Propagation of chaos for system of vortices in 2D

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The Navier-Stokes equation in 2D

In 2D, the NS equation

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad div u = 0, \qquad + \text{I.C.}$$

is oftently rewritten in terms of vorticity $\omega = \nabla^\perp \cdot u = \partial_1 u_2 - \partial_2 u_1$

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega \\ u(t,x) = K * \omega = \frac{x^\perp}{2\pi |x|^2} * \omega \end{cases} \qquad + \text{I.C.}, \tag{1}$$

where $K(x) = \frac{x^\perp}{2\pi |x|^2}$ is the **Biot-Savard** kernel $K \in L^{2,\infty}$.

Well-posedness theory : Leray ($u^0 \in L^2$), Giga-Miyakawa-Osada or Ben-Artzi ($\omega^0 \in L^1$), Cannone-Planchon or Meyer ($u^0 \in$ some Besov space), Gallagher-Gallay (ω^0 measure) and many others...

Less is known for the Euler equation ($\nu = 0$) : Yudovich (well-posed if $\omega \in L^\infty$), Delort (Existence if ω^0 positive measure), Scheffer, Schnirelman, De Lellis-Szekelyhidi (non-uniqueness).

The Vortex approximation

Idea : Approximate a "continuous" vorticity profile by a some of N Dirac masses, with position X_i and strength $\frac{a_i}{N} \in \mathbb{R}$.
The Euler Equation is transformed in a system of ODEs, and NS2D in a system of SDEs

$$\forall i \leq N, \quad dX_i = \left[\frac{1}{N} \sum_{j \neq i} a_j K(X_i - X_j) \right] dt + \sigma dB_i \tag{2}$$

sometimes called **Helmholtz-Kirchhoff** system (if $\nu = 0$).

Justification : Simulation of decaying 2D Turbulence

Theoritical justification given by Marchioro-Pulvirenti and Gallay.

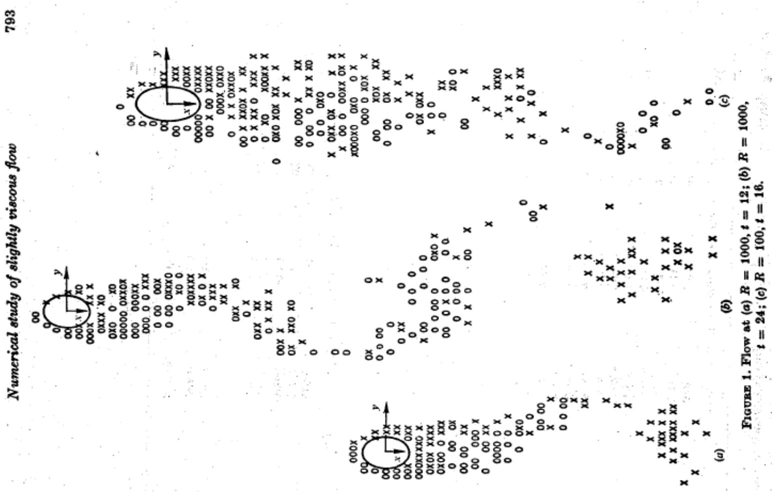
Well-posedness of the N vortex system :

- $\nu = 0$: Marchioro-Pulvirenti (OK for a.e. initial positions and vortices strengths).
- $\nu > 0$. Takanobu ($a_i > 0$), Osada ($a_i \in \mathbb{R}$), Fontbana-Martinez...

Simplification: From now, $a_i = 1$ for all i.

Numerical applications.

A simulation by Chorin in the '70.



The question of convergence as $N \to +\infty$.

A natural question.

- NS2D : Positive answer (for σ large enough) given by Osada in the '80.
- Euler: Very difficult.

In the viscous case, the difficulty is the singularity of the drift.
Goals of the talk :

- Review the general procedure (with an analyst? point if view).
- Explain some improvements we introduced.
- State and comment the result for the vortex system.

Limits of symmetric (exchangeable) N particles distributions

Two possible representations.

Here and below : $E = \mathbb{R}^d$ or $C([0,+\infty),\mathbb{R}^d)$ (Polish space).

Analyst: Let F^N be a sequence of symmetric proba on $\mathbf{P}(E^N)$.

Probabilist: Let $\mathcal{X}^N = (X_1^N, \ldots, X_N^N)$ be a sequence of exchangeable R. V.

What are the possible limit points?

- **1 : with empirical measures.**

$$\mu^N_{\mathcal{X}} := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N} \qquad \text{with law} \qquad \bar{F}^N$$

  converge to some R.V. f in $\mathbf{P}(E)$, with law $\bar{\pi} \in \mathbf{P}(\mathbf{P}(E))$.

- **2 : with infinite sequence of R.V.**

  F^N seen as probabilities on E^∞. They can converges towards some $\pi \in \mathbf{P}_{sym}(E^\infty)$.

In both cases, tightness is equivalent to tightness of $\mathcal{L}(X_1^N)$.

The two representations are the same.

Notations :

- Marginals of $\pi \in \mathbf{P}_{sym}(E^\infty)$ are denoted by π_N (law of the N first RV).
- For $\bar\pi \in \mathbf{P}(\mathbf{P}(E))$, $\bar\pi^N := \displaystyle\int \rho^{\otimes N} \pi(d\rho) \quad \in \mathbf{P}(E^N)$.

We can construct the following maps between $\mathbf{P}(\mathbf{P}(E))$ and $\mathbf{P}_{sym}(E^\infty)$.

$$\mathbf{P}(\mathbf{P}(E)) : \quad \begin{array}{ccc} \bar\pi & \xrightarrow{R} & \bar\pi_\infty := \int \rho^{\otimes\infty} \pi(d\rho) \\ \{\text{Limits of } \overline{\pi_N}\} & \xleftarrow{S} & \pi \end{array} \quad : \mathbf{P}_{sym}(E^\infty)$$

Theorem (De Finetti - Hewitt & Savage)

$$R \circ S = Id_{\mathbf{P}_{sym}(E^\infty)}, \qquad S \circ R = Id_{\mathbf{P}(\mathbf{P}(E))}$$

and S is univalent.

The algebraic relation $R \circ S = Id_{\mathbf{P}_{sym}(E^\infty)}$.

In fact, we can compute for instance with $j = 2$

$$
\begin{aligned}
(\bar{\pi_N})^2 &:= \int \rho^{\otimes 2} \bar{\pi_N}(d\rho) \\
&= \int (\mu^N_{\mathcal{X}})^{\otimes 2} \pi^N(d\mathcal{X}^N) \\
&= \frac{1}{N^2} \int \Big(\sum_{i \neq j} \delta_{X_i} \otimes \delta_{X_j} + \sum_i \delta_{X_i} \otimes \delta_{X_i} \Big) \pi^N(d\mathcal{X}^N) \\
&= \frac{N-1}{N} \pi^2 + \frac{1}{N} \pi^1 \delta_{X_1 = X_2} \\
\downarrow \quad & \qquad \downarrow \\
\big(R \circ S(\pi) \big)_2 &= \pi_2
\end{aligned}
$$

Do it for all $j \in \mathbb{N}$ and get $R \circ S(\pi) = \pi$.

$S \circ R = Id_{\mathbf{P}(\mathbf{P}(E))}$ is a consequence of concentration.

Here **concentration** means : Glivenko-Cantelli theorem or *empirical law of large number*.

Theorem (Varadarajan)

If the $(X_i)_{i\in\mathbb{N}}$ are i.i.d with law ρ, then $\mu^N_{\mathcal{X}}$ goes in law towards the constant ρ.

In other words,

$$S(\rho^\infty) = \text{limits of } \overline{\rho^{\otimes N}} = \delta_\rho$$

$$\text{but since} \quad R(\delta_\rho) = \int (\rho')^{\otimes\infty} \delta_\rho(\rho') = \rho^\infty,$$

$$\text{we get} \quad S[R(\delta_\rho)] = \delta_\rho$$

And by linearity and continuity $S\Big[R\Big(\int \delta_\rho \pi(d\rho)\Big)\Big] = \int \delta_\rho \pi(d\rho)$

To remember : Concentration implies that for N large, $\rho_1^{\otimes N}$ and $\rho_2^{\otimes N}$ have almost disjoints supports.

Two equivalent descriptions of convergence.

Going back to the original problem, we can give two *equivalent* definitions of convergence for $F^N \in \mathbf{P}_{sym}(E^N)$.

- $F^N \rightharpoonup \pi \in \mathbf{P}_{sym}(E^\infty)$, (usual sense for product space)

$$\forall j \in \mathbb{N}, \quad F_j^N \rightharpoonup \pi_j,$$

- $\bar{F}^N = \mathcal{L}(\mu_{\mathcal{X}}^N) \rightharpoonup \bar{\pi} \in \mathbf{P}(\mathbf{P}(E))$.

  Or better, the RV $\mu_{\mathcal{X}}^N$ goes in law toward some RV $\rho \in \mathbf{P}(E)$.

Chaotic sequences

We call F^N a chaotic sequence if the limit is an extremal point.

Corollary (of the previous theorem)

For $\pi \in \mathbf{P}_{sym}(E^\infty)$

$$\pi = \rho^\infty \Longleftrightarrow \pi^2 = \rho^{\otimes 2}.$$

"There cannot be three particles correlations if there is no two-particles correlations."

Exercice : Find a counter-example if $N = +\infty$ is replaced by $N = 3$.

Definition

For $\rho \in \mathbf{P}(E)$, F^N is a ρ-chaotic sequence if one of the three (equivalent) statements is true :

i) $\mu^N_{\mathcal{X}}$ *goes in law towards* ρ ,

ii) $\forall j \in \mathbb{N}, \quad F^N_j \rightharpoonup \rho^{\otimes j}$,

iii) $\quad F^N_2 \rightharpoonup \rho^{\otimes 2}$.

Propagation of chaos

Definition

$G^N(t)$ dynamical flow of a N particle system.
$G_\infty(t)$ "flow" the unique expected (non-linear) limit. Preservation of chaos holds in that case if with for all t

$$F^N(t) = F^N(0) \circ G^N(-t), \qquad \rho(t) = G_\infty(t)(\rho^0)$$

$$\begin{array}{ccc} F^N(0) & \textit{is} & \rho^0 - \textit{chaotic} \\ & \Downarrow & \\ F^N(t) & \textit{is} & \rho(t) - \textit{chaotic} \end{array}$$

Even better

Definition (Prop. of chaos II)

Trajectorial POC holds if for $\mathcal{X}^N$ that are ρ-chaotic, then the trajectories $\mathcal{X}^N([0,\infty))$ are $X([0,\infty))$-chaotic, where X stands for the unique solution of the expected non linear limit SDE.

Particles systems towards McKean-Vlasov non-linear eq.

A stochastic interacting particle system.

N vortices interacting via a 2 particles kernel $b(x, y)$.

Important : $b(x, x) = 0$.

$$\forall i \leq N, \quad dX_i = \left[\frac{1}{N} \sum_{j \neq i} b(X_i, X_j) \right] dt + \sigma dB_i \qquad (3)$$
$$= b(X_i, \mu^N_{\mathcal{X}})\, dt + \sigma dB_i$$

What is the expected limit?

If all the $\mu^N_{\mathcal{X}}$ remains close to the law $\rho(t)$ of $X_1(t)$ (i.e. the independence is approximately preserved in time ?), the X_i will look as N ind. copies of

$$d\mathcal{X}(t) = b(\mathcal{X}(t), \rho(t))dt + \sigma dB. \qquad (4)$$

where $\rho(t)$ is the law of $\mathcal{X}(t)$.

Compactness or tightness issue.

Notations : Bold letters for trajectorial quantities

$$\mathbf{X}(t) : t \mapsto X(t) \text{ on } [0,t], \qquad \boldsymbol{\mu}^N_{\mathcal{X}}(t) : t \mapsto \mu^N_{\mathcal{X}}(t) \text{ on } [0,t].$$

Proposition

The tightness of the sequence of RV $\boldsymbol{\mathcal{X}}^N = (\mathbf{X}^N_1, \dots, \mathbf{X}^N_N)$ *is equivalent to the tightness of* $\mathcal{L}(\boldsymbol{\mathcal{X}}^N_1)$.

Here we get for all $T > 0$, $\alpha + \beta = 1$, Hölder leads to Hölder

$$\mathbb{E}\left[\sup_{s \le t \le T} \frac{|\mathcal{X}^N_1(s) - \mathcal{X}^N_1(t)|}{|s-t|^\alpha}\right] \le \left(\int_0^T \mathbb{E}[b(\mathcal{X}^N_1(t), \mathcal{X}^N_2(t))^{\frac{1}{\beta}}]\, dt\right)^\beta + \mathbb{E}\left[\sup_{s \le t \le T} \frac{|B_1(s) - B_1(t)|}{|s-t|^\alpha}\right]$$

How to control the integral? Use uniform integrability on $\mathcal{L}(\mathcal{X}^N_1, \mathcal{X}^N_2)$.

Even better if $b(x,y) = b(x-y)$**.** Use uniform integrability on $\mathcal{L}(\mathcal{X}^N_1 - \mathcal{X}^N_2)$.

For analyst : ideas from deterministic equations.

In the case where there is no diffusion ($\sigma = 0$), then we do have ($b(0,0) = 0$)

$$\frac{d}{dt} X_i(t) = b(X_i(t), \mu^N_{\mathcal{X}}(t)).$$

So an R.V. $\mathbf{X}(t)$ with law (almost) any empirical measure $\boldsymbol{\mu}^N_{\mathcal{X}}$ is a solution of the NL limit ODE :

$$\frac{d}{dt} X(t) = b(X(t), \mu^N_{\mathcal{X}}(t)), \quad \text{for } \boldsymbol{\mu}^N_{\mathcal{X}} - \text{a.e. all } \mathbf{X}$$

If we simply rewrite the particle system, we get

$$\partial_t \mu^N_{\mathcal{X}} + \mathrm{div}(b(x, \mu^N_{\mathcal{X}}) \mu^N_{\mathcal{X}}) = 0$$

which is the associated forward Kolmogorov equation.

Consequence : The drift is not the issue here, even with diffusion.

Extend the idea to the case with diffusion

What is a “weak” solution of an SDE?
A law $\mathbb{P}$ on trajectories **AND** a coupling $\mathcal{Q}$ between the trajectories solution and the trajectories of the Brownian motion (law $\mathbf{B}(t)$).

Consequence : A trajectory $\boldsymbol{\mathcal{X}}^N$ of the N part system is coupled with N samples of Brownian motion $\boldsymbol{\mathcal{B}}^N$ (coupling $\mathcal{Q}$).

a.e w.r.t. $\mathcal{Q}$, we couple $\boldsymbol{\mu}^N_{\mathcal{X}}$ to the empirical measure $\boldsymbol{\mu}^N_{\mathcal{B}}$ with $\mathbb{Q}^N : B_i^N \mapsto X_i^N$. Then, we have $\mathcal{Q}$-almost surely

$$\text{for } \mathbb{Q}^N - a.e.\ X, B, \quad \forall t, \quad X_t - X_0 = \int_0^t b(X_s, \mu^N_{\mathcal{X}}(s))\, ds + B_t. \tag{5}$$

Warning :B_t is not a Brownian motion here. It is a variable : any trajectory in the Wiener space.

Thanks to the Glivenko-Cantelli theorem, $\boldsymbol{\mu}^N_{\mathcal{B}} \xrightarrow{\mathcal{L}} \mathcal{L}(Brownian)$.

We may expect, that the associated RV $\mathbb{Q}^N \xrightarrow{\mathcal{L}} \mathbb{Q}$, random variable, made of couples brownian-solutions of the expected NLSDE if we can pass in the limit in(5).

The non-linear SDE and martingale.

Definition

Given an intial condition ρ^0 , a weak solution of the non-linear SDE

$$dX(t) = b(X(t), \rho(t))dt + \nu dB(t), \qquad \rho(t) = \mathcal{L}(X(t)),$$

is a probability $\mathbb{P}$ on $E = C([0, +\infty), \mathbb{R}^d)$ such that there exists a Brownian motion $B(t)$ such that the previous relation holds (in the integral sense) $\mathbb{P}$-a.e., for all $t > 0$.

We define following functionals on $\mathbf{P}(E)$ by

$$\mathcal{F}(\mathbb{P}) := \iint_{E^2} \mathbb{P}(d\gamma)\mathbb{P}(d\bar{\gamma})\psi_s(\gamma)\Big[\varphi(x(t)) - \varphi(x(s)) - \int_s^t b(\gamma(u), \bar{\gamma}(u)) \cdot \nabla\varphi(\gamma(u))du - \frac{\sigma^2}{2}\int_s^t \Delta\varphi(\gamma(u))du\Big]$$

for all $s, t \in \mathbb{R}$, ψ_s smooth functions of the past (before s), and any smooth φ.

Proposition (Martingale formulation of the NL-SDE)

$\mathbb{P}$ is a weak solution of the NL-SDE iff $\mathcal{F}(\mathbb{P}) = 0$ for all $\mathcal{F}$.

Consistency : A rigourous justification following McKean, Sznitmann,...

Then the trajectorial empirical measures (R.V) are almost solutions of the NL-SDE. Precisely

Proposition

If we assume or set $b(0,0) = 0$, then for all $\mathcal{F}$

$$\mathbb{E}\left[\left|\mathcal{F}(\boldsymbol{\mu}^N_{\mathcal{X}})\right|^2\right] \leq \frac{C_{\mathcal{F}}}{N}$$

Consistency: what happens as $N \to +\infty$?

If b is bounded continuous, all the fonctional $\mathbb{P} \mapsto \mathcal{F}(\mathbb{P})$ are continuous. We then get

Proposition

Assume b is bounded continuous and that $\mathbb{P}$ is a random variable in $\mathbb{P}(C([0,+\infty),\mathbb{R}^2)$, limit point of some subsequence of the $\mu^N_{\mathcal{X}}$. Then P is concentrated on the subset $\mathcal{S}$

$$\mathcal{S} := \{\mathbb{P} \text{ solutions of the non linear SDE } \}$$

In the case were b is singular, there is a singular term in $\mathcal{F}$.

How to handle it? Use uniform integrability on $\mathcal{L}(X_1^N, X_2^N)$.

Even better if $b(x,y) = b(x-y)$. Use uniform integrability on $\mathcal{L}(X_1^N - X_2^N)$.

In fact it is more or less the same than for the tightness.

Uniqueness in the NL SDE needed to conclude.

If the interaction force b is bounded Lipschitz, then uniqueness of solution holds in the large class of measures.

Proposition

Assume that b si Lipschitz. Then for any initial condition $\rho^0 \in \mathbf{P}(\mathbb{R}^d)$*, there exists a unique* $\mathbb{P} \in \mathbf{P}(E)$ *solution of the NL SDE.*

We cannot obtain this uniqueness results if b is singular. We shall restrict to a smaller class of $\mathbb{P}$ satisfying some **a priori assumptions**.

Problem (maybe the most important one). How to obtain regularity of the possible limit R.V $\mathbb{P}$ of $\boldsymbol{\mu}_{\mathcal{X}}^N$?

To summarize : Problems for singular drift b.

We shall handle two problems :

- Provide some uniform smoothness or integrability estimates on $\mathcal{L}(X_1^N(t) - X_2^N(t))$. Useful in compactness and consistency steps.
- Provide smoothness and integrability estimates on the possible limit points of $\mu_{\mathcal{X}}^N(t)$.
- Get a uniqueness result for the limit NL SDE adapted to our problem.

Answer : Use extensively the bound on the **Fisher information** obtained from the dissipation on Entropy.

A comment about creation of correlation.

At fixed N, the interaction between particles created correlation. Propagation of chaos state more or less that they disappear in the limit $N \to +\infty$.

What can happen in the previous strategy if it is not true (correlations don not vanish)?

- There is no tightness. $\Rightarrow$ do something else.
- The consistency may fail if b is too singular. This seems to requires a large singularity.
- The limit problem NL SDE + regularity we can propagate may not have a unique solution. This seem to require less singularity.

Dissipation of entropy and uniform smoothness estimates.

Entropy, Dissipation and Fisher information

Start form the most simple heat equation $\partial_t f = \Delta f$. Then the dissipation of the **entropy** $H(f) := \int f \ln f$

is the **Fisher-information** $$\frac{d}{dt} H(f_t) = - \int \frac{|\nabla f_t|^2}{f_t}\, dx =: I(f_t)$$

Alternative definitions : $I(f) = 4 \int |\nabla \sqrt{f}|^2 = - \int \Delta f\, f$

In a probabilistic setting : If $dX_t = \sigma dB_t$, then with $\nu = \frac{\sigma^2}{2}$

$$H(X_t) + \nu \int_0^t I(X_s)\, ds = H(X_0).$$

Important : You can write the same dissipation equality for the equation

$$dX_t = a(X_t)\, dt + \sigma dB_t,$$

where a is a **divergence free** vector field.

Bound on the Fisher information in the N particles system

Here we have

$$H(\mathcal{X}^N(t)) + \int_0^t I(\mathcal{X}^N(s))\, ds \leq H(\mathcal{X}^N(0)).$$

And thanks control of some moments in x, we obtain

$$\sup_{n\in\mathbb{N}} \frac{1}{N} \int_0^t I(\mathcal{X}^N(s))\, ds \leq C_t.$$

All will follow from this last estimate.

Why we should use H, I and not L^2, H^1...?
Because of their extensiveness $H(f^{\otimes N}) = N\, H(f), I(f^{\otimes N}) = N\, I(f)$. To compare with $\|f^{\otimes N}\|_2 = \|f\|_2^N, \ldots$

Problem : Not so much extensive quantities available.

Properties of Entropy and Fisher information of different levels.

- Convexity
- Super-additivity If $F_\ell^N = \int F^N \, dx_{\ell+1} \dots dx_N$ and $F_{N-\ell}^N = \int F^N dx_1 \dots dx_\ell$,

$$H(F_\ell^N) + H(F_{N-\ell}^N) \leq H(F^N), \qquad I(F_\ell^N) + I(F_{N-\ell}^N) \leq I(F^N)$$

- Lower semi-continuity If $f_n \in \mathbb{P}(E)$ goes weakly towards f, then

$$H(f) \leq \liminf_{n\to+\infty} H(f_n), \qquad I(f) \leq \liminf_{n\to+\infty} I(f_n)$$

Consequence :

$$\int_0^t I(X_1(s) - X_2(s)) \, ds \leq \frac{2}{N} \int_0^t I(\mathcal{X}^N(s)) \, ds$$

Gagliardo-Nirenberg-Sobolev inequalities with FI and consequences.

With the notation p' for the conjugate exponent of p : $\frac{1}{p} + \frac{1}{p'} = 1$.

Proposition (G-N-S inequalities with Fisher.)

If $f \in \mathbb{P}(\mathbb{R}^2)$,

$$\forall\, p \in [1,\infty), \quad \|f\|_p \leq C_p\, I(f)^{1-1/p},$$
$$\forall\, q \in [1,2), \quad \|\nabla f\|_q \leq C_q\, I(f)^{3/2-1/q}.$$

With the **Hardy-Littlewood-Sobolev** inequality : $\|K * g\|_r \leq C\|g\|_q$, with $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$.
We get for any $p \in (1,2)$:

$$\int_0^t I(f_s)\, ds < +\infty \quad \xrightarrow{G-N} \quad f \in L_t^{p'}(L_x^p) \text{ and } \nabla f \in L_t^p(L_x^s), \quad \text{with } \frac{1}{s} = \frac{3}{2} - \frac{1}{p}$$
$$\xrightarrow{HLS} \quad f \in L_t^{p'}(L_x^p) \text{ and } K * \nabla f \in L_t^p(L_x^{p'}),$$
$$\xrightarrow{\text{Hölder}} \quad f(K * \nabla f) \in L^1_{t,x}$$

Important : The exponents are sharp in the vortex case.

Propagation of regularity in the limit.

Entropy and Fisher information on $\mathbf{P}_{sym}(E^\infty)$.

It is more natural than on $\mathbf{P}(\mathbf{P}(E))$.

Define the entropy and Fisher information on $\mathbf{P}_{sym}(E^\infty)$ by

$$\begin{aligned} \mathcal{H}(\pi) &:= \lim_{N\to+\infty} \tfrac{1}{N} H(\pi_N) = \sup_N \tfrac{1}{N} H(\pi_N) \\ \mathcal{I}(\pi) &:= \lim_{N\to+\infty} \tfrac{1}{N} I(\pi_N) = \sup_N \tfrac{1}{N} H(\pi_N) \end{aligned}$$

Then $\mathcal{H}$ and $\mathcal{I}$ are convex, l.s.c. **But also affine!!**

Idea : The support of $\rho_1^{\otimes N}$ and $\rho_2^{\otimes N}$ separate for large N, so that

$$\begin{array}{ccc} \frac{1}{N} I\left(\frac{1}{2}(\rho_1^{\otimes N} + \rho_2^{\otimes N})\right) & \approx & \frac{1}{2} I(\rho_1^{\otimes N}) + I(\rho_2^{\otimes N}) \\ \downarrow & & \| \\ \mathcal{I}\left(\frac{1}{2}(\rho_1^{\otimes \infty} + \rho_2^{\otimes \infty})\right) & = & \frac{1}{2} I(\rho_1) + I(\rho_2) \end{array}$$

and more generally that $\mathcal{I}$ is linear.
The same is true for $\mathcal{H}$ (Ruelle and Robinson).

Limits of N particles RV, entropy and Fisher info.

Theorem

$$\mathcal{I}\left(\int \rho^{\otimes\infty}\pi(d\rho)\right) = \int I(\rho)\,\pi(d\rho)$$

Corollary

If F^N goes in law to π, then

$$\int I(\rho)\,\pi(d\rho) \leq \liminf \frac{1}{N} I(F^N)$$

If a sequence $\mathcal{X}^N$ of exchangeable RVs is such that $\mu^N_{\mathcal{X}}$ goes in law towards some RV ρ in $\mathbf{P}(E)$, *then*

$$\mathbb{E}[I(\rho)] \leq \liminf \frac{1}{N} I(\mathcal{X}^N)$$

Need of *extensive* functionals if you want to obtain such things.

Uniqueness of NS2D under the a priori condition.

$\omega \in \mathcal{S} \iff \omega_t$ solves NS2D and $\int_0^t I(\omega_s)\,ds < +\infty$ for all $t > 0$.

Theorem

Assume that $\omega^0 \geq 0$, satisfy $H(\omega^0) < +\infty$. Then among the functions satisfying the a priori condition $\int_0^t I(\omega_s)\,ds < +\infty$ for all $t > 0$, there exists a unique ω_t solution of NS2D with initial condition ω^0.

Sketch of the argument.

- Use convolution the equation ($\omega^\varepsilon = \omega * \rho^\varepsilon$) and multiply by some smooth $\varphi'(\omega^\varepsilon)$.

$$\partial_t \varphi(\omega^\varepsilon) + (K * \omega) \cdot \nabla \omega_\varepsilon - \varphi'(\omega_\varepsilon)\Delta\omega_\varepsilon = \varphi'(\omega_\varepsilon)[(K * \omega)\nabla, \rho_\varepsilon *]\omega$$

- The bound on F.I. $\Longrightarrow \omega(K * \nabla\omega) \in L^1_{t,x}$.
- A commutator lemma (used by DiPerna-Lions) allows to pass to the limit and derive many dissipation estimates.
- They allow to prove that $\omega \in C\big((0, +\infty), L^1 \cap L^\infty\big)$ (note that 0 is not included).
- Use a theorem of Ben-Artzi which states uniqueness under the above continuity condition.

Uniqueness (in law) of Non linear SDE under the a priori condition.

From the previous uniqueness result on ω_t, it is enough to solve the linear SDE

$$X_t = X_0 + \int_0^t u_s(X_s)\,ds + \nu B_t, \qquad u_s = K * \omega_s, \quad \omega_s = \text{“given”}$$

Proposition

Assume that $\omega^0 = \mathcal{L}(X_0)$ satisfies $H(\omega_0) < +\infty$, and that ω_s is the unique solution of NS2D such that $\int_0^t I(\omega_s)\,ds < +\infty$ for all $t \geq 0$. Then, strong uniqueness for the previous linear SDE holds (and thus weak uniqueness by Yamada-Watanabe theorem).

Sketch of the proof

- Use argument used by Crippa-De Lellis for uniqueness in ODE with low regularity.
- Two solutions X and Y with same I.C. and brownian satisfies

$$\forall \delta > 0, \;\; \mathbb{E}\left[\ln\Big(1 + \frac{1}{\delta}\sup_{s\leq t}|X_s - Y_s|\Big)\right] \leq \mathbb{E}\left[\int_0^t [M\nabla u_s(X_s) + M\nabla u_s(Y_s)]\,ds\right]$$

  where M stands for maximal function.
- Standard estimates + bounds on F.I. helps to bound the r.h.s.
- A variant of Chebichev ineq. allows to conclude.

Conclusion : results on propagation of chaos.

The validity of the approximation.

Good agreement for numerical simulation, but what's about theoretical results.

- **Osada** : Ok for $\omega^0 \in L^\infty$ and a sufficiently large viscosity ν.
  The key argument : Nash-like estimates for convection-diffusion equation.
  A difficult result.
- **Méléard** : result with a cut-off $\varepsilon(N) \sim \ln(N)^{-1}$ (very large). Extended by Fontbana to 3D vortices.

Our result of propagation of chaos.

Hypothesis : ω_0, the initial condition is positive (for simplicity), entropic : $H(\omega^0) < +\infty$, and as a order one moment : $\int |x|\,\omega_0(dx) < +\infty$.

Theorem

Let $F_0^N = \omega_0^{\otimes N}$. Then there exists a unique law P^N of the N particles trajectories $\big(X(t \geq 0), \dots, X_N(t \geq 0)\big)$ solution of the N vortex problem (2), satisfying

$$\forall t \geq 0, \int_0^t I(F_s^N)\, ds < +\infty, \quad \text{with } F_t^N := \mathcal{L}\big(X_1^N(t), \dots, X_N^N(t)\big).$$

The sequence P^N is P-chaotic, where P is the **unique** *solution of the non-linear SDE, such that*

$$\forall t \geq 0, \int_0^t I(\omega_s)\, ds < +\infty, \quad \text{with } \omega_s := \mathcal{L}(X_s).$$

Moreover, the convergence is entropic in the sense that

$$\forall t \geq 0, \frac{1}{N} H(F_t^N) \xrightarrow[N \to +\infty]{} H(\omega_0),$$