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STABILIZATION OF COUPLED SYSTEMS

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Abstract. We characterize the stabilization for some coupled infinite dimensional systems. The proof of the main result uses the methodology introduced in Ammari and Tucsnak [2], where the exponential stability for the closed loop problem is reduced to an observability estimate for the corresponding uncontrolled system combined to a boundedness property of the transfer function of the associated open loop system and a result in [11].

1. Introduction and main results

Let H be a Hilbert space equipped with the norm $\|.\|_H$, and let $A_i : \mathcal{D}(A_i) \to H$, $i = 1,2$, be a self-adjoint, positive and boundedly invertible operator. We introduce the scale of Hilbert spaces $H_{i,\alpha}$, $\alpha \in \mathbb{R}$, $i = 1,2$, as follows: for every $\alpha \geqq 0$, $H_{i,\alpha} = \mathcal{D}(A_i^\alpha)$, $i = 1,2$, with the norm $\|z\|_{i,\alpha} = \|A_i^\alpha z\|_H$, $i = 1,2$. The space $H_{i,-\alpha}$, $i = 1,2$ is defined by duality with respect to the pivot space H as follows: $H_{i,-\alpha} = H_{i,\alpha}^*$, $i = 1,2$, for $\alpha > 0$. The operator A_i, $i = 1,2$ can be extended (or restricted) to each $H_{i,\alpha}$, $i = 1,2$, such that it becomes a bounded operator

$$A_i : H_{i,\alpha} \to H_{i,\alpha-1} \quad i = 1,2, \quad \forall\, \alpha \in \mathbb{R}. \tag{1.1}$$

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The second ingredient needed for our construction is a bounded linear operator $B_1 : U \to H_{1,-\frac{1}{2}}$ and a bounded linear operator $C : H \to H$, where U is another Hilbert space which will be identified with its dual.

The systems we consider are described by

$$\ddot{w}_1(t) + A_1 w_1(t) + B_1 B_1^* \dot{w}_1(t) + C\dot{w}_2 = 0, \quad \ddot{w}_2(t) + A_2 w_2(t) - C\dot{w}_1 = 0, \tag{1.2}$$

$$w_1(0) = w_1^0,\ \dot{w}_1(0) = w_1^1, \quad w_2(0) = w_2^0,\ \dot{w}_2(0) = w_2^1, \tag{1.3}$$

where $t \in [0, \infty)$ is the time. The equation (1.2) is understood as an equation in $H_{1,-\frac{1}{2}}$, i.e., all the terms are in $H_{1,-\frac{1}{2}}$. Most of the coupled linear equations modelling the damped vibrations of elastic structures (see [9]) can be written in the form (1.2), where (w_1, w_2) stands for the displacement field and the term $B_1 B_1^* \dot{w}_1(t)$, represents a viscous feedback damping. The system (1.2)–(1.3) is well-posed. More precisely, the following classical result (see, for instance, Weiss and Tucsnak [13]), holds:

Suppose that $\left(w_1^0, w_1^1, w_2^0, w_2^1\right) \in H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H$. Then the problem (1.2)–(1.3) admits a unique solution

$$(w_1, w_2) \in C\Big([0, \infty); H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}}\Big) \cap C^1\big([0, \infty); H \times H\big)$$

such that $B_1^* w_1(\cdot) \in H^1(0, T; U)$. Moreover (w_1, w_2) satisfies, for all $t \geqq 0$, the energy estimate

$$\begin{aligned} &\left\| (w_1^0, w_1^1, w_2^0, w_2^1) \right\|^2_{H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H} \\ &- \left\| \big(w_1(t), \dot{w}_1(t), w_2(t), \dot{w}_2(t)\big) \right\|^2_{H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H} = 2\int_0^t \left\| B_1^* \dot{w}_1(s) \right\|^2_U ds. \end{aligned} \tag{1.4}$$

In this paper we characterize the stabilization for some coupled infinite dimensional systems, see [1], [9] and [6]. The proof of the main result uses the methodology introduced in Ammari and Tucsnak [2] (see also [5] for the bounded case), where the exponential stability for the closed loop problem is reduced to an observability estimate for the corresponding uncontrolled system combined to a boundedness property of the transfer function of the associated open loop system and a result in [11].

Consider now the initial and boundary value problem

$$\ddot{w}(t) + A_1 w(t) = 0, \tag{1.5}$$

$$w(0) = w^0, \quad \dot{w}(0) = w^1 \tag{1.6}$$

and the unbounded linear operator

(1.7)
$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \to H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H, \quad \mathcal{A}_d = \begin{pmatrix} 0 & I & 0 & 0 \\ -A_1 & -B_1B_1^* & 0 & -C \\ 0 & 0 & 0 & I \\ 0 & C & -A_2 & 0 \end{pmatrix},$$

where

$$\mathcal{D}(\mathcal{A}_d) = \Big\{(u_1, u_2, v_1, v_2) \in H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H,$$
$$A_1u_1 + B_1B_1^*u_2 + Cv_2 \in H,\ u_2 \in H_{1,\frac{1}{2}},\ v_2 \in H_{2,\frac{1}{2}}\Big\}.$$

The main results of this paper are:

THEOREM 1.1. *Assume that A_1, A_2, C satisfy the following conditions:*

- *A_1 and A_2 have compact resolvent,*
- *C is compact,*
- *$\mathcal{A}_d$ has a pseudo-basis* of generalized eigenvectors whose eigenvalues are of finite type (see* [11], *for definition)*
- *for any $\gamma > 0$ we have*

$$\sup_{\operatorname{Re}\lambda=\gamma} \left\| \lambda B_1^*(\lambda^2 I + A_1)^{-1} B_1 \right\|_{\mathcal{L}(U)} < \infty. \tag{1.8}$$

The system described by (1.2)–(1.3) *is exponentially stable in $L \subset H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H = L \oplus M$, where M is finite dimensional, if there exists $T > 0$ such that the solution w of* (1.5)–(1.6) *satisfies*

$$\int_0^T \left\| B_1^*\dot{w}(t) \right\|_U^2 dt \asymp \left\| (w^0, w^1) \right\|^2_{H_{1,\frac{1}{2}} \times H}, \quad \forall\, (w^0, w^1) \in L_1, \tag{1.9}$$

where L_1 is a finite codimensional subspace of $H_{1,\frac{1}{2}} \times H$.

REMARK 1.2. We can have the same result for the weak stability (polynomial stability for example).

The paper is organized as follows. In the second section we give some results in the regularity for some infinite dimensional systems needed of the proof of the main result. Some application is given in Section 3.

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* $(f_k)_{k \geqq 1}$ is a pseudo-basis if $\operatorname{Vect}\{f_k\}$ is dense in $H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H$ and if, for every bounded sequence (x_k) such that $x_k \in \operatorname{Vect}\{f_j : j \geqq k\}$, we have $x_k \rightharpoonup 0$.

2. Stabilizability and regularity of some coupled systems

We consider the initial and boundary value problems

$$\ddot{\phi}_1(t) + A_1\phi_1(t) + C\dot{\phi}_2(t) = 0, \quad \ddot{\phi}_2(t) + A_2\phi_2(t) - C\dot{\phi}_1(t) = 0, \tag{2.10}$$

$$\phi_1(0) = w_1^0, \quad \dot{\phi}_1(0) = w_1^1, \quad \phi_2(0) = w_2^0, \quad \dot{\phi}_2(0) = w_2^1 \tag{2.11}$$

and

$$\ddot{\phi} + A_1\phi + C\dot{\psi} + B_1 g(t) = 0, \quad \ddot{\psi} + A_2\psi - C\dot{\phi} = 0, \tag{2.12}$$

$$\phi(0) = 0, \quad \dot{\phi}(0) = 0, \quad \psi(0) = 0, \quad \dot{\psi}(0) = 0. \tag{2.13}$$

Proposition 2.1. *Let $g \in L^2(0,T;U)$ and assume that the property* (1.8) *is satisfied. Then* (2.12)–(2.13) *admits a unique solution*

$$\big(\phi, \dot{\phi}, \psi, \dot{\psi}\big) \in C(0,T; H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H). \tag{2.14}$$

Moreover $B_1^*\dot{\phi} \in L^2(0,T;U)$ and there exists a constant $C > 0$ such that

$$\big\| B_1^*\dot{\phi}\big\|_{L^2(0,T;U)} \leqq C\|g\|_{L^2(0,T;U)}, \quad \forall\, g \in L^2(0,T;U). \tag{2.15}$$

For proving Proposition 2.1, we should study the conservative system (without dissipation) associated to problem (1.2)–(1.3). We have the following result.

Lemma 2.2. *For all $\big(w_1^0, w_1^1, w_2^0, w_2^1\big) \in H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H$ the system* (2.10)–(2.11) *admits a unique solution $\big(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2\big) \in C\big(0,T; H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H\big)$. Moreover, if we assume that the property* (1.8) *is satisfied then $B_1^*\dot{\phi}_1 \in L^2(0,T;U)$ and there exists a constant $C > 0$ such that*

$$\big\| B_1^*\dot{\phi}_1\big\|^2_{L^2(0,T;U)} \leqq C\big\|\big(w_1^0, w_1^1, w_2^0, w_2^1\big)\big\|^2_{H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H}, \tag{2.16}$$

$$\forall\, \big(w_1^0, w_1^1, w_2^0, w_2^1\big) \in H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H.$$

Proof. By the classical semi group theory, see [12], we prove that for all $\big(w_1^0, w_1^1, w_2^0, w_2^1\big) \in H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H$ the system (2.10)–(2.11) admits a unique solution $\big(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2\big) \in C\big(0,T; H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H\big)$.

Let $\phi_1 = p_1 + p_2$, where p_1 satisfies the following problem:

$$\ddot{p}_1 + A_1 p_1 = 0, \tag{2.17}$$

$$p_1(0) = w_1^0, \quad \dot{p}_1(0) = w_1^1, \tag{2.18}$$

and where p_2 satisfies

$$\ddot{p}_2 + A_1 p_2 + C\dot{\phi}_2 = 0, \quad \ddot{\phi}_2 + A_2\phi_2 - C\dot{\phi}_1 = 0, \tag{2.19}$$

$$p_2(0) = 0, \quad \dot{p}_2(0) = 0, \quad \phi_2(0) = w_2^0(x,y), \quad \dot{\phi}_2(0) = w_2^1. \tag{2.20}$$

For $T > 0$ we have, according to [2], that there exist constants $C_1, C_2 > 0$ such that

$$\left\| B_1^* \dot{p}_1 \right\|^2_{L^2(0,T;U)} \leqq C_1 \left\| \left(w_1^0, w_1^1 \right) \right\|^2_{H_{1,\frac{1}{2}} \times H}, \quad \forall \left(w_1^0, w_1^1 \right) \in H_{1,\frac{1}{2}} \times H,$$

$$\left\| B_1^* \dot{p}_2 \right\|^2_{L^2(0,T;U)} \leqq C_2 \left\| \left(w_1^0, w_1^1, w_2^0, w_2^1 \right) \right\|^2_{H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H}$$

thus,

$$\begin{aligned} \left\| B_1^* \dot{\phi}_1 \right\|^2_{L^2(0,T;U)} &\leqq 2\left(\left\| B_1^* \dot{p}_1 \right\|^2_{L^2(0,T;U)} + \left\| B_1^* \dot{p}_2 \right\|^2_{L^2(0,T;U)} \right) \\ &\leqq 2(C_1 + C_2) \left\| \left(w_1^0, w_1^1, w_2^0, w_2^1 \right) \right\|^2_{H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H}, \\ &\forall \left(w_1^0, w_1^1, w_2^0, w_2^1 \right) \in H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H, \end{aligned}$$

which implies the inequality (2.16). □

PROOF OF PROPOSITION 2.1. For the proof of (2.14) it suffices to use the transposition method (see Theorem 3.1 in [4, p. 173], [10] and [9]).

Now we will prove the trace regularity (2.15). Let $\phi = \varphi_1 + \varphi_2$ where φ_1 satisfies the following problem:

$$\ddot{\varphi}_1 + A_1\varphi_1 + B_1 g = 0, \tag{2.21}$$

$$\varphi_1(0) = 0, \quad \dot{\varphi}_1(0) = 0, \tag{2.22}$$

and φ_2 satisfies:

$$\ddot{\varphi}_2 + A_1\varphi_2 + C\dot{\psi} = 0, \tag{2.23}$$

$$\ddot{\psi} + A_2\psi - C\dot{\phi} = 0, \quad \Omega \times (0, +\infty), \tag{2.24}$$

$$\varphi_2(0) = 0, \quad \dot{\varphi}_2(0) = 0, \tag{2.25}$$

$$\psi(0) = 0, \quad \dot{\psi}(0) = 0. \tag{2.26}$$

For $T > 0$ we have, according to [2], that there exist constants $C_1, C_2 > 0$ such that

$$\| B_1^* \dot\varphi_1 \|^2_{L^2(0,T;U)} \leqq C_1 \|g\|^2_{L^2(0,T;U)}.$$

The energy method implies that

$$\| B_1^* \dot\varphi_2 \|^2_{L^2(0,T;U)} \leqq \frac{1}{4} \| B_1^* \dot\phi \|^2_{L^2(0,T;U)} + C_2 \| g \|^2_{L^2(0,T;U)}$$

thus

$$\| B_1^* \dot\phi \|^2_{L^2(0,T;U)} \leqq 2\big(\| B_1^* \dot\varphi_1 \|^2_{L^2(0,T;U)} + \| B_1^* \dot\varphi_2 \|^2_{L^2(0,T;U)} \big)$$

$$\leqq 2(C_1 + C_2) \|g\|^2_{L^2(0,T;U)} + \frac{1}{2} \| B_1^* \dot\phi \|^2_{L^2(0,T;U)}, \quad \forall\, g \in L^2(0,T;U),$$

which implies the inequality (2.15). □

Assume that the assumption (1.8) is satisfied. Then according to Proposition 2.1, Lemma 2.2 the solution (ϕ_1, ϕ_2) of (2.10)–(2.11) and the solution (ϕ, ψ) of (2.12)–(2.13) satisfy respectively (2.16) and (2.15) which implies according to [2] that the transfer function of the corresponding system is bounded in $\operatorname{Re} \lambda = \gamma$ for all $\gamma > 0$. More precisely we have the following result.

COROLLARY 2.3. *Assume that the assumption* (1.8) *is satisfied. Then we have for any* $\gamma > 0$

$$\sup_{\operatorname{Re}\lambda=\gamma} \big\| \lambda B_1^* \big[I + \lambda^2 (\lambda^2 I + A_1)^{-1} C (\lambda^2 I + A_2)^{-1} C \big]^{-1} \cdot (\lambda^2 I + A_1)^{-1} B_1 \big\|_{\mathcal{L}(U)} < \infty. \tag{2.27}$$

PROOF OF THEOREM 1.1. The results below generalize the results in [2] and in [5] and show that, under a certain regularity assumption, the exponential and polynomial stability of (1.2)–(1.3) is a consequence of some observability inequalities for (2.10)–(2.11). According to Corollary 2.3

$$\| B_1 \dot\phi_1 \|_{L^2(0,T;U)} \asymp \| B_1 \dot w_1 \|_{L^2(0,T;U)}.$$

Now the proof is a simple adaptation of the proof of [2, Theorem 2.2].

LEMMA 2.4. *Assume that for any* $\gamma > 0$ *we have condition* (1.8). *The system described by* (1.2)–(1.3) *is exponentially stable in* $H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H$ *if and only if there exist* $T, C > 0$ *such that*

$$\int_0^T \| B_1^* \dot\phi_1(t) \|^2_U \, dt \geqq C \| (w_1^0, w_1^1, w_2^0, w_2^1) \|^2_{H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H} \tag{2.28}$$

$$\forall\,(w_1^0, w_1^1, w_2^0, w_2^1) \in H_1 \times H_{1,\frac{1}{2}} \times H_2 \times H_{2,\frac{1}{2}}.$$

Assume that A_1, A_2, C satisfy the following conditions:
- A_1 and A_2 have compact resolvent,
- C is compact,
- $\mathcal{A}_d$ has a pseudo-basis of generalized eigenvectors whose eigenvalues are of finite type and condition (1.8) holds.

The system described by (1.2)–(1.3) is exponentially stable in $L \subset H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H = L \oplus M$, where M is finite dimensional, if there exist $T, C > 0$ such that the solution w of (1.5)–(1.6) satisfies

$$\int_0^T \left\| B_1^* \dot{w}(t) \right\|_U^2 dt \asymp \left\| (w^0, w^1) \right\|_{H_{1,\frac{1}{2}} \times H}^2, \quad \forall\,(w^0, w^1) \in L_1, \tag{2.29}$$

where L_1 is a finite codimensional subspace of $H_{1,\frac{1}{2}} \times H$. Then according to [11, Theorem 2.6] there exist $T, C > 0$ such that

$$\int_0^T \left\| B_1^* \dot{\phi}_1(t) \right\|_U^2 dt \geqq C \left\| (w_1^0, w_1^1, w_2^0, w_2^1) \right\|_{H_{1,\frac{1}{2}} \times H \times H_{2,\frac{1}{2}} \times H}^2, \tag{2.30}$$

$$\forall \left(w_1^0, w_1^1, w_2^0, w_2^1 \right) \in L.$$

Which, according to Lemma 2.4, implies the first assertion of Theorem 1.1. □

3. Application to stabilization for some coupled system

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset of boundary Γ. We consider the following system:

(3.31)
$$\begin{cases} \ddot{u}_1(t,x) - \Delta u_1(t,x) - \alpha(\Delta)^{-1}\dot{u}_2(t,x) + a(x)\dot{u}_1(t,x) = 0, \ (t,x) \in \mathbb{R} \times \Omega, \\ \ddot{u}_2(t,x) + \Delta^2 u_2(t,x) + \alpha(\Delta)^{-1}\dot{u}_1(t,x) = 0, \ (t,x) \in \mathbb{R} \times \Omega, \\ u_1(t,x) = 0, \ (t,x) \in \mathbb{R} \times \Gamma, \\ u_2(t,x) = \Delta u_2(t,x) = 0, \ (t,x) \in \mathbb{R} \times \Gamma, \\ u_i(0,x) = u_{i0}(x), \dot{u}_i(0,x) = u_{i1}(x), \ x \in \Omega, \ 1 \leqq i \leqq 2. \end{cases}$$

and the homogeneous counterpart system

$$
(3.32) \qquad \begin{cases} \ddot{u}_1(t,x) - \Delta u_1(t,x) + \alpha(\Delta)^{-1}\dot{u}_2(t,x) = 0, \ (t,x) \in \mathbb{R} \times \Omega, \\ \ddot{u}_2(t,x) + \Delta^2 u_2(t,x) - \alpha(\Delta)^{-1}\dot{u}_1(t,x) = 0, \ (t,x) \in \mathbb{R} \times \Omega, \\ u_1(t,x) = 0, \ (t,x) \in \mathbb{R} \times \Gamma, \\ u_2(t,x) = \Delta u_2(t,x) = 0, \ (t,x) \in \mathbb{R} \times \Gamma, \\ u_i(0,x) = u_{i0}(x), \dot{u}_i(0,x) = u_{i1}(x), \ x \in \Omega, \ 1 \leqq i \leqq 2, \end{cases}
$$

where $a \in L^\infty(\Omega)$, $a \geqq 0$, $(\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$.

We can verify by standard methods that, if $(u_{10}, u_{21}) \in H_0^1(\Omega) \times L^2(\Omega)$, and $(u_{20}, u_{21}) \in H_0^1(\Omega) \times H^{-1}(\Omega)$, then (3.32) has a unique weak solution $u = (u_1, u_2)$ which satisfies:

$$
u_1 \in C\big(\mathbb{R}, H_0^1(\Omega)\big) \cap C^1\big(\mathbb{R}, L^2(\Omega)\big), \quad u_2 \in C\big(\mathbb{R}, H_0^1(\Omega)\big) \cap C^1\big(\mathbb{R}, H^{-1}(\Omega)\big).
$$

Let E_0 be the *initial energy* of the solution defined by

$$
E_0 := \frac{1}{2}\big(\|u_{10}\|^2_{H_0^1(\Omega)} + \|u_{11}\|^2_{L^2(\Omega)} + \|u_{20}\|^2_{H_0^1(\Omega)} + \|u_{21}\|^2_{H^{-1}(\Omega)}\big).
$$

$L^2(\Omega)$ and $H_0^1(\Omega)$ are endowed with the norm:

$$
\|v\|^2_{L^2(\Omega)} = \int_\Omega |v|^2 \, dx, \qquad \|v\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla v|^2 \, dx
$$

and $H^{-1}(\Omega)$ is endowed with the dual norm of $H_0^1(\Omega)$. Denote by H the underlying Hilbert space

$$
H := H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega).
$$

Let I be an interval of $\mathbb{R}$. We look for the internal observability estimates

$$
(3.33) \qquad c_1 E_0 \leqq \int_I \int_{\operatorname{supp} a} |\dot{u}_1|^2 \, dx \, dt \leqq c_2 E_0,
$$

PROPOSITION 3.1. *Suppose that* (3.33) *holds for every solution* (u_1, u_2) *satisfying* (3.32) *with* $\alpha = 0$. *Then, given any other choice of* α, *for all initial data belonging to* H, (3.33) *holds for a solution* u *satisfying* (3.32) *with this particular choice of* α, *this initial data, and interval* J *instead of* I, J *containing the closure of* I *in its interior. As a consequence, the system* (3.31) *is exponentially stable.*

PROOF. We rewrite the problem (3.32) in the form $\dot{y} = (A + B)y$, $y(0) = y_0$ with $y = (u_1, \dot{u}_1, u_2, \dot{u}_2)$ and A corresponding to the case $\alpha = 0$. B then is a compact perturbation of A and A is a skew adjoint operator having a compact resolvent and it generates a group.

Let z_k be an orthonormal basis in $L^2(\Omega)$, satisfying $-\Delta z_k = \gamma_k^2 z_k$ in Ω, $z_k = 0$ on Γ. Since $Z_k := \{\beta \cdot z_k, \beta \in \mathbb{C}^2\}$ is stable by $A + B$, we obtain a Riesz basis of subspaces generated by generalized eigenvectors for $A + B$ and thus we can apply the abstract theorem in [11] with

$$p_1(x) := \left\| \dot{x}_1 \right\|_{L^2(\operatorname{supp} a)},$$

and verify that $M = \{0\}$ (see [11] and [7] for more details). $\square$

COROLLARY 3.2. *For $a \in C^\infty(\bar{\Omega})$, the system* (3.31) *is exponentially stable in the energy space if* $\operatorname{supp} a$ *satisfies the geometrical control condition (G.C.C).*

PROOF. The system (3.31) is exponentially stable in the energy space if the system (3.31) with $\alpha = 0$ is exponentially stable in the energy space and according to [8] if and only if $\operatorname{supp} a$ satisfies the geometrical control condition (G.C.C) (see [3]). $\square$

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