A VECTORIAL INGHAM–BEURLING TYPE THEOREM

By

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Dedicated to Professor László Simon on the occasion of his 70th birthday

Abstract. Baiocchi et al. generalized a few years ago a classical theorem of Ingham and Beurling by means of divided differences. The optimality of their assumption has been proven by the third author of this note. The purpose of this note is to extend these results to vector coefficient sums.

1. Introduction

Let $\Omega := (\omega_k)_{k\in\mathbb{Z}}$ be a family of real numbers satisfying the gap condition

$$\gamma := \inf_{k\neq n} |\omega_k - \omega_n| > 0. \tag{1.1}$$

Let us denote by $D^+ = D^+(\Omega)$ its Pólya upper density, defined by the formula $D^+ := \lim_{r\to\infty} r^{-1} n^+(r)$, where $n^+(r) = n^+(\Omega, r)$ denotes the largest number of terms of the sequence $(\omega_k)_{k\in\mathbb{Z}}$ contained in an interval of length r.

Let $(U_k)_{k\in\mathbb{Z}}$ be a corresponding family of unit vectors in some complex Hilbert space H and consider the sums

$$x(t) = \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}$$

with square summable complex coefficients x_k. We are interested in the validity of the estimates

$$\int_I |x(t)|_H^2 \, dt \asymp \sum_{k\in\mathbb{Z}} |x_k|^2 \tag{1.2}$$

AMS Subject Classification (2000): primary 42B99; secondary 42A99 where I is a bounded interval of length denoted by $|I|$ and where we write $A \asymp B$ if there exist two positive constants c_1, c_2 satisfying $c_1 A \le B \le c_2 A$.

The following result generalizes a theorem of Ingham [4]; for $d = 1$ it reduces to a theorem of Beurling [2].

THEOREM 1.1. *Let $\Omega := (\omega_k)_{k\in\mathbb{Z}}$ be a family of real numbers satisfying (1.1).*

(a) If $|I| > 2\pi D^+$, then the estimates (1.2) hold.

(b) If the estimates (1.2) hold true and H has a finite dimension d, then $|I| \ge 2\pi D^+/d$.

The optimality of Theorem 1.1 will be deduced from the following result:

THEOREM 1.2. *Let Ω be a set of real numbers with a finite upper density D^+ and let $\alpha_1, \alpha_2, \dots$ be a finite or infinite sequence of numbers in $[0,1]$ satisfying $\alpha_1 + \alpha_2 + \dots \ge 1$. Then there exists a partition $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots$ of Ω such that the upper density ${D^+}_j$ of Ω_j is equal to $\alpha_j D^+$ for every j.*

REMARK. It follows from the definition of the upper density that if $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots$ is a finite or infinite partition of Ω, then

$$\max\{D^+(\Omega_1), D^+(\Omega_2), \dots\} \le D^+(\Omega) \le D^+(\Omega_1) + D^+(\Omega_2) + \cdots. \tag{1.3}$$

This implies the necessity of the conditions $\alpha_j \le 1$ and $\alpha_1 + \alpha_2 + \dots \ge 1$ in the theorem.

Now we have the following corollary:

COROLLARY 1.3. *Let $\Omega := (\omega_k)_{k\in\mathbb{Z}}$ be a family of real numbers satisfying (1.1), and H a finite-dimensional Hilbert space. Given any real number $\frac{1}{d} \le \alpha \le 1$ where $d = \dim H$, there exists a family $(U_k)_{k\in\mathbb{Z}}$ of unit vectors in H such that the estimates (1.2) hold if $|I| > 2\pi\alpha D^+$, and they fail if $|I| < 2\pi\alpha D^+$.*

We prove Theorem 1.1 in the next section and we extend it to the case of a weakened gap condition in Section 3. Theorem 1.2 and Corollary 1.3 are proved in Sections 4–6.

We refer to [5] for various control theoretical applications of theorems of this type.

2. Proof of Theorem 1.1

Part (a) readily follows from the scalar case. Indeed, fixing an orthonormal basis $(E_n)_{n\in N}$ of the closed linear hull of $(U_k)_{k\in\mathbb{Z}}$ in H and developing the vectors U_k into Fourier series: $U_k = \sum_{n\in N} u_{kn}E_n$, for $|I| > 2\pi D^+$ we have

$$\int_I \left\| \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t} \right\|_H^2 dt = \sum_{n\in N} \int_I \left| \sum_{k\in\mathbb{Z}} x_k u_{kn} e^{i\omega_k t} \right|^2 dt \asymp$$

$$\asymp \sum_{n\in N} \sum_{k\in\mathbb{Z}} |x_k u_{kn}|^2 =$$

$$= \sum_{k\in\mathbb{Z}} |x_k|^2.$$

For the proof of part (b) we adapt the approach developed in [3] and [6]. We set $\gamma_k := 2\pi|I|^{-1}k$ for brevity. Given three real numbers y, r, R with $r, R > 0$, we introduce the orthogonal projections

$$P_{y,r} : L^2(I,H) \to V_{y,r} \quad \text{and} \quad Q_{y,r+R} : L^2(I,H) \to W_{y,r+R}$$

onto the finite-dimensional linear subspaces

$$V_{y,r} := \operatorname{Vect}\left\{ U_k e^{i\omega_k t} \ : \ |\omega_k - y| < r \right\}$$

and

$$W_{y,r+R} := \operatorname{Vect}\left\{ U e^{i\gamma_k t} \ : \ |\gamma_k - y| < r + R \quad \text{and} \quad U \in H \right\}.$$

Note that

$$n^+(2r) = \sup_y \dim V_{y,r} \tag{2.1}$$

and

$$(2r+2R)d\frac{|I|}{2\pi} \le \dim W_{y,r+R} \le (2r+2R+1)d\frac{|I|}{2\pi}. \tag{2.2}$$

Setting

$$S_{y,r,R} := P_{y,r} \circ Q_{y,r+R} \circ i$$

where i denotes the injection $i : V_{y,r} \hookrightarrow L^2(I,H)$, we obtain a linear map of $V_{y,r}$ into itself. We are going to study its trace.

LEMMA 2.1. *We have*

$$\left|\mathrm{tr}(S_{y,r,R})\right| \le \dim W_{y,r+R}.$$

PROOF. We have

$$\left\|S_{y,r,R}\right\| \le \left\|P_{y,r}\right\| \cdot \left\|Q_{y,r+R}\right\| \le 1.$$

Hence the eigenvalues of $S_{y,r,R}$ have modulus ≤ 1 and therefore

$$\left|\mathrm{tr}(S_{y,r,R})\right| \le \mathrm{rang}(S_{y,r,R}) \le \dim(W_{y,r+R}). \qquad \blacksquare$$

LEMMA 2.2. *Writing $e_k(t) := U_k e^{i\omega_k t}$ for brevity, there exists $(\varphi_k)_{k\in\mathbb{Z}}$ a bounded biorthogonal family to $(e_k)_{k\in\mathbb{Z}}$ in $L^2(I,H)$ and we have*

$$\mathrm{tr}(S_{y,r,R}) = \dim V_{y,r} + \sum_{|\omega_k - y|<r} ((Q_{y,r+R} - \mathrm{Id})e_k, P_{y,r}\varphi_k)_H.$$

PROOF. The existence of a bounded biorthogonal family comes from (1.2) (see [5] for a proof). We then write $S_{y,r,R}e_k = \sum_{|\omega_j - y|<r} S_{k,j}e_j$. Since (φ_k) is biorthogonal, we have $(S_{y,r,R}e_k, \varphi_k)_{L^2(I,H)} = S_{k,k}$ and thus

$$\mathrm{tr}(S_{y,r,R}) = \sum_{|\omega_j - y|<r} S_{k,k} = \sum_{|\omega_k - y|<r} (S_{y,r,R}e_k, \varphi_k)_{L^2(I,H)},$$

so that

$$\mathrm{tr}(S_{y,r,R}) = \sum_{|\omega_k - y|<r} (P_{y,r}e_k, \varphi_k)_{L^2(I,H)} +$$

$$+ \sum_{|\omega_k - y|<r} (P_{y,r}(Q_{y,r+R} - \mathrm{Id})e_k, \varphi_k)_{L^2(I,H)}.$$

Since $P_{y,r}e_k = e_k$, we have $(P_{y,r}e_k, \varphi_k)_{L^2(I,H)} = 1$ and the result follows.

LEMMA 2.3. *For $R \to \infty$ we have*

$$\left\|(Q_{y,r+R} - \mathrm{Id})e_k\right\| = O(1/\sqrt{R})$$

uniformly for all $y \in \mathbb{R}$, $r > 0$ and k satisfying $|\omega_k - y| < r$.

PROOF. Fixing an orthonormal basis $E_1, \ldots, E_d$ of H and setting

$$f_{n,j}(t) := |I|^{-1/2} E_j e^{i\gamma_n t}$$

we have

$$e_k = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{d}(e_k, f_{n,j})_{L^2(I,H)} f_{n,j}$$

and

$$Q_{y,r+R}e_k = \sum_{|\gamma_n - y|<r+R}\sum_{j=1}^{d}(e_k, f_{n,j})_{L^2(I,H)} f_{n,j}.$$

Applying Parseval's equality it follows that

$$\left\|(Q_{y,r+R} - \mathrm{Id})e_k\right\|^2 = \sum_{|\gamma_n - y|\geq r+R}\sum_{j=1}^{d}\left|(e_k, f_{n,j})_{L^2(I,H)}\right|^2.$$

Since

$$\left|(e_k, f_{n,j})_{L^2(I,H)}\right| = |I|^{-1/2}\left|\int_I (U_k, E_j)_H e^{i(\omega_k - \gamma_n)t}\ dt\right| \leq \frac{2|I|^{-1/2}}{|\omega_k - \gamma_n|}, \tag{2.3}$$

and $|\omega_k - y| < r$, then we obtain that

$$\left\|(Q_{y,r+R} - \mathrm{Id})e_k\right\|^2 \leq 4d|I|^{-1}\sum_{|\gamma_n - y|\geq r+R}\frac{1}{|\omega_k - \gamma_n|^2}$$

Note that from $|\gamma_n - y| \geq r + R$ and $|\omega_k - y| < r$, we get $|\gamma_n - \omega_k| > R$, and thus

$$\left\|(Q_{y,r+R} - \mathrm{Id})e_k\right\|^2 \leq 8d|I|^{-1}\sum_{n=0}^{\infty}\frac{1}{|2\pi|I|^{-1}n + R|^2}.$$

Since the last expression doesn't depend on r, y and is $O(1/R)$ as $R \to \to \infty$, the lemma follows. ∎

Now the proof of part (b) of Theorem 1.1 can be completed as follows. By the above lemmas we have

$$\begin{aligned}\dim W_{y,r+R} &\geq |\mathrm{tr}(S)| = \\ &= \left|\dim V_{y,r} + \sum_{|\omega_k - y|<r}((Q_{y,r+R} - \mathrm{Id})e_k, P_{y,r}\varphi_k)_H\right| \geq \\ &\geq \dim V_{y,r} - O(1/\sqrt{R})\dim V_{y,r}.\end{aligned}$$

Hence

$$\dim V_{y,r} \le (1 + O(1/\sqrt{R}))\dim W_{y,r+R}, \quad R \to \infty,$$

and using (2.1)–(2.2) we conclude that

$$n^+(2r) \le (1 + O(1/\sqrt{R}))d\frac{|I|}{2\pi}(2r + 2R + 1), \quad R \to \infty.$$

It follows that

$$\begin{aligned} D^+ &= \lim_{r\to\infty} \frac{n^+(2r)}{2r} \le \\ &\le (1 + O(1/\sqrt{R}))d\frac{|I|}{2\pi} \lim_{r\to\infty} \frac{2r + 2R + 1}{2r} = \\ &= (1 + O(1/\sqrt{R}))d\frac{|I|}{2\pi} \end{aligned}$$

for all $R > 0$. Letting $R \to \infty$ we conclude that $|I| \ge 2\pi D^+/d$.

3. The case of the divided differences

The gap condition (1.1) of the theorem may be weakened. Following [1] let $(\omega_k)_{k\in\mathbb{Z}}$ be a nondecreasing sequence of real numbers satisfying for some positive integer M and for some positive real number γ' the weakened gap condition

$$\omega_{k+M} - \omega_k \ge M\gamma' \quad \text{for all} \quad k \in \mathbb{Z}. \tag{3.1}$$

This implies that $D^+ < \infty$. We say that $\omega_m, \dots, \omega_{m+j-1}$ is a γ'-close exponent chain ($m \in \mathbb{Z}, j = 1, \dots, M$) if

$$\begin{cases} \omega_m - \omega_{m-1} \ge \gamma', \\ \omega_k - \omega_{k-1} < \gamma' \quad \text{for } k = m + 1, \dots, m + j - 1, \\ \omega_{m+j} - \omega_{m+j-1} \ge \gamma'. \end{cases}$$

Then we define the divided differences $f_\ell = [\omega_m, \dots, \omega_\ell]$ by the formula

$$[\omega_m](t) := \exp(i\omega_m t),\ [\omega_m, \omega_{m+1}](t) := it\int_0^1 \exp(i[s_m(\omega_{m+1} - \omega_m) + \omega_m]t)ds_m,$$

and for $\ell = m + 2, \dots, m + j - 1$,

$$[\omega_m, \dots, \omega_\ell](t) := (it)^{\ell-m} \int_0^1 \int_0^{s_m} \dots \int_0^{s_{\ell-2}}$$

$$\exp(i[s_{\ell-1}(\omega_\ell - \omega_{\ell-1}) + \dots + s_m(\omega_{m+1} - \omega_m) + \omega_m)]t)\ ds_{\ell-1} \dots\ ds_m.$$

We can now state a generalization of Theorem 1.1:

THEOREM 3.1. *Theorem 1.1 holds true if (1.1) is replaced by (1.3) and* $e^{i\omega_k t}$ *is replaced by* $f_k(t)$.

PROOF. Most of the proof of Theorem 1.1 may be easily adapted. For part (b) we have to replace the estimate (2.3) by the following:

$$\left|\int_I (U_k, E_j)_H f_k(t) e^{-i\gamma_n t}\,dt\right| \le \left|\int_I f_k(t) e^{-i\gamma_n t}\,dt\right| \le \frac{C}{|\omega_k - \gamma_n|}, \tag{3.2}$$

with a constant C depending only on γ', M and I. This is shown by arguing similarly as in [6]. We have

$$A := \int_I f_k(t) e^{-i\gamma_n t}\,dt = \int_I g(t) e^{i\omega_k t} e^{-i\gamma_n t}\,dt$$

with

$$g(t) = [\omega_m - \omega_k, \ldots, \omega_k - \omega_k](t).$$

Integrating by parts in $I = (a, b)$ we obtain that

$$A = \left[\frac{1}{i\omega_k - i\gamma_n} g(t) e^{i\omega_k t} e^{-i\gamma_n t}\right]_a^b - \int_I \frac{1}{i\omega_k - i\gamma_n} g'(t) e^{i\omega_k t} e^{-i\gamma_n t}\,dt.$$

Now a direct computation shows that for any real numbers $\mu_1, \ldots, \mu_r$ the divided differences satisfy the inequality

$$[\mu_1, \ldots, \mu_r]'(t) \le \frac{(r-1)t^{r-2}}{(r-1)!} + (|\mu_r - \mu_{r-1}| + \ldots + |\mu_2 - \mu_1| + |\mu_1|)\frac{t^{r-1}}{(r-1)!}.$$

Thus, in our case, thanks to the γ'-close exponent property, we have

$$|g'(t)| \le (k-m)\frac{t^{k-m-1}}{(k-m)!} + (k-m)\gamma' \frac{t^{k-m}}{(k-m)!}$$

and this yields (3.2). ∎

4. Proof of Theorem 1.2 for $\alpha_1 + \alpha_2 + \cdots = 1$

Assume that the theorem holds for sets Ω which are bounded from below. Then by changing Ω to $-\Omega$ we obtain that the theorem also holds if Ω is bounded from above. Finally, if $\inf \Omega = -\infty$ and $\sup \Omega = \infty$, then applying these two special cases of the theorem to

$$\Omega^- := \Omega \cap (-\infty, 0) \quad \text{and} \quad \Omega^+ := \Omega \cap [0, \infty)$$

we obtain two partitions

$$\Omega^- = {\Omega^-}_1 \cup {\Omega^-}_2 \cup \cdots \quad \text{and} \quad \Omega^+ = {\Omega^+}_1 \cup {\Omega^+}_2 \cup \cdots$$

satisfying

$$D^+({\Omega^-}_j) = \alpha_j D^+(\Omega^-) \quad \text{and} \quad D^+({\Omega^+}_j) = \alpha_j D^+(\Omega^+)$$

for all j. Then setting $\Omega_j := {\Omega^-}_j \cup {\Omega^+}_j$ we obtain a partition of Ω with the required properties. This follows by applying the following lemma for the partitions $\Omega := \Omega^- \cup \Omega^+$ and $\Omega_j := {\Omega^-}_j \cup {\Omega^+}_j$.

LEMMA 4.1. *For any set A of real numbers, setting $A^- := A \cap (-\infty, 0)$ and $A^+ := A \cap [0, \infty)$ we have*

$$D^+(A) = \max\{D^+(A^-), D^+(A^+)\}.$$

PROOF. The easy inequality $\ge$ follows from (1.3). Setting

$$M := \max\{D^+(A^-), D^+(A^+)\}$$

for brevity, for the converse inequality it is sufficient to show that

$$\limsup_{r,s,\ge 0, r+s\to\infty} \frac{\operatorname{Card}(A \cap [-r, s])}{r + s} \le M.$$

The case $M = \infty$ is obvious. Assume henceforth that $M < \infty$. For any fixed $\varepsilon > 0$ we may fix two positive numbers $r_\varepsilon, s_\varepsilon$ satisfying

$$\operatorname{Card}(A^- \cap [-r, 0]) \le (D^+(A^-) + \varepsilon) r \quad \text{for all} \quad r \ge r_\varepsilon$$

and

$$\operatorname{Card}(A^+ \cap [0, s]) \le (D^+(A^+) + \varepsilon) s \quad \text{for all} \quad s \ge s_\varepsilon.$$

Adding the two inequalities and setting $K := \operatorname{Card}(A \cap [-r_\varepsilon, s_\varepsilon])$ it follows that

$$\operatorname{Card}(A \cap [-r, s]) \le (M + \varepsilon)(r + s) + K$$

for all $r, s \ge 0$. Dividing by $r+s$ and letting $r+s \to \infty$ we conclude that

$$\limsup_{r,s,\ge 0, r+s\to\infty} \frac{\mathrm{Card}(A \cap [-r,s])}{r+s} \le M + \varepsilon$$

for all $\varepsilon > 0$, and the lemma follows. ∎

Henceforth we assume that Ω is bounded from below. If $D^+ = 0$, then every partition of Ω has the required property because all upper densities are equal to zero. Henceforth we assume also that $0 < D^+ < \infty$; then Ω is an unbounded set and we may enumerate the elements of Ω into a strictly increasing infinite sequence $\omega_1 < \omega_2 < \cdots$. Finally, by choosing $\Omega_j = \emptyset$ whenever $\alpha_j = 0$ we may assume without loss of generality that $0 < \alpha_j \le 1$ for all j.

In order to explain the idea of the proof first we consider the special case of a finite sequence $\alpha_1, \ldots, \alpha_d$ consisting of rational numbers. We fix a positive integer N such that $N\alpha_1, \ldots, N\alpha_d$ are all integers, and we represent Ω as the union of the disjoint blocks

(4.1) $B_n := \{\omega_k \in \Omega \ : \ k = nN+1, nN+2, \ldots, (n+1)N\}, \quad n = 0, 1, \ldots.$

Notice that each B_n has N elements. Therefore, since $N\alpha_1 + \cdots + N\alpha_d = N$, we may define a partition $\Omega = \Omega_1 \cup \cdots \cup \Omega_d$ of Ω such that

(4.2) $$\mathrm{Card}(B_n \cap \Omega_j) = N\alpha_j \quad \text{for all } n \text{ and } j.$$

We claim that ${D^+}_j \le \alpha_j D^+$ for each j. This is obvious if ${D^+}_j = 0$. If ${D^+}_j > 0$ for some j, then we choose a sequence of bounded intervals (I_m^j) satisfying $|I_m^j| \to \infty$ and

$$\frac{\mathrm{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} \to {D^+}_j.$$

Since ${D^+}_j > 0$, hence $\mathrm{Card}(\Omega_j \cap I_m^j) \to \infty$ and therefore $k_m \to \infty$ where k_m denotes the number of (consecutive) blocks B_n contained in I_m^j. Since $I_m^j \cap \Omega$ is contained in the union of at most $k_m + 2$ blocks B_n, using (4.2) it follows that

$$\begin{aligned}\frac{\operatorname{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} \le \frac{2N + k_m N\alpha_j}{|I_m^j|} &\le \\ \le \frac{2N + k_m N\alpha_j}{|I_m^j|} \cdot \frac{\operatorname{Card}(\Omega \cap I_m^j)}{k_m N} &\le \\ \le \frac{2 + k_m\alpha_j}{k_m} \cdot \frac{n^+(\Omega, |I_m^j|)}{|I_m^j|}&\end{aligned}$$

for every m. Letting $m \to \infty$ we conclude that ${D^+}_j \le \alpha_j D^+$.

In fact ${D^+}_j = \alpha_j D^+$ for all j. Indeed, if the inequalities ${D^+}_j \le \alpha_j D^+$ were not all equalities, then using (1.3) we would obtain a contradiction:

$$D^+ \le {D^+}_1 + \cdots + {D^+}_d < \alpha_1 D^+ + \cdots + \alpha_d D^+ = D^+.$$

Now we turn to the general case. We write $J = \{1, \dots, d\}$ in the finite case $\alpha_1 + \cdots + \alpha_d = 1$ and $J = \{1, 2, \dots\}$ in the infinite case.

Let

$$(k, j) : \{1, 2, \dots\} \to \{1, 2, \dots\} \times J$$

be the lexicographically strictly increasing enumeration of the pairs

$$(k, j) \in \{1, 2, \dots\} \times J \quad \text{satisfying} \quad [k\alpha_j] > [(k-1)\alpha_j],$$

where $[x]$ denotes the integer part of x. This enumeration is possible because for each fixed k only finitely many indices $j \in J$ may satisfy this inequality. Indeed, by the convergence of the series $\sum_{j \in J} \alpha_j$ we have $k\alpha_j < 1$ and thus $[k\alpha_j] = [(k-1)\alpha_j] = 0$ for all sufficiently large indices j.

Observe that

$$[k\alpha_j] - [(k-1)\alpha_j] = \begin{cases} 1 & \text{if } (k, j) \text{ belongs to the sequence;} \\ 0 & \text{otherwise.} \end{cases}$$

For each $j \in J$ we set

$$\Omega_j := \{\omega_s \ : \ j(s) = j\}.$$

We claim that ${D^+}_j = \alpha_j D^+$ for all $j \in J$.

First we prove that ${D^+}_j \le \alpha_j D^+$ for each fixed $j \in J$. The case of ${D^+}_j = 0$ is obvious. Let ${D^+}_j > 0$ and choose a sequence (I_m^j) of bounded intervals such that

$$|I_m^j| \to \infty \quad \text{and} \quad \frac{\operatorname{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} \to {D^+}_j.$$

Writing $\Omega \cap I_m^j = \{\omega_{s_m}, \ldots, \omega_{t_m}\}$ we have

$$\ell_m := k(t_m) - k(s_m) \geq \mathrm{Card}(\Omega_j \cap I_m^j) \to \infty;$$

the first inequality follows from the definition of Ω_j, while the second follows from our assumptions $D^+{}_j > 0$ and $|I_m^j| \to \infty$.

Now we have

$$\begin{aligned}\mathrm{Card}(\Omega_j \cap I_m^j) \leq \sum_{k=k(s_m)}^{k(t_m)} ([k\alpha_j] - [(k-1)\alpha_j]) &= \\ = [k(t_m)\alpha_j] - [(k(s_m)-1)\alpha_j] &\leq \\ \leq \ell_m \alpha_j + 1 &\end{aligned}$$

by the definition of Ω_j. Furthermore, we have

$$\begin{aligned}\mathrm{Card}(\Omega \cap I_m^j) &\geq \sum_{n \leq \sqrt{\ell_m}} \sum_{k=k(s_m)+1}^{k(t_m)-1} ([k\alpha_n] - [(k-1)\alpha_n]) = \\ &= \sum_{n \leq \sqrt{\ell_m}} ([(k(t_m)-1)\alpha_n] - [k(s_m)\alpha_n]) \geq \\ &\geq \sum_{n \leq \sqrt{\ell_m}} (\ell_m \alpha_n - 2) \geq \\ &\geq \ell_m \Big(\sum_{n \leq \sqrt{\ell_m}} \alpha_n \Big) - 2\sqrt{\ell_m}.\end{aligned}$$

Since $\ell_m \to \infty$ and $\sum \alpha_n = 1$, it follows from the above two estimates that

$$\mathrm{Card}(\Omega_j \cap I_m^j) \leq \ell_m \left(\alpha_j + o(1)\right)$$

and

$$\mathrm{Card}(\Omega \cap I_m^j) \geq \ell_m \,(1 - o(1))$$

as $m \to \infty$. Hence

$$\frac{\operatorname{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} \le \frac{\ell_m \left(\alpha_j + o(1)\right)}{|I_m^j|} \le$$

$$\le \frac{\ell_m \left(\alpha_j + o(1)\right)}{|I_m^j|} \cdot \frac{\operatorname{Card}(\Omega \cap I_m^j)}{\ell_m (1 - o(1))} =$$

$$= \frac{\alpha_j + o(1)}{1 - o(1)} \cdot \frac{\operatorname{Card}(\Omega \cap I_m^j)}{|I_m^j|}.$$

Letting $m \to \infty$ we conclude that

$$D^+{}_j \le \alpha_j \limsup \frac{\operatorname{Card}(\Omega \cap I_m^j)}{|I_m^j|} \le \alpha_j D^+.$$

It remains to show that none of the inequalities $D^+{}_j \le \alpha_j D^+$ is strict. However, in this case using (1.3) we would obtain the contradiction

$$D^+ \le D^+{}_1 + D^+{}_2 + \cdots < \alpha_1 D^+ + \alpha_2 D^+ + \cdots = D^+.$$

5. Proof of Corollary 1.3

Fix $1/d \le \alpha \le 1$ arbitrarily and then choose $\alpha_1, \dots, \alpha_d \ge 0$ such that

$$\alpha_1 + \cdots + \alpha_d = 1 \quad \text{and} \quad \max\{\alpha_1, \dots, \alpha_d\} = \alpha.$$

Applying the already proved part of Theorem 1.2 we obtain a partition $\Omega = \Omega_1 \cup \cdots \cup \Omega_d$ of Ω such that $D^+(\Omega_j) = \alpha_j D^+$ for all j. Fix an orthonormal basis $E_1, \dots, E_d$ of H and set $U_k = E_j$ if $\omega_k \in \Omega_j$. Then using the identity

$$\int_I \Big\| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \Big\|_H^2 dt = \sum_{j=1}^d \int_I \Big| \sum_{\omega_k \in \Omega_j} x_k e^{i\omega_k t} \Big|^2 dt$$

and applying the scalar case of the theorem we conclude that the estimates (1.2) hold if $|I| > 2\pi\alpha D^+$, and they do not hold if $|I| < 2\pi\alpha D^+$.

6. Proof of Theorem 1.2 for $\alpha_1+\alpha_2+\cdots>1$

By the same reasoning as at the beginning of Section 6, we may assume that $0<D^+<\infty$, $0<\alpha_j\le 1$ for all j, and that the elements of Ω form a strictly increasing infinite sequence $\omega_1<\omega_2<\cdots$.

First we choose a positive integer N such that

$$\sum_j \big([(n+1)N\alpha_j]-[nN\alpha_j]\big)\ge N \quad \text{for all} \quad n=0,1,\dots. \tag{6.1}$$

For this first we choose a positive integer k satisfying $\alpha_1+\cdots+\alpha_k>1$, and then a positive integer N such that

$$\frac{k}{N}<\alpha_1+\cdots+\alpha_k-1.$$

Then using the inequality $[x+y]\ge[x]+[y]$ we obtain the following estimate for all $n=0,1,\dots$:

$$\begin{aligned}\frac{1}{N}\sum_j\big([(n+1)N\alpha_j]-[nN\alpha_j]\big)&\ge\frac{1}{N}\sum_{j=1}^k[N\alpha_j]>\\&>\frac{1}{N}\sum_{j=1}^k(N\alpha_j-1)=\\&=\alpha_1+\cdots+\alpha_k-\frac{k}{N}>\\&>1.\end{aligned}$$

Note that

$$0\le[(n+1)N\alpha_j]-[nN\alpha_j]\le N \tag{6.2}$$

for all n and j because using the condition $\alpha_j\le1$ we have

$$[(n+1)N\alpha_j]=[nN\alpha_j+N\alpha_j]\le[nN\alpha_j+N]=[nN\alpha_j]+N.$$

Next we represent Ω again as the union of the disjoint blocks B_n of N elements as in (4.1). Since the upper density of Ω does not change if we remove a finite number of initial terms, we may define by recurrence a sequence of bounded intervals (I_m) having the following four properties:

$\sup I_m<\inf I_{m+1}$ for all m;

no block B_n belongs to more than one interval I_m;

$$|I_m| \to \infty;$$
$$\frac{\mathrm{Card}(\Omega \cap I_m)}{|I_m|} \to D^+. \tag{6.3}$$

Let us also introduce a sequence of positive integers containing each index j infinitely many times, for example

$$(b_m) := 1 \ldots d \ 1 \ldots d \ 1 \ldots d \ \ldots$$

in case of a finite sequence $\alpha_1, \ldots, \alpha_d$, and

$$(b_m) := 12 \ 123 \ 1234 \ 12345 \ 12 \ldots$$

in case of an infinite sequence $\alpha_1, \alpha_2, \ldots$.

Now, thanks to (6.1) and (6.2) we may define a partition $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots$ having the following properties, where we denote by B'_m the union of the k_m consecutive blocks $B_{n_m+1}, \ldots, B_{n_m+k_m}$ contained in I_m:

(6.4) $\mathrm{Card}(B_n \cap \Omega_j) \le [(n+1)N\alpha_j] - [nN\alpha_j]$ for all n and j;

(6.5) $\mathrm{Card}(B_n \cap \Omega_j) = [(n+1)N\alpha_j] - [nN\alpha_j]$ if $B_n \subset B'_m$ and $j = b_m$.

We claim that $D^+{}_j \le \alpha_j D^+$ for each j. The proof is similar to that in part (a). The case $D^+{}_j = 0$ is obvious. If $D^+{}_j > 0$ for some j, then let us choose a sequence of bounded intervals (I^j_m) satisfying $|I^j_m| \to \infty$ and

$$\frac{\mathrm{Card}(\Omega_j \cap I^j_m)}{|I^j_m|} \to D^+{}_j.$$

Since $D^+{}_j > 0$, hence $\mathrm{Card}(\Omega_j \cap I^j_m) \to \infty$ and therefore $k_m \to \infty$. Since $I^j_m \cap \Omega$ is contained in the union of the $k_m + 2$ blocks $B_{n_m}, \ldots, B_{n_m+k_m+1}$, using (6.4) and the inequality

$$[x+y] - [x] \le [y] + 1 \le y + 1 \le y + N$$

it follows that

$$\frac{\operatorname{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} \leq \frac{1}{|I_m^j|}\left(2N + \sum_{n=n_m+1}^{n_m+k_m} [(n+1)N\alpha_j] - [nN\alpha_j]\right) =$$

$$= \frac{2N + [(n_m + k_m + 1)N\alpha_j] - [(n_m+1)N\alpha_j]}{|I_m^j|} \leq$$

$$\leq \frac{3N + k_m N\alpha_j}{|I_m^j|} \leq$$

$$\leq \frac{3N + k_m N\alpha_j}{|I_m^j|} \cdot \frac{\operatorname{Card}(\Omega \cap I_m^j)}{k_m N} \leq$$

$$\leq \frac{3 + k_m\alpha_j}{k_m} \cdot \frac{n^+(\Omega, |I_m^j|)}{|I_m^j|}$$

for every m. Letting $m \to \infty$ we conclude that $D^+{}_j \leq \alpha_j D^+$.

We claim that $D^+{}_j \geq \alpha_j D^+$ and thus $D^+{}_j = \alpha_j D^+$ for each j. Indeed, for each fixed j there exist arbitrarily long intervals I_m for which $b_m = j$. For these intervals, using (6.5) and the inequality

$$[x+y] - [x] \geq [y] > y - 1 \geq y - N$$

we obtain

$$\operatorname{Card}(\Omega_j \cap I_m) \geq \operatorname{Card}(B'_m \cap \Omega_j) =$$

$$= \sum_{n=n_m+1}^{n_m+k_m} [(n+1)N\alpha_j] - [nN\alpha_j] =$$

$$= [(n_m + k_m + 1)N\alpha_j] - [(n_m+1)N\alpha_j] \geq$$

$$\geq k_m N\alpha_j - N$$

and therefore

$$\frac{\operatorname{Card}(\Omega_j \cap I_m)}{|I_m|} \geq \frac{k_m N\alpha_j - N}{|I_m|} \geq$$

$$\geq \frac{k_m N\alpha_j - N}{|I_m|} \cdot \frac{\operatorname{Card}(\Omega \cap I_m)}{(k_m+2)N} =$$

$$= \frac{k_m\alpha_j - 1}{k_m + 2} \cdot \frac{\operatorname{Card}(\Omega \cap I_m)}{|I_m|}.$$

It follows that

$$\frac{n^{+}{}_{j}(\Omega_j, |I_m|)}{|I_m|} \geq \frac{k_m\alpha_j - 1}{k_m + 2} \cdot \frac{\text{Card}(\Omega \cap I_m)}{|I_m|}.$$

Since $D^+ > 0$ and $|I_m| \to \infty$, by (6.3) we have $\text{Card}(\Omega \cap I_m) \to \infty$ and therefore $k_m \to \infty$. Therefore, letting $|I_m| \to \infty$ we conclude that $D^{+}{}_{j} \geq \alpha_j D^+$.

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