Semi-Lagrangian simulations of the diocotron instability

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Abstract

We consider a guiding center simulation on an annulus, following [2, 3]. There, the PIC method was used. We propose here to revisit this test case by using a classical semi-Lagrangian approach [4]. First, we obtain the conservation of the electric energy and mass for some adapted boundary conditions. Then we recall the dispersion relation from [1] and discussions on different boundary conditions are detailed. Finally, the semi-Lagrangian code is validated in the linear phase against analytical growth rates given by the dispersion relation. Also we have validated numerically the conservation of electric energy and mass. Numerical issues/difficulties due to the change of geometry can be tackled in such a test case which thus may be viewed as a first intermediate step between a classical guiding center simulation in a 2D cartesian mesh and a slab 4D drift kinetic simulation.

1   The guiding center model

We consider the guiding center model in polar coordinates for $\rho = \rho(t, r, \theta)$

$$\partial_t \rho - \frac{\partial_\theta \Phi}{r} \partial_r \rho + \frac{\partial_r \Phi}{r} \partial_\theta \rho = 0, \tag{1.1}$$

coupled with the Poisson equation for the potential $\Phi = \Phi(t, r, \theta)$

$$-\partial_r^2 \Phi - \frac{1}{r} \partial_r \Phi - \frac{1}{r^2} \partial_\theta^2 \Phi = \gamma \rho,$$

where γ is a number which is fixed to either 1 or -1. Generally, we take $\gamma = 1$ (see [1]); taking $\gamma = -1$ can be interpreted as looking for solutions backward in time. The domain is

$$(r, \theta) \in \Omega = [r_{\min}, r_{\max}] \times [0, 2\pi].$$

The initial and boundary conditions will be discussed later on. We assume for the moment that we have periodic boundary conditions in θ.

Proposition 1.1. *We define the electric energy*

$$\mathcal{E}(t) = \int_\Omega r |\partial_r \Phi|^2 + \frac{1}{r} |\partial_\theta \Phi|^2 dr d\theta, \tag{1.2}$$

1

and the mass

$$\mathcal{M}(t) = \int_\Omega r\rho \, dr d\theta. \tag{1.3}$$

We have

(i)

$$\mathcal{E}(t) = \int_0^{2\pi} [r\Phi\partial_r\Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta + \gamma \int_\Omega r\rho\Phi \, dr d\theta.$$

(ii)

$$\partial_t\mathcal{E}(t) = 2\int_0^{2\pi} [r\Phi\partial_t\partial_r\Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta + 2\gamma \int_0^{2\pi} [\Phi\rho\partial_\theta\Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta.$$

(iii)

$$\partial_t\mathcal{M}(t) = \int_0^{2\pi} [\Phi\partial_\theta\rho]_{r=r_{\min}}^{r=r_{\max}} d\theta$$

Proof. Note that the Poisson equation writes

$$-\partial_r(r\partial_r\Phi) - \frac{1}{r}\partial_\theta^2\Phi = \gamma r\rho. \tag{1.4}$$

We have, by integrating by parts the electric energy (1.2), in r for the first term and in θ for the second term,

$$\mathcal{E}(t) = \int_0^{2\pi} [r\Phi\partial_r\Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta - \int_\Omega \Phi\partial_r(r\partial_r\Phi) dr d\theta - \int_\Omega \frac{1}{r}\Phi \, \partial_\theta^2\Phi \, dr d\theta,$$

and get (i), by using the Poisson equation (1.4).
We then have

$$\partial_t\mathcal{E}(t) = I_1 + \gamma(I_2 + I_3), \tag{1.5}$$

with

$$I_1 = \int_0^{2\pi} [r\partial_t(\Phi\partial_r\Phi)]_{r=r_{\min}}^{r=r_{\max}} d\theta, \ I_2 = \int_\Omega \Phi\partial_t\rho \, r dr d\theta, \ I_3 = \int_\Omega \rho\partial_t\Phi \, r dr d\theta.$$

By using the guiding center equation (1.1) and integrating as above (in r for the first term and in θ for the second term), we compute

$$\begin{aligned}
I_2 &= \int_\Omega \Phi(\partial_\theta\Phi\partial_r\rho - \partial_r\Phi\partial_\theta\rho) dr d\theta \\
&= \int_0^{2\pi} [\Phi\rho\partial_\theta\Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta - \int_\Omega \partial_r(\Phi\partial_\theta\Phi)\rho \, dr d\theta + \int_\Omega \partial_\theta(\Phi\partial_r\Phi)\rho \, dr d\theta \\
&= \int_0^{2\pi} [\Phi\rho\partial_\theta\Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta,
\end{aligned}$$

since we have the relation $\partial_r(\Phi\partial_\theta\Phi) = \partial_\theta(\Phi\partial_r\Phi)$.
Using the Poisson equation (1.4), and integrating as usually, we have

$$\gamma I_3 = -\int_\Omega \partial_t\Phi\left(\partial_r(r\partial_r\Phi) + \frac{1}{r}\partial_\theta^2\Phi\right) dr d\theta = -\int_0^{2\pi} [r\partial_r\Phi\partial_t\Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta + I_4,$$

where

$$I_4 = \int_\Omega \partial_r \Phi \partial_t (r \partial_r \Phi) dr d\theta + \int_\Omega \partial_t \partial_\theta \Phi \frac{1}{r} \partial_\theta \Phi dr d\theta.$$

Integrating as usually and using then the Poisson equation (1.4), we get

$$\begin{aligned}
I_4 &= \int_0^{2\pi} [\Phi \partial_t (r \partial_r \Phi)]_{r=r_{\min}}^{r=r_{\max}} d\theta - \int_\Omega \Phi \partial_t \partial_r (r \partial_r \Phi) dr d\theta - \int_\Omega \Phi \partial_t \partial_\theta^2 \Phi \frac{1}{r} dr d\theta \\
&= \int_0^{2\pi} [\Phi \partial_t (r \partial_r \Phi)]_{r=r_{\min}}^{r=r_{\max}} d\theta + \gamma I_2.
\end{aligned}$$

Replacing the formulae for I_1, I_2, I_3 in (1.5) gives

$$\begin{aligned}
\partial_t \mathcal{E}(t) &= \int_0^{2\pi} [r \partial_t (\Phi \partial_r \Phi)]_{r=r_{\min}}^{r=r_{\max}} d\theta + 2\gamma \int_0^{2\pi} [\Phi \rho \partial_\theta \Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta \\
&\quad - \int_0^{2\pi} [r \partial_r \Phi \partial_t \Phi]_{r=r_{\min}}^{r=r_{\max}} d\theta + \int_0^{2\pi} [\Phi \partial_t (r \partial_r \Phi)]_{r=r_{\min}}^{r=r_{\max}} d\theta
\end{aligned}$$

which is (ii), using the relation $r \partial_t (\Phi \partial_r \Phi) = r \partial_t \Phi \partial_r \Phi + r \Phi \partial_t \partial_r \Phi$.

It remains to prove the formula (iii) for the mass. We use the guiding center equation (1.1) and integrate as usually

$$\begin{aligned}
\partial_t \mathcal{M}(t) = \int_\Omega \partial_t \rho \, r \, dr d\theta &= \int_\Omega (-\partial_\theta \Phi \partial_r \rho + \partial_r \Phi \partial_\theta \rho) \, dr d\theta \\
&= \int_0^{2\pi} [\Phi \partial_\theta \rho]_{r=r_{\min}}^{r=r_{\max}} d\theta + \int_\Omega (\Phi \partial_\theta \partial_r \rho - \Phi \partial_r \partial_\theta \rho) \, dr d\theta \\
&= \int_0^{2\pi} [\Phi \partial_\theta \rho]_{r=r_{\min}}^{r=r_{\max}} d\theta.
\end{aligned}$$

$\square$

Proposition 1.2. *We suppose Dirichlet boundary conditions at $r_{\min}$ and at $r_{\max}$:*

$$\Phi(t, r_{\min}, \theta) = \Phi(t, r_{\max}, \theta) = 0.$$

Then the electric energy and the mass are constant in time:

$$\partial_t \mathcal{E}(t) = 0, \ \partial_t \mathcal{M}(t) = 0.$$

Proof. We get immediately the result from Proposition 1.1. $\square$

Proposition 1.3. *We suppose the following boundary conditions:*

(i) *Dirichlet boundary condition at $r_{\max}$*

$$\Phi(t, r_{\max}, \theta) = 0$$

(ii) *Inhomogeneous Neumann boundary condition at $r_{\min}$ for the Fourier mode 0 in θ:*

$$\int_0^{2\pi} \partial_r \Phi(t, r_{\min}, \theta) d\theta = Q,$$

where Q is a given constant.

(iii) Dirichlet boundary condition at $r_{\min}$ for the other modes, which reads

$$\Phi(t, r_{\min}, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(t, r_{\min}, \theta') d\theta'$$

Then, the electric energy and the mass are constant in time:

$$\partial_t \mathcal{E}(t) = 0, \ \partial_t \mathcal{M}(t) = 0.$$

Proof. From Proposition 1.2 and using first the Dirichlet boundary condition at $r_{\max}$ (i), we get

$$\partial_t \mathcal{E}(t) = -2 \int_0^{2\pi} \Phi(t, r_{\min}, \theta) \left(r_{\min} \partial_t \partial_r \Phi(t, r_{\min}, \theta) + \gamma \rho(t, r_{\min}, \theta) \partial_\theta \Phi(t, r_{\min}, \theta) \right) d\theta.$$

Note that, by (iii), $\Phi(t, r_{\min}, \theta)$ does not depend on θ and thus, we get $\Phi(t, r_{\min}, \theta) = \Phi(t, r_{\min}, 0)$ and $\partial_\theta \Phi(t, r_{\min}, \theta) = 0$ which leads to

$$\partial_t \mathcal{E}(t) = -2\Phi(t, r_{\min}, 0) \int_0^{2\pi} r_{\min} \partial_t \partial_r \Phi(t, r_{\min}, \theta) d\theta.$$

We thus get, from (ii)

$$\partial_t \mathcal{E}(t) = -2\Phi(t, r_{\min}, 0) r_{\min} \partial_t \int_0^{2\pi} \partial_r \Phi(t, r_{\min}, \theta) d\theta = -2\Phi(t, r_{\min}, 0) r_{\min} \partial_t Q = 0.$$

For the mass, we get similarly

$$\partial_t \mathcal{M}(t) = -\Phi(t, r_{\min}, 0) \int_0^{2\pi} \partial_\theta \rho(t, r_{\min}, \theta) d\theta = 0.$$

$\square$

Remark 1. We do not know whether the electric energy and the mass remain constant in time when we consider the following boundary conditions:

(i) Dirichlet at $r_{\max}$: $\Phi(t, r_{\max}, \theta) = 0$.

(ii) Neumann at $r_{\min}$: $\partial_r \Phi(t, r_{\min}, \theta) = 0$.

If we impose that $\rho(t, r_{\min}, \cdot) = 0$, we can check that we get also conservation of mass and electric energy. In the semi-Lagrangian code, we have not imposed this condition (we have tried such a condition but got worser conservation for such boundary condition on Φ); if the foot of the characteristic is out of the domain at a point (r^*, θ^*), we take the value at $(r_{\min}, \theta^*)$, if $r^* < r_{\min}$ or at $(r_{\max}, \theta^*)$, if $r^* > r_{\max}$.

2 Study of the electrostatic eigenvalue equation and discretization

The eigenvalue equation which permits to obtain the growth rates of instability, has been derived in Davidson [1] (equation (6.28), p 296). We recall this derivation, in our notations. We begin with the following problem

$$\begin{cases} \partial_t \rho - \dfrac{\partial_\theta \Phi}{r} \partial_r \rho + \dfrac{\partial_r \Phi}{r} \partial_\theta \rho &= 0 \\[2mm] -\partial_{r^2}^2 \Phi - \dfrac{\partial_r \Phi}{r} - \dfrac{\partial_{\theta^2}^2 \Phi}{r^2} &= \gamma \rho \end{cases} \tag{2.6}$$

We consider at first an equilibrium $n_0(r)$ that does not depend on θ and corresponding potential $\Phi_0(r)$, which satisfies

$$-\partial_r(r\partial_r \Phi_0(r)) = r n_0(r).$$

As an equilibrium, the couple $(\rho, \Phi) = (n_0, \gamma \Phi_0)$ satisfies (2.6).
Then, we perturb this equilibrium and search an approximate solution of (2.6) in the form

$$\rho(t, r, \theta) \simeq n_0(r) + \varepsilon n_1(t, r, \theta)$$

We set also

$$\Phi(t, r, \theta) \simeq \gamma \Phi_0(r) + \varepsilon \gamma \Phi_1(t, r, \theta)$$

We inject these approximations in (2.6) to get

$$\begin{cases} \gamma \partial_t n_1 - \dfrac{\partial_\theta \Phi_1}{r} \partial_r n_0 + \dfrac{\partial_r \Phi_0}{r} \partial_\theta n_1 &= O(\varepsilon) \\[2mm] -\partial_{r^2}^2 \Phi_1 - \dfrac{\partial_r \Phi_1}{r} - \dfrac{\partial_{\theta^2}^2 \Phi_1}{r^2} &= n_1 + O(\varepsilon) \end{cases}$$

We now look for a particular solution of the linearized problem

$$\begin{cases} \gamma \partial_t n_1 - \dfrac{\partial_\theta \Phi_1}{r} \partial_r n_0 + \dfrac{\partial_r \Phi_0}{r} \partial_\theta n_1 &= 0 \\[2mm] -\partial_{r^2}^2 \Phi_1 - \dfrac{\partial_r \Phi_1}{r} - \dfrac{\partial_{\theta^2}^2 \Phi_1}{r^2} &= n_1 \end{cases}$$

in the form

$$n_1(t, r, \theta) = \widehat{n}_{1,\ell}(r) \exp(i\ell\theta) \exp(-i\omega t/\gamma), \quad \Phi_1(t, r, \theta) = \widehat{\Phi}_{1,\ell}(r) \exp(i\ell\theta) \exp(-i\omega t/\gamma).$$

This type of solution can be obtained through Laplace transform in time and Fourier transform in θ. We look for unstable equilibrium, that means that we want to have a solution with $\Im(\omega)/\gamma > 0$. We obtain

$$\begin{cases} -i\omega \widehat{n}_{1,\ell} - i\ell \dfrac{\partial_r n_0}{r} \widehat{\Phi}_{1,\ell} + i\ell \dfrac{\partial_r \Phi_0}{r} \widehat{n}_{1,\ell} &= 0 \\[2mm] -\partial_{r^2}^2 \widehat{\Phi}_{1,\ell} - \dfrac{\partial_r \widehat{\Phi}_{1,\ell}}{r} + \dfrac{\ell^2}{r^2} \widehat{\Phi}_{1,\ell} = \widehat{n}_{1,\ell} \end{cases}$$

We express $\widehat{n}_{1,\ell}$ in terms of $\widehat{\Phi}_{1,\ell}$ to get an equation on $\widehat{\Phi}_{1,\ell}$:

$$(-i\omega + i\ell\frac{\partial_r \Phi_0}{r})(-\partial^2_{r^2}\widehat{\Phi}_{1,\ell} - \frac{\partial_r\widehat{\Phi}_{1,\ell}}{r} + \frac{\ell^2}{r^2}\widehat{\Phi}_{1,\ell}) = i\ell\frac{\partial_r n_0}{r}\widehat{\Phi}_{1,\ell}, \tag{2.7}$$

where $r_i = r_{\min} + i\Delta r$ and $\Delta r = (r_{\max} - r_{\min})/N$ and $N \in \mathbb{N}^*$.
We can proceed in the numerical resolution of the problem by making the following discretization:

$$\widehat{\Phi}_{1,\ell}(r_i) = \phi_i, \ \partial_r\widehat{\Phi}_{1,\ell}(r_i) = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta r}, \ \partial^2_{r^2}\widehat{\Phi}_{1,\ell}(r_i) = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta r^2} \tag{2.8}$$

The problem can then be written as follows $A\phi = cB\phi$. We then have to find the eigenvalues c of the problem $B^{-1}A\phi = c\phi$, where $c = \frac{\omega}{\ell}$.
We look for an eigenvalue c such that c/γ has the greatest imaginary part which should be strictly positive.

3 Diocotron instability for an annular electron layer

We consider the following initial data

$$\rho(0, r, \theta) = \begin{cases} 0, \ r_{\min} \le r < r^-, \\ 1 + \varepsilon\cos(\ell\theta), \ r^- \le r \le r^+, \\ 0, \ r^+ < r \le r_{\max}, \end{cases}$$

where ε is a small parameter. In that specific case, we can do analytical computations.
The linear analysis is performed in [1].
We thus can find an explicit solution of (2.7):

$$\widehat{\Phi}_{1,\ell}(r) = \begin{cases} \Phi_{1,I}(r), \ r_{\min} \le r < r^-, \\ \Phi_{1,II}(r), \ r^- \le r < r^+, \\ \Phi_{1,III}(r), \ r^+ \le r < r_{\max}, \end{cases}$$

where

$$\Phi_{1,I}(r) = (r^-/r)^\ell(B(r^-)^\ell + C(r^-)^{-\ell})\frac{(r/r_{\min})^{2\ell} + \epsilon}{(r^-/r_{\min})^{2\ell} + \epsilon},$$

$$\Phi_{1,II}(r) = Br^\ell + Cr^{-\ell},$$

$$\Phi_{1,III}(r) = (r^+/r)^\ell(B(r^+)^\ell + C(r^+)^{-\ell})\frac{1 - (r/r_{\max})^{2\ell}}{1 - (r^+/r_{\max})^{2\ell}}.$$

Here, we have considered Dirichlet boundary condition at $r_{\max}$. At $r_{\min}$, we have Dirichlet boundary condition, if $\epsilon = -1$ and Neumann boundary condition if $\epsilon = 1$. The constants B and C are chosen so that (see [1])

$$r^-(\partial_r\Phi_{1,II}(r^-) - \partial_r\Phi_{1,I}(r^-)) + \ell\frac{\Phi_{1,I}(r^-)}{\omega - \ell\omega_m(r^-)} = 0,$$

and

$$r^+(\partial_r\Phi_{1,III}(r^+) - \partial_r\Phi_{1,II}(r^+)) + \ell\frac{\Phi_{1,II}(r^+)}{\omega - \ell\omega_m(r^+)} = 0,$$

with

$$\omega_m(r) = \frac{1}{2}\left(1 - \left(\frac{r^-}{r}\right)^2\right) + \omega_q\left(\frac{r^-}{r}\right)^2,$$

and

$$\omega_q = -2\frac{Q}{(r^-)^2}.$$

This leads to a 2×2 linear system for (B, C). We look for a solution with $(B, C) \neq (0, 0)$. This implies that the determinant of the corresponding matrix is zero. This is the dispersion relation.

In the case of boundary conditions given in Proposition 1.3, the dispersion relation can be explicitly given (see [1])

$$\omega^2 - b_\ell\omega + c_\ell = 0,$$

where

$$b_\ell = \ell(1 - (r^-/r^+)^2 + \omega_q(1 + (r^-/r^+)^2))(1 - (r_{\min}/r_{\max})^{2\ell})$$
$$+ (1 - (r^-/r^+)^{2\ell})\frac{(r^+/r_{\max})^{2\ell} - (r_{\min}/r^-)^{2\ell}}{1 - (r_{\min}/r_{\max})^{2\ell}} \quad (3.9)$$

and

$$c_\ell(1 - (r_{\min}/r_{\max})^{2\ell}) = \ell^2\omega_q(1 - (r^-/r^+)^2 + (\omega_q)(r^-/r^+)^2)(1 - (r_{\min}/r_{\max})^{2\ell})$$
$$- \ell\omega_q(1 - (r_{\min}/r^+)^{2\ell})(1 - (r^+/r_{\max})^{2\ell})$$
$$+ \ell(1 - (r^-/r^+)^2 + \omega_q(r^-/r^+)^2)(1 - (r^-/r_{\max})^{2\ell})(1 - (r_{\min}/r^-)^{2\ell})$$
$$- (1 - (r^+/r_{\max})^{2\ell})(1 - (r_{\min}/r^-)^{2\ell})(1 - (r^-/r^+)^{2\ell}) \quad (3.10)$$

and

$$\omega_q = -4\frac{Q}{(r^-)^2}.$$

The case of the boundary conditions of Proposition 1.2 is obtained by taking

$$Q = \frac{(r^-)^2 - (r^+)^2 + 2(r^-)^2\ln(r_{\max}/r^-) + 2(r^+)^2\ln(r^+/r_{\max})}{8\ln(r_{\min}/r_{\max})},$$

in the dispersion relation.

Remark 2. In [1], boundary conditions of Proposition 1.3 were considered. Here, we can also consider boundary conditions of Proposition 1.2 and Neumann boundary conditions (using $\epsilon = 1$). The computations have been done using Maple.

Remark 3. The system (2.6) for $\gamma = 1$ or $\gamma = -1$ leads to the same growth rate in the linear phase, since the dispersion relation has real coefficients.

Remark 4. By using the linear approximation $\rho = n_0 + \varepsilon n_1$ and $\Phi = \gamma(\Phi_0 + \varepsilon\Phi_1)$, which is supposed to be valid in the linear phase, we obtain for boundary conditions of Remark 1 (Neumann for all the modes at $r_{\min}$)

$$\partial_t \mathcal{E}(t) \simeq -2\gamma^3 \Phi_0(r_{\min})\varepsilon^2 \int_0^{2\pi} n_1(t, r_{\min}, \theta)\partial_\theta \Phi_1(t, r_{\min}, \theta)d\theta + O(\varepsilon^3),$$

which leads a priori to a loss of energy conservation for this linear approximation, which may even grow exponentially in time, with a growth rate dictated by the dispersion relation! On the other hand, we can check that the energy remains constant for the linear approximation when using boundary conditions of Propositions 1.2 and 1.3, as we have proven it for the initial non linear problem. This may explain the linear growth rate obtained by numerical approximations for the electric energy in [3].

4   Numerical results

4.1   Analytical and discretized dispersion relation

We obtain the following results in Tables 1, 2, 3 for the analytical dispersion relation.
We take

$$r_{\min} = 1, \ r_{\max} = 10, \ \gamma = 1.$$

The results are coherent with those given in [2].

Remark 5. We have taken the same modes as in [2]. For mode 5 of Table 3, we had to change r^+ in order to get an unstable mode.

We have also tested a discretization of the dispersion relation (2.8) with Dirichlet boundary conditions. The results are shown on Table 4.
The results are coherent with respect to the analytical case.

4.2   Semi-Lagrangian method

We use a classical backward semi-Lagrangian method [4]. A predictor corrector scheme is used as time scheme, together with a symplectic Verlet algorithm for the computation of the characteristics, The interpolation is done with cubic splines. For the Poisson equation, finite difference of order two are used in the radial direction and Fourier in the θ direction. We denote by N_r (resp. N_θ) the number of intervals in the radial (resp. θ) direction. The time step is Δt.
We give the growth rate obtained by the semi-Lagrangian method in Tables 5, 6, 7. We obtain results in agreement with those obtained with the dispersion relation, except for the mode 5, when Neumann boundary conditions are used for the mode 0 (or for all the modes). For this specific case, the obtained value is strongly dependent on the discretization parameters. By taking $N_r = 256$, $N_\theta = 512$ and $\Delta t = 0.025$, we get $\Im(\omega) = 0.051$ on the interval $[123, 241]$, which is closer to the theoretical value. An example of growth rate and of density function is given on Figure 1.
The time evolution of electric energy and of relative mass error is given on Figures 2,3,4,5, for different boundary conditions at $r_{\min}$ (we suppose Dirichlet boundary condition at $r_{\max}$).

We take here $\varepsilon = 0.5$ and $\ell = 3$. We observe convergence of energy and mass conservation for Dirichlet and Neumann mode 0 boundary conditions at $r_{\min}$. On the other hand, for Neumann boundary conditions at $r_{\min}$, these quantities are no more conserved in long time. We had to take a quite large amplitude of ε to see this phenomenon.

5 Conclusion

We have studied the conservation of electric energy and mass of the continuous model; we have enlightened adhoc boundary conditions for having such conservations. Then, we have detailed the dispersion relations for such boundary conditions. Finally, we have given some numerical results with a classical semi-Lagrangian method and validated the linear phase and the conservation of mass and electric energy. A comparison on numerical methods (PIC vs. semi-Lagrangian method) with regards to conservative quantities for example, as well as higher dimensional problem with velocity discretization are envisaged.

References

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ℓ	r^-	r^+	ω
2	4	5	$0.377298604860486 + 0.0717643545314240i$
3	4	5	$0.545674435361935 + 0.226724091815516i$
4	4	5	$0.721619095621660 + 0.298891133128847i$
5	7	9	$1.14786977697701 + 0.0578796847506166i$
7	6	7	$0.931570723109550 + 0.330764942142169i$

Table 1: Boundary conditions of Proposition 1.3 with $Q = 0$ (Neumann for mode 0 and Dirichlet for other modes)

ℓ	r^-	r^+	ω
2	4	5	$0.379601164883512 + 0.0673385706065534i$
3	4	5	$0.545854564457936 + 0.226665019518819i$
4	4	5	$0.721631794378213 + 0.298891598934172i$
5	7	9	$1.14786978019830 + 0.0578796993781420i$
7	6	7	$0.931570723120837 + 0.330764942148054i$

Table 2: Boundary conditions of Remark 1 (Neumann for all the modes).

ℓ	r^-	r^+	ω
2	4	5	$0.0582125254158010 + 0.288739227554270i$
3	4	5	$0.0670453161949078 + 0.367315895142460i$
4	4	5	$0.083446936732290 + 0.384081542249742i$
5	7	8	$0.457113044408886 + 0.323578172647371i$
7	6	7	$0.522330265240272 + 0.337573424025866i$

Table 3: Boundary conditions of Proposition 1.2 (Dirichlet for all the modes).

ℓ	r^-	r^+	$\Im(\omega)$, for $N = 256$, $N = 512$, $N = 1024$
2	4	5	0.29293, 0.29093, 0.28985
3	4	5	0.37043, 0.36899, 0.36817
4	4	5	0.38587, 0.38498, 0.38453
5	7	8	0.32706, 0.32539, 0.32450
7	6	7	0.33852, 0.33783, 0.33765

Table 4: Numerical resolution of the dispersion relation with boundary conditions of Proposition 1.2 (Dirichlet for all the modes).

ℓ	r^-	r^+	$\Im(\omega)$	time interval of validity
2	4	5	0.2875	$[26, 55]$
3	4	5	0.3667	$[26, 72]$
4	4	5	0.3852	$[26, 51]$
5	7	8	0.3222	$[22, 62]$
7	6	7	0.3424	$[26, 61]$

Table 5: Growth rate for the semi-Lagrangian method with boundary conditions of Proposition 1.2 (Dirichlet for all the modes). $N_r = N_\theta = 128$, $\Delta t = 0.05$

ℓ	r^-	r^+	$\Im(\omega)$	time interval of validity
2	4	5	0.0719	$[93, 138]$
3	4	5	0.2264	$[33, 87]$
4	4	5	0.2991	$[34, 69]$
5	7	9	0.1002	$[83, 165]$
7	6	7	0.3338	$[35, 53]$

Table 6: Growth rate for the semi-Lagrangian method with boundary conditions of Proposition 1.3 (Neumann for mode 0 and Dirichlet for the other modes). $N_r = N_\theta = 128$, $\Delta t = 0.05$

ℓ	r^-	r^+	$\Im(\omega)$	time interval of validity
2	4	5	0.0672	$[89, 151]$
3	4	5	0.2263	$[32, 89]$
4	4	5	0.2991	$[34, 69]$
7	6	7	0.3338	$[35, 53]$

Table 7: Growth rate for the semi-Lagrangian method with boundary conditions of Proposition 1.3 (Neumann for all the modes). $N_r = N_\theta = 128$, $\Delta t = 0.05$
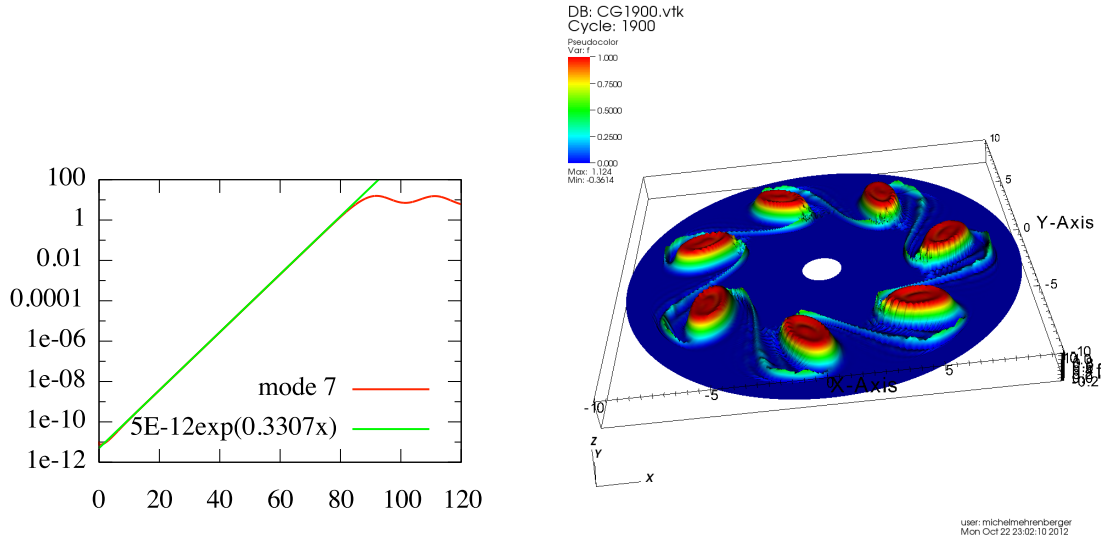
Figure 1: (Left) Square modulus of the 7th Fourier mode of $\int_{r_{\min}}^{r_{\max}} \Phi(t, r, \theta) dr$ vs time t for boundary condition of Proposition 1.3 with $Q = 0$ (Neumann for mode 0 and Dirichlet for other modes). (Right) Density ρ at $t = 95$. Discretization parameters are $N_r = 512$, $N_\theta = 256$ and $\Delta t = 0.05$.
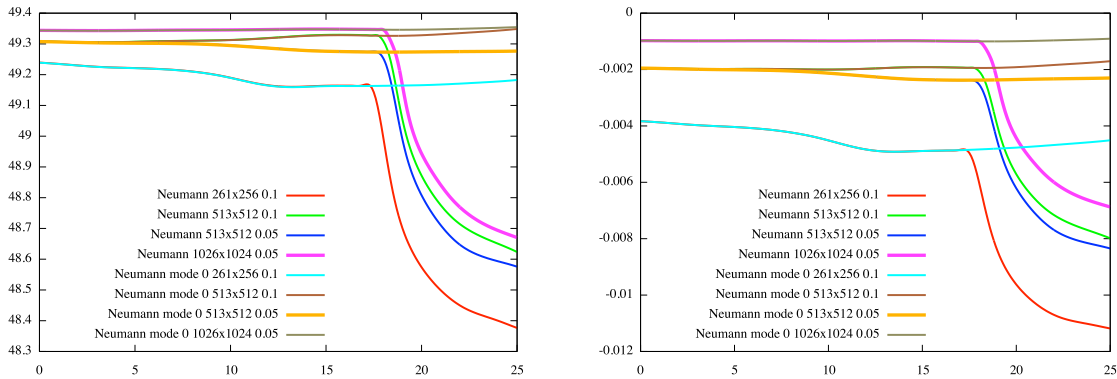

Figure 2: Time evolution of electric energy (left) and relative mass error (right) for Neumann and Neumann mode 0 boundary conditions, with different discretizations ($N_r \times N_\theta$ Δt on legend).
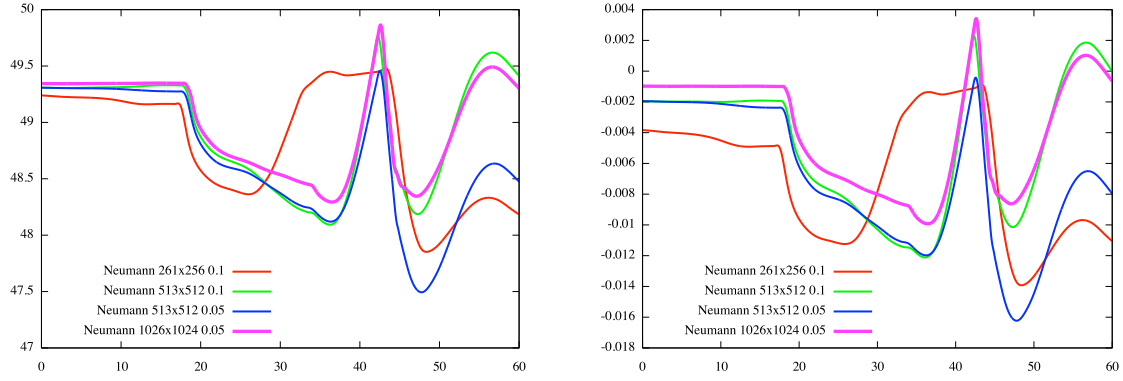
12

Figure 3: Long time evolution of electric energy (left) and relative mass error (right) for Neumann boundary conditions, with different discretizations ($N_r \times N_\theta$ Δt on legend).
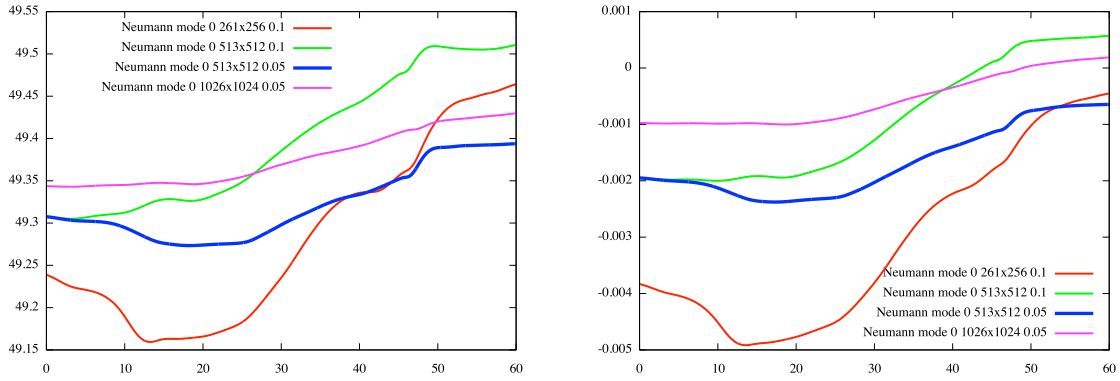

Figure 4: Long time evolution of electric energy (left) and relative mass error (right) for Neumann mode 0 boundary conditions, with different discretizations ($N_r \times N_\theta$ Δt on legend).
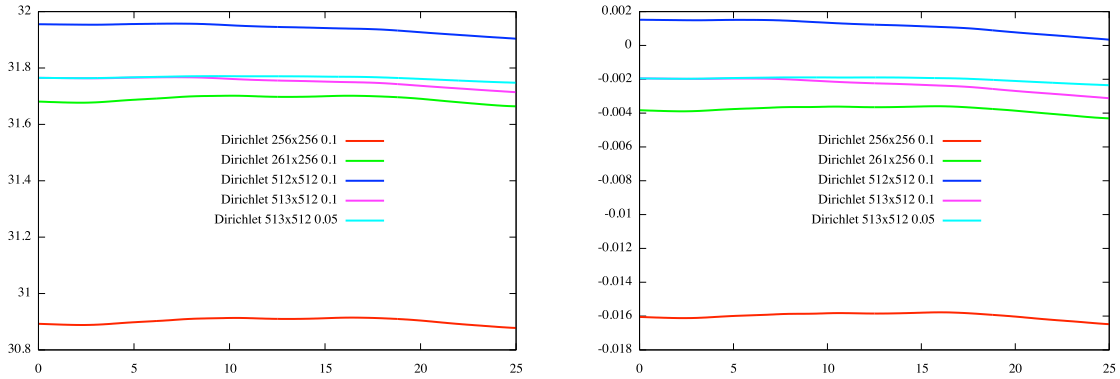

Figure 5: Long time evolution of electric energy (left) and relative mass error (right) for Dirichlet boundary conditions, with different discretizations ($N_r \times N_\theta$ Δt on legend).