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Study of the nodal feedback stabilization of a string-beams network

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Abstract We consider a stabilization problem for a string-beams network. We prove an exponential decay result. The method used is based on a frequency domain method and combine a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent. Moreover, we give a numerical illustration based on the methodology introduced in Ammari and Tucsnak (ESAIM Control Optim. Calc. Var. 6, 361–386, 2001) where the exponential stability for the closed loop problem is reduced to an observability estimate for the corresponding uncontrolled system combined to a boundedness property of the transfer function of the associated open loop system.

Keywords Numerical stabilization · Feedback stabilization · String-beams network

Mathematics Subject Classification (2000) 35B40 · 93D15 · 93D20 · 93B07 · 65N25

1 Introduction and main results

We study a network of N beams and one string, where $N \geq 1$, see [17] (pages: 80–81, for example) concerning the model. More precisely we consider the following initial

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and boundary value problem:

$$(S)\quad \begin{cases} \partial_t^2 u - \partial_x^2 u = 0, \quad x \in (0,\ell), \qquad \partial_t^2 u_i + \partial_x^4 u_i = 0, \quad x \in (0,\ell_i), \\ \partial_x u(t,0) = \partial_t u(t,0), \qquad \partial_x u_i(t,\ell_i) = 0, \qquad \partial_x^3 u_i(t,\ell_i) = \partial_t u_i(t,\ell_i), \\ \partial_x^2 u_i(t,0) = 0, \\ u(t,\ell) = u_i(t,0), \qquad \sum_{j=1}^N \partial_x^3 u_j(t,0) + \partial_x u(t,\ell) = 0, \\ u(0,x) = u^0(x), \qquad \partial_t u(0,x) = u^1(x), \qquad x \in (0,\ell), \\ u_i(0,x) = u_i^0(x), \qquad \partial_t u_i(0,x) = u_i^1(x), \quad x \in (0,\ell_i), t \in (0,\infty), \end{cases}$$

for all $i = 1, \ldots, N$, where ℓ_i denote the length of the beam number i and ℓ is the length of the string.

In the last years an important literature was devoted to the controllability and stabilizability of string network or beam network, see [1–3, 5, 6, 8–11, 13, 14] and [16]. In this paper, we study a stabilization problem for a string-beams network. By a resolvent method, we prove an exponential stability result of the energy of the system (S). Moreover, we give some numerical study for the stabilization problem, the method used is based an the methodology introduced in Ammari and Tucsnak [7] where the exponential stability for the closed loop problem is reduced to an observability estimate for the corresponding uncontrolled system combined to a boundedness property of the transfer function of the associated open loop system.

The paper is organized as follows. In this section we give precise statements of the main results. Section 2 contains the proof of the main results. Numerical illustration is given in the last section.

We define the energy of $\vec{u} = (u, u_1, \ldots, u_N)$ solution of (S) at instant t by

$$E_S(t) = \frac{1}{2}\int_0^\ell \left(|\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2\right) \mathrm{d}x + \frac{1}{2}\sum_{i=1}^N \int_0^{\ell_i} \left(|\partial_t u_i(t,x)|^2 + \left|\partial_x^2 u_i(t,x)\right|^2\right) \mathrm{d}x. \tag{1.1}$$

Let $\mathcal{H} = [V \times (L^2(0,\ell) \times \prod_{i=1}^N L^2(0,\ell_i))]^t$ and let the topological supplement $\tilde{\mathcal{H}}$ of $Span(\Phi_0)$ in $\mathcal{H}$, i.e.,

$$Span(\Phi_0) \oplus \tilde{\mathcal{H}} = \mathcal{H}, \tag{1.2}$$

where $\Phi_0 = (1, \ldots, 1, 0, \ldots, 0)^t$ and

$$V = \left\{ \vec{\phi} \in H^1(0,\ell) \times \prod_{i=1}^N H^2(0,\ell_i), \partial_x \phi_i(\ell_i) = 0, \phi(\ell) = \phi_j(0), \forall j = 1, \ldots, N \right\}.$$

Then, the wellposedness space for (S) is $\tilde{\mathcal{H}}$, equipped with the inner product

$$\langle(\vec{y},\vec{v});(\vec{z},\vec{w}))\rangle_{\tilde{\mathcal{H}}}=\int_0^\ell\left(\frac{dy}{dx}\overline{\frac{dz}{dx}}+v\overline{w}\right)dx+\sum_{j=1}^N\int_0^{\ell_j}\left(\frac{d^2y_j}{dx^2}\overline{\frac{d^2z_j}{dx^2}}+v_j\overline{w}_j\right)dx. \tag{1.3}$$

Denote

$$\begin{aligned}&\mathcal{D}(\mathcal{A}_S)\\&=\Bigg\{(\vec{u},\vec{v})^t\in\left(\left[V\cap\left(H^2(0,\ell)\times\prod_{i=1}^N H^4(0,\ell_i)\right)\right]\times V\right)^t\cap\tilde{\mathcal{H}},\ \partial_x u(0)=v(0),\\&\qquad\partial_x^3u_i(\ell_i)=v_i(\ell_i),\ \partial_x^2u_i(0)=0,\ i=1,\ldots,N,\ \sum_{j=1}^N\partial_x^3u_j(0)+\partial_x u(\ell)=0\Bigg\}.\end{aligned} \tag{1.4}$$

The corresponding operator $\mathcal{A}_S$ is defined by

$$\mathcal{A}_S(\vec{u},\vec{v})^t=\left(\vec{v},\partial_x^2u,-\partial_x^4u_1,\ldots,-\partial_x^4u_N\right)^t,\quad\forall(\vec{u},\vec{v})^t\in\mathcal{D}(\mathcal{A}_S). \tag{1.5}$$

For the well-posedness of the system (S), It can be seen easily that $\tilde{\mathcal{H}}$ endowed with this inner product, given by (1.3), is a Hilbert space. We show that the operator $(\mathcal{A}_S,\mathcal{D}(\mathcal{A}_S))$ defined by (1.4) and (1.5) generates a contraction semigroup on the Hilbert space $\tilde{\mathcal{H}}$.

We have the following fundamental result.

Theorem 1.1 *The operator $(\mathcal{A}_S,\mathcal{D}(\mathcal{A}_S))$ generates a strongly continuous contraction semigroup $(\mathcal{T}_S(t))_{t\geq 0}$ on $\tilde{\mathcal{H}}$.*

Proof To show the dissipativity of $\mathcal{A}_S$, let $(\vec{u},\vec{v})^t\in\mathcal{D}(\mathcal{A}_S)$. By the definition of $\mathcal{A}_S$ we have

$$\begin{aligned}&\left\langle\mathcal{A}_S(\vec{u},\vec{v})^t,(\vec{u},\vec{v})^t\right\rangle_{\tilde{\mathcal{H}}}\\&=\left\langle\left(\vec{v},\partial_x^2u,-\partial_x^4u_1,\ldots,-\partial_x^4u_N\right)^t,(\vec{u},\vec{v})^t\right\rangle_{\tilde{\mathcal{H}}}\\&=\int_0^\ell\partial_x v\overline{\partial_x u}\,dx+\int_0^\ell\partial_x^2u\overline{v}\,dx+\sum_{j=1}^N\left(\int_0^{\ell_j}\partial_x^2u_j\overline{\partial_x^2v_j}\,dx-\int_0^{\ell_j}\partial_x^4u_j\overline{v}_j\,dx\right)\\&=-\int_0^\ell v\overline{\partial_x^2u}\,dx+\int_0^\ell\partial_x^2u\overline{v}\,dx+\sum_{j=1}^N\left(\int_0^{\ell_j}\partial_x^4u_j\overline{v_j}\,dx-\int_0^{\ell_j}\partial_x^4u_j\overline{v}_j\,dx\right)\\&\quad+v(\ell)\overline{\partial_x u(\ell)}-v(0)\overline{\partial_x u(0)}\end{aligned}$$

$$+\sum_{j=1}^{N}\big(\partial_x^2 u_j(\ell_j)\overline{\partial_x v_j(\ell_j)} - \partial_x^2 u_j(0)\overline{\partial_x v_j(0)} - \partial_x^3 u_j(\ell_j)\overline{v_j(\ell_j)}$$
$$+\partial_x^3 u_j(0)\overline{v_j(0)}\big).$$

Since $\partial_x^2 u_j(0) = 0, \partial_x v_j(\ell_j) = 0, v_j(0) = v(\ell), \partial_x^3 u_j(\ell_j) = v(\ell_j), j = 1, \dots, N$, $\partial_x u(0) = v(0)$ and $\partial_x u(\ell) = -\sum_{j=1}^{N}\partial_x^3 u_j(0)$, we get

$$\Re\langle \mathcal{A}_S(\vec{u},\vec{v})^t, (\vec{u},\vec{v})^t\rangle_{\widetilde{\mathcal{H}}} = -|v(0)|^2 - \sum_{j=1}^{N}|v_j(\ell_j)|^2,$$

which shows that $\mathcal{A}_S$ is dissipative.

Next, we show that $(\lambda I - \mathcal{A}_S)$ is surjective for some $\lambda > 0$.

Given a vector $(\vec{f},\vec{g})^t \in \widetilde{\mathcal{H}}$, we look for $(\vec{u},\vec{v})^t \in \mathcal{D}(\mathcal{A}_S)$ such that

$$(\lambda I - \mathcal{A}_S)(\vec{u},\vec{v})^t = \left(\vec{f},\vec{g}\right)^t. \tag{1.6}$$

Hence by the definition of $\mathcal{A}_S$, we obtain

$$\begin{cases} \lambda u - v = f,\\ \lambda u_j - v_j = f_j,\\ \lambda v - \partial_x^2 u = g,\\ \lambda v_j + \partial_x^4 u_j = g_j, \quad j = 1,\dots,N. \end{cases}$$

This is clearly equivalent to

$$\begin{cases} v = \lambda u - f,\\ v_j = \lambda u_j - f_j,\\ \lambda^2 u - \partial_x^2 u = g + \lambda f,\\ \lambda^2 u_j + \partial_x^4 u_j = g_j + \lambda f_j, \quad j = 1,\dots,N. \end{cases}$$

Multiplying the third and the fourth identities by $\overline{\vec{w}} \in V$, and summing we get

$$\int_0^\ell (\lambda^2 u - \partial_x^2 u)\overline{w}\,dx + \sum_{j=1}^{N}\int_0^{\ell_j}(\lambda^2 u_j + \partial_x^4 u_j)\overline{w}_j\,dx$$
$$= \int_0^\ell (g + \lambda f)\overline{w}\,dx + \sum_{j=1}^{N}\int_0^{\ell_j}(g_j + \lambda f_j)\overline{w}_j\,dx, \quad \forall \vec{w} \in V.$$

By formal integrations by parts, the left hand side is equal to

$$\int_0^\ell (\lambda^2 u\overline{w} + \partial_x u\overline{\partial_x w})dx + \sum_{j=1}^{N}\int_0^{\ell_j}(\lambda^2 u_j\overline{w}_j + \partial_x^2 u_j\partial_x^2\overline{w}_j)\,dx$$

$$- \partial_x u(\ell)\overline{w(\ell)} + \partial_x u(0)\overline{w(0)}$$
$$+ \sum_{j=1}^{N} \big(\partial_x^3 u_j(\ell_j)\overline{w_j(\ell_j)} - \partial_x^3 u_j(0)\overline{w_j(0)} - \partial_x^2 u_j(\ell_j)\overline{\partial_x w_j(\ell_j)}$$
$$+ \partial_x^2 u_j(0)\overline{\partial_x w_j(0)}\big).$$

We thus find that

$$\Lambda(\vec{u}, \vec{w}) = F(\vec{w}), \quad \forall \vec{w} \in V, \tag{1.7}$$

where

$$\Lambda(\vec{u}, \vec{w}) = \int_0^\ell \left(\partial_x u \overline{\partial_x w} + \lambda^2 u \overline{w}\right)(x)\, dx + \sum_{j=1}^{N} \int_0^{\ell_j} (\partial_x^2 u_j \overline{\partial_x^2 w_j} + \lambda^2 u_j \overline{w}_j)\, dx$$
$$+ \lambda u(0)\overline{w}(0) + \lambda \sum_{j=1}^{N} u_j(\ell_j)\overline{w}_j(\ell_j)$$

and

$$F(\vec{w}) = \int_0^\ell (g + \lambda f)\overline{w}\, dx + \sum_{j=1}^{N} \int_0^{\ell_j} (g_j + \lambda f_j)\overline{w}_j\, dx + f(0)\overline{w}(0) + \sum_{j=1}^{N} f(\ell_j)\overline{w}_j(\ell_j).$$

Since Λ is a continuous bilinear coercive form on V and F is a continuous linear form V, by the Lax-Milgram lemma, problem (1.7) has a unique solution $\bar{u} \in V$. Using some integrations by parts, we easily check that $\vec{u}$ satisfies

$$u - \partial_x^2 u = g + \lambda f,$$
$$\lambda^2 u_j + \partial_x^4 u_j = g_j + \lambda f_j, \quad j = 1, \ldots, N,$$

as well as we have set $v = \lambda u - f$, and $v_j = \lambda u_j - f_j$, $j = 1, \ldots, N$. This means that (1.6) holds and consequently, $(\lambda I - \mathcal{A}_S)$ is surjective. The density of $\mathcal{D}(\mathcal{A}_S)$ in $\widetilde{\mathcal{H}}$ is clear. Finally, the Lumer-Phillips theorem (see [21]) leads to the claim. □

The above theorem provides the well-posedness of the evolution equation (S). More precisely, for every $(\vec{u}^0, \vec{u}^1)^t \in \widetilde{\mathcal{H}}$, the function $(\vec{u}(t), \partial_t \vec{u}(t))^t$ given by $\mathcal{T}_S(t)(\vec{u}^0, \vec{u}^1)^t$ is the mild solution of (S). In particular, for $(\vec{u}^0, \vec{u}^1)^t \in \mathcal{D}(\mathcal{A}_S)$, the problem (S) admits a unique classical solution

$$(\vec{u}, \partial_t \vec{u})^t \in C([0, \infty), \mathcal{D}(\mathcal{A}_S)) \cap C^1([0, \infty), \widetilde{\mathcal{H}}).$$

The well-posedness part follows from Theorem 1.1. In order to prove estimate (1.9) it suffices to remark that, by simple integrations by parts, it holds for regular solutions (i.e. $(\vec{u}, \partial_t \vec{u})^t \in C([0, +\infty); \mathcal{D}(\mathcal{A}_S))$. For mild solutions, we simply use the density of $\mathcal{D}(\mathcal{A}_S)$ in $\widetilde{\mathcal{H}}$.

Thus, we have the following proposition.

Proposition 1.2 *The following assertions hold true*

1. *If* $(\vec{u}^0, \vec{u}^1)^t \in \widetilde{\mathcal{H}}$ *then the problem* (S) *admits a unique solution*

$$\vec{u} \in C([0,+\infty); V) \cap C^1\left([0,+\infty); L^2(0,\ell) \times \prod_{i=1}^{N} L^2(0,\ell_i)\right)$$

such that $u(\cdot,0), u_i(\cdot,\ell_i) \in H^1(0,T), T > 0, i = 1,\dots,N$, *and*

$$\|u(\cdot,0)\|^2_{H^1(0,T)} + \sum_{i=1}^{N} \|u_i(\cdot,\ell_i)\|^2_{H^1(0,T)} \le C\|(\vec{u}^0,\vec{u}^1)\|^2_{\widetilde{\mathcal{H}}}, \tag{1.8}$$

for a constant $C > 0$.

Moreover $\vec{u}$ *satisfies the energy identity*:

$$E_S(0) - E_S(t) = \int_0^t |\partial_t u(s,0)|^2\,ds + \sum_{i=1}^{N} \int_0^t |\partial_t u_i(s,\ell_i)|^2\,ds. \tag{1.9}$$

2. *The estimate* $\lim_{t\to\infty} E_S(t) = 0$ *holds true for any finite energy solution of* (S).

Our main result can now be stated as follows.

Theorem 1.3 *There exist constants* $C, w > 0$ *such that for all* $(\vec{u}^0,\vec{u}^1)^t \in \widetilde{\mathcal{H}}$ *the solution* $\vec{u}$ *of the system* (S) *satisfies*:

$$E_S(t) \le Ce^{-wt}E_S(0), \quad \forall t > 0. \tag{1.10}$$

2 Proof of the main result

In order to prove the strong stability we need to verify the following property.

Lemma 2.1 *The spectrum of* $\mathcal{A}_S$ *contains no point on the imaginary axis.*

Proof Since $\mathcal{A}_S$ has compact resolvent, its spectrum $\sigma(\mathcal{A}_S)$, only consists of eigenvalues of $\mathcal{A}_S$. We will show that the equation

$$\mathcal{A}_S z = i\beta z \tag{2.1}$$

with $z = (\vec{y}, \vec{v})^t \in \mathcal{D}(\mathcal{A}_S)$ and $\beta \neq 0$ has only the trivial solution.

By taking the inner product of (2.1), with $z \in \widetilde{\mathcal{H}}$ and using

$$\Re\,\langle \mathcal{A}_S z, z\rangle_{\widetilde{\mathcal{H}}} = -|v(0)|^2 - \sum_{j=1}^{N} \left|v_j(\ell_j)\right|^2, \tag{2.2}$$

we obtain that $v(0)=0$, $v_j(\ell_j)=0$, $j=1,\dots,N$. From (2.1), we get

$$i\beta(\vec{y},\vec{v})^t=(\vec{v},\partial_x^2 y,-\partial_x^4 y_1,\dots,-\partial_x^4 y_N)^t.$$

Next, we eliminate $\vec{v}=i\beta\vec{y}$ to get:

$$\begin{cases} -\beta^2 y-\dfrac{d^2y}{dx^2}=0,\ (0,\ell), \\ -\beta^2 y_j+\dfrac{d^4y_j}{dx^4}=0,\ (0,\ell_j), \\ y(\ell)=y_j(0), \qquad \displaystyle\sum_{j=1}^{N}\frac{d^3y_j}{dx^3}(0)+\frac{dy}{dx}(\ell)=0, \qquad \dfrac{d^2y_j}{dx^2}(0)=0, \\ y(0)=0, \qquad y_j(\ell_j)=0, \qquad \dfrac{dy}{dx}(0)=0, \qquad \dfrac{dy_j}{dx}(\ell_j)=0, \\ \dfrac{d^3y_j}{dx^3}(\ell_j)=0, \qquad j=1,\dots,N. \end{cases} \tag{2.3}$$

By a simple calculation we show that the above system only has trivial solution. In fact, we have $y(x)=A\sin(\beta x)$, since $y(0)=0$ and $-\beta^2 y-\frac{d^2y}{dx^2}=0$. From $\frac{dy}{dx}(0)=0$, we get $A=0$ and thus $y(x)=0$. This implies that $y_j(0)=0$. We thus have

$$y_j(0)=\frac{d^2y_j}{dx^2}(0)=\frac{dy_j}{dx}(\ell_j)=\frac{d^3y_j}{dx^3}(\ell_j)=0, \tag{2.4}$$

together with $-\beta^2 y_j+\frac{d^4y_j}{dx^4}=0$. We can write

$$y_j(x)=A_1\exp(i\sqrt{\beta}x)+A_2\exp(-i\sqrt{\beta}x)+A_3\exp(\sqrt{\beta}x)+A_4\exp(-\sqrt{\beta}x).$$

By using the conditions (2.4), we get $A_1=A_2=A_3=A_4=0$, and thus $z=0$ □

Proof of Proposition 1.2 As the imbedding of $\mathcal{D}(\mathcal{A}_S)$ in $\widetilde{\mathcal{H}}$ is obviously compact, in order to prove the energy identity it suffices to remark that they hold true for regular solutions (i.e. $(\vec{u},\partial_t\vec{u})^t\in C([0,+\infty);\mathcal{D}(\mathcal{A}_S))$ and to use the density of $\mathcal{D}(\mathcal{A}_S)$ in $\widetilde{\mathcal{H}}$). Since $\mathcal{A}_S$ is a maximal-dissipative operator in $\widetilde{\mathcal{H}}$, $\mathcal{A}_S$ has no purely imaginary eigenvalues (see Lemma 2.1), and $\mathcal{A}_S$ has compact resolvent. Then, the strong stability estimate at the end of Proposition 1.2 can be obtained by applying the result in Sect. 5 of [18] (see also [19]). □

Proof of Theorem 1.3 By a classical result (see Huang [15] and Prüss [22]) it suffices to show that $\mathcal{A}$ satisfies the following two conditions:

$$\rho(\mathcal{A}_S)\supset\{i\beta \mid \beta\in\mathbb{R}\}\equiv i\mathbb{R}, \tag{2.5}$$

and

$$\limsup_{|\beta|\to+\infty} \|(i\beta I - \mathcal{A}_S)^{-1}\| < \infty, \tag{2.6}$$

where $\rho(\mathcal{A}_S)$ denotes the resolvent set of the operator $\mathcal{A}_S$.

By Lemma 2.1 the condition (2.5) is satisfied for all $\ell, \ell_j > 0$, $j = 1, \ldots, N$. Suppose that the condition (2.6) is false. By the Banach-Steinhaus Theorem (see [12]), there exist a sequence of real numbers $\beta_n \to \infty$ and a sequence of vectors $Z_n = \binom{\vec{y}_n}{\vec{v}_n} \in \mathcal{D}(\mathcal{A}_S)$ with $\|Z_n\|_{\widetilde{\mathcal{H}}} = 1$ such that

$$\|(i\beta_n I - \mathcal{A}_S)Z_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty, \tag{2.7}$$

i.e.,

$$i\beta_n \vec{y}_n - \vec{v}_n \equiv \vec{f}_n \to 0 \quad \text{in } V, \tag{2.8}$$

$$i\beta_n v_n - \frac{d^2 y_n}{dx^2} \equiv g_n \to 0 \quad \text{in } L^2(0,\ell), \tag{2.9}$$

$$i\beta_n v_{j,n} + \frac{d^4 y_{j,n}}{dx^4} \equiv k_{j,n} \to 0 \quad \text{in } L^2(0,\ell_j),\ j = 1, \ldots, N. \tag{2.10}$$

Our goal is to derive from (2.7) that $\|Z_n\|_{\widetilde{\mathcal{H}}}$ converges to zero, thus, a contradiction. The proof is divided in three steps:

First step. We notice that from (2.2) we have

$$\|(i\beta_n I - \mathcal{A})Z_n\|_{\widetilde{\mathcal{H}}} \geq \left|\Re\big(\langle (i\beta_n I - \mathcal{A})Z_n, Z_n\rangle_{\widetilde{\mathcal{H}}}\big)\right| = |v_n(0)|^2 + \sum_{j=1}^N \left|v_{j,n}(\ell_j)\right|^2.$$

Then, by (2.7)

$$v_n(0) \to 0, \qquad v_{j,n}(\ell_j) \to 0, \qquad \frac{d^3 y_{j,n}}{dx^3}(\ell_j) = v_{j,n}(\ell_j) \to 0,$$
$$\frac{dy_n}{dx}(0) = v_n(0) \to 0, \quad j = 1, \ldots, N.$$

This further leads to

$$\beta_n y_n(0) = -i f_n(0) - i v_n(0) \to 0, \beta_n y_{j,n}(\ell_j) = -i f_{j,n}(0) - i v_{j,n}(0) \to 0,$$
$$j = 1, \ldots, N, \tag{2.11}$$

due to (2.8) and the trace theorem.

Second step. We express now $v_n, v_{j,n}$ in terms of $y_n, y_{j,n}$, $j = 1, \ldots, N$, from (2.8)–(2.10) and substitute it into (2.8)–(2.10) to get

$$\left(-\beta_n^2 y_n - \frac{d^2 y_n}{dx^2}\right) = g_n + i\beta_n f_n, \tag{2.12}$$

$$\left(-\beta_n^2 y_{j,n} + \frac{d^4 y_{j,n}}{dx^4}\right) = k_{j,n} + i\beta_n f_{j,n}, \quad j = 1, \dots, N. \tag{2.13}$$

Next, we take the inner product of (2.12) with $q(x)\frac{dy_n}{dx}$ in $L^2(0,\ell)$ where $q(x) \in C^1([0,\ell])$ and $q(\ell) = 0$. We obtain that

$$\begin{aligned}
&\int_0^\ell \left(-\beta_n^2 y_n - \frac{d^2 y_n}{dx^2}\right) q(x) \frac{d\bar{y}_n}{dx}\, dx \\
&\quad = \int_0^\ell (g_n + i\beta_n f_n) q(x) \frac{d\bar{y}_n}{dx}\, dx \\
&\quad = \int_0^\ell g_n\, q(x) \frac{d\bar{y}_n}{dx}\, dx - i \int_0^\ell q \frac{df_n}{dx}\, \beta_n \bar{y}_n\, dx \\
&\qquad - i \int_0^\ell f_n \frac{dq}{dx}\, \beta_n \bar{y}_n\, dx - i f_n(0) q(0) \beta_n \bar{y}_n(0).
\end{aligned} \tag{2.14}$$

It is clear that the right-hand side of (2.14) converges to zero since $\bar{f}_n, g_n$ converge to zero in H^1 and L^2, respectively.

By a straight-forward calculation,

$$\Re\left\{\int_0^\ell -\beta_n^2 y_n q \frac{d\bar{y}_n}{dx}\, dx\right\} = \frac{1}{2} q(0) |\beta_n y_n(0)|^2 + \frac{1}{2} \int_0^\ell \frac{dq}{dx} |\beta_n y_n|^2 dx$$

and

$$\Re\left\{\int_0^\ell -\frac{d^2 y_n}{dx^2} q \frac{d\bar{y}_n}{dx}\, dx\right\} = \frac{1}{2} q(0) \left|\frac{dy_n}{dx}(0)\right|^2 + \frac{1}{2} \int_0^\ell \left|\frac{dy_n}{dx}\right|^2 \frac{dq}{dx}\, dx. \tag{2.15}$$

According to (2.11), we simplify (2.14), then take its real parts. This leads to

$$\int_0^\ell \frac{dq}{dx} |\beta_n y_n|^2\, dx + \int_0^\ell \frac{dq}{dx} \left|\frac{dy_n}{dx}\right|^2 dx \to 0. \tag{2.16}$$

By taking $q(x) = e^{x-\ell} - 1$, which satisfies $|q'(x)| \leq 1, x \in [0,\ell]$, we obtain that

$$\|\beta_n y_n\|_{L^2(0,\ell)} \to 0, \qquad \left\|\frac{dy_n}{dx}\right\|_{L^2(0,\ell)} \to 0, \qquad \|v_n\|_{L^2(0,\ell)} \to 0. \tag{2.17}$$

Similarly, we take the inner product of (2.12) with $q_j(x)\frac{dy_{j,n}}{dx}$ in $L^2(0,\ell_j)$ with $q_j \in C^2([0,\ell_j])$ and $q_j(0) = 0$, $j = 1, \dots, N$, then repeat the above procedure, this will give

$$\int_0^{\ell_j} \frac{dq_j}{dx} |\beta_n y_{2,n}|^2 dx + \int_0^{\ell_j} 3\frac{dq_j}{dx} \left|\frac{d^2 y_{j,n}}{dx^2}\right|^2 dx + 2\Re\left(\int_0^{\ell_j} \frac{d^2 y_{j,n}}{dx^2} \frac{d^2 q_j}{dx^2} \frac{d\bar{y}_{j,n}}{dx}\, dx\right)$$

$$- 2\left|\frac{d^2 y_{j,n}}{dx^2}(\ell_j)\right|^2 q_j(\ell_j) \to 0, \quad j = 1, \dots, N. \tag{2.18}$$

By integrating by parts, we get

$$\begin{aligned}
\int_0^{\ell_j} \left|\frac{dy_{j,n}}{dx}\right|^2 dx &= -\int_0^{\ell_j} y_{j,n}\overline{\frac{d^2 y_{j,n}}{dx^2}}\,dx - y_{j,n}(0)\overline{\frac{dy_{j,n}}{dx}}(0) \\
&= -\int_0^{\ell_j} y_{j,n}\overline{\frac{d^2 y_{j,n}}{dx^2}}\,dx - y_n(\ell)\overline{\frac{dy_{j,n}}{dx}}(0) \\
&= -\frac{1}{i\beta_n}\int_0^{\ell_j} v_{j,n}\overline{\frac{d^2 y_{j,n}}{dx^2}}\,dx - \frac{1}{i\beta_n}\int_0^{\ell_j}(i\beta_n y_{j,n} - v_{j,n})\overline{\frac{d^2 y_{j,n}}{dx^2}}\,dx \\
&\quad - \overline{\frac{dy_{j,n}}{dx}}(0)y_n(\ell).
\end{aligned}$$

Then, from the boundedness of $v_{j,n}$, $i\beta_n y_{j,n} - v_{j,n}$, $\frac{d^2 y_{j,n}}{dx^2}$, in $L^2(0,\ell_j)$, (2.17) and (2.11), we have $\frac{dy_{j,n}}{dx}$ converges to zero in $L^2(0,\ell_j)$, $j = 1, \dots, N$, which implies

$$\int_0^{\ell_j} \frac{dq_j}{dx}|\beta_n y_{2,n}|^2 dx + \int_0^{\ell_j} 3\frac{dq_j}{dx}\left|\frac{d^2 y_{j,n}}{dx^2}\right|^2 dx - 2\left|\frac{d^2 y_{j,n}}{dx^2}(\ell_j)\right|^2 q_j(\ell_j) \to 0, \quad j = 1, \dots, N. \tag{2.19}$$

Third step. Next, we show that $\frac{d^2 y_{j,n}}{dx^2}(\ell_j)$, $j = 1, \dots, N$, converge to zero. We take the inner product of (2.13) with $\frac{1}{\phi_n^{1/2}}e^{-\phi_n^{1/2}h_j(x)}$ in $L^2(0,\ell_j)$ where $\phi_n = |\beta_n|$, $h_j(x) = \ell_j - x$, $j = 1, \dots, N$.

This leads to

$$\int_0^{\ell_j}\left(\phi_n^{3/2}e^{-\phi_n^{1/2}h_j}y_{j,n} - \frac{1}{\phi_n^{1/2}}e^{-\phi_n^{1/2}h_j}\frac{d^4 y_{j,n}}{dx^4}\right)dx \to 0, \quad j = 1, \dots, N. \tag{2.20}$$

Performing integration by parts to the second term on the left-hand side of (2.20), we obtain

$$\begin{aligned}
&\int_0^{\ell_j}\left(\phi_n^{3/2}e^{-\phi_n^{1/2}h_j}y_{j,n} - \frac{1}{\phi_n^{1/2}}e^{-\phi_n^{1/2}h_j}\frac{d^4 y_{j,n}}{dx^4}\right)dx \\
&= \left[-\frac{1}{\phi_n^{1/2}}e^{-\phi_n^{1/2}(\ell_j - x)}\frac{d^3 y_{j,n}}{dx^3}\right]_{x=0}^{x=\ell_j} - \left[-e^{-\phi_n^{1/2}(\ell_j - x)}\frac{d^2 y_{j,n}}{dx^2}\right]_{x=0}^{x=\ell_j} \\
&\quad + \left[-\phi_n^{1/2}e^{-\phi_n^{1/2}(\ell_j - x)}\frac{dy_{j,n}}{dx}\right]_{x=0}^{x=\ell_j} - \left[-\phi_n e^{-\phi_n^{1/2}(\ell_j - x)}y_{j,n}\right]_{x=0}^{x=\ell_j}.
\end{aligned}$$

Note that $y^k_{j,n}(0)$ is bounded for $k \le 3$, since $y_{j,n} \in H^4(0,\ell_j)$, that $\phi_n \to \infty$, and also that $\frac{d^3 y_{j,n}}{dx^3}(\ell_j) \to 0$ and $\beta_n \frac{dy_{j,n}}{dx}(\ell_j) \to 0$. We thus get that

$$\int_0^{\ell_j} \left(\phi_n^{3/2} e^{-\phi_n^{1/2} h_j} y_{j,n} - \frac{1}{\phi_n^{1/2}} e^{-\phi_n^{1/2} h_j} \frac{d^4 y_{j,n}}{dx^4} \right) dx$$
$$= \frac{d^2 y_{j,n}}{dx^2}(\ell_j) + \phi_n y_{j,n}(\ell_j) + R_{j,n}, \quad j = 1, \ldots, N, \tag{2.21}$$

where $R_{j,n} \to 0, n \to +\infty, \forall j = 1, \ldots, N$. Thus, according to (2.11), we simplify (2.21) to

$$\frac{d^2 y_{j,n}}{dx^2}(\ell_j) \to 0, \quad j = 1, \ldots, N.$$

Consequently we have:

$$\int_0^{\ell_j} \frac{dq_j}{dx} |\beta_n y_{j,n}|^2 dx + \int_0^{\ell_j} 3 \frac{dq_j}{dx} \left| \frac{d^2 y_{j,n}}{dx^2} \right|^2 dx \to 0, \quad j = 1, \ldots, N. \tag{2.22}$$

Finally, we choose $q_j(x)$, $j = 1, \ldots, N$, so that $\frac{dq_j}{dx}$, $j = 1, \ldots, N$, is strictly negative. This can be done by taking

$$q_j(x) = e^{-x} - 1, \quad j = 1, \ldots, N.$$

Therefore, (2.22) implies

$$\|\beta_n y_{j,n}\|_{L^2(0,\ell_j)} \to 0, \qquad \left\| \frac{d^2 y_{j,n}}{dx^2} \right\|_{L^2(0,\ell_j)} \to 0, \quad j = 1, \ldots, N.$$

In view of (2.8), we also get

$$\|v_{j,n}\|_{L^2(0,\ell_j)} \to 0, \quad j = 1, \ldots, N,$$

which clearly contradicts the normalization condition $\|Z_n\|_{\tilde{H}} = 1$. □

3 Numerical illustration

Another way to study the stabililization problem is to adopt the strategy introduced in Ammari and Tucsnak [7] where the exponential stability for the closed loop problem is reduced to an observability estimate for the corresponding uncontrolled system combined to a boundedness property of the transfer function of the associated open loop system. We give here some numerical illustration of this approach. A theoretical study remains to be done.

3.1 Some background on a class of dynamical systems

Let H be a Hilbert space with the norm $\|.\|_H$, and let $A : \mathcal{D}(A) \to H$ be a self-adjoint, positive and boundedly invertible operator. We introduce the Hilbert space $H_{\frac{1}{2}} = \mathcal{D}(A^{\frac{1}{2}})$, with the norm $\|z\|_{\frac{1}{2}} = \|A^{\frac{1}{2}} z\|_H$. The space $H_{-\frac{1}{2}}$ is defined by duality with respect to the pivot space H.

Let the bounded linear operator $B : U \to H_{-\frac{1}{2}}$, where U is another Hilbert space which will be identified with its dual.

The system we consider is described by

$$\ddot{w}(t) + Aw(t) + By(t) = 0, \qquad w(0) = w_0, \qquad \dot{w}(0) = w_1, \quad t \in [0, \infty), \tag{3.1}$$

$$y(t) = B^* \dot{w}(t), \quad t \in [0, \infty). \tag{3.2}$$

The system (3.1)–(3.2) is well-posed:

For $(w_0, w_1) \in H_{\frac{1}{2}} \times H$, the problem (3.1)–(3.2) admits a unique solution

$$w \in C([0, \infty); H_{\frac{1}{2}}) \cap C^1([0, \infty); H)$$

such that $B^* w(\cdot) \in H^1_{loc}(0, \infty; U)$. Moreover, w satisfies the energy identity, for all $t \geq 0$

$$\|(w_0, w_1)\|^2_{H_{\frac{1}{2}} \times H} - \|(w(t), \dot{w}(t))\|^2_{H_{\frac{1}{2}} \times H} = 2 \int_0^t \left\| \frac{d}{dt} B^* w(s) \right\|^2_U ds. \tag{3.3}$$

For (3.3) we remark that the mapping $t \mapsto \|(w(t), \dot{w}(t))\|^2_{H_{\frac{1}{2}} \times H}$ is non-increasing.

Consider the initial value problem:

$$\ddot{\varphi}(t) + A\varphi(t) = 0, \tag{3.4}$$

$$\varphi(0) = w_0, \qquad \dot{\varphi}(0) = w_1. \tag{3.5}$$

It is well known that (3.4)–(3.5) is well posed in $H_1 \times H_{\frac{1}{2}}$ and in $H_{\frac{1}{2}} \times H$.

Now, we consider the unbounded linear operator

$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \to \left(H_{\frac{1}{2}} \times H\right)^t, \quad \mathcal{A}_d = \begin{pmatrix} 0 & I \\ -A & -BB^* \end{pmatrix}, \tag{3.6}$$

where

$$\mathcal{D}(\mathcal{A}_d) = \left\{ (u, v)^t \in \left(H_{\frac{1}{2}} \times H\right)^t, Au + BB^* v \in H, v \in H_{\frac{1}{2}} \right\}.$$

The result below, proved in [7], shows that, under a certain regularity assumption, the exponential stability of (3.1)–(3.2) is equivalent to a strong observability inequality for (3.4)–(3.5). More precisely, we have:

Theorem 3.1 (Ammari-Tucsnak [7]) *Assume that for any* $\gamma > 0$ *we have*

$$\sup_{\mathrm{Re}\lambda=\gamma} \left\| \lambda B^*(\lambda^2 I + A)^{-1} B \right\|_{\mathcal{L}(U)} < \infty. \tag{3.7}$$

Then, there exist constants $C, \delta > 0$ *such that for all* $t > 0$ *and for all* $(w^0, w^1) \in H_{\frac{1}{2}} \times H$, *we have*

$$\|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H} \leq C e^{-\delta t} \|(w^0, w^1)\|_{H_{\frac{1}{2}} \times H},$$

if and only if there exist constants $T, C > 0$ *such that*: *for all* $(w_0, w_1) \in H_1 \times H_{\frac{1}{2}}$, *we have*

$$\left\| B^* \varphi'(t) \right\|_{L^2(0,T;U)} \geq C \, \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}, \tag{3.8}$$

where $\varphi(t)$ *is the solution of the system* (3.4)–(3.5).

We will study the inequalities (3.7) and (3.8) numerically.

3.2 Transfert function

Let $A : V \to V'$,

$$A = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & \left(\frac{d^4}{dx^4}\right)_{i=1,\ldots,N} \end{pmatrix},$$

$$\mathcal{D}(A) = \left\{ \bar{y}^t = (y, y_1, \ldots, y_N)^t \in V^t \;\middle|\; \begin{array}{l} \frac{dy}{dx} \in H^1(0,\ell), \frac{d^2 y_i}{dx^2} \in H^2(0,\ell_i), \\ \frac{dy}{dx}(0) = 0, \frac{dy_i}{dx}(\ell_i) = 0, \frac{d^2 y_i}{dx^2}(0) = 0, \\ \frac{d^3 y_i}{dx^3}(\ell_i) = 0, \sum_{i=1}^N \frac{d^3 y_i}{dx^3}(0) + \frac{dy}{dx}(\ell) = 0 \end{array} \right\}$$

and $B : \mathbb{R}^{N+1} \to V'$, $B\bar{k}^t = AD\bar{k}^t$, $\forall \bar{k} = (k, k_1, \ldots, k_N) \in \mathbb{R}^{N+1}$, where V' is the dual space of V obtained by means of inner product in $L^2(0,\ell) \times \prod_{i=1}^N L^2(0,\ell_i)$, $D\bar{k}^t = \bar{W} = (W, W_1, \ldots, W_N)^t$ is the solution of

$$\begin{cases} \dfrac{d^2 W}{dx^2} = 0, (0,\ell), \\ \dfrac{d^4 W_i}{dx^4} = 0, (0,\ell_i), \\ \dfrac{dW_i}{dx}(\ell_i) = 0, \qquad \dfrac{d^2 W_i}{dx^2}(0) = 0, \\ W(\ell) = W_i(0), \\ \dfrac{dW}{dx}(0) = k, \qquad \dfrac{d^3 W}{dx^3}(\ell_i) = k_i, \\ \displaystyle\sum_{i=1}^N \dfrac{d^3 W_i}{dx^3}(0) + \dfrac{dW}{dx}(\ell) = 0. \end{cases}$$

We have: $B^*\bar{p}^t = (p(0), p_1(\ell_1), \ldots, p_N(\ell_N))^t, \forall \bar{p} \in V$.

Proposition 3.2 *Let $\gamma > 0$ and $C_\gamma = \{\lambda \in \mathbb{C} \mid \Re(\lambda) = \gamma\}$. Then the function $H(\lambda)\bar{k}^t = \lambda B^*(\lambda^2 + A)^{-1} B\bar{k}^t$, for $\bar{k} \in \mathbb{R}^{N+1}$, is given, for $\Re(\lambda) > 0$, by*

$$H(\lambda)\bar{k}^t = \left[(H_{i,j}(\lambda))_{0\le i,j\le N}\right]\bar{k}^t = \begin{pmatrix} H_0(\bar{k}^t) \\ \cdots \\ H_N(\bar{k}^t)\end{pmatrix},$$

$$\begin{aligned} &H_0(\bar{k}^t) \\ &\quad = -k + \frac{-\lambda k e^{-\lambda\ell} + k w^3 e^{-\lambda\ell}\sum_{j=1}^N h_j - \lambda\sum_{j=1}^N(-\frac{k_j}{2i}e^{iw\ell_j} - \frac{k_j}{4i\cosh(w\ell_j)}e^{iw\ell_j})h_j}{\lambda\sinh(\lambda\ell) + w^3\cosh(\lambda\ell)\sum_{j=1}^N h_j} \\ &\qquad + \frac{-\lambda\sum_{j=1}^N(\frac{k_j}{2}e^{w\ell_j} - \frac{k_j}{4\cosh(w\ell_j)}e^{w\ell_j})h_j - \lambda\sum_{j=1}^N(\frac{k_j}{2}e^{iw\ell_j} + \frac{k_j}{2}e^{w\ell_j})}{\lambda\sinh(\lambda\ell) + w^3\cosh(\lambda\ell)\sum_{j=1}^N h_j}, \end{aligned}$$

$$\begin{aligned} &H_p(\bar{k}^t) \\ &\quad = m_p\Bigg\{\frac{-\lambda\cosh(\lambda\ell)ke^{-\lambda\ell} + kw^3e^{-\lambda\ell}\sum_{j=1}^N h_j - \lambda\cosh(\lambda\ell)\sum_{j=1}^N(-\frac{k_j}{2i}e^{iw\ell_j} - \frac{k_j}{4i\cosh(w\ell_j)}e^{iw\ell_j})h_j}{\lambda\sinh(\lambda\ell) + w^3\cosh(\lambda\ell)\sum_{j=1}^N h_j} \\ &\qquad + \frac{-\lambda\cosh(\lambda\ell)\sum_{j=1}^N(\frac{k_j}{2}e^{w\ell_j} - \frac{k_j}{4\cosh(w\ell_j)}e^{w\ell_j})h_j - \lambda\cosh(\lambda\ell)\sum_{j=1}^N(\frac{k_j}{2}e^{iw\ell_j} + \frac{k_j}{2}e^{w\ell_j})}{\lambda\sinh(\lambda\ell) + w^3\cosh(\lambda\ell)\sum_{j=1}^N h_j}\Bigg\} \\ &\qquad - ke^{-\lambda\ell}m_p + \lambda\left(-\frac{k_p}{2iw^3}e^{iw\ell_p} - \frac{k_p}{4iw^3\cosh(w\ell_p)}e^{iw\ell_p}\right)m_p \\ &\qquad + \lambda\left(\frac{k_p}{2w^3}e^{w\ell_p} - \frac{k_p}{4w^3\cosh(w\ell_p)}e^{w\ell_p}\right)m_p \\ &\qquad + \lambda\left(\frac{k_p}{2iw^3} + \frac{k_p}{2w^3} + \frac{k_p}{4iw^3\cosh(wl_p)}e^{iwl_p} + \frac{k_p}{4w^3\cosh(wl_p)}e^{wl_p}\right), \quad p = 1, \ldots, N, \end{aligned}$$

where w is the unique complex number satisfying the conditions

$$\lambda = iw^2, \quad w = re^{i\theta}, \text{ with } r > 0 \text{ and } \theta \in \left]-\frac{\pi}{2}, 0\right[$$

and $h_p = \frac{2\sin(w\ell_j)\cosh(w\ell_j) - \cos(w\ell_j)\sinh(w\ell_j)}{4\cos(w\ell_j)\cosh(w\ell_j)}$, $m_p = \frac{2\cosh(w\ell_p) + \cos(w\ell_p)}{4\cos(w\ell_p)\cosh(w\ell_p)}$, $p = 1, \ldots, N$.

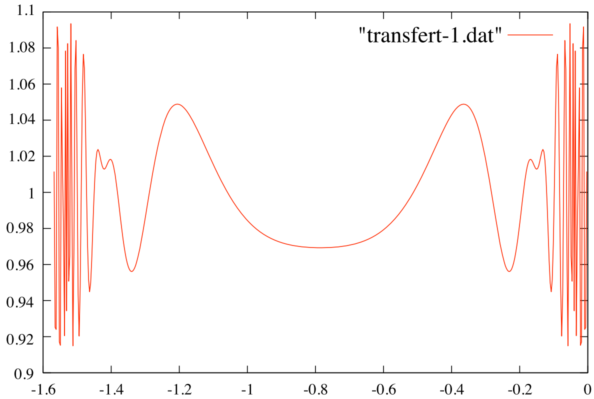
Proof Let $\bar{k} \in \mathbb{R}^{N+1}$. It can be easily checked that $\bar{y} = (\lambda^2 + A)^{-1} B\bar{k}^t$ satisfies:

$$\lambda^2 y(x) - \frac{d^2 y}{dx^2}(x) = 0, \quad x \in (0, \ell), \Re(\lambda) > 0, \tag{3.9}$$

$$\lambda^2 y_p(x) + \frac{d^4 y_p}{dx^4}(x) = 0, \quad x \in (0, \ell_p), \Re(\lambda) > 0, \tag{3.10}$$

$$\frac{dy}{dx}(0) = k, \qquad \frac{d^3 y_p}{dx^3}(\ell_p) = k_p, \qquad \frac{dy_p}{dx}(\ell_p) = 0, \qquad \frac{d^2 y_p}{dx^2}(0) = 0, \tag{3.11}$$

Fig. 1 $\|H(\lambda)\|$, for $\lambda = iw^2$ and $w = re^{i\theta}$, with $\Re(w) = \gamma$ against θ for $\gamma = 1$



$$y(\ell) = y_p(0), \qquad \frac{dy}{dx}(\ell) + \sum_{j=1}^{N} \frac{d^3 y_j}{dx^3}(0) = 0. \tag{3.12}$$

The solutions of (3.9)–(3.10) have the form

$$\begin{cases} y(x) = A\cosh(\lambda x) - \dfrac{k}{\lambda} e^{-\lambda x}, \quad x \in (0,\ell), \\ y_p(x) = A_p \cosh(w(x-\ell_p)) + B_p \cos(w(x-\ell_p)) + \dfrac{k_p}{2iw^3} e^{-iw(x-\ell_p)} \\ \qquad - \dfrac{k_p}{2w^3} e^{-w(x-\ell_p)}, \quad x \in (0,\ell_p), \end{cases}$$

where A, A_p, B_p are constants. Then, for $\Re(\lambda) > 0$, we get

$$\begin{pmatrix} \lambda y(0) \\ \lambda y_1(0) \\ \cdots \\ y_N(0) \end{pmatrix} = H(\lambda)\bar{k}^t.$$

□

We now investigate numerically the boundedness of the transfert function (3.7) and then also the observability inequality (3.8).

3.3 Boundedness of the transfert function

On Figs. 1–3, we see the value of $\|H(\lambda)\| := \max_{i,j}(|H_{i,j}(\lambda)|)$, for $\lambda = iw^2$ and $w = re^{i\theta}$, with $\Re(w) = \gamma$, for a given γ. The plot is against θ. We see that the maximum is finite; the value approaches 1 as γ increases and tends to ∞ as $\gamma > 0$ tends to 0. Note however that the function presents some pics for θ near 0 or $\pi/2$, as it is shown on Fig. 3 which can be difficult to see at a first glance.

We have taken $N = 2$, $\ell = 1.51$, $\ell_1 = 1.4$ and $\ell_2 = 1.2$.

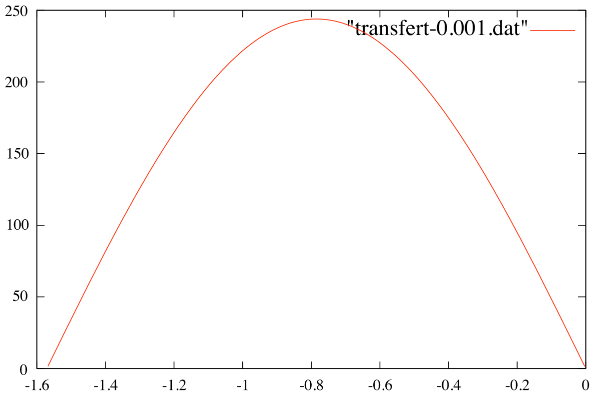


Fig. 2 $\|H(\lambda)\|$, for $\lambda = iw^2$ and $w = re^{i\theta}$, with $\Re(w) = \gamma$ against θ for $\gamma = 0.001$

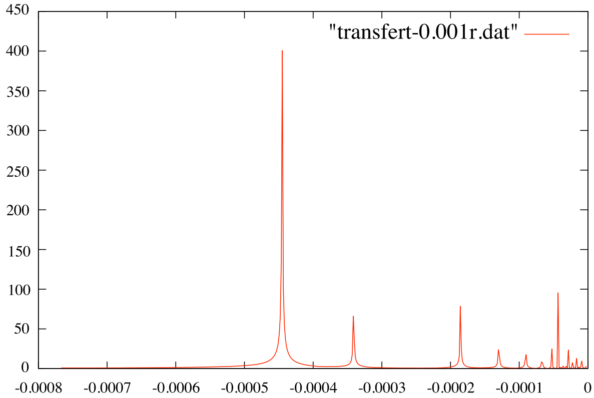


Fig. 3 $\|H(\lambda)\|$, for $\lambda = iw^2$ and $w = re^{i\theta}$, with $\Re(w) = \gamma$ against θ for $\gamma = 0.001$

3.4 Eigenvalues computation

In the case where the transfer function is bounded, the stabilization problem is reduced to study the associated conservative problem (see Sect. 3.1).

Thus, we investigate the observability inequality (3.8). Note that (3.8) concerns the conservative system. The eigenvalues of the conservative system are computed from the characteristic equation. Given the K smallest positive eigenvalues, we can then define the matrix $(m_{k,k'})_{k,k'=1}^K$ by

$$m_{k,k'} = \int_0^T e^{i(\lambda_k - \lambda_{k'})t} dt \left(\frac{\phi_{n,k}^1(0)\phi_{n,k'}^1(0) + \sum_{j=1}^N \phi_{n,k}^2(\ell_j)\phi_{n,k'}^2(\ell_j)}{\|\phi_{n,k}\|_{\tilde{\mathcal{H}}}\|\phi_{n,k'}\|_{\tilde{\mathcal{H}}}} \right),$$

and compute the minimal eigenvalue $\lambda_{\min,K}$ of this positive definite symmetric matrix. Note that we should have $\inf_K \lambda_{\min,K} > 0$ in order to have the observability inequality (3.8).

The eigenvalues are given by $\lambda_n = z_n^2$, where z_n is the n-th positive root of

$$\frac{z}{2}\cos(z^2\ell) \sum_{i=1}^N \left(\frac{\sin(z\ell_i)}{\cos(z\ell_i)} - \frac{\sinh(z\ell_i)}{\cosh(z\ell_i)} \right) - \sin(z^2\ell) = 0. \tag{3.13}$$

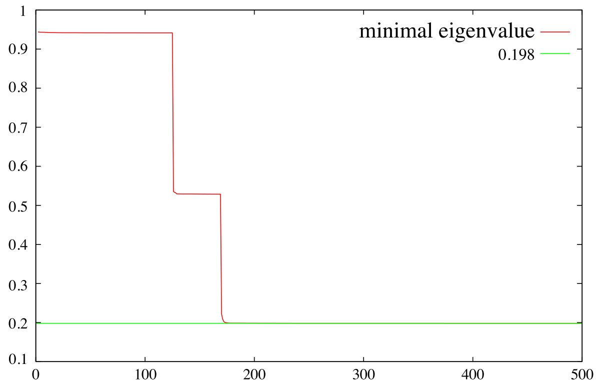


Fig. 4 Minimal eigenvalue against K, for the matrix $(m_{k,k'})_{k,k'=1}^{K}$, with $N = 2$, $\ell = 1.5$, $\ell_1 = 1.4$, $\ell_2 = 1.2$ and $T = 3.1$

The eigenfunctions are given by $\Phi_0 = (1, 1, \dots, 1, 0, 0, \dots, 0)$, $\Phi_n = \phi_n / \|\phi_n\|_{\tilde{\mathcal{H}}}$, $\forall n \in \mathbb{N}^*$ where

$$\phi_n = \left(\frac{1}{\lambda_n}\phi_n^1, \frac{1}{\lambda_n}\phi_{n,1}^2, \dots, \frac{1}{\lambda_n}\phi_{n,N}^2, \phi_n^1, \phi_{n,1}^2, \dots, \phi_{n,N}^2\right), \quad \forall n \in \mathbb{N}^*,$$

$$\phi_n^1(x) = \cos(z_n^2 x),$$

$$\phi_{n,i}^2(x) = \frac{\cos(z_n^2\ell)\cosh(z_n(x-\ell_i))}{2\cosh(z_n\ell_i)} + \frac{\cos(z_n^2\ell)\cos(z_n(x-\ell_i))}{2\cos(z_n\ell_i)},$$

and we have

$$\|\phi_n\|_{\tilde{\mathcal{H}}}^2 = \ell + \frac{\cos^2(z^2\ell)}{2}\sum_{i=1}^{N}\frac{\ell_i}{2}\left(\frac{1}{\cos^2(z\ell_i)} + \frac{1}{\cosh^2(z\ell_i)}\right) + \frac{\cos(z^2\ell)\sin(z^2\ell)}{2z}.$$

On Fig. 4, we compute the minimal eigenvalue in the case where $N = 2$, $\ell = 1.5$, $\ell_1 = 1.4$, $\ell_2 = 1.2$ and $T = 3.1$, by using Python, with the library Scipy. We check that the minimal eigenvalue is greater than 0.198 for $\lambda_n, n \leq 3772$, by using the test of Cholesky.

3.5 Related questions

As we have already mentioned, a theoretical study of this approach of stabilization by observability (for the conservative associated problem) remains to be done. Another related question is to generalize the results to a class of evolutions equations with unbounded feedbacks (see [23] and [20]), in particular to a general string-beam network [4], by using the theoretical approach used here or the methodology developed by Ammari-Tucsnak in [7].

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