Mean Field Limit and Propagation of chaos for particle systems quasi-neutral and gyro-kinetic limits for plasmas

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1. Limits of particle systems
   - Some definitions about particles systems
   - Stability analysis and quantitative estimates
   - Propagation of chaos via Functional Analysis

2. Quasi-neutral and gyrokinetic limits
   - Quasi-neutral limit alone
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3. Decoherence via a toy model
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What is a particle system?

- N particles described by their position X_i^N and velocity V_i^N, and possibly a parameter a_i^N,
- Satifying Newton's second law with an interaction force $F(X_i^N - X_j^N)$:

$$\frac{d}{dt}X_i^N(t) = V_i^N(t),$$
$$\frac{d}{dt}V_i^N(t) = \frac{1}{N}\sum_{j\neq i}^{N} a_j^N F\left(X_i^N(t) - X_j^N(t)\right) + \sigma\frac{dB_i(t)}{dt}$$

- The factor $\frac{1}{N}$ appears if time and position scale fit well.
- The B_i are (independant) Brownian motions:

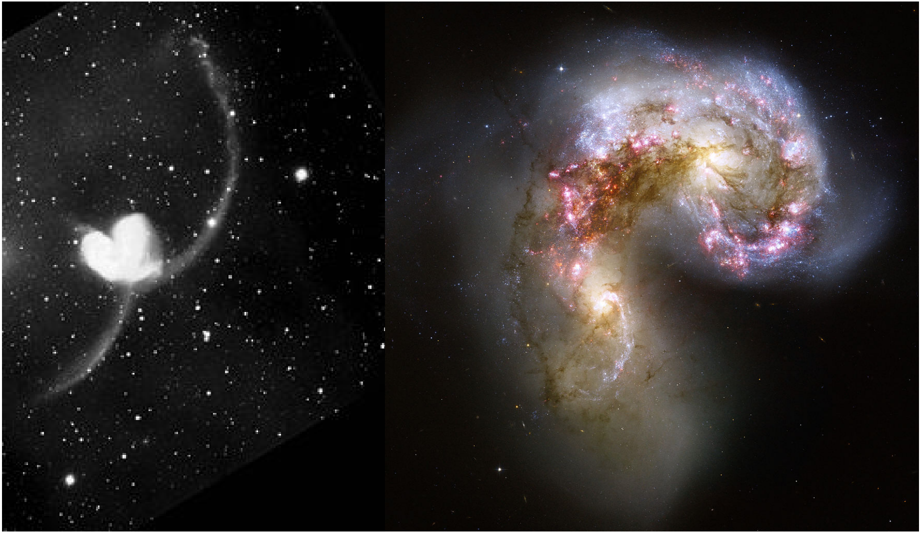
$$\sigma = 0 \Rightarrow \text{deterministic}, \quad \sigma > 0 \Rightarrow \text{stochastic}.$$

- The a_i^N are parameters (mass, charge,...). Here for simplicity, all $a_i^N = 1$.

Examples

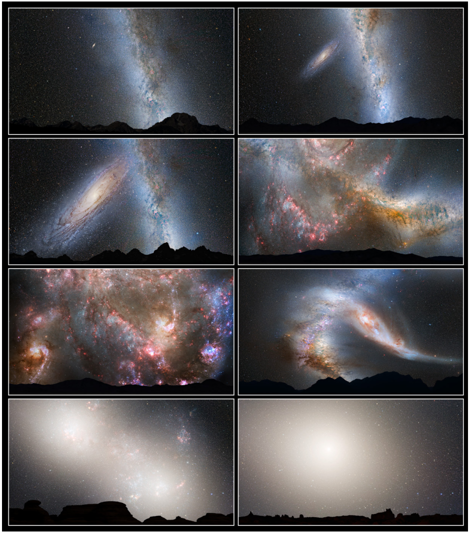
Stars in a galaxy, Galaxies in a cluster, ions or electrons in a plasma, insects in swarm,...

A complex particle system: Antennae Galaxies.



Credit: NASA, ESA, and the Hubble Heritage Team (STScI/AURA)-ESA/Hubble Collaboration – Under "Public domain" licence

Andromeda and Milky Way collision



Credit: NASA; ESA; Z. Levay and R. van der Marel, STScI; T. Hallas, and A. Mellinger – Under "Public domain" licence

Particle systems of order one

The previous system was second order (involving positions and velocities), but **first order models** also exist:

- N "particles" with positions X_i^N (and one parameter a_i^N)
- Satifying a system of ODE with interaction kernel K

$$\frac{d}{dt}X_i^N(t) = \frac{1}{N}\sum_{j\neq i}^{N} a_j^N K\left(X_i^N(t) - X_j^N(t)\right) + \sigma \frac{dB_i(t)}{dt}$$

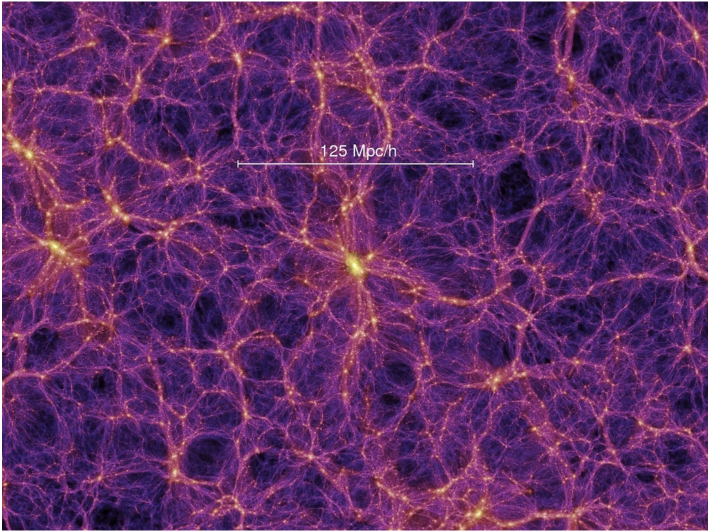
Examples

Vortex in fluids, bacterias and chemotaxis, ions in a homogeneous plasma...

Major issues

In all the above mentioned examples, N is too large to understand the behaviour of the system. Typically, $10^6 \leq N \leq 10^{23}$.
In almost all these example, the system can even not be numerically simulated.

Large scale structure in the universe



From the millenium run, done at the Max Planck Institute für Astrophysik.
Credit: Springel et al. (2005)

Limit for large N leads to *mean-field* equations

Replace the N particles by a distribution f of particles:

A time t, in a small volume $dx \times dv$ around (x, v),
you will find roughly $f(t, x, v)dxdv$ particles.

Then a typical limit trajectory satisfies

$$\frac{d}{dt}X(t) = V(t), \quad \frac{d}{dt}V(t) = E[f](t, X(t)) + \sigma\frac{dB}{dt}$$

$$\text{with} \quad E[f] = \int F(x-y)f(t,y,w)\,dydw$$

In particular, f satisfies the **colisionless Boltzmann equation** (a.k.a. Jeans-Vlasov)

$$\frac{df}{dt} + v \cdot \frac{df}{dx} + E[f]\frac{df}{dv} = \frac{\sigma^2}{2}\Delta_v f$$

Mean field equation (MFE)

It is called a mean-field equation, because the force field E is a kind of average of the interaction force F, with the weight f.

Mean-field equations for order one models

In order one systems, the limit trajectories are

$$\frac{d}{dt}X(t) = E[f](t, X(t)) + \sigma \frac{dB}{dt}, \quad E[f] = \int F(x-y) f(t,y)\, dy$$

or equivalently (with probabilistic notations)

$$dX(t) = \mathbb{E}_Y \left[K(X(t) - Y(t)) \right] dt + \sigma dB_t$$

where $Y(\cdot)$ is an indep. copy of $X(\cdot)$.

The associated MFE is

$$\frac{df}{dt} + E[f] \frac{df}{dx} = \frac{\sigma^2}{2} \Delta_x f$$

Some examples.

In 2D, $K(x) = \frac{x^\perp}{2\pi|x|^2}$ leads to **Navier-Stokes** (or Euler if $\sigma = 0$ equation).
In 2D, $K(x) = -\frac{x}{2\pi|x|^2}$,leads to the **Keller-Segel** model for bacteria aggregation.

Proving rigorously the limit

Main question

If the N particle are initially roughly distributed according to $f(0)$, does they are still roughly distributed according to $f(t)$ at time t.

Definition (Empirical measures)

For a given configuration $\mathcal{Z}^N = (X_i^N, V_i^N)_{i \le N}$ of particles, the associated empirical measure is

$$\mu^N_{\mathcal{Z}} := \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_i^N, V_i^N)}, \quad \text{where } \delta \text{ denotes the Dirac mass.}$$

New formulation: Is the following diagram commutative?

$$\begin{tikzcd} \mu^N_{\mathcal{Z}}(0) \arrow[r, squiggly, "\text{cvg}"] \arrow[d, "Npart"'] & f(0) \arrow[d, "\text{Mean-field}"] \\ \mu^N_{\mathcal{Z}}(t) \arrow[r, squiggly, "\text{cvg ?}"] & f(t) \end{tikzcd}$$

A good distance to quantify the limit

One interest of the empirical measures

Empirical measure $\mu^N_{\mathcal{Z}}$ and distribution $f\,dxdv$ are both **measures** on the phase space. We can compare them.

Definition (Monge Kantorovicth-Wasserstein distance of order one.)

The MKW distance of order one W_1 between two probabilities μ and ν is defined by

$$W_1(\mu,\nu) := \inf_{\Pi} \int |x-y|\,\Pi(dx,dy)$$

where the infimum is taken on the probability Π with first (resp. second) marginal μ (resp. ν).

Or equivalently with probabilistic notations

$$W_1(\mu,\nu) := \inf_{X\sim\mu,\, Y\sim\nu} \mathbb{E}\big[|X-Y|\big].$$

Convergence for smooth interaction when $\sigma = 0$

A second interest of empirical measures

Empirical measures are "weak" solutions of the limit equation, when $\sigma = 0$ and $(F(0) = K(0) = 0)$.

Very useful when the force F is Lipschitz and $\sigma = 0$ in view of

Theorem (Unconditionnal stability of mean-field equation)

If F is Lipschitz (and any $\sigma \geq 0$), any measure solutions μ and ν of the MFE satisfies

$$W_1\big(\mu(t), \nu(t)\big) \leq e^{2\|\nabla F\|_\infty t} W_1\big(\mu^0, \nu^0\big).$$

Apply it to $f(t)\,dxdv$ and $\mu^N_{\mathcal{Z}}$ and obtain:

Corollary (deterministic Mean-Field limit for smooth forces)

If F is Lipschitz and $\sigma = 0$, the convergence of PS towards MFE holds and

$$W_1\big(f(t), \mu^N_{\mathcal{Z}}(t)\big) \leq e^{2\|\nabla F\|_\infty t} W_1\big(f^0, \mu^{N,0}_{\mathcal{Z}}\big).$$

A result due to Braun & Hepp, Dobrushin, Neunzert & Wick.

The McKean-Vlasov diffusion in the smooth case

Problem

In the stochastic case, the empirical measures are not solutions of the limit equation.

Solution: Couple solutions (X_i^N, V_i^N) of the particle system to N copies of the limit equation with the same noises and same initial conditions:

$$dY_i^N(t) = W_i^N(t)\,dt, \qquad dW_i^N(t) = E[f](t, Y_i^N(t))\,dt + \sigma\,dB_i$$

Then, the same estimate than previously (the noises disappear) leads to

Theorem (Propagation of chaos for smooth interaction by McKean)

When F is Lipschitz, for some constant C_t

$$\mathbb{E}\Big[W_1\big(\mu^N_{\mathcal{Z}}(t), \mu^N_{\mathcal{Z}'}(t)\big)\Big] \leq \frac{C_t}{\sqrt{N}} e^{2\|\nabla F\|_\infty t}.$$

where $\mathcal{Z}^{N'} = (Y_i^N, W_i^N)_{i\leq N}$.

A first definition of propagation of molecular chaos

Important if there is some randomness, because of noise or/and initial conditions.

Definition (Propagation of chaos)

It holds when

$$\textit{if} \quad \mathbb{E}\big[W_1(\mu^{N,0}_{\mathcal{Z}}, f^0)\big] \to 0, \quad \textit{then} \quad \mathbb{E}\big[W_1(\mu^N_{\mathcal{Z}}(t), f(t))\big] \to 0$$

The previous theorem implies the propagation of chaos if the $(X_i^{N,0}, W_i^{N,0})$ are i.i.d. with law f^0 because of

Theorem (Ajtai-Komlós-Tusnády '84, Fournier-Guillin '14)

In dimension d if $(\mathcal{Z}^{N'}) = f^{\otimes N}$ (and technical assumptions), then

$$\mathbb{E}\big[W_1(\mu^N_{\mathcal{Z}'}, f)\big] \lesssim C\, N^{-\frac{1}{d}}$$

In fact, all in all

$$\mathbb{E}\Big[W_1\big(\mu^N_{\mathcal{Z}}(t), f(t)\big)\Big] \leq \frac{C_t}{\sqrt{N}} e^{2\|\nabla F\|_\infty t} + C_t\, N^{-\frac{1}{d}}.$$

Physical interactions are often singular

Problem

The previous results apply to smooth interaction: but in many physical situations, the interaction is singular

Examples:

- Gravitational or Coulombian force: $\pm c \frac{x}{|x|^{d-1}}$ in dimension d;
- The Navier-Stokes (or Euler) equation where $K(x) = c \frac{x^\perp}{|x|^2}$ in 2D;
- In Chemotaxis where $K(x) = -c \frac{x}{|x|^2}$ in 2D; ...

How to handle singularities?

A general strategy to go further

Study the stability of the limit MFE in MKW distance.

A first example: Vlasov-Poisson in 1D.

Here $d = 1$, the force $\mathbf{F}(\mathbf{x}) = \mathbf{sign}\,\mathbf{x}$ (not Lipschitz) and $\sigma = 0$. The stability result is

Theorem (Weak-strong stability, H. '13, Sem X)

If f_t solves VP1D with bounded density $\rho_t = \int f_t \, dv$. Then, any measure solution ν of VP1D satisfies for all $t \geq 0$

$$W_1(\nu_t, f_t) \leq e^{a(t)} W_1(\nu_0, f_0), \qquad \textit{with} \quad a(t) := \sqrt{2}\, t + 8 \int_0^t \|\rho_s\|_\infty \, ds.$$

In short, the stability holds provided that **one** solution is a **strong** one.

Consequences

- Apply it to $\nu_t = \mu_t^N$: $W_1(\mu_t^N, f_t) \leq e^{a(t)} W_1(\mu_0^N, f_0) \Rightarrow$ Mean-Field Limit
- Taking expectation: $\mathbb{E}\big[W_1(\mu_t^N, f_t)\big] \leq e^{a(t)} \mathbb{E}\big[W_1(\mu_0^N, f_0)\big] \Rightarrow$ Prop. of Chaos

Further : Also Propagation of entropic chaos, See [Hauray & Mischler, JFA '14].

Second example: Vlasov-Poisson-Fokker-Planck in 1D

Here $d = 1$, the force $\mathbf{F}(\mathbf{x}) = \mathbf{sign}\,\mathbf{x}$ (not Lipschitz) and $\sigma > 0$.

First idea:

Apply the same strategy than previously, but to the coupling with N i.i.d. of the limit McKean-Vlasov diffusion. It leads roughly to

$$\mathbb{E}\Big[W_1\big(\mu^N_{\mathcal{Z}}(t), \mu^N_{\mathcal{Z}'}(t)\big)\Big] \leq \frac{C_t}{\sqrt{N}}\mathbb{E}\big[e^{a(t)}\big], \quad a(t) := \sqrt{2}\,t + 8\int_0^t \|\mu^N_{\mathcal{Z}'}(s)\|_\infty \, ds.$$

Useless because $\|\mu^N_{\mathcal{Z}'}(s)\|_\infty = +\infty$.

Solution:

- Introduce discrete infinite norms $\|\mu\|_{\infty,\varepsilon} := \sup_{(x,v)} \frac{\mu\big(B_\varepsilon(x,v)\big)}{\mathrm{Vol}\big(B_\varepsilon(x,v)\big)}$,
- use a relaxed version of the weak-strong stability estimate for VPFP1D

$$\frac{d}{dt}W_1(\nu_t, \mu_t) \leq \big(\sqrt{2} + \|\nu_t\|_{\infty,\varepsilon}\big)\big(W_1(\nu_t, \mu_t) + \varepsilon\big).$$

Second example: Vlasov-Poisson-Fokker-Planck in 1D

- Look at the literature on deviation upper bounds for $\|\mu^N_{\mathcal{Z}'}(t)\|_{\infty,\varepsilon}$ at fixed time,
- Deduce some deviation upper bounds on $\displaystyle\int_0^t \|\mu^N_{\mathcal{Z}'}(s)\|_{\infty,\varepsilon}\, ds$

All in all, obtain a good deviation upper bound for the PS

Theorem (H. & Salem, WiP)

If f is a strong solution of the VPFP1D, then for some C_t,

$$\mathbb{P}\Big(W_1\big(\mu^N_{\mathcal{Z}}(t), \mu^N_{\mathcal{Z}'}(t)\big) \geq \varepsilon \Big) \leq C_t\, N^4\, e^{-\frac{1}{2}N\varepsilon^2}$$

and for some c'_t, C'_t

$$\mathbb{P}\Big(W_1\big(\mu^N_{\mathcal{Z}}(t), f(t)\big) \geq \varepsilon \Big) \leq C'_t\, N^4\, e^{-c'_t N\varepsilon^2}$$

which implies propagation of chaos.

Homogeneous Landau equation in dimension 3

The HL equation for moderately soft potentials ($\gamma \in (-1,0)$) reads

$$\frac{d}{dt} f(t,v) = \mathrm{div}\left(\int a(v-v')\left[f(t,v)\frac{df}{dv}(t,v') - f(t,v')\frac{df}{dv}(t,v)\right] dv' \right)$$

$$\text{where} \quad a(w) = |w|^{2+\gamma}\left(Id - \frac{w \otimes w}{|w|^2}\right)$$

An associated particle system contains:

- A non Lipschitz interaction kernel $K = \mathrm{div}\, b$,
- A diffusion with σ (non Lipschitz) dependent of the V_i^N .

The strategy used for the VPFP1D also work

Theorem (H. Fournier, WiP)

For any $t \geq 0$, any strong solution f, there exists a constant $C_{t,\gamma}$

$$\mathbb{E}[W_2^2(\mu_t^N, f_t)] \leq C_{t,\gamma} N^{-\alpha}.$$

for some $\alpha > 0$ depending explicitly on γ.

Homogeneous Landau equation in dimension 3

Additional difficulties:

- Since $\sigma \neq cst$, a non trivial coupling is necessary to get a weak-strong stability estimate,
- For the same reason, it is more difficult to get deviation upper bound.

4th example: Jeans-Vlasov in 3D, weakly singular case

Here $\sigma = 0$, and $F(x) = \dfrac{c}{|x|^\alpha}$, $\alpha \in (0,1)$.
Always deals with solution with compact support in v [Pfaffelmoser '95].

Theorem (Strong-strong stability, reformulation of [Loeper '05])

For two strong solutions f and g of the Jeans-Vlasov equation

$$\frac{d}{dt} W_p(f_t, g_t) \leq C \max(\|f_t\|_\infty, \|g_t\|_\infty) W_p(f_t, g_t)$$

for any MKW distance of order $p \in [1, +\infty]$.

Same strategy: Relax the stability estimate with one discrete infinite norm.
Tedious calculation leads to

$$\frac{d}{dt} W_p(f_t, \mu_t^N) \leq C \max(\|f_t\|_\infty, \|\mu_t^N\|_{\infty,\varepsilon}) \big(W_p(f_t, \mu_t^N) + R(\varepsilon)\big)$$

4th example: Jeans-Vlasov in 3D, weakly singular case

A new improvment

Take $p = \infty$ and $\varepsilon = W_\infty(f_t, \mu_t^N) =: W(t)$, because $\|\mu_t^N\|_{\infty, W(t)} \leq 2^d \|f_t\|_\infty$.
End up with a relaxed weak-strong stability estimate

$$\frac{d}{dt} W_\infty(f_t, \mu_t^N) \leq 2^d\, C\, \|f_t\|_\infty \big(W_\infty(f_t, \mu_t^N) + R(\varepsilon) \big)$$

Implies Mean Field limit and Propagation of Chaos. [H. & Jabin '15]

Some comments:

- Many difficulties hidden in $R(\varepsilon)$, which depends on

$$\min_{i \neq j} (|X_i^N - X_j^N| + |V_j^N - V_i^N|),$$

- The argument fails for $p \neq \infty$,
- The Prop. of Chaos is also entropic.

Dissipation of entropy for order one models

The entropy H and the Fisher information I are defined for a R.V. X or its law f by

$$H(X) = H(f) = \int f \ln f,\, dx \qquad I(X) = I(f) = \int \Big| \frac{df}{dx} \Big|^2 f \, dx.$$

When $\sigma > 0$ in order one models, entropy (or free energy) is dissipated:

For instance for the vortex systems (linked to NS2D)

$$\frac{d}{dt} X_i^N(t) = \frac{1}{N} \sum_{j \neq i}^{N} a_j^N \frac{\left(X_i^N(t) - X_j^N(t)\right)^{\perp}}{\left|X_i^N(t) - X_j^N(t)\right|^2} + \sigma \frac{dB_i(t)}{dt}$$

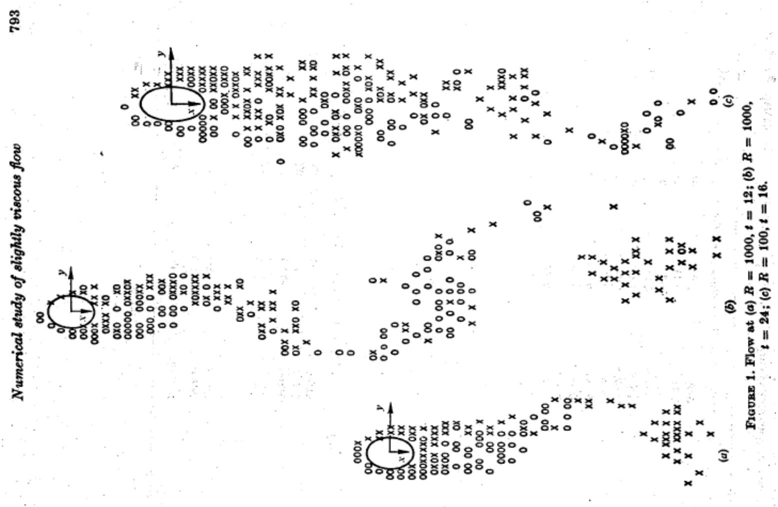
we get

$$H(\mathcal{X}_t^N) + \frac{\sigma^2}{2} \int_0^t I(\mathcal{X}_s^N)\, ds = H(\mathcal{X}_0^N).$$

But, how to use it?

Numerical applications.

A simulation by Chorin in the '70.



Fisher information and Prop. of chaos

Properties of the Fisher information (same for entropy).

1. I is convex an lower semi-continuous,
2. I is super-additive: $I(X_1, X_2) \leq I(X_1) + I(X_2)$
   with equality only if X_1, X_2 are independant.
3. I controls some L^p-norms thanks to GNS inequalities (good in low dim.),
4. "I go trough the limit": if the sequence of R.V. $(\mu^N_{\mathcal{X}})_N$ goes in law towards a R.V. g, then

$$\mathbb{E}\big[I(g)\big] \leq \liminf_{N\to\infty} \frac{1}{N} I(\mathcal{X}^N). \qquad \text{[H. Mischler '14]}$$

Then, apply Sznitman martingale method [St-Flour lecture notes '84]:

- Property 2 and 3 imply tightness and consistency,
- Property 4 implies the uniqueness of the limit.

Theorem (Osada '86; Fournier, H. & Mischler, '14)

The (entropic) Prop. of chaos holds for the stochastic vortex model, towards the NS2D equation, for any $\sigma > 0$.

Further applications of that qualitative method

Some comments:

- Here the results are qualitative (compactness technics), but allow to handle stronger singularities.
- In the previous part, the noise was a problem. Here it really helps.
- Work only for full noise, or almost full as in homogeneous Landau equation with $\gamma \in (-2, -1)$ [Fournier & H., WiP]
- Work also for relative entropy (or free energy) dissipation, as for sub-critical Keller-Segel model [Godinho & Quininao '14]

An important scale in Plasma, the Debye length.

After a nondimensionalization, The Vlasov-Poisson equation on the 1D torus $\mathbb{T}$ reads:

$$\frac{df_\varepsilon}{dt} + v\frac{df_\varepsilon}{dx} - \frac{dV_\varepsilon}{dx}(t,x)\frac{df_\varepsilon}{dv} = 0, \quad \text{with} \quad \varepsilon^2 \frac{d^2 V_\varepsilon}{dx^2} = \rho_\varepsilon - 1 = \int f_\varepsilon \, dv - 1$$

ε is here the ratio between the Debye length and the typical length of the system.

Langmuir waves or plasma oscillations.

An interesting wave phenomena is observed in plasma.
$J_\varepsilon = \int fv \, dv$ and $\varepsilon V_\varepsilon$ oscillate with frequency $\frac{1}{\varepsilon}$:

$$\partial_t[\varepsilon V_\varepsilon] = -\frac{J_\varepsilon}{\varepsilon}$$

$$\partial_t J_\varepsilon = \frac{\varepsilon V_\varepsilon}{\varepsilon} + \left(\frac{d}{dx}\right)^{-1}\left((\varepsilon \nabla_x V_\varepsilon)^2 - \int f_\varepsilon v^2 \, dv\right) + \frac{1}{2}\left(\varepsilon \frac{dV_\varepsilon}{dx}\right)^2$$

A very small scale in most situations.

Usually, the Debye length is much smaller than the typical scale of the system.

Plasma	Density $n_e(m^{-3})$	Electron temperature $T(K)$	Magnetic field $B(T)$	Debye length $\lambda_D(m)$
Solar core	10^{32}	10^{7}	--	10^{-11}
Tokamak	10^{20}	10^{8}	10	10^{-4}
Gas discharge	10^{16}	10^{4}	--	10^{-4}
Ionosphere	10^{12}	10^{3}	10^{-5}	10^{-3}
Magnetosphere	10^{7}	10^{7}	10^{-8}	10^{2}
Solar wind	10^{6}	10^{5}	10^{-9}	10
Interstellar medium	10^{5}	10^{4}	10^{-10}	10
Intergalactic medium	1	10^{6}	--	10^{5}

From a course by Kip Thorne at Caltech.

Problem:

Then the Langmuir waves are very fast.

An experimental observation of Langmuir waves in ionosphere

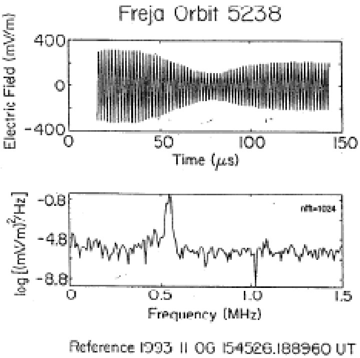


Figure 2. An example of narrow band Langmuir waves. The upper panel contains the waveform and the lower panel contains the power spectrum.

From Kintner, Holback & all, Cornell University and Swedish inst. of space phy. Geophy. Rev. Letters 1995. Record form Freja plasma wave instrument (alt. 1700 km).

Another experimental observation of Langmuir waves

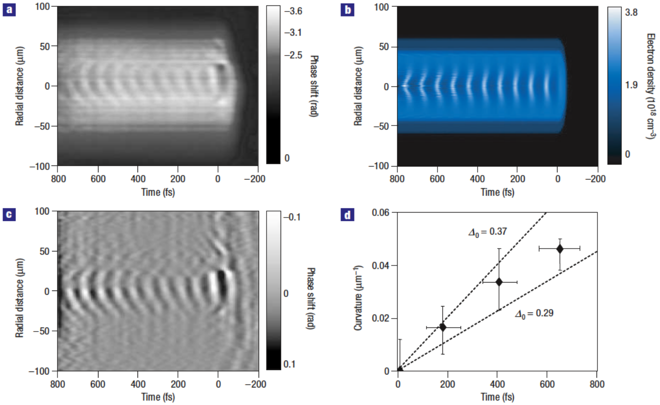


Figure 3 Strongly driven wake with curved wavefronts. a, Probe phase profile $\Delta\phi_{pr}(r, \zeta)$ for an ~30 TW pump, $\bar{n}_e^{max} = 2.2 \times 10^{18}$ cm^{-3} in the He^{2+} region. **b**, Simulated density profile $n_e(r, \zeta)$ near the jet centre. **c**, Same data as in **a**, with the background $\bar{n}_e$ subtracted to highlight the wake. **d**, Evolution of the reciprocal radius of wavefront curvature behind the pump (data points), compared with calculated evolution (dashed lines) for indicated wake potential amplitudes. Each data point (except at $\zeta = 0$) averages over three adjacent periods. The horizontal error bars extend over the three periods averaged, and the vertical error bars extend over the range of fitted curvature values averaged.

From Matlis, Downer & all, University of Texas and Michigan, Nature Phys 2006.

Quasi-neutrality: the formal limit when $\varepsilon = 0$.

$$\frac{df}{dt} + v\frac{df}{dx} - \frac{dV}{dx}(t,x)\frac{df}{dv} = 0, \quad \text{with} \quad \rho = 1$$

Problems:

- That model is probably ill-posed (mathematically). Very few results about it :

  existence of solutions for short time and analytical initial data [Grenier '96, Jabin, Tallay & all '13, Jabin & Nouri '13],
- The equation on V is implicit,
- What about the Langmuir waves?

However, some rigorous result of convergence when $\varepsilon \to 0$, in the zero temperature limit: when $f_\varepsilon^0 \rightharpoonup \rho^0 \delta_{v^0}$

- Use well-prepared initial data [Bernier '00],
- Filtrate the plasma oscillation [Grenier '96, Masmoudi '01].

Then $f_\varepsilon(t)$ converges to $\delta_{v(t)}$, where v is a solution of the **incompressible Euler equation**.

The case of general initial data

A note by Grenier [JEDP '96] explains two phenomena:

- Possible instabilities are instantaneous in the quasi-neutral limit,
- Around solutions of VQN with always **"one bump in v"** , the limit should holds.

In a joint work [H. Han-Kwan '15], we try to rigorously prove the above statements.

Theorem (Instantaneous instability)

Let $\mu(v)$ be a smooth profile satisfying the Penrose instability criterion. For any $N > 0$ and $s > 0$, there exists f_ε^0 such that

$$\|f_\varepsilon^0 - \mu\|_{W^{s,1}_{x,v}} \leq \varepsilon^N,$$

and for any $r \in \mathbb{Z}$, we have

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon]} \|f_\varepsilon(t) - \mu\|_{W^{r,1}_{x,v}} > 0.$$

The case of general initial data

Theorem (Stability after filtration of oscillations)

Let μ be a stable stationary profile. For any smooth potential V_0, we define an associated "modulated free energy"

$$\mathcal{L}_\varepsilon^O(t) := H_Q\left[f_\varepsilon\left(t, x, v - \partial_x V_0(x - \bar{v}t)\sin\frac{t}{\varepsilon}\right)\right] + \frac{1}{2}\int\left[\varepsilon\partial_x V_\varepsilon - \partial_x V_0(x - \bar{v}t)\cos\frac{t}{\varepsilon}\right]^2 dx.$$

Then,

$$\mathcal{L}_\varepsilon^O(t) \leq e^{2\|V_0''\|_{L^\infty} t}\left[\mathcal{L}_\varepsilon^O(0) + K\varepsilon\right],$$

where H_Q is a functional generalizing the entropy. It controls in most case somes L^p norm $(p = 1, 2, ...)$:

$$\|f - f^0\|_p \leq C\, H_Q(f).$$

Merging quasi-neutral and gyro-kinetic limit

Seems more complicated but in fact may help.

For instance, the **quasi-neutral gyro-system**

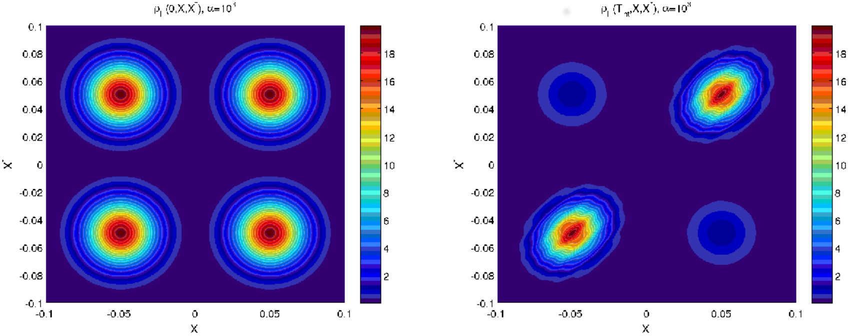
$$
\begin{aligned}
&\frac{df}{dt} + (J_u^0 \nabla_x V)^\perp \cdot \nabla_x f = \beta u \partial_u f + 2\beta f + \nu \left(\Delta_x f + \frac{1}{u} \partial_u (u \partial_u f) \right), \\
&\Phi = \rho, \\
&\rho(t,x) = \int (J_u^0 f(t,x,u) 2\pi u du),
\end{aligned}
$$

where J^0 is the gyro-average operator, is well posed [H. & Nouri '11].

In fact, as far as regularity is concerned it is very similar to the NS2D equation.

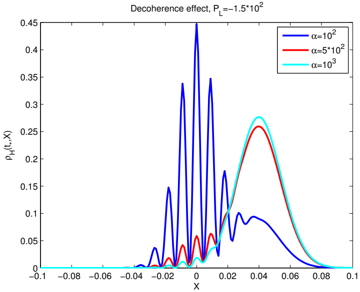
Decoherence: a basic model and it simulations

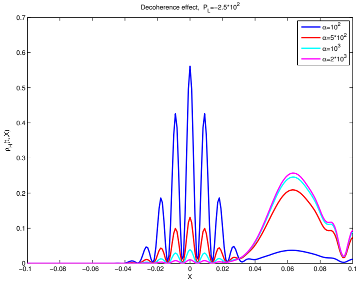
No time for much: only pictures from [Adami, H., Negulescu 201?].



Decoherence: a basic model and it simulation

Two other pictures.





The density $\rho^M(T^*, X, X)$ for different values of α, and p.

To do

A simple model for one collision. Get the limit master equation in a many interactions regime [Gomez & H. WiP]

Thanks!

Thanks for your attention !

Merci pour votre attention !

¡ Gracias por su attención !