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FEEDBACK STABILIZATION OF A COUPLED STRING-BEAM SYSTEM

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Abstract. We consider a stabilization problem for a coupled string-beam system. We prove some decay results of the energy of the system. The method used is based on the methodology introduced in Ammari and Tucsnak [2] where the exponential and weak stability for the closed loop problem is reduced to a boundedness property of the transfer function of the associated open loop system. Morever, we prove, for the same model but with two control functions, independently of the length of the beam that the energy decay with a polynomial rate for all regular initial data. The method used, in this case, is based on a frequency domain method and combine a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

1. **Introduction and main results.**

Let $\ell > 0$. We consider the following systems of equations:

$$(S.1)\begin{cases} (\partial_t^2 u_1 - \partial_x^2 u_1)(t,x) = 0,\ x \in (0,1),\ (\partial_t^2 u_2 + \partial_x^4 u_2)(t,x) = 0,\ x \in (1, 1+\ell), \\ u_1(t,0) = 0,\ u_2(t,1+\ell) = 0,\ \partial_x^2 u_2(t,1+\ell) = 0,\ \partial_x^2 u_2(t,1) = 0, \\ u_1(t,1) = u_2(t,1),\ \partial_x^3 u_2(t,1) + \partial_x u_1(t,1) = -\,\partial_{\mathbf{t}} \mathbf{u_1}(\mathbf{t},\mathbf{1}),\ t \in (0,\infty), \\ u_i(0,x) = u_i^0(x),\ \partial_t u_i(0,x) = u_i^1(x),\ i = 1,2, \end{cases}$$

$$(S.2)\begin{cases} (\partial_t^2 \vartheta_1 - \partial_x^2 \vartheta_1)(t,x) = 0,\ x \in (0,1),\ (\partial_t^2 \vartheta_2 + \partial_x^4 \vartheta_2)(t,x) = 0,\ x \in (1, 1+\ell), \\ \vartheta_1(t,0) = 0,\ \vartheta_2(t,1+\ell) = 0,\ \partial_x^2 \vartheta_2(t,1+\ell) = 0,\ \partial_x^2 \vartheta_2(t,1) = \partial^{\mathbf{2}}_{\mathbf{tx}} \vartheta_{\mathbf{2}}(\mathbf{t},\mathbf{1}), \\ \vartheta_1(t,1) = \vartheta_2(t,1),\ \partial_x^3 \vartheta_2(t,1) + \partial_x \vartheta_1(t,1) = -\partial_{\mathbf{t}} \vartheta_{\mathbf{1}}(\mathbf{t},\mathbf{1}),\ t \in (0,\infty), \\ \vartheta_i(0,x) = \vartheta_i^0(x),\ \partial_t \vartheta_i(0,x) = \vartheta_i^1(x), i = 1,2. \end{cases}$$

Models of the transient behavior of some or all of the state variables describing the motion of flexible structures have been of great interest in recent years, for details about physical motivation for the models see [6], [9] and the references therein. Mathematical analysis of coupled partial differential equations is detailed in [9].

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We study a feedback stabilization problem for a coupled string-beam system, see [1]-[3] and [9]. We prove some decay results of the energy of the system. The method used is based an the methodology introduced in Ammari and Tucsnak [2] where the exponential and weak stability for the closed loop problem is reduced to a boundedness property of the transfer function of the associated open loop system. Morever, we prove, for the same model but with two control functions, independently of the length of the beam that the energy decay with a polynomial rate for all regular initial data. The method used, in this case, is based on a frequency domain method and combine a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

The plan of the paper is as follows. In this Section 1 we give precise statements of the main results. Sections 2, 3 and 4 contain some spectral and regularity results needed in the following sections. In Section 5 we prove pointwise observability results for the associated undamped problem. The proof of the main results are given in Sections 5, 6.

We define the energy of a solution (u_1, u_2) of (S.1) and $(\vartheta_1, \vartheta_2)$ of (S.2) at the time t respectively by $E_1(t) = E(t, u_1, u_2)$ and $E_2(t) = E(t, \vartheta_1, \vartheta_2)$, with

$$2E(t,f,g) = \int_0^1 \left(|\partial_t f(t,x)|^2 + |\partial_x f(t,x)|^2\right) \mathrm{d}x + \int_1^{1+\ell} \left(|\partial_t g(t,x)|^2 + |\partial_x^2 g(t,x)|^2\right) \mathrm{d}x$$

By setting $\delta_i = E_i(t_2) - E_i(t_1),\ i = 1,2$, we can easily check that every sufficiently smooth solution of (S.1) and of (S.2) satisfies respectively the energy identity

$$\delta_1 = -\int_{t_1}^{t_2} \left|\partial_t u_1(s,1)\right|^2 \mathrm{d}s,\ \delta_2 = -\int_{t_1}^{t_2} \left(\left|\partial_t \vartheta_1(s,1)\right|^2 + \left|\partial_{tx}^2 \vartheta_2(s,1)\right|^2\right) \mathrm{d}s, \tag{1}$$

and therefore, these energies are nonincreasing functions of the time variable t. By denoting

$$V = \left\{(u,v) \in H^1(0,1) \times H^2(1,1+\ell),\ u(1) = v(1),\ u(0) = 0,\ v(1+\ell) = 0\right\},$$

we define $\mathcal{H} = V \times L^2(0,1) \times L^2(1,1+\ell)$, equipped with the inner product induced norm

$$\|(y_1,y_2,v_1,v_2)\|_{\mathcal{H}}^2 = \int_0^1 \left(|\partial_x y_1|^2 + |v_1|^2\right) \mathrm{d}x + \int_1^{1+\ell} \left(|\partial_x^2 y_2|^2 + |v_2|^2\right) \mathrm{d}x,$$

and the operators $\mathcal{A}_i$ in $\mathcal{H}$, by

$$\begin{aligned} \mathcal{D}(\mathcal{A}_1) = \{(y_1,y_2,v_1,v_2) \in \mathcal{H} | (v_1, v_2, \partial_x^2 y_1, \partial_x^4 y_2) \in V \times L^2(0,1) \times L^2(1,1+\ell), \\ \partial_x^2 y_2(1\ell) = 0, \partial_x^2 y_2(1) = 0, \partial_x^3 y_2(1) + \partial_x y_1(1) = -v_1(1)\}, \\ \mathcal{D}(\mathcal{A}_2) = \{(y_1,y_2,v_1,v_2) \in \mathcal{H} | (v_1, v_2, \partial_x^2 y_1, \partial_x^4 y_2) \in V \times L^2(0,1) \times L^2(1,1+\ell), \\ \partial_x^2 y_2(1+\ell) = 0, \partial_x^2 y_2(1) = \partial_x v_2(1), y_1(0) = 0, \partial_x^3 y_2(1) + \partial_x y_1(1) = -v_1(1)\}, \end{aligned}$$

$$\mathcal{A}_i(y_1,y_2,v_1,v_2) = (v_1, v_2, \partial_x^2 y_1, -\partial_x^4 y_2),$$

$$||(y_1,y_2,v_1,v_2)||_{\mathcal{D}(\mathcal{A}_i)}^2 = ||\mathcal{A}_i(y_1,y_2,v_1,v_2)||_{\mathcal{H}}^2 + ||(y_1,y_2,v_1,v_2)||_{\mathcal{H}}^2.$$

The result below concerns the well-posedness of the solutions of (S.1) and (S.2) and the behavior of $E_i(t), i = 1,2$ when $t \to +\infty$.

Proposition 1.
(i) For each $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{H}$, Problem (S.1) admits a unique finite energy solution such that, for all $T > 0$,

$$\int_0^T |\partial_t u_1(t,1)|^2 \mathrm{d}t \leq C \, \|(u_1^0, u_2^0, u_1^1, u_2^1)\|_{\mathcal{H}}^2,$$

where the constant $C > 0$ depends only on T. Moreover (u_1, u_2) satisfies (1).
(ii) We have $\lim_{t \to +\infty} E_1(t) = 0$, for each $(u_1^0, u_2^0, u_1^1, u_2^1)$ in $\mathcal{H}$ if and only if

$$\ell \notin \left\{ \sqrt{\frac{p^2}{q}\,\pi},\, p, q \in \mathbb{N}^* \right\}. \tag{2}$$

(iii) For each $(\vartheta_1^0, \vartheta_2^0, \vartheta_1^1, \vartheta_2^1) \in \mathcal{H}$, Problem (S.2) admits a unique finite energy solution such that, for all $T > 0$,

$$\int_0^T \left(|\partial_t \vartheta_1(t,1)|^2 + |\partial_{tx}^2 \vartheta_2(t,1)|^2 \right) \mathrm{d}t \leq C \|(\vartheta_1^0, \vartheta_2^0, \vartheta_1^1, \vartheta_2^1)\|_{\mathcal{H}}^2,$$

where the constant $C > 0$ depends only on T. Moreover $(\vartheta_1, \vartheta_2)$ satisfies (1).
(iv) We have $\lim_{t \to +\infty} E_2(t) = 0$, for each $(\vartheta_1^0, \vartheta_2^0, \vartheta_1^1, \vartheta_2^1)$ in $\mathcal{H}$ and for all $\ell > 0$.

Let $\mathcal{S}$ be the set of all numbers ρ such that $\rho \notin \mathbb{Q}$ (where $\mathbb{Q}$ denote the set of all rational numbers) and if $[0, a_1, \ldots, a_n, \ldots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. Let us notice that $\mathcal{S}$ is obviously uncountable and, by classical results on diophantine approximation, its Lebesgue measure is equal to zero. Roughly speaking the set $\mathcal{S}$ contains the irrationals which are"badly" approximable by rational numbers. According to a classical result if $\xi \in \mathcal{S}$ then there exists a constant $C_\xi > 0$ such that

$$|\sin(n\pi\xi)| \geq \frac{C_\xi}{n}, \, \forall \, n \geq 1. \tag{3}$$

The main result of this paper concerns the precise asymptotic behavior of the solutions of (S.1) and (S.2).

Theorem 1.1. *Suppose that ℓ satisfies condition (2).*
(i) For any $\ell > 0$ the system described by (S.1) is not exponentially stable in $\mathcal{H}$.
(ii) For all $\ell \in \mathcal{S}$ and for all $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}_1)$ the solution of the system (S.1) satisfies the estimate

$$E_1(t) \leq \frac{C}{1+t} \left\| (u_1^0, u_2^0, u_1^1, u_2^1) \right\|_{\mathcal{D}(\mathcal{A}_1)}^2, \forall t > 0. \tag{4}$$

(iii) For all $\varepsilon > 0$ there exists a set $B_\varepsilon \subset \mathbb{R}$, such that the Lebesgue measure of $\mathbb{R} \setminus B_\varepsilon$ is equal to zero, and a constant $C_\varepsilon > 0$ for which, if $\ell \in B_\varepsilon$, then for all $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}_1)$ the solution of the system (S.1) satisfies the estimate

$$E_1(t) \leq \frac{C_\varepsilon}{(1+t)^{\frac{1}{1+\varepsilon}}} \left\| (u_1^0, u_2^0, u_1^1, u_2^1) \right\|_{\mathcal{D}(\mathcal{A}_1)}^2, \forall t > 0. \tag{5}$$

Theorem 1.2.
(i) For any $\ell > 0$ the system described by (S.2) is not exponentially stable in $\mathcal{H}$.
(ii) There exists a constant $C > 0$ such that for all $\ell > 0$ and for all

$(\vartheta_1^0,\vartheta_2^0,\vartheta_1^1,\vartheta_2^1)\in\mathcal{D}(\mathcal{A}_2)$ *the solution of the system (S.2) satisfies the following estimate*

$$E_2(t)\leq C\,\frac{\ln^6(t)}{t^4}\,\left\|(\vartheta_1^0,\vartheta_2^0,\vartheta_1^1,\vartheta_2^1)\right\|^2_{\mathcal{D}(\mathcal{A}_2)},\forall t>0. \tag{6}$$

2. **Spectral analysis.** We define the operator $\mathcal{A}^c$ in $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A}^c)=\{(y_1,y_2,v_1,v_2)\in\mathcal{H}|\ (v_1,v_2)\in V,\partial_x y_1\in H^1(0,1),\ \partial_x^2 y_2\in H^2(1,1+\ell),$$
$$\partial_x^2 y_2(1+\ell)=0,\ \partial_x^2 y_2(1)=0,\ \partial_x^3 y_2(1)+\partial_x y_1(1)=0\},$$

$$\mathcal{A}^c(y_1,y_2,v_1,v_2)=(v_1,v_2,\partial_x^2 y_1,-\partial_x^4 y_2).$$

The embedding $\mathcal{D}(\mathcal{A}^c)\hookrightarrow\mathcal{H}$ is compact, the half plan $\Re(\lambda)>0$ is contained in the resolvent set of $\mathcal{A}^c$, and the spectrum of $\mathcal{A}^c$ is discret.
The eigenvalues of $\mathcal{A}^c$ are given by $\lambda_n,\ n\in\mathbb{Z}^*$, where for $n\in\mathbb{N}^*$, $\lambda_n=iz_n^2$, with z_n the n-th strictly positive root of

$$z\sin(z^2)\left(\cos(\ell z)\sinh(\ell z)-\cosh(\ell z)\sin(\ell z)\right)+2\cos(z^2)\sinh(\ell z)\sin(\ell z)=0, \tag{7}$$

and $\lambda_{-n}=-iz_n^2$. The eigenfunctions of $\mathcal{A}^c$ are given by $\Phi_n=\phi_n/\|\phi_n\|_{\mathcal{H}}$, where $\phi_n=(\phi_n^1,\phi_n^2,\lambda_n\phi_n^1,\lambda_n\phi_n^2)$, with

$$\phi_n^1=\sinh(\lambda_n x),$$
$$\phi_n^2=-\frac{\sinh(\lambda_n)}{2\sinh(\ell z_n)}\sinh((x-1-\ell)z_n)-\frac{\sinh(\lambda_n)}{2\sin(\ell z_n)}\sin((x-1-\ell)z_n).$$

In order to prove (10) we use a result in [8, p.120] to get that, for all $\varepsilon>0$ there exists a set $B_\varepsilon\subset\mathbb{R}$ such that the Lebesgue measure of $\mathbb{R}\setminus B_\varepsilon$ is equal to zero and a constant $C>0$, such that for any $\rho\in B_\varepsilon$

$$|\sin(n\rho)|\geq\frac{C}{n^{1+\varepsilon}},\forall n\geq 1.$$

Let us notice that by Roth's theorem B_ε contains all numbers having the property that is an algebraic irrational (see for instance [8, p.104]).

Proposition 2. *(i) There exists a constant $M>0$ such that for all $n\in\mathbb{N}^*$ we have*

$$|\lambda_{n+1}-\lambda_n|\geq M. \tag{8}$$

(ii) There exists $C>0$ such that for all $n\in\mathbb{N}^*$, $\ell\in\mathcal{S}$, we have

$$|\Phi_n^1(1)|\geq C/|z_n|^4. \tag{9}$$

(iii) For all $\varepsilon>0$, there exists $C_\varepsilon>0$ such that for all $n\in\mathbb{N}^*$, $\ell\in B_\varepsilon$, we have

$$|\Phi_n^1(1)|\geq C_\varepsilon/|z_n|^{4+\varepsilon}. \tag{10}$$

Proof. (i) We can restrict us to the the eigenvalues λ_n, with $n\in\mathbb{N}^*$. We multiply the equation (7) by $2z^{-1}\exp(-\ell z)$ and we note

$$\frac{e^{\ell z}}{2}g(z)=\sin(z^2)\left(\cos(\ell z)\sinh(\ell z)-\cosh(\ell z)\sin(\ell z)\right)+2\cos(z^2)\sinh(\ell z)\frac{\sin(\ell z)}{z},$$

so that z_n is the n-th strictly positive zero of g. Let $f(z)=\big(\cos(\ell z)-\sin(\ell z)\big)\sin(z^2)$.
There exists $C_0>0$ such that $\big|f(z)-g(z)\big|\leq C_0/\Re(z)$.
For all $x>0$, there exists a unique couple (n_x,γ_x) such that

$$n_x\in\mathbb{N},\ x\ell=\pi/4+n_x\pi+\gamma_x,\ \gamma_x\geq-\pi/2,\ \gamma_x<\pi/2.$$

For $C>0$, we note $A(C)$ the set of complex numbers z such that $x=\Re(z)>0$ and $|\gamma_x|\geq \frac{C}{n_x}$. For $z\in A(C)$, we have

$$|\cos(\ell z)-\sin(\ell z)|/\sqrt{2}=|\sin(\ell\Im(z)i+\gamma_{\Re(z)})|\geq|\sin(\gamma_{\Re(z)})|\geq\left|\frac{2}{\pi}(\gamma_{\Re(z)})\right|\geq\frac{2}{\pi}\frac{C}{n_{\Re(z)}}.$$

On the other hand, we have for $\alpha\neq 0$, $\inf_{\Im(z)=\alpha}|f(z)|>0$. Let $\pi/4>\epsilon>0$. By choosing C_ϵ large enough, we apply the Rouché theorem on $K=[\sqrt{k\pi-\epsilon},\sqrt{k\pi+\epsilon}]\times[-\alpha,\alpha]$ with k such that $K\subset A(C_\epsilon)$. We then obtain a unique real root $z_{\tilde{k}}$ of g satisfying $|z_{\tilde{k}}^2-k\pi|<\epsilon$.

We now fix $n\in\mathbb{N}$ large enough. Let k be the biggest integer such that $\sqrt{k\pi+\epsilon}\ell<\frac{\pi}{4}+n\pi-\frac{C_\epsilon}{n}$, and k' the smallest such that $\sqrt{k'\pi-\epsilon}\ell>\frac{\pi}{4}+n\pi+\frac{C_\epsilon}{n}$,

By applying the Rouché theorem on $K=[\sqrt{k\pi+\epsilon},\sqrt{k'\pi-\epsilon}]\times[-\alpha,\alpha]$, we see that there exists $k'-k$ real roots of g, the roots of f being $\sqrt{\pi(k+1)},\dots,\sqrt{\pi(k'-1)}$ and $\frac{\pi}{4\ell}+n\frac{\pi}{\ell}$. Thus, we have found the roots of g; it now remains to locate them more precisely. Let $|\delta|\leq C_\epsilon$ such that $g(\frac{\pi}{4}+n\pi+\frac{\delta}{n})=0$. There exists then α_n such that $\delta\tan(\alpha_n+\delta\frac{\pi}{2})=\sqrt{2}\frac{\ell}{\pi}+o(1)$. By looking at the intersecting of the curves $\sqrt{2}\frac{\ell}{\pi x}$ and $\tan(\alpha_n+x\frac{\pi}{2})$, we can thus find $\delta_{n,1},\dots,\delta_{n,k'-k}$ separated numbers independently of n such that $g(\frac{\pi}{4}+n\pi+\frac{\delta_{n,i}}{n})=0,\ i=1,\dots,k'-k$, by increasing if necessary the value of C_ϵ, and this gives the first point.

(ii)-(iii) We have at first

$$\|\phi_n\|^2_{\mathcal{H}}/z_n^4=1+\ell\frac{\sin^2(z_n^2)}{2}\Big(\int_0^1\frac{\sinh^2(\ell z_n s)}{\sinh^2(\ell z_n)}\mathrm{d}s+\int_0^1\frac{\sin^2(\ell z_n s)}{\sin^2(\ell z_n)}\mathrm{d}s\Big),$$

which implies that

$$z_n^2\,|\Phi_n^1(1)|>C\min(|\sin(\ell z_n)|,|\sin(z_n^2)|).$$

Note that we have

$$|\sin(z_n^2)\cos(\ell z_n+\pi/4)|\simeq|\cos(z_n^2)\sin(\ell z_n)/z_n|.$$

Let $\epsilon>0$, small enough. If we have $|\cos(\ell z)-\sin(\ell z)|<\epsilon$, we deduce that there exists $C>0$ such that $|\sin(\ell z)|>C$, and thus $|\sin(z_n^2)|\geq C|\cos(z_n^2)|/|z_n|$, which implies that $|\sin(z_n^2)|^2\geq C^2/(|z_n|^2+C^2)$, and we get (ii)-(iii).

Now, if we have $|\cos(\ell z)-\sin(\ell z)|>\epsilon$, we then get by using the theorem of Rouché that $z_n=\sqrt{k\pi}+\delta/k$, with $|\delta|\leq C'$, with a constant $C'>0$ large enough, and $k\in\mathbb{N}$. We conclude then by the following argument:

- If $\ell\in\mathcal{S}$, we have $|\sin(\ell\sqrt{k\pi})|\geq c_1/\sqrt{k}$, with a constant $c_1>0$.
- If $\ell\in B_\varepsilon$, we have $|\sin(\ell\sqrt{k\pi})|\geq c_2/\sqrt{k^{1+\epsilon}}$, with a constant $c_2>0$.

We thus obtain:

- If $\ell\in\mathcal{S}$, we have $z_n^4\,|\Phi_n^1(1)|>C$.
- If $\ell\in B_\varepsilon$, we have $z_n^{4+\varepsilon}\,|\Phi_n^1(1)|>C$.

□

3. **Some background in stabilization of a class of evolution systems.** Let H be a Hilbert space equipped with the norm $||.||_H$, and let $A_1:\mathcal{D}(A_1)\to H$ be a self-adjoint, positive and boundedly invertible operator with compact resolvent. We introduce the scale of Hilbert spaces H_α, $\alpha\in\mathbb{R}$, as follows: for every $\alpha\geq 0$, $H_\alpha=\mathcal{D}(A_1^\alpha)$, with the norm $\|z\|_\alpha=\|A_1^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows: $H_{-\alpha}=H_\alpha^*$ for $\alpha>0$. The operator

A_1 can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A_1 : H_\alpha \to H_{\alpha-1} \,\forall\, \alpha \in \mathbb{R}.$$

The second ingredient needed for our construction is a bounded linear operator $B_1 : U \to H_{-\frac{1}{2}}$, where U is another Hilbert space which will be identified with its dual.

The systems we consider are described by

$$\ddot{w}(t) + A_1 w(t) + By(t) = 0,\;\; w(0) = w_0,\;\; \dot{w}(0) = w_1, \tag{11}$$

$$y(t) = B_1^* \dot{w}(t), \tag{12}$$

where $t \in [0, \infty)$ is the time. The equation (11) is understood as an equation in $H_{-\frac{1}{2}}$, i.e., all the terms are in $H_{-\frac{1}{2}}$. Most of the linear equations modelling the damped vibrations of elastic structures can be written in the form (11), where w stands for the displacement field and the term $B_1 B_1^* \dot{w}(t)$, represents a viscous feedback damping. The system (11)-(12) is well-posed :
for $(w_0, w_1) \in \mathcal{D}(A_1^{\frac{1}{2}}) \times H$, the problem (11)-(12) admits a unique solution

$$w \in C([0, \infty); \mathcal{D}(A_1^{\frac{1}{2}})) \cap C^1([0, \infty); H)$$

such that $B_1^* w(\cdot) \in H^1(0, T; U)$. Moreover w satisfies, for all $t \geq 0$, the energy estimate

$$\|(w_0, w_1)\|^2_{\mathcal{D}(A_1^{\frac{1}{2}}) \times H} - \|(w(t), \dot{w}(t))\|^2_{\mathcal{D}(A_1^{\frac{1}{2}}) \times H} = 2 \int_0^t \|B_1^* \dot{w}(s)\|^2_U \,\mathrm{d}s\,. \tag{13}$$

From (13) it follows that the mapping $t \mapsto \|(w(t), \dot{w}(t))\|^2_{\mathcal{D}(A_1^{\frac{1}{2}}) \times H}$ is non increasing.

We consider the initial value problem

$$\ddot{\phi}(t) + A_1 \phi(t) = 0, \tag{14}$$

$$\phi(0) = w_0,\, \dot{\phi}(0) = w_1. \tag{15}$$

It is well known that (14)-(15) is well-posed in $\mathcal{D}(A_1) \times \mathcal{D}(A_1^{\frac{1}{2}})$ and in $\mathcal{D}(A_1^{\frac{1}{2}}) \times H$.

Consider now the unbounded linear operator

$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \to H_{\frac{1}{2}} \times H,\, \mathcal{A}_d = \begin{pmatrix} 0 & I \\ -A_1 & -B_1 B_1^* \end{pmatrix},$$

where

$$\mathcal{D}(\mathcal{A}_d) = \Big\{(u, v) \in H_{\frac{1}{2}} \times H,\, A_1 u + B_1 B_1^* v \in H,\, v \in H_{\frac{1}{2}}\Big\}.$$

The result below, proved in [2], show that, under a certain regularity assumption, the exponential stability of (11)-(12) is a consequence of an observability inequality. More precisely, we have:

Theorem 3.1. *Assume that for any $\gamma > 0$ we have*

$$\sup_{\Re(\lambda) = \gamma} \left\| \lambda B_1^* (\lambda^2 I + A_1)^{-1} B_1 \right\|_{\mathcal{L}(U)} < \infty. \tag{16}$$

Then the following assertions are equivalent:

1. *There exist $T, C > 0$ such that : for all $(w^0, w^1) \in \mathcal{D}(A_1) \times \mathcal{D}(A_1^{\frac{1}{2}})$ we have*

$$\int_0^T \left|\left| B_1^* \dot{\phi}(t) \right|\right|^2_U dt \geq C \left|\left|(w^0, w^1)\right|\right|^2_{\mathcal{D}(A_1^{\frac{1}{2}}) \times H}$$

2. There exist a constants $\beta, C_1 > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in \mathcal{D}(A_1^{\frac{1}{2}}) \times H$ we have

$$\|(w(t), \dot{w}(t))\|_{\mathcal{D}(A_1^{\frac{1}{2}})\times H} \leq C_1\, e^{-\beta t}\, ||(w^0, w^1)||_{\mathcal{D}(A_1^{\frac{1}{2}})\times H}.$$

The result below, proved in [2], show that, under a certain regularity assumption, the polynomial stability of (11)-(12) is a consequence of a weak observability inequality. More precisely, we have:

Theorem 3.2. *Assume that for any $\gamma > 0$ we have (16). Then the following assertion holds true:*
If there exist $T, C > 0, \alpha > -\frac{1}{2}$ such that : for all $(w^0, w^1) \in \mathcal{D}(A_1) \times \mathcal{D}(A_1^{\frac{1}{2}})$ we have

$$\int_0^T \left|\left|B_1^* \dot{\phi}(t)\right|\right|_U^2 \geq C\, \left|\left|(w^0, w^1)\right|\right|^2_{H_{-\alpha}\times H_{-\alpha-\frac{1}{2}}}.$$

then there exists a constant $C_1 > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in \mathcal{D}(\mathcal{A}_d)$ we have

$$\|(w(t), \dot{w}(t))\|_{\mathcal{D}(A_1^{\frac{1}{2}})\times H} \leq \frac{C_1}{(1+t)^{\frac{1}{4\alpha+2}}}\, ||(w^0, w^1)||_{\mathcal{D}(\mathcal{A}_d)}.$$

4. **Regularity property.** Let $A_1 : V \to V'$,

$$A_1 = \begin{pmatrix} -\dfrac{d^2}{dx^2} & 0 \\ 0 & \dfrac{d^4}{dx^4} \end{pmatrix},$$

$$\mathcal{D}(A_1) = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in V \;\middle|\; \begin{array}{l} \dfrac{dy_1}{dx} \in H^1(0,1),\ \dfrac{d^2y_2}{dx^2} \in H^2(1, 1+\ell), \\ \dfrac{d^2y_2}{dx^2}(1+\ell) = 0,\ \dfrac{d^2y_2}{dx^2}(1) = 0, \\ \dfrac{d^3y_2}{dx^3}(1) + \dfrac{dy_1}{dx}(1) = 0 \end{array} \right\}$$

and $B_1 : \mathbb{R} \to V'$, $B_1 k = A_1 D k$, $\forall\, k \in \mathbb{R}$,
where V' is the dual space of V obtained by means of inner product in $L^2(0,1) \times L^2(1, 1+\ell)$, $Dk = (W_1, W_2)$ is the solution of

$$\begin{cases} \frac{d^2W_1}{dx^2} = 0,\ (0,1), \\ \frac{d^4W2}{dx^4} = 0,\ (1,\ell), \\ W_1(0) = 0,\ W_2(1+\ell) = 0,\ \frac{d^2W_2}{dx^2}(1) = 0,\ \frac{d^2W_2}{dx^2}(1+\ell) = 0, \\ W_1(1) = W_2(1), \\ \frac{d^3W_1}{dx^3}(1) + \frac{dW_1}{dx}(1) = k. \end{cases}$$

We have : $(B_1)^* \begin{pmatrix} p \\ q \end{pmatrix} = p(1),\ \forall\, (p,q) \in V$.

Proposition 3. *Let $\gamma > 0$, be a fixed real number and $C_\gamma = \Big\{\lambda \in \mathcal{C} \mid \Re(\lambda) = \gamma\Big\}$. Then the function $H(\lambda) = \lambda\, (B_1)^*(\lambda^2 + A_1)^{-1} B_1$, is given, for $\Re(\lambda) > 0$, by*

$$H(\lambda) = \frac{\sinh(\lambda)\sinh(w\ell)\sin(w\ell)}{\cosh(\lambda)\sinh(w\ell)\sin(w\ell) - \frac{w\sinh(\lambda)}{2i}(\cosh(wl)\sin(w\ell) - \sinh(w\ell)\cos(w\ell))},$$

where w is the unique complex number satisfying the conditions

$$\lambda = iw^2,\ w = re^{i\theta},\ \textit{with } r > 0 \textit{ and } \theta \in [-\frac{\pi}{2}, 0].$$

Proof. Let $k \in \mathbb{R}$. It can be easily checked that $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (\lambda^2 + A_1^1)^{-1} B_1^1 k$ satisfies:

$$\lambda^2 y_1(x) - \frac{d^2 y_1}{dx^2}(x) = 0,\ x \in (0,1),\ \Re(\lambda) > 0, \tag{17}$$

$$\lambda^2 y_2(x) + \frac{d^4 y_2}{dx^4}(x) = 0,\ x \in (1, 1+\ell),\ \Re(\lambda) > 0, \tag{18}$$

$$y_1(0) = y_2(1+\ell) = \frac{d^2 y_2}{dx}(1) = \frac{d^2 y_2}{dx^2}(1+\ell) = 0, \tag{19}$$

$$y_1(1) = y_2(1),\ \frac{dy_1}{dx}(1) + \frac{d^3 y_2}{dx^3}(1) = k. \tag{20}$$

The solutions of (17)-(18) have the form

$$\begin{cases} y_1(x) = A\sinh(\lambda x), & x \in (0,1), \\ y_2(x) = A_1 \sinh(w(x-1-\ell)) + B_1 \sin(w(x-1-\ell)), & x \in (1, 1+\ell), \end{cases}$$

where A, A_1, B_1 are constants. Consequently, the solutions of (17)-(20) have the following form

$$\begin{cases} y_1(x) = \dfrac{2k \sinh(w\ell)\sin(w\ell)\sinh(\lambda x)}{\omega}, \\ \forall\, x \in (0,1), \\ \\ y_2(x) = \dfrac{k\sinh(\lambda)(\sin(w\ell)\sinh(w(x-1-\ell)) - \sinh(w\ell)\ \sin(w(x-1-\ell)))}{\omega}, \\ \forall\, x \in (1, 1+\ell), \end{cases}$$

with

$$\omega = 2\lambda \sinh(w\ell)\sin(w\ell)\cosh(\lambda) - w^3 \sinh(\lambda)(\cosh(w\ell)\sin(w\ell) - \cos(w\ell)\sinh(w\ell)).$$

Then, for $\Re(\lambda) > 0$, we get

$$H(\lambda) = \frac{\sinh(\lambda)\sinh(wl)\sin(w\ell)}{\cosh(\lambda)\sinh(w\ell)\sin(w\ell) - \frac{w\sinh(\lambda)}{2i}(\cosh(w\ell)\sin(w\ell) - \sinh(w\ell)\cos(w\ell))}.$$

□

Lemma 4.1. *For any $\gamma > 0$ we have*

$$\sup_{\Re(\lambda)=\gamma} |H(\lambda)| < \infty.$$

Proof. Let us suppose that H is not bounded on C_γ. In this case there exists a sequence $(\lambda_n = iw_n^2) \subset C_\gamma$ such that

$$\lim_{n\to+\infty} |H(\lambda_n)| = +\infty. \tag{21}$$

As H_1 is analytical in the open set $D = \Big\{ w \in \mathcal{C} \mid \Re(w)\Im(w) < 0 \Big\}$ and $C_\gamma \subset D$, relation (21) clearly implies that $|w_n| \to +\infty$. Due to the definition of C_γ, this can happen in the following two situations:

$$|\Re(w_n)| \to +\infty,\ |\Im(w_n)| = \frac{\gamma}{2\,|\Re(w_n)|} \to 0, \tag{22}$$

or

$$|\Im(w_n)| \to +\infty,\ |\Re(w_n)| = \frac{\gamma}{2\,|\Im(w_n)|} \to 0. \tag{23}$$

Suppose that (22) holds true.
In this case a simple calculation shows that

$$\lim_{n\to+\infty} 2i\,\frac{\cosh(-2\gamma + i(\Re(w_n))^2)}{\sinh(-2\gamma + i(\Re(w_n))^2)} - \Re(w_n) + \Re(w_n)\,\frac{\cos(\Re(w_n)\ell)}{\sin(\Re(w_n)\ell)} = 0, \tag{24}$$

$$\lim_{n\to+\infty} \Re\left\{\frac{\cosh(-2\gamma + i(\Re(w_n))^2)}{\sinh(-2\gamma + i(\Re(w_n))^2)}\right\} =$$

$$\lim_{n\to+\infty} -\frac{1}{2}\,\frac{\sinh(4\gamma)}{\sinh^2(2\gamma)\cos^2((\Re(w_n))^2) + \cosh^2(2\gamma)\sin^2((\Re(w_n))^2)} = 0 \tag{25}$$

Relations (21),(24) and (25) imply that

$$\lim_{n\to+\infty} \cos^2((\Re(w_n))^2) = \lim_{n\to+\infty} \sin^2((\Re(w_n))^2) = 0.$$

It follows that (21) and (22) cannot be both true.
By a similar method we can show that (21) and (23) cannot hold both true. This means that assumption (21) is false, i.e. H is bounded on C_γ.
The bounds are uniform with respect to ℓ since $\sup\limits_{\lambda\in C_\gamma} |H(\lambda)|$, depend continously on $\ell > 0$.

□

5. **Observability inequalities.** In this section we gather, for easy reference, some observability inequalities concerning the trace, at the point $x = 1$ and at the point $x = 0$ respectively, of the solutions of the following equations:

$$(S.3)\begin{cases} (\partial_t^2 v_1 - \partial_x^2 v_1)(t,x) = 0,\ x \in (0,1),\ (\partial_t^2 v_2 + \partial_x^4 v_2)(t,x) = 0,\ x \in (1, 1+\ell), \\ v_1(t,0) = 0,\ u_2(t, 1+\ell) = 0,\ \partial_x^2 u_2(t, 1+\ell) = 0,\ \partial_x^2 u_2(t,1) = 0, \\ v_1(t,1) = u_2(t,1),\ \partial_x^3 u_2(t,1) + \partial_x u_1(t,1) = 0, \\ v_i(0,x) = v_i^0(x),\ \partial_t v_i(0,x) = v_i^1(x),\ i = 1,2. \end{cases}$$

The result can be stated as follows

Proposition 4. *Let $T > 2$ be fixed, then we have the following estimate:*
(i) For all $\ell \in \mathcal{S}$ the solution (v_1, v_2) of (S.3) satisfies

$$\int_0^T |\partial_t v_1(1,t)|^2\,\mathrm{d}t \geq C_\ell \,\|(v_1^0, v_1^1, v_2^0, v_2^1)\|^2_{L^2(0,1)\times H^{-1}(0,1)\times L^2(1,1+\ell)\times H^{-2}(1,1+\ell)}, \tag{26}$$

for all $(v_1^0, v_2^0, v_1^0, v_2^1) \in \mathcal{H}$, where $C_\ell > 0$ is a constant depending only on ℓ.
(ii) For all $\varepsilon > 0$ and for almost all $\ell > 0$ the solution (v_1, v_2) of (S.3) satisfies

$$\int_0^T |\partial_t v_1(1,t)|^2\,dt \geq$$

$$C_{\ell,\varepsilon}\,\|(v_1^0, v_1^1, v_2^0, v_2^1)\|^2_{H^{-\varepsilon}(0,1)\times H^{-1-\varepsilon}(0,1)\times H^{-\varepsilon}(1,1+\ell)\times H^{-2-\varepsilon}(1,1+\ell)},$$

$$\forall (v_1^0, v_2^0, v_1^1, v_2^1) \in \mathcal{H}, \tag{27}$$

where $C_{\ell,\epsilon} > 0$ is a constant depending only on ℓ and ε.
(iii) The result in assertion 1 is sharp in the sense that, for all $\ell > 0$, there exists

a sequence $(v^0_{1,m}, v^0_{2,m}, v^1_{1,m}, v^1_{2,m}) \subset \mathcal{H}$ *such that the corresponding sequence of solutions* $(v_{1,m}, v_{2,m})$ *of (S.3) with initial data* $(v^0_{1,m}, v^0_{2,m}, v^1_{1,m}, v^1_{2,m})$ *satisfies for all* $\varepsilon > 0$

$$\lim_{m\to+\infty} \frac{\displaystyle\int_0^T \left|\frac{\partial v_{1,m}}{\partial t}(1,t)\right|^2 dt}{\|(v^0_{1,m}, v^1_{1,m}, v^0_{2,m}, v^1_{2,m})\|^2_{H^\varepsilon(0,1)\times H^{-1+\varepsilon}(0,1)\times H^\varepsilon(1,1+l)\times H^{-2+\varepsilon}(1,1+l)}} = 0. \tag{28}$$

Proof. Notice first that the left hand side of (26) is well defined and

$$\partial_t v_1(1,t) = \sum_{n=1}^{+\infty} \lambda_n \, a_n \, e^{\lambda_n t} \, \Phi^1_n(1), \quad \text{in } L^2(0,T), \tag{29}$$

provided that $(v^0_1, \; v^1_1, v^0_2, v^1_2) = \displaystyle\sum_{n=1}^{+\infty} a_n \, \Phi_n, \sum_{n=1}^{+\infty} |a_n|^2 < \infty$. Moreover from (29), the Ball-Slemrod generalization of Ingham's inequality, see [5], and from (8) (see Proposition 2) we obtain that, for all $T > 2$, there exists a constant $C_T > 0$ such that

$$\int_0^T |\partial_t v_1(1,t)|^2 \, dt \geq C_T \sum_{n=1}^{+\infty} |\lambda_n a_n|^2 |\Phi^1_n(1)|^2. \tag{30}$$

Suppose now that ℓ belongs to the set $\mathcal{S}$. Then relations (30) and (3) imply the existence of a constant $K_{T,\ell} > 0$ such that

$$\int_0^T |\partial_t v_1(1,t)|^2 \, dt \geq K_{T,l} \sum_{n=1}^{+\infty} |a_k|^2, \forall \ell \in \mathcal{S},$$

which is exactly (26).

In order to prove (27) we use a result in [8, p.120] to get that, for all $\varepsilon > 0$ there exists a set B_ε such that for any $\rho \in B_\varepsilon$

$$|\sin(n\pi\rho)| \geq \frac{C}{n^{1+\epsilon}}, \forall n \geq 1. \tag{31}$$

Let us notice that by Roth's theorem B_ε contains all numbers having the property that ℓ is an algebraic irrational (see for instance [8, p.104]). Inequalities (30) and (31) obviously imply (27).

We still have to show the existence of a sequence satisfying (28). By using continous fractions we can construct a sequence (q_m) such that $q_m \to \infty$ and

$$|\sin(q_m l)| \leq \frac{1}{q_m}, \forall m \geq 1. \tag{32}$$

Using (29) and (32) a simple calculation shows that the sequence $(v^0_{1,m}, v^0_{2,m}, v^1_{1,m}, v^1_{2,m}) = (\Phi^1_m, \Phi^2_m, 0, 0)$ satisfies (28).

□

Proof of Proposition 1 and of Theorem 1.1. The existence and uniqueness of finite energy solutions of (S.1) and of (S.2) can be obtained by standard semigroup method, see [13].

In order to prove assertion 2 it suffices to remark that they hold true for regular solutions (i.e. $\begin{pmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{pmatrix} \in C(0,T;\mathcal{D}(\mathcal{A}_1))$ (where $.' = \partial_t$.) and to use the density of $\mathcal{D}(\mathcal{A}_1)$ in $V \times L^2(0,1) \times L^2(1,1+\ell)$).

As the imbedding of $\mathcal{D}(\mathcal{A}_1)$ in $V \times L^2(0,1) \times L^2(1,1+\ell)$ is obviously compact. Since $\mathcal{A}_1$ is a maximal-dissipative operator in $V \times L^2(0,1) \times L^2(1,1+\ell)$, $\mathcal{A}_1$ has no purely imaginary eigenvalues if and only if ℓ satisfies condition (2) (see Lemma 6.2), and $\mathcal{A}_1$ has compact resolvent. Then, the strong stability estimate at the end of Proposition 1 can be obtained by applying the result in Section 5 of [11].

The proof of the assertion (iv) of Proposition (1) is a simple adaptation of the proof of assertion (ii) of Proposition (1), so we skip the details.

□

Proof of Theorem 1.1.
(i) *Proof of the first step of Theorem 1.1*
According to Theorem 3.1 (with A_1, B_1 defined in Section 4), the solutions of (S.1) satisfy the estimate

$$E(t) \leq Me^{-\omega t}E(0), \forall t \geq 0, \tag{33}$$

where $M, \omega > 0$ are constants depending only on ℓ, if and only if the solution (v_1, v_2) of (S.3) satisfies

$$\int_0^T |\partial_t v_1(1,s)|^2 \, ds \geq CE(0), \forall (v_1^0, v_2^0, v_1^1, v_2^1) \in \mathcal{H}.$$

The inequality above clearly contradicts assertion 3 in Proposition 4. So assumption (33) is false. We end up in this way the proof of the first assertion of Theorem 1.1.
(ii) *Proof of the second step of Theorem 1.1*
Let $\ell \in \mathcal{S}$. By Proposition 4, the solution (v_1, v_2) of (S.4) satisfies the inequality

$$\int_0^T \left| \frac{\partial v_1}{\partial t}(1,t) \right|^2 dt \geq K_1 \|(v_1^0, v_1^1, v_2^0, v_2^1)\|^2_{L^2(0,1)\times H^{-1}(0,1)\times L^2(1,1+l)\times H^{-2}(1,1+l)},$$
$$\forall (v_1^0, v_2^0, v_1^1, v_2^1) \in \mathcal{H},$$

where $K_1 > 0$ is a constant. The conclusion (5) follows now by simply using the Theorem 1.1 (with $\alpha = 0$).
(iii) *Proof of the third step of Theorem 1.1*
For $\varepsilon > 0$ let ℓ belongs to the set B_ε, introduced in Section 5. From (27), it follows (5) by Theorem 3.2 (with $\alpha = \frac{\varepsilon}{2}$).

□

6. Proof of Theorem 1.2.

Proof of first assertion of Theorem 1.2. From preceding results, by taking

$$v = (v_1, v_2, v_3, v_4) = \Phi_n e^{iz_n^2 t},$$

we have

$$\int_0^T \left(|\partial_t v_1(1)|^2 + |\partial_{xt}^2 v_2(1)|^2 \right) \mathrm{d}t \asymp W_n |a_n|^2, \tag{34}$$

where

$$W_n = z_n^4 \frac{|\phi_n^1(1)|^2 + |\partial_x \phi_n^2(1)|^2}{\|\phi_n\|^2_{\mathcal{H}}}. \tag{35}$$

For z_n large enough, we have

$$|W_n| \leq \sin^2(z_n^2)|1 + (1/4)(\text{cotan}(z_n \ell) + 3/2)^2|. \tag{36}$$

Now we can choose the indices n such that $\sin(z_n^2) \to 0$ and $\sin(z_n \ell) \not\to 0$, so that $W_n \to 0$, and this concludes the proof. □

Proof of second assertion of Theorem 1.2. We will employ the following frequency domain theorem for polynomial stability (see Liu-Rao [12]) of a C_0 semigroup of contractions on a Hilbert space:

Lemma 6.1. *A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space is $||e^{t\mathcal{L}}U_0|| \leq C\,\frac{\ln^{1+\frac{1}{\theta}}(t)}{t^{\frac{1}{\theta}}}||U||_{\mathcal{D}(\mathcal{L})}$ for some constant $C > 0$ and for $\theta > 0$ if*

$$\rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

$$\limsup_{|\beta|\to\infty} \frac{1}{\beta^\theta} \|(i\beta - \mathcal{L})^{-1}\| < \infty, \tag{37}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

Lemma 6.2. *The spectrum of $\mathcal{A}_2$ contains no point on the imaginary axis for all $\ell > 0$.*

Proof. Since $\mathcal{A}_2$ has compact resolvent, its will show that the equation

$$\mathcal{A}_2 Z = i\beta Z \tag{38}$$

with $Z = \begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix} \in \mathcal{D}(\mathcal{A}_2)$ and $\beta \neq 0$ has only the trivial solution.

By taking the inner product of (38) with Z and using

$$\Re\left(< \mathcal{A}_2 Z, Z >_{\mathcal{H}}\right) = -\,|v_1(1)|^2 - \left|\frac{dv_2}{dx}(1)\right|^2, \tag{39}$$

we obtain that $v_1(1) = 0$, $\frac{dv_2}{dx}(1) = 0$. Next, we eliminate v_1, v_2 in (38) to get a ordinary differential equation:

$$\begin{cases} \frac{d^2y_1}{dx^2} + \beta^2 y_1 = 0, \, (0,1), \\ \frac{d^4y_2}{dx^4} - \beta^2 y_2 = 0, \, (1, 1+\ell), \\ y_1(0) = y_1(1) = y_2(1+\ell) = \frac{d^2y_2}{dx^2}(1) = \frac{d^2y_2}{dx^2}(1+\ell) = 0, \\ \frac{d^3y_2}{dx^3}(1) + \frac{dy_1}{dx}(1) = 0, \, \frac{dy_2}{dx}(1) = 0. \end{cases}$$

Then, we can see easily that the above system has only trivial solution for all $\ell > 0$. □

Lemma 6.3. *The resolvent operator of $\mathcal{A}_2$ satisfies condition (37) for $\theta = \frac{1}{2}$.*

Proof of the second assertion of Theorem 1.1. By a result (see Liu-Rao [12]) it suffices to show that $\mathcal{A}_2$ satisfies the following two conditions:

$$\rho(\mathcal{A}_2) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{40}$$

and

$$\limsup_{|\beta|\to\infty} \frac{1}{\beta^{1/2}} \|(i\beta - \mathcal{A}_2)^{-1}\| < \infty, \tag{41}$$

where $\rho(\mathcal{A}_2)$ denotes the resolvent set of the operator $\mathcal{A}_2$.

By Lemma 6.2 the condition (40) is satisfied for all $\ell > 0$ satisifies (2). Suppose that the condition (41) is false. By the Banach-Steinhaus Theorem (see [7]), there exist a sequence of real numbers $\beta_n \to \infty$ and a sequence of vectors $Z_n = \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ v_{1,n} \\ v_{2,n} \end{pmatrix} \in \mathcal{D}(\mathcal{A}_2)$ with $\|Z_n\|_{\mathcal{H}} = 1$ such that

$$||\beta_n^{1/2}\,(i\beta_n I - \mathcal{A}_2)Z_n||_{\mathcal{H}} \to 0 \quad \text{as} \quad n \to \infty, \tag{42}$$

i.e.,

$$\beta_n^{1/2}\,(i\beta_n y_{1,n} - v_{1,n}, i\beta_n y_{2,n} - v_{2,n}) \equiv (f_n, h_n) \to 0 \quad \text{in } V, \tag{43}$$

$$\beta_n^{1/2}\left(i\beta_n v_{1,n} - \frac{d^2 y_{1,n}}{dx^2}\right) \equiv g_n \to 0 \quad \text{in } L^2(0,1), \tag{44}$$

$$\beta_n^{1/2}\left(i\beta_n v_{2,n} + \frac{d^4 y_{2,n}}{dx^4}\right) \equiv k_n \to 0 \quad \text{in } L^2(1, 1+\ell). \tag{45}$$

Our goal is to derive from (42) that $||Z_n||_{\mathcal{H}}$ converges to zero, thus, a contradiction. The proof is divided in four steps:

First step. We notice that from (39) we have

$$||\beta_n^{1/2}\,(i\beta_n I - \mathcal{A}_2)Z_n||_{\mathcal{H}} \geq |\Re\left(\langle \beta_n^{1/2}\,(i\beta_n I - \mathcal{A}_2)Z_n, Z_n\rangle_{\mathcal{H}}\right)| = |\beta_n|^{1/2}\left(|v_{1,n}(1)|^2 + \left|\frac{dv_{2,n}}{dx}(1)\right|^2\right).$$

Then, by (42)

$$|\beta_n|^{1/2}\,|v_{1,n}(1)| \to 0,\; |\beta_n|^{1/2}\left|\frac{dv_{2,n}}{dx}(1)\right| \to 0,\; |\beta_n|^{1/2}\left|\frac{d^3 y_{2,n}}{dx^3}(1) + \frac{dy_{1,n}}{dx}(1)\right| \to 0.$$

This further leads to

$$|\beta_n|^{\frac{3}{2}}\,|y_{1,n}(1)| \to 0,\; |\beta_n|^{3/2}\left|\frac{dy_{2,n}}{dx}(1)\right| \to 0,\; \left|\frac{d^2 y_{2,n}}{dx^2}(1)\right| \to 0 \tag{46}$$

due to (43) and the trace theorem.

Second step. We express now $v_{1,n}, v_{2,n}$ in function of $y_{1,n}, y_{2,n}$ from equation (43)-(45) and substitute it into (44)-(45) to get

$$\beta_n^{1/2}\left(-\beta_n^2 y_{1,n} - \frac{d^2 y_{1,n}}{dx^2}\right) = g_n + i\beta_n f_n, \tag{47}$$

$$\beta_n^{1/2}\left(-\beta_n^2 y_{2,n} + \frac{d^4 y_{2,n}}{dx^4}\right) = k_n + i\beta_n h_n. \tag{48}$$

Next, we take the inner product of (47) with $q(x)\dfrac{dy_{1,n}}{dx}$ in $L^2(0,1)$ where $q(x) \in C^1([0,1])$ and $q(0) = 0$. We obtain that

$$\int_0^1 \beta_n^{1/2}\left(-\beta_n^2 y_{1,n} - \frac{d^2 y_{1,n}}{dx^2}\right) q(x)\frac{d\bar{y}_{1,n}}{dx}\,dx = \int_0^1 \left(g_n + i\beta_n f_n\right) q(x)\frac{d\bar{y}_{1,n}}{dx}\,dx$$

$$= \int_0^1 g_n\, q(x) \frac{d\bar{y}_{1,n}}{dx}\, dx - i \int_0^1 q \frac{df_n}{dx}\, \beta_n \bar{y}_{1,n}\, dx$$
$$- i \int_0^1 f_n \frac{dq}{dx}\, \beta_n \bar{y}_{1,n}\, dx + i f_n(1) q(1) \beta_n \bar{y}_{1,n}(1). \quad (49)$$

It is clear that the right-hand side of (49) converges to zero since f_n, g_n converge to zero in H^1 and L^2, respectively.

By a straight-forward calculation,

$$\Re \left\{ \int_0^1 -\beta_n^2 y_{1,n}\, q \frac{d\bar{y}_{1,n}}{dx}\, dx \right\} = -\frac{1}{2} q(1) |\beta_n y_{1,n}(1)|^2 + \frac{1}{2} \int_0^1 \frac{dq}{dx} |\beta_n y_{1,n}|^2 dx$$

and

$$\Re \left\{ \int_0^1 -\frac{d^2 y_{1,n}}{dx^2}\, q \frac{d\bar{y}_{1,n}}{dx}\, dx \right\} = -\frac{1}{2}\, q(1) \left| \frac{dy_{1,n}}{dx}(1) \right|^2 + \frac{1}{2} \int_0^1 \left| \frac{dy_{1,n}}{dx} \right|^2 \frac{dq}{dx} dx. \quad (50)$$

According to (46), we simplify (49), then take its real parts. This leads to

$$\int_0^1 \frac{dq}{dx}\, |\beta_n y_{1,n}|^2\, dx + \int_0^1 \frac{dq}{dx} \left| \frac{dy_{1,n}}{dx} \right|^2 dx - q(1) \left| \frac{dy_{1,n}}{dx}(1) \right|^2 \to 0. \quad (51)$$

Similarly, we take the inner product of (47) with $q_1(x) \dfrac{dy_{2,n}}{dx}$ in $L^2(1, 1+\ell)$ with $q_1 \in C^2([1, 1+\ell])$ and $q_1(1+\ell) = 0$, then repeat the above procedure and since

$$\int_1^{1+\ell} \left| \frac{dy_{2,n}}{dx} \right|^2 dx = -\frac{1}{i\beta_n} \int_1^{1+\ell} v_{2,n} \frac{d^2 \bar{y}_{2,n}}{dx^2} - \frac{1}{i\beta_n} \int_1^{1+\ell} (i\beta_n\, y_{2,n} - v_{2,n}) \frac{d^2 \bar{y}_{2,n}}{dx^2} dx$$
$$- \left(\frac{1}{\beta_n} \frac{d\bar{y}_{2,n}}{dx}(1) \right) (\beta_n y_{2,n}(1)),$$

then from the boundedness of $v_{2,n}, i\beta_n y_{2,n} - v_{2,n}, \frac{d^2 y_{2,n}}{dx^2}$, in $L^2(1, 1+\ell)$ we have $\frac{dy_{2,n}}{dx}$ converges to zero in $L^2(1, 1+\ell)$. This will give

$$\int_1^{1+\ell} \frac{dq_1}{dx} |\beta_n y_{2,n}|^2 dx + \int_1^{1+\ell} 3 \frac{dq_1}{dx} \left| \frac{d^2 y_{2,n}}{dx^2} \right|^2 dx - 2\, \frac{d^3 y_{2,n}}{dx^3}(1) q_1(1) \frac{d\bar{y}_{2,n}}{dx}(1) \to 0. \quad (52)$$

Third step. Next, we show that $\dfrac{dy_{1,n}}{dx}(1)$, $\dfrac{d^3 y_{2,n}}{dx^3}(1)$, converge to zero. We take the inner product of (48) with $\frac{1}{\phi_n^{1/2}} e^{-\phi_n^{1/2} h(x)}$ in $L^2(1, 1+\ell)$ where $h(x) = x - 1$.

This leads to

$$\int_1^{1+\ell} \left(\phi_n^2 e^{-\phi_n^{1/2} h}\, y_{2,n} - e^{-\phi_n^{1/2} h}\, \frac{d^4 y_{2,n}}{dx^4} \right) dx \to 0. \quad (53)$$

Performing integration by parts to the second term on the left-hand side of (53), we obtain

$$\int_1^{1+\ell} \left(\phi_n^2 e^{-\phi_n^{1/2} h}\, y_{2,n} - e^{-\phi_n^{1/2} h}\, \frac{d^4 y_{2,n}}{dx^4} \right) dx = \frac{d^3 y_{2,n}}{dx^3}(1) + \phi_n\, \frac{dy_{2,n}}{dx}(1) +$$
$$\phi_n^{3/2} y_{2,n}(1) + o(1) \quad (54)$$

Thus, according to (46), we simplify (54) to

$$\frac{d^3 y_{2,n}}{dx^3}(1) \to 0$$

and consequently

$$\frac{dy_{1,n}}{dx}(1) \to 0. \tag{55}$$

Then

$$\frac{dy_{2,n}}{dx^2}(1)\,\frac{d^3y_{2,n}}{dx^3}(1) \to 0. \tag{56}$$

In view of (55)-(56), we simplify (51) and (52) to

$$\int_0^1 \frac{dq}{dx}|\beta_n y_{1,n}|^2 dx + \int_0^1 \frac{dq}{dx}\left|\frac{dy_{1,n}}{dx}\right|^2 dx \to 0. \tag{57}$$

$$\int_1^{1+\ell} \frac{dq_1}{dx}|\beta_n y_{2,n}|^2 dx + \int_1^{1+\ell} 3\frac{dq_1}{dx}\left|\frac{d^2y_{2,n}}{dx^2}\right|^2 dx \to 0 \tag{58}$$

respectively.

Fourth step. Finally, we choose $q(x)$ and $q_1(x)$ so that $\frac{dq}{dx}$ is strictly positive, $\frac{dq_1}{dx}$ is strictly negative. This can be done by taking

$$q(x) = e^x - 1, \qquad q_1(x) = e^{(1+\ell-x)} - 1.$$

Therefore, (57) and (58) imply

$$\|\beta_n y_{1,n}\|_{L^2(0,1)},\ \|\beta_n y_{2,n}\|_{L^2(1,1+\ell)} \to 0, \qquad \|(y_{1,n}, y_{2,n})\|_V \to 0.$$

In view of (43), we also get

$$\|v_{1,n}\|_{L^2(0,1)},\ \|v_{2,n}\|_{L^2(1,1+\ell)} \to 0,$$

which clearly contradicts (42).

□

7. **Related question.** A question related to the problem studied in this paper is the stabilization of a star and tree-shaped network of string-beams [4].

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