 - Id\|_\infty \left[\sup_x\left(\int \frac{\omega_2(y)\,dy}{|x - T^0\,y|^{\alpha+1}}\right) + \sup_x\left(\int \frac{\omega_2(y)\,dy}{|x-y|^{\alpha+1}}\right)\right]\\ &\le 4\|T^0 - Id\|_\infty \left[\sup_x\left(\int \frac{\omega_1(y)\,dy}{|x-y|^{\alpha+1}}\right) + \|\omega_2\|_\infty\left(\int \frac{dy}{|x-y|^{\alpha+1}}\right)\right]\\ &\le C\|T^0 - Id\|_\infty \max(\|\omega_1\|_\infty,\|\omega_2\|_\infty) && (30)\end{aligned}$$

In the last line, we use the fact that $1/|x|^{\alpha+1}$ is integrable on $\mathbb{T}^d$ if $\alpha < d-1$. Thus, we get the inequality announced at time 0. To get it for every time, we apply the previous calculations to the solutions $s \mapsto \omega_i(t+s,\cdot)$ of the Euler equation in VF.

For the case $\alpha = d-1$, we need to adapt a little bit this argument. Before using the bound (28), we split the integral (29) in two parts :

$$\begin{aligned} &\sup_x \int |K(T^0\,x - T^0\,y) - K(x-y)|\omega_2(y)\,dy\\ &\qquad \le \sup_x \int_{|x-y|\ge 2\|T^0 - Id\|_\infty} \cdots \quad + \sup_x \int_{|x-y|\le 2\|T^0 - Id\|_\infty} \cdots && (31)\\ &\qquad \le \quad S_1 \qquad\qquad + \qquad S_2\end{aligned}$$

For the first term S_1, we use the bound (28) and get:

$$
\begin{aligned}
S_1 &\leq C\|T^0 - Id\|_\infty \sup_x \int_{|x-y|\geq 2\|T^0-Id\|_\infty} \frac{\omega_1(y)+\omega_2(y)}{|x-y|^d}\, dy && (32)\\
&\leq C\max(\|\omega_1\|_\infty, \|\omega_2\|_\infty)\|T^0 - Id\|_\infty \max(1, -\ln(\|T^0 - Id\|_\infty)) && (33)
\end{aligned}
$$

For the second we simply use $|K(x) - K(y)| \leq |K(x)| + |K(y)|$ and get

$$
\begin{aligned}
S_2 &\leq \sup_x \int_{|y-x|\leq 2\|T^0-Id\|_\infty} \left(\frac{\omega_1(y)}{|T^0\, x - y|^{d-1}} + \frac{\omega_2(y)}{|x-y|^{d-1}} \right) dy \\
&\leq C\|T^0 - Id\|_\infty \max(\|\omega_1\|_\infty, \|\omega_2\|_\infty)
\end{aligned} \qquad (34)
$$

Adding S_1 and S_2 we get the inequality announced at time $t = 0$, and can also get it for all time. □

3.2 Proof of Theorem 1 in the periodic setting

In the following, ω will denote a L^∞ solution of the Euler equation in VF with initial vorticity ω^0, and ω_N will denote the empirical distribution associated to a solution of the system of equations (5) for the vortices system. We will often use the quantity $\eta(t)$ and $\eta_m(t)$ defined in equation (9). Our aim is to repeat the estimate done in section 3.1, with $\omega_1 = \omega_N$ and $\omega_2 = \omega$. The problem is that ω_N is a sum of Dirac masses and that we used $\|\omega_1\|_\infty$ many times in the previous proof. We will hedge this difficulty thanks to the introduction of the minimal distance between particules.

Proof. First, as ω_N and ω satisfy the compatibility conditions, we may pick up a transport map T^0 between ω^0 and ω_N^0. As previously, we use the flows Φ and Φ_N created by these distributions of vorticity to obtain a transport map at any time t:

$$T^t = \Phi_N(t, 0, \cdot) \circ T^0 \circ \Phi(0, t, \cdot)\ . \qquad (35)$$

To make the definition of the discrete flow Φ_N precise, we remark that the vector-field u_N created by the particles system is defined everywhere as long as no collision occurs, and our lower bound on $\eta_m(t)$ will imply that colisions never occurs. Following exactly the above proof of Theorem 4, we obtain as in (26) (erasing many references to the time for the clearness of the calculations, by instance $T = T^0$):

$$\frac{d\eta}{dt} \leq C\, \|u_N \circ T - u\|_\infty\,, \qquad (36)$$

where u (resp. u_N) is the speed field of the continuous (resp. discrete) system. And as in (27) we can rewrite this:

$$\frac{d\eta}{dt} \leq C \sup_x \left| \int (K(T\,x - T\,y) - K(x-y))\omega(y)\, dy \right| . \qquad (37)$$

Now we split the integral in two terms: the first one denoted I_1 will be the integral over the subset $J_1 = \{y|\ |x-y| \geq 4\eta\}$ and the second one denoted I_2 the integral over $J_2 = R^d - J_1$. For the first term I_1, we will use the bound

$$
\begin{aligned}
|K(T\,x - T\,y) - K(x-y)| &\leq \frac{2(|x - T\,x| + |y - T\,y|)}{\min(|x-y|, |T\,x - T\,y|)^{1+\alpha}} && (38)\\
&\leq \frac{2^{2+\alpha}}{|x-y|^{1+\alpha}}\eta\,, && (39)
\end{aligned}
$$

where we used that$|T\,x - T\,y| \geq |x-y| - |x - T\,x| - |y - T\,y| \geq |x-y| - 2\eta \geq |x-y|/2$.

But in dimension $d \geq 2$, $1/|x|^{1+\alpha}$ is integrable on the torus since $\alpha < d-1$ and we finally get:

$$I_1 \leq C\|\omega\|_\infty \eta\,. \qquad (40)$$

For the second term I_2, we use a simpler bound for the term under the integral:

$$|K(T\,x - T\,y) - K(x-y)| \leq |K(T\,x - T\,y)| + |K(x-y)| \leq \frac{1}{\eta_m^\alpha} + \frac{1}{|x-y|^\alpha}, \qquad (41)$$

where we used the fact that there is no self-interaction (i.e. $K(0)=0$), a property useful when $T\,x = T\,y$ for $x \neq y$. With this bound, we get:

$$I_2 \leq \int_{|x-y|\leq 4\eta} \frac{\omega(y)\,dy}{|x-y|^\alpha} + \int_{|x-y|\leq 4\eta} \frac{\omega(y)}{\eta_m^\alpha}\,dy \leq C\|\omega\|_\infty \eta^{d-\alpha} + C\|\omega\|_\infty \eta^d \eta_m^{-\alpha} \tag{42}$$

Now, we shall use the inequality $\eta_m \leq 2\eta$, that is true because the middle point between a vortex and its closest neighboor has to be transported to one of the vortices. Some exceptions may be found in very particular situations where all middle points are outside the support of ω (by instance consider the case where ω is a very small enlargement of ω_N), but once the continuous measure ω is fixed this will be true for N large enough. Adding I_1 and I_2, we get:

$$\frac{d\eta}{dt} \leq C\|\omega\|_\infty \eta(1+\eta^{d-1}\eta_m^{-\alpha}), \tag{43}$$

an equality that is also true for every times t.

We obtained a bound on the growth of η depending linearly on η, with a additionnal term involving η_m. To obtain a closed system of inequations, we need a bound on the decrease of η_m. For this, we pick up two indices i and j so that $|X_i - X_j| = \eta_m$. Then,

$$\frac{d}{dt}|X_i - X_j| \geq -|u_N(X_i) - u_N(X_j)| \geq -\int |K(X_i - y) - K(X_j - y)| d\omega_N(y), \tag{44}$$

which can be written, using the transport on the continuous distribution:

$$\frac{d}{dt}|X_i - X_j| \geq -\int |K(X_i - T\,y) - K(X_j - T\,y)|\omega(y)\,dy \tag{45}$$

Again, we cut the integral in two, the first denoted by I_3 performed on $J_3 = \{y|\ |X_i - y| \geq 2\eta \text{ and } |X_j - y| \geq 2\eta\}$ and the second I_4 performed on the complementary $J_4 = \mathbb{R}^d - J_3$. To bound I_3, we use

$$|X_i - T\,y| \geq |X_i - y| - \eta \geq \frac{1}{2}|X_i - y|, \tag{46}$$

and the equivalent for $|X_j - T\,y|$. We get for the term under the integral:

$$|K(X_i - T\,y) - K(X_j - T\,y)| \geq -\frac{2^{2+\alpha}}{\min(|X_j - y|, |X_i - y|)^{\alpha+1}}|X_i - X_j|, \tag{47}$$

a term that is integrable because $\alpha < d-1$ and finally:

$$I_3 \geq -C\|\omega\|_\infty \eta_m. \tag{48}$$

For the term I_4, remark that $T\,y$ is necessarily one of the X_k so that we get:

$$|K(X_i - T\,y) - K(X_j - T\,y)| \geq -\frac{1}{|X_j - T\,y|^\alpha} - \frac{1}{|X_i - T\,y|^\alpha} \geq -\frac{2}{\eta_m^\alpha}. \tag{49}$$

Moreover the Lebesgue measure of I_4 is bounded by: $\lambda(L) \leq C\eta^d$. It follows that:

$$I_4 \geq -C\|\omega\|_\infty \eta^d \eta_m^{-\alpha}\ , \tag{50}$$

and we obtain:

$$\frac{d\eta_m}{dt} \geq -C\|\omega\|_\infty \eta_m(1+\eta^d\eta_m^{-(1+\alpha)}) \tag{51}$$

That inequatily and the previous one obtained on η give the following system of differential inequalities stated in the Proposition 1:

$$\begin{cases} \dfrac{d\eta}{dt} \leq C\|\omega\|_\infty \eta(1+\eta^{d-1}\eta_m^{-\alpha}), \\ \dfrac{d\eta_m}{dt} \geq -C\|\omega\|_\infty \eta_m(1+\eta^d\eta_m^{-(1+\alpha)}) \end{cases} \tag{52}$$

We give now an heuristic argument explaining why this implies the conclusion of Theorem 1. The rigourous argument follows. Remark that if we delete the both products $\eta^{d-1}\eta_m^{-\alpha}$ and $\eta^d{\eta_m}^{-(1+\alpha)}$ in the left hand sides, the system is linear. In that case, η has exponential growth and η_m has exponential decay. If by instance η^0 is order $\varepsilon = N^{-1/d}$, $\eta(t)$ will remain of the same order $C(t)\varepsilon$ until every finite time, with a constant $C(t)$ depending only of the time, and $\eta_m(t) \geq C(t)^{-1}\eta_m^0$ for the same reason. Now, we take again into account the two neglected terms, but assume that they are initially very small. In that case, the system (52) is still linear (with the constant C replaced by $2C$), till the two products are smaller than 1 . But these products cannot become too large because their are bounded by a (time dependant) constant times their initial values. This closes the loop. Let us make this argument precise. We define

$$a(t) = \frac{\eta(t)}{\eta^0} \ , \qquad b(t) = \frac{\eta_m(t)}{\eta_m^0} \ , \tag{53}$$

together with the parameter $\nu = (\eta^0)^d(\eta_m^0)^{-(1+\alpha)}$, which depends only on the initial positions of the vortices (and then on the number of particles N). With these notations, the system becomes:

$$\begin{cases} \dfrac{da}{dt} \leq C\|\omega\|_\infty a(1 + \nu a^d b^{-(1+\alpha)}) \\ \dfrac{db}{dt} \geq -C\|\omega\|_\infty b(1 + \nu a^d b^{-(1+\alpha)}) \end{cases} , \tag{54}$$

where we have used another time the inequality $\eta_m \leq 2\eta$ to replace for clarity the term $\eta^{d-1}\eta_m^{-\alpha}$ in the first equation of (52) by $2\eta^d{\eta_m}^{-(1+\alpha)}$.

As long as $\nu a^d b^{-(1+\alpha)} \leq 1$, we have: $a \leq e^{2C\|\omega\|_\infty t}$ and $b \geq e^{-2C\|\omega\|_\infty t}$. And that give $\nu a^d b^{-(1+\alpha)} \leq \nu e^{(d+1+\alpha)2C\|\omega\|_\infty t}$. This ensure that $\nu a^d b^{-(1+\alpha)} \leq 1$ till the time $T^\star = -\ln(\nu)/(C\|\omega\|_\infty)$. Then, until this time:

$$\forall t \leq T^\star \ , \quad \eta(t) \leq \eta^0 e^{2C\|\omega\|_\infty t}, \text{ and } \quad \eta_m(t) \geq \eta_m^0 e^{-2C\|\omega\|_\infty t}. \tag{55}$$

Because $\lim_{N\to+\infty} T^\star = \lim -\ln(\nu) = +\infty$, this implies the weak convergence of $\omega_N(t)$ towards $\omega(t)$ for every time t, and gives the estimates on the rate of convergence announced in Theorem 1.

□

3.3 Adaptations for Theorems 2, 3, the whole space and half-space case.

3.3.1 The vortex-blob case: Theorem 2

For the vortex-blob method, it is possible to check directly that the previous calculations are still true when η_m is replaced by η_b. And there is no need to prove a bound by below on the decrease of η_b. We only get the following inequality:

$$\frac{d\eta}{dt} \leq C\|\omega\|_\infty \eta(1 + \eta^{d-1}{\eta_b}^{-\alpha}). \tag{56}$$

And this implies a bound on $\eta \leq e^{2C\|\omega\|_\infty t}\eta^0$, as long as $\eta^{d-1}{\eta_b}^{-\alpha} \leq 1$. We can conclude as above because one of the hypothesis in Theorem 2 is that this product converges initially to zero.

3.3.2 Short time result in the case $\alpha = d - 1$: Theorem 3

If we look carefully at the calculations done in the section 3.2, and apply the technics used in the section 3.1 for the case $\alpha = d - 1$ (see (34)) , we see that we can still get a system of differential inequalities satisfied by η and η_m. There are only two corrections to do. The first one is to replace the estimate (40) of I_1 by

$$I_1 \leq C\|\omega\|_\infty \eta(1 + |\ln(\eta)|), \tag{57}$$

and the second is to replace the estimate (48) of I_3 by:

$$I_3 \geq -C\|\omega\|_\infty \eta_m(1 + |\ln(\eta)|). \tag{58}$$

There is no correction to perform on the bound on I_2 and I_4. So that the final system of inequalities is:

$$\left\{\begin{array}{l} \dfrac{d\eta}{dt} \leq C\|\omega\|_\infty \eta \left(1 + |\ln(\eta)| + \left(\frac{\eta}{\eta_m}\right)^{d-1}\right), \\ \dfrac{d\eta_m}{dt} \geq -C\|\omega\|_\infty \eta_m \left(1 + |\ln(\eta)| + \left(\frac{\eta}{\eta_m}\right)^{d-1}\right) \end{array}\right. \tag{59}$$

Without the term in η/η_m and with the 1 in the parenthesis replaced by 2 (in other words as long as $\eta/\eta_m \leq 1$), we obtain two estimates: $\eta(t) \leq e^{e^{-2Ct}\ln(\eta^0)}$ and $\eta_m(t) \geq \eta_m^0 e^{-2C(1+\ln(\eta^0))t}$, as long as $\eta(t)$ is small with respect to one. But these estimates only give

$$\left(\frac{\eta}{\eta_m}\right)^d \leq \frac{\nu}{(\eta^0)^{Ct}}. \tag{60}$$

This will allow to control the linear term as long as $\ln(\nu) \leq C\,t\,\ln(\eta_0)$. But the inequality $\eta_m \leq 2\eta$ implies that $\nu = \eta^0/\eta_m^0 \geq 2$ and the previous conditions implies $\ln(2) \leq C\,t\,\ln(\eta_0)$. This is valid only if $t = 0$ since η^0 goes to 0 when $N \to +\infty$. So our technics will not provide any result for the true vortex case. The $x\ln(x)$ estimates on the speed field u do not allow us to preserve the minimal distance η_m.

However, in the vortex-blobs case, we have $\nu = \eta^0/\eta_b$. If the aproximation parameter η_b is choosen so that $\lim_{N\to+\infty}(\eta^0)^\gamma(\eta_b)^{-1} = 0$ for some $\gamma < 1$, which is the case under the hypothesis of theorem 5, then $\nu \leq C(\eta^0)^{1-\gamma}$ and a new suffisant condition is $(1-\gamma)\ln(\eta^0) \leq C\,t\,\ln(\eta_0)$. This is fulfilled provided $t \leq (1-\gamma)/C = T^\star$.

3.3.3 Adaptation to the whole space

Here we briefly explain how to adapt our proof in the whole space case. Indeed, we may use the same technics that in the proof of Section 3.2, replacing all the integrals performed on $\mathbb{T}^d$ by integrals performed on $\mathbb{R}^d$. Precisely, we have to estimate quantities like $\int_{\mathbb{R}^d} |x-y|^{-\beta}\omega(y)\,dy$, for $\beta = \alpha - 1, \alpha$ in the inequalities (40) and (48). But if we cut these integrals as follow, we get:

$$\begin{aligned} \int_{\mathbb{R}^d} |x-y|^\beta \omega(y)\,dy &\leq \int_{|x-y|\leq 1} \cdots + \int_{|x-y|\geq 1} \cdots \\ &\leq C\|\omega\|_\infty + C\|\omega\|_1, \end{aligned}$$

instead of a bound by $C\|\omega\|_\infty$. Using that technic, the only necessary adaptation is to replace all the occurences of $\|\omega\|_\infty$ by $\omega\|_\infty + \|\omega\|_1$. This works even in the adaptation to the true vortex case of these inequalities, because the additionnal splitting of the integral performed in that case is local.

3.3.4 Adaptation to the half-space

Here we explain how our results may be obtained on the half-space. We limit ourself to that particular domain, because in that case the Green function is explicitly known. It seems reasonnable to think that ours results remain true for more general regular domains, but proofs maybe much more difficult without explicit formulas for K.

On the half-space $(0, +\infty) \times \mathbb{R}^{d-1}$, we begin by a presentation in the true vortex case. The Green kernel of the half-space is know to be equal to:

$$G_+(x,y) = \left\{ \begin{array}{l} -\frac{1}{4\pi}(\ln(|x-y|) - \ln(|x-y'|)) \text{ if } d = 2 \\ \frac{c_d}{|x-y|^{d-2}} - \frac{c_d}{|x-y'|^{d-2}} \text{ if } d \geq 2 \end{array}\right. , \tag{61}$$

where the prime ' denote the symetrisation with respect to the hyperplane $\{x_1 = 0\}$. In terms of Hamiltonians, the Hamiltonian $\mathcal{H}_N$ on the whole space is equal to (with the notation $X^N = (X_1, \ldots, X_N)$):

$$\mathcal{H}_N(X^N) = \frac{1}{N^2}\sum_{i\neq j} \omega_i\omega_j G(X_i - X_j), \tag{62}$$

whereas the Hamiltonian of the half-space system is equal to

$$\mathcal{H}_N^{hs}(X^N) = \frac{1}{N^2}\sum_{i\neq j}\omega_i\omega_j G_+(X_i - X_j) + \frac{1}{N}\sum_i \omega_i^2\gamma(X_i)\,. \tag{63}$$

The last term comes from the interaction of a vortex with the wall and is given by

$$\gamma(x) = \begin{cases} -\frac{1}{4\pi}\ln(|x-x'|) \text{ if } d=2 \\ \frac{c_d}{|x-x'|^{d-2}} \text{ if } d\geq 2 \end{cases}, \tag{64}$$

exactly as if a single vortex interacts with its symetric. This new term γ will give rise to a supplementary term $\omega_i\nabla^\perp\gamma(X_i)$ in the ODE determining the speed of the vortex X_i. In the case of kernel satisfying only a (C_α) condition ($\alpha < d-1$), the dynamics on the half-space case may be defined similarly provided that there exists a function $\tilde{G}$ such that $\nabla^\perp\tilde{G} = K$ and that $K(x') = -K(x)'$. The last condition is satisfied by the Biot-Savard kernel (and also if $\tilde{G}$ is radial) and will be used just below. In that case, the Hamiltonian of the dynamics may be defined by (63), with G_+ replaced by $\tilde{G}(x-y) - \tilde{G}(x-y')$, and γ by $\tilde{G}(x-x')$.

In view of that, it is natural to introduce the N symetric vortices. So we define a new system of $2N$ vortices $(Y_1,\dots,Y_{2N})$ with vorticities μ_i and initial conditions defined by:

$$\text{for } i\leq N\,,\quad Y_i^0 = X_i^0\ ,\ \mu_i = 2\omega_i \quad \text{and} \quad Y_{i+N}^0 = (X_i^0)'\,,\ \mu_{i+N} = -2\omega_i. \tag{65}$$

The factor two is there only because the vorticities are now divided by $2N$, the number of vortices in the definition of the Hamiltonian. Now, these vortices will evolve according to the whole space system (5). This system will remains symetric ($Y_i(t)' = Y_{i+N}(t)'$) over time thanks to the choice of opposite vorticity for the symetric vortices (technically, this comes from the condition $K(x') = -K(x)'$). Then, we have

$$\mathcal{H}_{2N}(Y^{2N}) = \mathcal{H}_N^{hs}(X^N) + \mathcal{H}_N^{hs}(\bar{X}^N) \tag{66}$$

where $\bar{X}^N = (X_{N+1},\dots,X_{2N}) = (X_1',\dots,X_N')$. This implies that the evolution of the $(X_i)_{i\leq N}$ will be the one of the $(Y_i)_{i\leq N}$.

Similarly, we defined a symetrisation of ω^0 by: $\mu^0 = \omega^0 - (\omega^0)'$, where $(\omega^0)'$ is the measure ω^0 transported by the application $x\mapsto x'$. We denote by $\mu(t)$ the L^∞ solution of the Euler equation in VF on the whole space with initial condition μ^0. As for the vortex system, the measure $\omega(t)$ solution of the Euler Eq. in VF on the half-space will exactly be the restriction to the right half-space of the measure $\mu(t)$. Next, we may applied our result to the system of the vortices (Y_i) and the continuous vorticity $\mu(t)$, provided that the condition on η^0 and η_m^0 are fulfilled for the $(Y_i)_{i\leq 2N}$ and μ^0. But for the symetrized system we have $(\eta^0)' = 2\eta^0$, and

$$(\eta_m^0)' = \min(\inf_{i\neq j}|X_i - X_j|, 2\inf_i d(X_i,\{x_1 = 0\}))\,. \tag{67}$$

Eventually, the result of convergence applies to the half space case, provided the definition of η_m is replaced by the one just above (67).

4 Previously known results and interest of this new one

Here in a first section we briefly present and discuss the two previous methods of approximation of the original Euler equation in VF with vortex systems without regularization. Then, we explain the interests of our method and the novelties introduced with respect to our previous work with Jabin [13].

4.1 Strong convergence for vortices distributed on a grid

The first method was originally introduced in 1990 by J. Goodman , T. Hou and J. Lowengrub [11] and continued in [15], [14] and some more. We cite one of their most important theorem, valid for the 2D Euler equation:

Theorem 5 (Goodman, Hou, Lowengrub). *Choose an initial vorticity $\omega^0 \in C^2$ with bounded support. We choose a small positive real h and put a vortex at every vortices of the mesh $h\mathbb{Z}^2$. The strengh of the vortex initially at (ih, jh) is fixed at $\int_{D_{(i,j)}} \omega^0(x)\,dx$, where $D_{(i,j)}$ is the square of side-length h centered at (ih, jh). Then, the following estimates hold:*

$$\begin{cases} \|X_{i,j}(t) - \tilde{X_{i,j}}(t)\|_p \leq C(T)h^2 \\ \|u_N(X_{i,j}(t)) - u(\tilde{X_{i,j}}(t))\|_p \leq C(T)h^2 \end{cases} \tag{68}$$

where the $X_{i,j}(t)$ are the positions of the vortices at time t, u is the speed field created by the unique continuous solution ω of the Euler equation in VF with initial condition ω^0, the $\tilde{X}_i(t)$ are the positions of the mesh points transported by u, and $\|\cdot\|_p$ is a discrete equivalent of the usual L^p norm: $\|X_i\|_p = (\sum_{i,j} h^2 X_i^p)^{1/p}$.

This result has two great qualities: it provides strong estimates of convergence, and do not request any sign assumption on the initial vorticity. But it has two restrictions: it apllies only for sufficiently smooth initial conditions, and the positions and the initial vorticities are fixed once the mesh is choosen. From a numerical point of view, the second restriction is not a true one, but on the statistical point of view, the set of initial positions considered here is of zero measure and thus cannot provide a statistical interpretation if the smooth profil is choosen. Unfortunately, that strong initial assumption cannot be removed, because the proof rely on the symetry of the kernel K, a property that can be used only with vortices disposed in a "symmetric" way. Remark also that this approximation scheme was used by J.-G. Liu and Z. Xin [18] for H^{-1} positive initial vorticity. They showed (as S. Schochet did previously) that in that a case subsequence of the scheme converge towards weak solution of the Euler equation, but they of course lose the precise estimates of convergence of Theorem 5.

4.2 Weak convergence in bounded energy case

The second method was develop by Schochet in [23], using his simplification [22] of the proof by Delort [6] of existence of weak solutions to the Euler equation in VF with initial positive H^{-1} vorticity. Remark that results obtained in that setting cannot be used for Dirac masses, which are not in H^{-1} if $d = 2$ (even if it is almost true). Nevertheless, they can be used in the limits, again thanks to a clever symetrization of the formulation in the sense of distribution of Euler equation in VF. Before stating an important result of Schochet, we introduce the Hamiltonian of a positive measure of vorticity on the torus (that may be also applied to ω_N the empirical distribution of N vortices defined in (6)):

$$\mathcal{H}_N(\omega) = \int\int \left[\ln\left(\frac{1}{|x-y|}\right) + g(x-y) \right] d\omega(x) d\omega(y)\,,$$

where g is a smooth correction due to periodisation, and $\ln(1/|x|) = 0$ when $x = 0$ (There is no self-interaction for vortices).

Theorem 6 (S. Schochet). *Assume that initial positions of the vortices are choosen so that $\sup_N \mathcal{H}(\omega_N^0) < +\infty$ and that $\omega_N^0 \rightharpoonup \omega^0$ when $N \to +\infty$. Then, $\omega^0 \in H^{-1}$, and up to the extraction of a subsequence:*

$$\forall t \in \mathbb{R}, \quad \omega_N(t) \rightharpoonup \omega(t), \tag{69}$$

where ω is a weak solution of the Euler equation in VF with initial condition ω^0.

Statisticaly, this theorem may be used to obtain the corollary:

Corollary 1. *Assume that $\omega^0 \in L^\infty(\mathbb{T}^d)$ and that the initial positions of the vortex are choosen randomly and independantly according to the profil ω^0. Then, almost surely with respect to the probability $\otimes_{n=1}^\infty \omega^0$, and up to the extraction of a subsequence:*

$$\forall t \in \mathbb{R}, \quad \omega_N(t) \rightharpoonup \omega(t),$$

a H^{-1} weak solution of the Euler equation in VF with initial condition ω^0.

This corollary would be very interessant statistically, if we could say that the limiting solution ω would be in each case the unique L^∞ solution $\bar{\omega}$ of the Euler equation in VF with initial conditions ω_0. Unfortunately, we only know that our limiting solution is a priori H^{-1}. That do not allow us to conclude that $\omega = \bar{\omega}$.

This corollary is already proved in the article [23] where Schochet provides a lot of interesting large deviations results valid even for $L \log L$ initial vorticities. However, as the bounded setting is much simpler, we will give here a simple proof of this corollary.

Proof of the Corollary. Thanks to theorem 6 we only need to prove that the Hamiltonian is bounded with probability one in the limit, when the law on the initial position is $\otimes_{n=1}^{\infty}\omega^0$ (by this, we mean that for any N, the law of the initial positions is $\otimes_{n=1}^{N}\omega^0$). For this, we will use calculations similar to those used by Caglioti, Lions, Marchioro and Pulvirenti for the statistics of vortex in [3] and [4] to show that some exponential moments of $N\mathcal{H}_N$ increase at most exponentially. In others words, that for $\lambda > 0$ sufficiently small, there exists a numerical constant $L_0(\lambda)$ such that

$$\mathbb{E}(e^{\lambda N \mathcal{H}_N}) \leq e^{L_0 N}\,. \tag{70}$$

If it is true, we can use Markov inequality to get, for any $L > 0$:

$$\mathbb{P}(\mathcal{H}_N \geq L) \leq e^{-\lambda L N}\mathbb{E}(e^{\lambda N \mathcal{H}_N}) \leq e^{(L_0 - \lambda L)N}. \tag{71}$$

and we conclude by a classical Borel-Cantelli argument with a L sufficiently large. Thus, we need to show (70). For this, we compute (we use the notation $X^N = (X_1, \ldots, X_N)$)

$$\begin{aligned}
\mathbb{E}(e^{\lambda N \mathcal{H}_N}) &= \int e^{-\frac{\lambda}{N}\sum_{i\neq j}^{N} \ln(|X_i - X_j|)}(\omega^0)^{\otimes N}(X^N)\, dX^N && (72)\\
&\leq \|\omega^0\|_\infty^N \int \prod_{1\leq i\neq j\leq N} |X_i - X_j|^{-\frac{\lambda}{N}}\, dX^N && (73)\\
&\leq \|\omega^0\|_\infty^N \prod_{i=1}^{N}\left(\int \prod_{j\neq i} |X_i - X_j|^{-\lambda}\, dX^N\right)^{\frac{1}{N}}, && (74)
\end{aligned}$$

where in the last step we use an Hölder inequality for N functions. Under the integral in the last term, each X_i appears only one time, and we can use N times Fubini and integrate one variable after another. When $\lambda < d$ we have:

$$\int_{\mathbb{T}^d} |x|^{-\lambda} dx \leq \frac{C}{d-\lambda}, \tag{75}$$

for a numerical constant C, and we get

$$\mathbb{E}(e^{\lambda N \mathcal{H}_N}) \leq \|\omega^0\|_\infty^N \prod_{i=1}^{N}\left(\frac{C}{d-\lambda}\right)^{\frac{N-1}{N}} \leq \left(\|\omega^0\|\frac{C}{d-\lambda}\right)^N. \tag{76}$$

This is the result we need with with $L_0 = \ln\left(C\|\omega\|_\infty(d-\lambda)^{-1}\right)$. □

4.3 Interests and novelties of our technics.

We begin by listing some interest of the results of this article:

- It is the first time, at our knowledge, that Wasserstein distance are used for particles system with singular interaction. These distances have already proved to be useful in the case of regular interaction (See [9]), and for regular solution of mean-field equation with singular interaction [19]. Here, we show that the infinite Wasserstein distance can be used for systems of particles with singular interaction.
- We do not reach the good singularity, so that this result is in a certain sense weaker than the previously known results (discussed above) about the convergence of system of vortices. But, on the other hand, it is the only one that may be used to show statistical convergence of systems of vortices with singular interaction, against L^∞ solutions of the Euler equation in VF.

- Lastly, as it does not use the symetry of the kernel, the argument may be adapt, without overwhelming difficulties, to the case of systems of particles in interaction, converging towards the Vlasov equation (which possess a different symetry), contrarily to the technics of Goodman, Hou, Lowengrub [11], and Schochet [23]. It will be the aim of a future work [12], in which will provide more realistic initial conditions that in our previous work with Jabin [13].

Now, we explain what are the interests of the technics used in the proof, and what are the improvements with respect the previous work [13]. In that previous article, we studied convergence of system of particles in interaction towards the Vlasov equation. The convergence was shown for singular potential satifying a (C_α)-condition, with $\alpha < 1$, using estimates on discrete infinite norms on the empirical distribution μ_N of particles in position and space. Precisely, we estimate quantities defined by

$$\|\mu_N\|_{\infty,\varepsilon} = \sup_{z\in\mathbb{R}^{2d}} \frac{\mu_N(\prod[z_i-\varepsilon, z_i+\varepsilon])}{(2\varepsilon)^d}\,, \tag{77}$$

where $z=(x,v)$ and the z_i are the $2d$ components of z. ε is a scale parameter, choosen greater than $N^{-1/2d}$, the average distance (in position-speed space) between a particle and its closest neighbour. To estimates these quantities, we had to perform tedious estimates, which where all translations in the discrete setting of estimates simple to obtain in the Vlasov theory (continuous setting). By instance, tedious calculations provided estimates on the maximal speed of the discrete system, on a discrete analog of the gradient of the speed-field , quantities that are simply estimated in the continuous case. Here, the introduction of infinite Wasserstein distance allows to simplify these technics. Thanks to that distance, we do not perform any calculations on the discrete system and use that distance to compare the discrete value to the continuous one. Calculations are similar, but in one step we get all that previously requires estimates at different scales (a think that complicated a lot our previous work), and we get as a bonus precise estimates of convergence (it was not possible previously). These new estimates are also stronger than our previous ones because of the inequality

$$\|\omega_N\|_{\infty,\varepsilon} \leq \|\omega\|_\infty \left(1+\frac{W_\infty(\omega_N,\omega)}{\varepsilon}\right)^{2d}\,, \tag{78}$$

that relates discrete infinite norm to the infinite Wasserstein distance and the infinite norm of ω. Basically, for scale larger than the distance W_∞ you can obtain estimates of discrete norms (converse estimates are not so obvious). And additionnaly, the simplification in the calculations allows us to handle less restrictive initial conditions, taking η_m smaller by some order than η. The only inconvenient is now that the assumption on the initial infinite Wasserstein distance is stronger than the previous one on infinite discrete norms. But, as we do not know the expectation value of the W_∞ distance for large N, we do not know if it is a strong assumption or not. However, in a work in preparation [12], we will combine the technics used in this article with density kernel estimates in infinite norm (as obtained for instance by [10]) to weaken that assumption of convergence in W^∞ norm to an assumption of convergence in W^1 norm, that is known to be statistically reasonnable [8].

A Appendices

A.1 About the infinite Wasserstein distance

The Wasserstein distances are widely used in many domains of PDEs and especially for transport equations. As we deals with discrete approximations and singular potential, we will not use the classical Wasserstein distances of order one or two, but rather an infinite version of the Wasserstein distance. As there is not so many references on that topic (except the work of Champion, De Pascale and Juutinen [5]), we recall here its definition, after some preliminaries.

A transference plane between two probability measures μ and ν on $\mathbb{T}^d$ is a measure π on $\mathbb{T}^{2d}$ such that its first projection $p_1(\pi)=\mu$ and its second $p_2(\pi)=\nu$. The set of all transference planes between μ and ν is denoted by $\Pi(\mu,\nu)$:

$$\Pi(\mu,\nu) := \{\pi \in \mathcal{M}(\mathbb{T}^{2d}) |\ p_1(\pi)=\mu,\ p_2(\pi)=\nu\}\,. \tag{79}$$

Definition 1. *Let μ and ν be two measures on $\mathbb{T}^d$. Then, the infinite Wasserstein distance between μ and ν is:*

$$W_\infty(\mu,\nu) = \inf\{G_\infty(\pi) := \pi - \mathrm{esssup}_{(\mathrm{x},\mathrm{y})\in\mathbb{T}^{2d}}|\mathrm{x}-\mathrm{y}| \mid \pi \in \Pi(\mu,\nu)\} \tag{80}$$

The existence of an optimal transference plane, i.e. a transference plane γ such that $\gamma - \mathrm{esssup}_{(\mathrm{x},\mathrm{y})\in\mathbb{R}^2}|\mathrm{x}-\mathrm{y}| = \mathrm{W}_\infty(\mu,\nu)$, may be obtained by compactness. In [5], the authors prove the stronger result stated below:

Theorem 7 (Champion, De Pascale, Juutinen)**.** *Assume that, ν is continuous with respect to the Lebesgue measure, then there exist optimal transference plans, and at least one of them is given by a transport map, i.e. is of the form $(Id \otimes T_\infty)_{\#}\mu$, where T_∞ satisfies $(T_\infty)_{\#}\mu = \nu$ or*

$$\int f(T_\infty(x))\,d\mu(x) = \int f(y)\,d\nu(y), \qquad \forall f \;\; \nu - \textit{integrable}\ . \tag{81}$$

If moreover μ is a finite sum of Dirac masses, this optimal transport map is unique.

Technically, this theorem is not essential for our work, the existence of an optimal transference plan suffices, but the demonstration are simpler to write and to follow with the use of transport maps.

In our article, we used infinite Wasserstein distances and optimal transports for signed measures that satisfy the compatibility condition (8). For such measures ω_1 and ω_2, the infinite Wasserstein distance is defined by:

$$W_\infty(\omega_1,\omega_2) = W_\infty(\omega_1^+,\omega_2^+) + W_\infty(\omega_1^-,\omega_2^-) \tag{82}$$

and an optimal transport may be constructed gluing together and optimal transport T^+ of ω_2^+ onto ω_1^+ and a optimal transport T^- of ω_2^- onto ω_1^-.

We conclude this section by mentioning another result, obtained by Ambrosio ands Pratelli [1] which state (roughly speaking) that among all the optimal transports in term of W_1 distance, between two absolutely continuous measures, one is also optimal for all the distance W_p ($p < +\infty$). That transport is also optimal for W_∞, and that provide another way to find optimal tranport for W_∞.

A.2 Heuristic remarks about statistical estimates for η

In the mathematical litterature, there are some works dealing with the asymptotics of the Wasserstein distance W_1, which show that it scales like $N^{-1/d}$ [8], or that it satisfy a (very) large deviations principle [2]. Recently, R. Peyre states it in a very efficient way [21] (but in a unfortunately unpublished paper), and provides moderate deviations for that distance. We cite his result in the following proposition:

Proposition 2 (Peyre)**.** *Assume that μ is a probability measure on $\mathbb{T}^d$ (a similar result will be true for a probability with a bounded support on $\mathbb{R}^d$). Denote also $\mu_N = N^{-1}\sum_{i=1}^N \delta_{X_i}$ the empirical distribution of the $(X_i)_{1\le i\le N}$, independant random variables of law μ. Then there exists an increasing non-negative function Φ, independant of μ, such that for all $\lambda \ge 0$:*

$$\begin{cases} \textit{if } d \neq 2, & E_\beta[W_1(\mu_N,\mu)] \le \Phi\left(\frac{\beta}{\sqrt{N}}\right)\frac{1}{N^{1/\max(2,d)}}, \\ \textit{if } d = 2, & E_\beta[W_1(\mu_N,\mu)] \le \Phi\left(\frac{\beta}{\sqrt{N}}\right)\frac{\ln N}{\sqrt{N}}, \end{cases} \tag{83}$$

where $E_\beta[Y] = \beta^{-1}\ln(E[e^{\beta Y}])$ if $\beta > 0$ and $E_0[Y] = E[Y]$.

This result bounds the expectation of W_1 in the limit and give a good mean deviations result. For instance, if $d = 2$, it implies the existence of a constant c such that for all $L > 0$:

$$\mathbb{P}\left(W_1(\mu_N,\mu) \ge \frac{L\ln(N)}{N^{1/d}}\right) \le \frac{1}{N^{L-c}}. \tag{84}$$

Unfortunately, there is no similar result for W_∞ for all d (at least at our knowledge). Moreover, it is impossible to obtain a result of this type for W_∞ valid for general measure μ (for instance if the support of μ has two disjoint components separated by some distance). However, we may hope that the expectation of the infinite Wasserstein distance may be not much larger that the expectation of W_1 on $\mathbb{T}^2$ if the measure μ

satisfies $\mu \geq d_\mu \lambda > 0$. This because in the case of convex cost fonctions, the optimal transference plan selects many small deplacements rather than a large one (For the W_1 this "degenerate" in the sense that many small deplacements are equivalent to a large one, but the many deplacement option is still optimal). Of course, W_∞ is not given by a convex cost function (but may be seen as the limit of the W_p which are given by convex cost functions), but it also prefers the many deplacement option. More precisely, the result of Ambrosio and Pratelli already mentioned in Appendix (A.A.1) [1] shows that W_1 and W_∞ will be respectively the L^1-norm and the L^∞-norm of the same random function, and we may expect that the expectation of these two norms will not differ from many orders.

To illustrate the previous argument, we state below a result valid in dimension 1 for probabilities bounded by above, that show that W_∞ scale at most like $\ln(N)/\sqrt{N}$ in this case. The 1D case is more simple because the optimal transference plan is explicitely known (See below). Unfortunately we were not able to prove a result valid for $d \geq 2$, the cases that interests us, but this result may explain that the conjecture made above is reasonable.

Proposition 3. *There exists a purely deterministic function $\tilde{\gamma}$ such that for every $\mu \in \mathcal{P}(\mathbb{T})$ (the set of probability measure on $\mathbb{T}$), such that μ is bounded by below: $0 < d_\mu \lambda \leq \mu$ (where λ is the Lebesgue measure on the torus), we have:*

$$E_\beta(W_\infty(\mu_N, \mu)) \leq \frac{\ln(N) + 2}{2\beta} + \tilde{\gamma}\left(\frac{\beta}{d_\mu \sqrt{N}}\right) \frac{1}{d_\mu \sqrt{N}} \tag{85}$$

This result may provide mean deviations results exactly as the result of Proposition 2. Before begining the proof, we cite one exponential control lemma, and a corolary for binomial law which will be very useful in the following. Proof of these results may be found in classical books of probability theory.

Lemma 2. *There exists a purely deterministic function $\gamma : \mathbb{R}^+ \to (0, +\infty)$ such that for every real random variable with zero mean satisfying for some σ:*

$$\forall \beta \in \mathbb{R}, \quad \ln \mathbb{E}[e^{\beta X}] \leq \frac{\beta^2 \sigma^2}{2}, \tag{86}$$

we have:

$$\forall \beta \geq 0, \quad E[|X|] \leq \gamma(2\beta\sigma) 2\sigma \tag{87}$$

Corollary 3. *Assume $N \geq 1$ and $p \in [0, 1]$. Then for a random variable X with binomial law $\mathcal{B}(N, p)$:*

$$E_\beta(|\frac{X}{N} - p|) \leq \gamma(\beta/\sqrt{N}) \frac{1}{\sqrt{N}} \tag{88}$$

We have now stated all the results necessary for the proof of the proposition.

Proof. We will made a wide use of the cumulative distribution functions F (resp. F_N) associated to μ (resp. μ_N) defined on $[0, 1]$ by:

$$F(x) = \mu([0, x]) \tag{89}$$

Recall that for two probability measures μ and ν of cumulative distribution functions F and G:

$$W_1(\mu, \mu_N) = \int_0^1 |F^{-1}(t) - G^{-1}(t)|\, dt = \int_0^1 |F(x) - G(x)|\, dx, \tag{90}$$

and that similarly

$$W_\infty(\mu, \mu_N) = \|F^{-1} - G^{-1}\|_\infty. \tag{91}$$

where F^{-1} (and similarly G^{-1}) refers to the generalized inverse of F defined by $F^{-1}(t) = \inf\{x \in \mathbb{R}; F(x) > t\}$. The equality involving W_∞ is a consequence of the fact that the optimal transport of μ on ν is given by the monotone rearrangement $T = G^{-1} \circ F$ and that $\|T - Id\|_\infty = \|F^{-1} - G^{-1}\|_\infty$ (the monotone rearrangement selects small displacements and is optimal for all the distance W_p, $p \in [1 + \infty]$). We refer to the Chapter 2 of Villani's very clear book [25] for more details on that subject.

Now, we choose $m \leq 2(\sqrt{N} + 1)$ points $x_1 = 0 \leq \cdots \leq x_i \leq \cdots \leq x_m = 1$, such that $F(x_{i+1}) - F(x_i) \leq N^{-1/2}$. It is possible if there is no atoms a such that $\mu(a) > N^{-1/2}$. If there exists an atom a like that, there is a small adaptation to do. We shall consider also the ensemble $[0, a)$. We write with some misuse of notation $F(a^-) = \mu([0, a))$ and put the "point" a^- and a in the sequence $(x_i)_{i\leq m}$. We may have $F(a) - F(a^-) \geq N^{-1/2}$, but it will not raise any difficulties in the sequel since the "interval" $[a^-, a] = \{a\}$ is of length 0 and will not contribute in the transport. Remark that we need at most $2(\sqrt{N} + 1)$ to do that, the worst cases being measures concentrated on $(1 - \varepsilon)\sqrt{N}$ atoms (with ε small), with mass larger than $1/\sqrt{N}$, which requires $2(1 - \varepsilon)(\sqrt{N} + 1)$ points (a^- and a for each atom).

In the sequel we note $S_N = \sup_{i\leq m} |F_N(x_i) - F(x_i)|$. Now, for any $x \in [0, 1]$, there exist $i \leq m - 1$ such that $x_i \leq x \leq x_{i+1}$. If $x_i = a^-$, with a one of the atom such that $\mu(a) \geq N^{-1/2}$, then $x_{i+1} = a$ and $x = a$. Thus $|F_N(x) - F(x)| \leq S_N$ because a and a^- are in the sequence $(x_i)_{i\leq m}$. Next, we assume that x_i is not one of a^-. Then, $F(x_{i+1}) - F(x_i) \leq N^{-1/2}$ and:

$$F_N(x) \leq F_N(x_{i+1}) \leq F(x_{i+1}) + S_N \quad \leq \quad F(x) + S_N + \frac{1}{\sqrt{N}}, \tag{92}$$

$$F_N(x) \geq F_N(x_i) \geq F(x_i) - S_N \quad \geq \quad F(x) - S_N - \frac{1}{\sqrt{N}}, \tag{93}$$

so that:

$$\|F_N - F\|_\infty \leq S_N + N^{-1/2}. \tag{94}$$

Now we will use the hypothesis that μ is bounded from zero. Remark that the graph of F_N is included in the following set, where $r = \|F_N - F\|_\infty$:

$$V_r = \{(x, y) \in [0, 1]^2\,; F(x) - r \leq y \leq F(x) + r\}. \tag{95}$$

But our hypothesis $\mu \geq d_\mu\lambda$ implies $F' \geq d_\mu$ and then vertical enlarging V_r of the graph of F is included in a horizontal enlarging of this graph (the atoms do not raise any difficulties for that):

$$H_{r/d_\mu} = \{(x, y) \in [0, 1]^2\,; F^{-1}(y) - r\, d_\mu^{-1} \leq x \leq F^{-1}(y) + r\, d_\mu^{-1}\}\,. \tag{96}$$

This implies that:

$$\|F^{-1} - F_N^{-1}\|_\infty \leq \|F_N - F\|_\infty / d_\mu. \tag{97}$$

Putting all together, we obtain:

$$W_\infty(\mu_N, \mu) \leq \frac{S_N + N^{-1/2}}{d_\mu}\ . \tag{98}$$

So that

$$\begin{aligned}
E_\beta(W_\infty) &\leq \frac{1}{d_\mu} E_{\beta/d_\mu}(S_N + N^{-1/2}) && (99)\\
&\leq \frac{1}{d_\mu\sqrt{N}} + \frac{1}{d_\mu} E_{\beta/d_\mu}(\sup_{i\leq m} |F_N(x_i) - F(x_i)|) && (100)\\
&\leq \frac{1}{d_\mu\sqrt{N}} + \frac{1}{\beta} \ln\left(\mathbb{E}(\sum_{i\leq m} e^{\beta|F_N(x_i) - F(x_i)|/d_\mu})\right) && (101)\\
&\leq \frac{1}{d_\mu\sqrt{N}} + \frac{1}{\beta} \ln(m e^{\beta\gamma(\cdot)/d_\mu\sqrt{N}}) && (102)\\
&\leq \frac{1}{d_\mu\sqrt{N}} \left(1 + \gamma(\beta/(d_\mu\sqrt{N}))\right) + \frac{\ln(N) + 2}{2\beta}\ . && (103)
\end{aligned}$$

This is the inequality announced if we define $\tilde{\gamma} = 1 + \gamma$. □

A.3 Statistical estimates for η_m

For the distance between the two closest vortices η_m^0, heuristic arguments show that η_m^0 will be of order $N^{-2/d}$ with great probability. We made it precise in the following proposition:

Proposition 4. *Assume that $\mu \in \mathbb{P}(\mathbb{T}^d)$, and that $(X_1,\dots,X_N)$ are independant random variables in $\mathbb{T}^d$ with law μ. Let η_m denote the distance between the two closest of these N particles: $\eta_m = \inf_{i\neq j}|X_i - X_j|$, and denote by c_d the volume of the unit ball in dimension d. Then, if μ is bounded below by some constant, $\mu \geq d_\mu\lambda > 0$, we have:*

$$P\left(\eta_m \geq \frac{L}{N^{\frac{2}{d}}}\right) \leq e^{-\frac{c_d d_\mu L^d}{2^{d+2}}}. \tag{104}$$

If μ is bounded by above $\mu \leq D_\mu \lambda$, then for all L such that $0 < L^d \leq N/(2c_d D_\mu)$:

$$e^{-D_\mu c_d L^d} \leq P\left(\eta_m \geq \frac{L}{N^{\frac{2}{d}}}\right). \tag{105}$$

These two inequalities imply for a measure bounded above and below that

$$\lim_{L\to 0}\lim_{N\to+\infty} P\left(L \leq N^{2/d}\eta_m \leq \frac{1}{L}\right) = 1. \tag{106}$$

and show that $N^{2/d}\eta_m$ "concentrate" on compact set of $\mathbb{R}^{+*}$. We give a proof of this Proposition after a short discussion about the boundness hypothesis.

Remark 6. *The hypothesis on the bound by below for the measure could be removed, because the lower bound is essentially a local problem. If the measure μ satisfies the condition $\mu \geq d_\mu\lambda > 0$ only on a small open subset of $\mathbb{T}^d$, then the first inequality is still be true (with a smaller constant in front of L^d in the exponential). However, the bound by above on μ in the second inequality is essential. Here, the important quantity is $J(s) = \sup\{\mu(A) | A \subset \mathbb{T}^d, |A| = s\}$. If $\mu \in L^\infty$, then $J(s) = O(s)$ near 0 and the result stated is true. But if $\mu \in L^2$, then $J(s) = O(\sqrt{s})$ near 0 and we will obtain similar estimates but with $N^{-3/d}$ in place of $N^{-2/d}$.*

Proof. We will need an estimate by above of the probability $P(\eta_m \geq LN^{-2/d})$. For this, remark that given $(X_1,\dots,X_n)$ such that $\eta_m \geq LN^{-2/d}$, the balls of center X_i and radius $(L/2)N^{-2/d}$ are disjoint. Thus, when the first k balls are choosen, the point X_{k+1} must be choosen in $F_k = \mathbb{T}^d - \cup_{i\leq k} B(X_i,(L/2)N^{-2/d})$. But, since all these balls are disjoints and μ is bounded by below, $\mu(F_k) \leq 1 - c_d d_\mu k(L/2)^d N^{-2}$. Then, using the independance of the X_i, we obtain:

$$\begin{aligned}
P\left(\eta_m \geq \frac{L}{N^{\frac{2}{d}}}\right) &\leq \prod_{k=1}^{n-1}\left(1 - \frac{kd_\mu c_d L^d}{2^d N^2}\right) && (107)\\
\ln\left(P\left(\eta_m \geq \frac{L}{N^{\frac{2}{d}}}\right)\right) &\leq \sum_{i=1}^{n-1}\ln\left(1 - \frac{kc_d d_\mu L^d}{2^d N^2}\right) && (108)\\
\dots &\leq -\sum_{i=1}^{n-1}\frac{kc_d d_\mu L^d}{2^d N^2} && (109)\\
\dots &\leq -\frac{c_d d_\mu L^d}{2^{d+2}}. && (110)
\end{aligned}$$

Which give the first estimate announced in the proposition.

For the second one, we will need the upper bound on μ. The raisoning will be similar to the precedent. First, remark that if for all $k \in [1,N]$, $X_k \in G_k = \mathbb{T}^d - \cup_{i\leq k-1} B(X_i, LN^{-2/d})$, then $\eta_m \geq LN^{-2/d}$. Moreover, the sets G_k satify $\mu(G_k) \geq 1 - c_d D_\mu k L^d N^{-2}$. Then, using the independance of the X_i and the fact that $-2x \leq \ln(1-x)$ if $x \in [0, 1/2]$, we get:

$$
\begin{aligned}
P\left(\eta_m \geq \frac{L}{N^{\frac{2}{d}}}\right) &\geq \prod_{k=1}^{n-1}\left(1 - \frac{kc_d D_\mu L^d}{N^2}\right) && (111)\\
\ln\left(P\left(\eta_m \geq \frac{L}{N^{\frac{2}{d}}}\right)\right) &\geq \sum_{i=1}^{n-1} \ln\left(1 - \frac{kc_d D_\mu L^d}{N^2}\right) && (112)\\
\dots &\geq -\sum_{i=1}^{n-1} \frac{2kc_d D_\mu L^d}{N^2} && (113)\\
\dots &\geq -c_d D_\mu L^d, && (114)
\end{aligned}
$$

where we have used the fact that $L^d \leq N/(2c_d D_\mu)$. □

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