Energy and entropy in the Quasi-neutral limit.

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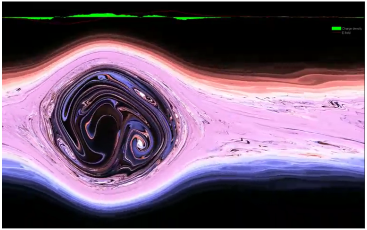

Outline

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1. Introduction to the problem
2. The existing mathematical literature on the quasi neutral limit.
3. The stability of homogeneous equilibria in VP.
4. Strong instability and stability in the quasi-neutral limit ($d = 1$).

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Section 1

Introduction to the problem

The Debye (- Hückel) length.

- **Debye (- Hückel) length :** The scale of "charge separation", plasma oscillations.

$$\lambda_D := \left(\frac{\varepsilon_0 k_B T}{\sum_j \rho_j^0 Z_j^2 e^2} \right)^{\frac{1}{2}}$$

- Relatively small (*with respect to typical length*) in many physical situation.

Plasma	Density $n_e(m^{-3})$	Electron temperature $T(K)$	Magnetic field $B(T)$	Debye length $\lambda_D(m)$
Solar core	10^{32}	10^7	--	10^{-11}
Tokamak	10^{20}	10^8	10	10^{-4}
Gas discharge	10^{16}	10^4	--	10^{-4}
Ionosphere	10^{12}	10^3	10^{-5}	10^{-3}
Magnetosphere	10^7	10^7	10^{-8}	10^2
Solar wind	10^6	10^5	10^{-9}	10
Interstellar medium	10^5	10^4	10^{-10}	10
Intergalactic medium	1	10^6	--	10^5

From a course by Kip Thorne at Caltech.

A quick explanation of its origin.

- Start with the density of e^- (charge $Z = 1$) in a fixed background of ions.
- Write the **Poisson equation** on the potential Φ

$$\Delta\Phi = \frac{Z\,e(\rho_e - \rho^0)}{\varepsilon_0}.$$

- Assume that the e^- are at thermal equilibrium with large temperature : $Ze\Phi << k_B T$

$$\rho_e(x) = \rho^0 e^{\frac{Z\,e\Phi(x)}{k_B T}} \approx \rho^0 + \rho^0 \frac{Z\,e\Phi(x)}{k_B T}.$$

- We end up with the linearised Poisson-Boltzman equation

$$\Delta\Phi = \left(\frac{\varepsilon_0 k_B T}{\rho^0 Z^2 e^2}\right)\Phi = \lambda_D^{-2}\Phi.$$

- $\Rightarrow$ Φ varies at the scale λ_D.

More rigorously : the nondimensionalization of Vlasov-Poisson equation.

- Start from the Vlasov-Poisson eq. for the density $f(t,x,v)$ of e^- (fixed ions background)

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial v} + \frac{e}{m_e}\frac{\partial \Phi}{\partial x} \cdot \frac{\partial f}{\partial v} = 0,$$

$$\text{with} \qquad \Delta \Phi = \frac{e(\rho_e - \rho^0)}{\varepsilon_0}.$$

- Introduce the typical scales and associated new variables without dimension (*with prime*)

$$t = Tt', \quad x = Lx', \quad v = V_{th}v', \quad n_0 f'(t,x',v')\,dx'dv' = f(t,x,v)\,dxdv$$

n_0 number of moles at size L, i.e. $\rho^0 = \frac{n_0}{L^d}$. Also assume $V_{th}T = L$.

- This leads to the **nondimensional equation**

$$\frac{\partial f'}{\partial t'} + v' \cdot \frac{\partial f'}{\partial v'} + \frac{\partial \Phi'}{\partial x'} \cdot \frac{\partial f'}{\partial v'} = 0,$$

$$\text{with} \qquad \frac{\lambda_D^2}{L^2}\Delta \Phi = \rho' - 1.$$

- Again $\lambda_D^2 = \dfrac{\varepsilon_0 m_e V_{th}^2}{\rho^0 e^2}$. The important parameter is the ratio $\boxed{\varepsilon = \dfrac{\lambda_D}{L}}$.

The related Plasma oscillations, a.k.a. "Langmuir Waves".

- Rewrite the previous system (for convenience) as

$$\partial_t f_\varepsilon + v \cdot \partial_x f_\varepsilon - \partial_x \Phi_\varepsilon \cdot \partial_v f_\varepsilon = 0, \qquad \text{with} \quad -\varepsilon^2 \Delta \Phi_\varepsilon = \rho_\varepsilon - 1.$$

- The energy is

$$\mathcal{E}_\varepsilon[f_\varepsilon] := \frac{1}{2} \int v^2 f_\varepsilon \, dxdv + \frac{1}{2} \int \left| \nabla [\varepsilon \Phi_\varepsilon] \right|^2 dx.$$

- Decompose the current $j_\varepsilon = \int f_\varepsilon v \, dv$ in divergence free j^d_ε and gradient part $\partial_x J_\varepsilon$. The equations for J_ε and $\varepsilon \Phi_\varepsilon$ are

$$\partial_t [\varepsilon \Phi_\varepsilon] = -\frac{J_\varepsilon}{\varepsilon}$$

$$\partial_t J_\varepsilon = \frac{\varepsilon \Phi_\varepsilon}{\varepsilon} + \Delta^{-1} \operatorname{ddiv} \left([\varepsilon \nabla_x \Phi_\varepsilon] \otimes [\varepsilon \nabla_x \Phi_\varepsilon] - \int f_\varepsilon v \otimes v \, dv \right) + \frac{1}{2} |\varepsilon \nabla \Phi_\varepsilon|^2$$

- Setting $\mathcal{O}_\varepsilon = J_\varepsilon + i \varepsilon \Phi_\varepsilon$, $\qquad \partial_t \mathcal{O}_\varepsilon = \frac{i}{\varepsilon} \mathcal{O}_\varepsilon +$ something of order one.

  $\Rightarrow$ **Strong oscillations** of period $\frac{2\pi}{\varepsilon}$ in Φ_ε, J_ε and also ρ_ε.

Experimental observation of Langmuir Waves inionosphere.

- Very fast phenomena ⇒ Quite difficult to observe.

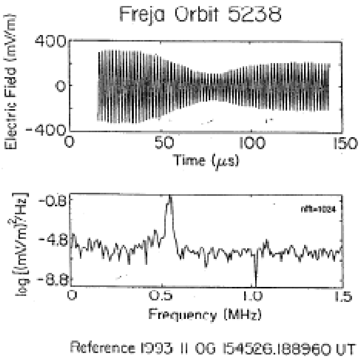

Figure 2. An example of narrow band Langmuir waves. The upper panel contains the waveform and the lower panel contains the power spectrum.

From Kintner, Holback & all, Cornell University and Swedish inst. of space phy. Geophy. Rev. Letters 1995. Record form Freja plasma wave instrument (alt. 1700 km).

Experimental observation of Langmuir Waves in plasma.

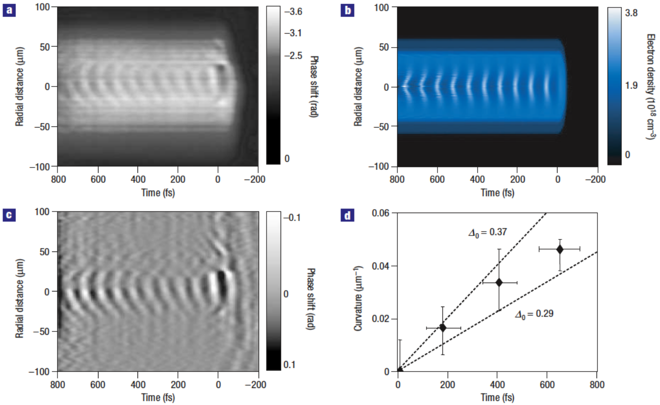

Figure 3 Strongly driven wake with curved wavefronts. a, Probe phase profile $\Delta\phi_{pr}(r,\zeta)$ for an ~30 TW pump, $\bar{n}_e^{max} = 2.2 \times 10^{18}$ cm^{-3} in the He^{2+} region. **b**, Simulated density profile $n_e(r,\zeta)$ near the jet centre. **c**, Same data as in **a**, with the background $\bar{n}_e$ subtracted to highlight the wake. **d**, Evolution of the reciprocal radius of wavefront curvature behind the pump (data points), compared with calculated evolution (dashed lines) for indicated wake potential amplitudes. Each data point (except at $\zeta = 0$) averages over three adjacent periods. The horizontal error bars extend over the three periods averaged, and the vertical error bars extend over the range of fitted curvature values averaged.

From Matlis, Downer & all, University of Texas and Michigan, Nature Phys 2006.

Heuristic on the Quasi-neutral limit $\varepsilon \to 0$.

- *Neglect* the problem of the "plasma oscillations". Very formally, the expected limit is

$$\partial_t f + v \cdot \partial_x f - \partial_x \Phi \cdot \partial_v f = 0,$$
$$\text{with} \quad \rho = 1.$$

- Using $\varepsilon = 0$ in the equation for J_ε and Π_ε, we get very formally (false)

$$\Phi := \Delta^{-1} \operatorname{ddiv} \int f_\varepsilon v \otimes v \, dv.$$

  This is correct only if $\rho(0) = 1$ and $J(0) = 0$, i.e. **well prepared** case.

- The previous "neutral" Vlasov system is very singular. We known only
  - A *Cauchy-Kowalevsky* type result : local in time existence for analytic initial data [Bossy, Fontbana, Jabin, Jabir in CPDE '13].
  - Same analytic setting, but with a plasma seen as a **superposition of fuilds** [Grenier, CPDE '96].
    Similar result but in H^s for (very) particular initial data [Besse, ARMA'11] [Bardos, Besse, Work in progress].

Section 2

The existing mathematical literature on the quasi neutral limit.

Early results in the '90 by Grenier (and Brenier)

- Defect measures used in [Brenier, Grenier, CRAS '94] and [Grenier, CPDE '95] :

  The 2 first moments will satisfy the expected equation
  with **defect measures** in the r.h.s.

- **Deep result with the fluid point of view** [Grenier, CPDE '96].
  Write the plasma as a collection of many fluids (μ some measure)

$$f_\varepsilon(t,x,v) = \int \rho_\theta^\varepsilon(t,x)\delta_{v_\theta^\varepsilon(t,x)}(v)\,\mu(d\theta).$$

- The family $(\rho_\theta, v_\theta)_\theta$ satifies **coupled Euler-Poisson**

$$\partial_t \rho_\theta^\varepsilon + \mathrm{div}(\rho_\theta^\varepsilon v_\theta^\varepsilon) = 0, \qquad \partial_t v_\theta^\varepsilon + (v_\theta^\varepsilon \cdot \nabla) v_\theta^\varepsilon = -\nabla V,$$
$$\Delta V_\varepsilon = \int \rho_\theta^\varepsilon \mu(d\theta) - 1$$

- The expected limit model : **coupled incompressible Euler** equation :

$$\partial_t \rho_\theta + \mathrm{div}(\rho_\theta v_\theta) = 0, \qquad \partial_t v_\theta + (v_\theta \cdot \nabla) v_\theta = -\nabla p,$$
$$\int \rho_\theta^\varepsilon \mu(d\theta) = 1$$

Grenier: : convergence after filtration of the Plasma oscillations.

Theorem (Grenier, CPDE '96)

Assume that

- *the family* $(\rho^\epsilon_\theta, v^\epsilon_\theta)_{\epsilon,\theta}$ *satisfies uniform* H^s *estimates (s large).*
- $\varepsilon V_\varepsilon(0) \to V_0$ *and* $j_\varepsilon \to \nu_0 + \nabla J_0$ *with* $\operatorname{div} \nu_0 = 0$.

Then

$$\left(\rho^\varepsilon_\theta, v^\varepsilon_\theta - \nabla J^\varepsilon\right)$$

converges towards solution of the expected coupled inc. Euler equation, with a corrector J^ϵ *defined by* $J^\varepsilon(t,x) = \mathrm{Re}\left[e^{i\frac{t}{\varepsilon}}\mathcal{U}(t,x)\right]$, *and* $\mathcal{U}$ *is solution of*

$$\mathcal{U}_0 = J_0 + iV_0, \qquad \partial_t \mathcal{U} + \left(\int \rho_\theta v_\theta \mu(d\theta)\right) \cdot \nabla \mathcal{U} = 0$$

- Contains almost everything but the formalism is unusual
- Not simple to pass from f formalism to the superposition of plasma.
- To summarize, Convergence possible only
  - Under good a priori estimates.
  - After filtration of the Plasma oscillation.

Later results : The Quasi-neutral and zero temperature limit.

- **Zero temperature limit** : Assume that for some $\bar{v}(t,x)$

$$f_\varepsilon(0,x,v) \rightharpoonup \delta_{j_0(x)}(v) \quad \text{i.e.} \quad \int |v - j_0(x)|^2 f_\varepsilon(0,x,v)\, dxdv.$$

- We denote $j_0 = \nu_0 + \nabla J_0$, with $\operatorname{div} j_0 = 0$, and

$$V_0 = \lim_{\varepsilon \to 0} \varepsilon V_\epsilon(0) \quad \left(= \Delta^{-1} \frac{\rho_\epsilon(0) - 1}{\varepsilon}\right) \quad \text{in} \quad H^1.$$

- First result in well prepared case [Brenier, CPDE '00] :

Theorem

Assume that $J_0 = 0$ and $V_0 = 0$.
Then j_ε converges weakly towards a **dissipative solution** *to the inc. Euler equation with initial data ν_0.*

- Based on the use of the "modulated energy"

$$E_u^\varepsilon(t) = \frac{1}{2} \int |v - u(t,x)|^2 f_\varepsilon\, dxdv + \frac{1}{2} \int |\varepsilon \nabla V_\varepsilon|^2\, dx$$

Another quasi-neutral and zero temperature limit by Masmoudi.

- Next result in the "ill-prepared" case [Masmoudi, CPDE '01]

Theorem

Assume that ν is a sufficiently smooth (in some H^s) solution of the inc. Euler eq. with initial data ν_0.
Define $\mathcal{U}$ by $\mathcal{U}(0) = J_0 + iV_0$ *and* $\partial_t \mathcal{U} + \nu \cdot \nabla \mathcal{U} = 0.$
Define

$$E^\varepsilon_\nu(t) = \frac{1}{2}\int \left| v - \nu(t,x) - \mathrm{Re}\left(e^{i\frac{t}{\varepsilon}}\mathcal{U}(t,x)\right)\right|^2 f(t,x,v)\,dxdv + \frac{1}{2}\int \left| \varepsilon \nabla V_\varepsilon(t,x) - \mathrm{Im}\left(e^{i\frac{t}{\varepsilon}}\right)\mathcal{U}(t,x)\right|^2 dx$$

Then if $E^\varepsilon_\nu(0) \to 0$, we have $E^\epsilon_\nu(t) \to 0$ for any $t \geqslant 0$.

- Compatible with Grenier's result (and more or less included in it).
- Based on a control of the increase of E^ε_ν.

$$E^\varepsilon_\nu(t) \leqslant C_t\left(E^\epsilon_\nu(0) + \varepsilon\right).$$

Section 3

The stability of homogeneous equilibria in VP.

The Penrose criterion for existence of Growing model.

- In 1D. Study the linearized Vlasov equation around $f(v)$

$$\partial_t g + v\partial_x g - \partial_x V_g \partial_v f = 0, \qquad -\varepsilon^2 \partial_x^2 V_g = g \tag{1}$$

- Ansatz : $g(t,x,v) = e^{ikx+\omega t} h(v)$ (or Use Fourier-Laplace transform).
- It satisfies (1) iff

$$F\left(i\frac{\omega}{k}\right) := \int \frac{\partial_v f}{v - i\frac{\omega}{k}}\, dv = (\varepsilon k)^2 \quad \text{with} \quad h(v) = \frac{1}{(\varepsilon k)^2} \int \frac{\partial_v f}{v - i\frac{\omega}{k}}\, dv$$

- If exists z with $\operatorname{Im} z \neq 0$ satisfying $F(z) \in \mathbb{R}^+$, then

$$k = \pm\sqrt{\frac{F(z)}{\varepsilon}}, \quad \omega = \mp iz\sqrt{\frac{F(z)}{\varepsilon}} \Longrightarrow \text{Growing mode}$$

- $F(\bar{z}) = \overline{F(z)} \Longrightarrow$ consider only the case $\operatorname{Im} z > 0$.

The Penrose criterion : a story of contour.

- Introduce $F_+(\xi) := \lim_{\eta \to 0^+} F(\xi + i\eta) = PV\left(\int \frac{\partial_v f(v)}{v-\xi}\,dv\right) + i\pi\partial_v f(\xi)$
- $F(z) \in \mathbb{R}^+$ for some z with $\operatorname{Im} z > 0$

  $\iff$ the contour $F_+(\mathbb{R})$ circles (↺) some part of $\mathbb{R}^+$

  $\iff$ $F_+(\mathbb{R})$ cross $\mathbb{R}^+$ from below at some point.

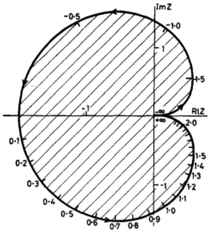

Fig. 1. The curve $Z(R)$ for the Maxwell distribution $F(u) = (\omega^2/\alpha\sqrt{\pi})\exp(-u^2/\alpha^2)$. The axes are marked in units of ω^2/α^2. Values of u/α are shown on the curve itself. The image of the upper half plane is shaded, and includes no positive real values.

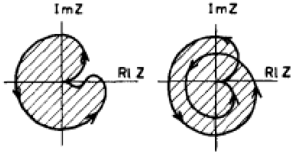

Fig. 2. Possible $Z(R)$ curves which do enclose positive real values. The image of the upper half plane is shaded.

Left : Contour of a Maxwellian distribution. **Right** : Contour of unstable profiles. From O. Penrose, Phys of Fluids 1960.

The Penrose criterion : A condition on local minimum.

- Now $F_+(\xi_0) \in \mathbb{R} \Longrightarrow \xi_0$ a **critical point** of f.
- F_+ cross $\mathbb{R}$ by below at $\xi_0 \Longrightarrow$ is a **local minimum**.
- At this local minimum $\mathrm{Re}\, F_+(\xi_0) > 0$.

Definition (Penrose criterion on $\mathbb{R}$.)

A homogeneous profile f with sufficient regularity and moments satisfy the Penrose criterion iff there exists a local minimum ξ_0 such that

$$PV\left(\int \frac{\partial_v f(v)}{v-\xi}\,dv\right) = \int \frac{f(v)-f(\xi_0)}{(v-\xi_0)^2}\,dv > 0$$

- The criterion is slightly different on a torus.
  The contour should circle some part of $\{\varepsilon k^2, k \in \mathbb{N}^*\}$ and not $\mathbb{R}^+$.
- **Non-linear instability :** If f satisfy the PC is symetric, then it is non-linearly unstable in H^s with some weight [Guo, Strauss, ANIHP '95]
- **"One-humped" profiles**, with no local minima do not satisfy the criterion.

The non-linear stability for symetric profile.

- The so called **"Energy-Casimir" method** introduced by Arnold [Dokl. USSR '65] [IVUZM'66] may be used in VP.
- The adaptation to plasmas is done in [Holm, Mardsen, Ratiu, Weinstein Phy Rep '85] and [Rein, MMAS '95].
- **Idea** : Use the invariant to construct a *convex* functional that is *minimal* at some profil f.
- In VP on the 1D torus, ε fixed, the invariants are :
  - The total energy

$$\mathcal{E}_\epsilon[f_\varepsilon] := \frac{1}{2}\int f_\varepsilon |v|^2\, dxdv + \frac{\epsilon}{2}\int |\partial_x V_\varepsilon[f_\varepsilon]|^2 dx,$$

  - the total quantity of mvt :

$$P[f_\varepsilon] := \int f_\varepsilon v\, dxdv,$$

  - The integral I_Q below for any smooth enough Q. (Requires strong solutions)

$$I_Q[f_\varepsilon] = \int Q(f_\varepsilon)\, dxdv$$

Construction of an appropriate Casimir functional.

- At which condition does F_Q^ε defined by

$$F_Q^\epsilon[f_\varepsilon] := \int Q(f_\varepsilon\, dxdv) + \mathcal{E}_\epsilon[f_\varepsilon]$$

  admits f as **critical point**.

  Answer: possible only if $Q'(f) = -\frac{|v|^2}{2}$.
- $\Longrightarrow f$ is radial: $f(v) = \varphi(|v|^2/2)$ with an injective φ, and $Q = -\varphi^{-1}$.
- At which condition is Q and then F_Q^ε convex?

  Answer : φ is decreasing.
- The momentum invariance allows to replace $|v|$ by $|v - \bar{v}|^2$.

Generalized "entropy" for a model without collision.

- In the previous situation : $f(v) = \varphi(|v|^2/2)$ and $Q = \varphi^{-1}$, define

$$H_Q[g] := \int [Q(g) - Q(f) - Q'(f)(g-f)]\, dxdv$$
$$= \int Q(g)\, dxdv + \frac{1}{2}\int g|v|^2\, dxdv + C^{st}$$

- H_Q is convex (often strictly).
- H_Q **strictly convex** $\Longrightarrow$ **Non-linear stability of** f **in** L^2**.**
- H_Q is the usual relative entropy if f is a Maxwellian dist.

$$f(v) = e^{\frac{|v|^2}{2T}} \Longrightarrow H_Q(g) = TH(g|f) = T\int g \ln g + \frac{T}{2}\int |v|^2 g$$

- H_Q is a kind of relative entropy. $H_Q + E_{pot}$ a kind of free energy.
- H_Q is not uniquely defined for a fixed φ.

Non-linear stability of VP via rearrangment inequality.

- **Notation** : $f \sim g$ if their symmetric rearrangement are equals $f^* = g^*$.
- **Basic idea** :
  - The Vlasov equation preserves the rearrangement : $g(t)^* = g(0)^*$.
  - By conservation of the total energy

$$\int [g(t) - g(0)^*]|v|^2 \, dxdv \leqslant \mathcal{E}_\epsilon[g(0)] - \int g(0)^* |v|^2 \, dxdv$$

  - “If g as kinetic energy close to g^*, they should be close”.
- [Marchioro, Pulvirenti, MMAS '86] : Precise the later idea in dim $d \geqslant 2$

$$\|g - g^*\|_1^2 \leqslant C \int [g - g^*]|v|^2 \, dxdv$$

  C depends on $\|g\|_\infty$,..
- $\Longrightarrow$ Non-linear stability in L^1.

Section 4

Strong instability and stability in the quasi-neutral limit ($d = 1$).

Plasma oscillations in dimension one.

- Assume $f_\varepsilon(0) \approx f_0(v) = g_0((v-\bar{v})^2)$, an homogeneous profile, symetric w.r.t. $\bar{v}$.
- In dim 1, j^d_ε is a constant. The equations on J_ε and $\varepsilon V_\varepsilon$ are simpler

$$\begin{cases} \partial_t(\varepsilon V_\varepsilon) &= \dfrac{J_\varepsilon}{\varepsilon} \\ \partial_t(J_\varepsilon) &= -\dfrac{\varepsilon V_\varepsilon}{\varepsilon} + \dfrac{1}{2}|\partial_x(\sqrt{\varepsilon} V_\varepsilon)|^2 - \displaystyle\int f_\varepsilon v^2 \, dv \end{cases}.$$

- Setting as before $\mathcal{O}_\varepsilon = J_\varepsilon + i\varepsilon\Phi_\varepsilon$ leads to

$$\partial_t \mathcal{O}_\varepsilon = \frac{i}{\varepsilon}\mathcal{O}_\varepsilon + |\partial_x(\operatorname{Im}\mathcal{O}_\varepsilon)|^2 - \int f_\varepsilon v^2 \, dv.$$

- Due to the fast variation of $\partial_x J_\varepsilon$, we cannot have

$$f_\varepsilon(t,x,v) \approx f_0(v), \quad \text{but maybe} \quad f_\varepsilon(t,x,v) \approx f_0(v - \partial_x J_\varepsilon(x))$$

- If the later is true, then

$$\int f_\varepsilon(t,x,v) v^2 \, dv \approx 2T + |\bar{v} + \partial_x J_\varepsilon(t,x)|^2,$$

Plasma oscillations in dimension one, part II.

- So that the equation for $\mathcal{O}_\varepsilon$ may be approximated by (erase the constants)

$$\partial_t \mathcal{O}_\varepsilon = \frac{i}{\varepsilon}\mathcal{O}_\varepsilon + \left|\partial_x (\operatorname{Im} \mathcal{O}_\varepsilon)\right|^2 - \left|\bar{v} + \partial_x \operatorname{Re} \mathcal{O}_\varepsilon\right|^2.$$

- Setting $\mathcal{U}_\varepsilon = e^{-i\frac{t}{\varepsilon}}\mathcal{O}_\varepsilon$, it comes

$$\partial_t \mathcal{U}_\varepsilon = e^{-i\frac{t}{\varepsilon}}\left|\partial_x (\operatorname{Im} e^{i\frac{t}{\varepsilon}}\mathcal{U}_\varepsilon)\right|^2 - e^{-i\frac{t}{\varepsilon}}\left|\bar{v} + \partial_x \operatorname{Re} e^{i\frac{t}{\varepsilon}}\mathcal{U}_\varepsilon\right|^2$$

- Using $\operatorname{Im} z = \frac{1}{2}(z - \bar{z})$, expanding and keeping only the non-oscillating terms

$$\partial_t \mathcal{U}_\varepsilon = -\bar{v}\partial_x \mathcal{U}_\varepsilon + \text{quickly oscillating terms}$$

- $\mathcal{U}_\varepsilon$ should converge towards $\mathcal{U}$, solution of

$$\partial_t \mathcal{U}_\varepsilon + \bar{v}\partial_x \mathcal{U} = 0, \qquad \mathcal{U}(0) := \lim_{\varepsilon\to 0} J_\varepsilon(0) + i\varepsilon V_\varepsilon(0) = i \lim_{\varepsilon\to 0} \varepsilon V_\varepsilon(0)$$

- $V_\varepsilon(t,x) \approx V_0(x - \bar{v}t)\cos\left(\frac{t}{\varepsilon}\right)$ and $J_\varepsilon(t,x) \approx -V_0(x - \bar{v}t)\sin\left(\frac{t}{\varepsilon}\right)$.

A rigorous stability result in the "ill"-prepared case.

- Again around $f(v) = \varphi\big(|v - \bar{v}|^2\big)$, and H_Q the associated Casimir functional.
- Assume that $\lim_{\varepsilon \to 0} \varepsilon V_\varepsilon(0) = V_0 \in W^{3,\infty}$.
- Use the energy Casimir method together with the filtration of the oscillations. Define the functional

$$\mathcal{L}^O_\varepsilon(t) := H_Q\left[f_\varepsilon\big(t, x, v - \partial_x V_0(x - \bar{v}t)\sin\frac{t}{\varepsilon}\big)\right] + \frac{1}{2}\int\left[\varepsilon\partial_x V_\epsilon - \partial_x V_0(x - \bar{v}t)\cos\frac{t}{\varepsilon}\right]^2 dx$$

Theorem (Han-Kwan, Hauray, '13)

Under the above assumptions, and also

$$\mathcal{E}_\varepsilon(f_\varepsilon(0)) \leqslant C_0, \quad \int\left(|Q|(f_\varepsilon(0)) + \frac{Q^2(f_\varepsilon(0))}{f_\varepsilon(0)}\right) dvdx \leqslant C_0,$$

there is $C > 0$, *such that*

$$\forall t \geqslant 0, \quad \mathcal{L}^O_\varepsilon(t) \leqslant e^{2\|\partial_{xxx}V_0\|_\infty t}\mathcal{L}^O_\varepsilon(0) + C\varepsilon(e^{2\|\partial_{xxx}V_0\|_\infty t} + 1).$$

Very fast instability around unstable profil.

- (VP_ε) in the variables $(t', x') = \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$ is (VP_1).
- $\Longrightarrow$ The possible instabilities are much faster.
- **Notation :** H^s_γ is the space "H^s with weight $(1+|v|)^\gamma$".

Theorem (Han-Kwan, Hauray '13)

Assume that f is a symmetric profile, unstable in the sense of Penrose. Then, for some $\gamma > 0$ and any $s > 0$ and $N \in \mathbb{N}$, there exists a family of initial conditions $[f_\varepsilon(0)]_\varepsilon$ such that

- $\|f_\epsilon(0) - f\|_{H^s_\gamma} \leqslant C\epsilon^N$,
- $\lim_{\varepsilon \to 0} \sup_{t \leqslant \varepsilon |\ln \varepsilon|} \|f_\varepsilon(0) - f\|_{L^2_\gamma} > 0.$

- Uses a technic introduced by Grenier (again!) for Euler and Prandtl equation [CPAM '00].
- $\Longrightarrow$ **General stability is not possible**, except in analytic framework.

Conclusion

- Our stability results requires symmetry, but the only symmetric solutions are the homogeneous equilibria.
- **Plasma oscillation are not damped** (in our setting). No initial boundary layer.
- May leads to fast instabilities.
- **Open problems :** Non symmetric equilibria? Non stationary solutions???

Lot's of inspiration from [Grenier, JEDP '99].

Thanks (him and you)!