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ÉQUATIONS DE LIOUVILLE, LIMITES EN GRAND NOMBRE DE PARTICULES.

THÈSE

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Résumé. Cette thèse est consacrée aux équations différentielles ordinaires (EDO) et aux équations de transport associées, pour des champs de vecteurs peu réguliers, et contient quatre travaux.
Le premier traite la résolution des EDO et des équations de transport pour des champs de vecteurs dans $L^2(\mathbb{R}^2)$ à divergence nulle, vérifiant une condition de régularité sur la direction du champ. Les résultats sont obtenus dans le cadre de la théorie développée par R. DiPerna et P.L. Lions pour la résolution des équations de transport à coefficients $W^{1,1}$. On abaisse les conditions de régularité nécessaires dans le cas de la dimension deux.
Le second travail concerne l'équation de Liouville, qui gouverne le comportement d'une densité de N particules en interaction, dans le cadre de champs peu réguliers. Les résultats de R. DiPerna et P.-L. Lions, déjà étendus au cas cinétique par F. Bouchut, sont adaptés pour permettre de prendre en compte une singularité à l'origine.
Dans le troisième travail, nous nous intéressons à la convergence des systèmes de particules en interaction vers l'équation de Vlasov. La convergence est obtenue grâce à des estimations discrètes précises, dans le cas de forces d'interactions en $1/|x|^\alpha$, pour $\alpha < 1$. Cela améliore le résultat connu précedemment pour des forces C^1.
Le quatrième utilise le même type de technique pour l'approximation d'Euler par des vortex. On y prouve la convergence pour tout temps quand l'interaction est a peine moins singulières que pour les vortex. On donne aussi des bornes uniformes sur le champ et son acrroissement dans le cas des vrais vortex.

Abstract. This thesis is devoted to the study of ordinary differential equations and associated transport equations with low regularity vector fields. It contains four works.
The first deals with the resolution of the ODEs and associated transport equations for divergence free vector field in $L^2(\mathbb{R}^2)$, satisfying a regularity condition on the direction of the field. The results obtained extend those of P. L. Lions and R. DiPerna for vector fields of low regularity in any dimension in the particular case of the dimension two.
The second work concerns the equation governing the behaviour of a density of N interacting particles, known as the Liouville equation. The results of P.-L. Lions and R. DiPerna, already extended to the kinetic case by F. Bouchut are adapted here to allow to treat a singularity at the origin.
The third one is devoted to the convergence of interacting particles system towards the Vlasov equation. The convergence is obtained by careful discrete estimates for interaction forces in $1/|x|^\alpha$, for $\alpha < 1$.The fourth one use the same technics for the approximation of the Euler equation by simili vortex systems allowing the use of different signs.

Chapter 1

Introduction

1.1 Généralités.

On sait résoudre depuis Cauchy les équations différentielles ordinaires (EDO) dans le cas où le champ est localement Lipschitzien. Rappelons que par EDO, on entend l'équation suivante:

$$Y'(t) = b(t, Y) \tag{1.1}$$

où le temps t varie dans un intervalle de $\mathbb{R}$. Le résultat plus précis est le suivant:

Théorème 1.1 (Théorème de Cauchy-Lipschitz). *Soit $b : I \times O \to \mathbb{R}^n$ un champ de vecteurs localement Lipschitzien en espace (O est un ouvert de $\mathbb{R}^n$). Soit t^0 un temps dans I et x^0 une position dans O. Alors il existe une unique solution maximale définie sur un intervalle $J \subset I$ à l'équation (1.1) vérifiant la condition $x(t^0) = x^0$.*

Remark Un champ localement Lipschitzien b sur un sous-ensemble de $\mathbb{R}^n$ étant un champ vérifiant la propriété suivante:
Pour tout compact K de $\mathbb{R}^n$, il existe une constante C_K telle que $\forall x,\, y \in K,\ |b(x) - b(y)| \leq C_K |x - y|$.
Ce théorème est connu depuis longtemps (voir par exemple les Élements d'analyse de J. Dieudonné [Die68]). Il a été amélioré pour des champs non plus Lipschitzien, mais satisfaisant localement la condition suivante

$$|b(t,x) - b(t,y)| \leq C|x-y| \ln\left(\frac{1}{|x-y|}\right)$$

(même référence [Die68]) . Peut-on encore abaisser les hypothèses uniformes sur l'accroissement de b? En fait, on sait que si le champ est continu, il y a toujours une solution au problème de Cauchy. Mais celle-ci n'est plus forcement unique si le champ n'est que continu ou Hölderien. Donc il semble difficile d'aller plus loin avec des bornes uniformes sur l'accroissement de b. Mais, alors que peut-on dire si on sait que le champ a sa dérivée dans un espace L^p? Dans ce cas, le champ n'est plus défini partout, mais presque partout, et on ne peut donc pas s'attendre à obtenir une solution pour toute condition initiale, mais plutôt pour presque toute condition initiale. On introduit donc la notion de flot pour pouvoir parler plus globalement de solutions de l'EDO. Dans le cas régulier, elle est la suivante:

Définition 1.1. *Le flot associé à l'ODE* (1.1) *est la fonction X définie sur un sous-ensemble de $\mathbb{R} \times \mathbb{R} \times O$ dans O telle que*

i. $\forall t, x \quad X(t,t,x) = x$

ii. $X(\cdot, s, x)$ *est solution de l'EDO,*

L'ensemble de définition est l'union $\cup_{x^0 \in O, t^0 \in \mathbb{R}} (t^-(x^0,t^0), t^+(x^0,t^0)) \times \{(x^0,t^0)\}$, *où* (t^-, t^+) *est l'intervalle maximal sur lequel est définie la solution vérifiant* $x(t^0) = x^0$. *Si une particule située en x au temps t_0 suit le flot, elle sera en* $X(t, t_0, x_0)$ *au temps t.*

Ce flot permet de résoudre facilement l'équation de transport associée à un champ de vecteurs.

1.1.1 Équation de transport associée

À un champ de vecteur b, on associe l'équation suivante appelée *équation de transport*,

$$\partial_t f + b(t,x) \cdot \nabla f = 0$$

On cherche à la résoudre sur $\mathbb{R} \times O$ avec une condition initiale $f(0,x) = f^0(x)$ pour tout $x \in O$.
Pour la résoudre dans le cas régulier, il suffit de remarquer que dans l'équation, est juste écrit que la solution est constante le long des trajectoires solutions de l'EDO. En un point de l'espace temps (t,x), il faut remonter la trajectoire passant par ce point jusqu'au temps $t = 0$ pour avoir la valeur de f. La solution avec conditions initiales f^0 est donc donnée par

$$f(t,x) = f^0(X(0,t,x))$$

1.1.2 Résolution des EDO pour des champs $W^{1,p}$

Les techniques qui vont être évoquées ici ont été introduite en 1989 par Ronald DiPerna et Pierre-Louis Lions (voir [DL89]), puis développées par les mêmes, François Bouchut, Nicolas Lerner, Ferrucio Colombini, Benoît Desjardins et Luigi Ambrosio.
Ici, nous parlerons des cas où nous pouvons démontrer des résultats globaux en temps sur les équations. Les cas où les solutions tendent vers l'infini en un temps fini sont plus difficiles à traiter. Nous introduirons les notions nécessaires pour des champs définis sur le tore. Toute la théorie reste valable sur l'espace tout entier, ou sur un ouvert borné suffisamment régulier. Dans le premier cas, il faut introduire des conditions sur la croissance du champ à l'infini, pour justement ne pas avoir d'explosion des solutions en temps fini pour certaines conditions initiales. Dans le second, il faut imposer un champ tangent sur le bord.
Sur le tore $\mathbb{T}^n$, muni d'un flot $b(t,x)$ dans $L^1_t(W^{1,p})$ on définit la notion de flot solution de l'EDO (1.1).

Définition 1.2. *Un flot presque partout solution de l'équation* (1.1) *est une application X de $\mathbb{R} \times \mathbb{T}^n$ dans $\mathbb{T}^n$ satisfaisant les propriétés suivantes:*

i. $X \in C(\mathbb{R}, L^1(\mathbb{T}^n)) \cap L^\infty_{loc}(\mathbb{R} \times \mathbb{T}^n)$

ii. $\forall \phi \in C_0^\infty, \quad \forall t \in \mathbb{R}, \quad \int \phi(X(t,x))\,dx = \int \phi(x)\,dx$

iii. $\forall s,t \in \mathbb{R}$ $X(t+s,x) = X(t, X(s,x))$ *presque partout en* x.

iv. $\dot{X} = b(t,X)$ *est satisfait au sens des distributions, et* $X(0,x) = x$ *p.p. sur* $\mathbb{T}^n$. *Plus précisément,pour tout* $\phi \in C_0^\infty([0,\infty) \times \mathbb{T}^n, \mathbb{R})^n$,

$$\int_{[0,\infty)\times\mathbb{T}^n} X(t,x)\cdot\left(\frac{\partial\phi}{\partial t} - b(t,X(t,x)\phi)\right)\,dt\,dx = \int_{\mathbb{T}^n} X(0,x)\cdot\phi(0,x)\,dx$$

R. DiPerna et P.-L. Lions ont démontré dans [DL89] le théorème suivant.

Théorème 1.2 (DiPerna - Lions). *Si b appartient à $L^1(\mathbb{R}, W^{1,1}(\mathbb{T}^n))$, alors il existe un unique flot presque partout solution de (1.1)*

Schéma de la démonstration. Contrairement au cas lipschitzien, on ne connaît pas de méthode de résolution directe de l'EDO. Il suffit d'abord de résoudre l'équation de transport associée, et montrer que les solutions vérifient la propriété clé énoncée ci-dessous:
Propriété de renormalisation. Une solution f de l'équation de transport (1.1.1) est dite renormalisée si pour tout $\beta \in C^1_b(\mathbb{R},\mathbb{R})$, $\beta(f)$ est aussi solution de (1.1.1) avec pour condition initiale $\beta(f_0)$.

Cette propriété est une condition suffisante pour l'existence d'un flot. Plus exactement, si on note (R) la propriété suivante:

(R) Toute solution de l'équation (1.1.1) est une solution renormalisée.

On a la proposition suivante:

Proposition 1.1 (P.-L. Lions). *Soit b un champ de vecteur satisfaisant (R). Alors il existe un unique flot presque partout solution de (1.1).*

Il reste donc à vérifier cette condition pour les champs $W^{1,1}$. Cette vérification repose principalement sur le lemme suivant:

Lemme 1.1. *Soit f une fonction dans L^∞ et b un champ de vecteur dans $W^{1,1}$, vérifiant aussi $div(b) \in L^\infty$. Soit ρ_ε une approximation de l'identité. Alors*

$$\rho_\varepsilon * div(f\,b) - div(f\rho_\varepsilon * b) \to 0 \quad dans\ L^1 \quad quand\ \varepsilon \to 0$$

On pourra trouver une démonstration dans l'article de R. DiPerna et P.-L. Lions [DL89].

Remarque 1.1. *Par approximation de l'identité on entend une suite ρ_n de fonctions positives a support compact sur $\mathbb{R}^n$ telle que $\int_{\mathbb{R}^n} \rho_n = 1$ satisfaisse* $\sup |x||x \in supp(\rho_n)$ *tends vers 0 quand n tends vers $+\infty$.*

Pour la démonstration, voir l'article de R. Diperna et Lions [DL89] ou celui de F. Bouchut [Bou01].
Pour un champ de vecteurs régulier, la propriété de renormalisation est évidente, vu que les solutions sont données par $f^0(X(t,x))$. Ce lemme permet, quand on écrit qu'une solution de l'équation de transport associée au champ approchée $\rho_\varepsilon * b$ est renormalisée, de passer à la limite et d'obtenir la propriété de renormalisation pour la solution limite avec le champ initial. □

Ce résultat a été amélioré depuis. Pierre-Louis Lions a étendu ce résultat à une classe plus large de champs de vecteurs dans une note aux *Comptes Rendus de l'Académie des Sciences* (c.f. [Lio98]). Il y définit des champs de vecteurs $W^{1,1}$ par morceaux et montre que le résultat reste valable pour ces champs. L'autre résultat de cette note est de montrer que la propriété (R) est générique dans L^1, c'est-à-dire qu'elle est vérifié sur une intersection d'ouverts denses de L^1.
Plus recemment, Francois Bouchut dans a traité dans [Bou01] le cas particulier le cas d'une équation de transport cinétique. Celles-ci sont de la forme:

$$\partial_t f + v \cdot \nabla_x f + E(t,x) \cdot \nabla f = 0$$

où f est définie sur un espace du type $O \times \mathbb{R}^n$, avec O ouvert de $\mathbb{R}^n$, x est la position et v la vitesse d'un élément. L'EDO associée est

$$\begin{cases} \dot{X} = V \\ \dot{V} = E(t, X). \end{cases}$$

Ce cas, est bien évidemment un cas particulier du cas précédent, le champ b étant alors donné par $b(x, v) = (v, E(x))$.
F. Bouchut a étendu le résultat de R. DiPerna et P.-L. Lions au cas où le champ E est dans BV_{loc}, en adaptant le lemme de convolution à ce cas. Pour cela, il choisit une nouvelle approximation de l'identité du type $\rho(x/\varepsilon)\rho(v/\eta)$ et utilise le fait que b dépend très régulierement de v pour faire tendre η vers 0 plus vite que ε.
Très récemment, L. Ambrosio a démontré le résultat du cas général pour des b dans BV_{loc} (voir [Amb03]). Il utilise pour cela plus precisement les propriétés géométriques des fonctions BV pour adapter le lemme de convolution à ces fonctions. Ce cas englobe celui des champs de vecteurs localement $W^{1,1}$ et celui des équations cinétiques avec champ de force BV. Citons aussi les travaux de N. Lerner et F. Colombini qui avait démontré un résultat intermédiaire avant L. Ambrosio. Plusieurs exemples semblent indiquer que l'on aurait atteint ou presque la régularité limite pour l'unicité des solutions. Certes, N. Lerner a récemment encore un peu améliorée le résultat en étudiant une classe de champs particuliers seulement partiellement dans BV (Voir [Ler04]). Mais N. Depauw ainsi que F. Colombini et J. Rauch ont étudié des champs presque BV pour lesquelles les solutions de l'équation de transport associée sont non uniques (voir [DP] ou [CR03]). Et R. DiPerna et P.-L. Lions introduisent dans [DL89] un champ de vecteur qui est dans tous les $W^{1,s}$, pour $s < 1$, mais pas dans BV. Ce contre-exemple est étudié en détail à la fin de la seconde partie.

1.1.3 Le cas de la dimension deux avec des champs à divergence nulle

La deuxième partie de cette thèse est consacrée principalement au cas particulier des equations de transport en dimension deux, plus précisément à l'afaiblissement des hypothèse dans le résultats suivant, obtenu par Fraçois Bouchut et Laurent Desvillettes.

Théorème 1.3 (Bouchut - Desvillettes). *Soit b un champ de vecteurs continu à divergence nulle sur $\mathbb{R}^2$. Notons Z l'ensemble des point ou il s'annule, et H un hamiltonien associé. Si $m(Z)$ et $m(H(Z))$ sont nulles, m désignant la mesure de Lesbegue sur $\mathbb{R}^2$ ou $\mathbb{R}$, alors il existe un unique* *flot associé à b. Cela induit aussi l'unicité des solutions de l'équation de transport.*

Ce résultat, même s'il utilise les techniques de DiPerna et Lions en diffère par l'utilisation de champs continus. Remarquons que les hypothèses sur les mesures de $m(Z)$ et $m(H(Z))$ sont juste là pour assurer que le changement de variables va être possible sur un ensemble suffisamment grand. Le premier théorème démontré dans la seconde partie de cette thèse consiste à adapter ce résultat pour des b seulement L^2. Les résultats obtenus ont fait l'objet d'un article publié dans Les Annales d'Analyse Non-linéaire de l'Institut Henri Poincaré [Hau03] reproduit dans le deuxième chapitre de cette thèse. Ayant perdu la continuité, on introduit la condition suivante:

$$(D_x) \qquad \exists \eta,\, \alpha > 0, \xi \in \mathbb{R}^2, \forall y \in B(x,\eta) \quad b(y) \cdot \xi \geq \alpha,$$

avec la notation $B(x,\eta)$ pour la boule de centre x et rayon η. Cette condition impose au champ de garder localement une direction constante (en un sens peu contraignant). Elle est utile pour demontrer le théorème suivant:

Théorème 1.4 (Hauray). *Soit b un champ de vecteurs dans $L^2_{loc}(\mathbb{R}^2)$. Soit Z l'ensemble des points x où (D_x) n'est pas vérifié, et H un hamiltonien associé à b. Si les mesures (de Lesbegue) $m(Z)$ et $m(H(Z))$ sont nulles, alors il existe un unique flot presque partout associé à b.*

Idée de la démonstration. A tout champ b à divergence nulle sur $\mathbb{R}^2$, on peut associer un Hamiltonien, c'est-à-dire une fonction H telle que $\nabla^\perp H = b$. En effet, écrire la condition de divergence nulle revient à écrire que les dérivées croisées sont nulles. Cette fonction H est conservée le long des trajectoires. On peut en particulier essayer de l'utiliser pour construire un changement de variable du type $(x,y) \to (H(x,y), \xi(x,y))$ pour un bon ξ. Dans ce cas, la première composante du nouveau champ s'annule et on peut donc se ramener à un problème de transport en dimension un, qui est bien plus facile à résoudre. La difficulté est dans la justification de ces étapes quand la régularité est faible.
On peut remarquer la correspondance entre les cas critiques dans ce cadre et dans celui de DiPerna et Lions. Ici, le résultat est vrai pour les champs dans L^p,pour $p \geq 2$. Sur $\mathbb{R}^n$, il est connu pour les champs $W^{1,p}$ pour $p \geq 1$. Or les injections de Sobolev nous disent que $W^{1,p}$ s'injecte dans L^{p^*} avec $1/p^* = 1/p - 1/n$. Le cas limite L^2 correspond donc au cas limite $W^{1,1}$ et BV , car en dimension deux, $1^* = 2$.
J'ai aussi traité le cas cinétique en dimension un. J'entends par là une dimension d'espace et une dimension pour la vitesse, ce qui forme un espace

des phases de dimension 2. Dans ce cas, le Hamiltonien est de la forme:

$$H(x, v) = \frac{v^2}{2} + V(x)$$

où V est un potentiel. Le résultat est le suivant:

Théorème 1.5 (Hauray). *Supposons que la dérivée de V est dans L^1_{loc}. Alors il existe un unique flot presque partout associé au champ $(v, -V'(x))$.*

1.1.4 Problème ouvert

La méthode de résolution des EDO et équations de transport pour des champs de vecteurs $W^{1,1}$ ou BV diffère du cadre régulier car on commence par résoudre l'équation de transport, et on en déduit que celle-ci est gouvernée par un flot. Cela paraît moins naturel et on peut se demander s'il n'existe pas une méthode permettant de résoudre directement l'EDO sans passer par l'équation de transport. Mais ce problème est toujours ouvert à l'heure actuelle.

1.2 EDO et équation de transport associée pour N particules en interaction

Je résume ici mon second travail, un article paru dans *Communication in Partial and Differential Equation* [Hau04], qui concerne une équation de transport particulière, l'équation de Liouville, et son champ associé. Cet article forme la troisième partie de cette thèse.
Considérons N particules interagissant avec un potentiel V. On note leur position $X_1, \dots, X_N$ et leur vitesse $V_1, \dots, V_N$. Chaque particule est supposée de masse $1/N$ pour que la masse totale soit égale à 1. Les équations de la mécanique classique s'écrivent:

$$\forall\ 1 \le i \le n \begin{cases} \dot{X}_i(t) = V_i \\ \dot{V}_i(t) = -\sum_{i \neq j} \nabla V(X_i(t) - X_j(t)) \end{cases} \tag{2.1}$$

Et l'équation de transport associée est:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} v_i \cdot \nabla_{x_i} f - \sum_{i \neq j} \nabla V(x_i - x_j) \cdot \nabla_{v_i} f = 0 \tag{2.2}$$

avec les conditions intiales:

$$f(0, x_1, \dots, x_n, v_1, \dots, v_n) = f^0(x_1, \dots, x_n, v_1, \dots, v_n) \tag{2.3}$$

D'après les théorèmes précédents, on sait résoudre ces deux problèmes dans le cas où la dérivée du potentiel est dans BV_{loc}. Mais, ce cas ne comprend pas la plupart des situations physiquement intéressantes, dans lesquelles le champ a en général toujours une singularité en 0, comme lorsque V est un potentiel d'interaction coulombien en $1/|x|$. Dans le but de rendre ces résultats utilisables avec des potentiels physiques, je les ai adapté au cas où le champ possède une singularité à l'origine pour obtenir les deux théorèmes suivants, valables en dimension $d \geq 2$:

Théorème 1.6 (Hauray). *Supposons que $\nabla V \in BV_{loc}(\mathbb{R}^d \backslash 0)$, que $\nabla V \in L^1$ près de l'origine et que V satisfait $V(x) \geq C(1+|x|^2)$ p.p.. Alors pour toute condition initiale dans L^1_{loc}, il existe une unique solution renormalisée de (2.2).*

Théorème 1.7 (Hauray). *Supposons que $\nabla V \in BV_{loc}(\mathbb{R}^d \backslash 0)$, que V est localement borné sur $\mathbb{R}^d \backslash 0$ que V satisfait $V(x) \geq C(1+|x|^2)$ p.p. et que $\lim_{|x|\to 0} V = +\infty$. Alors, pour toute condition initiale dans L^1_{loc}, il existe une unique solution renormalisée de (2.2).*

Remarque 1.2.

- *Dans le cas d'un potentiel coulombien, le potentiel est presque dans BV_{loc} et vérifie en tout cas les hypothèses du premier théorème.*

- *Dans le deuxième théorème, vu que l'on a aucune condition d'intégrabilité sur V au voisinage de l'origine, on ne peut intègrer que contre des fonctions à support en dehors de $I = \{(X,V) | \exists i \leq j \quad \text{tels que} \quad x_i = x_j\}$*

Dans ces deux cas on peut démontrer l'existence d'un unique flot presque partout associé.

1.3 Approximation de l'équation de Vlasov par des systèmes de particules

Après avoir augmenté la dimension, le problème naturel devint celui du passage à la limite vers les équations continues quand le nombre de particules tends vers l'infini. J'ai commencé à travailler avec Pierre-Emmanuel Jabin sur la convergence des sytèmes de particules vers Vlasov avec des potentiels singuliers. Les résultats obtenus ont fait l'objet d'un article accepté sous conditions.

L'équation de Vlasov-Poisson modélise le comportement d'une densité macroscopique f définie sur $\mathbb{R}^{2d}$ de particules interagissant entre elles. Si le potentiel d'interaction est V, l'équation est la suivante:

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0 \tag{3.1}$$

$$\text{avec} \quad F(x) = \nabla V(x) = \nabla(\int_{x,v} \frac{\rho(t,y)}{|x-y|}\, dy) \tag{3.2}$$

$$\text{et} \quad \rho(t,y) = \int_v f(t,x,v)\, dv. \tag{3.3}$$

En physique, cette équation modélise en général bien l'évolution de la densité de particules dans un plasma, si on néglige le champ magnétique. Elle devient l'équation de Vlasov-Maxwell si on ajoute le champ magnétique. Ces équations modélisent la densité d'ions dans la haute atmospère, celle au coeur d'un Tokamak...
Pour écrire cette équation, on suppose que les particules sont assez bien réparties pour que la densité de particules ait un sens. De toute façon, il est impossible de résoudre le système de N particules pour les cas physiquement intéressants. En général, les plasmas auquels on s'intéresse contiennent au minimum 10^8 particules par unité de volume. Les calculs numériques avec un tel nombre de particules sont exclus. Par contre, il existe des algorithmes permettant de calculer les solutions de l'équations de Vlasov avec une bonne précision. Certains remplacent les 10^8 molécules ou ions par un nombre plus petit de particules "virtuelles" dont on peut cette fois-ci calculer le mouvement, en espérant que cela ne modifie pas qualitativement le résultat.
Mais d'un point de vue mathématique, on peut s'intéresser à la validité de cette équation. Notamment pour en dégager les phénomènes microscopiques pertinents. Si on part du système de particules, est-il légitime de faire cette moyenne? Plus précisement, partons d'une condition intiale f^0. On cherche à l'approcher par un système de N particules qui se partagent la masse totale du système. Comme nous la supposerons normalisée à 1, chaque particule sera de masse $1/N$. Les N particules sont de positions-vitesses initiales $(X_1^0, V_1^0, \ldots, X_N^0, V_0^N)$. Elles évoluent alors suivant l'EDO (2.1). La distribution des particules est une mesure définie par

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_i(t), V_i(t))},$$

où δ désigne la mesure de Dirac. On peut remarquer que, dans le cas où le potentiel V est régulier et la force s'annule en zéro, μ_N vérifie l'équation de

Vlasov au sens des distributions, avec la condition initiale

$$\mu_N(0) = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_i^0, V_i^0)}.$$

En effet, choisissons une fonction test $\phi(x, v)$ régulière.

$$\frac{d}{dt}\langle \mu_N(t), \phi(x,v)\rangle = \frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\phi(X_i(t), V_i(t)) \quad (3.4)$$

$$\frac{1}{N}\sum_{i=1}^{N}\left(\nabla_{x_i}\phi \cdot v_i + \nabla_{v_i}\phi \cdot \left(\sum_{j\neq i}\nabla V(x_i - x_j)\right)\right)(X_i(t), V_i(t)) \quad (3.5)$$

$$\sum_{i=1}^{N}\nabla_{x_i}\langle \mu_N(t), v_i\phi(x,v)\rangle + \sum_{i=1}^{N}\nabla_{v_i}\langle \mu_N(t), (\mu_N(t) * \nabla V)\cdot \phi(x,v)\rangle \quad (3.6)$$

La question plus précise est la suivante: si $\mu_N(0)$ tend faiblement au sens des mesures vers la densité initiale f^0, a-t-on convergence faible de $\mu_N(t)$ vers $f(t)$, la (ou une, à préciser) solution de l'équation de Vlasov?

1.3.1 Le cas d'un potentiel régulier

Dans le cas où le potentiel est régulier de classe C_b^2 (le b en indice pour dérivée bornée) et ne dépend que de la distance entre particules, le problème à été résolu à la fin des années 70 indépendamment par Braun et Hepp (cf. [BH77]) et Neunzert et Wick (cf. [NW80]) et Dobrushin (cf. [Dob79]). Voir aussi pour cela le cours de Mathématiques sur les équations de transports donné par Pierre-Louis Lions à l'École Polytechnique [Lio89], ou le chapitre 4 du livre "Large scale dynamics of interacting particles" de Herbert Spohn [Spo91]. Le principe est qu'avec une force d'interaction continue et bornée, on peut résoudre le problème avec des conditions initiales mesures, ce qui englobe les sommes de petits Dirac et les fonctions densité. Reste donc à montrer la continuité de la solution obtenue en fonction des conditions initiales. Celle-ci peut-être obtenue si la force est C^1 à dérivée bornée. C'est le théorème suivant.

Théorème 1.8 (Braun et Hepp). *Soit V un potentiel défini sur $\mathbb{R}^N$ borné et à dérivée seconde bornée. Soit f^0 une fonction bornée, et $\mu_N(0)$ une suite de mesure telle que μ_N tend faiblement au sens des mesures vers f^0 quand N tend vers $+\infty$. Alors, pour tout temps t, $\mu_N(t)$ la densité du système de particules tend aussi vers $f(t)$ la solution de l'équation de Vlasov.*

1.3.2 Le cas de potentiels moins singuliers que Coulomb

Bien sûr, la condition de régularité imposée plus haut sur les champs est hautement non physique. La plupart des potentiels physiques comme celui de Coulomb en $1/|x|$ admettent une singularité à l'origine. Dans ces cas, le résultat ci-dessus ne s'applique plus, et on ne peut plus espérer l'étendre car les hypothèses de régularité nécessaires pour faire fonctionner un théorème de point fixe ne sont plus satisfaites. Et il n'est pas non plus possible de définir une notion de solution du problème qui engloberait des conditions initiales continues et celles qui sont somme de masses de Dirac. Il faut donc agir différemment.
L'objectif initial de ma thèse était de démontrer un résultat de convergence vers Vlasov-Poisson pour un système de particules. Malheureusment, la singularité assez forte du potentiel à rendu cette tâche inabordable. Mais, avec Pierre-Emmanuel Jabin, nous avons réussi à traiter le cas de singularités plus petites, c'est-à-dire des forces en $1/|x|^\alpha$, pour $\alpha < 1$.
Le résultat que nous avons obtenu est le suivant.

Théorème 1.9 (Hauray-Jabin). *Supposons que la force d'interaction F vérifie les estimations suivantes:*

$$|F(x)| \leq \frac{1}{|x|^\alpha}, \quad |\nabla F(x)| \leq \frac{1}{|x|^{\alpha+1}} \text{et} \quad |\nabla^2 F(x)| \leq \frac{1}{|x|^{\alpha+2}}$$

Soit f^0 une condition initiale pour l'équation de Vlasov bornée à support compact. Pour tout N, on choisit des positions initiales $(X_1^0, V_1^0, \ldots, X_N^0, V_N^0)$ telles que la distribution μ_N^0 satisfasse:

i. *$\mu_N^0 \rightharpoonup f^0$ faiblement au sens des mesures quand $N \to +\infty$*

ii. *Il existe un M indépendant de N tel que $\|(X_i, V_i) - (X_j, V_j)\| \geq \varepsilon/M$ pour tout $i \neq j$.*

où $\varepsilon = N^{-1/6}$. Alors, pour tout temps t,

$$\mu_N(t) \rightharpoonup f(t) \quad \text{quand} \quad N \to +\infty$$

Avant d'énoncer notre résultat principal, introduisons des quantités qui nous serons utiles.

Définition 1.3. *Soit $\varepsilon > 0$ et μ une mesure sur $\mathbb{R}^{2d}$. On définit la norme L^∞ discrète à l'échelle ε de μ noté $\|\mu\|_{\infty,\varepsilon}$ par*

$$\|\mu\|_{\infty,\varepsilon} = \sup_{(x,v)\in\mathbb{R}^{2d}} \frac{1}{\varepsilon^{2d}} |\mu(B((x,v), \varepsilon/2))|$$

où $B((x,v),\varepsilon/2)$ désigne la boule de centre (x,v) et de diamètre ε pour la distance $\|(x,v)\| = \sup(|x,|,|v|)$.

Cette quantité compte la masse contenue dans toutes les boules de rayon ε. Ici, on observe en quelque sorte la mesure à la précision ε. On ne voit pas les détails plus précis de la distribution. On pourrait aussi la définir par la formule suivante

$$\|\mu\|_{\infty,\varepsilon} = \|\mu \star \chi_{B(0,\varepsilon)}\|_\infty,$$

où $\chi_{B(0,\varepsilon)}$ désigne la fonction caractéristique de la boule $B(0,\varepsilon)$.
La démonstration du théorème se déroule en plusieurs étapes. On prouve d'abord que le champ, moyenné sur des intervalles de temps de taille ε, le support des vitesses, le support des positions, l'inverse de la distance minimale divisée par ε, restent bornés sur un petit intervalle de temps. Grâce à ces estimations, on montre qu'en fait, la norme $\|\mu_N(t)\|_{\infty,\eta}$ reste elle bornée par 2 fois la valeur de $\|\mu_N^0\|_{\infty,\varepsilon}$, pour des η qui tendent vers 0 en restant grands devant ε. Cela permet d'obtenir de nouvelles estimations qui permettent de contrôler le champ moyenné en temps et les autres quantités. Et ensuite, les bornes sur $\|\mu_N^0\|_{\infty,\varepsilon}$ permettent de passer à la limite dans l'équation.
Remarquons que la condition (ii.) implique que les normes L^∞ discrètes $\|\mu\|_{\infty,\varepsilon}$ sont bornées indépendamment de N, par $(2M)^6$. Malheureusement, cette condition est peu naturelle au sens où quand on tire N particules au hasard suivant la loi f^0, la probabilité que la distribution μ_N^0 obtenue satisfasse l'estimation décroit en e^{-cN}. Ce résultat demande de bien préparer les conditions initiales. Mais peut-être est-il possible d'améliorer le résultat dans cette direction au prix d'une complexification des calculs pour que l'ensemble des conditions initiales prises en compte soit de taille non négligeable si on tire les données initiales aléatoirement suivant f^0.

1.3.3 Vers Vlasov-Poisson

Le théorème précédent n'est pas valable dans le cas du potentiel de Coulomb ou du potentiel gravitationnel. Dans ce cas, la singularité est trop forte et nos estimations sont impossibles à adapter. La résolution (en positif ou en négatif) de ce cas nécessite donc d'introduire de nouvelles techniques. Remarquons que dans le cas du paragraphe précédent, la conservation de l'énergie (potentielle + cinétique) n'était d'aucune utilité dans la démonstration. Le potentiel $c|x|^{1-\alpha}$ n'explose pas à l'origine et ne nous donne pas d'information sur la distance minimale entre les particules, par exemple. Mais, en même temps, la faiblesse de la singularité fait que le rapprochement de deux particules n'a pas trop d'incidence sur leur comportement. Par contre, dans le cas d'un potentiel coulombien répulsif, la singularité du potentiel interdit les collisions entre particules. Cela n'est pas un avantage car, si deux particules se rapprochent beaucoup, elles se repoussent dans des directions qui dépendent très fortement des conditions initiales. Au sens où une petite modification de celles-ci peut entraîner une grande différence après un tel "choc".

1.3.4 Hiérarchie de Vlasov

Reprenons l'équation de Liouville pour N particules. Dans celle-ci, on peut s'intéresser à la probabilité de trouver les s premières particules aux positions $(X_1, V_1, \ldots, X_s, V_s)$. On la notera $f_N^s(X_1, V_1, \ldots, X_s, V_s)$. En intégrant l'équation de Liouville, on obtient pour f_N^s l'équation suivante:

$$\frac{\partial f_N^s}{\partial t} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_N^s + \frac{1}{N} \sum_{i,j=1, i\neq j}^{s} F(x_i - x_j) \cdot \nabla_{v_i} f_N^s + \frac{N-s}{N} \sum_{i=1}^{N} \int \nabla_{v_i} f_N^{s+1} F(x_i - x_{s+1})\, dx_{s+1} = 0, \quad (3.7)$$

à condition que la condition initiale f_N^0 soit invariante par permutation des variables, ce qui revient à supposer que les particules sont indicernables. On a écrit N équations. Remarquons que f_N^N est juste la solution de l'équation de Liouville. Que se passe-t-il quand le nombre de particules tend vers $+\infty$? Formellement, on obtient une hiérarchie infinie d'équation que l'on appellle la hiérarchie de Vlasov:

$$\frac{\partial f^s}{\partial t} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f^s \sum_{i=1}^{N} \int \nabla_{v_i} f^{s+1} F(x_i - x_{s+1})\, dx_{s+1} = 0, \quad (3.8)$$

Il est assez simple d'obtenir l'existence de solutions satisfaisant une condition initiale donnée à cette hiérarchie. Il serait intéressant de connaître l'unicité de la solution de la hiérarchie. Spohn a montré dans [Spo91] que les solutions étaient bien uniques pour des potentiels réguliers, mais on ne sait pas s'il y a unicité dans le cas de Vlasov-Poisson. Par contre,on le connait dans l'analogue quantique (c.f. [BGGM03]). Dans ce cas, l'équation de Liouville est remplacée par l'équation de Schrödinger à N particules, et l'équation de Vlasov, par l'équation de Hartree, appelée aussi équation de Schrödinger-Poisson, dans le cas particulier du potentiel de Coulomb. Dans ce cas, les outils sont beaucoup plus complexes, mais le champ devient beaucoup moins irrégulier car les particules ne sont plus localisées en un point.

Ce résultat d'unicité serait intéressant, car, il nous dirait que partant d'une condition initiale factorisée, c'est-à-dire du type

$$f_\infty^0 = \prod_{i=1}^{\infty} f^0(x,v),$$

la solution de la hiérarchie serait

$$f_\infty(t) = \prod_{i=1}^{\infty} f(t,x,v),$$

où f est la solution de l'équation de Vlasov pour la condition intiale f^0. Connaissant la convergence de Liouville vers la hiérarchie, on en déduirait celle que l'on cherche pour presque toute condition initiale.

1.3.5 Approximation d'Euler par des vortex

La dernière partie de cette thèse est consacrée à l'approximation d'Euler, écrite en vorticité, par des systèmes de vortex. L'équation d'Euler est la suivante:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ u(x) = \int K(x-y) w(y)\, dy \end{cases} \tag{3.9}$$

Un vortex est une masse de Dirac de vorticité. L'évolution d'un système de N vortex de force ω_i/N où $\omega_i \in [-1,1]$ est donnée par le système d'EDO suivant:

$$\text{pour} \quad i=1,\dots,N, \quad \dot{X}_i(t) = u(X_i(t)) = \frac{1}{N}\sum_{j\neq i}^{N} \omega_i K(X_i(t) - X_j(t)) \tag{3.10}$$

où K est le noyau d'interaction. Pour l'équation d'Euler de dimension 2, il est de la forme

$$K(x) = \frac{x^\perp}{|x|^2},$$

Mais j'ai aussi étudié des noyaux un peu moins singuliers que celui d'Euler, des noyaux vérifiant la condition (a) suivante

$$\text{(a)} \qquad |K(x)| \le C\frac{1}{|x|^\alpha} \quad \text{et} \quad |\nabla K(x)| \le \frac{C}{|x|^{\alpha+1}}$$

pour un $\alpha < 1$. La singularité un peu plus petite permet pour l'instant d'obtenir des résultats plus forts.

Pour le système discret, la distribution de la vorticité à alors la forme suivante:

$$\omega_N = \frac{1}{N}\sum_{i=1}^{N} \omega_i \delta_{X_i}$$

Cette vorticité mesure vérifie l'équation d'Euler au sens des distributions. Une question naturelle est de se demander si le système des vortex est une bonne approximation de l'équation d'Euler. Plus précisément, choisissons pour tout N, N vortex, leur force et leur position initiale de manière à ce que la distribution $\omega_N(0)$ tendent faiblement vers une vorticité $\omega(0)$ dans L^∞. On sait que la solution $\omega(t)$ de l'équation d'Euler est unique dans ces conditions. La distribution $\omega_N(t)$ tend-elle vers $\omega(t)$? La réponse est oui, pour des vortex toujours du même signe. Ce résultat a été démontré par Steve Schochet dans [Sch95] et complété dans [Sch96], sous certaines hypothèses sur la répartition locale des vortex. D'autres résultats plus numériques avec de bons ordres de convergence ont été obtenus par J. Goodman, T.Y. Hou and J. Lowengrub dans [GHL90]. Leurs vortex sont placés initialement aux sommets d'une grille. Le but de la cinquième partie est d'adapter les techniques utilisés dans la convergence vers Vlasov du modèle de particules à l'équation d'Euler. Les techniques permettent d'obtenir un bon résultat dans le cas des simili-vortex.

Théorème 1.10. *Choisissons pour tout N, N vortex tels que la suite des distributions ω_N converge vers un ω dans L^∞ et que $\|\omega_N(0)\|_{\infty,\varepsilon}$ soit bornée, où $\varepsilon^2 = N^{-1}$, et que la distance minimale inter-particule soit uniformément d'ordre ε. Alors, pour un noyau d'interaction vérifiant la condition (i), la convergence de $\omega_N(t)$ vers ω est vraie pour tout temps. Il y a aussi préservation asymptotique de $\|\omega_N\|_{\infty,\eta}$. C'est-à-dire que localement uniformement pour tout temps t et pour tout η et $\eta' \in [\varepsilon,\eta]$ tels que η/η' tende vers l'infini,*

$$\liminf_{N\to\infty} \|\omega_N(t)\|_{\infty,\eta} \leq \|\omega(0)\|_{\infty,\eta'}$$

Ce résultat est obtenu par les mêmes techniques que celles utilisées dans l'article sur l'approximation de l'équation de Vlasov (cf. Chap 4). Pourquoi ne s'adaptent-ils pas dans le cas des vrais vortex? Parce que dans ce cas, on ne peut obtenir qu'une borne $x|\log(x)|$ sur l'acroissement du champ limite, qu'on arrive à adapter au cas disret. Mais la technique pour obtenir la dérivation pour le temps grand est toujours de prouver une préservation de la norme à plus grande échelle, et pour cela il faut une borne sur la dérivée du champ, et même pouvoir le linéariser. Cette borne $x|\log(x)|$ ne permet pas cela. Ce champ déforme-t-il pourtant les boîtes de manière régulière? Question à résoudre.

1.3.6 Perspectives de recherche

Développement des techniques des deux dernières parties

Les techniques utilisées dans les deux dernières parties peuvent se résumer de la manière suivante. On approche une densité initiale L^∞ par des données initiales bien réparties au sens suivant. Si ε est la distance moyenne inter-particules, alors la densité reste bornée en norme L^∞-ε, et la distance minimale est minorée par ε/m_0. Dans ce cas, on prouve la convergence au sens faible de la distribution des particules vers la solution de l'équation continue pour tout temps. Cela se fait en propagant les quantités caractéristiques du support de la distribution et de la distance minimale inter-particules (dans l'espace des phases dans le cas de Vlasov). Pour obtenir cela, il faut des estimations sur le champ pour contrôler l'accroissement du support, et la dérivée du champ pour controler la diminution de la distance inter-particules. On obtient toujours ces estimations en adaptant les techniques du continu à notre système discret. Cela ne peut se faire que si une incertitude ε sur la position de la particule n'a qu'une incidence négligeable dans le calcul du champ. On remarque aussi que la préservation de la norme ne peut s'obtenir pour nos systèmes discrets qu'à une échelle plus grande que ε, et que la preuve de cette préservation est une étape non triviale de nos démonstrations, alors que son obtention dans le cas continue est immédiate. C'est le cas dans nos deux systèmes, et il semble fort possible d'adapter notre technique à toute équation qui vérifiera cette propriété. On obtiendra toujours une convergence valable pour toutes les conditions initiales bien réparties, et qui dit aussi que toutes les particules suivent un flot qui tend vers le flot limite.
D'autre part, il semble possible d'obtenir grâce à ces techniques des estimations sur le taux de convergence de la distributions des particules en termes de normes L^∞ discrètes à une taille toujours supérieure à la distance moyenne inter-particules, c'est-à-dire à un niveau macroscopique pour nous.

Vers Vlasov-Poisson

Que pourrait-on obtenir sur l'équation de Vlasov-Poisson? On sait que le résultat est vrai si on remplace le potentiel exact par le potentiel approché suivant $V_\varepsilon(x) = 1/(|x| + \varepsilon^\beta)$ pour tout β strictement plus petit que 1. Mais cela n'est pas satisfaisant car il serait souhaitable que la forme du potentiel à l'origine ne joue pas de rôle vu que la limite est une équation de champ moyen sans noyau de collision. De plus, le ε^β est beaucoup plus grand que la distance moyenne inter-particules dans l'espace physique qui est de l'ordre de ε^2. Nos particules étalées se recoupent beaucoup trop. En fait, ici le problème est que la singularité est trop forte pour se contenter d'une erreur d'ordre ε sur

la position des particules. Quand deux particules se rapprochent à moins de ε dans l'espace physique, il est très important de savoir si elles rapprochent à une distance d'ordre ε ou beaucoup plus proches. Pour cela, nos bornes sur $\|\mu_N\|_{\infty,\varepsilon}$ ne suffisent pas. Il est possible que l'on soit obligé d'introduire des probabilités sur les données initiales pour rendre compte des phénomènes qui ont lieu à une échelle plus petite que ε. Autre difficulté, il est peut être possible que des couples ou des petits ensembles de particules divergent trop par rapport au comportement moyen, ce qui voudrait dire qu'on ne pourra obtenir la convergence pour toutes les particules.
Il reste donc beaucoup de travail à faire avec ces techniques. D'une part les utiliser plus en détail dans les cas déjà connus. D'autre part les améliorer pour les adapter au cas de Vlasov-Poisson, voire d'autres équations.

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Chapter 2

Equations de tranports pour des champs de vecteurs L^2 en dimension deux

Abstract. We consider two-dimensional autonomous divergence free vector-fields in L^2_{loc}. Under a condition on direction of the flow and on the set of critical points, we prove the existence and uniqueness of a stable a.e. flow and of renormalized solutions of the associated transport equation.

Résumé. On considère ici des champs de vecteurs à divergence nulle et à coefficients dans L^2_{loc}. Avec une condition locale sur la direction du champ de vecteur, on prouve l'existence et l'unicité d'un flot presque partout et des solutions renormalisées de l'équation de transport associée.

AMS Subject Classification: 35R05, 35F99

2.1 Introduction

We consider the following transport equation,

$$\frac{\partial u}{\partial t}(t,x) + b(x)\cdot\nabla_x u(t,x) = 0 \tag{1.1}$$

with initial conditions

$$u(0,x) = u^o(x) \tag{1.2}$$

where $t \in \mathbb{R}$, $x \in \Omega$, $u^o : \Omega \to \mathbb{R}$, $b : \Omega \to \mathbb{R}^2$ satisfies $div\, b = 0$ and $u : \mathbb{R}\times\Omega \to \mathbb{R}$. The domain Ω is the torus Π^2, or $\mathbb{R}^2$ but in that case we

must assume that b satisfies some natural growth conditions, or a bounded open regular subset of $\mathbb{R}^2$ and b is then required to be tangent to the surface $\partial\Omega$. We assume that $u^0 \in L^p$ for some $p \in [1, \infty]$.
As is well known, this transport equation is in some sense equivalent to the ODE

$$\dot{X}(t) = b(X(t)) \tag{1.3}$$

Let us begin with some definitions and a proposition in which we always assume that b belongs at least to L^1_{loc}.

Definition 2.1. *Given an initial condition in L^∞, a solution of (1.1)-(1.2) is a function in $L^\infty([0,\infty) \times \Omega)$ satisfying for all $\phi \in C_c^\infty([0,\infty) \times \Omega)$*

$$\int_{[0,\infty)\times\Omega} u(\frac{\partial\phi}{\partial t} + b \cdot \nabla_x \phi) = -\int_\Omega u^o \phi(0,\cdot) \tag{1.4}$$

Definition 2.2. *We shall call renormalized solution a function u in $L^1_{loc}([0,\infty)\times \Omega)$ such that $\beta(u)$ is a solution of (1.1) with initial value $\beta(u^o)$, for all $\beta \in C^1_b(\mathbb{R})$, the set of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ with bounded continuous derivative.*

Remark In this definition, we do not ask u to be a solution because if u only belongs to L^1_{loc}, we cannot give a sense to the product uv. This is one of the reasons why we introduce this definition. But, this is of course an extension of the notion of solution. If $u \in L^\infty$ is a renormalized solution, it may be shown using good β that u is a solution.
We will give the next definition only for the case where $\Omega = \Pi^2$ or a bounded open subset of $\mathbb{R}^2$ in order to simplify the presentation. We refer to [DL89] for the adaptation to the case of $\mathbb{R}^n$.

Definition 2.3. *A flow defined almost everywhere (or a.e. flow) solving (1.3) is a function X from $\mathbb{R} \times \Omega$ to Ω satisfying*

i. $X \in C(\mathbb{R}, L^1)^2$

ii. $\int_\Omega \phi(X(t,x))\,dx = \int_\Omega \phi(x)\,dx \quad \forall\phi \in C^\infty, \quad \forall t \in \mathbb{R}$ *(preservation of the Lebesgue's measure)*

iii. $X(s+t,x) = X(t, X(s,x))$ *a.e. in x, $\forall s,t \in \mathbb{R}$*

iv. (1.3) is satisfied in the sense of distributions.

These properties implies that for almost all x, $\forall t \in \mathbb{R}$, $\quad X(t,x) = x + \int_0^t b(X(s,x))\,ds$.

Moreover, the useful following result is stated in [Lio98].

Proposition 2.1. *The two following statements are equivalent*

i. For all initial condition $u^o \in L^1$, there exists a unique stable renormalized solution of (1.1).

ii. There exists a unique stable a.e. flow solution of (1.3).

Moreover the following condition (R) implies these two equivalent statements

(R) Every solution of (1.1) belonging to $L^\infty(\mathbb{R} \times \Omega)$ is a renormalized solution.

This method of resolution of ODE's and associated transport equations was introduced by R.J. DiPerna and P.L. Lions in [DL89]. In this article, they show that if $b \in W^{1,1}_{loc}$, the problem (1.1)-(1.2) has a unique renormalized solution u. In fact, even if it is not stated in these terms in their article, we can adapt the method used in it to prove Proposition 1 and the fact that (R) is true when $b \in W^{1,1}_{loc}$. In our paper we will show that (R) holds provided that $b \in L^2_{loc}$ and that the following condition (P_x) on the local direction of b is true for a sufficiently large set of points x

$$(P_x) \qquad \exists \xi \in \mathbb{R}^2,\ \alpha > 0,\ \epsilon > 0 \quad \text{such that for almost all} \quad y \in B(x,\epsilon) \quad b(y)\cdot\xi \geq \alpha$$

This is a local condition and the quantities ξ, α, ϵ depend on x.
We will also show that (R) still holds in the case of a physical Hamiltonian $H(x,y) = y^2/2 + V(x)$ with $V' \in L^1_{loc}$.
This paper is a extension of L. Desvillettes and F. Bouchut [BD01], in which similar results are shown when b is continuous. The authors use the fact that since we have an Hamiltonian, we can integrate the ODE to obtain a one dimensionnal problem, that we are able to solve. We will adapt this method with less regularity on b.

2.2 Main result

Since we are in dimension two and that $div(b) = 0$, there exists a scalar function H (the hamiltonian) such that $\nabla H^\perp = b$. If b belongs to L^p, then H is in $W^{1,p}$.

Theorem 2.1. *Let Ω' be an open subset of Ω. Assume that $b \in L^2_{loc}(\Omega')$ and (P_x) holds for every $x \in \Omega'$, Then the condition (R) holds in Ω'.*

Remarks

i. Two is the critical exponent. It corresponds to the critical case $W^{1,1}$ in [DL89] since in two dimension we have the Sobolev embedding from $W^{1,1}$ to L^2. In the fourth paragraph, we shall describe a flow which is in L^p for all $p < 2$, which satisfy the condition (P_x) everywhere but for which uniqueness is false.

ii. This theorem does not extend the result in [DL89] in this particular case because a vector-fields in $W^{1,1}$ does not necessary satisfy the condition (P_x). We can construct divergence free vector-fields in $W^{1,1}$ which does not satisfy the condition (P_x) at any point x.

iii. Our method allow to prove the existence and the uniqueness directly (i.e. without using (R)), but it raises many difficulties concerning localisation and the addition of critical points.

iv. Here we state a result for a subset of Ω. Of course, a particular case of interest is the case when $\Omega' = \Omega$, where we may then use proposition 1 to obtain the existence and the uniqueness of an a.e. flow and of the solution of the transport equation. But the case $\Omega' \subsetneq \Omega$ will be useful when we will shall take into account some points where (P_x) is not true.

Proof. We shall prove this result in several steps. First, we shall state and prove some results about a change of variables. Then, we shall justify its application in formula (1.1), and obtain a new transport equation. Finally, we reduces this problem to a one dimensionnal one, that we are able to solve.
Step 1. A change of variable.
It is sufficient to show the result stated in Theorem 1 locally. Then, we shall work in a bounded neighbourhood U of x_0, in which we assume that $b \cdot \xi > \alpha$ a.e. as in (P_x). We define Φ on U by

$$\Phi(x) = ((x - x_0) \cdot \xi, H(x))$$

We wish to use Φ as a change of variable. For this, we use the following lemma

Lemma 2.1. *Assume that $H \in W^{1,p}(U)$ for $p \geq 2$, then there exist a bounded connected open set V containing $(0,0)$ and $\Phi^{-1} \in W^{1,p}(V)$ such that*

$$\text{for almost all } x \in U,\ \Phi(x) \in V \text{ and } \Phi^{-1} \circ \Phi(x) = x,$$
$$\text{for almost all } y \in V,\ \Phi^{-1}(y) \in U \text{ and } \Phi \circ \Phi^{-1}(y) = y,$$

Φ and Φ^{-1} leave invariant zero-measure sets.
Moreover, we have for $f \in L^\infty(V)$ the following formula:

$$\int_U f \circ \Phi(x) |D\Phi(x)| \, dx = \int_V f(y) \, dy \tag{2.1}$$

Proof of the lemma. Without loss of generality, we may assume that $x_0 = 0$, $\xi = (-1, 0)$, $U = (-\eta, \eta) \times (-\eta, \eta)$. According to [Zie89] we can assume, since H is $W^{1,p}$, that H is absolutely continuous on almost all lines parallel to the coordinate axes and that it is true in particular for the lines $\{y = \pm\eta\}$. Then we define a open set V by

$$V = \{(y_1, y_2) \in \mathbb{R}^2 \mid H(y_1, -\eta) < y_2 < H(y_1, \eta)\}$$

To show that V is connected we have to show that $H(x_1, -\eta) < H(x_1, \eta)$ for all $x_1 \in (-\eta, \eta)$. But we have $|b_1(x)| > \alpha$ for almost all $x \in U$ then $H(x_1, \eta) - H(x_1, -\eta) > 2\eta\alpha$ for almost all $x_1 \in (-\eta, \eta)$, then for all those x_1 by continuity.
Φ preserve the first coordinate, and for almost all $x_1 \in (-\eta, \eta)$, $H(x_1, \cdot)$ is a strictly increasing homeomorphism from $(-\eta, \eta)$ to $(H(x_1, -\eta), H(x_1, \eta))$. Hence we can define a suitable mesurable Φ^{-1}.
Now, we can prove the equation (2.1) using Fubini's theorem. First we consider the case when f is continuous. Then, we have

$$\int_U f \circ \Phi(x) |D\Phi(x)| \, dx = \int_{-\eta}^{\eta} \left(\int_{-\eta}^{\eta} f(x_1, H(x_1, x_2)) |b_1(x_1, x_2)| \, dx_2 \right) dx_1$$

Next, if F is $C^1(\mathbb{R}, \mathbb{R})$ and ϕ is in $W^{1,p}([a,b])$, then $F \circ \phi$ is in $W^{1,p}([a,b])$ and $(F \circ \phi)' = (F' \circ \phi)\phi'$. We now use this fact with F a primitive of f. Therefore, we can write

$$\int_{-\eta}^{\eta} f(x_1, H(x_1, x_2)) |b_1|(x_1, x_2) \, dx_2 = \int_{H(x_1,-\eta)}^{H(x_1,\eta)} f(x_1, y) \, dy$$

And if we use Fubini's theorem again, we obtain the result.
Now, we prove (2.1) for an arbitrary function in L^∞. If O is an open subset of V, we choose a sequence of f_n continuous such that $f_n \to \chi_O$ everywhere when n goes to ∞. By increasing convergence, the result is still true for χ_O. We have it for the caracteristic function of an open set. If we use the fact that $|b_1| \geq \alpha$, we obtain the inequality

$$\lambda(\Phi^{-1}(O)) \leq \frac{1}{\alpha} \lambda(O)$$

where λ is the Lebesgue measure. Next, if E is a zero measure subset of V, we obtain (using the above inequality with open set of small measure containing E) that $\Phi^{-1}(E)$ is also a zero-measure set.
Now, if we approximate a L^∞-function f by a sequence of continuous functions f_n converging to f a.e., then the sequence $f_n \circ \Phi$ converges to $f \circ \Phi$ a.e. and with the dominated convergence theorem, we obtain the result for f.

The formula (2.1) may be rewritten as follows

$$\int_U f(x)|D\Phi(x)|\,dx = \int_V f\circ\Phi^{-1}(y)\,dy$$

By approximation, it is always true provided the left hand side is meaningful, as it is the case, for instance when f belongs to $\mathbb{L}^q(U)$, with q the conjugate exposant of p ($p^{-1}+q^{-1}=1$). And if $f\in\mathbb{L}^a(U)$, then $f\circ\Phi^{-1}$belongs to $\mathbb{L}^b(V)$ with $b=a/q$.
To show that Φ^{-1} belongs to $W^{1,p}(V)$, and that $D(\Phi^{-1})=(D\Phi)^{-1}\circ\Phi^{-1}$, the most difficult case is to show that

$$\frac{\partial\Phi_2^{-1}}{\partial x_1} = -(\frac{b_2}{|b_1|})\circ\Phi^{-1} \tag{2.2}$$

First, since $b_2\in L^p(U)$ and $|b_1|>\alpha$, we can use the change of variables to deduce

$$\int_V |\frac{b_2}{b_1}|^p\circ\Phi^{-1} = \int_U b_2^p b_1^{1-p}$$

Hence, the right handside of (2.2) belongs to L^p.
Then, let ϕ be in $C_o^\infty(V)$, we have

$$\begin{aligned}
\int_V \Phi_2^{-1}(y)\frac{\partial\phi}{\partial x_1}\,dy &= \int_U x_2\frac{\partial\phi}{\partial x_1}\circ\Phi(x)\,|b_1(x)|\,dx \\
&= \int_U x_2\Big(\frac{\partial(\phi\circ\Phi)}{\partial x_1}(x)|b_1(x)| - \frac{\partial(\phi\circ\Phi)}{\partial x_2}(x)b_2(x)\Big)\,dx \\
&= \int_U \phi\circ\Phi(x)b_2(x)\,dx \\
&= \int_V \phi(y)\frac{b_2}{|b_1|}\circ\Phi^{-1}(x)\,dx
\end{aligned}$$

and this is the expected result. To obtain the second line from the first, we write

$$\partial_{x_1}(\phi\circ\Phi) = \partial_{x_1}\phi\circ\Phi - b_2\partial_{x_2}\phi\circ\Phi$$

$$\partial_{x_2}(\phi\circ\Phi) = b_1\partial_{x_2}\phi\circ\Phi$$

And when $p\geq 2$, these two quantities belong to L^2 and we may multiply the first by b_1, the second by $-b_2$ and add them to obtain the desired identity. To obtain the third line from the second, we use an integration by parts and the fact that $div\, b=0$. □

Step 2. Equivalence with a new transport equation.
We now wish to apply the change of variables in the formula (we recall that we assume that $\xi = (-1, 0)$)

$$\int_{[0,\infty)\times U} u(\partial_t \phi + b.\nabla\phi) = -\int_U u^o \phi^o \tag{2.3}$$

Since u belongs to $L^\infty(U)$, this expression make sense for ϕ in $W_0^{1,q}([0,\infty)\times U)$ (here and below q is always the conjugate exponent of p). But, we want to apply (2.3) with $\phi(t,x) = \psi(t, \Phi(y))$, where $\psi \in C_0^\infty([0,\infty)\times V)$. In this case ϕ will belong to $W^{1,p}([0,\infty)\times U)$ and will also have a compact support because of the form of Φ. In addition, since $p \geq 2$, we may write

$$\begin{aligned} b\cdot\nabla\phi &= b_1(\partial_{x_1}\psi\circ\Phi - b_2\partial_{x_2}\psi\circ\Phi) + b_2\, b_1\partial_{x_2}\psi\circ\Phi \\ &= b_1\partial_{x_1}\psi\circ\Phi \end{aligned}$$

and we obtain, denoting by $v(t,y) = u(t,\Phi^{-1}(y))$ and $J = |b_1|\circ\Phi^{-1}$

$$\int_{[0,\infty)\times V} v\left(\frac{1}{J(y)}\partial_t\psi(y) + \partial_{x_1}\psi(y)\right) = -\int_V \frac{v^o\psi^o}{J}$$

for all ψ in $C_0^\infty([0,\infty)\times V)$. In other words, v is solution in V (in the sense of the distributions) of

$$\partial_t\left(\frac{v}{J}\right) + \partial_{x_1} v = 0 \tag{2.4}$$

with the initial condition $(v/J)(0,\cdot) = v^o/J$.
Conversely, if $v \in L^\infty([0,\infty)\times V)$ is a solution of (2.4), we may test it against functions ψ in $W_0^{1,1}([0,\infty)\times V)$, and if ϕ is in $C_0^\infty([0,\infty)\times U)$ then $\phi\circ\Phi^{-1}$ is in $W_0^{1,1}([0,\infty)\times V)$. Thus we may follow the above argument backwards, and we obtain that (2.4) is equivalent to (1.1).
Step 3. Solution of the one dimensionnal problem.
In view of the precedent steps, it is sufficient for us to show that (R) hold for the equation (2.4). But, in this equation there is no derivative with respect to y_2. Therefore, it is equivalent to say that for almost all y_2, $\partial_t(v/J) + \partial_{x_1} v = 0$ on the set $\mathbb{R}\times V_{y_2}$ with the good initial conditions (here $V_{y_2} = \{y \in \mathbb{R} \,|(y, y_2) \in V\}$). This would be obvious if V were of the form $(a,b)\times(c,d)$, but we can always see V as a countable union of such rectangular sets. And since an open subset of $\mathbb{R}$ is a countable union of open intervals, we just have to show that the property (R) is true for the equation (2.4) on an interval $I = (a,b)$ of $\mathbb{R}$, with $J \geq \alpha$ a.e. on I.
Let F be a primitive of $1/J$. F is continuous, strictly increasing on (a,b) onto $(F(a), F(b))$, and its inverse F^{-1} belongs to $W^{1,1}(F(a), F(b))$. Again,

we may performe the change of variables $y \mapsto z = F(y)$ and we obtain that the equation (2.4) on I is equivalent to

$$\partial_t w + \partial_z w = 0 \qquad \text{on} \quad [0,\infty) \times (F(a), F(b)) \tag{2.5}$$

where $w(t,z) = v(t, F^{-1}(z))$. For this equation the property (R) is true. In fact we have a flow $X(t,x) = F^{-1}(F(x)+t)$ for (2.4), but we need to be careful because we are not exactly on the whole line and so this quantity is not defined for all t. $\square$

2.3 Critical points

In the preceding result, we assumed that the condition (P_x) was true for all x. We want here to take into account possible critical points. However, since we only assume that $b \in L^1_{loc}$, we cannot define critical points (of the Hamiltonian) as points where b vanishes (the usual notion when the flow is continuous). In some sense, critical points mean for us all those points where (P_x) is not true. In fact, this yields a "larger" set of critical points.

2.3.1 Isolated critical points

Our first result is the following

Corollary 2.1. *If b satisfies (P_x) everywhere in Ω except on a set of isolated points, then the (R) hypothesis holds.*

Proof. Without loss of generality we may assume that $\Omega = \mathbb{R}^2$, that (P_x) holds everywhere except at the origin $(0,0)$ and that $b \in L^2$. We take $\psi \in C_0^\infty(\mathbb{R})$ so that $\psi \equiv 1$ on a neighbourhood of $(0,0)$ and vanishes outside the ball B_1 of radius 1. We define $\psi_\epsilon = \psi(\frac{\cdot}{\epsilon})$.
Let $\phi \in C_0^\infty([0,\infty) \times \mathbb{R}^2)$, $\beta \in C^1(\mathbb{R})$ and u be a solution of the transport equation on $\mathbb{R}^2$, then $(1-\psi_\epsilon)\phi \in C_0^\infty(\mathbb{R}^2 \backslash \{(0,0)\}$, and since u is a renormalized solution on $\mathbb{R}^2 \backslash \{(0,0)\}$, we may write

$$\langle \partial_t \beta(u) + div(b\beta(u)), (1-\psi_\epsilon)\phi \rangle = \int_{\mathbb{R}^2} \beta(u^o)(1-\psi_\epsilon)\phi^o \tag{3.1}$$

i.e.
$$\int_{[0,\infty)\times\mathbb{R}^2} \beta(u)(1-\psi_\epsilon)(\partial_t\phi + b\cdot\nabla\phi) - \int_{[0,\infty)\times\mathbb{R}^2} \beta(u)\phi b\cdot\nabla\psi_\epsilon = -\int_{\mathbb{R}^2} \beta(u^o)(1-\psi_\epsilon)\phi^o \tag{3.2}$$

When ϵ goes to 0, the first integral converges to $\langle \partial_t \beta(u) + div(b\beta(u)), \phi \rangle$, the second one converges to 0 since

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \beta(u)\phi\nabla\psi_\epsilon \right| &\leq C\|b\|_{L^2(B_\epsilon)}\|\nabla\psi_\epsilon\|_{L^2} \\ &\leq C\|\nabla\psi\|_{L^2}\|b\|_{L^2(B_\epsilon)} \end{aligned}$$

and the right hand side converges to $-\int_{\mathbb{R}^2} \beta(u^o)\phi^o$.
We conclude that

$$\langle \partial_t \beta(u) + div(b\beta(u)), \phi \rangle = \int_{\mathbb{R}^2} \beta(u^o)\phi^o$$

for all $\phi \in C_0^\infty([0,\infty) \times \mathbb{R}^2)$. Hence u is a renormalized solution. □

2.3.2 A result with more regularity on H

The above result, of course, does not allow for many critical points. But we can allow much more with stronger conditions on H. First, points where there exists a neighbourhood on which b vanishes, are obviously easy to handle. We shall call O the set of all these points, and P the set of the points where (P_x) is true. We denote $Z = (O \cup P)^c$ (this complementary is taken in Ω). It is closed, since O and P are open . Then we have the following corolary.

Corollary 2.2. *Assume that H is continuous, Z is a set of zero-measure in $\mathbb{R}^2$ and $H(Z)$ is a set of zero-measure in $\mathbb{R}$. Then (R) still holds.*

Remarks

i. These conditions where introduced by L. Desvillettes and F. Bouchut in [BD01] in the case when b is continuous. Here, we only rewrite their proof in a less regular case.

ii. If $p > 2$, according to Sobolev embeddings, H is automatically continuous.

iii. We do not know if $H(Z)$ has zero-measure since we cannot apply Sard's lemma.

Proof. Let u be a solution of the transport equation (1.1) in Ω, $\beta \in C^1(\mathbb{R})$, ϕ a C^∞-test function, and K_o a compact set containing the support of $\phi(t,\cdot)$ for all t. We define $Z_o = Z \cap K_o$ and $K = H(Z_o)$. Then K is a zero-measure compact set. Then, we can find functions $\chi_n \in C_0^\infty(\mathbb{R})$ such that, $0 \leq \chi_n \leq 1$, $\chi_n \equiv 1$ on a neighbourhood of K and $\chi_n \to \chi_K$, the caracteristic function of

K, when n goes to ∞. We set $\Psi_n = \chi_n \circ H$. Ψ_n is continuous, belongs to $W^{1,p}(\mathbb{R}^2)$ and $\Psi_n \equiv 1$ on a neighbourhood of Z_o.
By theorem 1, $\beta(u)$ is a solution of (1.1) in P, and is also a solution in O, because on this set u is independent of the time. Since this two sets are open, $\beta(u)$ is a solution in $P \cup O$. $(1-\Psi_n)\phi$ is continuous and belongs to $W^{1,p}([0,\infty) \times K_o)$ and has its support in $[0,\infty) \times (K_o \backslash Z_o)$. Hence, since $(K_o \backslash Z_o) \subset P \cup O$ we can use it as a test fonction. We have

$$\langle \partial_t \beta(u) + div(b\beta(u)), (1-\Psi_n)\phi \rangle = \int_\Omega \beta(u^o)(1-\Psi_n)\phi^o$$

$$\text{i.e.} \qquad \int_{[0,\infty)\times\Omega} \beta(u)(1-\Psi_n)(\partial_t \phi + b \cdot \nabla \phi) - \int_{[0,\infty)\times\Omega} \beta(u) b \cdot \nabla \Psi_n = -\int_\Omega \beta(u^o)(1-\Psi_n)\phi^o \quad (3.3)$$

The second integral vanishes because $\nabla \Psi_n = (\Phi'_n \circ H)\nabla H$ and $b = \nabla H^\perp$.
The first integral converges by dominated convergence to

$$\int_{[0,\infty)\times H^{-1}(K)^c} \beta(u)(\partial_t \phi + b \cdot \nabla \phi)$$

while the left hand side goes to $-\int_{H^{-1}(K)^c} \beta(u^o)\phi^o$.
Then, to prove that (1.4) holds, we just have to show that

$$\int_{[0,\infty)\times H^{-1}(K)} \beta(u)(\partial_t \phi + b \cdot \nabla \phi) = -\int_{H^{-1}(K)} \beta(u^o)\phi^o \qquad (3.4)$$

But $H \in W^{1,p}(\mathbb{R}^2)$ and K is a zero-measure set, and this is a classical result that in that case $\nabla H = 0$ a.e. on $H^{-1}(K)$ (see for instance [Bou01]). Then, $b = \nabla H^\perp = 0$ a.e. on this set, and

$$\int_{[0,\infty)\times H^{-1}(K)} \beta(u)(\partial_t \phi + b \cdot \nabla \phi) = \int_{[0,\infty)\times H^{-1}(K)} \beta(u)\partial_t \phi$$

Moreover, $H^{-1}(K) \cap P$ is a set of zero-measure because on P, $\nabla H \neq 0$ a.e.. Hence the following quantity will not change if we integrate only on $H^{-1}(K) \cap (O \cup Z)$ or on $H^{-1}(K) \cap O$ since Z has zero-measure. But we already know that u is independent of the time on this set, then we can integrate in time to obtain the equality (3.4). □

As a conclusion to this section we just wanted to say that we do not know what happens when the condition (P_x) is not true on a sufficiently large set. Of course, we can construct divergence free vector-fields which do not satisfy (P_x) at every point, but it seems difficult to work with such flows because their definition is complex.

2.4 One example

We observe in this section that the example introduced by R. DiPerna and P.L. Lions in [DL89] provides an example of an divergence free vector fields b such that $b \in L^\infty_{loc}(\mathbb{R}^2 \backslash (0,0))$, b is in L^p in a neighbourhood of the origin for all $p < 2$ but not for $p = 2$, b satisfies the condition (P_x) everywhere, but there exist several solutions to the transport equations and several a.e. flows solving the associated ODE.

2.4.1 Definition of the vector-field

We define the hamiltonian H as follows (see fig 2.2)

$$H(x) = \begin{cases} -\frac{x_1}{|x_2|} & \text{if } |x_1| \leq |x_2| \\ -(x_1 - |x_2| + 1) & \text{if } x_1 > |x_2| \\ -(x_1 + |x_2| - 1) & \text{if } x_1 < -|x_2| \end{cases}$$

Then, b is given by

$$b_1(x) = -\frac{\partial H}{\partial x_2} = -sign(x_2) \left(\frac{x_1}{|x_2|^2} 1_{|x_1| \leq |x_2|} + sign(x_1) 1_{|x_1| > |x_2|} \right)$$

$$b_2(x) = \frac{\partial H}{\partial x_1} = -\left(\frac{1}{|x_2|} 1_{|x_1| \leq |x_2|} + 1_{|x_1| > |x_2|} \right)$$

2.4.2 Form of the solutions

First, we construct an a.e. flow X solution of the associated ODE. Since it would be symmetric in relation to the x_2-axis, we only defined it for $x_1 \geq 0$. We also define it just for $t \geq 0$.
In the case when $0 \leq x_2 \leq x_1$, we set

$$X(t,x) = (x_1 - t, x_2 - t) \qquad \text{for} \quad t \leq x_2$$
$$X(t,x) = (x_1 - 2x_2 + t, x_2 - t) \qquad \text{for} \quad t \geq x_2$$

while for $0 \leq -x_2 \leq x_1$, we set

$$X(t,x) = (x_1 + t, x_2 - t)$$

In the case when $0 \leq x_1 < x_2$, we set

$$X(t,x) = \sqrt{1 - \frac{2t}{(x_2)^2}}\,(x_1, x_2) \qquad \text{for} \quad t \leq \frac{(x_2)^2}{2}$$
$$X(t,x) = \sqrt{\frac{2t}{(x_2)^2} - 1}\,(x_1, -x_2) \qquad \text{for} \quad t \geq \frac{(x_2)^2}{2}$$

And if $0 \leq x_1 < -x_2$,

$$X(t,x) = \sqrt{1 + \frac{2t}{(x_2)^2}}\,(x_1, x_2)$$

In the sequels, we denote $I = \{x \in \mathbb{R}^2 | 0 < x_1 < -x_2\}$ and $J = \{x \in \mathbb{R}^2 | 0 < x_1 < x_2\}$. For an initial condition u^o, some tedious computation easily shows that the solutions of the transport equation (we use the fact that $u(t, X(t,x))$ is independent of t as long as X(t,x) does not reach the origin, and then we use the change of variable $(t,x) \to (t, X(t,x))$ on all the space, paying attention to what happens at the origin). They are of the form

$$u(t,x) = \begin{cases} u^o(X(-t,x)) & \text{if} \quad x \notin I \quad \text{or} \quad x \in I \quad \text{and} \quad t \leq \frac{(x_2)^2}{2} \\ \tilde{u}(X(-t,x)) & \text{if} \quad x \in I \quad \text{and} \quad t \geq \frac{(x_2)^2}{2} \end{cases} \tag{4.1}$$

where $\tilde{u}$ is any function defined on J satisfying the condition

$$\forall x_2 > 0, \quad \int_{-x_2}^{x_2} \tilde{u}(x_1, x_2)\, dx_1 = \int_{-x_2}^{x_2} u^o(x_1, x_2)\, dx_1 \tag{4.2}$$

Indeed, we use here the flow X for simplicity but these solutions are not defined according to this flow when a trajectory pass through the origin. We will try to explain what happens at the origin. For $x_2 > 0$, if the quantity u represent a density of mass, all the mass on the segment $\{(x, x_2) | x \in (-x_2, x_2)\}$ reaches the origin at the time $(x_2)^2/2$. After this time it continues to move in I always on segments parallel to the x_1-axis, but it can be redistributed on them in any way provided the total mass on this segment in conserved. This is what means the condition (4.2).
The renormalized solutions are always of this form, but the condition (4.2) should be replaced by

$$\forall x_2 > 0, \forall \beta \in C^1(\mathbb{R}) \quad \int_{-x_2}^{x_2} \beta(\tilde{u})(x_1, x_2)\, dx_1 = \int_{-x_2}^{x_2} \beta(u^o)(x_1, x_2)\, dx_1 \tag{4.3}$$

This condition (4.3) is equivalent to the fact that for all $x_2 \in \mathbb{R}$, we have a measure-preserving transformation Φ from $(-x_2, x_2)$ into itself such that $\tilde{u} = u^o \circ \Phi$. We refer to [Roy63] for this point.
Moreover, we can also find all the flows solutions of the associated ODE. Choosing a mesurable measure-preserving transformation Ψ from $(-1,1)$ into itself, we defined a flow X_Ψ by

$$X_\Psi(t,x) = \begin{cases} X(t,x) & \text{if} \quad x \notin J \quad \text{or} \quad t \leq \frac{(x_2)^2}{2} \\ \Psi(x)X(t,x) & \text{if} \quad x \in J \quad \text{and} \quad t \geq \frac{(x_2)^2}{2} \end{cases}$$

To see that this defined a a.e. flow, we use the property stated in the definition of an a.e. flow and the fact that an a.e. flow is measure preserving. Let us try to illustrate this definition. A particle with an initial position x^o in J moving according to X_Ψ behaves as follow. It moves on the half-line $\{x | x_1/x_2 = \lambda, \quad x_2 > 0\}$ (with $\lambda = x_1^o/x_2^o$) until it reaches the origin. Then it continues to move in I but on the half-line $\{x | x_1/x_2 = \Psi(\lambda), \quad x_2 < 0\}$. Indeed, Ψ may be seen as a mapping between the upper half-lines and the lower ones.
We can thus see that in this case, we have renormalized solutions that are not defined according to an a.e. flow. Indeed, for a renormalized solution, we can choose different mappings between the upper half-lines and the lower ones for each x_2 (in other words $\tilde{u} = u^o(\Psi_{x_2}(x_1/x_2)x_2, x_1)$ where Ψ_{x_2} is measure-preserving transformation from $(-1,1)$ into itself depending on x_2), while for a solution defined according to an a.e. flow, this correspondance will be independant of x_2 .

2.4.3 Remark about the uniqueness of the solution

First we remark that the flow X is a specific one. It is the only one for which the hamiltonian remains constant on all the trajectories. Moreover, we observe that the solution $\overline{u}$ defined according to X is specific among all the others. This is the only one which satisfies also the above family of equations (4.4), which says that the hamiltonian remains constant on the trajectories.

$$\forall f \in C(\mathbb{R}, \mathbb{R}) \qquad \partial_t(f(H)u) + \operatorname{div}(f(H)bu) = 0 \tag{4.4}$$

Indeed, we do the same computation that leads to (4.2) with these equations and we obtain the following conditions

$$\forall x_2 > 0,\; f \in C(\mathbb{R}, \mathbb{R}), \qquad \int_{-x_2}^{x_2} \tilde{u}(x_1, x_2) f(x_1)\, dx_1 = \int_{-x_2}^{x_2} u^o(x_1, x_2) f(x_1)\, dx_1 \tag{4.5}$$

This implies that $\tilde{u} = u^o$ and then that $u = \overline{u}$.
Hence, adding the conditions (4.4) in the definition of a solution, we are able to define it uniquely. Moreover, if we try to solve this problem by approximation, choosing a sequence of divergence free vector-field b_m converging to b in all L^p_{loc}, for $p < 2$, (this implies that H_m converge to H up to a constant in all $W^{1,p}_{loc}$, for $p < 2$), we obtain a sequence of solutions u_m, that satisfy all the equations (4.4) with the initial condition u^o. This sequence u_m is weakly compact in L^∞. Extracting a converging subsequence, we see that the equations (4.4) are always true at the limit and then the sequence u_m converge to $\overline{u}$. This solution is therefore the only that we can construct by approximation.

2.5 The case of a particule moving on a line

We consider here a classical Hamiltonian

$$H(x,y) = y^2/2 + V(x) \quad \text{or} \quad b(x,y) = (y, -V'(x))$$

with V a potential in $W^{1,p}_{loc}(\mathbb{R})$. Then V' belongs to $L^p_{loc}(\mathbb{R})$. In this case, b satisfies the (P_x) assumption in $\mathbb{R}^2 \backslash \{y = 0\}$. The set of criticals points Z has then zero-measure. We can apply the preceding results, if H satisfies $m(H(Z)) = 0$. When V' is continuous, this is true because we can apply the Sard Lemma. But this is false for a general $V' \in L^1_{loc}$. If V' oscillates very quickly, Z may even be the whole line. And then $H(Z)$ is an interval because H is continuous. However, we will show that the result is always true in this case. Moreover, we can only assume that V' belongs to $L^1_{loc}(\mathbb{R})$, since the other composant of b is in $L^\infty_{loc}(\mathbb{R})$.
The transport equation we are considering has the form

$$\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial x} - V'(x)\frac{\partial u}{\partial y} = 0 \tag{5.1}$$

Here we can solve the differential equation $x' = y$, $y' = -a(x)$ directly if we use the fact that the Hamiltonian is constant on a trajectory and integrate the system. But this flow is not regular, and we do not know how to work directly with it in order to solve the transport equation.

Theorem 2.2. *For a flow $b(x,y) = (y, -V'(x))$ with $V' \in L^1_{loc}(\mathbb{R})$ and $1/\sqrt{max(1, -V(x))}$ not integrable at $\pm\infty$, the transport equation has an unique renormalized solution.*

Remarks

i. The condition of integrability on V is there to insure that a point does not reach $\pm\infty$ in a finite time. It could be replaced by a stronger condition like $V(x) \geq -C(1+x^2)$.

ii. This result can be adapted to the case of two particles moving on a line according to a interaction potential in $W^{1,1}_{loc}$. In order to do so this we just have to use a change of variable which follows the classical way of reducing this two-body problem to a one-body problem.

Proof. We can use our previous theorem in the neighboorhood of a point with $y \neq 0$. This will give us "the result" on two half-planes, but we need to "glue" together the information available on this two half-planes. Then we need to work differently, and we shall follow the same sketch of proof as in our first theorem.

Step 1. A change of variables.

First we define

$$\Phi_+(x,y) = (x, y^2/2 + V(x)) \qquad \text{from} \quad \mathbb{R}\times(0,\infty) \quad \text{to} \quad B$$

$$\Phi_-(x,y) = (x, y^2/2 + V(x)) \qquad \text{from} \quad \mathbb{R}\times(-\infty,0) \quad \text{to} \quad B$$

where $B = \{(x,E) \in \mathbb{R}^2 | V(x) < E\}$. Then Φ_+ and Ψ_- are continuous and belong to $W^{1,1}_{loc}$ with

$$D\Phi_\pm = \begin{pmatrix} 1 & 0 \\ V'(x) & y \end{pmatrix}$$

and the same for Φ_-.

These transformations are one-to-one and onto and

$$\Phi_\pm^{-1}(x,E) = (x, \pm\sqrt{2(E-V(x))})$$

.

$$D\Phi_+^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{V'(x)}{\sqrt{2(E-V(x))}} & \frac{1}{\sqrt{2(E-V(x))}} \end{pmatrix},$$

with a similar formula for Φ_-^{-1}.

The following change of variables is true

$$\int_{y>0} f(x,y)\,dxdy = \int_B \frac{f\circ\Phi_+^{-1}(x,E)}{\sqrt{2(E-V(x))}}\,dxdE$$

for f in L^∞ and even in L^1.

Before going further, we state some properties about the set $W^{1,1}(B)$. We define $C^\infty(\overline{B})$ (resp.$C_0^\infty(\overline{B})$) the space of restriction to B of C^∞-functions on $\mathbb{R}^2$ (resp. such functions with compact support). We recall that $\partial B = \{(x,V(x))|x\in\mathbb{R}\}$.

Proposition 2.2.

i. $C_0^\infty(\overline{B})$ *is dense in* $W^{1,1}(B)$.

ii. *The trace of a function in* $W^{1,1}(B)$ *has a sense in* L^1. *More precisely, there exist a continuous application* $Tr : W^{1,1}(B) \to L^1(\mathbb{R})$ *such that* $Tr(\phi) = \phi(\cdot, V(\cdot))$ *if* $\phi \in C_0^\infty(\overline{B})$.

iii. $\overline{C_0^\infty(B)} = Ker(Tr)$, *the kernel of the trace, also denoted by* $W_o^{1,1}(B)$.

iv. *The same results are true for* $\mathbb{R} \times B$ *and* $[0, \infty) \times B$ *as well as locally.*

Proof of the proposition. We refer to theorem 3.18 in [Ada75] for a complete proof. But we may adapt the proof of this theorem to this simpler case. For a $f \in W^{1,1}(B)$, we define for $\epsilon > 0$

$$f_\epsilon = \rho_\epsilon * (f(\cdot + 2\epsilon e)\chi_{\{E>V(x)-2\epsilon\}})$$

where $e = (0, 1)$ and $\rho_\epsilon(x, E) = \rho(x/\epsilon, E/\epsilon)$ with $\rho \in C_0^\infty(\mathbb{R}^2)$ satisfying $\int \rho = 1$. Then, the functions f_ϵ belong to $C_0^\infty(\overline{B})$ and converges to f in $W^{1,1}(B)$ as $\epsilon \to 0$.
For the second point, we choose $f \in C_0^\infty(\overline{B})$. Then

$$f(x, V(x)) = -\int_{V(x)}^{\infty} \frac{\partial f}{\partial E}(x, E)\, dE$$

taking the absolute value and integrating in x leads to

$$\int_{\mathbb{R}} |f(\cdot, V(\cdot))| \le \|\nabla f\|_{L^1(B)}$$

Then, the trace is a contraction from $C_0^\infty(\overline{B})$ with the $W^{1,1}$-norm into $L^1(\mathbb{R})$, and since $C_0^\infty(\overline{B})$ is dense in $W^{1,1}(B)$, we may extend this application to $W^{1,1}(B)$.
For the third point, we take $f \in Ker(Tr)$ and extend it by zero outside B. We obtain a $\tilde{f}$ in $W^{1,1}(\mathbb{R}^2)$. Then if we translate $\tilde{f}$ in the direction of $e = (1, 0)$ and smooth it by convolution, we can construct C^∞-approximations of f with support in B. □

Step 2. Equivalence with a simpler transport equation.
Now, let u be a solution of the transport equation. We may write

$$\int_{[0,\infty)\times\mathbb{R}^2} u(\partial_t\phi + y\partial_x\phi - V'(x)\partial_y\phi)\, dxdy = -\int_{\mathbb{R}^2} u^o\phi^o \qquad (5.2)$$

for all $\phi \in W^{1,1}([0,\infty) \times \mathbb{R}^2)$ with compact support (in the sense of distributions) satisfying moreover $\partial_y \phi \in L^\infty([0,\infty) \times \mathbb{R}^2)$.
Let Ψ_+ and Ψ_- be in $C_0^\infty([0,\infty) \times \overline{B})$, and Ψ_+ and Ψ_- satisfy the compatibility condition $\Psi_{+|[0,\infty)\times\partial B} = \Psi_{-|[0,\infty)\times\partial B}$. We define ϕ from $[0,\infty) \times \mathbb{R}^2$ to $\mathbb{R}$ with

$$\phi(t,x,y) = \begin{cases} \Psi_+(t, \Phi_+(x,y)) & \text{if} \quad y > 0 \\ \Psi_-(t, \Phi_-(x,y)) & \text{if} \quad y < 0 \end{cases}$$

Then, ϕ belongs to $W^{1,1}([0,\infty) \times \mathbb{R}^2)$, has a compact support, and ϕ, $\partial_t \phi$, $\partial_y \phi$ are in L^∞. Moreover,

$$\partial_x \phi = (\partial_x \Psi_+) \circ \Phi_+ + E(x)(\partial_y \Psi_+) \circ \Phi_+ \quad \text{for } y > 0$$
$$\partial_y \phi = y(\partial_y \Psi_+) \circ \Phi_+$$
$$\partial_t \phi = (\partial_t \Psi_+) \circ \Phi_+$$

Then we have $\quad \partial_t \phi + y \partial_x \phi - E(x) \partial_y \phi = (\partial_t \Psi_+) \circ \Phi_+ + y(\partial_x \Psi_+) \circ \Phi_+$ for all $y > 0$.
We write $v_\pm = u \circ \Phi_\pm^{-1}$, defined on $[0,\infty) \times B$. Then, (5.2) may be written as follows.

$$\int_{[0,\infty)\times\{y>0\}} [v_+(\partial_t \Psi_+ + \sqrt{2(E - V(x))}\partial_x \Psi_+)] \circ \Phi_+ + \int_{[0,\infty)\times\{y<0\}} [v_-(\partial_t \Psi_- - \sqrt{2(E - V(x))}\partial_x \Psi_-)] \circ \Phi_- = \int_{y>0} (v_+^0 \Psi_+^0) \circ \Phi_+ + \int_{y<0} (v_-^0 \Psi_-^0) \circ \Phi_- \quad (5.3)$$

We can apply the change of variables, and we obtain

$$\int_{[0,\infty)\times B} v_+ \left(\frac{\partial_t \Psi_+}{\sqrt{2(E - V(x))}} + \partial_x \Psi_+\right) + \int_{[0,\infty)\times B} v_- \left(\frac{\partial_t \Psi_-}{\sqrt{2(E - V(x))}} - \partial_x \Psi_-\right) = \int_B \frac{v_+^0 \Psi_+^0 + v_-^0 \Psi_-^0}{\sqrt{2(E - V(x))}} \quad (5.4)$$

It is difficult to work with Φ_+ and Φ_- because of the compatibility condition. But we may make the particular choice $\Phi_+ = \Phi_-$ (below we will omit the indices $\pm$). Then (5.4) becomes

$$\int_{[0,\infty)\times B} (v_+ + v_-) \frac{\partial_t \Psi}{\sqrt{2(E - V(x))}} + (v_+ - v_-)\partial_x \Psi = \int_B \frac{(v_+^0 + v_-^0}{\sqrt{2(E - V(x))}} \Psi^0 \quad (5.5)$$

Now, we choose $\Psi_+ = -\Psi_-$ and $\Psi_{|[0,\infty)\times\partial B} = 0$ (we omit the indices $\pm$). In this case, (5.4) becomes

$$\int_{[0,\infty)\times B} (v_+ - v_-)\frac{\partial_t \Psi}{\sqrt{2(E-V(x))}} + (v_+ + v_-)\partial_x \Psi = \int_B \frac{(v_+^0 - v_-^0)}{\sqrt{2(E-V(x))}}\Psi^0 \tag{5.6}$$

Then, (5.2) implies (5.5) for all Ψ in $C_0^\infty([0,\infty)\times\overline{B})$, and (5.6) for all $\Psi \in C_0^\infty([0,\infty)\times\overline{B})$ with $\Psi_{|[0,\infty)\times\partial B} = 0$, or equivalently for all $\Psi \in C_0^\infty([0,\infty)\times B)$ since $C_0^\infty([0,\infty)\times B)$ is dense in $W_o^{1,1}([0,\infty)\times B)$. And conversly, these two statements are equivalent with (5.4) for all Ψ_+ and Ψ_- in $C_0^\infty([0,\infty)\times\overline{B})$ having the same trace on the boundary.
Thus, we have to solve

$$\partial_t(\frac{v_+ + v_-}{\sqrt{2(E-V(x))}}) + \partial_x(v_+ - v_-) = 0 \qquad \text{on} \quad \mathcal{D}'([0,\infty)\times\overline{B}) \tag{5.7}$$

$$\partial_t(\frac{v_+ - v_-}{\sqrt{2(E-V(x))}}) + \partial_x(v_+ + v_-) = 0 \qquad \text{on} \quad \mathcal{D}'([0,\infty)\times B) \tag{5.8}$$

with the convenient initial conditions. In (5.7), $\mathcal{D}'([0,\infty)\times\overline{B})$ means that we allow test functions in $C^\infty([0,\infty)\times\overline{B})$.
We can do the same arguments backwards. Therefore, solving (5.7)-(5.8) is equivalent to solve (5.1)
Step 3. Reduction to one dimension.
These two equations do not contain any derivative in E. As in the proof of the first result, we want to reduce them to equations in one dimension of space. For the second equation (5.8), we can make the same argument and we obtain that this equation holds on B_E, for almost all E in $\mathbb{R}$. (with $B_E = \{x \in \mathbb{R} | (x,E) \in B\}$).
For the first equation (5.7) we can still apply the argument. We shall be more precise since it is a little bit more involved. We choose a test function ϕ of the form $\phi_1\phi_2$ with ϕ_1 depending only on (t,x) and ϕ_2 depending on E. We obtain

$$\int_{[0,\infty)\times B}\left((v_+ + v_-)\frac{\partial_t\phi_1}{\sqrt{2(E-V(x))}} + (v_+ - v_-)\partial_x\phi_1\right)\phi_2 = \int_B \frac{(v_+^0 + v_-^0)}{\sqrt{2(E-V(x))}}\phi_1^o\phi_2 \tag{5.9}$$

Since the linear combinaisons of functions of the form $\phi_1\phi_2$ are dense in $C_0^\infty([0,\infty))$ with the $W^{1,1}$-norm, (5.9) for all C_0^∞ ϕ_1 and ϕ_2 is equivalent with (5.5) for all C_0^∞ Ψ. Moreover, since $W^{1,1}([0,\infty)\times\mathbb{R})$ is separable, it

is sufficient (and necessary) to write (5.9) for ϕ_1 choosen among a countable subset F_1 of C_0^∞-functions.
Now, using Fubini's theorem (5.9) may be rewritten

$$\int_{\mathbb{R}} \left(\int_{[0,\infty)\times B_E} (v_+ + v_-) \frac{\partial_t \phi_1}{\sqrt{2(E - V(x))}} + (v_+ - v_-)\partial_x \phi_1 \, dtdx \right) \phi_2 \, dE$$
$$= \int_{\mathbb{R}} \left(\int_{B_E} \frac{(v_+^0 + v_-^0)}{\sqrt{2(E-V(x))}} \phi_1^o \, dx \right) \phi_2 \, dE \quad (5.10)$$

for a fixed ϕ_1. Since it is satisfied for all C_0^∞-ϕ_2, we obtain that

$$\int_{[0,\infty)\times B_E} (v_+ + v_-) \frac{\partial_t \phi}{\sqrt{2(E - V(x))}} + (v_+ - v_-)\partial_x \phi = \int_{B_E} \frac{(v_+^0 + v_-^0)}{\sqrt{2(E-V(x))}} \phi^o \quad (5.11)$$

for all $E \in \mathbb{R}\backslash N$ where N is a zero-measure set depending on ϕ_1. Now, if we write this equation for all $\phi_1 \in F_1$, we obtain that (5.7) is satisfied, but this time in $[0,\infty) \times B_E$ for allmost all $E \in \mathbb{R}$. And we can do the argument backwards to show that this is equivalent to the initial problem. Finally, we just have to solve (5.7)-(5.8) on B_E instead of B.
Step 4. Solution of the one dimensionnal problem.
B_E is a countable union of disjoint open intervals. We denote $B_E = \cup_n (a_n, b_n)$, where a_n, b_n are disjoints reals. But, since we shall also work on $\overline{B_E}$ we want that these open intervals are not to "close" to each other. For instance, if there exist n, m such that $b_n = a_m$ and if $1/\sqrt{2(E - V(x))}$ is integrable on a neighboorhood of b_n, a particle reaching b_n from the left may continue to go further right or may change direction and go backwards. This will give rise to distinct solutions of the transport equation. But, we shall show that for almost all E, we have some "free zone" around each (a_n, b_n). More precisely, for almost all E there exists an $\epsilon_n > 0$ such that $B_E \cap (a_n - \epsilon_n, b_n + \epsilon_n) = (a_n, b_n)$. If we admit this point, we see that we just have to solve (5.7)-(5.8) on an interval of the type (a', b'), where a' belongs to $[-\infty, +\infty)$ and b' to $(-\infty, +\infty]$. Before going further, we prove the

Lemma 2.2. *For almost all E, if we write $B_E = \cup_n (a_n, b_n)$ then, for each n, there exists some $\epsilon_n > 0$ such that $B_E \cap (a_n - \epsilon_n, b_n + \epsilon_n) = (a_n, b_n)$*

Proof of the lemma. First we recall that since V belongs to $W^{1,1}_{loc}$, the image by V of a zero-measure set is a zero-measure set. Then, we state a result similar to the Sard's lemma for V. Let Z be the set were V' vanishes. We claim that $V(Z)$ has zero-measure. Of course , Z is defined up to a zero-measure set, but this is irrelevant for our claim in view of the fact recalled above. In order to prove our claim, we choose a sequence of open sets O_n such that $Z \subset O_n$ and $\lambda(O_n \backslash Z)$ goes to 0 as n goes to ∞. Here and below λ denotes the Lesbegue measure on $\mathbb{R}$ or $\mathbb{R}^2$. We may write $O_n = \cup_m I_{n,m}$ where the $I_{n,m}$ are disjoint intervals of $\mathbb{R}$. Then,

$$\begin{aligned}
\lambda(V(O_n)) &= \lambda(V(\cup_m I_{n,m})) \leq \sum_m \lambda(V(I_{n,m})) \\
&\leq \sum_m \int_{I_{n,m}} |V'| = \int_{O_n} |V'| \\
&\leq \int_{O_n \backslash Z} |V'|
\end{aligned}$$

and the last quantity goes to 0 as $n \to \infty$ since $\lambda(O_n \backslash Z)$ goes to 0 as $n \to \infty$ and our claim is shown.

Next, we denote by Z_1 the set such that Z_1^c is the set of Lebesgue points of V' (i.e. the set of points such that $1/(2\epsilon) \int_{x-\epsilon}^{x+\epsilon} |V'(y) - V'(x)| \, dy$ goes to zero as $\epsilon \to 0$). Then, $\lambda(Z_1) = 0$. According to what we proved above, we know that $\lambda(V(Z \cup Z_1)) = 0$. Now, if we choose $E \in V(Z \cup Z_1)^c$,and write $B_E = \cup_n (a_n, b_n)$ as above, we know that a_n and b_n are Lebesgue's point of V' with $V'(a_n) \neq 0$ and $V'(b_n) \neq 0$. Then necessarily, $V'(b_n) > 0$ and V is strictly increasing in a neighboorhood of b_n because it is a Lebesgue's point. Since we may make the same argument near a_n, we have then shown the existence of ϵ_n as stated in the lemma. $\square$

To solve (5.7)-(5.8) on (a', b') we use the change of variable $x \mapsto z = F(x)$ where F is a primitive of $1/\sqrt{2(E - V(x))}$ from (a', b') to (a, b). We can because this quantity is locally integrable on almost all lines (this result is easily seen using Fubini's theorem). Then, we obtain the two following equations

$$\partial_t (w_+ + w_-) + \partial_z (w_+ - w_-) = 0 \qquad \text{on} \quad [0, \infty) \times [a, b] \tag{5.12}$$

$$\partial_t (w_+ - w_-) + \partial_z (w_+ + w_-) = 0 \qquad \text{on} \quad [0, \infty) \times (a, b) \tag{5.13}$$

with appropriate initial conditions. And as before, in (5.12) we use test functions in $C_0^\infty([0, \infty) \times [a, b])$ (in others words the tests functions do not necessarily vanish on $\{z = a\}$ and $\{z = b\}$ when a and b are finite).

Here, if $a' = -\infty$ or $b' = +\infty$ we need the assumption of non-integrability on V. If it is not verified, a (or b) will be finite, and we cannot use test functions which do not vanish on $\{z = a\}$ (or $\{z = b\}$) in (5.7). And we

shall not have the uniqueness of solutions of the equivalent problem (as will become clearerr below).
Adding and substracting the two equations in $\mathcal{D}'([0,\infty)\times(a,b))$ yields

$$\partial_t w_+ + \partial_z w_+ = 0 \qquad \text{in} \quad \mathcal{D}'([0,\infty)\times(a,b))$$

$$\partial_t w_- - \partial_z w_- = 0 \qquad \text{in} \quad \mathcal{D}'([0,\infty)\times(a,b))$$

Hence, the solutions are of the form $w_+(t,z) = \Phi_+(z-t)$ and $w_-(t,z) = \Phi_-(z+t)$ with Φ_+ and Φ_- belonging to $L^\infty(\mathbb{R})$. but we have not used yet the fact that (5.12) is true on [a,b]. This tells us formally that $w_+(t,a) = w_-(t,a)$ when $a \neq -\infty$ and $w_+(t,b) = w_-(t,b)$ when $b \neq +\infty$. This can be justified. Indeed, let us assume that $b \neq +\infty$ and let we choose some $\phi \in C_0^\infty((0,\infty))$, an $\epsilon \in (0, b-a)$ and $\chi_\epsilon \in C^\infty(\mathbb{R})$ increasing such that $\chi_\epsilon(z) = 0$ for $z < b-\epsilon$ and some $\chi_\epsilon(z) = 1$ for $z > b$. We use $\phi\chi_\epsilon$ as a test function in (5.12). We then obtain

$$\int_{[0,\infty)\times(b-\epsilon,b)} (\Phi_+(z-t) + \Phi_-(z+t))\partial_t\phi(t)\chi_\epsilon(z)\,dtdz \\ + \int_{[0,\infty)\times(b-\epsilon,b)} (\Phi_+(z-t) - \Phi_-(z+t))\phi(t)\partial_z\chi_\epsilon(z)\,dtdz = 0$$

When $\epsilon \to 0$, the first integral goes to 0. The second integral goes to $\int_{[0,\infty)}(\Phi_+(b-t) - \Phi_-(b+t))\phi(t)\,dt$. Since it holds for all $\phi \in C_0^\infty((0,\infty))$, we obtain that $\Phi_+(b-t) = \Phi_-(b+t)$. We can prove similary that $\Phi_+(a-t) = \Phi_-(t+a)$ if $a \neq -\infty$.
Now, we shall assume that a and b are both finite (the other cases are similar and simpler) and we define $l = b-a$. Without using the boundary conditions, the initial conditions on w_+ and w_- impose the value of Φ_+ and Φ_- on the interval (a,b). Of course, it should be understood in sense of functions defined almost everywhere, but here it does not raise any difficulty and we will omit to specify it afterwards. Using the boundary condition $\Phi_+(b-t) = \Phi_-(b+t)$, we see that Φ_+ and Φ_- are determined in $(b, b+l)$. And the condition $\Phi_+(a-t) = \Phi_-(t+a)$ determines Φ_+ and Φ_- in $(a-l, a)$. If we continue to use this symmetry argument further, we see that Φ_+ and Φ_- are uniquely determined in $\mathbb{R}$, provided we know them in (a,b) (we remark here that it is not the case if one of the boundary counditions is missing, as it is the case when $a = -\infty$ or $b = +\infty$ and the assumption of non-integrability on V is not satisfied).Then, for every intial condition (on w_+ and w_-) in L^∞, there exists a unique solution to the system (5.12)-(5.13). And in view of the form of those solutions, we see that they are renormalized ones. This concludes the proof.

□

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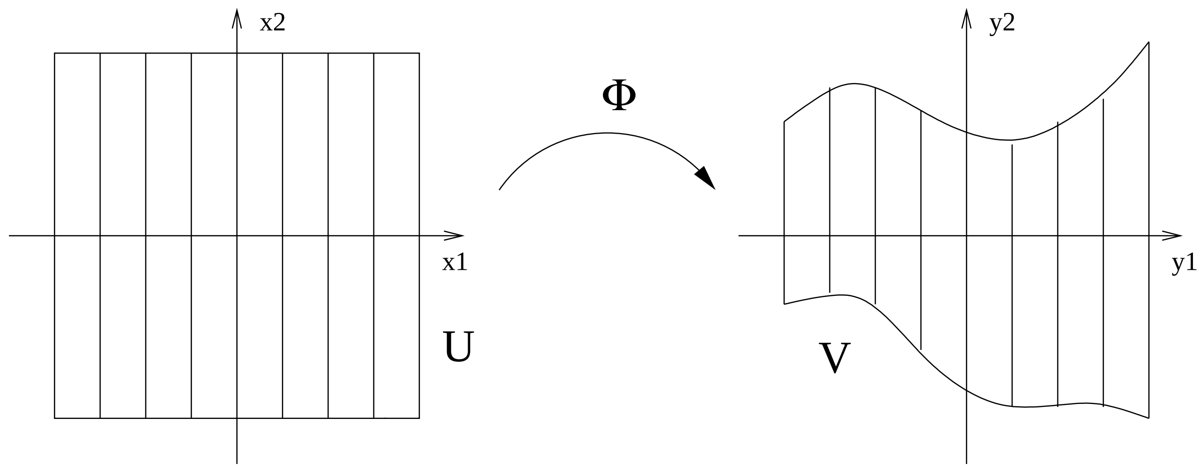

Figure 2.1: The Φ map

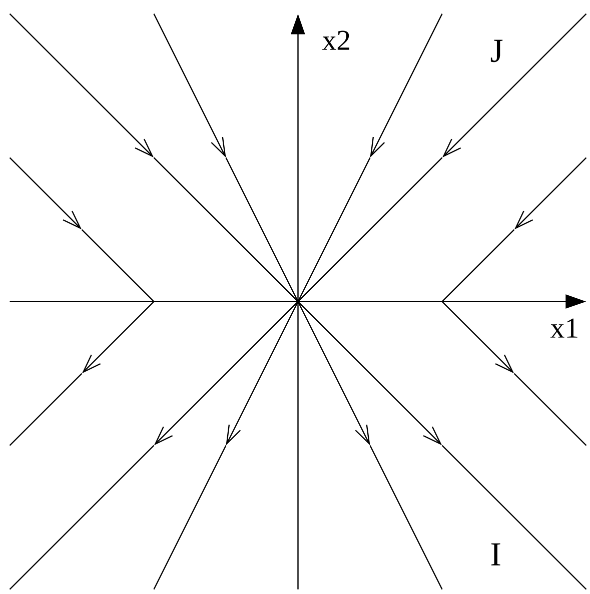

Figure 2.2: Flow lines of the example

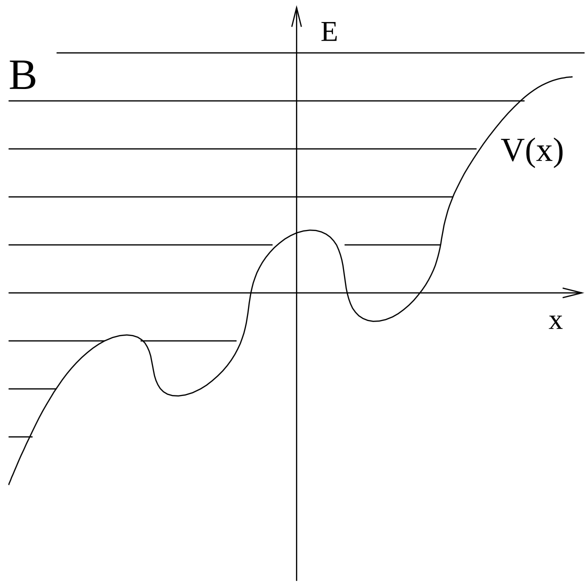

Figure 2.3: The domain B

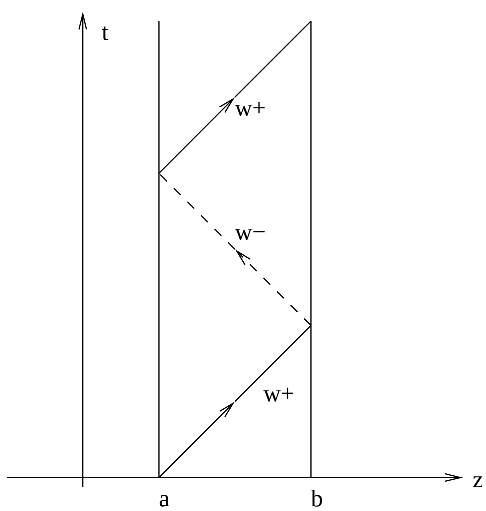

Figure 2.4: behaviour of w_+ and w_-

Chapter 3

Résolution de l'Equation de Liouville pour des forces d'interactions BV_{loc} avec singularité à l'origine.

Abstract We prove the existence and uniqueness of renormalized solutions of the Liouville equation for n particles with a interaction potential in BV_{loc} execpt at the origin. This implies the existence and uniqueness of a a.e. flow solution of the associated ODE.

We consider the Liouville (or transport) equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} v_i \cdot \nabla_{x_i} f - \sum_{i \neq j} \nabla V(x_i - x_j) \cdot \nabla_{v_i} f = 0 \tag{0.1}$$

with initial conditions

$$f(0, x_1, \ldots, x_n, v_1, \ldots, v_n) = f^0(x_1, \ldots, x_n, v_1, \ldots, v_n) \tag{0.2}$$

Here $n \in \mathbb{N}$, each x_i and v_i belongs to $\mathbb{R}^d$ for some $d \geq 1$, f is a real function defined on $[0, \infty) \times \mathbb{R}^{2dn}$. We shall always assume that the interaction potential V is such that $\nabla V \in BV_{loc}(\mathbb{R}^d - 0)$. Our goal here is to show the existence and the uniqueness of solutions (in a sense to be made more precise) of (0.1)-(0.2) for each $f^0 \in L^1_{loc}(\mathbb{R}^{2dn})$.

As is well know, this equation is in some sense equivalent to the system of ODE

$$\begin{cases} \dot{X}_i(t) = V_i \\ \dot{V}_i(t) = -\sum_{i\neq j} \nabla V(X_i(t) - X_j(t)) \end{cases} \qquad \forall\, 1 \le i \le n \tag{0.3}$$

We shall also prove the existence and the uniqueness of a flow solution of (0.3) (in a sense to be made more precise).
Here, we will use the method of resolution of transport equations and associated ODE introduced by R. DiPerna and P.L. Lions in 1989 in [DL89]. In this paper, they prove the existence and the uniqueness of the solution of a transport equation when the vector field belongs to $W^{1,1}_{loc}$, and use this to obtain a unique flow solution of the ODE. In the note [Lio98], P.L. Lions extend this result to piecewise $W^{1,1}$ vector-field and give a clearer proof of the equivalence between the existence and uniqueness of a solution of the transport equation and the existence and the uniqueness of a flow solution of the ODE. In [Bou01], F. Bouchut extend the result to the kinetic case with a force field in BV_{loc}. We will often use this result in this article (see theorem 1). In the case of two dimensionnal vector-field, we also refer to the work of F. Bouchut and L. Desvillettes [BD01] in which the case of divergence free vector-field with continuous coefficient is treated, and to my precedent work [Hau03] in which this result is extented to vector-field with L^2_{loc} coefficients with a condition of regularity on the direction of the vector-field, and to the one dimensionnal kinetic case with a force in L^1_{loc}. The most recent work [Amb03] in this domain is from L. Ambrosio and extend the existence and uniqueness result to BV_loc vector field.
Let us now define precisely what we mean by a solution of (0.1)-(0.2).

Definition 3.1. *Given an initial condition in L^∞, a solution of (0.1)-(0.2) is a function $f \in L^\infty([0,\infty)\times\mathbb{R}^{2dn})$ satisfying for all $\phi \in C_0^\infty([0,\infty)\times\mathbb{R}^{2dn} - I)$*

$$\int_{[0,\infty)\times\mathbb{R}^{2dn}} f\left(\frac{\partial\phi}{\partial t} + \sum_{i=1}^{n} v_i\cdot\nabla_{x_i}\phi - \sum_{i\neq j}\nabla V(x_i - x_j)\cdot\nabla_{v_i}\phi\right) = -\int_{\mathbb{R}^{2dn}} f^0\phi(0,\cdot) \tag{0.4}$$

where $I = \{(x_1,\ldots,x_n,v_1,\ldots,v_n) | \exists i \neq j, \quad x_i = x_j\}$, the set of all configurations in which at least two particles are at the same place.
We will also use the notion of solution on the whole space. By this we mean a function $f \in L^\infty([0,\infty)\times\mathbb{R}^{2dn})$ satisfying (0.4) for all $\phi \in C_0^\infty([0,\infty)\times\mathbb{R}^{2dn})$.

Remark that usually, the definition of solutions is the second one. Indeed, as we shall see later, this two definitions are equivalent if $\nabla V \in L^1$ near the origin. But we shall also deal with potentials that do not satisfy this condition, and then the quantities in (0.4) are not defined for any $\phi \in C_0^\infty$. This is why we introduce this two notions of solutions. Moreover, we want to find solutions for every initial conditions in L^1_{loc}, but with this assumption, the products in (0.4) are not necessary well defined. Thus, we introduced as in the work of P.L. Lions and R. DiPerna the notion of renormalized solution defined below.

Definition 3.2. *We shall say that a mesurable function f is a renormalized solution (resp. a renormalized solution on the whole space) if $\beta(f)$ is a solution (resp. a solution on the whole space) of (0.1) with initial conditions $\beta(f^0)$, for all $\beta \in C^1_b(\mathbb{R})$, the set of differentiable functions from $\mathbb{R}$ into $\mathbb{R}$ with a bounded continuous derivative.*

In our proof, we will often use the following result, proved by F. Bouchut in [Bou01],

Theorem 3.1. *Let $f \in L^\infty$ be a solution of the following equation on $\Omega \times \mathbb{R}^m$, where Ω is an open subset of $\mathbb{R}^m$*

$$\partial_t f + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = 0 \tag{0.5}$$

where the force field F belongs to $BV_{loc}(\Omega)$. Then, this solution is also a renormalized one. In other words, $\beta(f)$ is also a solution of the equation (0.5) for every $\beta \in C^1_b(\mathbb{R})$.

This kind of result is very useful, because it implies the existence and the uniqueness of the solution of the transport equation and of a flow solution of the associated ODE (in a sense which will be defined later on). Here, we shall extend the result of F. Bouchut to vector-fields with one singularity at the origin.

3.1 Existence and uniqueness of the solutions of the Liouville equation

Theorem 3.2. *Assume that $d \geq 2$, $\nabla V \in BV_{loc}(\mathbb{R}^d - 0)$, $\nabla V \in L^1_{loc}$ near the origin, and that there exists a positive constant C such that $V(x) \geq -C(1+|x|^2)$ a.e.. Then, for every initial condition in L^1_{loc}, there exists a unique renormalized solution to (0.1)-(0.2).*

Remark that with our assumptions, the potentials is bounded by below. So, we do not deal with the case of the attractive coulombian potential, by instance.

Proof. First, we will prove that in this case, a bounded solution is always a solution on the whole space.
Step 1. Equivalence between the two notions of solutions.
Let $f \in L^\infty(\mathbb{R} \times \mathbb{R}^{2dn})$ be a solution of (0.1). We want to prove that it is also a solution on the whole space. In order to show this fact, we choose a $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $(1-\phi)$ has his support in $B(0,1)$, the open ball of radius one centered at the origin, and that $\int(1-\phi) = 1$, and we defined, for every $\epsilon > 0$, $\phi_\epsilon = \phi(\cdot/\epsilon)$. Moreover, we denote

$$\Phi(x_1, \ldots, x_n) = \prod \phi_{\epsilon_{i,j}}(x_i - x_j)$$

where the product run over all the set of two indices $\{i,j\}$ except the set $\{1,2\}$, and where $\epsilon_{i,j}$ depends of $\{i,j\}$. We also choose an $\epsilon_{1,2}$ that we will denote by μ for simplification in the following.
Next, we choose a test function $\Psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{2dn})$. Since $\Psi\Phi\phi_\mu(x_1 - x_2)$ has a compact support in $\mathbb{R} \times (\mathbb{R}^{2dn} - I)$, we can write,

$$\int_{\mathbb{R}\times\mathbb{R}^{2dn}} f\phi_\mu(x_1-x_2)\left(\frac{\partial\Psi}{\partial t}\Phi + \sum_{i=1}^n v_i \cdot \nabla_{x_i}(\Psi\Phi) + \sum_{i\neq j} \Phi\nabla V(x_i - x_j)\cdot\nabla_{v_i}\Psi\right) + \int_{\mathbb{R}\times\mathbb{R}^{2dn}} f\Phi\Psi\frac{1}{\mu}\phi\left(\frac{x_1-x_2}{\mu}\right)\cdot(v_1 - v_2) = 0 \quad (1.1)$$

When μ goes to 0, the first integral goes to

$$\int_{\mathbb{R}\times\mathbb{R}^{2dn}} f\left(\frac{\partial\Psi}{\partial t}\Phi + \sum_{i=1}^n v_i \cdot \nabla_{x_i}(\Psi\Phi) + \sum_{i\neq j} \Phi\nabla V(x_i - x_j)\cdot\nabla_{v_i}\Psi\right)$$

For the second integral, we may write

$$\begin{aligned}\left|\int_{\mathbb{R}\times\mathbb{R}^{2dn}} f\Phi\Psi\frac{1}{\mu}\phi\left(\frac{x_1-x_2}{\mu}\right)\cdot(v_1 - v_2)\right| &\leq \frac{M}{\mu}\int_{|x_1-x_2|\leq\mu}\Psi \\ &\leq M'\mu^{d-1}\end{aligned}$$

and since $d \geq 2$, the second integral goes to zero.
Then, we have that

$$\int_{\mathbb{R}\times\mathbb{R}^{2dn}} f\left(\frac{\partial\Psi}{\partial t}\Phi + \sum_{i=1}^n v_i \cdot \nabla_{x_i}(\Psi\Phi) - \sum_{i\neq j} \Phi\nabla V(x_i - x_j)\cdot\nabla_{v_i}\phi\right) = 0 \quad (1.2)$$

Next, we can write $\Phi = \Phi'\phi_{\epsilon 1,3}$. It is possible only if $n \geq 3$, but in the case $n = 2$, $\Phi = 1$ and we have already prove what we want. We make the same argument. Let as above $\epsilon_{1,3}$ going to zero and obtain (1.2) with Φ replaced by Φ'. At this point, we can go on and do this with all the couple (i,j), with $i \neq j$. At the end, we can delete Φ in the equality (1.1). We obtain the equation (0.4). Then, f is a solution on the whole space.

Step 2. Every L^∞-solution is a renormalized solution on the whole space.
Let $f \in L^\infty$ be a solution of (0.1). We choose a $\beta \in C_b^1(\mathbb{R})$. By using the theorem 1 and because the notion of renormalisation is local, we obtain that $\beta(f)$ is a solution of (0.1). But by the step one, we know that $\beta(f)$ is a solution on the whole space. Since this is true for every $\beta \in C_b^1(\mathbb{R})$, f is a renormalized solution on the whole space.

Step 3. Uniqueness for solution in $L^\infty([0,+\infty) \times \mathbb{R}^{2dn})$.
We choose two solutions f and $g \in L^\infty([0,+\infty) \times \mathbb{R}^{2dn})$ of (0.1) with the same initial condition, and a $\beta \in C_b^1(\mathbb{R},\mathbb{R})$, non-negative, with $\beta(0) = 0$. By step 2 and the linearity of the equation, $h = \beta(f-g)$ is also a solution on the whole space of (0.1) with vanishing initial conditions.
Next, we choose a function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on $(-\infty, 1)$ and $\psi \equiv 0$ on $(2,+\infty)$. We also define on $\mathbb{R}^{2dn}$ the energy E of a configuration which is given by

$$E(x_1,\dots,x_n,v_1,\dots,v_n) = \frac{1}{2}\sum_{i\neq j} V(x_i - x_j) + \sum_{i=1}^{n} \frac{|v_i|^2}{2}$$

Remark that the assumption $V(x) \geq -C(1+|x|^2)$ implies that there exists another constant $C > 0$ such that if $E \leq R^2$ and all the $|x_i| \leq R$ for all i, then $|v_i| \leq C(1+R)$ for all i. Roughly, if our particles are initially in a bounded region, their speeds will remain bounded on every compact interval of time. We will use this fact to prove the uniqueness. For every $T > 0$ and $R \geq 0$, we define $\phi_{R,T} = \psi(\sqrt{1+\sum |x_i|^2} - (R+1)e^{C'(T-t)} - 2)\psi(E/R^2)$, with $C' = nC$. $\partial_t \phi_{R,T} \in L^\infty_{loc}$, $\nabla_{x_i}\phi_{R,T} \in L^1$ and $\nabla_{v_i}\phi_{R,T} \in L^\infty_{loc}$, so we may multiply the distribution h by the function $\phi_{R,T}$. We compute

$$\frac{\partial(h\phi_{R,T})}{\partial t} = \phi_{R,T}\frac{\partial h}{\partial t} + \frac{\partial \phi_{R,T}}{\partial t}h \tag{1.3}$$

and we obtain

$$\frac{\partial(h\phi_{R,T})}{\partial t} = -\phi_{R,T}\left(\sum_{i=1}^{n} v_i \cdot \nabla_{x_i} h - \sum_{i\neq j} \nabla V(x_i - x_j)\cdot \nabla_{v_i} h\right) + \frac{\partial \phi_{R,T}}{\partial t}h \tag{1.4}$$

Then, if we integrate this equation with respect to x,v and use integration by parts, we obtain

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^{2dn}} h(t,\cdot)\phi_{R,T}\right) = \int_{\mathbb{R}^{2dn}} h(t,\cdot)\psi'(\sqrt{1+\sum|x_i|^2}-(R+1)e^{C'(T-t)}-2)\psi(E/R^2)\times\ldots$$
$$(\sum v_i\cdot B_i - C(R+3)e^{C(T-t)}) \quad (1.5)$$

where the term $|B_i| = |\partial_i(\sqrt{(1+\sum|x_i|^2})| = |x_i|/\sqrt{(1+\sum|x_i|^2}$ is bounded by 1. It is useful there to use the energy in the test function because many terms vanish when we perform the computation, since E is invariant by the flow. Now, when $\Phi_{R,T}$ do not vanish, it means that $E \le R^2$ and $|x_i| \le CR$ for all i. Then, we deduce that we have

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^{2dn}} h(t,\cdot)\phi_{R,T}\right) \le 0 \quad (1.6)$$

because when $\Phi_{R,T}$ do not vanish, $E \le R^2$ and $|x_i| \le R$ for all i. And in this conditions we have $\sum v_i\cdot B_i - C(R+3)e^{C(T-t)} \le 0$ and ψ' is nonpositive. Since h vanishes at $t=0$, this means that

$$\int_{\mathbb{R}^{2dn}} h(T,\cdot)\phi_{R,T} = 0$$

Since this is true for every R and every T, and since h is nonnegative, we obtain that h vanishes almost everywhere on $[0,\infty)\times\mathbb{R}^{2dn}$. This is true for every $\beta \in C^1_b(\mathbb{R})$ satisfying $\beta(0)=0$. Therefore, $f=g$ a.e..

Step 4. Existence and uniqueness for initial conditions in L^1_{loc}.

First, we remark that if $f^0 \in L^\infty$, it is easy to obtain a solution on the whole space of (0.1)-(0.2) by regularisation of the force field and the use of weak limits. In addition, in view of the result obtained in step 3, we obtain that, if $f^0 \in L^\infty$, there exists a unique solution of (0.1)-(0.2) which is also a renormalized solution on the whole space.

Next, let $f^0 \in L^1_{loc}$. For $m \in \mathbb{N}$, we define $\beta_m(x) = (x \wedge m) \vee -m$, and we remark that $\beta_m \circ \beta_p = \beta_p$, if $p \ge m$. For all $m \in \mathbb{N}$, there exists a unique solution of (0.1)-(0.2) corresponding to the initial condition $\beta_m(f^0)$. We denote it by f_m. For every $p \in \mathbb{N}$, f_p is a renormalized solution on the whole space, then $\beta_m(f_p)$ is a solution with intitial conditions $\beta_m(\beta_p(f^0)) = \beta_m(f^0)$. Of course, β_m do not belongs to $C^1_b(\mathbb{R})$ but it can be shown that the renormalisation property is still true for Lipschitz function by regularisation of those functions (see [Bou01]). Then, by the uniqueness of the solution of

(0.1)-(0.2) when the initial condition belongs to L^∞, we obtain that $\beta_m(f_p) = f_m$, for all $p \geq m$. This allows us to define almost everywhere

$$f = lim_{m\to\infty} f_m$$

This measurable function f satisfies $\beta_m(f) = f_m$ for all $m \in \mathbb{N}$. And f is a renormalized solution corresponding to the initial condition f^0 because for every $\beta \in C_b^1(\mathbb{R})$, $\beta \circ \beta_m(x)$ goes to $\beta(x)$ a.e. in x when m goes to $+\infty$. Then, the solution $\beta \circ \beta_m(f)$ of (0.1)-(0.2) with the initial condition $\beta \circ \beta_m(f^0)$ goes a.e. to $\beta(f)$ which is still a solution of (0.1)-(0.2) with the initial condition $\beta(f^0)$, because this linear equation is always satisfied by a weak limit of solutions. This shows the existence of the solution. For the uniqueness, if there existed two solutions f, g for the same initial conditions f^0, there would be a $m \in \mathbb{N}$ such that $\beta_m(f) \neq \beta_m(g)$, and $\beta_m(f)$,$\beta_m(g)$ would be two distinct solutions in L^∞ with the same initial conditions. This would contradict the uniqueness of solutions already proved in that case.
Finally, we remark that we can say something about the integrability of the solution f. Since the speed of propagation is finite on the sets of bounded energy, $f\chi_{E<m} \in L^\infty_{loc}(\mathbb{R}, L^1_{loc})$, for all $m \in \mathbb{R}$, where $\chi_{E<m}$ denote the characteristic function of the set of all the configurations with an energy less than m. □

Theorem 3.3. *Assume now that $\nabla V \in BV_{loc}(\mathbb{R}^d - 0)$, that V is bounded on all compact sets of $\mathbb{R}^d - 0$, that V satisfies $V(x) \geq C(1 + |x|^2)$ a.e. and that V goes to $+\infty$ when $|x|$ goes to 0. Then, there exists a unique renormalized solution of (0.1)-(0.2).*

Proof. The proof will follow the same sketch that the one of the theorem 2, but the difficulties are at others places. First, the existence of solution by regularisation is not so obvious here, because we cannot work on the whole space.
Step 1. Existence of solution with initial condition in L^∞.
We choose a smooth $f^0 \in L^\infty$. We shall show the existence of a solution with this initial condition by regularisation. We choose a regularisation kernel $\rho \in C_0^\infty(\mathbb{R}^d)$, such that $Supp(\rho) \subset B_1$, and that $\int \rho = 1$. We also choose a smooth function α from $\mathbb{R}^n$ into $\mathbb{R}$ satisfying $\alpha(x) \leq min(1, |x|/2)$ for all x. We denote $\rho_\epsilon = \rho(\cdot/\epsilon)$ and define for all integer $n \geq 1$

$$V_n(x) = \int_{\mathbb{R}^d} V(y)\rho_{2^{-n}\alpha(x)}(x-y)\,dy$$

It is a sort of convolution, in which the radius of the ball on which we average V depends on x so that 0 is never in that ball. Hence V_n is well defined in

$\mathbb{R}^d - 0$, belongs to $C_0^\infty(\mathbb{R}^d - 0)$ and satisfies also $V_n(x) \to +\infty$ when $|x| \to 0$. Moreover, $\nabla V_n \to \nabla V$ in $BV_{loc}(\mathbb{R}^d - 0)$ when $n \to \infty$.
Then, if $Y = (X, V)$ is such that $X \notin I$, there exists a unique maximal solution to the ODE with value (X, V) at time $t = 0$. Because of the conservation of the energy, it cannot reaches I and because of property of V_n, it cannot go to infinity in a finite time. Then, this maximal solution is defined for every time. This allows us to define a smooth flow $Y_n(t, \cdot)$ in $\mathbb{R}^{2dn} - I$. And $f_n = f^0(Y_n)$ satisfies the Liouville equation in the classical sense on $\mathbb{R}^{2dn} - I$. Then, f_n also satisfies (0.4), for all test functions $\phi \in C_0^\infty(\mathbb{R}^{2dn} - I)$.
Moreover, the sequence (f_n) is bounded by $\|f^0\|_\infty$ in L^∞, then, up to an extraction, we can assume that $f_n \to f$ weakly in $L^\infty - w*$. And we can pass to the limit in (0.4) and obtain that f is a solution of (0.1)-(0.2). For non smooth initial condition $f^0 \in L^\infty$, we obtain the existence of the solution by regularistion of f^0 and by taking weak limit.
Step 2. Uniqueness of solution for initial conditions in L^∞.
Here we choose an h solution of (0.1) with vanishing intial conditions. Now, with our assumption that $V(x) \to +\infty$ when $|x| \to 0$, the support of function is included in $\mathbb{R}^{2dn} - I$. As in the proof of the theorem 2, we may write the equations (1.3) and (1.4), and not only for test functions vanishing on I, but for every smooth functions with compact support in $\mathbb{R}^{2dn}$, because of the property of the support of $\phi_{R,T}$. So, we obtain (1.5), and then that,

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^{2dn}} h(t,\cdot)\phi_{R,T}\right) \leq 0$$

And this implies the uniqueness of the solution.
The last step about existence and uniqueness of the solution with initial conditions in L^∞_{loc} is the same that in the theorem 2. □

3.2 Resolution of the ordinary differential equation

We are now looking for a solution to the ODE associated to the transport equation. Since the vector-field used in this ODE is not defined everywhere, we cannot solve this ODE for every initial condition. Then, we will solve it globally with a flow, namely a application Y from $\mathbb{R} \times \mathbb{R}^{2dn}$ to $\mathbb{R}^{2dn}$ such that $Y(t, Y_0)$ is the position in the phase space at time t when we start from Y_0 at time 0. Of course, this flow will be defined only almost everywhere. Here we use the notation $Y = (X, V) = (x_1, \ldots, x_n, v_1, \ldots, v_n)$, where X,$V \in \mathbb{R}^{dn}$

and x_i,$v_i \in \mathbb{R}^d$ for all i. This flow shall solve the following system

$$\begin{cases} \dot{x}_i(t,Y) = v_i \\ \dot{v}_i(t,Y) = -\sum_{j\neq i} \nabla V(x_i - x_j) \\ Y(0,X,V) = (X,V) \end{cases} \tag{2.1}$$

If we denote by B the vector-field defined below on $\mathbb{R}^{2dn}$, we may rewrite the two first equations

$$\dot{Y} = B(Y)$$

$$\text{where} \quad B(Y) = (v_1, \dots, v_n, -\sum_{j\neq 1} \nabla V(x_1 - x_j), \dots, -\sum_{j\neq n} \nabla V(x_n - x_j))$$

In our situation of a vector field with low regularity, we have to say more precisely what we will mean by a flow, and in which sense we look at the equation (2.1). This is the aim of the following definition, in which $\chi_{E<m}$ denote the characteristic function of the set of all the configurations with energy less than m.

Definition 3.3. *A flow defined almost everywhere (a.e. flow) solution of the ODE (0.3) is a function Y from $\mathbb{R} \times \mathbb{R}^{2dn}$ to $\mathbb{R}^{2dn}$ such that*

i. $Y\chi_{E<m} \in C(\mathbb{R}, L^1_{loc})^{2dn} \cap L^\infty_{loc}(\mathbb{R}^{2dn+1})$, $\forall m \in \mathbb{R}$

ii. $\int \phi(Y(t,X,V))\,dXdV = \int \phi(X,V)\,dXdV, \quad \forall \phi \in C_0^\infty, \quad \forall t \in \mathbb{R}$

iii. $Y(t+s,Y') = Y(t,Y(s,Y'))$ *a.e. in* Y', $\quad \forall s,t \in \mathbb{R}$

iv. $E \circ Y(t,Y') = E(Y')$ *(the energy is preserved by the flow).*

v. $\dot{Y}\chi_{E<m} = B(Y)\chi_{E<m}$ *is satisfied in the sense of the distributions for all* $m \in \mathbb{R}$, *and* $Y(0,X,V) = (X,V)$ *a.e. on* $\mathbb{R}^{2dn}$.

Remark. We use the truncation $\chi_{E<m}$ because in the region where E is large, the particles may go to infinity very quickly and we cannot expect Y to be integrable. It has the avantage to allow us to give a sense to the EDO without using renormalization, like in [DL89]. But, this definition is not completely satisfactory because we like part iv. to be a consequence of the others points, but I do not know how to do this.
Using the results of the first section, we will prove the existence and the uniqueness of an a.e. flow in the two case seen above. For this, we use the method introduced by R. DiPerna and P.L. Lions in [DL89]. Indeed, we just adapt the argument introduced in [Lio98] for periodic vector-fields.

Theorem 3.4. *Under the two kind of assumptions made in the section 1, there exists a unique a.e. flow solution of (2.1).*

Proof. We first remark, that in the case when V is smooth, a flow solution of an ODE is also a solution of the transport equation (more precisely each component Y_i is the unique solution corresponding to the initial condition $f^0(Y) = Y_i$). Here, we will use this remark, and the fact that we know how to solve the transport equation. We thus denote by Y the solution of the transport equation (0.1) for the initial condition $f^0(Y) = Y$. We will prove that this defined an a.e. flow solution of (2.1).
In the first section, we have shown that Y is a renormalized solution of (0.1). Let us recall that it means that $\beta(f)$ is a solution of the Liouville equation, for every $\beta \in C^1$. But here the initial condition belongs to L^∞_{loc}. And we point out that in both cases of section 1, we have proved that the speed of propagation is finite on the set where the energy is bounded. Then, assume that f is a solution with an initial condition given by $f^0 \in L^\infty_{loc}$. We choose a smooth function $\psi \in C_0^\infty(\mathbb{R})$. We may prove adapting the argument made in section 1, that for every R and $T \in \mathbb{R}$, there exists a constant $R' > 0$ such that

$$\int_{|x_i|,|v_i| \leq R} \beta(f(t,Y))^n \psi^n(E)\, dY \leq \int_{|x_i|,|v_i| \leq R'} \beta(f^0)^n \psi^n(E)\, dY$$

for all $\beta \in C^1_b$, and all $n \in \mathbb{N}$. Since this is true for all n and all β we obtain that for every $m \in \mathbb{R}$, $f\chi_{E<m} \in L^\infty_{loc}(\mathbb{R}^{2dn+1})$. This implies that $f\chi_{E<m}$ is a solution (not only a renormalized solution) of (0.1).
Next, we shall show that we can extend the renormalisation property to functions of several variables. More precisely, if $G \in C(\mathbb{R}^k)$ and $f_1, \dots, f_k$ are solution in L^∞_{loc}, then $G(f_1, \dots, f_k)$ is also a solution of the same equation, with initial conditions $G(f_1^0, \dots, f_k^0)$. Let us show the proof for $k = 2$ for example.
Thus, take f and $g \in L^\infty_{loc}$ two solutions of the transport equation (0.1). Next, $(f+g)$, $(f-g)$ are also solutions by linearity, and so are $(f+g)^2$,$(f-g)^2$, and finally $fg = (1/4)[(f+g)^2 - (f-g)^2]$. Doing this again, we can show that P(f,g) is also a solution for all P polynomial in two variables. And using the density of the polynomials, we finally obtain that G(f,g) is a solution for every continious G.
Then, for all $f^0 \in C_0^\infty(\mathbb{R}^{2dn})$, $f^0(Y(t,Y'))\chi_{E<m}$ is the solution with initial conditions $f^0\chi_{E<m}$. Letting m going to ∞, we obtain that $f^0(Y(t,Y'))$ is the solution with initial conditions f^0. And this is true for every $f^0 \in L^\infty$ by approximation. Next, since the Liouville equation preserves the total mass, we obtain that $\int f(Y(t,Y'))\, dY' = \int f(Y')\, dY'$, for every smooth f. This implies the part ii. of the definition of an a.e. flow (the conservation of the Lebesgue measure).

For the group property $Y(s+t,Y') = Y(t,Y(s,Y'))$ a.e. in Y', we choose a fixed t and a sequence of smooth function going to $Y(t,\cdot)$ in L^1_{loc}. Because of the part ii. of the definition, $f(Y(s,\cdot))$ goes to $Y(t,Y(s,\cdot))$. But, since f goes to $Y(t,\cdot)$ in L^1_{loc}, $f(t,\cdot)\chi_{E<m}$ goes in L^1_{loc} to the solution of (0.1) with initial conditions $Y(t,\cdot)\chi_{E<m}$ at time s. This is $Y(s+t,\cdot)\chi_{E<m}$. And the group properties follows.
To show that the energy E is invariant by the flow (part iv.), remark that $E\circ Y$ and E are two solutions of (0.1) with the same initial conditions E. Then, they are equal.
In order to show the part v., we choose $\phi \in C_0^\infty(\mathbb{R}^{2dn})$ and $\psi \in C_0^\infty(\mathbb{R})$. We will use the function $\phi\psi$ as test function. It is sufficient to use only this type of functions to show that f satisfy the equation, because linear combinations of such functions are dense in the space $C_o^1(\mathbb{R}\times\mathbb{R}^{2dn})$. We compute for all $i \le 2dn$, where the index i denote the $i-th$ component of vector in $\mathbb{R}^{2dn}$

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}^{2dn}} Y_i(t,Y')\chi_{E<m}\phi(Y')\frac{\partial\psi}{\partial t}(t)\,dYdt \\
&\qquad = \int_{\mathbb{R}\times\mathbb{R}^{2dn}} Y_i(-t,Y')\chi_{E<m}\phi(Y')\frac{\partial\psi}{\partial t}(-t)\,dY'dt \\
&\qquad = \int_{\mathbb{R}\times\mathbb{R}^{2dn}} \phi(Y(t,Y'))\chi_{E<m}Y_i'\frac{\partial\psi}{\partial t}(-t)\,dY'dt
\end{aligned}$$

To obtain the second equation from the first, we use the change of variable $Y(t,\cdot)$. And we remark that $\chi_{E<m}\circ Y = \chi_{E<m}$, since the energy is invariant by the flow.
Moreover, we know that $\phi(Y(t,Y'))$ is the solution of the transport equation (0.1) with initial conditions ϕ. We use this to write

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}^{2dn}} Y_i(t,Y')\chi_{E<m}\phi(Y')\frac{\partial\psi}{\partial t}(t)\,dY'dt \\
&\qquad = -\int_{\mathbb{R}\times\mathbb{R}^{2dn}} \phi(Y(t,Y'))\chi_{E<m}B_i(Y')\psi(-t)\,dY'dt \\
&\qquad = -\int_{\mathbb{R}\times\mathbb{R}^{2dn}} B_i(Y(-t,Y'))\chi_{E<m}\phi(Y')\psi(-t)\,dY'dt \\
&\qquad = -\int_{\mathbb{R}\times\mathbb{R}^{2dn}} B_i(Y(t,Y'))\chi_{E<m}\phi(Y')\psi(t)\,dY'dt
\end{aligned}$$

And this shows the part iv.. Therefore, the existence of such a solution is proven. Remark that in the second case we can delete $\chi_{E<m}$ if we only use test functions whose support does not contain 0.
For the uniqueness of the a.e. flow, we will show that the five properties satisfied by this flow implies that all his components are solutions of the Liouville

equation. This is sufficient to show the uniqueness of an a.e. flow because we already know the uniqueness of the solution of the Liouville equation.
We choose $\phi \in C_0^\infty(\mathbb{R}^{2dn})$ and $\psi \in C_0^\infty(\mathbb{R})$ and use $\phi\psi$ as test function. We have for all $i \leq 2dn$ that

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}^{2dn}} & Y_i(t,Y')\phi(Y')\chi_{E<m}\frac{\partial\psi}{\partial t}(t)\,dY'dt \\ &= \int_{\mathbb{R}\times\mathbb{R}^{2dn}} Y_i'\phi(Y(-t,Y'))\chi_{E<m}\frac{\partial\psi}{\partial t}(t)\,dY'dt \\ &= -\int_{\mathbb{R}\times\mathbb{R}^{2dn}} Y_i'\chi_{E<m}\frac{\partial}{\partial t}\big(\phi(Y(-t,Y'))\big)\psi(t)\,dY'dt\end{aligned}$$

In the first equality we use the change of variable $Y' = Y(t, Y')$, and the second one is deduced by an integration by parts. Remark that we use the preservation of the energy by the flow in every change of variable. But we can show that in a L^1_{loc}-sense,

$$\frac{\partial}{\partial t}\big(\phi(Y(-t,Y'))\chi_{E<m}\big) = -\nabla\phi(Y(-t,Y'))\cdot B(Y(-t,Y'))\chi_{E<m}$$

This, because for t fixed, $\frac{Y(t+h,Y')-Y(t,Y')}{h}\chi_{E<m} \to B(Y(t,Y'))\chi_{E<m}$ in $L^1_{loc}(\mathbb{R}^{2dn})$ when $h \to 0$. Let us show this fact. Indeed, if we look at the five properties satistied by an a.e. flow, we can show that

$$Y(t,Y')\chi_{E<m} = Y'\chi_{E<m} + \int_0^t B(s,Y(s,Y')\,ds \quad \text{a.e. in Y', } \forall t \in \mathbb{R}.$$

It remains to show that $\chi_{E<m}B(Y) \in C(\mathbb{R}, L^1_{loc})$ to obtain the result. For this, if B is replaced by a smooth and bounded B_ϵ, this is true, because $Y\chi_{E<m} \in C(\mathbb{R}, L^1_{loc})$. And this is still true for B because Y preserves the Lebesgue measure and because the energy is preserved by the flow.
Then, we obtain if we use the change of variables backwards

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}^{2dn}} & Y_i(t,Y')\chi_{E<m}\phi(Y')\frac{\partial\psi}{\partial t}(t)\,dY'dt \\ &= \int_{\mathbb{R}\times\mathbb{R}^{2dn}} Y_i(t,Y')\chi_{E<m}B(x)\cdot\nabla\phi(x)\psi(t)\,dY'dt \quad (2.2)\end{aligned}$$

And $Y_i(t, Y')$ satisfies (0.1) and the proof is complete.

□

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Chapter 4

Approximation de l'équation de Vlasov par des systèmes de particules

Abstract. We prove the convergence in any time interval of a point-particle approximation of the Vlasov equation by particles initially equally separated for a force in $1/|x|^\alpha$, with $\alpha \leq 1$. We introduce discrete versions of the L^∞ norm and time averages of the force field. The core of the proof is to show hat these quantities are bounded and that consequently the minimal distance between particles in the phase space is bounded from below.
Key words. Derivation of kinetic equations. Particle methods. Vlasov equations.

4.1 Introduction

We are interested here by the validity of the modeling of a continuous media by a kinetic equation, with a density of presence in space and velocity. In other words, do the trajectories of many interacting particles follow the evolution given by the continuous media if their number is sufficiently large? This is a very general question and this paper claims to give a (partial) answer only for the mean field approach.

Let us be more precise. We study the evolution of N particles, centered at $(X_1, \ldots, X_n)$ in $\mathbb{R}^d$ with velocities $(V_1, \ldots, V_n)$ and interacting with a central force $F(x)$. The positions and velocities satisfy the following system of ODEs

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = E(X_i) = \displaystyle\sum_{j \neq i} \frac{\alpha_i\, \alpha_j}{m_i}\, F(X_i - X_j), \end{cases} \tag{1.1}$$

where the initial conditions $(X_1^0, V_1^0, \ldots, X_n^0, V_n^0)$ are given. The prime example for (1.1) consists in charged particles with charges α_i and masses m_i, in which case $F(x) = -x/|x|^3$ in dimension three.
To easily derive from (1.1) a kinetic equation (at least formally), it is very convenient to assume that the particles are identical which means $\alpha_i = \alpha_j$. Moreover we will rescale system (1.1) in time and space to work with quantities of order one, which means that we may assume that

$$\frac{\alpha_i\,\alpha_j}{m_i} = \frac{1}{N}, \quad \forall i, j. \tag{1.2}$$

We now write the Vlasov equation modelling the evolution of a density f of particles interacting with a radial force in $F(x)$. This is a kinetic equation in the sense that the density depends on the position and on the velocity (and of course on the time)

$$\begin{aligned}
&\partial_t f + v\cdot\nabla_x f + E(x)\cdot\nabla_v f = 0, \quad t\in\mathbb{R}_+,\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d,\\
&E(x) = \int_{\mathbb{R}^d} \rho(t,y)\,F(x-y)\,dy,\\
&\rho(t,x) = \int_v f(t,x,v)\,dv.
\end{aligned} \tag{1.3}$$

Here ρ is the spatial density and the initial density f^0 is given.
When the number N of particles is large, it is obviously easier to study (or solve numerically) (1.3) than (1.1). Therefore it is a crucial point to determine whether (1.3) can be seen as a limit of (1.1).
Remark that if $(X_1, \ldots, X_N, V_1, \ldots, V_n)$ is a solution of (1.1), then the measure

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^n \delta(x - X_i(t)) \otimes \delta(v - V_i(t))$$

is a solution of the Vlasov equation in the sense of distributions. And the question is whether a weak limit f of μ_N solves (1.3) or not. If F is C^1 with compact support, then it is indeed the case (it is proved in the book by Spohn [Spo91] for example). The purpose of this paper is to justify this limit if

$$|F(x)| \leq \frac{C}{|x|^\alpha}, \quad |\nabla F(x)| \leq \frac{C}{|x|^{1+\alpha}} \quad |\nabla^2 F(x)| \leq \frac{C}{|x|^{2+\alpha}}, \quad \forall x \neq 0, \tag{1.4}$$

for $\alpha < 1$, which is the first rigorous proof of the limit in a case where F is not necessarily bounded.
Before being more precise concerning our result, let us explain what is the meaning of (1.1) in view of the singularity in F. Here we assume either

that we restrict ourselves to the initial configurations for which there are no collisions between particles over a time interval $[0,\ T]$ with a fixed T, independent of N. Or we assume that F is regular or regularized but that the norm $\|F\|_{W^{1,\infty}}$ may depend on N; This procedure is well presented in [Bat01] and it is the usual one in numerical simulations (see [VA91] and [Wol00]). In both cases, we have classical solutions to (1.1) but the only bound we may use is (1.4).
Other possible approaches would consist in justifying that the set of initial configurations $X_1(0),\ldots,X_N(0),V_1(0),\ldots,V_N(0)$ for which there is at least one collision, is negligible or that it is possible to define a solution (unique or not) to the dynamics even with collisions.
Finally notice that the condition $\alpha < 1$ is not unphysical. Indeed if F derives from a potential, $\alpha = 1$ is the critical exponent for which repulsive and attractive forces seem very different. In other words, this is the point where the behavior of the force when two particles are very close takes all its importance.

4.1.1 Important quantities

The derivation of the limit requires a control on many quantities. Although some of them are important only at the discrete level, many were already used to get the existence of strong solutions to the Vlasov-Poisson equation (we refer to [Hor81], [Hor82] and [Pfa92], [Sch91] as being the closest from our method).
The first two are quite natural and are bounds on the size of the support of the initial data in space and velocity,

$$R(T) = \sup_{t\in[0,T],\, i=1,\ldots N} |X_i(t)|, \quad K(T) = \sup_{t\in[0,T],\, i=1,\ldots N} |V_i(t)|. \tag{1.5}$$

Of course R is trivially controlled by K since

$$R(T) \leq R(0) + T\,K(T). \tag{1.6}$$

Now a very important and new parameter is the discrete scale of the problem denoted ε. This quantity represents roughly the minimal distance between two particles or the minimal time interval which the discrete dynamics can see. We fix this parameter from the beginning and somehow the main part of our work is to show that it is indeed correct, so take

$$\varepsilon = \frac{R(0)}{N^{1/2d}}. \tag{1.7}$$

At the initial time, we will choose our approximation so that the minimal distance between two particles will be of order ε.
The force term cannot be bounded at every time for the discrete dynamics (a quantity like $F\star\rho_N$ is not bounded even in the case of free transport), but we can expect that its average on a short interval of time will be bounded. So we denote

$$\overline{E}(T)=\sup_{t\in[0,\,T-\varepsilon],i=1,\ldots,N}\left\{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}|E(X_i(s))|\,ds\right\},\tag{1.8}$$

with for $T<\varepsilon$

$$\overline{E}(T)=\sup_{i=1,\ldots,N}\left\{\frac{1}{\varepsilon}\int_0^{T}|E(X_i(s))|\,ds\right\},\tag{1.9}$$

thus obtaining a unique and consistant definition for all $T>0$. Moreover we denote by E^0 the supremum over all i of $|E(X_i(0))|$.
This definition comes from the following intuition. The force is big when two particles are close together. But if their speeds are different, they will not stay close for a long time. So we can expect the interaction force between these two particles to be integrable in time even if they "collide". There just remains the case of two close particles with almost the same speed. To estimate the force created by them, we need an estimate on their number. One way of obtaining it is to have a bound on

$$m(T)=\sup_{t\in[0,T],i\neq j}\frac{\varepsilon}{|X_i(t)-X_j(t)|+|V_i(t)-V_j(t)|}.\tag{1.10}$$

The control on m requires the use of a discretized derivative of E, more precisely, we define for any exponent $\beta\in\,]1,\ d-\alpha\,[$, which also satisfies $\beta<2d-3\alpha$ ($\beta=1$ would be enough for short time estimates)

$$\Delta\overline{E}(T)=\sup_{t\in[0,\,T-\varepsilon]}\ \sup_{i,j=1,\ldots,N,}\left\{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\frac{|E(X_i(s))-E(X_j(s))|}{\varepsilon^\beta+|X_i(s)-X_j(s)|}\,ds\right\},\tag{1.11}$$

with as for $\overline{E}$, when $T<\varepsilon$

$$\Delta\overline{E}(T)=\sup_{i,j=1,\ldots,N}\left\{\frac{1}{\varepsilon}\int_0^{T}\frac{|E(X_i(s))-E(X_j(s))|}{\varepsilon^\beta+|X_i(s)-X_j(s)|}\,ds\right\}.\tag{1.12}$$

Now, we introduce what we called the discrete infinite norm of the distribution of the particle μ_N. This quantity is the supremum over all the boxes of size ε of the total mass they contain divided by the size of the box. That is, for a measure μ we denote

$$\|\mu\|_{\infty,\varepsilon} = \frac{1}{(2\varepsilon)^{2d}} \sup_{(x,v)\in\mathbb{R}^{2d}} \{\mu(B_\infty((x,v),\varepsilon))\}. \tag{1.13}$$

where $B_\infty((x,v),\varepsilon)$ is the ball of radius ε centered at (x,v) for the infinite norm. Note that we may bound $\|\mu_N(T,\cdot)\|_{\infty,\varepsilon}$ by

$$\|\mu_N(T,\cdot)\|_{\infty,\varepsilon} \leq (4\, m(T))^{2d}. \tag{1.14}$$

We may also introduce discrete L^∞ norm at other scales by defining in general

$$\|\mu\|_{\infty,\eta} = \frac{1}{(2\eta)^{2d}} \sup_{(x,v)\in\mathbb{R}^{2d}} \{\mu(B_\infty((x,v),\eta))\}. \tag{1.15}$$

The quantities R, K, m will always be assumed to be bounded at the initial time $T=0$ uniformly in N.

4.1.2 Main results

The main point in the derivation of the Vlasov equation is to obtain a control on the previous quantities. We first do it for a short time as given by

Theorem 4.1. *If $\alpha < 1$, there exists a time T and a constant c depending only on $R(0)$, $K(0)$, $m(0)$ but not on N such that for some $\alpha < \alpha' < 3$*

$$R(T) \leq 2\,(1+R(0)), \quad K(T) \leq 2\,(1+K(0)), \quad m(T) \leq 2\,m(0),$$
$$\overline{E}(T) \leq c\,(m(0))^{2\alpha'}\,(K(0))^{\alpha'}\,(R(0))^{\alpha'-\alpha}, \quad \sup_{t\leq T} \|\mu_N(t,\cdot)\|_{\infty,\varepsilon} \leq (8\,m(0))^{2d}.$$

Remark

The constant 2, which appears in the bounds, is of course only a matter of convenience. This means that another theorem could be written with 3 instead of 2 for instance; The time T would then be larger. However increasing this value is not really helpful because the kind of estimates which we use for this theorem blow up in finite time, no matter how large the constant in the bounds is.

This theorem can, in fact, be extended on any time interval

Theorem 4.2. *For any time $T > 0$, there exists a function $\tilde{N}$ of $R(0)$, $K(0)$, $m(0)$ and T and a constant $C(R(0),K(0),m(0),T)$ such that if $N \geq \tilde{N}$ then*

$$R(T),\ K(T),\ m(T),\ \overline{E}(T) \leq C(R(0),K(0),m(0),T).$$

From this last theorem, it is easy to deduce the main result of this paper, which reads

Theorem 4.3. *Consider a time T and sequence $\mu_N(t)$ corresponding to solutions to* (1.1) *such that $R(0)$, $K(0)$ and $m(0)$ are bounded uniformly in N. Then any weak limit f of $\mu_N(t)$ in $L^\infty([0,\ T],\ M^1(\mathbb{R}^{2d}))$ belongs to $L^\infty([0,\ T],\ L^1 \cap L^\infty(\mathbb{R}^{2d}))$, has compact support and is a solution to* (1.3).

Of course the main limitation of our results is the condition $\alpha < 1$ and the main open question is to know what happens when $\alpha \geq 1$. However this condition is not only technical and new ideas will be needed to prove something for $\alpha \geq 1$. It would also be interesting to extend our result to more complicated forces like the ones found in the formal derivation of [JP00].
The second important limitation is that $m(0)$ be uniformly bounded. The two applications of Theorem 4.3 concern the numerical simulation of kinetic equations and a justification of the model through the derivation of the equation in statistical mechanics. Concerning numerical simulation, the approximations of the initial data which are usually chosen imply a bound on $m(0)$. For statistical mechanics, determining the initial data is more of a problem. A natural way would be to take identically distributed particles; In that case, the average distance in the phase space between one particle and the closest one, is of the order of $\varepsilon \sim N^{-1/2d}$. However the probability that the minimal distance between any particles be always at least ε decreases exponentially fast with N, making the assumption on $m(0)$ much more restrictive.
Finally the two conditions of compact support in space and velocity are very usual, for instance to prove the existence of "strong solutions" to Vlasov equations. In the case $\alpha < 1$ which we consider here, getting strong solutions is rather easy which explains why passing from Theorem 4.1 to Theorem 4.2 "only" requires the proof of the almost preservation of discrete L^∞ bounds. For the sake of completeness, we recall the proof of existence of strong solutions in an appendix at the end of the paper.
The derivation of kinetic equations is an important question both for numerical and theoretical aspects. The first results for Vlasov equations are due to Neunzert and Wick [NW80], Dobrushin [Dob79] and Braun and Hepp [BH77]. We also refer to works of Batt [Bat01], Spohn [Spo91], Victory and Allen [VA91] and Wollmann [Wol00]. Another interesting case concerns Boltzmann equation, for which we refer to the book by Cercignani, Illner and Pulvirenti [CIP94] and the paper by Illner and Pulvirenti [IP89].
On the other hand, the derivation of hydrodynamic equations is somewhat different and some results are already known (although not since a very long time) even in cases with singularity. In particular and that is more or less the hydrodynamic equivalent of our result, the convergence of the

point vortex method for $2 - D$ Euler equations was obtained by Goodman, Hou and Lowengrub [GHL90] (see also the works by Schochet [Sch95] and [Sch96]). The main part of the proof for hydrodynamic systems consists in controlling the minimal distance between two particles in the physical space (as it is also clear in [JO]). The situation for kinetic equations is different: First of all, such a control is impossible to obtain. And then, having it is not necessary as the two particles could still be far away in the phase space. On the other hand, for a hydrodynamic system, the velocity of a particle only depends on its position in the physical space and therefore two particles with the same position, at a given time, still have the same position at any latter time. As a consequence preventing collisions is really a necessity for a hydrodynamic system; This more or less implies that the proofs are simpler but more demanding for hydrodynamic systems and that a more complex approach is required for kinetic equations.

Our method of proof makes full use of the method of characteristics developed for the Vlasov-Poisson equation in dimension two and three. This method was introduced by Horst in [Hor81] and [Hor82] with the aim of obtaining strong solutions in large time and was, eventually and successfully, used to do that in [Pfa92] and [Sch91]. These results were extended to the periodic case by Batt and Rein in [BR91]. At about the same time strong solutions were obtained by Lions and Perthame in [LP91] with a different method (see also [GJP00] for a slightly simpler proof and [Per96] for an application to the asymptotic behavior of the equation). Their method controls the moments, *i.e.* quantities of the kind $\int |v|^k f\, dv$ with f the solution, and is therefore closer to the notion of weak solutions. It was then applied to the Vlasov-Poisson-Fokker-Planck equation by Bouchut in [Bou95]. Still for the Vlasov-Poisson-Fokker-Planck equation, L^∞ bounds were obtained by Pulvirenti and Simeoni in [PS00], this time with the method of characteristics. The proof is interesting because it also shows the need to integrate in time to control the oscillations of the force. For a given problem, choosing between the method of characteristics and the control of the moments is obviously not easy and could simply be a matter of "taste". The reason why we opted for the characteristics is that it seems more appropriate for a discrete setting. Finally we refer to the book by Glassey [Gla96] for a general discussion of the existence theory for kinetic equations.

In the rest of the paper, C will denote a generic constant, depending maybe on $R(0)$, $K(0)$, or $m(0)$ but not on N or any other quantity. We first prove Theorem 4.1, then we show a preservation of discrete L^∞ norms which proves Theorem 4.2. In the last section we explain how to deduce Theorem 4.3, the appendix being devoted to the proof of existence of strong solutions to (1.3).

4.2 Proof of Theorem 4.1

The first steps are to estimate all quantities in terms of themselves. Then if this is done correctly it is possible to deduce bounds for them on a short interval of time.

4.2.1 Estimate on $\overline{E}$

In this section we will prove a usefull estimate on $\overline{E}$. As explain above, we will decomposate the force that a particle see in the force created by the distants particles, at an order larger than ε, the close particles but with a different speed, again at order ε, and the particles with almost the same position and speed at order ε. So we have three terms to estimate. As we will often have to estimate terms of the same type in the rest of the article, we will in a first lemma prove estimate for all this terms, and unite it in the second lemma.

Lemma 4.1. *We choose an δ in $(0,d)$ and a particles i and assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\,K(t_0)\,\Delta\overline{E}(t_0)}.$$

We defined three subsets of $\{1,\dots,N\}\backslash\{i\}$, G_i, B_i and U_i by

$$G_i = \{j||X_i(t)-X_j(t)| \geq 2K(t)\varepsilon\}$$

$$B_i = \{j||X_i(t)-X_j(t)| \leq 2K(t)\varepsilon \quad \textit{and} \quad |V_i(t)-V_j(t)| \geq 2\overline{E}(t)\varepsilon\}$$

$$U_i = \{j||X_i(t)-X_j(t)| \leq 2K(t)\varepsilon \quad \textit{and} \quad |V_i(t)-V_j(t)| \leq 2\overline{E}(t)\varepsilon\}$$

Then, for any δ' satisfying $\delta \leq \delta' \leq d$, we have the following estimates

$$i.\ \frac{1}{N}\sum_{j\in G_i}\int_{t-\varepsilon}^{t}\frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds \leq \|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\,K^{\delta'}\,R^{\delta'-\delta}$$

If we assume moreover that δ and δ' satisfy $\delta<\delta'<1$, we have the following estimates

$$ii.\ \frac{1}{N}\sum_{j\in B_i}\int_{t-\varepsilon}^{t}\frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds \leq \varepsilon^{d-\delta}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\delta}$$

$$iii.\ \frac{1}{N}\sum_{j\in U_i}\int_{t-\varepsilon}^{t}\frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds \leq \varepsilon^{2d-3\delta}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\delta}\overline{E}^d$$

Proof. The first estimate. For the first point, we denote

$$I_1 = \frac{1}{N} \sum_{j \in G_i} \int_{t-\varepsilon}^{t} \frac{1}{|X_i(s) - X_j(s)|^\delta}\, ds$$

and divided again G_i in

$$G_{i,k} = \left\{ i \,\middle|\, 3\,\varepsilon\, K(t_0)\, 2^{k-1} < |X_i(t_1) - X_1(t_1)| \le 3\,\varepsilon\, K(t_0)\, 2^k \right\}. \tag{2.1}$$

Remark that after $k_0 = (\ln(R(t)/4\,\varepsilon\, K(t_0)))/\ln 2$, the set $G_{i,k}$ is empty.

Approximate stability of the $G_{i,k}$.
Given their definition, the $G_{i,k}$ enjoy the following property: For any $i \in G_{i,k}$ with $k > 1$, we have for any $t \in [t_1,\ t_0]$

$$|X_1(t) - X_i(t)| \ge \varepsilon\, K(t_0)\, 2^{k-1}.$$

Indeed, we of course know that

$$\left| \frac{d}{dt}(X_i(t) - X_1(t)) \right| = |V_i(t) - V_1(t)| \le 2\, K(t_0),$$

and then

$$\begin{aligned} |X_j(t) - X_i(t)| &\ge |X_j(t_1) - X_i(t_1)| - 2\,(t_0 - t_1)\, K(t_0) \\ &\ge 3\,\varepsilon\, K(t_0)\, 2^{k-1} - 2\,\varepsilon\, K(t_0), \end{aligned}$$

with the corresponding result since $k \ge 1$. Of course the same argument also shows that if $i \in B_i$ then for any $t \in [t_1,\ t_0]$,

$$|X_j(t) - X_i(t)| \le 5\,\varepsilon\, K(t_0).$$

This prove also show that B_i is approximately stable.

Sommation over the $G_{i,k}$ Using the result from the previous step, we deduce that for any $j \in G_{i,k}$ with $k \ge 1$,

$$\frac{1}{|X_i(t) - X_1(t)|^\delta} \le \frac{C\, 2^{-\delta k}}{\varepsilon^\delta\, (K(t_0))^\delta}.$$

On the other hand, we have of course $|G_{i,k}| \le N$. Moreover the set of points (x, v) in the phase space with $3\varepsilon K(t_0)\, 2^{k-1} < |x - X(t_1)| < 3\varepsilon K(t_0)\, 2^k$, can be covered by $K^d \times \varepsilon^{-2d} \times (3K(t_0)\, 2^k)^d$ balls of radius ε in the phase space.

According to the definition of the discrete L^∞ norm (1.13), this implies that $|G_{i,k}| \le C\,\varepsilon^{-d}\,K^{2d}\,2^{d\,k} \times \|\mu_N\|_{\infty,\varepsilon}$.
Consequently for any $\delta' < d$, since $\varepsilon^{2d} = C/N$, interpolating between these two values, we get

$$|G_{i,k}| \le C\,N\,(K(t_0))^{2\delta'}\,\varepsilon^{\delta'}\,2^{\delta' k} \times \|\mu_N(t_0,.)\|_{\infty,\varepsilon}^{\delta'/d}.$$

Now we can use this two bounds to compute I_1.

$$\begin{aligned} I_1 &\le \sum_{k=1}^{k_0}\sum_{j\in G_{i,k}} \int_{t-\varepsilon}^{t} \frac{1}{N|X_j(s)-X_i(s)|^\alpha}\,ds \\ &\le \sum_{k=1}^{k_0} |G_{i,k}| \times N^{-1}\,(K(t_0))^{-\delta}\,\varepsilon^{-\delta}\,2^{-\delta\,k} \\ &\le C\,\|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\,K^{2\delta'-\delta}\,\varepsilon^{\delta'-\delta}\sum_{k=1}^{k_0} 2^{(\delta'-\delta)k}. \end{aligned}$$

Eventually for any $\delta < \delta' < 1$, we deduce that

$$I_1 \le C\,\|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\,K^{2\delta'-\delta}\,\varepsilon^{\delta'-\delta}\,2^{(\delta'-\delta)k_0} \le C\,\|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\,r^{\delta'-\delta}\,K^{\delta'}, \qquad (2.2)$$

all the values being taken at t. This gives the point $i.$ in Lemma 4.1.

The second estimate. We denote

$$I_2 = \frac{1}{N}\sum_{j\in B_i}\int_{t-\varepsilon}^{t} \frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds$$

and decompose again the set B_i in

$$B_{i,l} = \left\{ j \in B_i \mid 3\,\varepsilon\,\overline{E}(t_0)\,2^{l-1} < |V_1(t_1) - V_j(t_1)| \le 3\,\varepsilon\,\overline{E}(t_0)\,2^{l} \right\}, \qquad (2.3)$$

for $l \ge 1$. Remark that the set $B_{i,l}$ is empty if $l > l_0 = \ln(K(t_0)/(\varepsilon\,\overline{E}(t_0)))/\ln 2$.
As before we decompose I_2 in

$$I_2 = \sum_{l=1}^{l_0}\sum_{j\in Q_l} \frac{1}{\varepsilon}\int_{t_1}^{t_0} \frac{dt}{N\,|X_j(t)-X_1(t)|^\delta}, \qquad (2.4)$$

The idea behind this new decomposition is that although the particles in $B_{i,l}$ with $l \ge 1$ are close to X_i, their speed is different from V_i. So even if they come very close to X_i they will stay close only for a very short time. Since the singularity of the potential is not too high, we will be able to bound the force.

Approximate stability of the $B_{i,l}$. Just as for the $G_{i,k}$, we may prove that for any time t in $[t_1,\ t_0]$ and any $j \in B_{i,l}$ with $l \geq 1$

$$|V_j(t) - V_i(t)| > \varepsilon\, \overline{E}(t_0)\, 2^{l-1}.$$

This is again due to the fact that

$$|V_j(t) - V_j(t_1)| \leq \int_{t_1}^{t_0} |E(X_j(s))|\, ds \leq \varepsilon\, \overline{E}(t_0),$$

so that in fact the result is even more precise in the sense that the relative velocity $V_j(t) - V_i(t)$ remains close to $V_j(t_1) - V_i(t_1)$ up to exactly $\varepsilon\, \overline{E}(t_0)$. We also remind that B_i was approximately stable and so that $\forall l,\ \forall j \in B_{i,l}$ and $\forall t \in [t_1,\ t_0]$

$$|X_j(t) - X_i(t)| \leq 5\, \varepsilon\, K(t_0).$$

Therefore all the particles which now concern us are in a spatial box of size $C\, \varepsilon\, K(t_0)$.

Control of I_2. Together with the next one, this is the only step which uses the condition $\delta < 1$. Given this previous point, for any $j \in Q_l$ with $l > 0$ and any $t \in [t_1,\ t_2]$, we have, denoting by t_m the time in the interval $[t_1,\ t_0]$ where $|X_j(t) - X_i(t)|$ is minimal

$$|X_i(t) - X_j(t)| \geq \left| |X_i(t_m) - X_j(t_m)| - \frac{1}{2}(t - t_m)|V_1(t_m) - V_j(t_m)| \right|.$$

Then,

$$\begin{aligned} \frac{1}{\varepsilon} \int_{t_1}^{t_0} \frac{1}{|X_1(t) - X_j(t)|^\delta}\, dt \ &\leq\ \frac{C}{\varepsilon}\, |V_i(t_m) - V_j(t_m)|^{-\delta} \varepsilon^{1-\delta} \\ &\leq\ C\, \varepsilon^{-2\delta}\, (\overline{E}(t_0))^{-\delta}\, 2^{-\delta l}. \end{aligned}$$

Summing up on l, we obtain

$$|I_2| \leq C \sum_{l=1}^{l_0} |B_{i,l}| \frac{1}{N}\, \varepsilon^{-2\delta}\, (\overline{E}(t_0))^{-\delta}\, 2^{-\delta l}.$$

We bound $|B_{i,l}|$ by $|B_{i,l}| \leq C\, \|\mu\|_{\infty,\varepsilon}\, (K(t_0)\, \varepsilon)^d (2^l\, \overline{E}(t_0)\, \varepsilon)^d$ using again the definition of the discrete L^∞ norm and recalling that $Q_l \subset C_0$. It gives us the inequality

$$\begin{aligned} I_2 &\leq C\,(K(t_0))^d\,(\overline{E}(t_0))^{d-\delta}\,\varepsilon^{2d-2\delta}\,\|\mu_N\|_{\infty,\varepsilon}\times\sum_{l=2}^{l_0}2^{(d-\delta)l}\\ &\leq C\,(K(t_0))^d\,(\overline{E}(t_0))^{d-\delta}\,\|\mu_N\|_{\infty,\varepsilon}\,\varepsilon^{2d-2\delta}\,\left(\frac{K(t_0)}{\overline{E}(t_0)\,\varepsilon}\right)^{d-\delta}\\ &\leq C\,(K(t_0))^{2d-\delta}\,\|\mu_N\|_{\infty,\varepsilon}\,\varepsilon^{d-\delta}, \end{aligned}$$

which is the point ii. of the Lemma 4.1.

The point iii. We denote

$$I_3=\frac{1}{N}\sum_{j\in U_i}\int_{t-\varepsilon}^{t}\frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds$$

We will also uses the condition $\delta<1$ for this step and it is the only one where m is needed. The first point to note is that for any $j\in U_i$ and any $t\in[t_1,\ t_0]$, as $U_i\subset B_i$ we have that

$$|X_j(t)-X_i(t)|\leq 5\,\varepsilon\,K(t_0).$$

Consequently, by the definition (1.11) of $\Delta\overline{E}$

$$|V_j(t)-V_i(t)-V_j(t_1)-V_i(t_1)|\leq 5\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0).$$

It is thus logical to decompose (again) U_i in $U_i'\cup U_i''$ and I_3 in the corresponding $I_3'+I_3''$ with

$$U_i'=\left\{j\in Q_0\,\Big|\,|V_j(t_1)-V_i(t_1)|\geq 6\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0)\right\},$$

U_i'' the remaining part of U_i and I_3', I_3'' the sums on the corresponding indices. Then for any $j\in U_i'$, the same computation as in the fifth step, shows that

$$\frac{1}{\varepsilon}\int_{t_1}^{t_0}\frac{dt}{N\,|X_j(t)-X_1(t)|^\delta}\leq C\,\varepsilon^{2d-3\delta}\,(K(t_0))^{-\delta}\,(\Delta\overline{E}(t_0))^{-\delta}.$$

The cardinal of U_i' is bounded by the one of U_i and using as always the discrete L^∞ bound

$$|U_i'|\leq C\,(K(t_0))^d\,(\overline{E}(t_0))^d\,\|\mu_N\|_{\infty,\varepsilon}.$$

Eventually that gives

$$\begin{aligned} I_3' &\le |U_i'| \times \sup_{j\in U_i'} \frac{1}{\varepsilon}\int_{t_1}^{t_0} \frac{dt}{N\,|X_j(t)-X_1(t)|^\delta} \\ &\le C\,\varepsilon^{2d-3\delta}\,(K(t_0))^{d-\delta}\,(\overline{E}(t_0))^d\,\|\mu_N(t_0,.)\|_{\infty,\varepsilon} \times (\Delta\overline{E}(t_0))^{-\delta} \\ &\le C\,\varepsilon^{2d-3\delta}\,(K(t_0))^{d-\delta}\,(\overline{E}(t_0))^d\,\|\mu_N(t_0,.)\|_{\infty,\varepsilon}, \end{aligned}$$

as $\Delta\overline{E}(t_0)\ge\Delta\overline{E}(0)$ and this last quantity is bounded easily in terms of $m(0)$, $K(0)$ and $R(0)$.
Let us conclude the proof with the bound on I_3''. Of course if $j\in U_i''$ then for any $t\in[t_1,\ t_0]$,

$$\begin{aligned} |V_j(t)-V_i(t)| &\le |V_j(t_1)-V_i(t_1)| + |V_j(t)-V_i(t)-V_j(t_1)+V_i(t_1)| \\ &\le (6+5)\,\varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0). \end{aligned}$$

Now we use the definition (1.10) of m and the assumption in the lemma to deduce that

$$|X_j(t)-X_i(t)| \ge \frac{\varepsilon}{m(t_0)} - |V_j(t)-V_i(t)| \ge \varepsilon^2\,K(t_0)\,\Delta\overline{E}(t_0).$$

We bound $|U_i''|$ by $|U_i|$ which is the best we can do since the discrete L^∞ norm cannot see the scales smaller than ε and we obtain

$$I_3'' \le C\,\varepsilon^{2d-2\delta}\,(K(t_0))^{d-\delta}\,(\overline{E}(t_0))^d\,\|\mu_N(t_0,.)\|_{\infty,\varepsilon},$$

which is dominated by the bound which we have just obtained on I_3'. This give the point iii. □

We will now just state a corrolary that will be usefull in the last section.

Corollary 4.1. *We choose an δ in $(0,d)$ and a particle i and a real $r>0$ and assume that*

$$m(t_0) \le \frac{1}{12\,\varepsilon\,K(t_0)\,\Delta\overline{E}(t_0)},$$

We defined the subset G_i^r of $\{1,\dots,N\}\backslash\{i\}$,

$$G_i^r = \{j|2K(t)\varepsilon \le |X_i(t)-X_j(t)| \le r\}$$

Then, for any δ' satisfying $\delta\le\delta'\le d$, we have the following estimate $\displaystyle\frac{1}{N}\sum_{j\in G_i^r}\int_{t-\varepsilon}^t \frac{1}{|X_i(s)-X_j(s)|^\delta}\,ds \le \|\mu_N\|_{\infty,\varepsilon}^{\delta'/d}\,K^{\delta'}\,r^{\delta'-\delta}$

Proof. We only have to replace $R(t)$ by r in the proof of the point i. of the preceding lemma 4.1. □

Now we can use this lemma to get an estimate on $\overline{E}$.

Lemma 4.2. *For any α' with $\alpha < \alpha' < 1$, and any $t_0 > 0$, if*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\, K(t_0)\, \Delta\overline{E}(t_0)},$$

then there exist a constant $C(\alpha')$ so that

$$\overline{E}(t_0) \leq C\,(\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\, K^{\alpha'}\, R^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} + \varepsilon^{2d-3\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha}\overline{E}^{d}),$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

Of course if any of the above quantity is infinite then the result is obvious. This lemma could appear stupid since we control $\overline{E}(t_0)$ by itself (and with a power larger than 1 in addition). But the point is that except for the first term, the other two are very small because of the ε in front of them so that they almost do not count.

Proof. We choose a particles i and apply the preceding lemma 4.1. We separate the remaining particles in the three set G_i,B_i, and U_i. Combining the three estimates in which we use $\delta = \alpha$ and $\delta' = \alpha'$, we obtain

$$\int_{t_0-\varepsilon}^{t_0} |E(X_i(s))|\, ds \leq C\,(\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\, K^{\alpha'}\, R^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} + \varepsilon^{2d-3\alpha}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha}\overline{E}^{d}). \quad (2.5)$$

Since this is independant of the particle we choose, we get the estimate on $\overline{E}(t_0)$. □

4.2.2 Estimate on $\Delta\overline{E}$

We may show the following with the same remarks as for Lemma 4.2,

Lemma 4.3. *For any α' with $\alpha < \alpha' < 1$, and for any t_0, if*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\, K(t_0)\, \Delta\overline{E}(t_0)},$$

then there exist a constant $C(\alpha')$

$$\Delta\overline{E}(t_0) \leq C\,(\|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d} K^{1+\alpha'} R^{\alpha'-\alpha} + \varepsilon^{d-\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{2d-\alpha} + \varepsilon^{2d-3\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{d-\alpha}\overline{E}^d),$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

Proof. We choose a time t, two particles i and j and introduce the sets G_i, G_j, B_i, B_j, U_i and U_j. We decomposed the term in sums on these sets:

$$\Delta I = \frac{1}{\varepsilon}\int_{t_1}^{t_0} \frac{|E(X_i(t)) - E(X_j(t))|}{\varepsilon^\beta + |X_i(t) - X_j(t)|}\,dt.$$

$$\begin{aligned}\Delta I \leq \frac{1}{N}\sum_{k\in G_i\cap G_j}\int_{t_1}^{t_0} &\frac{|F(X_i(t) - X_k(t)) - F(X_j(t) - F(X_k(t)|}{\varepsilon^\beta + |X_i(t) - X_j(t)|}\,dt \\ &+ \frac{1}{\varepsilon^\beta}\sum_{k\in B_i\cup U_i} |F(X_i(t) - X_k(t))| \\ &+ \frac{1}{\varepsilon^\beta}\sum_{k\in B_j\cup U_j} |F(X_j(t) - X_k(t))| \\ &+ \frac{1}{\varepsilon^\beta}\sum_{k\in B_i\cup U_i} |F(X_j(t) - X_k(t))| \\ &+ \frac{1}{\varepsilon^\beta}\sum_{k\in B_j\cup U_j} |F(X_i(t) - X_k(t))|. \end{aligned} \tag{2.6}$$

We denote the term of the right hand side, keeping the order

$$\Delta I \leq \Delta I_1 + \Delta I_2 + \Delta I_3 + \Delta I_4 + \Delta I_5$$

Both the term ΔI_2 and ΔI_3 can be bounded by $C\varepsilon^{d-\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{2d-\alpha} + C\varepsilon^{2d-3\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{d-\alpha}\overline{E}^d$ using point ii. and iii. of the lemma (4.1).
The term ΔI_4 and ΔI_5 are of the same form (just exchange the indices i and j). So we will give a bound for ΔI_4 which will be valid for ΔI_5. For this, we decompose again ΔI_4 in the sum on the index in $C' = (B_i \cup U_i) \cap (B_j \cup U_j)$ denoted $\Delta I_4'$ and the sum on the rest $C'' = (B_i \cup U_i)\backslash(B_j \cup U_j)$ denoted $\Delta I_4''$. The first one is bounded by the sum on $B_j \cup U_j)$ which is bounded by $C\varepsilon^{d-\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{2d-\alpha} + C\varepsilon^{2d-3\alpha-\beta} \|\mu_N\|_{\infty,\varepsilon} K^{d-\alpha}\overline{E}^d$ according to points ii. and iii. of the lemma (4.1). For the second term $\Delta I_4'$, if $k \in C''$, then $|X_k(t) - X_j(t)| \geq 2K(t)\varepsilon$. Since, $B_i \cup U_i$ and C'' can be cover by ε-balls of

total volume $4(K(t)\varepsilon)^d$, we can bound $\Delta I_4''$ by $CK(t)^{d-\alpha}\varepsilon^{d-\alpha-\beta}$ a term that will be bounded by the one bounding ΔI_3 if K is greater than one.
Now for ΔI_1, we observe that for $i \notin B$ and for any t

$$|F(X_1(t)-X_i(t)) - F(X_2(t)-X_i(t))| \leq C|X_1(t)-X_2(t)| \times \left(\frac{1}{N\,|X_1(t)-X_i(t)|^{\alpha+1}} + \frac{1}{N\,|X_1(t)-X_i(t)|^{\alpha+1}}\right),$$

since it is always possible to find a regular path $x_t(s)$ of length less than $2\,|X_1(t)-X_2(t)|$ such that $x_t(0)=X_1(t)$, $x_t(1)=X_2(t)$ and $|x_t(s)-X_i(t)|$ is always larger than the minimum between $|X_1(t)-X_i(t)|$ and $|X_2(t)-X_i(t)|$. The only problem if we always choose the direct line between X_1 and X_2 arises when X_i is almost on this line, because $F(x-X_i)$ has a singularity at X_i. So,

$$\Delta I_1 \leq C \sum_{k\in B_i\cap B_j} \left(\frac{1}{N\,|X_k(t)-X_i(t)|^{\alpha+1}} + \frac{1}{N\,|X_k(t)-X_i(t)|^{\alpha+1}}\right)$$

This two sums can be bounded thanks to the point i. of the lemma (4.1) with $\delta = \alpha+1$ by

$$\Delta I_1 \leq C\|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d}\, K^{1+\alpha'}\, R^{\alpha'-\alpha}$$

putting all the bound together we get the result of the lemma. □

4.2.3 Control on m and K

We prove the

Lemma 4.4. *Assume that for a given $t>0$*

$$m(t) \leq \frac{1}{\varepsilon^{\beta-1}},$$

then we also have that

$$m(t) \leq m(0) \times e^{Ct + C\varepsilon\,\Delta\overline{E}(t) + C\int_0^t \Delta\overline{E}(s)\,ds},$$

and we may eliminate the $\varepsilon\,\Delta\overline{E}(t)$ term if $t>\varepsilon$.

Note that we still need an assumption on m but it is a bit different (and somewhat "harder" to satisfy) than the corresponding one for Lemmas 4.2 and 4.3. And note also that by definition $m(t)$ is a non decreasing quantity therefore if $m(t) \geq \varepsilon^{1-\beta}$ then it is true for all $s<t$.

Proof. We consider any two indices $i \neq j$. Then we write

$$\frac{d}{ds}\left(\frac{\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}\right)=\frac{\varepsilon}{(|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|)^2}$$
$$\times\left(\frac{X_i-X_j}{|X_i-X_j|}\cdot(V_i-V_j)+\frac{V_i-V_j}{|V_i-V_j|}\cdot(E(X_i)-E(X_j))\right)$$
$$\leq\frac{\varepsilon\,(|V_i(s)-V_j(s)|+|E(X_i(s))-E(X_j(s))|)}{(|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|)^2}.$$

Since $m(t)\leq\varepsilon^{1-\beta}$, the same is true of $m(s)$ and at least one of the quantities $|X_i(s)-X_j(s)|$ and $|V_i(s)-V_j(s)|$ is larger than $\varepsilon^\beta/2$, therefore

$$\frac{d}{ds}\left(\frac{\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}\right)\leq\frac{C\,\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}$$
$$\times\left(1+\frac{|E(X_i(s))-E_(X_j(s))|}{\varepsilon^\beta+|X_i(s)-X_j(s)|}\right).$$

But by the definition of $\Delta\overline{E}$, see (1.11), we know that for $t>\varepsilon$

$$\int_\varepsilon^t\frac{|E(X_i(s))-E_(X_j(s))|}{\varepsilon^\beta+|X_i(s)-X_j(s)|}\,ds\leq\int_0^t\Delta\overline{E}(s)\,ds,$$

and of course for $t<\varepsilon$

$$\int_0^t\frac{|E(X_i(s))-E_(X_j(s))|}{\varepsilon^\beta+|X_i(s)-X_j(s)|}\,ds\leq\varepsilon\Delta\overline{E}(t).$$

Hence, integrating in time, we find

$$\frac{\varepsilon}{|X_i(s)-X_j(s)|+|V_i(s)-V_j(s)|}\leq\frac{\varepsilon}{|X_i(0)-X_j(0)|+|V_i(0)-V_j(0)|}$$
$$\times\,e^{Ct+C\varepsilon\,\Delta\overline{E}(t)+C\int_0^t\Delta\overline{E}(s)\,ds},$$

wich after taking the supremum in i and j is precisely the lemma. □

As for K, using the equation that $\dot{V}_i(t)=E_i(X_i(t))$, we may prove by the same method which we do not repeat, the result

Lemma 4.5. *We have that for any t*

$$K(t)\leq K(0)+Ct+C\varepsilon\,\overline{E}(t)+C\int_0^t\overline{E}(s)\,ds.$$

4.2.4 Conclusion on the proof of Theorem 4.1

Here (but only in this subsection) for a question of clarity, we keep the notation C for the constants appearing in Lemmas 4.2, 4.3, 4.4 and 4.5 and we denote by $\tilde{C}$ any other constant depending only on $R(0)$, $K(0)$ and $m(0)$. We assume that on a time interval $[0,\ T]$, we have (for a given α') for a constant k

$$\begin{aligned} &m(t) \leq k\, m(0), \quad \overline{E}(t) \leq kC\, k^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha}, \\ &K(t) \leq k\,(1+K(0)), \quad R(0) \leq k\,(1+R(0)), \quad \forall\, t \in [0,\ T], \end{aligned} \tag{2.7}$$

which we may always do since all these quantities are continuous in time (although they may a priori increase very fast). The constant k is chosen to be equal to 2, however we keep the notation k in order to let the reader keep more easily track of this constant.
Then we show that if T is too small we have in fact the same inequalities but with a $3k/4$ constant instead of k. By contradiction this of course shows that we can bound T from below in terms of only $R(0)$, $K(0)$ and $m(0)$ and it proves Theorem 4.1 with $c = C \times k^{8\alpha'-\alpha+1}$.

First of all, we note that since $m(t) \leq k\, m(0)$, we may apply Lemmas 4.2, 4.3, and 4.4. Furthermore we immediately know from (1.14) that

$$\|\mu_N(t,.)\|_{\infty,\varepsilon} \leq (k^3\, m(0))^{2d}.$$

Let us start with Lemma 4.2, using the assumption (2.7) we deduce that for any $t \in [0,\ T]$,

$$\overline{E}(t) \leq C\, k^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha} + \tilde{C}\, \varepsilon^{d-a} + \tilde{C}\, \varepsilon^{2d-3\alpha}.$$

For ε small enough this proves that

$$\overline{E}(t) \leq \frac{3\,k\,C}{4}\, k^{8\alpha'-\alpha}\, (m(0))^{2\alpha'}\, (K(0))^{\alpha'}\, (R(0))^{\alpha'-\alpha},$$

which is the first point.

Next applying Lemma 4.3, we deduce that for any $t \in [0,\ T]$

$$\Delta\overline{E}(t) \leq \tilde{C}.$$

From Lemma 4.4, we obtain that

$$m(t) \leq m(0) \times e^{\tilde{C}T},$$

so if T is such that $\tilde{C}\,T < \ln(3\,k/4)$ then we get

$$m(t) \leq \frac{3\,k}{4}\,m(0).$$

Lemma 4.5 implies that for $t \in [0,\ T]$

$$K(t) \leq K(0) + \tilde{C}\,T,$$

so that again for T small enough

$$K(t) \leq \frac{3\,k}{4}\,(1 + K(0)).$$

Eventually thanks to relation (1.6), we know that for $t \in [0,\ T]$

$$R(t) \leq R(0) + T\,K(t) \leq R(0) + \tilde{C}\,T,$$

hence the corresponding estimate for R provided $\tilde{C}\,T \leq 3\,k/4$.
In conclusion we have shown that if (2.7) holds and if T is smaller than a given time depending only on $R(0)$, $K(0)$ and $m(0)$ then the same inequalities are true with $3/2$ instead of $k = 2$. By the continuity of R, K, m and $\overline{E}$ this has for consequence that (2.7) is indeed valid at least on this time interval thus proving Theorem 4.1.

4.3 Preservation of $\|\mu_N\|_{\infty,\eta}$

From the form of the estimate on m in Lemma 4.4, it is clear that with this estimate we will never get a result for a long time. Indeed, even assuming that we have bounded before K and R, we would have the equivalent of $\dot{m} \leq m \times \Delta\overline{E} \leq C\,m \times m^{2+2\alpha'}$.
On the other hand this suggests the possibility that we did not use enough the structure of the equation since, in the limit, the L^∞ norm is conserved. And this preservation is very useful in the proof of the existence and uniqueness of the solution of the Vlasov equation, see for instance [LP91] or the appendix. But, how to obtain the analog of this in the discrete case? At this time, we just have a bound on $\|\mu_N\|_{\infty,\varepsilon}$ on a small time, and the bound is too huge to allow us to prove convergence results for long time. Of course, this norm is not preserved at all because we are looking at the scheme at the scale of the discretization. And in our calculation we do not use the fact that the flow is divergence free, a property that is the key for the preservation of the L^∞ norm.

So what else can we do? One of the solutions is to look at a scale $\eta > \varepsilon$, with ε/η going to zero as ε goes to zero. At this scale, we have many more particles in a cell and we will be able to obtain the asymptotical preservation of this norm. This will be very useful because it will allow us to sharpen our estimate on E and ΔE. And with this we will obtain long time convergence results.

4.3.1 Sketch of the proof

Now, we will try to give roughly the idea of the proof in dimension 1 before beginning the genuine calculations. We choose a time t and a box S_t in the phase space of size ε centered at (X_t, V_t). The field $(v, E(t,x))$ is divergence free, so it preserves the volume; Heuristically speaking because this field is not regular. This will be the first problem we will have to resolve. If it is solved, we can deform the set S_t backwards in time according to the flow. We obtain at time 0 the set S_0, which is of the same volume than S_t. Our question is: “How many particles contains S_0?”. Remember that we only control the norm $L_{\infty,\varepsilon}$ of μ_N^0. So we need to recover the set S_0 by balls of size ε. In order to obtain a not too huge number of balls, we need a control on the shape of S_0. By instance, if S_0 is the set $\{(x,v) | |x| \leq \varepsilon^2, |v| \leq (\eta/\varepsilon)^2\}$, then we need $(\eta/\varepsilon)^{2d} \times (1/\varepsilon)^d$ balls to recover it. It will give us

$$\|\mu_N(t)\|_{\infty,\eta} \geq \frac{1}{\eta^{2d}} \mu_N^0(S_0) \geq \frac{1}{\varepsilon^d} \|\mu_N^0\|_{\infty,\varepsilon},$$

which is a very bad estimate.
For the control of the shape, we will move backwards with steps of size ε in time. So first, we look at $S_{t-\varepsilon}$. Assume that a particle is in S_t at time t. Since

$$\begin{aligned} X_i(t) &\equiv X_i(t-\varepsilon) + \varepsilon V(t-\varepsilon), \\ V_i(t) &\equiv V_i(t-\varepsilon) + \varepsilon E(t-\varepsilon, X_i(t-s\varepsilon)), \end{aligned}$$

if we assume that the field E is Lipschitz, we obtain approximatively that

$$|X_i(t-\varepsilon) - X_t - \varepsilon V_i(t-\varepsilon)| \leq \eta,$$

$$\begin{aligned} \big| V_i(t-\varepsilon) &- (V_t - \varepsilon E(t-\varepsilon, X_t - \varepsilon V_t)) \\ &- \nabla E(t-\varepsilon, X_t - \varepsilon V_t)) \cdot (X_i(t-\varepsilon) - X_t - \varepsilon V_t)| \leq \eta. \end{aligned}$$

We denote $X_{t-\varepsilon} = X_t - \varepsilon V_t$ and $V_{t-\varepsilon} = V_t - \varepsilon E(t-\varepsilon, X_t - \varepsilon V_t)$, the approximate positions of the center of the balls at time $t-\varepsilon$. This two equations may be rewritten

$$|X_i(t-\varepsilon) - X_{t-\varepsilon} - \varepsilon(V_i(t-\varepsilon) - V_{t-\varepsilon})| \leq \eta,$$

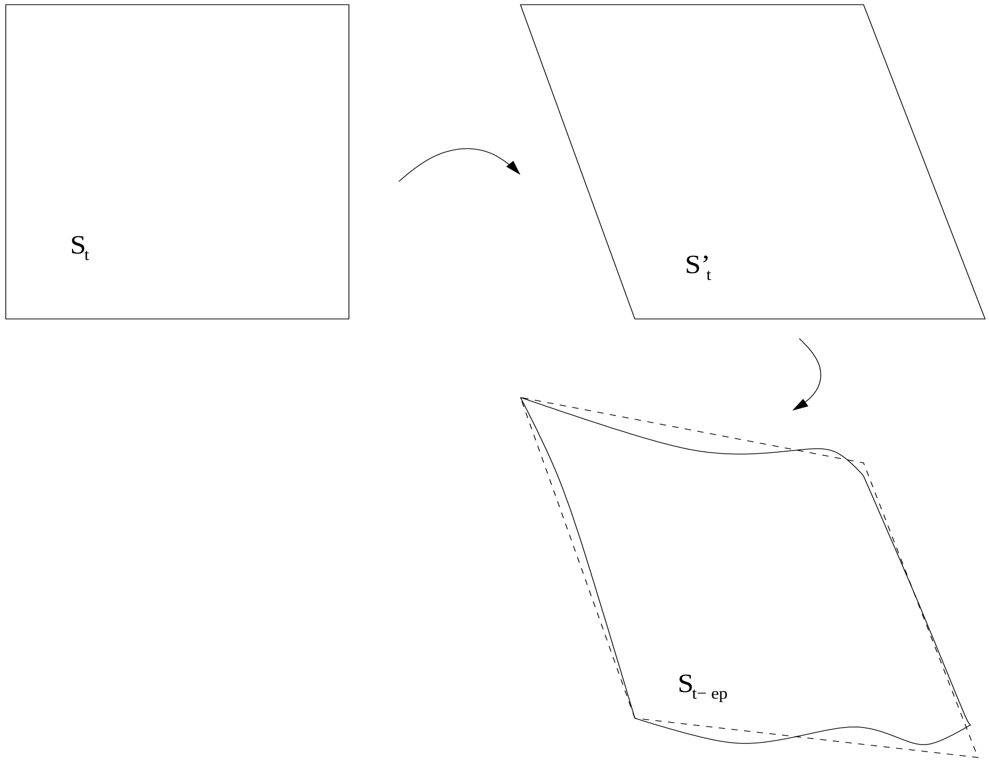

Figure 4.1: Evolution of S_t.

$$|V_i(t-\varepsilon) - V_{t-\varepsilon} - \varepsilon\nabla E(t-\varepsilon, X_t - \varepsilon V_t)) \cdot (X_i(t-\varepsilon) - X_{t-\varepsilon})| \leq \eta.$$

So the particles are at time $t-\varepsilon$ in the set

$$S_{t-\varepsilon} = \left\{ (x,v) \left| \begin{array}{c} |X_i(t-\varepsilon) - X_{t-\varepsilon} - \varepsilon(V_i(t-\varepsilon) - V_{t-\varepsilon})| \leq \eta \\ \big|V_i(t-\varepsilon) - V_{t-\varepsilon} \\ -\varepsilon\nabla E(t-\varepsilon, X_t - \varepsilon V_t)) \cdot (X_i(t-\varepsilon) - X_{t-\varepsilon})\big| \leq \eta \end{array} \right. \right\}.$$

If $d=1$, this set is a paralellogram (see the Figure 1), and for commodity we will still call it paralellogram in higher dimension.
If we define the matrix $M_{t-\varepsilon}$ of dimension $2d \times 2d$ by

$$M_{t-\varepsilon} = \begin{pmatrix} I & \varepsilon I \\ \nabla E(t-\varepsilon, X_t - \varepsilon V_t)) & I \end{pmatrix},$$

$$\text{then,} \qquad S_{t-\varepsilon} = \left\{ (x,v) \left| \left\| M_{t-\varepsilon} \cdot \begin{pmatrix} x - X_{t-\varepsilon} \\ v - V_{t-\varepsilon} \end{pmatrix} \right\| \leq \rho \right. \right\}.$$

This definition involving the matrix M will considerably simplify our work.

Definition 4.1. *We call parallelogram a subset S of $\mathbb{R}^{2d}$ defined as above:*

$$S = \left\{ (x,v) \left| \left\| M \cdot \begin{pmatrix} x - X \\ v - V \end{pmatrix} \right\| \leq \rho \right. \right\},$$

where (X,V) in $\mathbb{R}^{2d}$ is the center of the paralellogram, ρ in $\mathbb{R}$ is the size, M is a matrix in $\mathcal{M}(\mathbb{R}^{2d})$. The norm used is defined by $\|(x,v)\| = \max(|x|,|v|)$. We will always decompose the matrix M in four block

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Because we need to control the deformation of a paralellogram, we introduce the following definition

Definition 4.2. *A paralellogram S will be called not too stretched if the corresponding matrix M satisfies $|\det(M) - 1| \leq 1/2$ and*

$$\|A - Id\|, \|B\|, \|C\|, \|D - Id\| \leq \frac{1}{3}.$$

4.3.2 The notion of ε-volume

Now, we need to control the number of ε-balls needed to cover a paralellogram S. For this, we introduce the following definition:

Definition 4.3. *The ε-volume, denoted $Vol_\varepsilon(S)$, of a subset S of $\mathbb{R}^{2d}$ is the volume of the minimal number of balls of size ε needed to recover S times $(2\varepsilon)^{2d}$ (the volume of a ball).*

Notice that the ε-volume can be very different from the volume. For instance the set

$$T = \{(x,v)|\ |x| \leq \varepsilon^2,\ |v| \leq 1\}$$

has volume of order ε^{2d}, but ε-volume ε^d. We can also see that, up to a constant, the ε-volume is the volume of the set $S_{+\varepsilon/2} = \{(x,v)|d((x,v),S) \leq \varepsilon\} = S + B(0,\varepsilon/2)$.

This notion is usefull to compute the number of particles in a set at time 0. At this time, we only control the number of particles by balls of size ε. Then, the best estimate we can obtain on the total mass $\mu_N(S)$ of particles in a set S is

$$\mu_N(S) \leq Vol_\varepsilon(S) \times \|\mu_N\|_{\infty,\varepsilon}.$$

We need this estimation of the ε-volume of the set S_0. Roughly speaking, we know its volume and want to show that its ε-volume is closed to its volume. Thanks to the following lemma, we will be able to proof this if S_0 is a not too stretched paralellogram.

Lemma 4.6. *Let S be a not too stretched paralellogram, then we have the following inequality:*

$$Vol_\varepsilon(S) \leq Vol(S) \times (1 + \frac{2\varepsilon}{\rho})^{2d}.$$

Proof of the lemma. We define, for all positive integer k

$$S^+_{k\varepsilon} = \left\{ (x,v) | \left\| M \cdot \begin{pmatrix} x - X \\ v - V \end{pmatrix} \right\| \leq \rho + k\varepsilon \right\} ,$$

and $P = \varepsilon\mathbb{Z} \cap S^+_{2\varepsilon}$. Here, ρ, M, (X,V) stands for the size, the matrix and the center of the paralellogram as in the definition. We look at the set $P_+\varepsilon$ consisting of the union of all the balls of size ε centered at points of P, that is $P_\varepsilon = P + B(0,\varepsilon)$. We will show that this set is included in $S^+_{4\varepsilon}$. For this, we choose $(x,v) \in P_\varepsilon$ and a couple (m,n) in $\mathbb{Z}^2$ such that $\|(x - \varepsilon m, v - \varepsilon n)\| \leq \varepsilon/2$. Then,

$$\begin{aligned} \left\| M \cdot \begin{pmatrix} x \\ v \end{pmatrix} \right\| &\leq \left\| M \cdot \begin{pmatrix} x - \varepsilon m \\ v - \varepsilon n \end{pmatrix} \right\| + \left\| M \cdot \begin{pmatrix} \varepsilon m \\ \varepsilon n \end{pmatrix} \right\| \\ &\leq \|M\| \frac{\varepsilon}{2} + \rho + \varepsilon \\ &\leq \eta + 2\varepsilon. \end{aligned}$$

In the last line, we use $\|M\| \leq 2$. This inequalitiy is implied by the condition in the definition of a not too streched paralellogram. Therefore we have the inclusion $P_\varepsilon \subset S^+_{2\varepsilon}$.
Moreover, if we choose a point $(x,v) \in S$, we can find a point $(\varepsilon m, \varepsilon n)$of $\varepsilon\mathbb{Z}^{2d}$ such that $\|(x - \varepsilon m, v - \varepsilon n)\| \leq \varepsilon/2$. As above, we have

$$\left\| M \cdot \begin{pmatrix} \varepsilon m \\ \varepsilon n \end{pmatrix} \right\| \leq \eta + 2\varepsilon.$$

Thus, $\varepsilon(m,n) \in P$. That proves that $S \subset P_{+\varepsilon}$. So, we have the inclusions

$$S \subset P_{+\varepsilon} \subset S^+_{2\varepsilon}.$$

The first is the recovering we want. The second gives us an estimate on the cardinal of P. Comparing the volume of P_ε and $S^+_{2\varepsilon}$ we obtain

$$(\varepsilon)^{2d}|P| \leq Vol(S^+_{2\varepsilon}) = det(M)^{-1}(\rho + 2\varepsilon)^{2d}.$$

Since $Vol(S) = det(M)^{-1}\rho^{2d}$ we obtain

$$Vol_\varepsilon(S) \leq Vol(S) \times (1 + \frac{2\varepsilon}{\rho})^{2d}.$$

□

4.3.3 Asymptotic preservation of $\|\mu\|_{\infty,\eta}$ for small time

Now, given a box S_t, our goal is to find a not too stretched paralellogram S_0 which contains at time 0 all the particles that are in S_t at time t. For this, we will go from t to $t-\varepsilon$ using the following lemma:

Lemma 4.7. *Assume as before that*

$$m(t) \leq \frac{1}{12\varepsilon\, K(t)\Delta\overline{E}(t)}.$$

Then, for any $1 < \beta < d-1$, there exists a constant K_1 depending on t, R, K, $\overline{E}$, $\|\mu\|_{\infty,\varepsilon}$ such that for all not too stretched paralellogram S_t, of center (X_t, V_t), matrix M_t (decomposed in A_t,B_t,C_t,D_t) and size ρ_t, there exists a paralellogram $S_{t-\varepsilon}$ of center $(X_{t-\varepsilon}, V_{t-\varepsilon})$ and so on, satisfying the following conditions

i. $\|A_{t-\varepsilon} - A_t\|, \|B_{t-\varepsilon} - B_t\|, \|C_{t-\varepsilon} - C_t\|, \|D_{t-\varepsilon} - D_t\| \leq K_1\varepsilon$

ii. $|det(M_{t-\varepsilon}) - \det(M)| \leq K_1\varepsilon^2$

iii. $\rho_{t-\varepsilon} \leq \rho_t + K_1\varepsilon(\rho_t^\beta + \varepsilon)$

and that contains at time $t-\varepsilon$ all the particles that are in S_t at time t.

Remarks

- We always use the heavy expression " contains at time t' all the particles that are in S at time t" because here we can not speak of the reverse image by the flow. There is not a flow that all the particles follow because a particle do not see the force-field it creates.

- What is important here is that $\mu_N(t, S_t) \leq \mu_N(t-\varepsilon, S_{t-\varepsilon})$.

Proof. We want to rewrite our inequalities involving $X_j(t)$, $V_j(t)$, X_t and V_t in inequalities involving $X_j(t-\varepsilon)$, $V_j(t-\varepsilon)$, $X_{t-\varepsilon}$ and $V_{t-\varepsilon}$ (and we have to choose the last position and speed). Of course, the center of the paralellogram will approximately move according to the flow created by all the particles. We write approximately because particles close from the center will induce pertubation in his trajectory (these perturbation are however negligible). The best way to do this is to regularise the flow at order ε. So, we introduce

$$E_\varepsilon(t,x) = \sum_{i=1}^{n} F * \xi_{B(0,\varepsilon)}(x - X_i(t))$$

Remark that the kernel $F_\varepsilon = F * \xi_{B(0,\varepsilon)}$ satisfy the same assumptions that the kernel ∇F, it means

$$F_\varepsilon, |x||\nabla F_\varepsilon|, |x|^2|\nabla^2 F_\varepsilon| \leq C|x|^{-\alpha}$$

At this point, we define the center $(X_{t-\varepsilon}, V_{t-\varepsilon})$ of the paralellogram $S_{t-\varepsilon}$. It will be the center (X_t, V_t) moved backward to the time $t-\varepsilon$ according to E_ε. Moreover, all the estimates on $\overline{E}$, $\Delta\overline{E}$ can be applied to this virtual particle. More precisely, the two first point of the lemma (4.1) are true even for a virtual particle because for this two estimation we do not use the minimal distance between particles m. The last one is more easy to obtain because the approximate kernel F_ε is bounded by ε. We wanted an estimate of $|X_j(t-\varepsilon) - X_{t-\varepsilon}|$ and $|V_j(t-\varepsilon) - V_{t-\varepsilon}|$. We will begin with the second and integrate it.
Step 1: Estimation of $|V_j(t-\varepsilon) - V_{t-\varepsilon}|$.
For the particle j, we have

$$\begin{aligned} V_j(t-\varepsilon) - V_{t-\varepsilon} &= V_j(t) - V_t - \varepsilon \int_0^1 (E(X_j(t-s\varepsilon)) - E_\varepsilon(X_{t-s\varepsilon}))\, ds \\ &= V_j(t) - V_t - \varepsilon \int_0^1 (E(X_j(t-s\varepsilon)) - E_\varepsilon(X_j(t-s\varepsilon)))\, ds \\ &\qquad + \varepsilon \int_0^1 (E_\varepsilon(X_j(t-s\varepsilon)) - E_\varepsilon(X_{t-s\varepsilon}))\, ds \\ &= \varepsilon(J_1 + J_2). \end{aligned}$$

We need to bound the first term J_1. The approximation error is

$$\begin{aligned} J_1 &= \int_0^1 E(X_j(t-s\varepsilon)) - E_\varepsilon(X_j(t-s\varepsilon))\, ds \\ &= \frac{1}{N}\sum_{k\neq j}\int_0^1 \big(F(X_j(t-s\varepsilon) - X_k(t-s\varepsilon)) - F_\varepsilon(X_j(t-s\varepsilon) - X_k(t-s\varepsilon))\big)\, ds. \end{aligned}$$

We can bound this term using the two bounds $|F(x) - F_\varepsilon(x)| \leq C\varepsilon/|x|^{\alpha+1}$ and $|F(x) - F_\varepsilon(x)| \leq C/|x|^\alpha$. We write, recalling the notation $G_j = \{k||X_k(t) - X_j(t)| \geq 2K(t)\varepsilon\}$ of the lemma (4.1)

$$\begin{aligned} J_1 \leq \frac{C\varepsilon}{N}\sum_{k\in G_j}\int_{t-\varepsilon}^t \frac{1}{|X_k(s) - X_j(s)|^{1+\alpha}}\, ds \\ + \frac{C}{N}\sum_{k\notin G_j}\int_{t-\varepsilon}^t \frac{1}{|X_k(s) - X_j(s)|^{\alpha}}\, ds \quad (3.1) \end{aligned}$$

We choose an α' so that $\alpha < \alpha' < 1$. Using the point i. of the lemma (4.1) with $\delta = 1 + \alpha$ and $\delta' = 1 + \alpha'$, we can bound the first term of the right hand side by $C\varepsilon^2 \|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d} K^{1+\alpha'} R^{\alpha'-\alpha}$. Using the point ii. and iii., we can bound the second term by $\varepsilon^{d-\alpha} \|\mu_N\|_{\infty,\varepsilon} K^{2d-\alpha} + \varepsilon^{2d-3\alpha} \|\mu_N\|_{\infty,\varepsilon} K^{d-\alpha} \overline{E}^d K^d)$. So, without forgetting is dependance, we can write

$$J_1 \leq K_2(\varepsilon + \varepsilon^{d-\alpha}).$$

The term J_2 contains only term using the approximate field. In that case the estimates are simpler because we do not need to integrate it over a small interval of time. We state them in the following lemma

Lemma 4.8. *We assume that the force K_ε satifyies*

$$K_\varepsilon(x) \leq \frac{C}{(\varepsilon + |x|)^\delta}.$$

We choose a particle i and define two set

$$G_{\varepsilon,i} = \{j || X_j(t) - X_i(t)| \geq \varepsilon\}$$

$$B_{\varepsilon,i} = \{j || X_j(t) - X_i(t)| < \varepsilon\}$$

then for any δ' satisfying $\delta < \delta' < d$, there exist a numerical constant C such that the two following inequality are true.

i.) $$\left| \frac{1}{N} \sum_{j \in G_i} K_\varepsilon(X_i(t) - X_j(t) \right| \leq C \|\mu_N\|_{\infty,\varepsilon}^{\delta'/d} R^{\delta'-\delta}$$

ii.) $$\left| \frac{1}{N} \sum_{j \in B_i} K_\varepsilon(X_i(t) - X_j(t) \right| \leq \|\mu_N\|_{\infty,\varepsilon} K^d \varepsilon^{d-\delta}$$

Proof. The point i. This is exactly the estimate i. of the lemma (4.1) with $K = 1$, so we will not write it again.
the point ii. The ε-volume of the set $B_{\varepsilon,i}$ is $(K\varepsilon)^d$. So the mass in it is less than $\|\mu_N\|_{\infty,\varepsilon}(K\varepsilon)^d$. Moreover, K_ε is bounded by C/ε^δ. So we get

$$\frac{1}{N} \sum_{j \in B_i} K_\varepsilon(X_i(t) - X_j(t)| \leq \|\mu_N\|_{\infty,\varepsilon} K^d \varepsilon^{d-\delta}$$

□

Now, we want to approximate J_2 by $\int_{t-\varepsilon}^t \nabla E_\varepsilon(X_s)ds \cdot (X_j(t-\varepsilon) - X_{t-\varepsilon})$. First, we can replace $X_j(s) - X_s$ by $X_j(t-\varepsilon) - X_{t-\varepsilon}$ in the expression of J_2, because for all $s \in (t-\varepsilon, t)$

$$|(X_j(t-\varepsilon) - X_{t-\varepsilon}) - (X_j(s) - X_s)| \leq 2K\varepsilon,$$

and then,

$$\Big| \int_{t-\varepsilon}^t \nabla E_\varepsilon(X_s)ds \cdot (X_j(t-\varepsilon) - X_{t-\varepsilon}) - \dots \\ \dots \int_{t-\varepsilon}^t \nabla E_\varepsilon(X_s) \cdot (X_j(s) - X_s)\, ds \Big| \leq 2K\Delta\overline{E}\varepsilon^2. \quad (3.2)$$

We write $\Delta\overline{E}$ even for the approximate field because it is bounded by the bound of lemma (4.3) like the true field, eventually addind a numerical factor in front of the bound.
Now, the main term is

$$J_2' = \int_{t-\varepsilon}^t E_\varepsilon(X_j(s)) - E_\varepsilon(X_s) - \nabla E_\varepsilon(X_s)\, ds \cdot (X_j(s) - X_s)\, ds.$$

This is a sum of terms of the form

$$\int_{t-\varepsilon}^t \Big(F_\varepsilon(X_j(s) - X_i(s)) - \dots \\ \dots F_\varepsilon(X_s - X_i(s)) - \nabla F_\varepsilon(X_s - X_i(s)) \cdot (X_j(s) - X_s) \Big)\, ds. \quad (3.3)$$

So, for each i, j and s, we choose a path $I(s,\cdot)$ between $X_j(s)$ and X_s so that its length is less than $4|X_j(s) - X_s|$ and so that $|I(s,u) - X_i(s)|$ always stays between in the interval between $|X_j(s) - X_i(s)|$ and $|X_s - X_i(s)|$. We can rewrite the previous term as

$$\int_{t-\varepsilon}^t \int_0^1 \nabla F_\varepsilon(I(s,u) - X_i(s)) - \nabla F_\varepsilon(X_s - X_i(s)) \cdot (X_j(s) - X_s)\, du\, ds.$$

The integrand may be bounded in two ways. First by

$$\frac{C|X_j(s) - X_s|}{(\varepsilon + \min(|I(s,u) - X_i(t)|, |X_s - X_i(t)|))^{1+\alpha}},$$

if we bound it by the sum of the two terms and also by

$$C\frac{|I(s,u) - X_s|^2}{(\varepsilon + \min(|I(s,u) - X_i(s)|, |X_s - X_i(s)|))^{2+\alpha}},$$

if we use the derivative. We need a majoration by a term with a small power of $|I(s,u) - X_s|$ on the top, and an exponant sufficiently small below. For this, we pick a γ in $(0,1)$ and bound the integrand by the first bound at the power $1-\gamma$ and the second at the power γ. So, we bound the term by

$$J_2' \leq \sum_{i\neq j} \int_{t-\varepsilon}^{t} \int_0^1 \frac{|X_j(s) - X_s|^{1+\gamma}}{\varepsilon + \min(|X_j(s) - X_i(s)|, |X_s - X_i(s)|)^(1+\alpha+\gamma)}\, ds.$$

First, as $|X_j(s) - X_s| \leq |X_j(t-\varepsilon) - X_{t-\varepsilon}| + K\varepsilon$ for all $s \in [t-\varepsilon]$, we can write

$$\begin{aligned} J_2' \leq C\frac{K\,\varepsilon}{N} \sum_{i\neq j} \int_{t-\varepsilon}^{t} \int_0^1 \frac{1}{\varepsilon + \min(|X_j(s) - X_i(s)|, |X_s - X_i(s)|)^{1+\alpha+\gamma}} \\ + \left(\frac{1}{N} \sum_{i\neq j} \int_{t-\varepsilon}^{t} \int_0^1 \frac{|X_j(t-\varepsilon) - X_{t-\varepsilon}|^{1+\gamma}\, ds\, du}{5\varepsilon + \min(|X_j(s) - X_i(s)|, |X_s - X_i(s)|))^{1+\alpha+\gamma}} \right) \end{aligned} \quad (3.4)$$

Now we can use the estimates of the lemma (4.8) with $\delta = 1+\alpha+\gamma$ to bound J_2'. We obtain

$$\begin{aligned} J_2' \leq C(K\varepsilon + |X_j(t-\varepsilon) - X_{t-\varepsilon}|^{1+\gamma})\|\mu_N\|_{\infty,\varepsilon}^{\delta'/d} R^{\delta'-\delta} \\ + C|X_j(t-\varepsilon) - X_{t-\varepsilon}|^{1+\gamma}\|\mu_N\|_{\infty,\varepsilon} K^d \varepsilon^{d-\delta}. \end{aligned} \quad (3.5)$$

This gives us a nice bound if $\gamma < d-1-\alpha$. In this case, defining $\beta = 1+\gamma$, we may rewrite it as

$$J_2' \leq \tilde{K}_2 \varepsilon (|X_j(t) - X_t|^\beta + \varepsilon)$$

without forgetting the dependance of K_2. Now, putting everything together and denoting $\widetilde{\nabla E_\varepsilon} = (1/\varepsilon)\int_{t-\varepsilon}^t \nabla E_\varepsilon(X_s)\, ds$, we have:

$$\begin{aligned} |(V_j(t-\varepsilon) - V_{t-\varepsilon})) - (V_j(t) - V_t) - \varepsilon\widetilde{\nabla E_\varepsilon} \cdot (X_j(t-\varepsilon) - X_{t-\varepsilon})| \\ \leq K_1\varepsilon(|X_j(t) - X_t|^\beta + \varepsilon + \varepsilon^{d-1-\alpha}). \end{aligned}$$

This is the estimation we will use.
Step 2: Estimation of $|X_j(t-\varepsilon) - X_{t-\varepsilon}|$
The bound on the position is easier to obtain. We have

$$(X_j(t-\varepsilon) - X_{t-\varepsilon}) = (X_j(t) - X_t) - \varepsilon(V_j(t-\varepsilon) - V_{t-\varepsilon}) - \varepsilon^2 R_\varepsilon,$$

with

$$R_\varepsilon = \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} (s-t+\varepsilon)(E(X_j(s) - E(X_s))\, ds.$$

Here, a bound on R_ε will be sufficient. And we have $R_\varepsilon \leq 2\overline{E}$ which gives

$$|(X_j(t-\varepsilon) - X_{t-\varepsilon}) - (X_j(t) - X_t) + \varepsilon(V_j(t-\varepsilon) - V_{t-\varepsilon})| \leq K_3\varepsilon^2.$$

Step 3: The new paralellogram
Consequently if we apply $A\times$ 4.3.3 $-B\times$ 4.3.3 and use the fact that $|A_t \cdot (X_j(t) - X_t) - B_t \cdot (V_j(t) - V_t)| \leq \rho$, we obtain

$$\begin{aligned}|(A_t + \varepsilon B_t \widetilde{\nabla E_\varepsilon}) \cdot (X_j(t_\varepsilon) - X_{t-\varepsilon}) - (B_t \varepsilon A_t) \cdot (V_j(t-\varepsilon) - V_{t-\varepsilon})| \\ \leq \rho_t + C(\|A_t\| + \|B\|)\,\varepsilon\,(\rho_t^\beta + \varepsilon).\end{aligned}$$

So, if we denote $A_{t-\varepsilon} = A_t + \varepsilon B_t \widetilde{\nabla E_\varepsilon}$, $B_{t-\varepsilon} = B_t \varepsilon A_t$ and $\rho_{t-\varepsilon} = \rho_t + C(\|A_t\| + \|B\|)\varepsilon(\rho_t^{1+\gamma} + \varepsilon + \varepsilon^{d-1-\alpha})$, we obtain that

$$|A_{t-\varepsilon} \cdot (X_j(t-\varepsilon) - X_{t-\varepsilon}) - B_{t-\varepsilon} \cdot (V_j(t-\varepsilon) - V_{t-\varepsilon})| \leq \rho_{t-\varepsilon}.$$

We can do the same for the second line of the matrix. If we denote $C_{t-\varepsilon} = C_t + \varepsilon D_t \widetilde{\nabla E_\varepsilon}$ and $D_{t-\varepsilon} = D_t \varepsilon C_t$, and

$$M_{t-\varepsilon} = \begin{pmatrix} A_{t-\varepsilon} & B_{t-\varepsilon} \\ C_{t-\varepsilon} & D_{t-\varepsilon} \end{pmatrix}$$

we obtain that $(X_j(t-\varepsilon, V_j(t-\varepsilon)) \in S_{t-\varepsilon}$, the paralellogram of center $(X_{t-\varepsilon}, V_{t-\varepsilon})$, matrix $M_{t-\varepsilon}$ and size $\rho_{t-\varepsilon}$.
It remain to prove the estimates on $M_{t-\varepsilon}$. For this, remark that, $M_{t-\varepsilon} = M_t J_t$ with

$$J_t = \begin{pmatrix} I & \varepsilon I \\ \varepsilon \nabla \tilde{E}_\varepsilon & I \end{pmatrix} = I + \varepsilon N_t.$$

Then, $\det M_{t-\varepsilon} = \det(M_t)\det(I + \varepsilon N_t)$. And $|det(I + \varepsilon N_t) - 1 - \varepsilon tr(N_t)| \leq C\|N_t\|^2\varepsilon^2$. Moreover, $tr(N_t) = 0$. Remark that this is here that we use the fact that our field in the phase space is divergence free. And we obtain

$$|\det(M_{t-\varepsilon} - \det(M_t)| \leq C \det(M_t)\varepsilon^2,$$

where C is of the form $K(\overline{E} + \Delta\overline{E})$. And of course, $\|A_{t-\varepsilon} - A_t\| \leq K\Delta\overline{E}\varepsilon$, $\|B_{t-\varepsilon} - B_t\| \leq K\varepsilon$ and so on. This is all we needed to prove. □

Now, we need to go from a time t to time 0, by backward jumps in time of size ε. At each step we obtain a new paralellogram. We can go on till this paralellogram is too stretched. This will happens in a time of order $1/\Delta\overline{E}$, because of the inequality $\|A_{t-\varepsilon} - A_t\| \leq \Delta\overline{E}\varepsilon$. We would be able to conclude if we had a bound on ρ_0, the size of the paralellogram obtain at time 0. The following lemma provides it.

Lemma 4.9. *Assume that $t' = t - M\varepsilon$, that S'_t is obtain from S_t by iteration of the lemma 4.7 and that $3^\beta K_1 (t-t')(\rho_t^\beta + \varepsilon) \leq \rho_t$. Then, the folowing inequality holds*

$$\rho_{t'} \leq \rho_t + 3^\beta K_1 (t-t') (\rho_t^\beta + \varepsilon).$$

Proof. We recall that $\rho_{t-\varepsilon} = \rho_t + K_1 \varepsilon (\rho_t^\beta + \varepsilon)$. From these formulas, we expect that $\rho_{t-n\varepsilon} \approx \rho_t + K_1 n\varepsilon(\rho_t^\beta + \varepsilon)$.
To prove this rigourously, we define $\alpha_n = (\rho_{t-n\varepsilon} - \rho_t - 3^\beta K_1 n\varepsilon (\rho_t^\beta + \varepsilon))_+$. We have

$$\alpha_{n+1} - \alpha_n \leq K_1 \varepsilon (\rho_t + 3^\beta nK_1\varepsilon\rho_t^\beta + 3^\beta nK_1 \varepsilon^2 + \alpha_n)^\beta - 3^\beta K_1 \varepsilon \rho_t^\beta - (3^\beta - 1) K_1 \varepsilon^2.$$

Provided $3^\beta K_1 \varepsilon n (\rho_t^{\beta-1} + \varepsilon/\rho_t) \leq 1$ and $\alpha_n \leq \rho_t$, we have that

$$\alpha_{n+1} - \alpha_n \quad \leq \quad 3^\beta K_1 \varepsilon \rho_t^\beta - 3^\beta K_1 \leq 0$$

Therefore α_n remains equal to 0 which gives the corresponding result for $\rho_{t'}$. ☐

Now that we control the growth of ρ_t, we are able to prove the following theorem

Theorem 4.4. *There exists a numerical constant K_2 such that if*

$$m(t) \leq \frac{1}{12\, \varepsilon\, K(t)\, \Delta\overline{E}(t)},$$

$t \leq 1/(2\Delta\overline{E})$ and ε is small enough, then the following inequality holds:

$$\|\mu_N(t)\|_{\infty,\eta} \leq \|\mu\|_{\infty,\varepsilon} + K_2 \left(\eta^\beta + \frac{\varepsilon}{\eta} \right).$$

Proof. We start at time t from a box $S_t = \{(x,v) \mid \|x - X_t, v - V_t\| \leq \eta\}$. It means that $\rho_t = \eta$ and

$$M_t = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

We define backward all the $S_{t-n\varepsilon}$ till S_0. If t is not a multiple of ε, we use a last step less than ε, but all our estimates are still true for a step like this. As $\|A_{t-(n+1)\varepsilon} - A_{t-n\varepsilon}\| \leq 2\Delta\overline{E}$ and as $A_t = I$, we have $\|A_s - I\| \leq 1/2$. And the same estimates hold for B, C, and D. That means that all our parallelograms are always not too streched. We may then apply the previous lemma to get the corresponding estimate on ρ_0. Using the definition of the discrete L^∞ norm at ε, we control the number of particles in S_0 which is also the numer of particles in S_t. This last number is the bound on the discrete L^∞ norm at η and at time t. ☐

4.3.4 New estimates on $\overline{E}$ and $\Delta\overline{E}$

The almost preservation of the $\|\mu_N\|_{\infty,\eta}$ norms will enable us to prove a new estimate on $\overline{E}$, namely

Lemma 4.10. *For any α' with $\alpha < \alpha' < d$, assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\,K(t_0)\,\Delta\overline{E}(t_0)},$$

then the following inequality holds

$$\overline{E}(t_0) \leq C\,(\|\mu_N\|_{\infty,\eta}^{\alpha'/d}\,K^{\alpha'}\,R^{\alpha'-\alpha} + \|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,K^{2\alpha'-\alpha}\,\eta^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\alpha} + \varepsilon^{2d-3\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\alpha}\overline{E}^d),$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

The only non-negligable term in this estimate is sub-linear if α' is chosen sufficiently close to α.

Proof. The idea is very similar to Lemma 4.1. We do the same separation of the position space in dyadic cells, but we begin with cells $\tilde{C}_k$ satisfying

$$\tilde{C}_k = \left\{ i \,\middle|\, 3\,\eta\,K(t_0)\,2^{k-1} < |X_i(t_1) - X_1(t_1)| \leq 3\,\eta\,K(t_0)\,2^k \right\},$$

with k between 0 and $k_0 = \ln(R/(3\eta\,K))/\ln 2$.
For $\tilde{C}_0$, we apply estimate (4.1) with $r = 3\eta\,K(t_0)$ which gives

$$I_{\tilde{C}_0} \leq \|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d}\,K^{2\alpha'-\alpha}\,\eta^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{2d-\alpha} + \varepsilon^{2d-3\alpha}\,\|\mu_N\|_{\infty,\varepsilon}\,K^{d-\alpha}\overline{E}^d).$$

Next, notice that $\tilde{C}_k$ can be covered by at most $C\,(K(t_0))^d\,2^{kd} \times \eta^{-d}\,(K(t_0))^d$ balls of radius η and therefore by the definition of $\|\mu_N\|_{\infty,\eta}$, we have that

$$|\tilde{C}_k| \leq C\,N\,2^{kd}\,(K(t_0))^{2d}\,\eta^d\,\|\mu_N\|_{\infty,\eta}.$$

On the other hand $|\tilde{C}_k| \leq N$ so for any $\alpha' < d$

$$|\tilde{C}_k| \leq C\,N\,\eta^{\alpha'}\,2^{k\alpha'}\,K^{2\alpha'}\,\|\mu_N\|_{\infty,\eta}^{\alpha'/d}.$$

Of course the $\tilde{C}_k$ are also approximatly stable in the sense that if $i \in C_k$ then $|X_i(t) - X_1(t)| \geq \eta\, K(t_0)\, 2^{k-1}$ for any $t \in [t_1,\ t_0]$. Therefore

$$\begin{aligned} I_1 &= \sum_{k=1}^{k_0} \sum_{i \in \tilde{C}_k} \frac{1}{\varepsilon} \int_{t_1}^{t_0} \frac{1}{N\,|X_1(t) - X_i(t)|^\alpha}\, dt \\ &\leq C \sum_{k=1}^{k_0} |\tilde{C}_k|\; N^{-1}\, \eta^{-\alpha}\, K^{-\alpha}\, 2^{-k\alpha} \\ &\leq C\, \eta^{\alpha'-\alpha}\, K^{2\alpha'-\alpha} \sum_{k=1}^{k_0} 2^{k(\alpha'-\alpha)} \\ &\leq C\, \eta^{\alpha'-\alpha}\, K^{2\alpha'-\alpha}\, 2^{k_0(\alpha'-\alpha)}, \end{aligned}$$

provided that $\alpha' > \alpha$. Therefore

$$I_1 \leq C\, R^{\alpha'-\alpha}\, K^{\alpha'}\, \|\mu_N\|_{\infty,\eta}^{\alpha'/d}.$$

Summing I_1 with $I_{\tilde{C}_0}$ proves the lemma. □

Of course we can perform the same changes for the estimates on $\Delta\overline{E}$ to get

Lemma 4.11. *For any α' with $\alpha < \alpha' < 3$, assume that*

$$m(t_0) \leq \frac{1}{12\,\varepsilon\, K(t_0)\, \Delta\overline{E}(t_0)},$$

then the following inequality holds

$$\begin{aligned} \Delta\overline{E}(t_0) \leq C\,(&\|\mu_N\|_{\infty,\eta}^{(1+\alpha')/d}\, K^{1+\alpha'}\, R^{\alpha'-\alpha} + \|\mu_N\|_{\infty,\varepsilon}^{(1+\alpha')/d}\, K^{1+2\alpha'-\alpha}\, \eta^{\alpha'-\alpha} \\ &+ \varepsilon^{d-\alpha-\beta}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{2d-\alpha} + \varepsilon^{2d-3\alpha-\beta}\, \|\mu_N\|_{\infty,\varepsilon}\, K^{d-\alpha}\overline{E}^d), \end{aligned}$$

where we use the values of $\|\mu_N\|_{\infty,\varepsilon}$, R, K, m and $\overline{E}$ at the time t_0.

4.3.5 Proof of Theorem 4.2

Let us fix any time $T > 0$. The aim is to show that we have bounds for R, K, $\overline{E}$ and m, uniform in N on $[0,\ T]$.
Next we choose $\eta_0 = \varepsilon^{1/2}$ for instance and $\eta' = \varepsilon^{1/4}$.
Since for any N the quantities R, K, $\overline{E}$ and m are continuous in time, we may define $T_N < T$ as the first time t (if it exists) such that one of the

following inequality at least is not true for some integer M to be chosen after

$$
\begin{aligned}
&T'=T(R(t),K(t),\overline{E}(t),\sup_{s\le t}\|\mu_N\|_{\infty,\varepsilon})\ge\frac{T}{M},\\
&m(t)\le\frac{1}{12\,\varepsilon\,K(t)\,\Delta\overline{E}(t)},\quad C(R(t),K(t),\overline{E}(t),\sup_{s\le t}\|\mu_N\|_{\infty,\varepsilon})\\
&\varepsilon^{d-\alpha}(m(t))^{-2d}\,(K(t))^{2d-\alpha}\le\varepsilon^{\beta},\\
&\varepsilon^{2d-3\alpha}(m(t))^{-2d}\,(\overline{E}(t))^{d}\,(K(t))^{d-\alpha}\le\varepsilon^{\beta}.
\end{aligned}
\tag{3.6}
$$

The quantity T' and C are the time and constant defined in Theorem 4.4. Therefore on $[0,\ T_N]$ all inequalities (3.6) are true and we may apply both Theorem 3.1 and Lemma 4.10.

We define $t_i=i\,T'$ and $\eta_i=\eta_0\times r^i$ with $r=\varepsilon^{-1/4M}$ so that $\eta_M=\eta'$. We are going to apply M times Theorem 4.4, once on every interval $[t_{i-1},\ t_i]$ (of size less than T') and with $\eta=\eta_i$ and ε replaced by η_{i-1}. That gives

$$\sup_{t\in[t_{i-1},t_i]}\|\mu_N(t)\|_{\infty,\eta_i}\le\|\mu_N\|_{\infty,\eta_{i-1}}+C(\overline{E}(t_i),\Delta\overline{E}(t_i))\,(\eta_i^{\gamma}+\varepsilon^{1/4M}),$$

and consequently thanks to (3.6)

$$\sup_{t\le T_N}\|\mu_N(t)\|_{\infty,\eta'}\le\|\mu_N\|_{\infty,\varepsilon}+C(\overline{E}(T_N),\Delta\overline{E}(T_N))\,M\,\varepsilon^{1/4M}\le 2\,\|\mu_N^0\|_{\infty,\varepsilon},\tag{3.7}$$

independently of N (and T_N). Now we apply Lemma 4.10 at time T_N and because of (3.6), we obtain

$$
\begin{aligned}
\overline{E}(T_N)&\le C\,\|\mu_N(T_N)\|_{\infty,\eta}^{\alpha'/d}\,(K(T_N))^{\alpha'}\,(R(T_N))^{\alpha'-\alpha}\\
&\le C\,(K(T_N))^{\alpha'}\,(R(T_N))^{\alpha'-\alpha},
\end{aligned}
\tag{3.8}
$$

using (3.7). As $T_N>\varepsilon$, Lemma 4.5 implies that

$$K(T_N)\le K(0)+C\int_0^{T_N}\overline{E}(t)\,dt\le K(0)+C\,T_N\,\overline{E}(T_N).$$

From this inequality, we immediately deduce that

$$R(T_N)\le R(0)+T_N\,K(0)+C\,T_N^2\,\overline{E}(T_N)\le C\,T+C\,T^2\,\overline{E}(T_N).$$

Inserting these last two inequalities in (3.8), we find

$$\overline{E}(T_N)\le C\,T+C\,T^2\,(\overline{E}(T_N))^{2\alpha'-\alpha}.$$

Since $2\alpha' - \alpha < 1$, there exists a constant $C(T)$ depending only on T and the initial distribution such that

$$\overline{E}(T_N) \le C(T), \quad K(T_N) \le C(T), \quad R(T_N) \le C(T). \tag{3.9}$$

We are almost ready to conclude, we only need to apply once Lemma 4.11 and by (3.6), (3.7) and (3.9)

$$\Delta\overline{E}(T_N) \le C(T). \tag{3.10}$$

Inserting (3.10) in Lemma 4.4, we eventually get

$$m(T_N) \le C(T). \tag{3.11}$$

Together (3.9), (3.10) and (3.11) imply that all the inequalities of (3.6) are true with a factor 1/2 at time T_N, provided N and M are large enough. Therefore (3.6) is still true on at least a short time interval after T_N and that means that necessarily $T_N = T$. The consequence is that (3.9), (3.10) and (3.11) are true on any time interval $[0,\ T]$ which is exactly Theorem 4.2.
Finally note that we have implicitly used the short time result when we said that $T_N > \varepsilon$.

4.4 Convergence of the density in the approximation

The existence of the bounds on R, K, $\overline{E}$, $\Delta\overline{E}$ and $\|\mu_N\|_{\infty,\eta}$ implies the weak convergence of the distribution μ_N to a weak solution of the Vlasov equation and Theorem 4.3 is only a consequence of Theorem 4.2 and the following proposition

Proposition 4.1. *Let μ_N be the distributions associated with the solutions to (1.1). We assume that the initial conditions μ_N^0 converges weakly in $M^1(\mathbb{R}^{2d})$ to some $f_0 \in L^1 \cap L^\infty(\mathbb{R}^{2d})$. We choose a time $T > 0$. Assume furthermore that there exists a constant $C(T)$ independent of N such that*

$$\sup_{\varepsilon>0}(R(T),\ K(T),\ \overline{E}(T),\ \Delta\overline{E}(T),\ \|\mu_{\infty,\eta}\|) < +\infty\ ,$$

where η depends on ε and N and goes to zero when ε goes to zero. Then, $\mu_N(t)$ converges weakly to $f(t)$, a solution to the Vlasov equation with initial conditions f^0.

Proof. We recall that the distribution of the particles μ_N satisfies the Vlasov equation in the sense of distribution provided the force field is correctly written. Moreover, the sequence μ_N is bounded in $C([0,T], M^1(\mathbb{R}^{3d}))$. Up to an extraction, we may assume that μ_N converges weakly to some $f \in L^\infty([0,T], M^1(\mathbb{R}^{2d}))$. In addition, the fact that $\|\mu_N\|_{\infty,\eta}$ is bounded implies that $f \in L^\infty$. To see this, we choose a regular test function Φ with compact support. We have

$$\langle \mu_N, \Phi\rangle = \frac{1}{N}\sum_{i=1}^{N} \Phi(X_i(t), V_i(t)).$$

Now, we define $\rho_\eta(x,v) = \chi_C(x/\eta, v/\eta)$ where χ_C is the characteristic function of the set $C = \{(x,v) | \|(x,v)\| \leq 1\}$ and we write

$$\begin{aligned}\langle \mu_N, \Phi\rangle &= \frac{1}{N}\sum_{i=1}^{N} \Phi * \rho_\eta(X_i(t), V_i(t)) \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}(\Phi(X_i(t), V_i(t)) - \Phi * \rho_\eta(X_i(t), V_i(t))).\end{aligned}$$

The first term is $\int \phi * \rho_\eta(x,v)\, d\mu_N(x,v) = \int \phi(\mu_N * \rho_\eta)\, dxdv$. So it is bounded by $\|\phi\|_1 \times \|\mu_N * \rho_\eta\|_\infty$. But $\|\mu_N * \rho_\eta\|_\infty$ is exactly $\|\mu_N\|_{\infty,\eta}$. The second term is easily bounded by $\eta\|\nabla\Phi\|_\infty$. Putting all together, we obtain that

$$\langle \mu_N, \Phi\rangle \leq \|\mu_N\|_{\infty,\eta}\|\Phi\|_1 + \eta\|\nabla\Phi\|_\infty.$$

At the limit,

$$\langle f, \Phi\rangle \leq \liminf_{N\to\infty} \|\mu_N\|_{\infty,\eta}\|\Phi\|_1,$$

which means that $f \in L^\infty$ and that $\|f\|_\infty \leq \liminf_{N\to\infty} \|\mu_N\|_{\infty,\eta}$.

The passage to the limit in the linear part of the equation does not raise any difficulty. For the term in $F \cdot \nabla_v f$, we need a strong convergence in the force. We denote by F_∞ the force induced by f and by F_N the force induced by μ_N

$$\begin{aligned}F_\infty(x) &= \int \frac{x-y}{|x-y|^{1+\alpha}}\, dydw, \\ F_N(x) &= \frac{1}{N}\sum_{i=1}^{N} \frac{x - X_i(t)}{|x - X_i(t)|^{1+\alpha}}.\end{aligned}$$

We have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} F_N(X_i(t)) - F_\infty(X_i(t))\,dt = I_1 + I_2 + I_3$$
$$= \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|y-X_i(s)|\geq r}\frac{y-X_i(s)}{|y-X_i(s)|^{\alpha+1}}\,d(\mu_N - f)(y)\,ds$$
$$+ \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|y-X_i(t)|\leq r}\frac{y-X_i(s)}{|y-X_i(s)|^{\alpha+1}}\,d\mu_N(y)\,ds$$
$$- \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|y-X_i(t)|\leq r}\frac{y-X_i(s)}{|y-X_i(s)|^{\alpha+1}}\,df(y)\,ds,$$

for all $r > 0$. The first term I_1 in the right hand side always goes to zero because μ_N converges weakly to f. The second term is dominated by $\|f\|_\infty \int_{B(0,R)} dy/|y|^\alpha$, a quantity which is less than $C\|f\|_\infty r^{d-\alpha}$. The last one is the field created by the close particles in the discrete case. To estimate it, we use estimate (4.1), which gives

$$I_3 \leq C\left(\|\mu_N\|_{\infty,\varepsilon}^{\alpha'/d} K^{\alpha'} r^{\alpha'-\alpha} + \varepsilon^{d-\alpha}\|\mu_N\|_{\infty,\varepsilon} K^{2d-\alpha} + \varepsilon^{2d-3\alpha}\|\mu_N\|_{\infty,\varepsilon} K^{d-\alpha}\overline{E}^d K^d\right) \leq C\, r^{\alpha'-\alpha}.$$

And these bounds are independent of N or i.
Then, letting ε going to 0 and then r, we find that

$$\sup_{i,t}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}|F_N(X_i(s)) - F_\infty(X_i(s))|\,ds \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.1}$$

With this strong convergence, we are able to prove the convergence of the term $F_N \cdot \nabla_v \mu_N$ towards $F_\infty \cdot \nabla_v f$ in the sense of distributions. We choose a smooth test function ϕ with compact support and compute

$$J = \int_0^T \Big(\int_{x,v} F_\infty(t,x)\cdot\nabla_v\phi(t,x,v) f(t,x,v)\,dxdv - \sum_{i=1}^N F_N(t, X_i(t), V_i(t))\cdot\nabla_v\phi(t, X_i(t), V_i(t))\Big)\,dt. \tag{4.2}$$

We separate J in $J_1 + J_2$, with

$$J_1 = \int_0^T\int_{x,v} F_\infty(t,x)\cdot\nabla_v\phi(t,x,v) d(f-\mu_N)(.,x,v)\,dt,$$

and

$$J_2 = \int_0^T \Big(\sum_{i=1}^N F_\infty(t,X_i(t),V_i(t)) - F_N(t,X_i(t),V_i(t))\cdot\nabla_v\phi(t,X_i(t),V_i(t))\Big).\,dt$$

Because of the continuity of F_∞, J_1 vanishes as ε goes to zero. To show that J_2 vanishes as well, we decompose it in $M = [T/\varepsilon]+1$ integrals on M intervals of time with length ε. The last interval is of length less than ε, but that does not create any difficulty and we do as if it were of length ε. We obtain,

$$\begin{aligned} J_2 = &\sum_{k=1}^M \int_{k\varepsilon}^{(k+1)\varepsilon} \Big(\sum_{i=1}^N \big(F_\infty(t,X_i(t),V_i(t)) - F_N(t,X_i(t),V_i(t))\big) \\ &\qquad \cdot \nabla_v\phi(t,X_i(t),V_i(t))\Big)\,dt \\ \le{}& C\sum_{k=1}^M \int_{k\varepsilon}^{(k+1)\varepsilon} \Big(\sum_{i=1}^N \big|F_\infty(t,X_i(t),V_i(t)) - F_N(t,X_i(t),V_i(t))\big|\Big)\,dt. \end{aligned} \tag{4.3}$$

This sum may be bounded by

$$CT\sup_{i,t}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |F_N(X_i(s)) - F_\infty(X_i(s))|\,ds,$$

a quantity which goes to zero according to (4.1). Thus, J goes to zero when ε goes to zero and the proof is done. □

Appendix : Existence of strong solutions to Equation (1.3)

We mean by strong solution on a time interval $[0,\ T]$, a function $f \in L^\infty([0,\ T]\times\mathbb{R}^{2d})$ with compact support in space and velocity and which satisfies (1.3) in the sense of distributions.

Obtaining such solutions for any time was a major issue for the Vlasov-Poisson system (finally solved in [LP91], [Sch91] and [Pfa92]) because from strong solutions it is easy to get uniqueness or classical solutions. However if the potential is not as singular (and it is the case here), the issue of strong solutions is relatively simple

Theorem 4.5. *Assume that* (1.4) *with* $\alpha < 1$. *Let* $f^0 \in L^\infty(\mathbb{R}^{2d})$ *with compact support and* $T > 0$. *Then there exists* $f \in L^\infty([0,\ T]\times\mathbb{R}^{2d})$ *with compact support, satisfying* (1.3) *in the sense of distribution.*

Corollary 4.2. *This solution is unique.*

The proof of the corollary is immediate as the theorem implies that $E(t,x) = F \star_x f$ is lipschitz thanks to (1.4).
The core of the proof of the theorem is the following estimate

Lemma 4.12. *Let $f \in L^\infty([0,\ T],\ \mathbb{R}^{2d})$ with compact support be a solution to (1.3) in the sense of distribution with (1.4) and $\alpha < d-1$. Then if we denote by $R(t)$ and $K(t)$ the size of the supports of f in space and velocity, they satisfy for a numerical constant C*

$$R(t) \le R(0) + \int_0^t K(s)\,ds,$$
$$K(t) \le K(0) + C\,\|f(t=0,.,.)\|_{L^\infty}^{\alpha/d}\,\|f(t=0,.,.)\|_{L^1}^{1-\alpha/d} \times \int_0^t (R(s))^\alpha\,ds.$$

Proof of the lemma. Given the estimate on f, ρ also belongs to L^∞ with the bound

$$\|\rho(t,.)\|_{L^\infty(\mathbb{R}^d)} \le C\,(K(t))^d\,\|f(t,.,.)\|_{L^\infty(\mathbb{R}^{2d})}.$$

As we have (1.4) with $\alpha < d-1$, $E = F \star_x \rho$ is lipschitz. Therefore the solution to (1.3) is unique and is given by the characteristics. Namely, we define X and V the unique solutions to

$$\partial_t X(t,s,x,v) = V(t,s,x,v), \quad \partial_t V(t,s,x,v) = E(t, X(t,s,x,v)),$$
$$X(s,s,x,v) = x, \quad V(s,s,x,v) = v.$$

The solution f is now given by

$$f(t,x,v) = f(0, X(0,t,x,v), V(0,t,x,v)),$$

with the consequence that

$$R(t) \le R(0) + \int_0^t K(s)\,ds, \quad K(t) \le K(0) + \int_0^t \|E(s,.)\|_{L^\infty}\,ds.$$

Then

$$\|E\|_{L^\infty} \le \|\rho\|_{L^1}^{1-\alpha/d}\,\|\rho\|_{L^\infty}^{\alpha/d},$$

and it is enough to notice that the L^1 and L^∞ norms of f are preserved in this case. □
From Lemma 4.12, one may obtain very easily Theorem 4.5 with a standard approximation procedure. The only thing to check is that the estimates on the support are independent of the parameter of the approximation and this is ensured by Lemma 4.12 in the case $\alpha < 1$ thanks to Gronwall Lemma.

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Chapter 5

Approximation de l'équation d'Euler et quasi-Euler par des systèmes de vortex

Abstract. We prove the weak convergence for any time of a system of quasi-vortex with positive and negative signs to the solution of the Euler equation. Quasi-vortex means that the kernel has a singularity in $1/|x|^\alpha$ with $\alpha \leq 1$ instead of diverging in $1/|x|$ near the origin. We also give some bounds on the force field for the true vortex case.
Key words. Derivation of kinetic equations. Particle methods. Euler equation.

5.1 Introduction

We shall consider here system of N generalized vortex ($N \in \mathbb{N}$) evolving in $\mathbb{R}^2$. We denote their position by $(X_1, \ldots, X_N)$ and the strengh opf the i-th vortex by ω_i/N with ω_i in $[-1, 1]$. These vortices are governed by the following system of differential equations:

$$\dot{X}_i = u(X_i(t)) = \frac{1}{N} \sum_{j \neq i}^{N} \omega_i \, K(X_i - X_j) \quad \text{for} \quad i = 1, \ldots, N, \tag{1.1}$$

where $K : \mathbb{R} \to \mathbb{R}$ is the kernel of interaction. Here the kernel will be C^∞ except at the origin. So, once we precise the initial conditions $(X_1^0, \ldots, X_N^0)$, for which we assume there are not two vortices at the same place, there is a unique solution defined till the first collision.

This equation was originally introduced in the case of the Biot-Savart kernel $K(x) = x^\perp/|x|^2$ as a discrete approximation of the Euler equation written just above.
The continuous equation associated to that system (for one time, this is the discrete model that is derivated from the continuous one) is the Euler equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ u(x) = \int K(x-y)\omega(y)\, dy \end{cases} \tag{1.2}$$

with a given initial vorticity ω^0. Remark that the distribution of vorticity $\omega_N(t) = (1/N)\sum_{i=1}^N \omega_i \delta_{X_i(t)}$ solve the Euler equation in the sense of distribution if the kernel is regular.
What kind of kernel K is used? As we said above, for a system of vortex, K is given by the Biot Savard law:

$$K(x) = \frac{x^\perp}{|x|^2}$$

where $x^\perp$ is the vector of same length as x so that $(x, x^\perp)$ is a direct orthogonal basis. This is the case of physical interest. However, in the first part of our article we will state theorems for forces satifying

$$|K(x)| \leq C\frac{1}{|x|^\alpha}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{\alpha+1}}, \quad \mathrm{div}(K) = 0.$$

In this case, we will obtain estimates that will allow us to conclude that convergence occurs. Our proof will be true if the divergence of K is only assumed to be bounded.

5.1.1 Resolution of the ODEs

Before speaking of the limit, we need to talk a little bit about the resolution of the system of ordinary equations. A good reference for this problem is the book of C. Marchioro and M. Pulvirenti [MP94]. They do a careful analysis of the system to show that the problem of the singularity can be solved and that the set of the initial conditions for which collision occurs is negligible. Here, thanks to the hypotheses we will put on the initial conditions, we will show that collisions never occur for the system of simili-vortex system , and that there are not any collisions till a time T for the system of true vortices. We also refer to [MP94] for the existence and the uniqueness of solution of the Euler equation when the vorticity is in L^∞. We will show some result of convergence of system of vortex to the continuous equation it approximates, the Euler equation. In the next section, we present the already known results.

5.1.2 Known results

The weak convergence for the vortex system was proven by Jean-Marc Delort in [Del91]. Steve Schochet simplified the proof in [Sch96] and [Sch95]. The principle of their demonstration is to lower the singularity with a symmetrisation of the kernel. This can only yield weak convergence. With numerical techniques, J. Goodman, T.Y. Hou and J. Lowengrub prove the convergence of the vortex method with well-distributed initial conditions to the Euler system [GHL90], [GH91]. They use vortices initially distributed on a regular grid, but that allows them to obtain a good order of convergence.
Here, We shall use vortices initially distributed with some regularity, but our assumptions are not so strong than those in [GHL90]. We will not obtain order of convergence, but the result of convergence is stronger than the one of S. Schochet [Sch95]. Our results will use the techniques introduced by Pierre-Emmanuel Jabin and myself in [HJ03].

5.2 Main results

To state our result, we introduce a notion of L^∞ discrete norm.

Definition 5.1 (L^∞ discrete norm). *Choosen an η in $\mathbb{R}$, an a signed measure ω in $\mathbb{R}^2$, the L^∞- discrete norm of μ at scale η is*

$$\|\omega\|_{\infty,\eta} = \sup_{x\in\mathbb{R}^2} \frac{1}{\eta^2}|\omega|(B(x,\eta)).$$

where $B(0,\eta)$ denotes the ball of center 0 and diameter η, $|\omega|(A)$ is the total variation of ω in the set A.

Remark 5.1. *This is the definition for $\mathbb{R}^2$ because we will only work on $\mathbb{R}^2$, but this can of course be extended for all the $\mathbb{R}^n$. If ω is a sum of N vortex of strengh $\pm 1/N$, $\|\omega\|_{\infty,\eta}$ compute the number of vortex in a ball of size η.*

Using this norm is like saying that we do not want to look too close at the vortices. If we know their position with an incertude of η, it will be sufficient for us. It allows us to get rid of the singularity of the Dirac masses. But on the other hand, we have to prove that the uncertainty on the position of the Dirac masses will not have any consequences in the calculation.
We want to look for every N at a scale that goes to zero when ε goes to infinity, because at the limit a uniform bound on such L^∞-discrete norm will give us true L^∞ bounds. Now, at which scale η can we look at? We have vortex of strengh $1/N$. We define ε by $N = \varepsilon^{-2}$. Remark that since we have

ε^{-2} vortex in dimension 2, ε is the order of the average distance between vortex. We want to look at scale of type ε^κ. If κ is strictly greater than 1, the initial ε^κ norm of the distribution of vortex automatically blows up as $\varepsilon \to 0$. So ε is the smallest scale we can look at, our microscopical scale. At this scale and roughly speaking, we deal with a finite number of particles per box. If we look at a scale ε^κ with a $\kappa \in (0,1)$, the number of particles by box goes to infinity and maybe we will be able to observe macroscopical comportement at this scale. The use of that norm at different scales will be crucial in the rest of this article.

5.2.1 The system of simili vortices

By this we mean the system of ODE when the force is of type $1/|x|^\alpha$, or more precisely satisfy the conditions (5.1). In this case, we define two quantities. The first $R(t)$ is the size of the support of μ_N:

$$R(t) = \sup_{i=1,\dots,N} |X_i(t)|.$$

The second quantity $m(t)$ control the minimal distance between two vortices.

$$m(t) = \sup_{i \neq j} \frac{\varepsilon}{|X_i(t) - X_j(t)|}$$

We recall that we define the distribution of vorticity ω_N by

$$\omega_N(t) = \frac{1}{/} N \sum_{i=1}^{N} \omega_i \delta_{X_i(t)}$$

For this system, we obtain the following result of convergence:

Theorem 5.1. *For each N, choose the initial positions of N vortices so that $R(0)$, $m(0)$ and $\|\omega_N\|_{\infty,\varepsilon}$ are uniformly bounded. Moreover, assume that $\mu_N(0)$ goes weakly in the sense of measure to a function ω^0, which is then in L^∞. Then, for every t in $\mathbb{R}$, $\omega_N(t)$ goes weakly to $\omega(t)$, the unique solution of the Euler equation (1.2) with initial conditions ω^0.*

5.2.2 The vortex system

For the vortex system we introduce the same quantities R and m. We will only get bounds on the field and its growth, that does not imply the convergence of the system.

Warnings. We will often erase the time t or the subscript N in our calculation, but the reader should keep in mind that we always do calculations at a fixed time t and for a fixed N. We will also use many time C as a numerical constant whose value can change from one line to another. An equation with a C in it means that there exists a numerical constant so that... Non-numerical constants will be denoted by other letters as K...

5.3 The system of simili-vortices

This section is devoted to the proof of Theorem 5.1. We shall begin by proving that the convergence holds on a small interval of time.

5.3.1 Convergence for short time

Step 1. Estimate of the speeds. We define

$$U(t) = \sup_{i=1,\dots,N} u(X_i(t))$$

Then we have the following result

Lemma 5.1. *We have the following bound*

$$U \leq C\|\omega\|_{\infty,\varepsilon}^{1/2} R^{1-\alpha} + C\varepsilon^{2-\alpha}\|\omega\|_{\infty,\varepsilon}\, m(t)^\alpha$$

Proof of the lemma. We choose one i and decompose the space $\mathbb{R}^2$ in the subset $I_0 = \{|x - X_i| \leq \varepsilon\}$ and $I_k = \{|x - X_i| \in [2^k\varepsilon, 2^{k+1}\varepsilon)\}$ for all $k \in \mathbb{N}$. The greatest k such that I_k may contain at least one particle is $k_{max} = [R/\varepsilon]+1$, where the brackets denoted the integer part. We denote by u_k the part of the speed of X_i due to the vortices in I_k. It is easy to bound u_0 by

$$|u_0| \leq \|\omega\|_{\infty,\varepsilon}\,\varepsilon^2 \left(\frac{m(t)}{\varepsilon}\right)^\alpha = \|\omega\|_{\infty,\varepsilon}\,\varepsilon^{2-\alpha} m(t)^\alpha$$

using the lowest bound $\varepsilon/m(t)$ for the minimal inter-vortices distance. For the remaining terms, we use a discrete analog of an Hölder inequality. For this, remark that

$$|\omega|(I_k) \leq 1 \quad \text{and that} \quad |\omega|(I_k) \leq \|\omega\|_{\infty,\varepsilon}(2^{k+1}\varepsilon)^2$$

If we take the first inequality at the power $1/2$ multiplied by the second at the power $1/2$, we obtain

$$|\omega|(I_k) \leq 4\|\omega\|_{\infty,\varepsilon}^{1/2}\, 2^{k+1}\varepsilon$$

Moreover, for each $X_j \in I_k$, we have $|X_j - X_i| \geq 2^k\varepsilon$. So we can bound $|u_k|$ by $|u_k| \leq 8\|\omega\|_{\infty,\varepsilon}^{1/2}(2^{k+1}\varepsilon)^{1-\alpha}$. Now, we sum from $k=0$ to k_{max}. We obtain

$$\sum_{k=0}^{k_{max}} |u_k| \leq 16\|\omega\|_{\infty,\varepsilon}^{1/2}\varepsilon^{1-\alpha}\sum_{k=1}^{k_{max}} 2^{k(1-\alpha)} \quad (3.1)$$

$$\leq C\|\omega\|_{\infty,\varepsilon}^{1/2}R^{1-\alpha} \quad (3.2)$$

Adding the bound on u_0 to this one give the expected result. □

Step 2. Estimation of the derivative of the speed.
We define

$$\nabla U(t) = \sup_{i\neq j}\frac{|u(X_i(t)) - u(X_j(t))|}{|X_j(t) - X_i(t)|}$$

We use this definition instead of a true bound on $\bar{\nabla U} = \omega_N * \nabla K$ because a vortex does not interact with itself and then this quantity is not of real interest. The following lemma will give us a bound on $\bar{\nabla U}(t)$

Lemma 5.2. *We have the following bound*

$$\bar{\nabla U}(t) \leq C\|\omega(t)\|_{\infty,\varepsilon}R(t)^{1-\alpha} + C\varepsilon^{1-\alpha}(1 + \|\omega(t)\|_{\infty,\varepsilon})m(t)^{1+\alpha}$$

Proof. We pick an i and a j. Then,

$$\frac{|u(X_i) - u(X_j)|}{|X_j - X_i|} \leq \frac{|K(X_i - X_j) - K(X_j - X_i)|}{N|X_i - X_j|} + \frac{1}{|X_j - X_i|}\sum_{k\neq i,j}|K(X_k - X_j) - K(X_k - X_i)| \quad (3.3)$$

The first term is bounded by $C\varepsilon^2|X_i - X_j|^{-(1+\alpha)} \leq C\varepsilon^{1-\alpha}m(t)^{1+\alpha}$. To bound the others, we use

$$\frac{|K(X_k - X_j) - K(X_k - X_i)|}{|X_j - X_i|} \leq \frac{C}{\min(|X_k - X_i|, |X_k - X_j|)^{1+\alpha}}.$$

This inequality come from the condition (5.1) on the derivative of K. We decompose the space into the subsets $I_l = \{x|\min(|x - X_i|, |x - X_j|) \in [l\varepsilon, (l+1)\varepsilon]\}$ for l equals 0 to $l_{\max} = [R/2\varepsilon]$. Remark that we do not use the same decomposition that in the proof of the estimate for the speed field. The absolute vorticity in I_l is bounded by $C\|\omega\|_{\infty,\varepsilon}l\varepsilon^2$, and for every vortex k in I_l, $\min(|X_k - X_j|, |X_i - X_k|) \geq l\varepsilon$. The, the sum on all vortices in I_l is bounded by $1/(l\varepsilon)^{1+\alpha}$ for $l \geq 1$. Thus, the contribution of I_l is bounded by $C\|\omega\|_{\infty,\varepsilon}\varepsilon^{1-\alpha}l^{-\alpha}$. Separetely, the contribution of I_0 can be bounded by $C\|\omega\|_{\infty,\varepsilon}\varepsilon^{1-\alpha}m(t)^{1+\alpha}$. Adding all these contributions, we get

$$\bar{\nabla U}(t) \leq C\|\omega\|_{\infty,\varepsilon}\,\varepsilon^{1-\alpha}\sum_{l=1}^{l_{max}} l^{-\alpha} + C(\|\omega\|_{\infty,\varepsilon}+1)\varepsilon^{1-\alpha}m(t)^{1+\alpha}$$

We may bound the sum by $Cl_{max}^{1-\alpha} = CR^{1-\alpha}\varepsilon^{\alpha-1}$. This is the expected result.

□

Step 3. A system of differential inequalities.
Now we want to control the growth of R and m. For R, we can obtain

$$\begin{aligned}
\dot{R}(t) &= \frac{\partial}{\partial t}(\sup_{i\leq N} X_i(t)) && (3.4)\\
&\leq \sup\left(\frac{\partial}{\partial t}|X_i(t)|\right) && (3.5)\\
&\leq \sup|\dot{X}_i(t)| \leq U(t). && (3.6)
\end{aligned}$$

For m, we do as above:

$$\begin{aligned}
\dot{m}(t) &= \frac{\partial}{\partial t}\left(\sup_{i\neq j}\frac{\varepsilon}{|X_i(t)-X_j(t)|}\right) && (3.7)\\
&\leq \sup\frac{C\varepsilon\,|U(X_i(t))-U(X_j(t))|}{|X_i(t)-X_j(t)|^2} && (3.8)\\
&\leq m(t)\bar{\nabla U}(t). && (3.9)
\end{aligned}$$

We finally obtain the following system of ordinary differential equations:

$$\begin{cases} \dot{m}(t) \leq \bar{\nabla U}(t)\,m(t) \\ \dot{R}(t) \leq U(t) \end{cases} \tag{3.10}$$

With this and the bounds (5.3.1) and (5.2), we can bound $\dot{m}$ and $\dot{R}$ in function of m, R, and $\|\omega\|_{\infty,\varepsilon}$. Moreover, it is possible to bound $\|\omega\|_{\infty,\varepsilon}$ in terms of $m(t)$. Indeed, we can not put more than CM^2 particles in a ball of size ε if we want that the minimal distance between particles to be greater than ε/M. So, we can bound $\|\omega\|_{\infty,\varepsilon}$ by $\|\omega\|_{\infty,\varepsilon} \leq Cm(t)^2$. Putting all together, we get the following system of differential inequalities for m and R:

$$\begin{cases} \dot{m}(t) \leq Cm(t)^3R(t)^{1-\alpha} + C\varepsilon^{1-\alpha}m(t)^{4+\alpha} \\ \dot{R}(t) \leq Cm(t)^2R(t)^{1-\alpha} + C\varepsilon^{2-\alpha}m(t)^{2+\alpha} \end{cases} \tag{3.11}$$

As long as the two quantities containing ε, that is $C\varepsilon^{1-\alpha}m(t)^{4+\alpha}$ and $C\varepsilon^{2-\alpha}m(t)^{2+\alpha}$ are less than one, a condition that is true at time $t=0$ if ε is sufficiently small, we may write

$$\begin{cases} \dot{m}(t) \le Cm(t)^3R(t)^{1-\alpha} + 1 \\ \dot{R}(t) \le Cm(t)R(t)^{1-\alpha} + 1 \end{cases} \tag{3.12}$$

Now, we choose m_0 and R_0 so that $m_N(0) \le m_0$ and $R_N(0) \le R_0$ for all N. We denote also (m_t, R_t) the solution of the ODE

$$\begin{cases} \dot{m}(t) = Cm(t)^3R(t)^{1-\alpha} + 1 \\ \dot{R}(t) = Cm(t)R(t)^{1-\alpha} + 1 \end{cases}$$

with initial conditions (m_0, R_0). It exists till a time of explosion T^*. Since the right hand side terms are increasing in R and m, we can write $m_N(t) \le m_t$ and $R_N(t) \le R_t$ provided that the conditions $C\varepsilon^{1-\alpha}m_t^{4+\alpha} \le 1$ and $C\varepsilon^{2-\alpha}m_t^{2+\alpha} \le 1$ are true. This will be the case for any time t less than T^* if ε is small enough. So, we get uniform bound on m_N and R_N for $t \le T^*$.

Step 4. Conclusion of the convergence.
So, we have uniform bounds on $\|\omega(t)\|_{\infty,\varepsilon}$, $m_N(t)$ and $R_N(t)$ for all t. This will imply strong convergence results for the field of speeds, and allow us to take the limit in the equation. First, if we take a subsequence of $\omega_N(t)$ that goes weakly in the sense of measure to ω, this ω belongs to L^∞. This is proved in the following lemma:

Lemma 5.3. *Take a sequence of probability measure ω_n on $\mathbb{R}^2$ that converge weakly to ω, and such that there exists a sequence ρ_n of positive real going to zero so that $\|\omega_n\|_{\infty,\rho_n}$ is uniformly bounded. Then, ω belongs to L^∞.*

Proof of the lemma. We denote by ξ_n the caracteristic function of the ball $B(0,\rho_n)$ divided by is volume ρ_n^2: $\xi_n = 1/\rho_n^2 \xi_{B(0,\rho_n)}$. We choose a smooth test function ϕ. We have

$$\int \phi(x)\, d\mu_n(x) = \int \phi(x)\, d(\omega_n - \omega_n * \xi_n)(x) + \int \phi(x) d(\omega_n * \xi_n)(x) \tag{3.13}$$

$$= \int (\phi(x) - \phi * \xi_n(x))\, d\omega_n(x) + \int \phi(x) d(\omega_n * \xi_n)(x) \tag{3.14}$$

The first integral is bounded by $\|\nabla\phi\|_\infty \rho_n$ because ω_n is of total mass one, and the second is bounded by $\|\phi\|_1 \|\omega_n\|_{\infty,\rho_n}$. We get,

$$\left| \int \phi(x)\, d\omega_n(x) \right| \le \|\nabla\phi\|_\infty \rho_n + \|\phi\|_1 \|\omega_n\|_{\infty,\rho_n}$$

Taking the limit when n goes to $+\infty$, we obtain

$$\left|\int \phi(x)\, d\omega_n(x)\right| \leq \liminf_{n\to\infty} \|\phi\|_1 \|\omega_n\|_{\infty,\rho_n}$$

Since this is true for every smooth ϕ, this means that ω belongs to L^∞ and that $\|\omega\|_\infty \leq \liminf_{n\to\infty} \|\omega_n\|_{\infty,\rho_n}$ □

Moreover, with those uniform bounds, we can obtain the strong convergence for the speed field. So we extract a subsequence in the ω_N that we will still denote by ω_N which converges to a ω in L^∞. We denote by u_N the speed field created by ω_N and by u the field created by ω. Then, for every positive r

$$\begin{aligned}|u_n(x) - u(x)| &= |\int K(x-y)\, d(\omega_N - \omega)(y)| \\ &\leq \int_{|x-y|\geq r} |K(x-y)|\, d(\omega_N - \omega)(y) + \int_{|x-y|\leq r} |K(x-y)|\, d(\omega_N + \omega)(y)\end{aligned}$$

The first term goes to 0 when N goes to $+\infty$ because of the weak convergence of ω_N to ω and the continuity of the kernel outside the origin. The second can be bounded by $(\|\omega\|_\infty + C\|\omega_N\|_{\infty,\varepsilon})r^{2-\alpha} + \varepsilon^{2-\alpha}\|\omega_N\|_{\infty,\varepsilon} m(t)^\alpha$ using the usual decomposition for the term due to ω_N or equivalently by replacing $R(t)$ by r in the proof of Lemma 5.1. So this term goes to zero when r goes to zero and we get the pointwise convergence of u_N to u. Moreover, the sequence u_N is uniformly bounded in L^∞. This allows us to pass to the limit in the Euler equation, satified by all the ω_N and we obtain that ω is also a solution of the Euler equation, with initial conditions ω_0. Since the solution of the Euler equation is unique in L^∞, we get that the whole sequence ω_N goes weakly to ω, the solution of the Euler equation.

5.3.2 Long time convergence

How could we get a convergence for long time? For this, we need bounds on R and m for any time. This could be done if our system of inequalities were sub-linear. Remark that, it could be so if we could write that $\|\omega(t)\|_{\infty,\varepsilon} = \|\omega(0)\|_{\infty,\varepsilon}$. It this case, we won't have to replace $\|\omega\|_{\infty,\varepsilon}$ by $m(t)^2$ in the sytem (3.10), and instead of (3.12), we will obtain a system of the form below:

$$\begin{cases} \dot{m}(t) \leq C\|\omega 0\|_{\infty,\varepsilon} m(t)\, R(t)^{1-\alpha} + 1 \\ \dot{R}(t) \leq C\|\omega 0\|_{\infty,\varepsilon} R(t)^{1-\alpha} + 1 \end{cases}$$

This system do not explode in a finite time. The second line give us a polynomial growth for R and once we obtain this growth, we obtain an exponential growth for m by replacing R by its bound in the first line. So we will get bound for our two quantities for every time. Remark also that this preservation of the L^∞ norm is obvious in the continuous model, the Euler equation and that here we will need work to obtain. But how can we obtain bound of L^∞ discrete norms. It seems that the L^∞ norm at scale ε is not preserved. But, the answer is to look at larger scale. We can obtain the asymptotic preservation of the discrete L^∞ norm at a macroscopic level. The following proposition states it more precisely:

Proposition 5.1. *Set T_1 to be the time so that $\int_0^{T_1} \nabla\bar{U}(t)\,dt = 1/4$ and fix a $t \leq T_1$. Choose a $\gamma in(0,1)$. Then, there exist two constants K_1 and K_2 depending on m, R such that for all $\eta \geq \varepsilon$*

$$\|\omega_N(t)\|_{\infty,\eta} \leq (1 + K_2\eta^\gamma + K_1\frac{\varepsilon}{\eta})\|\omega_N(0)\|_{\infty,\varepsilon}$$

To prove this proposition, we first introduce the following definitions:

Definition 5.2 (Parallelogram). *A parallelogram in $\mathbb{R}^2$ is a set S defined by*

$$S = \{x | \|A(x - x_0)\| \leq \rho\}$$

where A is a 2×2 matrix of determinant 1, ρ is a positive real and x_0 belongs to $\mathbb{R}^2$. A, ρ, x_0 will be called respectively the matrix, the size and the center of the parallelogram.

Definition 5.3 (Not too stretched parallelogram). *A parallelogram is not to stretched if*

$$\|A - Id\| = \sup_{\|x\|=1} \|Ax - x\| \leq 1/2.$$

Remark 5.2. *Roughly speaking, this definition means that our parallelogram as a shapre close from the one of a square and that it is not too stretched in one direction.*

At a time t, we will look at a box $S_t = \{x||x - x_0| \leq \rho\}$ and let it evolves backward according to the field of velocity u till time $t' \leq t$ not too far from t. We obtain a set denoted $\tilde{S}_{t'}$. We will show that this set could be included in a not too streched parallelogram with almost the same volume than the initial box. The fact that the volume is the same is not sufficient because we only control the distribution at scale ε and we could need too many ε balls to cover the parallelogram. We need to control the shape of the parallelogram. The definition and the lemma below will help us to bypass this difficulty.

Definition 5.4 (ε-volume). *The ε-volume of a set S, denoted by $Vol_\varepsilon(S)$ is the minimal number of balls of diameter ε that we need to cover it, divided by the volume of such a ball.*

$$Vol_\varepsilon(S) = \inf\{Vol(\cup_i B(x_i, \varepsilon/2)) | N \in \mathbb{N}, x_1, \dots, x_n \in \mathbb{R}^2 \dots \\ \dots \textit{so that} \quad S \subset \cup_i B(x_i, \varepsilon/2)\} \quad (3.15)$$

Lemma 5.4. *Let S be a not too streched parallelogram of size ρ. Then*

$$Vol_\varepsilon(S) \leq (1 + 2\frac{\varepsilon}{\rho})^2 Vol(S)$$

Proof of the lemma. We suppose that $x_0 = 0$ for simplicity. We set $S_{+\varepsilon} = \{x | |Ax| \leq \rho + \varepsilon\}$ and $S_{+2\varepsilon} = \{x | |Ax| \leq \rho + 2\varepsilon\}$. We define $P = S_{+\varepsilon} \cap \varepsilon\mathbb{Z}^2$ and $P_\varepsilon = P + B(0, \varepsilon/2)$, the union of all the balls of diameter $\varepsilon/2$ with center in P.
First, we will show that $S \subset P_\varepsilon$. For this, we choose an x in S and associate to it a couple m of $\mathbb{Z}^2$ so that $|x - \varepsilon m| \leq \varepsilon/2$. Then,

$$|A(\varepsilon m)| \leq |A(\varepsilon m - x)| + |Ax| \leq \|A\| |x - \varepsilon m| + \rho.$$

Since $\|A\| \leq 2$, we obtain that εm belongs to $S_{+\varepsilon}$, and then to P.
Next, we shall show that $P_\varepsilon \subset S_{+2\varepsilon}$. We choose a y in P_ε and associate to it a couple m of $\mathbb{Z}^2$ so that $|y - \varepsilon m| \leq \varepsilon/2$. Then,

$$|Ay| \leq |A(y - \varepsilon m)| + \varepsilon |Am| \leq \|A\| |y - \varepsilon m| + \rho + \varepsilon$$

Again, we obtain $|Ay| \leq \rho + 2\varepsilon$ and then y belongs to $S_{+2\varepsilon}$. Now, we can compare the ε-volume of S and the volume of the two others sets. We have

$$Vol_\varepsilon(S) \leq Vol(P_\varepsilon) \leq Vol(S_{+2\varepsilon})$$

And $Vol(S_{+2\varepsilon}) = (\det(A))^{-1}(\rho + 2\varepsilon)^2 = (1 + 2\frac{\varepsilon}{\rho})^2 Vol(S)$. This concludes the proof. □

Remark 5.3. *We also remark that the norm of the matrix of a not too stretched parallelogram and of its inverse are always less than 2 because $\|A - Id\| \leq 1/2$. The inequality on the inverse implies that a not too streched parallelogram of center x_0 is always included in the ball centered at x_0 of radius twice its size.*

Now we need to control the evolution of the vortices that are in a parallelogram at time t. For this, we will state the following lemma. But before, we introduce notations which will be usefull below. We will denote parallelogram by S_t, it means that it is related to the time t. And then, its size, center, and matrix will always be denoted by ρ_t, x_t and A_t. We will also use an approximation of the field of the form

$$u_\varepsilon(x) = \sum_{i=1}^{N} \omega_i K_\varepsilon(x - X_j(t)), \tag{3.16}$$

where K_ε is an approximation of K given by $K_\varepsilon = K \star \xi_\varepsilon$. ξ_ε is a classical approximation of the identity, it means that $\xi_\varepsilon = (1/\varepsilon^2)\xi(\cdot/\varepsilon)$ with a ξ C^∞ with suport in $B(0,1)$ of total mass 1. This approximated field satisfy the same estimate than u, because K_ε satisfies the same conditions (5.1) than K.

Lemma 5.5. *Choose a time $t > 0$ and a $\gamma \in (0,1)$. There exists two positive constants K_1 and K_2, such that for every not too streched parallelogram S_t, there exists a time $t^* < t$ and a family of not too streched parallelogram $(S_{t'})_{t^*<t'<t}$ such all the vortices that are in S_t at time t are in S'_t at time t'. The parallelograms $S_{t'}$ satisfy:*

i. their center $x_{t'}$ is the point x_t transported backward in time by u_ε,

ii. their matrix $A_{t'}$ are always of determinant 1 and satisfy the ODE

$$\dot{A}_s = -A_s \, \nabla u_\varepsilon(x_s)$$

iii. their size $\rho_{t'}$ satisfies

$$\dot{\rho}_s = -K_1\varepsilon - K_2\rho_s^{1+\gamma}.$$

The time t^* before which no control by a not too streched parallelogram is possible is the time when $\|A_t - Id\|$ becomes greater than $1/2$ and is of the order of $(1/2 - .\|A_t - Id\|)/\bar{\nabla U}(t)$.

Proof of the lemma. We will use an approximation of the field because we will need to control the movement of a virtual vortex without knowing its distance to the other vortices. First, we control the difference between the approximation and the true field for a particle.

$$|u(X_i) - u_\varepsilon(X_i)| \leq \frac{1}{N} \sum_{j \neq i} |K(X_j - X_i) - K_\varepsilon(X_j - X_i)|.$$

Using the derivative of K, we can show that

$$|K(x) - K_\varepsilon(x)| \leq \frac{C\varepsilon}{|x|^{1+\alpha}}$$

Using this bound and our usual division of the space, we can compute the difference due to the particles at distance greater than ε. For the rest, we do not use the last bound and bound the difference by the sum of the two terms and bound it as above:

$$|K(x) - K_\varepsilon(x)| \leq \frac{C}{|x|^{\alpha}}$$

We obtain at the end

$$\sup_{i=1,\ldots,N} |u(X_i) - u_\varepsilon(X_i)| \leq C\|\omega\|_{\infty,\varepsilon} R(t)^{1-\alpha}\varepsilon + C\varepsilon^{2-\alpha}\|\omega\|_{\infty,\varepsilon} m(t)^\alpha$$

So if we define

$$K_1 = C\|\omega\|_{\infty,\varepsilon}(R(t)^{1-\alpha} + m(t)^\alpha),$$

we obtain

$$\sup_{i=1,\ldots,N} |u(X_i) - u_\varepsilon(X_i)| \leq K_1\varepsilon;$$

Now, we want to bound the derivative with respect to the time of $|A_t(X_i(t) - x_t)|$. We have

$$\begin{aligned}\frac{d}{dt}|A_t(X_i(t) - X_t)| &\geq -|A_t(-\nabla u_\varepsilon(x_s)(X_i(t)) - x_t) + u(X_i(t)) - u_\varepsilon(x_t))| \\ &\geq -\|A_t\||u(X_i(t)) - u_\varepsilon(X_i(t))| - \ldots \\ &\qquad \ldots \|A_t\||u_\varepsilon(X_i(t)) - u_\varepsilon(x_t) - \nabla u_\varepsilon(x_t)(X_i(t) - x_t)|\end{aligned} \tag{3.17}$$

The first contribution is controlled using the bound (5.3.2) just above by $\|A_t\|K_1\varepsilon$. The second term A_2 is the error between the field u_ε and its linearization near x_s. To bound this term, we remark that

$$|K_\varepsilon(x) - K_\varepsilon(y) - \nabla K_\varepsilon(y)(x-y)| \leq \frac{C|x-y|^2}{\min(|x|,|y|)^{2+\alpha}}$$

if we use a Taylor inequality. Moreover,

$$\begin{aligned}&|K_\varepsilon(x) - K_\varepsilon(y) - \nabla K_\varepsilon(y)(x-y)| = \\ &\qquad \int_0^1 ((1-u)\nabla K_\varepsilon((1-u)x + uy) - K_\varepsilon(y)) \cdot (x-y)\, du \\ &\qquad\qquad \leq \frac{C|x-y|}{\min(|x|,|y|)^{1+\alpha}}.\end{aligned} \tag{3.18}$$

And we can get many inequalities between this two. We fix a positive γ smaller than $1-\alpha$. If we take the first inequality at the power γ, and the second at the power $(1-\alpha)$ and multiply them, we obtain

$$|K_\varepsilon(x) - K_\varepsilon(y) - \nabla K_\varepsilon(y)(x-y)| \leq \frac{C|x-y|^{1+\gamma}}{\min(|x|,|y|)^{1+\alpha+\gamma}}. \tag{3.19}$$

Thanks to this inequality, we get that

$$A_2 \geq -\frac{C|X_i(t)-x_t|^{1+\gamma}}{N} \sum_{j\neq i} \frac{1}{(\min(|X_j(t)-x_t|, |X_j(t)-X_i(t)|)^{1+\alpha+\gamma}}$$

We can bound this sum exactly as we do for the derivative of u. The only difference is that $1+\alpha$ is replaced by $1+\alpha+\gamma$. We obtain

$$A_2 \geq -C|X_i(t)-x_t|^{1+\gamma}\left(\|\omega\|_{\infty,\varepsilon} R(t)^{1-\alpha-\gamma} + \varepsilon^{1-\alpha-\gamma}(\|\omega\|_{\infty,\varepsilon}+2) m(t)^{1+\alpha+\gamma}\right)$$

bound that we will also write that bound $A_2 \geq -K_2|X_i(t)-x_t|^{1+\gamma}$, with

$$K_2 = C\|\omega(\|_{\infty,\varepsilon} R(t)^{1-\alpha-\gamma} + m(t)^{1+\alpha+\gamma}$$

Finally, we obtain

$$\frac{d}{dt}|A_t(X_i(t)-x_t)| \geq -\|A_t\|(K_1\varepsilon + K_2|X_i(t)-x_t|^{1+\gamma})$$

Using the remark 5.3, we can bound $|X_i(t)-x_t|$ by $2\rho_t$ so that

$$\frac{d}{dt}|A_t(X_i(t)-x_t)| \geq -(K_1\varepsilon + K_2\rho_t^{1+\gamma}),$$

This is true till $S_{t'}$ is not too streched because in that case $\|A_{t'}\| \leq 2$. For this, we need to multiply K_1 and K_2 by a numerical constant. And this inequality implies that if $X_i(t)$ belongs to S_t, then $X_{t'}$ will also belongs to $S_{t'}$.

There only remains to prove the ODE satisfied by the determinant to finish the proof. For this, we just derivate classically the determinant:

$$d/ds(\det(A_s)) = tr(A_s^{-1}\dot{A}_s) = tr(\nabla u_\varepsilon(x_s)) = \operatorname{div}(u_\varepsilon(x_s)) = 0,$$

if K and then K_ε are divergence free. □

Thanks to this lemma, we will get the asymptotic preservation of the L^∞ discrete norm on the interval of time $[0, T_1]$, with T_1 the time so that $\int_0^{T_1} \nabla\bar{U}(t)\, dt = 1/4$. This is the aim of the following proposition:

Proposition 5.2. *We fix a* $\gamma \in (0, 1-\gamma)$ *and a* T_1 *so that* $\int_0^{T_1} \nabla \bar{U}(t)\,dt = 1/4$. *Then,there exist two constant* K_1 *and* K_2 *depending on* γ*,* R *and also* m *such that for every time* $t \leq T_1$*, we have the following bound on* $\|\omega(t)\|_{\infty,\eta}$*:*

$$\|\omega(t)\|_{\infty,\eta} \leq \|\omega(0)\|_{\infty,\varepsilon}\left(1 + CK_2\eta^\gamma + K_1\frac{\varepsilon}{\eta}\right)^2$$

Proof. We choose an x and denote $S_t = B(x,\eta)$. Since, $B(x,\eta)$ is a not too streched parallelogram with matrix Id, we can make it evolve backwards according to the preceding Lemma 5.5. We obtain a family $(S_{t'})_{t^* \leq t' < t}$ of parallelograms. The matrix of the family of parallelogram we obtain is

$$A_{t'} = e^{\int_{t'}^t \nabla u_\varepsilon(x_s)\,ds}$$

Since $|e^x - 1| \leq xe^x$ and since $e^{1/4}/4 \leq 1/2$, the condition $\int_0^{T_1} \nabla \bar{U}(t)\,dt = 1/4$ ensures that $\|A_{t'} - id\| \leq 1/2$ up to $t' = 0$. So it means that the family is define till time $t^* = 0$. The paralellogram S_0 will help us to bound $\omega_N(t, S_t)$ by

$$|\omega_N(t, S_t)| \leq |\omega_N^0(S_0)| \leq \|\omega(0)\|_{\infty,\varepsilon} Vol_\varepsilon(S_0)$$

Now, we only need a bound on the size ρ_0 of S_0 to control is ε-volume. To be precise, we define $\rho'_s = \rho_s - K_1\varepsilon(t-s)$.With this notation, the ODE satisfied by ρ_t may be rewritten if ρ_s is greater than $K_1\varepsilon$

$$\dot{\rho}'_s \geq -K_2 \rho_s'^{1+\gamma}.$$

If we integrate this equation, we obtain

$$\rho'_s \leq \frac{\rho'_t}{(1 - K_2(t-s)\rho'^{\gamma}_t)^{1/\gamma}}.$$

This inequation give the following one if we replace ρ'_u by $\rho_u - K_1(t-u)$ in it:

$$\rho_s \leq \frac{\rho_t}{(1 - K_2(t-s)\rho_t^\gamma)^{1/\gamma}} + K_1(t-s)\varepsilon$$

If $\rho_t^\gamma T_1$ is choosen sufficiently small, we may rewrite it

$$\rho_s \leq \rho_t(1 + K_2(t-s)\rho_t^\gamma) + K_1(t-s)\varepsilon.$$

So, at time $t' = 0$, our vortices were localized in a not too stretched parallelogram of size smaller than $\rho_t(1 + CK_2 t\rho_t^\gamma) + K_1 t\varepsilon$. Using Lemma (5.4) to control its ε-volume, we obtain that

$$\omega_N(B(x_t, \rho_t)) \leq \rho_t^2\left(1 + K_2 t\rho_t^\gamma + K_1\frac{\varepsilon}{\rho_t}\right)^2$$

Now, if we choose for S_t every ball of size η, we obtain

$$\|\omega(t)\|_{\infty,\eta} \leq \|\omega(0)\|_{\infty,\varepsilon} \left(1 + K_2 t \eta^\gamma + K_1 \frac{\varepsilon}{\eta}\right)^2$$

□

The bound we obtain at scale η are better than the one we obtain at scale ε. Thanks to them, the system of inequalities on R and m will be sublinear up to some negligeable term. The only problem is that this new bound is only valid till the time T_1. Wed can bypass this difficulty by iterating the previous Proposition 5.2. This give the following Lemma:

Lemma 5.6. *Let t be a time $t \in \mathbb{R}+$ and η_N a sequence of scale going to 0 so that η/ε goes to $+\infty$. We fix a N and choose a integer k so that $\int_0^t \bar{\nabla} U_N(s)\, ds \leq k/4$. We also define*

$$\delta = \left(\frac{\varepsilon}{\eta}\right)^{1/k}.$$

Then the following inequality holds if N is large enough.

$$\|\omega(t)\|_{\infty,\eta_k} \leq \|\omega(0)\|_{\infty,\varepsilon} (1 + K_2 \eta^\gamma + K_1 \delta)^{2k}$$

Remark 5.4. *Remark first that we have written the subscript N to simplify the presentation and that δ goes to 0 when N goes to infinity.*

Proof. We only use the proposition 5.5 k times. The first time between 0 and T_1 with the scale ε and ε/δ. The second time between T_1 and T_2, where T_2 defined by $\int_{T_1}^{T_2} \bar{\nabla} U_N(s)\, ds \leq 1/4$, replacing ε by ε/δ and ε/δ by ε/δ^2 in Proposition 5.5. We do it k times, the last time with the scale ε/δ^{k-1} and $\varepsilon/\delta^k = \eta$ and obtain that:

$$\|\omega(t)\|_{\infty,\eta_k} \leq \|\omega(0)\|_{\infty,\varepsilon} \prod_{l=1}^{k} \left(1 + K_2 \left(\frac{\varepsilon}{\delta^l}\right)^\gamma + K_1 \delta\right)^2$$

And this product can be bounded by the right hand side of (5.6). □

New estimates on U and $\bar{\nabla} U$.

Now we can use these bounds on the norms $\|\omega\|_{\infty,\eta}$ to refine our bound on U and $\bar{\nabla} U$. Using decomposition of the space at scale η and then at scale ε for what is close from the discontinuity, we get the bounds of the following lemma

Lemma 5.7.

$$U(t) \leq C\|\omega\|_{\infty,\eta}^{1/2} R^{1-\alpha} + C\|\omega\|_{\infty,\varepsilon}^{1/2} \eta^{1-\alpha} + \varepsilon^{2-\alpha}\|\omega\|_{\infty,\varepsilon} m(t)^{\alpha}$$

Lemma 5.8.

$$\bar{\nabla U}(t) \leq C\|\omega\|_{\infty,\eta} R(t)^{1-\alpha} + C\|\omega\|_{\infty,\varepsilon}\eta^{1-\alpha} + C\varepsilon^{1-\alpha}(\|\omega\|_{\infty,\varepsilon} + 2)m(t)^{1+\alpha}$$

A new sub-linear system of differential inequalities

We will now introduce a system of differential equation that will be able to give us the bound we need. this is the following one:

$$\begin{cases} \dot{\tilde{m}}(t) = C\|\omega(0)\|_{\infty,\varepsilon}\tilde{m}(t)\,\tilde{R}(t)^{1-\alpha} + 1 \\ \dot{\tilde{R}}(t) = C\|\omega(0)\|_{\infty,\varepsilon}\tilde{R}(t)^{1-\alpha} + 1 \end{cases}$$

with the initial conditions R^0 and m^0 so that $m_N(0) \leq m^0$ and $R_N(0) \leq R^0$ for all N. We also define

$$\tilde{\nabla U} = C\|\omega(0)\|_{\infty,\varepsilon}\tilde{R}(t)^{1-\alpha},$$

where we use for C the same constant that in (5.8). This is a bound for the derivative of the field if we use $\tilde{R}$ and $\tilde{m}$ in the bound and neglect the term with power of ε.
We now choose a time t, and fix a k such that $\int_0^t \tilde{\nabla U}(s)\,ds \leq k/4$. Using the bounds (5.7), (5.8) and (5.6) with this k, we get the following system of inequalities

$$\begin{cases} \dot{R}(t) \leq \|\omega(0)\|_{\infty,\varepsilon}^{1/2} R(t)^{1-\alpha} + S_N^1(R(t), m(t)) \\ \dot{m}(t) \leq m(t)(\|\omega(0)\|_{\infty,\varepsilon} R(t)^{1-\alpha} + S_N^2(R(t), m(t)) \end{cases},$$

where S^1 and S^2 are two polynoms with positive coefficients using $\|\omega(0)\|_{\infty,\varepsilon}$ containing all a positive power of ε, *eta* or ε/η. This means that they will become small if N is chosen large enough. More precisely, we choose N_0 such that for $N \geq N_0$,

$$\sup_{s\leq t} \max(S_N^1(\tilde{R}(s), \tilde{m})(t), S_N^2(\tilde{R}(s), \tilde{m})) \leq \frac{1}{2}.$$

We will show that from this rank, m_N and R_N are bounded by $\tilde{m}$ and $\tilde{R}$ till time t. We fix a N greater than N_0 and define τ_N to be the first time where either $R_N(s) \geq \tilde{R}$ or $m_N(t) \geq \tilde{m(t)}$. Till this time

$$\max_i(S_N^i(R_N(s), m_N)(t),) \leq \frac{1}{2},$$

because the S_i have positive coefficients. Moreover, if $\tau_N \leq t$, we have

$$\int_0^{\tau_N} \nabla \bar{U}_N(s)\, ds \leq \int_0^t \nabla \tilde{U}(s)\, ds \leq k/4$$

and then we have the right to use the the bound of Lemma 5.6, and the system (5.3.2). But at time τ_N, $\max_i(S_N^i(R_N(\tau_N), m_N(\tau_N)) \leq 1/2$. Thanks to this, and the fact (5.3.2), the bounds $R_N(s) \geq \tilde{R}$ and $m_N(t) \geq \tilde{m(t)}$ will remains true a little after τ_N. so τ_N is necessarily greater than t. And we have the uniform bound we need till t. The asymptotic preservation of $\|\omega(t)\|_{\infty,\eta}$ also occurs.
Thanks to the uniform bounds on $\mathbb{R}$, m and $\|\omega\|_{\infty,\eta}$, we can end the proof as in the result for small time. We choose an sequence η so that η and ε/η goes to zero when N goes to $+\infty$. And we replace ε y η in the estimates we use in the conclusion of the proof for small times.

5.4 The vortex system

To try to obtain results for the system of true vortices, we should look at the demonstration for the simili-vortices and look for the point where the hypothesis $\alpha < 1$ is necessary. They are the following:

i. In the estimation (5.2) of the gradient of the field of speed, the term in $1/|x|^2$ will not be integrale in the neighbourhood of the origin.

ii. In the estimation we use to linearize the field of speed in a box, we use an Hölder bound on $\nabla \bar{U}_\varepsilon$ for which the fact that we can find a γ strictly positive and less than $1-\alpha$ is crucial (see (3.19)).

The first difficulty is understandable because for in the Euler equation, the continuous model towards which our system should go, the field of speed created by a vorticity in L^∞ is not Lipschitz, but $x\log(x)$. We mean by this that there exists K_1 and K_2 depending on the three quantities such that

$$\sup_{x \neq y} |u(x) - u(y)| \leq K_1|x-y| + K_2\big||x-y|\log|x-y|\big|$$

So one possibility is to get bound of this kind for our system. We will obtain some bounds analog to the bound we can get in the continuous model (the Euler equation), but at this time do not seems sufficient to get convergence result.

As in the simili vortex case, we want to obtain estimate on U and its derivative. The first estimate is the same that in the simili-vortex case.

$$U(t) \le C\|\omega\|_{\infty,\varepsilon} R(t) + C\varepsilon\|\omega\|_{\infty,\varepsilon} m(t)$$

The second is given in the following lemma

Lemma 5.9. *There exists constants C so that for all distincts i and j:*

$$\begin{aligned} |u(X_i(t)) - u(X_j(t))| \le C\|\omega\|_{\infty,\varepsilon} \log(R)|X_i - Y_j| \\ + C\|\omega\|_{\infty,\varepsilon}(1 + m(t)\log(\frac{\varepsilon}{m(t)}))\big||X_i - X_j|\log(|X_i - X_j|)\big| \end{aligned} \tag{4.1}$$

Proof of the lemma. We choose a couple (i,j) to estimate the difference $|u(X_i) - u(X_j)|$. We will of course decompose the space in cells of size ε. We pick one integer L depending on ε that we will fix later and decompose the sum

$$\begin{aligned} |u(X_i) - u(X_j)| \le \frac{1}{N} \sum_{\min(|X_k - X_i|, |X_k - X_j|) \ge 2^L\varepsilon} \frac{|X_i - X_j|}{\min(|X_k - X_i|, |X_k - X_j|)^2} \\ + \frac{2}{N} \sum_{\varepsilon \le \min(|X_k - X_i|, |X_k - X_j|) \le 2^L\varepsilon} \frac{1}{\min(|X_k - X_i|, |X_k - X_j|)} \\ + \frac{2}{N} \sum_{\min(|X_k - X_i|, |X_k - X_j|) \le \varepsilon} \frac{1}{\min(|X_k - X_i|, |X_k - X_j|)} \end{aligned} \tag{4.2}$$

We decompose the first sum $\bar{\nabla U}_1$ in

$$\bar{\nabla U}_1 = \frac{1}{N} \sum_{l=1}^{l_{max}} \sum_{k \in I_l} \frac{|X_i - X_j|}{\min(|X_k - X_i|, |X_k - X_j|)^2}$$

where $I_l = \{k | \min(|X_k - X_i|, |X_k - X_j|) \in [2^{K+l}\varepsilon, 2^{K+l+1}\varepsilon]\}$ and $l_{max} = [\log_2(R/2^K\varepsilon)] + 1$. The total vorticity in I_k can be bounded $\omega_N(I_k) \le C(2^{K+l}\varepsilon)^2\|\omega\|_{\infty,\varepsilon}$. We get:

$$\bar{\nabla U}_1 \le |X_i - X_j| \sum_{l=1}^{l_{max}} \frac{C\|\omega\|_{\infty,\varepsilon}(2^{L+l}\varepsilon)^2}{(2^{L+l}\varepsilon)^2} \tag{4.3}$$

$$\le C\|\omega\|_{\infty,\varepsilon}|X_i - X_j| \sum_{l=1}^{l_{max}} 1 \tag{4.4}$$

As $l_{max} = \log(R/2^L\varepsilon)$, we get

$$\bar{\nabla U}_1 \le C\|\omega\|_{\infty,\varepsilon}|X_i - X_j| \log(\frac{R}{2^L\varepsilon})$$

For the second sum $\bar{\nabla U}_2$, we use the decomposition $J_m = \{k | \min(|X_k - X_i|, |X_k - X_j|) \in [2^m\varepsilon, 2^{m+1}\varepsilon]\}$, for $m = 1$ to $L - 1$. The total vorticity in J_m satisfy $\omega(J_m) \leq \|\omega\|_{\infty,\varepsilon}(2^l\varepsilon)^2$. So,

$$\bar{\nabla U}_2 \leq C\|\omega\|_{\infty,\varepsilon} \sum_{m=1}^{L-1} \frac{(2^m\varepsilon)^2}{2^m\varepsilon} \leq C\|\omega\|_{\infty,\varepsilon} 2^L\varepsilon$$

And the last term $\bar{\nabla U}_3$ is bounded by

$$\bar{\nabla U}_3 \leq 2\|\omega\|_{\infty,\varepsilon}\varepsilon^2 \frac{m(t)}{\varepsilon} \frac{m(t))|X_i - X_j| \log(|X_i - X_j|)}{\varepsilon \log(\varepsilon/m(t))}$$

In the last fraction we uses the facts that $-x\log(x)$ is increasing near the origin and that $|X_i - X_j| \geq \varepsilon/m(t)$. This to introduce the factor $|X_i - X_j| \log(|X_i - X_j|)$. We obtain for $\bar{\nabla U}_3$ the bound

$$\bar{\nabla U}_3 \leq 2\|\omega\|_{\infty,\varepsilon} log(\frac{\varepsilon}{m(t)})m(t)\big||X_i - X_j| \log|X_i - X_j|\big|$$

Putting the three bound together, we obtain

$$\begin{aligned}\bar{\nabla U} &\leq C\|\omega\|_{\infty,\varepsilon}|X_i - X_j| \log(\frac{R}{2^K\varepsilon}) \\ &+ C\|\omega\|_{\infty,\varepsilon}2^K\varepsilon + 2\|\omega\|_{\infty,\varepsilon} log(\frac{\varepsilon}{m(t)})^{-1} m(t)|X_i - X_j| \log(|X_i - X_j|) \quad (4.5)\end{aligned}$$

Now, we just have to choose L in order to obtain a $x\log(x)$ bound. We take

$$L = [\log\left(|X_i -_j|\right)/\varepsilon)]$$

Where $[\cdot]$ stands for the integer part of. The error between this L and those without the integer part will only add a constant 2 in the bound.
So, with choice, the second sum give a bound in x. The first term becomes something of the form $x(\log(R) - \log(x))$, if $x = |X_i - X_j|$. As This give us

$$\begin{aligned}\sup_{i\neq j} |u(X_i) - u(X_j)| &\leq C\|\omega\|_{\infty,\varepsilon}(1 + \log(R))|X_i - Y_j| \\ &+ C\|\omega\|_{\infty,\varepsilon}(1 + m(t)\log(\frac{\varepsilon}{m(t)})^{-1})\big|(X_i -_j)\log(|X_i - X_j|)\big| \quad (4.6)\end{aligned}$$

□

So we can get a estimate on the growth of u. But, it do not exclude collision in a vanishing time. Indeed, the solution of the ODE $x' = x|\log x|$ is of the form $x_0^{e^t}$. So if we begin with $x_0 = \varepsilon$, after a time $\log(2)$, $x(\log(2)) = \varepsilon^2$. For our system, it means that our control on the minimal distance is immediately of order less than ε, so all our bounds explod immediately. We need a better result to obtain a result of convergence.

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