

# On the Boltzmann equation for charged particle beams under the effect of strong magnetic fields

Mihai Bostan \*

(March 8, 2015)

## Abstract

The subject matter of this paper concerns the paraxial approximation for the transport of charged particles. We focus on the magnetic confinement properties of charged particle beams. The collisions between particles are taken into account through the Boltzmann kernel. We derive the magnetic high field limit and we emphasize the main properties of the averaged Boltzmann collision kernel, together with its equilibria.

**Keywords:** Particle beams, Finite Larmor radius regime, Boltzmann equation,  $H$ -theorem.

**MSC:** 35Q20, 35Q83.

## 1 Introduction

The charged particle beams play a major role in many applications : particle physics experiments, particle therapy, astrophysics, etc. The main mathematical model for studying beam propagation is the Vlasov-Poisson or Vlasov-Maxwell system. The numerical resolution of these systems requires huge computational efforts and therefore

---

\*Aix Marseille Université, CNRS, Centrale Marseille, LAMP, UMR 7353 13453 Marseille France.

E-mail : [bostan@cmi.univ-mrs.fr](mailto:bostan@cmi.univ-mrs.fr) [mihai.bostan@univ-amu.fr](mailto:mihai.bostan@univ-amu.fr)

simplified models have been derived. One of the reduced model which is often used in accelerator physics is the paraxial approximation [10], [11], [22], [19], [12]. This model was designed for beams which possess an optical axis, assuming that the particles remain close to the optical axis, having about the same kinetic energy. This paper is devoted to the study of the confinement properties of charged particle beams, under the action of strong magnetic fields parallel to the optical axis. We neglect the self-consistent electro-magnetic field but we take into account the collisions between particles. If we denote by  $\mathcal{F} = \mathcal{F}(t, \mathcal{X}, \mathcal{V}) \geq 0$  the presence density of the charged particles in the position-velocity phase space  $(\mathcal{X}, \mathcal{V})$ , we are led to the problem

$$\partial_t \mathcal{F} + \mathcal{V} \cdot \nabla_{\mathcal{X}} \mathcal{F} + \frac{q}{m} (\mathcal{V} \wedge \mathcal{B}^\varepsilon) \cdot \nabla_{\mathcal{V}} \mathcal{F} = Q(\mathcal{F}, \mathcal{F}), \quad (t, \mathcal{X}, \mathcal{V}) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (1)$$

$$\mathcal{F}(0, \mathcal{X}, \mathcal{V}) = \mathcal{F}^{\text{in}}(\mathcal{X}, \mathcal{V}), \quad (\mathcal{X}, \mathcal{V}) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad (2)$$

where  $\mathcal{B}^\varepsilon$  is the external magnetic field,  $m$  is the particle mass,  $q$  is the particle charge and  $Q$  stands for the collision kernel. Let us consider that the optical axis is parallel to  $\mathcal{X}_3$  and that the magnetic field is stationary, uniform and strong

$$\mathcal{B}^\varepsilon = \left( 0, 0, \frac{B}{\varepsilon} \right)$$

for some constant  $B \neq 0$  and a small parameter  $\varepsilon > 0$ . If we take as observation time  $T_{\text{obs}} \sim m/qB$ , the parameter  $\varepsilon$  appears as the ratio between the cyclotronic period  $T_c^\varepsilon = 2\pi/\omega_c^\varepsilon = \varepsilon 2\pi m/qB$  and  $T_{\text{obs}}$ . Therefore we deal with a two time scale problem, coupling a slow time variable, associated to the reference time  $T_{\text{obs}}$ , and a fast time scale, coming from the fast cyclotronic motion. We assume that the typical velocity in the parallel direction is much larger than that in the perpendicular directions. Since the particles remain close to the optical axis, we take a space unit in the perpendicular directions much smaller than that in the parallel direction. Finally we search for a presence density of the form

$$\mathcal{F}^\varepsilon(t, \mathcal{X}, \mathcal{V}) = \frac{1}{\varepsilon^3} f^\varepsilon \left( t, \frac{\mathcal{X}_1}{\varepsilon^2}, \frac{\mathcal{X}_2}{\varepsilon^2}, \mathcal{X}_3, \frac{\mathcal{V}_1}{\varepsilon}, \frac{\mathcal{V}_2}{\varepsilon}, \frac{\mathcal{V}_3 - u_3}{\varepsilon} \right) \quad (3)$$

where  $u_3 = u_3(t, \mathcal{X}_1/\varepsilon^2, \mathcal{X}_2/\varepsilon^2, \mathcal{X}_3)$  is about the mean parallel velocity

$$\frac{\int_{\mathbb{R}^3} \mathcal{V}_3 \mathcal{F}^\varepsilon(t, \mathcal{X}, \mathcal{V}) \, d\mathcal{V}}{\int_{\mathbb{R}^3} \mathcal{F}^\varepsilon(t, \mathcal{X}, \mathcal{V}) \, d\mathcal{V}} = \frac{\int_{\mathbb{R}^3} (u_3 + \varepsilon v_3) f^\varepsilon(t, x, v) \, dv}{\int_{\mathbb{R}^3} f^\varepsilon(t, x, v) \, dv} = u_3 + \varepsilon \frac{\int_{\mathbb{R}^3} v_3 f^\varepsilon(t, x, v) \, dv}{\int_{\mathbb{R}^3} f^\varepsilon(t, x, v) \, dv}. \quad (4)$$

Observe that the Larmor radius scales like  $\varepsilon^2$  since both the typical perpendicular velocity and the cyclotronic period are of orders  $\varepsilon$ . This explains our choice for the space unit in the perpendicular directions in (3) : we focus on the finite Larmor radius regime *i.e.*, the space unit in the perpendicular directions and the Larmor radius are of the same order [14], [16].

The collisions between the particles are taken into account through the Boltzmann kernel [9], [23], [24], which writes

$$Q(\mathcal{F}, \mathcal{F}) = \int_{S^2} \int_{\mathbb{R}^3} \sigma(\mathcal{V} - \mathcal{V}', \omega) \{ \mathcal{F}(\mathcal{V} - \omega \otimes \omega(\mathcal{V} - \mathcal{V}')) \mathcal{F}(\mathcal{V}' + \omega \otimes \omega(\mathcal{V} - \mathcal{V}')) - \mathcal{F}(\mathcal{V}) \mathcal{F}(\mathcal{V}') \} d\mathcal{V}' d\omega$$

with  $\sigma(z, \omega) = |z|^\gamma b(z/|z| \cdot \omega)$ . For the presence density  $\mathcal{F}^\varepsilon$  in (3) we obtain

$$\begin{aligned} Q(\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon) &= \frac{\varepsilon^\gamma}{\varepsilon^3} \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) \{ f^\varepsilon(t, x, v - \omega \otimes \omega(v - v')) f^\varepsilon(t, x, v' + \omega \otimes \omega(v - v')) \\ &\quad - f^\varepsilon(t, x, v) f^\varepsilon(t, x, v') \} dv' d\omega \\ &= \varepsilon^{\gamma-3} Q(f^\varepsilon, f^\varepsilon) \end{aligned}$$

and the equation (1) becomes

$$\begin{aligned} \partial_t f^\varepsilon + (u_3 + \varepsilon v_3) \partial_{x_3} f^\varepsilon - v_3 \partial_{x_3} u_3 \partial_{v_3} f^\varepsilon + \frac{1}{\varepsilon} (\bar{v} \cdot \nabla_{\bar{x}} f^\varepsilon + \omega_c {}^\perp \bar{v} \cdot \nabla_{\bar{v}} f^\varepsilon) \\ - \frac{1}{\varepsilon} (\partial_t u_3 + u_3 \partial_{x_3} u_3) \partial_{v_3} f^\varepsilon - \frac{1}{\varepsilon^2} \bar{v} \cdot \nabla_{\bar{x}} u_3 \partial_{v_3} f^\varepsilon = \varepsilon^\gamma Q(f^\varepsilon, f^\varepsilon) \end{aligned} \quad (5)$$

where  $\bar{x} = (x_1, x_2)$ ,  $\bar{v} = (v_1, v_2)$ ,  ${}^\perp \bar{v} = (v_2, -v_1)$ ,  $\omega_c = qB/m$ . For the sake of simplicity we focus on the Maxwell molecule case (*i.e.*,  $\gamma = 0$ ) but other cases can be analyzed as well. Actually the key point when considering any  $\gamma$  model consists in gyroaveraging the Boltzmann collision kernel. In the perspective of possible treatments of other cases with  $\gamma \neq 0$ , we prefer to perform the gyroaverage of the Boltzmann kernel for any  $\gamma$ , such that this step could be used unchanged for further developpements. In the Maxwell molecule case (5) writes

$$\begin{aligned} \partial_t f^\varepsilon + (u_3 + \varepsilon v_3) \partial_{x_3} f^\varepsilon - v_3 \partial_{x_3} u_3 \partial_{v_3} f^\varepsilon + \frac{1}{\varepsilon} (\bar{v} \cdot \nabla_{\bar{x}} f^\varepsilon + \omega_c {}^\perp \bar{v} \cdot \nabla_{\bar{v}} f^\varepsilon) \\ - \frac{1}{\varepsilon} (\partial_t u_3 + u_3 \partial_{x_3} u_3) \partial_{v_3} f^\varepsilon - \frac{1}{\varepsilon^2} \bar{v} \cdot \nabla_{\bar{x}} u_3 \partial_{v_3} f^\varepsilon = Q_0(f^\varepsilon, f^\varepsilon) \end{aligned} \quad (6)$$

where the scattering section entering the kernel  $Q_0$  is given by  $\sigma_0(z, \omega) = b(z/|z| \cdot \omega)$ .

We complete the model by the initial condition

$$f^\varepsilon(0, x, v) = f^{\text{in}}(x, v), \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

A formal expansion

$$f^\varepsilon = f + \varepsilon f^1 + \varepsilon^2 f^2 + \dots \quad (7)$$

leads to the equality

$$\bar{v} \cdot \nabla_{\bar{x}} u_3 \partial_{v_3} f = 0. \quad (8)$$

Multiplying (8) by  $v_3$  and integrating with respect to  $v_3 \in \mathbb{R}$  yield

$$\bar{v} \cdot \nabla_{\bar{x}} u_3 \int_{\mathbb{R}} f(t, x, v) dv_3 = 0$$

and thus we are led to consider  $u_3 = u_3(t, x_3)$ . In that case, the leading order term in (6) is

$$\frac{1}{\varepsilon} (\bar{v} \cdot \nabla_{\bar{x}} f^\varepsilon + \omega_c \perp \bar{v} \cdot \nabla_{\bar{v}} f^\varepsilon) - \frac{1}{\varepsilon} (\partial_t u_3 + u_3 \partial_{x_3} u_3) \partial_{v_3} f^\varepsilon$$

and therefore we obtain

$$\bar{v} \cdot \nabla_{\bar{x}} f + \omega_c \perp \bar{v} \cdot \nabla_{\bar{v}} f - (\partial_t u_3 + u_3 \partial_{x_3} u_3) \partial_{v_3} f = 0.$$

Multiplying by  $v_3$  and integrating with respect to  $v_3 \in \mathbb{R}$  we deduce as before that

$$\operatorname{div}_{\bar{x}} \int_{\mathbb{R}} v_3 \bar{v} f dv_3 + \omega_c \operatorname{div}_{\bar{v}} \int_{\mathbb{R}} v_3 \perp \bar{v} f dv_3 + (\partial_t u_3 + u_3 \partial_{x_3} u_3) \int_{\mathbb{R}} f dv_3 = 0.$$

Integrating the previous equality with respect to  $(\bar{x}, \bar{v})$  implies

$$(\partial_t u_3 + u_3 \partial_{x_3} u_3) \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f dv d\bar{x} = 0$$

and therefore we expect that

$$\partial_t u_3 + u_3 \partial_{x_3} u_3 = 0.$$

After these observations (6) writes

$$\partial_t f^\varepsilon + (u_3 + \varepsilon v_3) \partial_{x_3} f^\varepsilon - v_3 \partial_{x_3} u_3 \partial_{v_3} f^\varepsilon + \frac{1}{\varepsilon} (\bar{v} \cdot \nabla_{\bar{x}} f^\varepsilon + \omega_c \perp \bar{v} \cdot \nabla_{\bar{v}} f^\varepsilon) = Q(f^\varepsilon, f^\varepsilon) \quad (9)$$

and thus the dominant density in (7) satisfies

$$\bar{v} \cdot \nabla_{\bar{x}} f + \omega_c \perp \bar{v} \cdot \nabla_{\bar{v}} f = 0 \quad (10)$$

$$\partial_t f + u_3 \partial_{x_3} f - v_3 \partial_{x_3} u_3 \partial_{v_3} f + \bar{v} \cdot \nabla_{\bar{x}} f^1 + \omega_c \perp \bar{v} \cdot \nabla_{\bar{v}} f^1 = Q(f, f). \quad (11)$$

Since we expect that  $\lim_{\varepsilon \searrow 0} f^\varepsilon = f$ , in order to get a good approximation for  $f^\varepsilon$ , we need to compute  $f$ . That is, we have to eliminate  $f^1$  in (11), thanks to the constraint

(10). This can be done by averaging along the characteristic flow of the transport operator  $\mathcal{T} = \bar{v} \cdot \nabla_{\bar{x}} + \omega_c {}^\perp \bar{v} \cdot \nabla_{\bar{v}}$ . Indeed, as the transport term  $\mathcal{T}f^1$  represents the derivative of  $f^1$  along this flow, its average will vanish, while the density  $f$  is left invariant by the same average (because, by (10),  $f$  is constant along this flow). The difficult task consists in averaging the Boltzmann collision kernel.

Averaged collision operators have been proposed by many authors [26], [7], [8], [17], [21]. Most of them have been obtained by linearization around Maxwellians, expecting that the Maxwellians belong to the equilibria of the averaged collision kernels. It happens that this fails to be true, at least in the finite Larmor radius regime.

The main goal of this paper is to derive the expression of the averaged version of the Boltzmann collision operator. Under strong magnetic fields, the particles turn fast on the Larmor circles and the collisions will be assimilated to interactions between pairs of Larmor circles. Only pairs of Larmor circles having non empty intersection will be in interaction, and the velocity collisions occur when the particles occupy the same position *i.e.*, a intersection point between circles. We also characterize the equilibria of the averaged Boltzmann collision kernel. In particular we will see that these equilibria are special products of Maxwellians, parametrized by six moments. We extend the averaging techniques employed in [4], [5], [6] where the relaxation Boltzmann operator, the Fokker-Planck and Fokker-Planck-Landau operators have been studied.

Our paper is organized as follows. In Section 2 we present the main results : the finite Larmor radius regime for particle beams interacting through the collision Boltzmann kernel. The averaged Boltzmann kernel is computed in Section 3. The equilibria of the averaged kernel follow thanks to a  $H$  type theorem, see Section 4. Fluid models around these equilibria are investigated as well. Some technical proofs involving similar computations to those in Section 3 are postponed to Appendix A.

## 2 Presentation of the main results

We appeal to the Boltzmann collision kernel for characterizing the interactions between particles

$$Q(f, f)(v) = \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) \{f(V)f(V') - f(v)f(v')\} dv' d\omega \quad (12)$$

where for any pre-collisional velocities  $v, v' \in \mathbb{R}^3$ , the functions  $V, V'$  stand for the post-collisional velocities

$$V(v, v', \omega) = v - (v - v', \omega)\omega, \quad V'(v, v', \omega) = v' + (v - v', \omega)\omega. \quad (13)$$

The function  $\sigma$  denotes the scattering section and has the form cf. [25]

$$\sigma(z, \omega) = |z|^\gamma b(z/|z| \cdot \omega), \quad \gamma = \frac{s-5}{s-1} \quad (14)$$

the number  $s$  characterizing the inverse power law of the interaction potential (the interaction force between particles being of order  $1/|z|^s$ ). Here  $b : [-1, 1] \rightarrow \mathbb{R}$  is a non negative even function. For simplicity we make the Grad angular cut-off hypothesis *i.e.*,  $b \in L^1(-1, 1)$ , saying that for any  $e \in S^2$

$$\int_{S^2} b(e \cdot \omega) d\omega = 2\pi \int_{-1}^1 b(u) du < +\infty.$$

As usual we distinguish between the gain and loss part of  $Q$

$$Q(f, f) = Q_+(f, f) - Q_-(f, f)$$

$$Q_+ = \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(V) f(V') dv' d\omega, \quad Q_- = \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(v) f(v') dv' d\omega.$$

We will compute the average of the gain and loss parts. For this we need first to introduce the definition and properties of the average operator along a characteristic flow. We introduce the linear operator  $\mathcal{T}$  defined in  $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$  by

$$\mathcal{T}u = \operatorname{div}_{x,v}(u b), \quad b = (\bar{v}, 0, \omega_c^\perp \bar{v}, 0), \quad \omega_c = \frac{qB}{m}$$

for any function  $u$  in the domain

$$D(\mathcal{T}) = \{u(x, v) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) : \operatorname{div}_{x,v}(u b) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\}.$$

The constraint (10) says that at any time  $t$  the density  $f(t, \cdot, \cdot)$  remains constant along the flow  $(X, V)(s; x, v)$  associated to the transport operator  $\bar{v} \cdot \nabla_x + \omega_c^\perp \bar{v} \cdot \nabla_v$

$$\frac{d\bar{X}}{ds} = \bar{V}(s), \quad \frac{dX_3}{ds} = 0, \quad \frac{d\bar{V}}{ds} = \omega_c^\perp \bar{V}(s), \quad \frac{dV_3}{ds} = 0, \quad (X, V)(0; x, v) = (x, v). \quad (15)$$

Therefore the density  $f(t, \cdot, \cdot)$  depends only on the invariants of (15)

$$f(t, x, v) = g(t, \omega_c x_1 + v_2, \omega_c x_2 - v_1, x_3, v_3, r = |\bar{v}|).$$

In order to determine the evolution of  $f$ , we need to eliminate the density  $f^1$ . For doing that it is enough to notice that  $\mathcal{T}$  is skew adjoint on  $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$  and therefore  $\mathcal{T}f^1$  belongs to the orthogonal of  $\ker \mathcal{T}$ . Therefore, taking the orthogonal projection of (11) onto  $\ker \mathcal{T}$  will allow us to get rid of  $f^1$

$$\text{Proj}_{\ker \mathcal{T}} \{ \partial_t f + u_3 \partial_{x_3} f - v_3 \partial_{x_3} u_3 \partial_{v_3} f \} = \text{Proj}_{\ker \mathcal{T}} \{ Q(f, f) \}, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \quad (16)$$

It is easily seen that taking the orthogonal projection on  $\ker \mathcal{T}$  reduces to averaging along the characteristic flow of  $\mathcal{T}$  in (15) cf. [1], [2], [3], [13], [15], [18]. This flow is  $T_c = \frac{2\pi}{\omega_c}$  periodic and writes

$$\bar{V}(s) = \mathcal{R}(-\omega_c s) \bar{v}, \quad \bar{X}(s) = \bar{x} + \frac{\perp \bar{v}}{\omega_c} - \frac{\perp \bar{V}(s)}{\omega_c}, \quad X_3(s) = x_3, \quad V_3(s) = v_3$$

where  $\mathcal{R}(\alpha)$  stands for the rotation of angle  $\alpha$

$$\mathcal{R}(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$

For any function  $u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ , the average operator is defined by

$$\begin{aligned} \langle u \rangle(x, v) &= \frac{1}{T_c} \int_0^{T_c} u(X(s; x, v), V(s; x, v)) \, ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} u \left( \bar{x} + \frac{\perp \bar{v}}{\omega_c} - \frac{\perp \{ \mathcal{R}(\alpha) \bar{v} \}}{\omega_c}, x_3, \mathcal{R}(\alpha) \bar{v}, v_3 \right) \, d\alpha. \end{aligned} \quad (17)$$

We introduce the notation  $e^{i\varphi}$  for the  $\mathbb{R}^2$  vector  $(\cos \varphi, \sin \varphi)$ . If the vector  $\bar{v}$  writes  $\bar{v} = |\bar{v}| e^{i\varphi}$ , then  $\mathcal{R}(\alpha) \bar{v} = |\bar{v}| e^{i(\alpha+\varphi)}$  and the expression for  $\langle u \rangle$  becomes

$$\begin{aligned} \langle u \rangle(x, v) &= \frac{1}{2\pi} \int_0^{2\pi} u \left( \bar{x} + \frac{\perp \bar{v}}{\omega_c} - \frac{\perp \{ |\bar{v}| e^{i(\alpha+\varphi)} \}}{\omega_c}, x_3, |\bar{v}| e^{i(\alpha+\varphi)}, v_3 \right) \, d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} u \left( \bar{x} + \frac{\perp \bar{v}}{\omega_c} - \frac{\perp \{ |\bar{v}| e^{i\alpha} \}}{\omega_c}, x_3, |\bar{v}| e^{i\alpha}, v_3 \right) \, d\alpha. \end{aligned}$$

The properties of the average operator (17) are summarized below (see Propositions 2.1, 2.2 in [3] for proof details). We denote by  $\| \cdot \|$  the standard norm of  $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ .

**Proposition 2.1** *The average operator is linear and continuous. Moreover it coincides with the orthogonal projection on the kernel of  $\mathcal{T}$  i.e.,*

$$\langle u \rangle \in \ker \mathcal{T} \quad \text{and} \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle) \varphi \, dv dx = 0, \quad \forall \varphi \in \ker \mathcal{T}. \quad (18)$$

**Remark 2.1** Notice that  $(\bar{X}, \bar{V})$  depends only on  $s$  and  $(\bar{x}, \bar{v})$  and thus the variational characterization in (18) holds true at any fixed  $(x_3, v_3) \in \mathbb{R}^2$ . Indeed, for any  $\varphi \in \ker \mathcal{T}$ ,  $(x_3, v_3) \in \mathbb{R}^2$  we have

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u\varphi)(x, v) \, d\bar{v}d\bar{x} &= \frac{1}{T_c} \int_0^{T_c} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(x, v) \varphi(\bar{X}(-s; x, v), x_3, \bar{V}(-s; x, v), v_3) \, d\bar{v}d\bar{x}ds \\ &= \frac{1}{T_c} \int_0^{T_c} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(\bar{X}(s; x, v), x_3, \bar{V}(s; x, v), v_3) \varphi(x, v) \, d\bar{v}d\bar{x}ds \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle u \rangle (x, v) \varphi(x, v) \, d\bar{v}d\bar{x}. \end{aligned}$$

We have the orthogonal decomposition of  $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$  into invariant functions along the characteristics (15) and zero average functions

$$u = \langle u \rangle + (u - \langle u \rangle), \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle) \langle u \rangle \, dvdx = 0.$$

Notice that  $\mathcal{T}^* = -\mathcal{T}$  and thus the equality  $\langle \cdot \rangle = \text{Proj}_{\ker \mathcal{T}}$  implies

$$\ker \langle \cdot \rangle = (\ker \mathcal{T})^\perp = (\ker \mathcal{T}^*)^\perp = \overline{\text{Range } \mathcal{T}}.$$

In particular  $\text{Range } \mathcal{T} \subset \ker \langle \cdot \rangle$ . We show that  $\text{Range } \mathcal{T}$  is closed, which will give a solvability condition for  $\mathcal{T}u = w$  (cf. [3], Propositions 2.2).

**Proposition 2.2** *The restriction of  $\mathcal{T}$  to  $\ker \langle \cdot \rangle$  is one to one map onto  $\ker \langle \cdot \rangle$ . Its inverse belongs to  $\mathcal{L}(\ker \langle \cdot \rangle, \ker \langle \cdot \rangle)$  and we have the Poincaré inequality*

$$\|u\| \leq \frac{2\pi}{|\omega_c|} \|\mathcal{T}u\|, \quad \omega_c = \frac{qB}{m} \neq 0$$

for any  $u \in D(\mathcal{T}) \cap \ker \langle \cdot \rangle$ .

The average operator can be defined in any Lebesgue space  $L^p$ , with  $1 \leq p \leq +\infty$  cf. [2]. A straightforward computation shows that if  $\mathcal{T}f(t) = 0$ , that is  $f(t)$  depends only on the invariants of (15), then  $\partial_t f(t), \partial_{x_3} f(t), \partial_{v_3} f(t)$  belong to  $\ker \mathcal{T}$ , since all these functions depend only on  $\omega_c \bar{x} + {}^\perp \bar{v}, x_3, |\bar{v}|, v_3$ . We deduce that at any time  $\partial_t f + u_3 \partial_{x_3} f - v_3 \partial_{x_3} u_3 \partial_{v_3} f \in \ker \mathcal{T}$  and thus (16) reduces to

$$\partial_t f + \partial_{x_3} \{u_3 f\} - \partial_{v_3} \{v_3 \partial_{x_3} u_3 f\} = \partial_t f + u_3 \partial_{x_3} f - v_3 \partial_{x_3} u_3 \partial_{v_3} f = \langle Q(f, f) \rangle. \quad (19)$$

For any  $r, r' \in \mathbb{R}_+$ , we denote by  $\chi(r, r', \cdot)$  the probability density on  $\mathbb{R}^2$  given by

$$\chi(r, r', \bar{z}) = \frac{\mathbf{1}_{\{|r-r'| < |\bar{z}| < r+r'\}}}{\pi^2 \sqrt{|\bar{z}|^2 - (r-r')^2} \sqrt{(r+r')^2 - |\bar{z}|^2}}, \quad r, r' \in \mathbb{R}_+, \quad \bar{z} \in \mathbb{R}^2.$$



The probability  $\chi$  charges only pairs of Larmor circles having non empty intersection and, as we will see below, only such pairs of Larmor circles will interact through the averaged Boltzmann collision kernel. The average of the loss part is given by

**Proposition 2.3** *For any non negative densities  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$ ,  $f'(\bar{x}, v) = g'(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  the following equality holds true*

$$\begin{aligned} & \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v) f'(\bar{x}, v') dv' d\omega \right\rangle (\bar{x}, v) \\ &= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g'(y', r') \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de. \end{aligned} \quad (20)$$

Notice that the averaged loss part has similar structure with the Boltzmann loss part : it is an integral operator with respect to the pre-collisional quantities  $(y', r')$  and a collision parameter  $e \in S^2$ .

The average of the gain part will express in terms of post-collisional quantities. For any  $e \in S^2$  we introduce the transformation mapping  $(y, r), (y', r')$  to  $(Y, R), (Y', R')$  given by

$$\begin{aligned} Y &= y - (y - y', e)e, \quad Y' = y' + (y - y', e)e \\ R &= \left| r \mathcal{R}(-\psi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} - (y - y', e)\bar{e} \right|, \quad R' = \left| r' \mathcal{R}(-(\psi - \varphi)) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} + (y - y', e)\bar{e} \right| \end{aligned} \quad (21)$$

where  $\varphi, \psi$  stand for the unique angles in  $(0, \pi)$  such that

$$|\bar{y} - \bar{y}'|^2 = r^2 + (r')^2 - 2rr' \cos \varphi, \quad (r')^2 = r^2 + |\bar{y} - \bar{y}'|^2 + 2r|\bar{y} - \bar{y}'| \cos \psi.$$

Here  $y = (\omega_c \bar{x} + {}^\perp \bar{v}, v_3), r = |\bar{v}|, y' = (\omega_c \bar{x}' + {}^\perp \bar{v}', v_3'), r' = |\bar{v}'|$  are the pre-collisional Larmor centers and radii (up to the factor  $\omega_c$ ) and parallel velocities and  $(Y, R), (Y', R')$  are the corresponding post-collisional quantities. The average of the gain part writes.

**Proposition 2.4** *For any non negative densities  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$ ,  $f'(\bar{x}, v) = g'(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  the following equality holds true*

$$\begin{aligned} & \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, V(v, v', \omega)) f'(\bar{x}, V'(v, v', \omega)) dv' d\omega \right\rangle (\bar{x}, v) \\ &= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g'(Y', R') \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de. \end{aligned} \quad (23)$$

In the Maxwell molecule case, the expression for the averaged Boltzmann collision kernel is

$$\begin{aligned} \langle Q_0 \rangle (f, f')(\bar{x}, v) &:= \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma_0(v - v', \omega) \{f(\bar{x}, V) f'(\bar{x}, V') - f(\bar{x}, v) f'(\bar{x}, v')\} dv' d\omega \right\rangle \\ &= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma_0(y - y', e) \{g(Y, R) g'(Y', R') - g(y, r) g'(y', r')\} \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de. \end{aligned}$$

After computing in detail the average of the Boltzmann kernel, we obtain, at least formally, the following high magnetic field limit

**Theorem 2.1** *Let  $u_3 = u_3(t, x_3)$  be a smooth function satisfying*

$$\partial_t u_3 + u_3(t, x_3) \partial_{x_3} u_3 = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

and  $f^{\text{in}} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$  be a non negative density. For any  $\varepsilon > 0$  we denote by  $f^\varepsilon$  the solution of the problem

$$\begin{aligned} \partial_t f^\varepsilon + (u_3 + \varepsilon v_3) \partial_{x_3} f^\varepsilon - v_3 \partial_{x_3} u_3 \partial_{v_3} f^\varepsilon + \frac{1}{\varepsilon} (\bar{v} \cdot \nabla_{\bar{x}} f^\varepsilon + \omega_c \perp \bar{v} \cdot \nabla_{\bar{v}} f^\varepsilon) \\ = \int_{S^2} \int_{\mathbb{R}^3} \sigma_0(v - v', \omega) \{f^\varepsilon(\bar{x}, V) f^\varepsilon(\bar{x}, V') - f^\varepsilon(\bar{x}, v) f^\varepsilon(\bar{x}, v')\} dv' d\omega \\ f^\varepsilon(0, \cdot, \cdot) = f^{\text{in}}. \end{aligned}$$

Therefore the limit density  $f = \lim_{\varepsilon \searrow 0} f^\varepsilon$  belongs to  $\ker \mathcal{T}$  at any time  $t \in \mathbb{R}_+$  and satisfies

$$\partial_t f + \partial_{x_3} \{u_3 f\} - \partial_{v_3} \{v_3 \partial_{x_3} u_3 f\} = \langle Q_0 \rangle (f, f)$$

and

$$f(0, \cdot, \cdot) = \langle f^{\text{in}} \rangle.$$

Once we have determined the averaged Boltzmann kernel, it is worth investigating its equilibria and collision invariants. This can be done thanks to a  $H$  type theorem, cf. Theorem 2.2, or equivalently, based on the equilibria and collision invariants of the Boltzmann kernel, see Theorem 4.2.

**Theorem 2.2**

1. For any function  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + \perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  and non negative

density  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  we have

$$\begin{aligned} \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m(\bar{x}, v) \langle Q \rangle (f, f) \, dv d\bar{x} &= -\pi^2 \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) \\ &\times \{n(Y, R) + n(Y', R') - n(y, r) - n(y', r')\} \\ &\times \{g(Y, R)g(Y', R') - g(y, r)g(y', r')\} \chi(r, r', \bar{y} - \bar{y}') \, r dr dy' dr' dy' de \end{aligned} \quad (24)$$

2. For any positive density  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  we have the inequality

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f(\bar{x}, v) \langle Q \rangle (f, f) \, dv d\bar{x} \leq 0$$

with equality iff

$$\ln g(Y, R) + \ln g(Y', R') = \ln g(y, r) + \ln g(y', r'), \quad |r - r'| < |\bar{y} - \bar{y}'| < r + r'. \quad (25)$$

3. The positive equilibria  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  of the averaged Boltzmann kernel i.e.,  $f > 0, \langle Q \rangle (f, f) = 0$ , are the positive densities satisfying (25).

4. The collision invariants, i.e., the functions  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  such that  $\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m(\bar{x}, v) \langle Q \rangle (f, f) \, dv d\bar{x} = 0$  for any non negative density  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  are the functions  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  satisfying

$$n(Y, R) + n(Y', R') = n(y, r) + n(y', r'), \quad |r - r'| < |\bar{y} - \bar{y}'| < r + r'. \quad (26)$$

We prove that the equilibria of the averaged Boltzmann kernel are local with respect to the parallel space coordinate  $x_3$  and that they are parametrized by six moments which correspond to the collision invariants  $1, \omega_c \bar{x} + {}^\perp \bar{v}, v_3, |v|^2/2, \{|\omega_c \bar{x} + {}^\perp \bar{v}|^2 - |\bar{v}|^2\}/2$

$$\begin{aligned} \rho(x_3) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f(x, v) \, dv d\bar{x} \\ \rho(x_3) w(x_3) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (\omega_c \bar{x} + {}^\perp \bar{v}, v_3) f(x, v) \, dv d\bar{x} \\ \rho(x_3) K(x_3) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|\bar{v}|^2 + (v_3 - w_3)^2}{2} f(x, v) \, dv d\bar{x} \\ \rho(x_3) G(x_3) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|\omega_c \bar{x} + {}^\perp \bar{v} - \bar{w}|^2 - |\bar{v}|^2}{2} f(x, v) \, dv d\bar{x}. \end{aligned}$$

**Proposition 2.5** For any  $(\rho, w, K, G) \in \mathbb{R}^6$ ,  $\rho > 0, K > 0, K + G > 0$  there is a unique equilibrium  $f = f(\bar{x}, v)$  for  $\langle Q \rangle$  satisfying

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f \, dv d\bar{x} = \rho, \quad \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (\omega_c \bar{x} + {}^\perp \bar{v}, v_3) f \, dv d\bar{x} = \rho w$$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|v|^2}{2} f \, dv d\bar{x} = \rho \frac{(w_3)^2}{2} + \rho K, \quad \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|\omega_c \bar{x} + {}^\perp \bar{v}|^2 - |\bar{v}|^2}{2} f \, dv d\bar{x} = \rho \frac{|\bar{w}|^2}{2} + \rho G.$$

This equilibrium is given by

$$f = \frac{\rho \omega_c^2}{(2\pi)^{5/2} \frac{\mu^2 \theta^{3/2}}{\mu - \theta}} \exp\left(-\frac{|\bar{v}|^2 + (v_3 - w_3)^2}{2\theta}\right) \exp\left(-\frac{|\omega_c \bar{x} + {}^\perp \bar{v} - \bar{w}|^2 - |\bar{v}|^2}{2\mu}\right)$$

where  $\theta, \mu$  satisfy

$$\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} = K, \quad \mu - \frac{\mu\theta}{\mu - \theta} = G, \quad \mu > \theta > 0.$$

The averaged Boltzmann kernel requires a huge computational effort. But simpler fluid models can be derived, at least when the collisions dominate the transport.

**Theorem 2.3** Assume that  $u_3 = u_3(t, x_3)$  is a smooth function satisfying

$$\partial_t u_3 + u_3(t, x_3) \partial_{x_3} u_3 = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

and let  $f^{\text{in}} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}$  be a non negative density. For any  $\tau > 0$  the density  $f^\tau$  stands for the solution of the problem

$$\partial_t f^\tau + \partial_{x_3} \{u_3 f^\tau\} - \partial_{v_3} \{v_3 \partial_{x_3} u_3 f^\tau\} = \frac{1}{\tau} \langle Q_0 \rangle (f^\tau, f^\tau), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$$
(27)

$$f^\tau(t = 0, x, v) = f^{\text{in}}(x, v) > 0, \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Therefore  $(f^\tau)_{\tau > 0}$  converges, at least formally when  $\tau \searrow 0$ , towards a local equilibrium  $f$  parametrized by the functions  $\rho = \rho(t, x_3) > 0, w = w(t, x_3), \theta = \theta(t, x_3) > 0, \mu = \mu(t, x_3) > \theta(t, x_3) > 0$ , which satisfy the system of conservation laws

$$\partial_t \rho + \partial_{x_3} (u_3 \rho) = 0, \quad \partial_t (\rho w) + \partial_{x_3} (u_3 \rho w) + (0, 0, \partial_{x_3} u_3 \rho w_3) = (0, 0, 0), \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

$$\partial_t \left[ \rho \left( \frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} + \frac{(w_3)^2}{2} \right) \right] + \partial_{x_3} \left[ u_3 \rho \left( \frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} + \frac{(w_3)^2}{2} \right) \right] + \partial_{x_3} u_3 \rho [(w_3)^2 + \theta] = 0$$

$$\partial_t \left[ \rho \left( \mu - \frac{\mu\theta}{\mu - \theta} + \frac{|\bar{w}|^2}{2} \right) \right] + \partial_{x_3} \left[ \rho u_3 \left( \mu - \frac{\mu\theta}{\mu - \theta} + \frac{|\bar{w}|^2}{2} \right) \right] = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

and the initial conditions

$$\begin{aligned}\rho(0, x_3) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f^{\text{in}}(x, v) \, dv d\bar{x}, \quad \rho(0, x_3) w(0, x_3) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (\omega_c \bar{x} + {}^\perp \bar{v}, v_3) f^{\text{in}}(x, v) \, dv d\bar{x} \\ \rho(0, x_3) \left( \frac{\mu(0, x_3) \theta(0, x_3)}{\mu(0, x_3) - \theta(0, x_3)} + \frac{\theta(0, x_3)}{2} \right) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|\bar{v}|^2 + (v_3 - w_3(0, x_3))^2}{2} f^{\text{in}}(x, v) \, dv d\bar{x} \\ \rho(0, x_3) \left( \mu(0, x_3) - \frac{\mu(0, x_3) \theta(0, x_3)}{\mu(0, x_3) - \theta(0, x_3)} \right) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|\omega_c \bar{x} + {}^\perp \bar{v} - \bar{w}(0, x_3)|^2 - |\bar{v}|^2}{2} f^{\text{in}} \, dv d\bar{x}.\end{aligned}$$

### 3 The averaged Boltzmann collision operator

In this section we determine the explicit form of the averaged Boltzmann kernel. As indicated in the introduction, we treat the Maxwell molecule case *i.e.*,  $\gamma = 0, s = 5$  and thus the scattering section has the form  $\sigma_0(z, \omega) = b(z/|z| \cdot \omega)$ . It is easily seen that in this case the Boltzmann collision kernel is a bilinear operator mapping  $L^1(\mathbb{R}^3) \times L^1(\mathbb{R}^3)$  to  $L^1(\mathbb{R}^3)$  and

$$\|Q_0(f, f')\|_{L^1(\mathbb{R}^3)} \leq 4\pi \|b\|_{L^1(-1,1)} \|f\|_{L^1(\mathbb{R}^3)} \|f'\|_{L^1(\mathbb{R}^3)}, \quad f, f' \in L^1(\mathbb{R}^3).$$

Recall that the underlying structure of the Boltzmann collision kernel relies on the parametrization of the collisions between particles. The post-collisional velocities  $V, V'$  of any two particles occupying at the time  $t$  the same position  $x$ , and having the pre-collisional velocities  $v, v'$  are given by

$$V(v, v', \omega) = v - (v - v', \omega)\omega, \quad V'(v, v', \omega) = v' + (v - v', \omega)\omega, \quad \omega \in S^2.$$

The post-collisional velocities (6 components) are obtained by imposing the momentum and kinetic energy conservations (4 conditions) and thus they are described using two parameters, that is a direction  $\omega \in S^2$ . It is easily seen that

$$V - V' = (I_3 - 2\omega \otimes \omega)(v - v')$$

saying that the post-collisional relative velocity appears as the symmetry of the pre-collisional relative velocity with respect to the plane orthogonal to  $\omega$ . We expect that the averaged Boltzmann kernel possesses a similar structure, but with respect to a larger phase space. The densities belong to the kernel of  $\mathcal{T}$  and collisions will be observed between pairs of Larmor circles rather than particles. Indeed, any density

$f \in \ker \mathcal{T}$  writes  $f(x, v) = g(\omega_c \bar{x} + {}^\perp \bar{v}, x_3, v_3, |\bar{v}|)$  and we are looking for collisions transforming the Larmor center  $\bar{x} + {}^\perp \bar{v}/\omega_c$  and radius  $|\bar{v}|/\omega_c$ , and the parallel velocity  $v_3$ .

### 3.1 Collision parametrization of the averaged Boltzmann kernel

We introduce the notation  $y = (\omega_c \bar{x} + {}^\perp \bar{v}, v_3), r = |\bar{v}|$ . Collisions will occur only between pairs of Larmor circles having non empty intersection

$$|r - r'| = ||\bar{v}| - |\bar{v}'|| < |(\omega_c \bar{x} + {}^\perp \bar{v}) - (\omega_c \bar{x}' + {}^\perp \bar{v}')| = |\bar{y} - \bar{y}'| < |\bar{v}| + |\bar{v}'| = r + r'.$$

Let us see how we can construct such collisions. We fix a direction  $d \in S^2$  and take a pre-collision pair  $(\bar{x}, x_3, v), (\bar{x}', x_3, v')$  having the same parallel position  $x_3$ . We assume that the corresponding Larmor circles have non empty intersection. We denote by  $o, o'$  the centers of the circles

$$c = \{\bar{z} : |\omega_c \bar{x} + {}^\perp \bar{v} - \bar{z}| = |\bar{v}|\}, \quad c' = \{\bar{z}' : |\omega_c \bar{x}' + {}^\perp \bar{v}' - \bar{z}'| = |\bar{v}'|\}$$

and by  $I$  the intersection point between these circles such that the oriented angle  $o\hat{I}o'$  has positive measure  $\varphi \in (0, \pi)$ . Let us consider the characteristics  $(\bar{X}(s), V(s))$  and  $(\bar{X}'(s'), V'(s'))$  starting from  $(\bar{x}, v), (\bar{x}', v')$ . After some times  $s, s'$  these characteristics meet in  $I$

$$\omega_c \bar{X}(s) = \omega_c \bar{X}'(s')$$

and we denote by

$$v_I = V(s) = (\mathcal{R}(-\omega_c s)\bar{v}, v_3), \quad v'_I = V'(s') = (\mathcal{R}(-\omega_c s')\bar{v}', v'_3)$$

the associated velocities. Since the particles share the same position, it makes sense to perform a velocity collision parametrized by the direction  $d$ , according to

$$V_I = v_I - (v_I - v'_I, d)d, \quad V'_I = v'_I + (v_I - v'_I, d)d. \quad (28)$$

It is easily seen that

$$\bar{y}' - \bar{y} = (\omega_c \bar{x}' + {}^\perp \bar{v}') - (\omega_c \bar{x} + {}^\perp \bar{v}) = o\vec{o}' = \vec{I}o' - \vec{I}o = {}^\perp \bar{v}' - {}^\perp \bar{v}$$

$$y'_3 - y_3 = v'_3 - v_3 = v'_{I3} - v_{I3}$$

and thus

$$\begin{aligned} (v_I - v'_I, d) &= ({}^\perp\bar{v}_I - {}^\perp v'_I, {}^\perp\bar{d}) + (v_{I3} - v'_{I3})d_3 \\ &= (\bar{y} - y', {}^\perp\bar{d}) + (v_3 - v'_3)d_3 = (y - y', \tilde{d}) \end{aligned} \quad (29)$$

where  $\tilde{d} = ({}^\perp\bar{d}, d_3)$ . The post-collision pair  $(\bar{X}, x_3, V), (\bar{X}', x_3, V')$  follows moving backwards on the characteristics, during the times  $s, s'$ , starting from  $(\bar{x}_I, V_I), (\bar{x}_I, V'_I)$ . More exactly, the perpendicular velocities are given by

$$\begin{aligned} \bar{V} &= \mathcal{R}(\omega_c s)\bar{V}_I = \mathcal{R}(\omega_c s)[\bar{v}_I - (y - y', \tilde{d})\bar{d}] = \bar{v} - (y - y', \tilde{d})\mathcal{R}(\omega_c s)\bar{d} \\ \bar{V}' &= \mathcal{R}(\omega_c s')\bar{V}'_I = \mathcal{R}(\omega_c s')[\bar{v}'_I + (y - y', \tilde{d})\bar{d}] = \bar{v}' + (y - y', \tilde{d})\mathcal{R}(\omega_c s')\bar{d}. \end{aligned}$$

For the parallel velocities we get

$$V_3 = V_{I3} = v_3 - (y - y', \tilde{d})d_3, \quad V'_3 = V'_{I3} = v'_3 + (y - y', \tilde{d})d_3. \quad (30)$$

It remains to determine the perpendicular positions. For this we use the conservation of the Larmor centers. We have

$$\bar{y} = \omega_c \bar{x} + {}^\perp\bar{v} = \omega_c \bar{x}_I + {}^\perp\bar{v}_I, \quad \bar{y}' = \omega_c \bar{x}' + {}^\perp\bar{v}' = \omega_c \bar{x}_I + {}^\perp\bar{v}'_I$$

and the backwards motion gives

$$\omega_c \bar{x}_I + {}^\perp\bar{V}_I = \omega_c \bar{X} + {}^\perp\bar{V} \quad (31)$$

$$\omega_c \bar{x}_I + {}^\perp\bar{V}'_I = \omega_c \bar{X}' + {}^\perp\bar{V}'. \quad (32)$$

Eliminating the perpendicular position of the intersection point  $I$ , we obtain

$$\bar{X} = \bar{x} + \frac{{}^\perp(\bar{v} - \bar{V})}{\omega_c} + \frac{{}^\perp(\bar{V}_I - \bar{v}_I)}{\omega_c} = \bar{x} - \frac{(y - y', \tilde{d}){}^\perp\bar{d}}{\omega_c} - \frac{{}^\perp(\bar{V} - \bar{v})}{\omega_c} \quad (33)$$

and

$$\bar{X}' = \bar{x}' + \frac{{}^\perp(\bar{v}' - \bar{V}')}{\omega_c} + \frac{{}^\perp(\bar{V}'_I - \bar{v}'_I)}{\omega_c} = \bar{x}' + \frac{(y - y', \tilde{d}){}^\perp\bar{d}}{\omega_c} - \frac{{}^\perp(\bar{V}' - \bar{v}')}{\omega_c}. \quad (34)$$

We claim that the invariants of the post-collision pair  $(\bar{X}, V), (\bar{X}', V')$  depend only on the invariant of the pre-collision pair  $(\bar{x}, v), (\bar{x}', v')$ . Indeed, we have

$$\bar{Y} := \omega_c \bar{X} + {}^\perp\bar{V} = \omega_c \bar{x}_I + {}^\perp\bar{V}_I = \bar{y} - {}^\perp\bar{v}_I + {}^\perp\bar{V}_I = \bar{y} - (y - y', \tilde{d}){}^\perp\bar{d}$$

$$Y_3 := V_3 = v_3 - (y - y', \tilde{d})d_3 = y_3 - (y - y', \tilde{d})d_3$$

and similarly

$$\overline{Y'} := \omega_c \overline{X'} + {}^\perp \overline{V'} = \omega_c \overline{x_I} + {}^\perp \overline{V'_I} = \overline{y'} - {}^\perp \overline{v'_I} + {}^\perp \overline{V'_I} = \overline{y'} + (y - y', \tilde{d})^\perp \overline{d}$$

$$Y'_3 := V'_3 = v'_3 + (y - y', \tilde{d})d_3 = y'_3 + (y - y', \tilde{d})d_3.$$

Notice that the previous four equalities write

$$Y = y - (y - y', \tilde{d})\tilde{d}, \quad Y' = y' + (y - y', \tilde{d})\tilde{d}. \quad (35)$$

We also need to express the modulus of the perpendicular velocities

$$R := |\overline{V}| = |\mathcal{R}(\omega_c s)\overline{V}_I| = |\overline{V}_I| = |{}^\perp \overline{V}_I| = |{}^\perp \overline{v}_I - (y - y', \tilde{d})^\perp \overline{d}|$$

$$R' := |\overline{V'}| = |\mathcal{R}(\omega_c s')\overline{V}'_I| = |\overline{V}'_I| = |{}^\perp \overline{V}'_I| = |{}^\perp \overline{v}'_I + (y - y', \tilde{d})^\perp \overline{d}|.$$

We denote by  $\psi \in (0, \pi)$  the positive exterior angle corresponding to the vertex  $o$  of the triangle  $oIo'$ . The velocities  ${}^\perp \overline{v}_I, {}^\perp \overline{v}'_I$  come easily, observing that

$${}^\perp \overline{v}_I = \vec{Io} = r\mathcal{R}(-\psi)\frac{\overline{y'} - \overline{y}}{|\overline{y'} - \overline{y}|}, \quad {}^\perp \overline{v}'_I = \vec{Io'} = r'\mathcal{R}(-(\psi - \varphi))\frac{\overline{y'} - \overline{y}}{|\overline{y'} - \overline{y}|}$$

and finally

$$R = \left| r\mathcal{R}(-\psi)\frac{\overline{y'} - \overline{y}}{|\overline{y'} - \overline{y}|} - (y - y', \tilde{d})^\perp \overline{d} \right|, \quad R' = \left| r'\mathcal{R}(-(\psi - \varphi))\frac{\overline{y'} - \overline{y}}{|\overline{y'} - \overline{y}|} + (y - y', \tilde{d})^\perp \overline{d} \right|. \quad (36)$$

Observe that the post-collision Larmor circles (up to a factor  $\omega_c$ ), whose centers are denoted by  $O, O'$

$$C = \{ \overline{Z} : |\omega_c \overline{X} + {}^\perp \overline{V} - \overline{Z}| = |\overline{V}| \}, \quad C' = \{ \overline{Z}' : |\omega_c \overline{X}' + {}^\perp \overline{V}' - \overline{Z}'| = |\overline{V}'| \}$$

have non empty intersection, since both of them contain the point  $I$ , thanks to (31), (32). Therefore any pair of colliding Larmor circles will generate another pair of colliding Larmor circles.

The collision (35), (36) can be parametrized with respect to  $e = \tilde{d} = ({}^\perp \overline{d}, d_3) \in S^2$ , rather than  $d \in S^2$ . Therefore, for any  $e \in S^2$  and any pair of Larmor circles which collide *i.e.*,

$$|r - r'| = | |\overline{v}| - |\overline{v}'| | < |(\omega_c \overline{x} + {}^\perp \overline{v}) - (\omega_c \overline{x}' + {}^\perp \overline{v}')| = |\overline{y} - \overline{y}'| < |\overline{v}| + |\overline{v}'| = r + r'$$



we introduce the map transforming  $(y, r), (y', r')$  to  $(Y, R), (Y', R')$

$$Y = y - (y - y', e)e, \quad Y' = y' + (y - y', e)e \quad (37)$$

$$R = \left| r\mathcal{R}(-\psi) \frac{\bar{y}' - \bar{y}}{|y' - \bar{y}|} - (y - y', e)\bar{e} \right|, \quad R' = \left| r'\mathcal{R}(-(\psi - \varphi)) \frac{\bar{y}' - \bar{y}}{|y' - \bar{y}|} + (y - y', e)\bar{e} \right| \quad (38)$$

where the notations  $\varphi$  and  $\psi$  stand for the unique angles in  $(0, \pi)$  such that

$$|\bar{y} - \bar{y}'|^2 = r^2 + (r')^2 - 2rr' \cos \varphi, \quad (r')^2 = r^2 + |\bar{y} - \bar{y}'|^2 + 2r|\bar{y} - \bar{y}'| \cos \psi.$$

In the sequel we will need some computations, which we detail here. Notice that the definition of  $\varphi$  ensures that  $|\bar{y} - \bar{y}'| = |r'e^{i\varphi} - (r, 0)|$  and therefore there is  $\alpha$  such that

$$\bar{y}' - \bar{y} = {}^\perp \{ \mathcal{R}(\alpha) (r'e^{i\varphi} - (r, 0)) \}.$$

It is immediately seen, using the geometry of the triangle whose vertices are  $(0, 0), (r, 0), r'e^{i\varphi}$  that

$$\mathcal{R}(-\psi) \frac{r'e^{i\varphi} - (r, 0)}{|r'e^{i\varphi} - (r, 0)|} = (1, 0)$$

and thus

$$\begin{aligned} r\mathcal{R}(-\psi) \frac{\bar{y}' - \bar{y}}{|y' - \bar{y}|} &= r {}^\perp \left\{ \mathcal{R}(-\psi)\mathcal{R}(\alpha) \frac{r'e^{i\varphi} - (r, 0)}{|r'e^{i\varphi} - (r, 0)|} \right\} \\ &= r {}^\perp \left\{ \mathcal{R}(\alpha)\mathcal{R}(-\psi) \frac{r'e^{i\varphi} - (r, 0)}{|r'e^{i\varphi} - (r, 0)|} \right\} \\ &= r {}^\perp \{ \mathcal{R}(\alpha)(1, 0) \} = {}^\perp \{ r e^{i\alpha} \}. \end{aligned} \quad (39)$$

Similarly we have

$$\mathcal{R}(-(\psi - \varphi)) \frac{r'e^{i\varphi} - (r, 0)}{|r'e^{i\varphi} - (r, 0)|} = e^{i\varphi}$$

implying that

$$\begin{aligned} r'\mathcal{R}(-(\psi - \varphi)) \frac{\bar{y}' - \bar{y}}{|y' - \bar{y}|} &= r' {}^\perp \left\{ \mathcal{R}(-(\psi - \varphi))\mathcal{R}(\alpha) \frac{r'e^{i\varphi} - (r, 0)}{|r'e^{i\varphi} - (r, 0)|} \right\} \\ &= r' {}^\perp \left\{ \mathcal{R}(\alpha)\mathcal{R}(-(\psi - \varphi)) \frac{r'e^{i\varphi} - (r, 0)}{|r'e^{i\varphi} - (r, 0)|} \right\} \\ &= r' {}^\perp \{ \mathcal{R}(\alpha)e^{i\varphi} \} = {}^\perp \{ r' e^{i(\alpha+\varphi)} \}. \end{aligned} \quad (40)$$

### 3.2 Conservations through the collisions of the averaged Boltzmann kernel

In the case of the Boltzmann kernel, the pre/post-collision velocities satisfy the conservations of mass, momentum and kinetic energy. Similarly, the pre/post-collision quantities (37), (38) satisfy several conservation laws, summarized below.

**Proposition 3.1** *For any pre-collision pair  $(y, r), (y', r')$ , whose post-collision pair is  $(Y, R), (Y', R')$  (cf. (37), (38)) we have*

$$Y + Y' = y + y' \quad (\text{Larmor center and parallel velocity conservation})$$

$$\frac{R^2 + (Y_3)^2}{2} + \frac{(R')^2 + (Y'_3)^2}{2} = \frac{r^2 + (y_3)^2}{2} + \frac{(r')^2 + (y'_3)^2}{2} \quad (\text{kinetic energy conservation})$$

$$\frac{|\bar{Y}|^2 - R^2}{2} + \frac{|\bar{Y}'|^2 - (R')^2}{2} = \frac{|\bar{y}|^2 - r^2}{2} + \frac{|\bar{y}'|^2 - (r')^2}{2} \quad (\text{Larmor circle power conservation}).$$

**Proof.** Obviously, for any fixed  $e \in S^2$  we have

$$Y + Y' = y - (y - y', e)e + y' + (y - y', e)e = y + y'$$

which express the conservation of the Larmor center and parallel velocity. Notice also that

$$|Y|^2 + |Y'|^2 = |y - (y - y', e)e|^2 + |y' + (y - y', e)e|^2 = |y|^2 + |y'|^2. \quad (41)$$

With the notations used in the definition of the transformation  $\{(y, r), (y', r')\} \rightarrow \{(Y, R), (Y', R')\}$  we have

$$|V_I|^2 + |V'_I|^2 = |v_I|^2 + |v'_I|^2$$

which guarantee the kinetic energy conservation

$$\begin{aligned} \frac{R^2 + (Y_3)^2}{2} + \frac{(R')^2 + (Y'_3)^2}{2} &= \frac{|\bar{V}|^2 + (V_3)^2}{2} + \frac{|\bar{V}'|^2 + (V'_3)^2}{2} \\ &= \frac{|\bar{V}_I|^2 + (V_{I3})^2}{2} + \frac{|\bar{V}'_I|^2 + (V'_{I3})^2}{2} \\ &= \frac{|\bar{v}_I|^2 + (v_{I3})^2}{2} + \frac{|\bar{v}'_I|^2 + (v'_{I3})^2}{2} \\ &= \frac{|\bar{v}|^2 + (v_3)^2}{2} + \frac{|\bar{v}'|^2 + (v'_3)^2}{2} \\ &= \frac{r^2 + (y_3)^2}{2} + \frac{(r')^2 + (y'_3)^2}{2}. \end{aligned}$$

The last conservation is obtained thanks to the equalities

$$|\bar{Y}|^2 + (Y_3)^2 + |\bar{Y}'|^2 + (Y_3')^2 = |\bar{y}|^2 + (y_3)^2 + |\bar{y}'|^2 + (y_3')^2$$

$$R^2 + (Y_3)^2 + (R')^2 + (Y_3')^2 = r^2 + (y_3)^2 + (r')^2 + (y_3')^2$$

which yield

$$|\bar{Y}|^2 - R^2 + |\bar{Y}'|^2 - (R')^2 = |\bar{y}|^2 - r^2 + |\bar{y}'|^2 - (r')^2. \quad (42)$$

Notice that  $(|\bar{y}|^2 - r^2)/\omega_c^2$  represents the power of the Larmor circle of center  $\bar{x} + {}^\perp\bar{v}/\omega_c$  and radius  $|\bar{v}|/|\omega_c|$  with respect to the origin and thus (42) expresses the conservation of the Larmor circle power with respect to the origin.  $\square$

### 3.3 Average of velocity convolutions

The average computation for both the gain and loss parts relies on the general result stated in Proposition 3.2. We present formal computations leading to an explicit formula for the average of velocity convolutions. Nevertheless, the lines below provide rigorous arguments at least in the Maxwell molecule case and under Grad cut-off angular hypothesis, since in that situation  $Q_\pm$  map  $L^1(\mathbb{R}^3) \times L^1(\mathbb{R}^3)$  to  $L^1(\mathbb{R}^3)$ . In this case the average operator should be understood in the  $L^1$  setting cf. [2].

**Proposition 3.2** *Consider  $F, F' : \mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \rightarrow \mathbb{R}$ ,  $\Sigma : \mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \rightarrow \mathbb{R}$  three functions which are left invariant by any rotation around  $e_3$ , that is for any  $v, v' \in \mathbb{R}^3, \omega \in S^2$*

$$F(\mathcal{O}v, \mathcal{O}v', \mathcal{O}\omega) = \mathcal{O}F(v, v', \omega), \quad F'(\mathcal{O}v, \mathcal{O}v', \mathcal{O}\omega) = \mathcal{O}F'(v, v', \omega)$$

$$\Sigma(\mathcal{O}v, \mathcal{O}v', \mathcal{O}\omega) = \Sigma(v, v', \omega), \quad \mathcal{O} = \mathcal{O}_\alpha := \begin{pmatrix} \mathcal{R}(\alpha) & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

We assume also that

$$\bar{F}(v, v', \omega) + \bar{F}'(v, v', \omega) = \bar{v} + \bar{v}', \quad v, v' \in \mathbb{R}^3, \quad \omega \in S^2.$$

Then for any non negative densities  $f = f(\bar{x}, v), f' = f'(\bar{x}, v) \in \ker \mathcal{T}$  i.e.,

$$f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp\bar{v}, y_3 = v_3, r = |\bar{v}|), \quad f'(\bar{x}, v) = g'(\bar{y} = \omega_c \bar{x} + {}^\perp\bar{v}, y_3 = v_3, r = |\bar{v}|)$$

the following equality holds true

$$\left\langle \int_{S^2} \int_{\mathbb{R}^3} \Sigma(v, v', \omega) f(\bar{x}, F(v, v', \omega)) f'(\bar{x}, F'(v, v', \omega)) dv' d\omega \right\rangle (\bar{x}, v) = I_+ + I_-$$

where

$$I_{\pm} = \pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \Sigma(\varphi_{\pm}, \omega) g(\bar{Y}(\alpha_{\pm}, \varphi_{\pm}, \omega), F_3(\varphi_{\pm}, \omega), |\bar{F}(\varphi_{\pm}, \omega)|) \\ \times g'(\bar{Y}'(\alpha_{\pm}, \varphi_{\pm}, \omega), F'_3(\varphi_{\pm}, \omega), |\bar{F}'(\varphi_{\pm}, \omega)|) \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' d\omega$$

$$\bar{Y}(\alpha, \varphi, \omega) = \bar{y} -^{\perp} \{r e^{i\alpha}\} +^{\perp} \{\mathcal{R}(\alpha) \bar{F}(\varphi, \omega)\}, \quad \alpha \in (0, 2\pi), \quad \varphi \in (-\pi, \pi)$$

$$\bar{Y}'(\alpha, \varphi, \omega) = \bar{y}' -^{\perp} \{r e^{i\alpha}\} +^{\perp} \{\mathcal{R}(\alpha) \bar{F}'(\varphi, \omega)\}, \quad \alpha \in (0, 2\pi), \quad \varphi \in (-\pi, \pi)$$

$$\bar{y} = \omega_c \bar{x} +^{\perp} \bar{v}, \quad r = |\bar{v}|, \quad \bar{y}' - \bar{y} = -^{\perp} \{\mathcal{R}(\alpha_{\pm})(r' e^{i\varphi_{\pm}} - (r, 0))\}, \quad \alpha_{\pm} \in (0, 2\pi), \quad \pm \varphi_{\pm} \in (0, \pi)$$

$$F(\varphi, \omega) = F(r, 0, v_3, r' e^{i\varphi}, v'_3, \omega), \quad F'(\varphi, \omega) = F'(r, 0, v_3, r' e^{i\varphi}, v'_3, \omega)$$

$$\Sigma(\varphi, \omega) = \Sigma(r, 0, v_3, r' e^{i\varphi}, v'_3, \omega).$$

**Proof.** By the definition of the average operator we have

$$I := \left\langle \int_{S^2} \int_{\mathbb{R}^3} \Sigma(v, v', \omega) f(\bar{x}, F(v, v', \omega)) f'(\bar{x}, F'(v, v', \omega)) dv' d\omega \right\rangle \\ = \frac{1}{2\pi} \int_0^{2\pi} \int_{S^2} \int_{\mathbb{R}^3} \Sigma(r e^{i\alpha}, v_3, v', \omega) \\ \times g(\bar{y} -^{\perp} \{r e^{i\alpha}\} +^{\perp} \bar{F}(r e^{i\alpha}, v_3, v', \omega), F_3(r e^{i\alpha}, v_3, v', \omega), |\bar{F}(r e^{i\alpha}, v_3, v', \omega)|) \\ \times g'(\bar{y}' -^{\perp} \{r e^{i\alpha}\} +^{\perp} \bar{F}'(r e^{i\alpha}, v_3, v', \omega), F'_3(r e^{i\alpha}, v_3, v', \omega), |\bar{F}'(r e^{i\alpha}, v_3, v', \omega)|) dv' d\omega d\alpha.$$

For any fixed  $\alpha \in (0, 2\pi)$  we perform the change of variable  $\omega \rightarrow \mathcal{O}\omega$  and  $v' \rightarrow \mathcal{O}v'$ , with  $\mathcal{O} = \mathcal{O}_{\alpha}$ . Since  $F, F'$  and  $\Sigma$  are left invariant by  $\mathcal{O}$  we obtain

$$I = \frac{1}{2\pi} \int_0^{2\pi} \int_{S^2} \int_{\mathbb{R}^3} \Sigma(\mathcal{O}(r, 0, v_3), \mathcal{O}v', \mathcal{O}\omega) \\ \times g(\bar{y} -^{\perp} \{r e^{i\alpha}\} +^{\perp} \bar{F}(\mathcal{O}(r, 0, v_3), \mathcal{O}v', \mathcal{O}\omega), (F_3, |\bar{F}|)(\mathcal{O}(r, 0, v_3), \mathcal{O}v', \mathcal{O}\omega)) \\ \times g'(\bar{y}' -^{\perp} \{r e^{i\alpha}\} +^{\perp} \bar{F}'(\mathcal{O}(r, 0, v_3), \mathcal{O}v', \mathcal{O}\omega), (F'_3, |\bar{F}'|)(\mathcal{O}(r, 0, v_3), \mathcal{O}v', \mathcal{O}\omega)) dv' d\omega d\alpha \\ = \frac{1}{2\pi} \int_0^{2\pi} \int_{S^2} \int_{\mathbb{R}^3} \Sigma(r, 0, v_3, v', \omega) \\ \times g(\bar{y} -^{\perp} \{r e^{i\alpha}\} +^{\perp} \{\mathcal{R}(\alpha) \bar{F}(r, 0, v_3, v', \omega)\}, (F_3, |\bar{F}|)(r, 0, v_3, v', \omega)) \\ \times g'(\bar{y}' -^{\perp} \{r e^{i\alpha}\} +^{\perp} \{\mathcal{R}(\alpha) \bar{F}'(r, 0, v_3, v', \omega)\}, (F'_3, |\bar{F}'|)(r, 0, v_3, v', \omega)) dv' d\omega d\alpha.$$

We use cylindrical coordinates for  $v'$ , that is

$$v' = (r'e^{i\varphi}, v'_3), \quad r' \in \mathbb{R}_+, \quad \varphi \in (-\pi, \pi), \quad v'_3 \in \mathbb{R}$$

and we introduce the short cuts

$$(F, F', \Sigma)(\varphi, \omega) = (F, F', \Sigma)(r, 0, v_3, r'e^{i\varphi}, v'_3, \omega)$$

leading to  $I = I_+ + I_-$

$$I_{\pm} = \pm \frac{1}{2\pi} \int_0^{2\pi} \int_{S^2} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_0^{\pm\pi} \Sigma(\varphi, \omega) g(\bar{y} - {}^\perp \{r e^{i\alpha}\} + {}^\perp \{\mathcal{R}(\alpha) \bar{F}(\varphi, \omega)\}, F_3(\varphi, \omega), |\bar{F}(\varphi, \omega)|) \\ \times g'(\bar{y} - {}^\perp \{r e^{i\alpha}\} + {}^\perp \{\mathcal{R}(\alpha) \bar{F}'(\varphi, \omega)\}, F'_3(\varphi, \omega), |\bar{F}'(\varphi, \omega)|) d\varphi r' dr' dv'_3 d\omega d\alpha$$

The key point is to replace the variables  $(\alpha, \varphi)$  by a new variable  $\bar{y}' \in \mathbb{R}^2$  such that the quantities

$$\bar{Y}(\alpha, \varphi, \omega) := \bar{y} - {}^\perp \{r e^{i\alpha}\} + {}^\perp \{\mathcal{R}(\alpha) \bar{F}(\varphi, \omega)\}, \quad \alpha \in (0, 2\pi), \quad \varphi \in (-\pi, \pi)$$

$$\bar{Y}'(\alpha, \varphi, \omega) := \bar{y} - {}^\perp \{r e^{i\alpha}\} + {}^\perp \{\mathcal{R}(\alpha) \bar{F}'(\varphi, \omega)\}, \quad \alpha \in (0, 2\pi), \quad \varphi \in (-\pi, \pi)$$

verify the conservation

$$\bar{Y}(\alpha, \varphi, \omega) + \bar{Y}'(\alpha, \varphi, \omega) = \bar{y} + \bar{y}'. \quad (43)$$

By the hypothesis we have  $\bar{F}(\varphi, \omega) + \bar{F}'(\varphi, \omega) = (r, 0) + r'e^{i\varphi}$  and (43) becomes

$$2\bar{y} - 2 {}^\perp \{r e^{i\alpha}\} + {}^\perp \{\mathcal{R}(\alpha) ((r, 0) + r'e^{i\varphi})\} = \bar{y} + \bar{y}'$$

leading to the new variable  $\bar{y}'$  defined by

$$\bar{y}' - \bar{y} = {}^\perp \{\mathcal{R}(\alpha) (r'e^{i\varphi} - (r, 0))\}. \quad (44)$$

Notice that the above relation defines a change of coordinates between  $(\alpha, \varphi) \in (0, 2\pi) \times (0, \pi)$  and  $\{\bar{y}' \in \mathbb{R}^2 : |r - r'| < |\bar{y}' - \bar{y}| < r + r'\}$  and also between  $(\alpha, \varphi) \in (0, 2\pi) \times (-\pi, 0)$  and  $\{\bar{y}' \in \mathbb{R}^2 : |r - r'| < |\bar{y}' - \bar{y}| < r + r'\}$ , which is the reason why the integral  $I$  was separated in  $I_+$  and  $I_-$ . It is easily seen that

$$\det \left( \frac{\partial \bar{y}'}{\partial (\alpha, \varphi)} \right) = -rr' \sin \varphi$$

and thus in both cases ( $\varphi \in (0, \pi), \varphi \in (-\pi, 0)$ ), thanks to the equality

$$|\bar{y}' - \bar{y}|^2 = |r'e^{i\varphi} - (r, 0)|^2 = r^2 + (r')^2 - 2rr' \cos \varphi$$

we obtain

$$\left| \det \left( \frac{\partial(\alpha, \varphi)}{\partial \bar{y}'} \right) \right| = \frac{2}{\sqrt{|\bar{y}' - \bar{y}|^2 - (r - r')^2} \sqrt{(r + r')^2 - |\bar{y}' - \bar{y}|^2}}.$$

Therefore the integrals  $I_{\pm}$  become

$$\begin{aligned} I_{\pm} = & \frac{1}{2\pi} \int_{S^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \Sigma(\varphi_{\pm}, \omega) g(\bar{Y}(\alpha_{\pm}, \varphi_{\pm}, \omega), F_3(\varphi_{\pm}, \omega), |\bar{F}(\varphi_{\pm}, \omega)|) \\ & \times g'(\bar{Y}'(\alpha_{\pm}, \varphi_{\pm}, \omega), F_3'(\varphi_{\pm}, \omega), |\bar{F}'(\varphi_{\pm}, \omega)|) 2\pi^2 \chi(r, r', \bar{y} - \bar{y}') r' dr' dv_3' d\bar{y}' d\omega \end{aligned} \quad (45)$$

where  $(\alpha_{\pm}, \pm\varphi_{\pm}) \in (0, 2\pi) \times (0, \pi)$  are given by

$$\bar{y}' - \bar{y} = {}^{\perp} \{ \mathcal{R}(\alpha_{\pm})(r'e^{i\varphi_{\pm}} - (r, 0)) \}. \quad (46)$$

□

The previous result allows us to average  $Q_{\pm}(f, f')$ , picking as functions  $\Sigma, F, F'$  the scattering section  $\sigma(v - v', \omega)$  and the pre/post-collisional velocities, which are left invariant by any rotation and satisfy the conservation

$$F(v, v', \omega) + F'(v, v', \omega) = v + v', \quad v, v' \in \mathbb{R}^3, \quad \omega \in S^2.$$

**Proof.** (of Proposition 2.3)

Take  $\Sigma(v, v', \omega) = \sigma(v - v', \omega) = |v - v'|^{\gamma} b((v - v')/|v - v'| \cdot \omega)$ ,  $F(v, v', \omega) = v$ ,  $F'(v, v', \omega) = v'$ . In this case we have  $\bar{Y}(\alpha, \varphi, \omega) = \bar{y}$ ,  $\bar{Y}'(\alpha, \varphi, \omega) = \bar{y}'$ ,  $F(\varphi, \omega) = (r, 0, v_3)$ ,  $F'(\varphi, \omega) = (r'e^{i\varphi}, v_3')$  implying that

$$\begin{aligned} & \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v) f'(\bar{x}, v') dv' d\omega \right\rangle (\bar{x}, v) = I_+ + I_- \\ & = \pi \int_{S^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}_+} (\Sigma(\varphi_+, \omega) + \Sigma(\varphi_-, \omega)) g(\bar{y}, v_3, r) g'(\bar{y}', v_3', r') \chi(r, r', \bar{y} - \bar{y}') r' dr' dv_3' d\bar{y}' d\omega. \end{aligned}$$

Notice that  $\varphi_+ = \varphi, \varphi_- = -\varphi$ , where  $\varphi$  is the unique angle in  $(0, \pi)$  such that  $|\bar{y} - \bar{y}'|^2 = r^2 + (r')^2 - 2rr' \cos \varphi$ ,  $|r - r'| < |\bar{y} - \bar{y}'| < r + r'$ . We are done if we express  $\Sigma(\varphi_{\pm}, \omega)$  in terms of  $y, r, y', r', \omega$ . Observe that

$$|(r, 0, v_3) - (r'e^{i\varphi_{\pm}}, v_3')|^2 = |\bar{y} - \bar{y}'|^2 + (y_3 - y_3')^2 = |y - y'|^2$$

and

$$\begin{aligned} [(r, 0, v_3) - (r'e^{i\varphi\pm}, v'_3)] \cdot \omega &= {}^\perp \{ \mathcal{R}(\alpha_\pm) ( (r, 0) - r'e^{i\varphi\pm} ) \} \cdot {}^\perp \{ \mathcal{R}(\alpha_\pm) \bar{\omega} \} + (v_3 - v'_3) \omega_3 \\ &= (\bar{y} - \bar{y}') \cdot {}^\perp \{ \mathcal{R}(\alpha_\pm) \bar{\omega} \} + (v_3 - v'_3) \omega_3 = (y - y', e_\pm) \end{aligned}$$

where  $e_\pm = ({}^\perp \{ \mathcal{R}(\alpha_\pm) \bar{\omega} \}, \omega_3) \in S^2$ . We deduce that

$$\Sigma(\varphi_\pm, \omega) = |y - y'|^\gamma b \left( \frac{y - y'}{|y - y'|} \cdot e_\pm \right) = \sigma(y - y', e_\pm)$$

and next we intend to replace the integration variable  $\omega \in S^2$  by  $e_\pm \in S^2$ . Indeed, this is possible since, thanks to Fubini theorem, we can fix  $y', r'$  (and therefore the angles  $\alpha_\pm$  coming from  $\bar{y}' - \bar{y} = {}^\perp \{ \mathcal{R}(\alpha_\pm) ( r'e^{i\varphi\pm} - (r, 0) ) \}$ ) and integrate first with respect to  $\omega \in S^2$ , or  $e_\pm \in S^2$ , observing that  $d\omega = de_\pm$ . Therefore (20) holds true.  $\square$

**Proof.** (of Proposition 2.4)

Take  $\Sigma(v, v', \omega) = \sigma(v - v', \omega) = |v - v'|^\gamma b((v - v')/|v - v'| \cdot \omega)$  and

$$F(v, v', \omega) = V(v, v', \omega) = v - (v - v', \omega)\omega, \quad F'(v, v', \omega) = V'(v, v', \omega) = v' + (v - v', \omega)\omega.$$

As before, for any  $\alpha \in (0, 2\pi), \omega \in S^2$ , the notation  $e$  stands for

$$e = ({}^\perp \{ \mathcal{R}(\alpha) \bar{\omega} \}, \omega_3) \in S^2.$$

Having in mind the change of coordinates (44), notice that each time we have

$$\bar{y}' - \bar{y} = {}^\perp \{ \mathcal{R}(\alpha) ( r'e^{i\varphi} - (r, 0) ) \}, \quad v = (r, 0, v_3), \quad v' = (r'e^{i\varphi}, v'_3)$$

one gets

$$\begin{aligned} (v - v', \omega) &= {}^\perp \{ \mathcal{R}(\alpha) ( (r, 0) - r'e^{i\varphi} ) \} \cdot {}^\perp \{ \mathcal{R}(\alpha) \bar{\omega} \} + (v_3 - v'_3) \omega_3 \\ &= (\bar{y} - \bar{y}') \cdot {}^\perp \{ \mathcal{R}(\alpha) \bar{\omega} \} + (v_3 - v'_3) \omega_3 = (y - y', e) \end{aligned}$$

and  $\sigma(v - v', \omega) = \sigma(y - y', e)$ . We need to express  $\bar{Y}(\alpha, \varphi, \omega), \bar{Y}'(\alpha, \varphi, \omega), F(\varphi, \omega), F'(\varphi, \omega)$  in terms of  $y = (\omega_c \bar{x} + {}^\perp \bar{v}, v_3), y' = (\bar{y}', v'_3), r = |\bar{v}|, r', \omega$ . We have

$$\bar{Y}(\alpha, \varphi, \omega) = \bar{y} + {}^\perp \{ \mathcal{R}(\alpha) (\bar{F}(\varphi, \omega) - (r, 0)) \} = \bar{y} - {}^\perp \{ \mathcal{R}(\alpha) (y - y', e) \bar{\omega} \} = \bar{y} - (y - y', e) \bar{e}$$

and

$$F_3(\varphi, \omega) = v_3 - (y - y', e) e_3 = y_3 - (y - y', e) e_3.$$

Similarly we obtain

$$\begin{aligned}
\overline{Y}'(\alpha, \varphi, \omega) &= \overline{y} + {}^\perp \{ \mathcal{R}(\alpha)(\overline{F}'(\varphi, \omega) - (r, 0)) \} \\
&= \overline{y} + {}^\perp \{ \mathcal{R}(\alpha)(r'e^{i\varphi} - (r, 0) + (y - y', e)\overline{\omega}) \} \\
&= \overline{y}' + (y - y', e)\overline{e}
\end{aligned}$$

and

$$F'_3(\varphi, \omega) = v'_3 + (y - y', e)e_3 = y'_3 + (y - y', e)e_3.$$

Therefore

$$(\overline{Y}, F_3) = y - (y - y', e)e = Y, \quad (\overline{Y}', F'_3) = y' + (y - y', e)e = Y'$$

that is, we recognize here the formula (37) giving the post-collisional Larmor centers and parallel velocities. It remains to analyse  $|\overline{F}(\varphi_\pm, \omega)|, |\overline{F}'(\varphi_\pm, \omega)|$ . We consider first the case  $\varphi_+ \in (0, \pi)$ . Recall that

$$\overline{y}' - \overline{y} = {}^\perp \{ \mathcal{R}(\alpha_+)(r'e^{i\varphi_+} - (r, 0)) \}, \quad \alpha_+ \in (0, 2\pi), \quad \varphi_+ \in (0, \pi)$$

and thus  $\varphi_+$  is the unique angle  $\varphi \in (0, \pi)$  satisfying  $|\overline{y}' - \overline{y}|^2 = r^2 + (r')^2 - 2rr' \cos \varphi$ . In order to express  $|\overline{F}(\varphi_+, \omega)|, |\overline{F}'(\varphi_+, \omega)|$  in terms of  $y, y', r, r'$  we appeal to the geometry of the triangle whose vertices are  $(0, 0), (r, 0), r'e^{i\varphi}$ , see (39), (40), leading to

$$\begin{aligned}
|\overline{F}(\varphi_+, \omega)| &= |(r, 0) - (y - y', e)\overline{\omega}| \\
&= |{}^\perp \{ r e^{i\alpha_+} \} - (y - y', e)\overline{e}| \\
&= \left| r \mathcal{R}(-\psi) \frac{\overline{y}' - \overline{y}}{|\overline{y}' - \overline{y}|} - (y - y', e)\overline{e} \right| = R
\end{aligned}$$

and

$$\begin{aligned}
|\overline{F}'(\varphi_+, \omega)| &= |r'e^{i\varphi} + (y - y', e)\overline{\omega}| \\
&= |{}^\perp \{ r' e^{i(\alpha_+ + \varphi)} \} + (y - y', e)\overline{e}| \\
&= \left| r' \mathcal{R}(-(\psi - \varphi)) \frac{\overline{y}' - \overline{y}}{|\overline{y}' - \overline{y}|} + (y - y', e)\overline{e} \right| = R'
\end{aligned}$$

where  $\psi$  is the unique angle in  $(0, \pi)$  satisfying  $(r')^2 = r^2 + |\overline{y} - \overline{y}'|^2 + 2r|\overline{y} - \overline{y}'| \cos \psi$ . We have obtained the post-collisional perpendicular velocities  $R, R'$  in (38) and therefore,



thanks to Proposition 3.2, one gets

$$\begin{aligned} I_+ &= \pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g'(Y', R') \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' d\omega \\ &= \pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g'(Y', R') \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de. \end{aligned}$$

The last equality was obtained by Fubini theorem, keeping  $r', y'$  fixed (and therefore keeping  $\alpha_+$  fixed), integrating first with respect to  $\omega \in S^2$  and observing that  $d\omega = de$ , since  $e = (\perp\{\mathcal{R}(\alpha_+)\bar{\omega}\}, \omega_3)$ . We focus now on  $I_-$ . Notice that  $\varphi_- = -\varphi$ . In this case we work in the triangle of vertices  $(0, 0), (r, 0), r'e^{-i\varphi}$  which yields

$$\begin{aligned} |\bar{F}(\varphi_-, \omega)| &= |(r, 0) - (y - y', e)\bar{\omega}| \\ &= |\perp\{re^{i\alpha_-}\} - (y - y', e)\bar{e}| \\ &= \left| r\mathcal{R}(\psi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} - (y - y', e)\bar{e} \right| =: R_- \end{aligned}$$

and

$$\begin{aligned} |\bar{F}'(\varphi_-, \omega)| &= |r'e^{-i\varphi} + (y - y', e)\bar{\omega}| \\ &= |\perp\{r'e^{i(\alpha_- - \varphi)}\} + (y - y', e)\bar{e}| \\ &= \left| r'\mathcal{R}(\psi - \varphi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} + (y - y', e)\bar{e} \right| =: R'_-. \end{aligned}$$

Observing as before that  $d\omega = de$ , we obtain the formula

$$I_- = \pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R_-) g'(Y', R'_-) \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de.$$

We are done if we prove that the contributions  $I_{\pm}$  agree. For checking that let us fix the variables  $y', r'$  and perform the change  $e \rightarrow \mathcal{S}e$  where  $\mathcal{S}$  is the orthogonal symmetry with respect to the plane spanned by  $(\bar{y}' - \bar{y}, 0)$  and  $(0, 0, 1)$  that is,  $(\mathcal{S}e)_3 = e_3$  and  $\bar{\mathcal{S}}e$  is the image of  $\bar{e}$  by the orthogonal symmetry in  $\mathbb{R}^2$  with respect to  $\bar{y}' - \bar{y}$ . It is easily seen that  $\mathcal{R}(\pm\psi)(\bar{y}' - \bar{y}), \mathcal{R}(\pm(\psi - \varphi))(\bar{y}' - \bar{y})$  are symmetric with respect to  $\bar{y}' - \bar{y}$  in  $\mathbb{R}^2$ . Obviously we have  $de = d\mathcal{S}e$ ,  $(y - y', e) = (y - y', \mathcal{S}e)$ ,  $\sigma(y - y', e) = \sigma(y - y', \mathcal{S}e)$

$$\begin{aligned} R_-(e) &= \left| r\mathcal{R}(\psi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} - (y - y', e)\bar{e} \right| \\ &= \left| r\mathcal{R}(-\psi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} - (y - y', \mathcal{S}e)\bar{\mathcal{S}}e \right| := R(\mathcal{S}e) \end{aligned}$$

$$\begin{aligned}
R'_-(e) &= \left| r' \mathcal{R}(\psi - \varphi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} + (y - y', e) \bar{e} \right| \\
&= \left| r' \mathcal{R}(-(\psi - \varphi)) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} + (y - y', \mathcal{S}e) \overline{\mathcal{S}e} \right| =: R'(\mathcal{S}e)
\end{aligned}$$

leading to

$$\begin{aligned}
I_- &= \pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', \mathcal{S}e) g(Y, R(\mathcal{S}e)) g'(Y', R'(\mathcal{S}e)) \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de \\
&= \pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R(e)) g'(Y', R'(e)) \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de = I_+.
\end{aligned}$$

□

As for the Boltzmann kernel, we get more information about the collision mechanism considering the reverse collision, obtained by interchanging  $(y, r)$  with  $(y', r')$ . More exactly, the previous proof leads to (47), cf. remark below.

**Remark 3.1** *The proof of Proposition 2.4 also established that*

$$2I_+ = \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, V(v, v', \omega)) f'(\bar{x}, V'(v, v', \omega)) dv' d\omega \right\rangle(\bar{x}, v) = 2I_-$$

where

$$\begin{aligned}
I_+ &= \pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g'(Y', R') \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de \\
I_- &= \pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R_-) g'(Y', R'_-) \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de.
\end{aligned}$$

Moreover, the equalities above have an important consequence, that comes immediately.

In fact it is easily seen that

$$Y(y', r', y, r, e) = Y'(y, r, y', r', e), \quad Y'(y', r', y, r, e) = Y(y, r, y', r', e)$$

and, by taking into account that interchanging  $(y, r)$  with  $(y', r')$  reverses the orientation, we also obtain

$$\begin{aligned}
R(y', r', y, r, e) &= \left| r' \mathcal{R}(\psi - \varphi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} + (y - y', e) \bar{e} \right| = R'_- \\
R'(y', r', y, r, e) &= \left| r \mathcal{R}(\psi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} - (y - y', e) \bar{e} \right| = R_-
\end{aligned}$$

Finally we deduce the new formula

$$\begin{aligned}
& 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma g((Y, R)(y, r, y', r', e)) g'((Y', R')(y, r, y', r', e)) \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de \\
&= \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, V(v, v', \omega)) f'(\bar{x}, V'(v, v', \omega)) dv' d\omega \right\rangle (\bar{x}, v) \\
&= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma g((Y', R')(y', r', y, r, e)) g'((Y, R)(y', r', y, r, e)) \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de
\end{aligned} \tag{47}$$

The averaged Boltzmann collision kernel follows combining Propositions 2.3, 2.4.

**Corollary 3.1** *For any non negative densities*

$$f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|), \quad f'(\bar{x}, v) = g'(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$$

the following equality holds true

$$\begin{aligned}
& \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) \{f(\bar{x}, V) f'(\bar{x}, V') - f(\bar{x}, v) f'(\bar{x}, v')\} dv' d\omega \right\rangle (\bar{x}, v) \\
&= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) \{g(Y, R) g'(Y', R') - g(y, r) g'(y', r')\} \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de
\end{aligned} \tag{48}$$

where the expressions for  $(Y, R), (Y', R')$  are given by (37), (38). In particular, the same formula holds true in the Maxwell molecule case  $\sigma_0(v - v', \omega) = b((v - v')/|v - v'| \cdot \omega)$ .

The result stated in Theorem 2.1 comes immediately combining the formal computations in the introduction, see (19), and Corollary 3.1.

## 4 The equilibria of the averaged Boltzmann collision kernel

We intend to determine the equilibria of the averaged Boltzmann collision kernel. For doing that, the main tool will be a  $H$  type theorem. We need first a weak representation formula for the averaged kernel.

### 4.1 Preliminary computations for the weak formulation

Let us take a test function  $m \in \ker \mathcal{T}$  i.e.,

$$m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$$

and let us use the well known property of the Boltzmann kernel

$$\begin{aligned}
& \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, V) f'(\bar{x}, V') m(\bar{x}, v) \, dv' d\omega dv \\
&= \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v) f'(\bar{x}, v') m(\bar{x}, V) \, dv' d\omega dv \\
&= \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v') f'(\bar{x}, v) m(\bar{x}, V') \, dv' d\omega dv
\end{aligned}$$

where  $V, V'$  are the post-collisional velocities cf. (13). Integrating the previous equalities with respect to  $\bar{x}$  yields

$$\begin{aligned}
& \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m(\bar{x}, v) \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, V) f'(\bar{x}, V') \, dv' d\omega \, dv d\bar{x} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f(\bar{x}, v) \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) m(\bar{x}, V) f'(\bar{x}, v') \, dv' d\omega \, dv d\bar{x} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f'(\bar{x}, v) \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v') m(\bar{x}, V') \, dv' d\omega \, dv d\bar{x}
\end{aligned}$$

and thus we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m(\bar{x}, v) \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, V) f'(\bar{x}, V') \, dv' d\omega \right\rangle \, dv d\bar{x} \quad (49) \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f(\bar{x}, v) \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) m(\bar{x}, V) f'(\bar{x}, v') \, dv' d\omega \right\rangle \, dv d\bar{x} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f'(\bar{x}, v) \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v') m(\bar{x}, V') \, dv' d\omega \right\rangle \, dv d\bar{x}.
\end{aligned}$$

We use the arguments in the proof of Proposition 3.2 for averaging (see Appendix A for details)

$$\int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) m(\bar{x}, V) f'(\bar{x}, v') \, dv' d\omega, \quad \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v') m(\bar{x}, V') \, dv' d\omega.$$

The notations  $(Y, R), (Y', R')$  stand for the quantities introduced in (37), (38).

**Proposition 4.1** *For any function  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  and non negative density  $f'(\bar{x}, v) = g'(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  the following equality holds true*

$$\begin{aligned}
& \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) m(\bar{x}, V) f'(\bar{x}, v') \, dv' d\omega \right\rangle (\bar{x}, v) \\
&= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) n(Y, R) g'(y', r') \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de.
\end{aligned}$$

**Proposition 4.2** For any function  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  and non negative density  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  the following equality holds true

$$\begin{aligned} & \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v') m(\bar{x}, V') \, dv' d\omega \right\rangle (\bar{x}, v) \\ &= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y', r') n(Y', R') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de. \end{aligned}$$

Coming back to (49), combined with the conclusion of Proposition 2.4

$$\begin{aligned} & \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, V) f'(\bar{x}, V') \, dv' d\omega \right\rangle (\bar{x}, v) \\ &= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g'(Y', R') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de \end{aligned}$$

and Propositions 4.1, 4.2

$$\begin{aligned} & \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) m(\bar{x}, V) f'(\bar{x}, v') \, dv' d\omega \right\rangle (\bar{x}, v) \\ &= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) n(Y, R) g'(y', r') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de \end{aligned}$$

$$\begin{aligned} & \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v') m(\bar{x}, V') \, dv' d\omega \right\rangle (\bar{x}, v) \\ &= 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y', r') n(Y', R') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de \end{aligned}$$

yields

$$\begin{aligned} & \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} n(y, r) \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g'(Y', R') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de \, r dr dy \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} g(y, r) \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) n(Y, R) g'(y', r') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de \, r dr dy \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} g'(y, r) \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y', r') n(Y', R') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de \, r dr dy. \end{aligned} \tag{50}$$

In particular, if  $n = 1$  we deduce

$$\begin{aligned} & \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g'(y', r') \chi(r, r', \bar{y} - \bar{y}') \, r dr dy r' dr' dy' de \\ &= \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g'(Y', R') \chi(r, r', \bar{y} - \bar{y}') \, r dr dy r' dr' dy' de \end{aligned} \tag{51}$$

The above equalities will allow us to write a weak formulation for the averaged Boltzmann kernel, which can be used to determine its equilibria and collision invariants.

## 4.2 $H$ theorem for the averaged Boltzmann kernel

We prove now the  $H$  type Theorem 2.2 stated in Section 2. It follows by adapting the standard arguments to the new collision mechanism. We denote by  $\langle Q \rangle$  the averaged Boltzmann collision kernel, that is  $\langle Q \rangle = \langle Q_+ \rangle - \langle Q_- \rangle$  where

$$\begin{aligned}\langle Q_+ \rangle (f, f) &= \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, V) f(\bar{x}, V') \, dv' d\omega \right\rangle \\ \langle Q_- \rangle (f, f) &= \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v) f(\bar{x}, v') \, dv' d\omega \right\rangle.\end{aligned}$$

**Proof.** (of Theorem 2.2)

1. Thanks to Proposition 2.4 and (50) we have

$$\begin{aligned}\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_+ \rangle m(\bar{x}, v) \, dv d\bar{x} &= 2\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \langle Q_+ \rangle (f, f) n(y, r) \, r dr dy \quad (52) \\ &= 4\pi^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} n(y, r) \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g(Y', R') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de \, r dr dy \\ &= 2\pi^2 \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g(y', r') \{n(Y, R) + n(Y', R')\} \\ &\quad \times \chi(r, r', \bar{y} - \bar{y}') \, r dr dy r' dr' dy' de.\end{aligned}$$

By Proposition 2.3 we know that

$$\begin{aligned}\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_- \rangle (f, f) m(\bar{x}, v) \, dv d\bar{x} &= 2\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \langle Q_- \rangle (f, f) n(y, r) \, r dr dy \quad (53) \\ &= 4\pi^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} n(y, r) \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g(y', r') \chi(r, r', \bar{y} - \bar{y}') \, r' dr' dy' de \, r dr dy \\ &= 2\pi^2 \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g(y', r') \{n(y, r) + n(y', r')\} \\ &\quad \times \chi(r, r', \bar{y} - \bar{y}') \, r dr dy r' dr' dy' de.\end{aligned}$$

Combining (52), (53) yields

$$\begin{aligned}\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q \rangle (f, f) m(\bar{x}, v) \, dv d\bar{x} &= 2\pi^2 \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g(y', r') \\ &\quad \times \{n(Y, R) + n(Y', R') - n(y, r) - n(y', r')\} \chi(r, r', \bar{y} - \bar{y}') \, r dr dy r' dr' dy' de. \quad (54)\end{aligned}$$

By the formula (51) we have

$$\begin{aligned}&\int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g(y', r') n(y, r) \chi(r, r', \bar{y} - \bar{y}') \, r dr dy r' dr' dy' de \\ &= \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g(Y', R') n(Y, R) \chi(r, r', \bar{y} - \bar{y}') \, r dr dy r' dr' dy' de\end{aligned} \quad (55)$$

and

$$\begin{aligned}
& \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g(y', r') n(y', r') \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de \\
&= \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g(Y', R') n(Y', R') \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de.
\end{aligned} \tag{56}$$

By the first equality in (50) we obtain

$$\begin{aligned}
& \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g(y', r') n(Y, R) \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de \\
&= \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g(Y', R') n(y, r) \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de.
\end{aligned} \tag{57}$$

We are done if we prove that the following equality holds true

$$\begin{aligned}
& \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y, r) g(y', r') n(Y', R') \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de \\
&= \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g(Y', R') n(y', r') \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de
\end{aligned} \tag{58}$$

since in this case, (54), (55), (56), (57), (58) yield

$$\begin{aligned}
\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q \rangle (f, f) m \, dv d\bar{x} &= -\pi^2 \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma \{ g(Y, R) g(Y', R') - g(y, r) g(y', r') \} \\
&\times \{ n(Y, R) + n(Y', R') - n(y, r) - n(y', r') \} \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de.
\end{aligned}$$

It remains to establish (58). Thanks to (50), the formula (58) is equivalent to

$$\begin{aligned}
& \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g(Y', R') n(y, r) \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de \\
&= \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g(Y', R') n(y', r') \chi(r, r', \bar{y} - \bar{y}') r dr dy' dr' dy' de
\end{aligned} \tag{59}$$

and also writes

$$\begin{aligned}
& \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} n(y, r) \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma g((Y, R)(y, r, y', r', e)) g((Y', R')(y, r, y', r', e)) \chi r' dr' dy' de \, r dr dy \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} n(y, r) \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma g((Y', R')(y', r', y, r, e)) g((Y, R)(y', r', y, r, e)) \chi r' dr' dy' de \, r dr dy.
\end{aligned}$$

Therefore (59) holds true, thanks to Remark 3.1.

2. We pick as test function  $m = \ln f$  and by the weak formulation one gets

$$\begin{aligned} \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f \langle Q \rangle (f, f) \, dv d\bar{x} &= -\pi^2 \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma \{g(Y, R)g(Y', R') - g(y, r)g(y', r')\} \\ &\times \{\ln(g(Y, R)g(Y', R')) - \ln(g(y, r)g(y', r'))\} \chi(r, r', \bar{y} - \bar{y}') \, r dr dy' dr' dy' de \leq 0. \end{aligned}$$

We have equality in the above inequality iff

$$\chi(r, r', \bar{y} - \bar{y}') \{\ln g(Y, R) + \ln g(Y', R')\} = \chi(r, r', \bar{y} - \bar{y}') \{\ln g(y, r) + \ln g(y', r')\}$$

which is equivalent to

$$\ln g(Y, R) + \ln g(Y', R') = \ln g(y, r) + \ln g(y', r'), \quad |r - r'| < |\bar{y} - \bar{y}'| < r + r'.$$

3. Consider  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  a positive equilibrium of  $\langle Q \rangle$ .

Therefore we have the equality

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f \langle Q \rangle (f, f) \, dv d\bar{x} = 0$$

and by the previous assertion we deduce (25). Conversely, let  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  be a positive density satisfying (25). Then for any function  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  we have, thanks to (24)

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m(\bar{x}, v) \langle Q \rangle (f, f) \, dv d\bar{x} = 0$$

implying that  $\langle Q \rangle (f, f) = 0$ .

4. Clearly, any function  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$ , satisfying (26), is a collision invariant thanks to (24). Conversely, let  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  be a collision invariant. In particular we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln e^{m(\bar{x}, v)} \langle Q \rangle (e^m, e^m) \, dv d\bar{x} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m(\bar{x}, v) \langle Q \rangle (e^m, e^m) \, dv d\bar{x} = 0$$

and by the second statement we obtain

$$n(Y, R) + n(Y', R') = n(y, r) + n(y', r'), \quad |r - r'| < |\bar{y} - \bar{y}'| < r + r'.$$

□



### 4.3 Equilibria and collision invariants of $\langle Q \rangle$

The previous theorem gives us necessary and sufficient conditions for determining the equilibria and collision invariants of the averaged Boltzmann collision kernel. Nevertheless working with these conditions, see (25), (26), is not obvious, since they hold true only on the support of  $\chi$ . But it is possible to identify the equilibria and collision invariants of  $\langle Q \rangle$  thanks to the fact that we know the equilibria and collision invariants of  $Q$ . It is well known that  $f(v)$  is a positive equilibrium for  $Q$  iff  $\ln f(v)$  is a collision invariant for  $Q$ , or iff  $\ln f(v)$  is a linear combination of  $1, v, |v|^2/2$ . More exactly

#### Theorem 4.1

1. For any function  $m(v)$  and non negative density  $f(v)$  we have

$$\int_{\mathbb{R}^3} m(v)Q(f, f) dv = -\frac{1}{4} \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(v - v', \omega) \{m(V) + m(V') - m(v) - m(v')\} \\ \times \{f(V)f(V') - f(v)f(v')\} dv' dv d\omega.$$

2. For any positive density  $f(v)$  we have the inequality

$$\int_{\mathbb{R}^3} \ln f(v)Q(f, f) dv \leq 0$$

with equality iff

$$\ln f(V) + \ln f(V') = \ln f(v) + \ln f(v'), \quad v, v' \in \mathbb{R}^3. \quad (60)$$

3. The positive equilibria  $f(v)$  of the Boltzmann kernel i.e.,  $f > 0, Q(f, f) = 0$ , are the positive densities satisfying (60).

4. The collision invariants, i.e., the functions  $m(v)$  such that  $\int_{\mathbb{R}^3} m(v)Q(f, f) dv = 0$  for any non negative density  $f(v)$ , are the functions  $m(v)$  satisfying

$$m(V) + m(V') = m(v) + m(v'), \quad v, v' \in \mathbb{R}^3. \quad (61)$$

Combining Theorems 2.2, 4.1 provides the following characterization for the equilibria and collision invariants of  $\langle Q \rangle$ .

#### Theorem 4.2

1. A function  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  is a collision invariant for the averaged Boltzmann collision kernel  $\langle Q \rangle$  iff for any  $\bar{x}$ ,  $m(\bar{x}, \cdot)$  is a collision

invariant for the Boltzmann kernel  $Q$ .

2. A positive density  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  is a equilibrium for the averaged Boltzmann collision kernel  $\langle Q \rangle$  iff for any  $\bar{x}$ ,  $f(\bar{x}, \cdot)$  is a equilibrium for the Boltzmann kernel  $Q$ .

**Proof.** 1. Assume that  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  is a collision invariant for  $\langle Q \rangle$  i.e.,

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m(\bar{x}, v) \langle Q \rangle (f, f) \, dv d\bar{x} = 0$$

for any non negative density  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$ . Using the variational characterization of the average operator we obtain

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m Q(f, f) \, dv d\bar{x} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m \langle Q(f, f) \rangle \, dv d\bar{x}.$$

By the definition we have  $\langle Q \rangle (f, f) = \langle Q(f, f) \rangle$  and therefore, since  $m$  is a collision invariant for  $\langle Q \rangle$ , we deduce

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m Q(f, f) \, dv d\bar{x} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m \langle Q \rangle (f, f) \, dv d\bar{x} = 0.$$

In particular, taking  $f = e^m$ , we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m(\bar{x}, v) Q(e^{m(\bar{x}, \cdot)}, e^{m(\bar{x}, \cdot)}) \, dv d\bar{x} = 0. \quad (62)$$

By the second statement of Theorem 4.1, for any  $\bar{x}$  we have the inequality

$$\int_{\mathbb{R}^3} m(\bar{x}, v) Q(e^{m(\bar{x}, \cdot)}, e^{m(\bar{x}, \cdot)}) \, dv \leq 0. \quad (63)$$

Combining (62), (63) we deduce that we have equality in (63), saying that  $m(\bar{x}, \cdot)$  is a collision invariant for  $Q$ ,  $\bar{x} \in \mathbb{R}^2$ . Conversely, let  $m(\bar{x}, v) = n(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$ , be a function such that  $m(\bar{x}, \cdot)$  is a collision invariant for  $Q$ ,  $\bar{x} \in \mathbb{R}^2$ .

Therefore

$$\int_{\mathbb{R}^3} m(\bar{x}, v) Q(e^{m(\bar{x}, \cdot)}, e^{m(\bar{x}, \cdot)}) \, dv = 0, \quad \bar{x} \in \mathbb{R}^2$$

and

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m \langle Q \rangle (e^m, e^m) \, dv d\bar{x} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m \langle Q(e^m, e^m) \rangle \, dv d\bar{x} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} m Q(e^m, e^m) \, dv d\bar{x} = 0.$$

By the second statement in Theorem 2.2 we deduce that

$$n(Y, R) + n(Y', R') = n(y, r) + n(y', r'), \quad |r - r'| < |\bar{y} - \bar{y}'| < r + r'$$

saying that  $m$  is a collision invariant for  $\langle Q \rangle$ , cf. to the fourth statement in Theorem 2.2.

2. By Theorem 2.2, a positive density  $f(\bar{x}, v) = g(\bar{y} = \omega_c \bar{x} + {}^\perp \bar{v}, y_3 = v_3, r = |\bar{v}|)$  is a equilibrium for  $\langle Q \rangle$  iff  $\ln f$  is a collision invariant for  $\langle Q \rangle$ . By the previous statement,  $\ln f$  is a collision invariant for  $\langle Q \rangle$  iff for any  $\bar{x}$ ,  $\ln f(\bar{x}, \cdot)$  is a collision invariant for  $Q$ . Using now Theorem 4.1 we deduce that  $\ln f(\bar{x}, \cdot)$  is a collision invariant for  $Q$  iff  $f(\bar{x}, \cdot)$  is a equilibrium for  $Q$  and our conclusion follows.  $\square$

#### 4.4 Parametrization of the equilibria of $\langle Q \rangle$

The previous result allows us to express the equilibria of the averaged Boltzmann collision kernel in terms of six moments (see Appendix A for the proof).

**Proposition 4.3** *The positive densities  $f$  in the kernel of  $\mathcal{T}$  satisfying  $\langle Q \rangle(f, f) = 0$  are of the form*

$$\begin{aligned} \ln f(x, v) = & \frac{\alpha(x_3)}{2} \left| \bar{x} + \frac{{}^\perp \bar{v}}{\omega_c} \right|^2 + \bar{\beta}(x_3) \cdot \left( \bar{x} + \frac{{}^\perp \bar{v}}{\omega_c} \right) + \gamma(x_3) \frac{|\bar{v}|^2}{2} \\ & + \left( \gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2} \right) \frac{(v_3)^2}{2} + \delta(x_3) v_3 + \eta(x_3) \end{aligned} \quad (64)$$

for some functions  $\alpha, \gamma, \delta, \eta : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\bar{\beta} : \mathbb{R} \rightarrow \mathbb{R}^2$ .

Notice that the right hand side in (64) is a linear combination (with coefficients depending on  $x_3$ ) of the quantities

$$1, \omega_c \bar{x} + {}^\perp \bar{v}, v_3, \frac{|v|^2}{2}, \frac{|\omega_c \bar{x} + {}^\perp \bar{v}|^2 - |\bar{v}|^2}{2}$$

which are collision invariants for  $\langle Q \rangle$ , thanks to the first statement in Theorem 4.2. Indeed, the above quantities satisfy (26) as shown in Proposition 3.1.

Up to a factor depending on  $x_3$ , the equilibrium  $f$  writes

$$\exp \left( - \frac{|\bar{v}|^2 + (v_3 - w_3(x_3))^2}{2\theta(x_3)} \right) \exp \left( - \frac{|\omega_c \bar{x} + {}^\perp \bar{v} - \bar{w}(x_3)|^2 - |\bar{v}|^2}{2\mu(x_3)} \right)$$

for some functions  $w(x_3) = (w_1, w_2, w_3)(x_3)$ ,  $\theta(x_3)$ ,  $\mu(x_3)$ , or equivalently (up to another factor depending on  $x_3$ ) as a product of three Maxwellians

$$\frac{1}{2\pi \frac{\mu\theta}{\mu-\theta}} \exp \left( - \frac{|\bar{v}|^2}{2 \frac{\mu\theta}{\mu-\theta}} \right) \frac{1}{(2\pi\theta)^{1/2}} \exp \left( - \frac{(v_3 - w_3)^2}{2\theta} \right) \frac{1}{2\pi\mu} \exp \left( - \frac{|\omega_c \bar{x} + {}^\perp \bar{v} - \bar{w}|^2}{2\mu} \right).$$

Finally we parametrize the equilibria of  $\langle Q \rangle$  by six functions  $\rho > 0, w = (w_1, w_2, w_3), \mu > \theta > 0$  uniquely determined by the moments of  $f$  with respect to

$$1, \omega_c \bar{x} + \bar{v}, v_3, \frac{|v|^2}{2}, \frac{|\omega_c \bar{x} + \bar{v}|^2 - |\bar{v}|^2}{2} \quad (65)$$

that is

$$\begin{aligned} f(x, v) &= \frac{\rho(x_3) \omega_c^2}{(2\pi)^{5/2} \frac{\mu^2 \theta^{3/2}}{\mu - \theta}} \exp\left(-\frac{|\bar{v}|^2 + (v_3 - w_3(x_3))^2}{2\theta(x_3)}\right) \exp\left(-\frac{|\omega_c \bar{x} + \bar{v} - \bar{w}(x_3)|^2 - |\bar{v}|^2}{2\mu(x_3)}\right) \\ &= \frac{\rho(x_3)}{2\pi \frac{\mu\theta}{\mu - \theta}} \exp\left(-\frac{|\bar{v}|^2}{2 \frac{\mu\theta}{\mu - \theta}}\right) \frac{1}{(2\pi\theta)^{1/2}} \exp\left(-\frac{(v_3 - w_3(x_3))^2}{2\theta}\right) \\ &\quad \times \frac{\omega_c^2}{2\pi\mu} \exp\left(-\frac{|\omega_c \bar{x} + \bar{v} - \bar{w}(x_3)|^2}{2\mu}\right). \end{aligned} \quad (66)$$

The expressions of the functions  $\theta, \mu$  in terms of the moments of  $f$  come easily.

**Proof.** (of Proposition 2.5)

Observe that for any equilibrium  $f$  in (66) we have

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|\bar{v}|^2 + (v_3 - w_3)^2}{2} f \, dv d\bar{x} &= \rho \left( \frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} \right) \\ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|\omega_c \bar{x} + \bar{v} - \bar{w}|^2 - |\bar{v}|^2}{2} f \, dv d\bar{x} &= \rho \left( \mu - \frac{\mu\theta}{\mu - \theta} \right) \end{aligned}$$

and that for any  $K > 0, K + G > 0$ , the system

$$\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} = K, \quad \mu - \frac{\mu\theta}{\mu - \theta} = G$$

has a unique solution  $\theta, \mu$  satisfying  $\mu > \theta > 0$  (solve for  $\nu := \mu/\theta$ , observing that  $2\nu(\nu - 2)/(3\nu - 1) = G/K > -1$ ).  $\square$

The equilibrium  $f$  in (66) also writes

$$f = \frac{\rho(x_3) \omega_c^2}{2\pi \frac{\mu^2}{\mu - \theta}} \exp\left(-\frac{|\omega_c \bar{x} - \bar{w}|^2}{2 \frac{\mu^2}{\mu - \theta}}\right) \frac{1}{(2\pi\theta)^{3/2}} \exp\left(-\frac{\left|\bar{v} - \frac{\theta}{\mu} (\omega_c \bar{x} - \bar{w})\right|^2 + (v_3 - w_3)^2}{2\theta}\right)$$

which is a local Maxwellian of density

$$\frac{\rho(x_3) \omega_c^2}{2\pi \frac{\mu^2}{\mu - \theta}} \exp\left(-\frac{|\omega_c \bar{x} - \bar{w}|^2}{2 \frac{\mu^2}{\mu - \theta}}\right)$$

mean velocity  $(\frac{\theta}{\mu}(\omega_c \bar{x} - \bar{w}), w_3)$  and temperature  $\theta$ . We observe that the mean parallel velocity and the temperature depend only on the parallel position. The mean perpendicular velocity vanishes at the mean Larmor center  $\bar{x} = \frac{\bar{w}}{\omega_c}$ , where the density attains its maximum with respect to the perpendicular directions.

The averaged Boltzmann collision kernel is even more complex than the original one. But once we have determined its equilibria, we can simplify it, using a BGK approximation, that is, we replace  $\langle Q_0 \rangle(f, f)$  by  $-(f - \mathcal{E}_f)$ , where  $\mathcal{E}_f$  stands for the equilibrium of  $\langle Q_0 \rangle$ , having the same moments as  $f$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (f - \mathcal{E}_f) \varphi(\bar{x}, v) \, dv d\bar{x} = 0$$

for any collision invariant  $\varphi$  in (65).

## 4.5 Fluid approximation

The fluid approximation comes immediately. In the strongly collisional regimes, the density  $f$  remains close to local equilibria whose parameters satisfy a system of conservation laws cf. [20].

**Proof.** (of Theorem 2.3)

Formally we have  $\lim_{\tau \searrow 0} f^\tau = f$ , where the density  $f$  satisfies

$$\langle Q_0 \rangle(f, f) = 0 \tag{67}$$

and

$$\partial_t f + \partial_{x_3} \{u_3 f\} - \partial_{v_3} \{v_3 \partial_{x_3} u_3 f\} = \lim_{\tau \searrow 0} \frac{\langle Q_0 \rangle(f^\tau, f^\tau)}{\tau}. \tag{68}$$

For any  $(t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$  the density  $(\bar{x}, v) \rightarrow f(t, x, v)$  is a local equilibrium cf. (66), parametrized by  $\rho(t, x_3), w(t, x_3), \theta(t, x_3), \mu(t, x_3)$ . The evolution of the functions  $(t, x_3) \rightarrow (\rho, w, \theta, \mu)(t, x_3)$  comes by appealing to the collision invariants (65). Integrating (68) with respect to  $(\bar{x}, v)$  gives the continuity equation

$$\partial_t \rho + \partial_{x_3} (u_3 \rho) = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}.$$

Similarly, multiplying (68) by  $\omega_c \bar{x} + {}^\perp \bar{v}$ ,  $(|\omega_c \bar{x} + {}^\perp \bar{v}|^2 - |\bar{v}|^2)/2$  and integrating with respect to  $(\bar{x}, v)$  yield

$$\partial_t (\rho \bar{w}) + \partial_{x_3} (u_3 \rho \bar{w}) = (0, 0), \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

$$\partial_t \left[ \rho \left( \mu - \frac{\mu\theta}{\mu - \theta} + \frac{|\bar{w}|^2}{2} \right) \right] + \partial_{x_3} \left[ \rho u_3 \left( \mu - \frac{\mu\theta}{\mu - \theta} + \frac{|\bar{w}|^2}{2} \right) \right] = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

The parallel velocity balance writes

$$\partial_t(\rho w_3) + \partial_{x_3}(u_3 \rho w_3) + \partial_{x_3} u_3 \rho w_3 = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

and the kinetic energy balance gives, using that  $\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (v_3)^2 f \, dv d\bar{x} = \rho((w_3)^2 + \theta)$

$$\partial_t \left[ \rho \left( \frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} + \frac{(w_3)^2}{2} \right) \right] + \partial_{x_3} \left[ u_3 \rho \left( \frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} + \frac{(w_3)^2}{2} \right) \right] + \partial_{x_3} u_3 \rho [(w_3)^2 + \theta] = 0.$$

□

## A Proofs of Propositions 4.1, 4.2, 4.3

When establishing the weak representation formula for  $\langle Q \rangle$ , we have used some technical results stated in Propositions 4.1, 4.2. They can be obtained as particular cases of Proposition 3.2.

**Proof.** (of Proposition 4.1)

We intend to apply Proposition 3.2 with the functions  $\Sigma(v, v', \omega) = \sigma(v - v', \omega)$ ,  $F(v, v', \omega) = V(v, v', \omega) = v - (v - v', \omega)\omega$ ,  $F'(v, v', \omega) = v'$ . Observe that  $\bar{F} + \bar{F}' \neq \bar{v} + \bar{v}'$  and thus we can not directly apply the conclusion of Proposition 3.2. Nevertheless we can follow the main lines of its proof. As  $F, F'$  are left invariant by rotations we obtain

$$I := \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) m(\bar{x}, V) f'(\bar{x}, v') \, dv' d\omega \right\rangle (\bar{x}, v) = I_+ + I_-$$

with

$$I_{\pm} := \pm \frac{1}{2\pi} \int_0^{2\pi} \int_{S^2} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_0^{\pm\pi} \Sigma(\varphi, \omega) n(\bar{y} - {}^{\perp}\{r e^{i\alpha}\} + {}^{\perp}\{\mathcal{R}(\alpha)\bar{F}(\varphi, \omega)\}, F_3(\varphi, \omega), |\bar{F}(\varphi, \omega)|) \\ \times g'(\bar{y} - {}^{\perp}\{r e^{i\alpha}\} + {}^{\perp}\{r' e^{i(\alpha+\varphi)}\}, v'_3, r') \, d\varphi r' dr' dv'_3 d\omega d\alpha.$$

Here  $F(\varphi, \omega), \Sigma(\varphi, \omega)$  are the same short-cuts as before

$$F(\varphi, \omega) = F(r, 0, v_3, r' e^{i\varphi}, v'_3, \omega), \quad \Sigma(\varphi, \omega) = \Sigma(r, 0, v_3, r' e^{i\varphi}, v'_3, \omega).$$

We replace the angles  $(\alpha, \varphi)$  by the variable  $\bar{y}' \in \mathbb{R}^2$  given by

$$\bar{y}' - \bar{y} = {}^{\perp}\{\mathcal{R}(\alpha)(r' e^{i\varphi} - (r, 0))\}$$

with  $(\alpha, \varphi) = (\alpha_+, \varphi_+) \in (0, 2\pi) \times (0, \pi)$  for the integral  $I_+$  and  $(\alpha, \varphi) = (\alpha_-, \varphi_-) \in (0, 2\pi) \times (-\pi, 0)$  for the integral  $I_-$ . We obtain

$$\Sigma(\varphi, \omega) = \sigma(y - y', e), \quad \bar{y} -^\perp \{r e^{i\alpha}\} +^\perp \{r' e^{i(\alpha+\varphi)}\} = \bar{y}'$$

and

$$\bar{Y}(\alpha, \varphi, \omega) := \bar{y} -^\perp \{r e^{i\alpha}\} +^\perp \{\mathcal{R}(\alpha) \bar{F}(\varphi, \omega)\} = \bar{y} - (y - y', e) \bar{e}, \quad F_3(\varphi, \omega) = v_3 - (y - y', e) e_3$$

saying that  $(\bar{Y}(\alpha, \varphi, \omega), F_3(\varphi, \omega))$  are the post-collisional Larmor center and parallel velocity  $Y$  in (37). The last argument of the density  $n$  in  $I_\pm$  writes

$$|\bar{F}(\varphi_\pm, \omega)| = \left| r \mathcal{R}(\mp \psi) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} - (y - y', e) \bar{e} \right|$$

and using the orthogonal symmetry with respect to the plane spanned by  $(\bar{y}' - \bar{y}, 0)$  and  $(0, 0, 1)$  one gets

$$I = I_+ + I_- = 2I_+ = 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) n(Y, R) g'(y', r') \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de.$$

□

**Proof.** (of Proposition 4.2)

We proceed as in the proof of Proposition 3.2 with the functions  $\Sigma(v, v', \omega) = \sigma(v - v', \omega)$ ,  $F(v, v', \omega) = v'$ ,  $F'(v, v', \omega) = V'(v, v', \omega) = v' + (v - v', \omega) \omega$ . These functions are left invariant by rotations and we obtain

$$I := \left\langle \int_{S^2} \int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\bar{x}, v') m(\bar{x}, V') dv' d\omega \right\rangle (\bar{x}, v) = I_+ + I_-$$

with

$$I_\pm := \pm \frac{1}{2\pi} \int_0^{2\pi} \int_{S^2} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_0^{\pm\pi} \Sigma(\varphi, \omega) g(\bar{y} -^\perp \{r e^{i\alpha}\} +^\perp \{\mathcal{R}(\alpha) r' e^{i\varphi}\}, v'_3, r') \\ \times n(\bar{y} -^\perp \{r e^{i\alpha}\} +^\perp \{\mathcal{R}(\alpha) \bar{F}'(\varphi, \omega)\}, F'_3(\varphi, \omega), |\bar{F}'(\varphi, \omega)|) d\varphi r' dr' dv'_3 d\omega d\alpha$$

where

$$F'(\varphi, \omega) = F'(r, 0, v_3, r' e^{i\varphi}, v'_3, \omega), \quad \Sigma(\varphi, \omega) = \Sigma(r, 0, v_3, r' e^{i\varphi}, v'_3, \omega).$$

We replace the angles  $(\alpha, \varphi)$  by the variable  $\bar{y}' \in \mathbb{R}^2$  given by

$$\bar{y}' - \bar{y} = ^\perp \{\mathcal{R}(\alpha) (r' e^{i\varphi} - (r, 0))\}$$

with  $(\alpha, \varphi) = (\alpha_+, \varphi_+) \in (0, 2\pi) \times (0, \pi)$  for the integral  $I_+$  and  $(\alpha, \varphi) = (\alpha_-, \varphi_-) \in (0, 2\pi) \times (-\pi, 0)$  for the integral  $I_-$ . We have

$$\Sigma(\varphi, \omega) = \sigma(y - y', e), \quad \bar{y} -^\perp \{r e^{i\alpha}\} +^\perp \{r' e^{i(\alpha+\varphi)}\} = \bar{y}'$$

and

$$\bar{Y}'(\alpha, \varphi, \omega) := \bar{y} -^\perp \{r e^{i\alpha}\} +^\perp \{\mathcal{R}(\alpha) \bar{F}'(\varphi, \omega)\} = \bar{y}' + (y - y', e) \bar{e}, \quad F'_3(\varphi, \omega) = v'_3 + (y - y', e) e_3$$

saying that  $(\bar{Y}'(\alpha, \varphi, \omega), F'_3(\varphi, \omega))$  are the post-collisional Larmor center and parallel velocity  $Y'$  in (37). For the last argument of the density  $n$  in  $I_\pm$  write

$$|\bar{F}'(\varphi_\pm, \omega)| = \left| r' \mathcal{R}(\mp(\psi - \varphi)) \frac{\bar{y}' - \bar{y}}{|\bar{y}' - \bar{y}|} + (y - y', e) \bar{e} \right|$$

and use the orthogonal symmetry with respect to the plane spanned by  $(\bar{y}' - \bar{y}, 0)$  and  $(0, 0, 1)$  which lead to

$$I = I_+ + I_- = 2I_+ = 2\pi \int_{S^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \sigma(y - y', e) g(y', r') n(Y', R') \chi(r, r', \bar{y} - \bar{y}') r' dr' dy' de.$$

□

We present now in detail the computation of the equilibria of  $\langle Q \rangle$ . It reduces to determine all local Maxwellians, depending only on  $\omega_c \bar{x} + ^\perp \bar{v}, x_3, |\bar{v}|, v_3$ .

**Proof.** (of Proposition 4.3)

Clearly any positive density  $f$  in (64) is a local Maxwellian with respect to  $v$ , satisfying the constraint  $\mathcal{T}f = 0$  (since  $f$  depends only on  $\omega_c \bar{x} + ^\perp \bar{v}, x_3, |\bar{v}|, v_3$ ) and

$$\langle Q \rangle (f, f) = \langle Q(f, f) \rangle = \langle 0 \rangle = 0.$$

Conversely, let us consider a positive density  $f$  satisfying  $\mathcal{T}f = 0, \langle Q \rangle (f, f) = 0$ . By Theorem 4.2 we deduce that for any  $x = (\bar{x}, x_3)$ ,  $f(x, \cdot)$  is a equilibrium for  $Q$ , that is a local Maxwellian

$$\ln f(x, v) = \frac{A(x)}{\omega_c^2} \frac{|v|^2}{2} + \bar{B}(x) \cdot \frac{^\perp \bar{v}}{\omega_c} + \delta(x) v_3 + C(x)$$

for some functions  $A, B_1, B_2, \delta, C : \mathbb{R}^3 \rightarrow \mathbb{R}$ . We have to determine the structure of the previous functions, such that the constraint  $\mathcal{T}f = 0$  holds true. Observe that

$$0 = \mathcal{T} \ln f = \frac{^\perp \bar{v} \cdot \nabla_{\bar{x}} A}{\omega_c^2} \frac{|v|^2}{2} - \frac{\partial_{\bar{x}} ^\perp \bar{B} : \bar{v} \otimes \bar{v}}{\omega_c} - \bar{B} \cdot \bar{v} + \bar{v} \cdot \nabla_{\bar{x}} \delta v_3 + \bar{v} \cdot \nabla_{\bar{x}} C.$$



Clearly, the third (higher) order term in velocity vanishes, saying that  $\nabla_{\bar{x}}A = 0$ , or equivalently  $A = A(x_3)$  and

$$-\frac{\partial_{\bar{x}} \perp \bar{B} : \bar{v} \otimes \bar{v}}{\omega_c} - \bar{B} \cdot \bar{v} + \bar{v} \cdot \nabla_{\bar{x}} \delta v_3 + \bar{v} \cdot \nabla_{\bar{x}} C = 0.$$

Similarly  $\delta = \delta(x_3)$  and the second order term in  $\bar{v}$  vanishes

$$\partial_{\bar{x}} \perp \bar{B} : \bar{v} \otimes \bar{v} = 0$$

implying that  $\partial_{\bar{x}} \perp \bar{B}$  is antisymmetric

$$\partial_{x_1} B_2 = \partial_{x_2} B_1 = 0, \quad \partial_{x_1} B_1 = \partial_{x_2} B_2, \quad \nabla_{\bar{x}} C = \bar{B}.$$

There is a function  $\alpha = \alpha(x_3)$  such that

$$\partial_{x_1} B_1(x_1, x_3) = \alpha(x_3) = \partial_{x_2} B_2(x_2, x_3)$$

and thus  $\bar{B} = \bar{\beta}(x_3) + \alpha(x_3)\bar{x}$  for some functions  $\bar{\beta} = (\beta_1(x_3), \beta_2(x_3))$ . The function  $C$  writes

$$C(x) = \bar{\beta}(x_3) \cdot \bar{x} + \alpha(x_3) \frac{|\bar{x}|^2}{2} + \eta(x_3)$$

and finally

$$\begin{aligned} \ln f &= \frac{A(x_3)}{\omega_c^2} \frac{|v|^2}{2} + \bar{\beta}(x_3) \cdot \left( \bar{x} + \frac{\perp \bar{v}}{\omega_c} \right) + \alpha(x_3) \bar{x} \cdot \frac{\perp \bar{v}}{\omega_c} + \delta(x_3) v_3 + \alpha(x_3) \frac{|\bar{x}|^2}{2} + \eta(x_3) \\ &= \frac{\alpha(x_3)}{2} \left| \bar{x} + \frac{\perp \bar{v}}{\omega_c} \right|^2 + \bar{\beta}(x_3) \cdot \left( \bar{x} + \frac{\perp \bar{v}}{\omega_c} \right) + \frac{A(x_3) - \alpha(x_3)}{\omega_c^2} \frac{|v|^2}{2} + \frac{A(x_3)}{\omega_c^2} \frac{(v_3)^2}{2} \\ &\quad + \delta(x_3) v_3 + \eta(x_3) \end{aligned}$$

which is the form in (64), with  $\gamma(x_3) = (A(x_3) - \alpha(x_3))/\omega_c^2$ . □

## References

- [1] M. Bostan, The Vlasov-Poisson system with strong external magnetic field. Finite Larmor radius regime, *Asymptot. Anal.* 61(2009) 91-123.
- [2] M. Bostan, Transport equations with disparate advection fields. Application to the gyrokinetic models in plasma physics, *J. Differential Equations* 249(2010) 1620-1663.

- [3] M. Bostan, Gyrokinetic Vlasov equation in three dimensional setting. Second order approximation, *SIAM J. Multiscale Model. Simul.* 8(2010) 1923-1957.
- [4] M. Bostan, C. Caldini-Queiros, Finite Larmor radius approximation for collisional magnetized plasmas, *C. R. Math. Acad. Sci. Paris Sér. I* 350(2012) 879-884.
- [5] M. Bostan, C. Caldini-Queiros, Finite Larmor radius approximation for collisional magnetic confinement. Part I: the linear Boltzmann equation, *Quart. Appl. Math.* Vol. LXXII(2014) 323-345.
- [6] M. Bostan, C. Caldini-Queiros, Finite Larmor radius approximation for collisional magnetic confinement. Part II: the Fokker-Planck-Landau equation, to appear in *Quart. Appl. Math.*
- [7] A.J. Brizard, A guiding-center Fokker-Planck collision operator for nonuniform magnetic fields, *Phys. Plasmas* 11(2004) 4429-4438.
- [8] A.J. Brizard, T.S. Hahm, Foundations of nonlinear gyrokinetic theory, *Rev. Modern Phys.* 79(2007) 421-468.
- [9] C. Cercignani, R. Illner, M. Pulvirenti, *The mathematical theory of dilute gases*, Applied Mathematical Sciences, 106, Springer-Verlag New-York 1994.
- [10] R.C. Davidson, H. Qin, *Physics of charged particle beams in high energy accelerators*, Imperial College Press, World Scientific Singapore, 2001.
- [11] P. Degond, P.-A. Raviart, On the paraxial approximation of the stationary Vlasov-Maxwell system, *Math. Models Meth. Appl. Sci.* 3(1993) 513-562.
- [12] F. Filbet, E. Sonnendrücker, Modeling and numerical simulation of space charge dominated beams in the paraxial approximation, *Math. Models Methods Appl. Sci.* 16, no. 5(2006) 763-791.
- [13] E. Frénod, Application of the averaging method to the gyrokinetic plasma, *Asymptot. Anal.* 46(2006) 1-28.

- [14] E. Frénod, A. Mouton, Two-dimensional finite Larmor radius approximation in canonical gyrokinetic coordinates, *J. Pures Appl. Math. Adv. Appl.* 4(2010) 135-169.
- [15] E. Frénod, E. Sonnendrücker, Homogenization of the Vlasov equation and of the Vlasov-Poisson system with strong external magnetic field, *Asymptotic Anal.* 18(1998) 193-213.
- [16] E. Frénod, E. Sonnendrücker, The finite Larmor radius approximation, *SIAM J. Math. Anal.* 32(2001) 1227-1247.
- [17] X. Garbet, G. Dif-Pradalier, C. Nguyen, Y. Sarazin, V. Grandgirard, Ph. Ghendrih, Neoclassical equilibrium in gyrokinetic simulations, *Phys. Plasmas* 16(2009).
- [18] F. Golse, L. Saint-Raymond, The Vlasov-Poisson system with strong magnetic field, *J. Math. Pures Appl.* 78(1999) 791-817.
- [19] G. Laval, S. Mas-Gallic, P.-A. Raviart, Paraxial approximation of ultra-relativistic intense beams, *Numer. Math.* 1(1994) 33-60.
- [20] D. Levermore, Moment closure hierarchies for kinetic theories. *J. Statist. Phys.* 83(1996) 1021-1065.
- [21] J. Madsen, Gyrokinetic linearized Landau collision operator, *Phys. Review E* 87(2013).
- [22] P.-A. Raviart, Paraxial approximation of the stationary Vlasov-Maxwell equations, *Nonlinear partial differential equations and their applications. Collège de France Seminar, vol XIII Paris (1991-1993); Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow* 302 (1994) 158-171.
- [23] G. Toscani, C. Villani, Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas, *J. Statist. Phys.* 94, 3/4(1999) 619-637.
- [24] G. Toscani, C. Villani, Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation, *Comm. Math. Phys.* 203, 3(1999) 667-706.

- [25] C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations, *Arch. Rational Mech. Anal.* 143(1998) 273-307.
- [26] X.Q. Xu, M.N. Rosenbluth, Numerical simulation of ion-temperature-gradient-driven modes, *Phys. Fluids, B* 3(1991) 627-643.