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FINITE LARMOR RADIUS APPROXIMATION FOR COLLISIONAL MAGNETIC CONFINEMENT. PART II: THE FOKKER-PLANCK-LANDAU EQUATION

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Abstract. This paper is devoted to the finite Larmor radius approximation of the Fokker-Planck-Landau equation, which plays a major role in plasma physics. We obtain a completely explicit form for the gyroaverage of the Fokker-Planck-Landau kernel, accounting for diffusion and convolution with respect to both velocity and (perpendicular) position coordinates. We show that the new collision operator enjoys the usual physical properties ; the averaged kernel balances the mass, momentum, kinetic energy and dissipates the entropy, globally in velocity and perpendicular position coordinates.

1. Introduction.

Motivated by the energy production through thermonuclear fusion, the magnetic confinement represents an important issue in plasma physics. A tokamak plasma is controlled when subject to a strong magnetic field. We neglect the self-consistent electro-magnetic

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field with respect to the external one, given by

$$E = -\nabla_x \phi, \;\; B^\varepsilon = \frac{B(x)}{\varepsilon}\, b(x), \;\; |b| = 1$$

when $\varepsilon > 0$ is a small parameter, destinated to converge to 0, in order to describe strong magnetic fields. The scalar function ϕ stands for the electric potential, $B(x) > 0$ is the rescaled magnitude of the magnetic field and $b(x)$ denotes its direction. The presence density $f^\varepsilon = f^\varepsilon(t, x, v) \geq 0$ of the population of charged particles with mass m and charge q satisfies

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m}(E + v \wedge B^\varepsilon) \cdot \nabla_v f^\varepsilon = Q_{FPL}(f^\varepsilon), \;\; (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (1.1)$$

$$f^\varepsilon(0, x, v) = f^{\mathrm{in}}(x, v), \;\; (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad (1.2)$$

where Q_{FPL} denotes the bilinear Fokker-Planck-Landau collision kernel. In the collisionless case the limit model as $\varepsilon \searrow 0$ comes by averaging with respect to the fast cyclotronic motion [14, 16, 10, 1, 2, 3, 11, 12]. Linear collision kernels (relaxation, Fokker-Planck) have been gyroaveraged in [5], [4], [17], [13].

We study here the finite Larmor radius scaling *i.e.*, the typical perpendicular spatial length is of the same order as the Larmor radius and the parallel spatial length is much larger. Assuming that the magnetic field is homogeneous and stationary

$$B^\varepsilon = \left(0, 0, \frac{B}{\varepsilon}\right)$$

for some constant $B > 0$, the equation (1.1) writes

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v_1 \partial_{x_1} f^\varepsilon + v_2 \partial_{x_2} f^\varepsilon) + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon + \frac{\omega_c}{\varepsilon}(v_2 \partial_{v_1} f^\varepsilon - v_1 \partial_{v_2} f^\varepsilon) = Q_{FPL}(f^\varepsilon) \quad (1.3)$$

where $\omega_c = qB/m$ stands for the rescaled cyclotronic frequency. The density f^ε is decomposed into a dominant density f and fluctuations of orders $\varepsilon, \varepsilon^2, ...$

$$f^\varepsilon = f + \varepsilon f^1 + \varepsilon^2 f^2 + ... \quad (1.4)$$

Combining (1.3), (1.4) yields, with the notations $\overline{x} = (x_1, x_2), \overline{v} = (v_1, v_2)$, ${}^\perp\overline{v} = (v_2, -v_1)$

$$\mathcal{T} f := \overline{v} \cdot \nabla_{\overline{x}} f + \omega_c\, {}^\perp\overline{v} \cdot \nabla_{\overline{v}} f = 0 \quad (1.5)$$

$$\partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 = Q_{FPL}(f) \tag{1.6}$$

$$\vdots$$

The equation (1.5) appears as a divergence constraint

$$\mathrm{div}_{x,v}\{f(\overline{v}, 0, \omega_c\, {}^\perp\overline{v}, 0)\} = 0.$$

It says that at any time t the density $f(t,\cdot,\cdot)$ is left invariant by the flow associated to $\overline{v} \cdot \nabla_{\overline{x}} + \omega_c\, {}^\perp\overline{v} \cdot \nabla_{\overline{v}}$

$$\frac{\mathrm{d}\overline{X}}{\mathrm{d}s} = \overline{V}(s), \quad \frac{\mathrm{d}X_3}{\mathrm{d}s} = 0, \quad \frac{\mathrm{d}\overline{V}}{\mathrm{d}s} = \omega_c\, {}^\perp\overline{V}(s), \quad \frac{\mathrm{d}V_3}{\mathrm{d}s} = 0 \tag{1.7}$$

and therefore, at any time t, the density $f(t,\cdot,\cdot)$ depends only on the invariants of (1.7)

$$f(t,x,v) = g\left(t, x_1 + \frac{v_2}{\omega_c}, x_2 - \frac{v_1}{\omega_c}, x_3, r = |\overline{v}|, v_3\right).$$

The dynamics for f comes by eliminating f^1 in (1.6), after projecting onto the kernel of $\mathcal{T}$. This projection appears as the average along the characteristic flow (1.7). If $\langle\cdot\rangle$ stands for this projection, we obtain

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = \langle Q_{FPL}(f)\rangle, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \tag{1.8}$$

Averaging $\partial_t + v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$ leads to another transport operator

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = \partial_t f + \frac{\langle {}^\perp\overline{E}\rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f.$$

We also need to average the collision kernel. Our main motivation concerns the bilinear Fokker-Planck-Landau equation, more exactly how to average kernels like

$$Q_{FPL}(f,f)(v) = \mathrm{div}_v \left\{ \int_{\mathbb{R}^3} \sigma(|v-v'|) S(v-v')[f(v')\nabla_v f(v) - f(v) \nabla_{v'} f(v')] \,\mathrm{d}v' \right\}$$

where σ denotes the scattering cross section and $S(w) = I - \frac{w \otimes w}{|w|^2}$ is the orthogonal projection on the plane of normal w, cf. [15]. Recall that Q_{FPL} satisfies the mass, momentum and kinetic energy balances

$$\int_{\mathbb{R}^3} Q_{FPL}(f,f)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} v Q_{FPL}(f,f)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} \frac{|v|^2}{2} Q_{FPL}(f,f)\,\mathrm{d}v = 0.$$

Moreover it decreases the entropy $f \ln f$ since, by standard computations, we obtain

$$\int_{\mathbb{R}^3} \ln f \; Q_{FPL}(f,f) \;\mathrm{d}v$$
$$= -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sigma f(v) f(v') \frac{|(v-v')\wedge(\nabla_v \ln f(v) - \nabla_{v'} \ln f(v'))|^2}{|v-v'|^2} \;\mathrm{d}v'\mathrm{d}v \leq 0.$$

We expect that the averaged Fokker-Planck-Landau operator satisfies the same properties. Nevertheless we will see that all of them hold true only globally in velocity and perpendicular position coordinates. Indeed, the averaged collision kernel will account for the interactions between Larmor circles (characterized by the center $\overline{x} + {}^\perp\overline{v}/\omega_c$ and the radius $|\overline{v}|/|\omega_c|$) rather than between particles. We show that the averaged Fokker-Planck-Landau kernel has the form

$$\begin{aligned} \langle Q_{FPL}\rangle (f,f) : &= \langle Q_{FPL}(f,f)\rangle (x,v) \\ &= \mathrm{div}_{\omega_c x, v}\left\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma\chi f(\overline{x'}, x_3, v') A^+ \nabla_{\omega_c x, v} f(x,v)\;\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\right\} \\ &- \mathrm{div}_{\omega_c x, v}\left\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma\chi f(x,v) A^- \nabla_{\omega_c x', v'} f(\overline{x'}, x_3, v')\;\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\right\} \end{aligned} \tag{1.9}$$

with $z = \omega_c \overline{x} + {}^\perp\overline{v} - (\omega_c\overline{x'} + {}^\perp\overline{v'})$, $\sigma = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\;)$, $\chi = \chi(|\overline{v}|, |\overline{v'}|, z)$ and

$$\chi(r,r',z) = \frac{\mathbf{1}_{\{|r-r'|<|z|<r+r'\}}}{\pi^2\sqrt{|z|^2-(r-r')^2}\sqrt{(r+r')^2-|z|^2}}, \quad r,r' \in \mathbb{R}_+, \quad z \in \mathbb{R}^2.$$

$$\sigma\chi A^+(r, v_3, r', v_3', z) = \sum_{i=1}^4 \xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x}, v, \overline{x'}, v')$$

$$\sigma\chi A^-(r, v_3, r', v_3', z) = \sum_{i=1}^4 \varepsilon_i \xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x'}, v', \overline{x}, v)$$

for some vector fields $(\xi^i)_{1\leq i\leq 4}$ and $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1$, see Proposition 4.10. Actually A^+, A^- have only zero entries on the third line and column and therefore, averaging the Fokker-Planck-Landau kernel leads to diffusion (and convolution) with respect to velocity but also perpendicular position coordinates. To the best of our knowledge, this is the first completely explicit result on this topic. In particular, the above collisional kernel decreases the entropy $f \ln f$ since, by standard computations we obtain (see

Proposition 4.11)

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f \left\langle Q_{FPL}\right\rangle(f,f)\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = -\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} ff'$$

$$\times\left(\xi^i\cdot\nabla\ln f - \varepsilon_i(\xi^i)'\cdot\nabla'\ln f'\right)^2\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \le 0,\ x_3\in\mathbb{R}.$$

Here, for any $\xi,\eta\in\mathbb{R}^6$, the notations $\xi\otimes\eta$ stands for the matrix whose entries are $(\xi\otimes\eta)_{kl}=\xi_k\eta_l$, $1\le k,l\le 6$. We obtain formally the following stability result

Theorem 1.1. Let us consider $f^{\mathrm{in}}\ge 0$, $(1+|\ln f^{\mathrm{in}}|)f^{\mathrm{in}}\in L^1(\mathbb{R}^3\times\mathbb{R}^3)$ and denote by f^ε the solution of (1.3), (1.2), for any $\varepsilon>0$. Then the limit $f=\lim_{\varepsilon\searrow 0} f^\varepsilon$ satisfies

$$\partial_t f + \frac{\left\langle {}^\perp\overline{E}\right\rangle}{B}\cdot\nabla_{\overline{x}} f + v_3\partial_{x_3} f + \frac{q}{m}\left\langle E_3\right\rangle\partial_{v_3} f = \left\langle Q_{FPL}\right\rangle(f,f),\ \ (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3 \tag{1.10}$$

$$f(0,x,v) = \left\langle f^{\mathrm{in}}\right\rangle(x,v),\ \ (x,v)\in\mathbb{R}^3\times\mathbb{R}^3 \tag{1.11}$$

where the averaged Fokker-Planck-Landau operator is given by (1.9).

Our paper is organized as follows. In Section 2 we recall briefly the notion of average along a characteristic flow. Its main properties are stated in Section 3. The gyroaverage of the Fokker-Planck-Landau kernel is discussed in Section 4, Theorem 1.1. We give a explicit form of its average and check the main physical properties. We prove the mass, momentum and total energy conservations for smooth solutions of the averaged Fokker-Planck-Landau equation coupled to the Poisson equation for the electric field. We also show that the mean Larmor circle center and power (with respect to the origin) are left invariant. Up to our knowledge this has not been reported yet.

2. Average operator. Consider the transport operator $\mathcal{T}$ in the $L^2(\mathbb{R}^3\times\mathbb{R}^3)$ setting, defined by

$$\mathcal{T}u = \mathrm{div}_{x,v}(u\,b),\ \ b=(\overline{v},0,\omega_c\,{}^\perp\overline{v},0),\ \ \omega_c=\frac{qB}{m}$$

for any function u in the domain

$$\mathrm{D}(\mathcal{T}) = \{u(x,v)\in L^2(\mathbb{R}^3\times\mathbb{R}^3)\ :\ \mathrm{div}_{x,v}(u\,b)\in L^2(\mathbb{R}^3\times\mathbb{R}^3)\}.$$

We denote by $\|\cdot\|$ the standard norm of $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$.The characteristics $(X,V)(s;x,v)$ associated to $\overline{v}\cdot\nabla_{\overline{x}} + \omega_c\,{}^\perp\overline{v}\cdot\nabla_{\overline{v}}$, see (1.7), satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\overline{X} + \frac{{}^\perp\overline{V}}{\omega_c}\right\} = 0,\quad \frac{\mathrm{d}\overline{V}}{\mathrm{d}s} = \omega_c\,{}^\perp\overline{V},\quad \frac{\mathrm{d}X_3}{\mathrm{d}s} = 0,\quad \frac{\mathrm{d}V_3}{\mathrm{d}s} = 0$$

implying that

$$\overline{V}(s) = R(-\omega_c s)\overline{v},\ \ \overline{X}(s) = \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\overline{V}(s)}{\omega_c},\ \ X_3(s) = x_3,\ \ V_3(s) = v_3$$

where $R(\alpha)$ stands for the rotation of angle α

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

All the trajectories are $T_c = 2\pi/\omega_c$ periodic and we introduce the average operator, see [2], for any function $u \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$

$$\begin{aligned} \langle u\rangle(x,v) &= \frac{1}{T_c}\int_0^{T_c} u(X(s;x,v),V(s;x,v))\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_0^{2\pi} u\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\{R(\alpha)\overline{v}\}}{\omega_c}, x_3, R(\alpha)\overline{v}, v_3\right)\,\mathrm{d}\alpha. \end{aligned}$$

It is convenient to introduce the notation $e^{i\varphi}$ for the $\mathbb{R}^2$ vector $(\cos\varphi,\sin\varphi)$. Assume that the vector $\overline{v}$ writes $\overline{v} = |\overline{v}|e^{i\varphi}$. Then $R(\alpha)\overline{v} = |\overline{v}|e^{i(\alpha+\varphi)}$ and the expression for $\langle u\rangle$ becomes

$$\begin{aligned} \langle u\rangle(x,v) &= \frac{1}{2\pi}\int_0^{2\pi} u\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\{|\overline{v}|e^{i(\alpha+\varphi)}\}}{\omega_c}, x_3, |\overline{v}|e^{i(\alpha+\varphi)}, v_3\right)\,\mathrm{d}\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi} u\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\{|\overline{v}|e^{i\alpha}\}}{\omega_c}, x_3, |\overline{v}|e^{i\alpha}, v_3\right)\,\mathrm{d}\alpha. \end{aligned} \tag{2.1}$$

Notice that $\langle u\rangle$ depends only on the invariants $\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}, |\overline{v}|, x_3, v_3$ and therefore belongs to $\ker\mathcal{T}$. The following two results are borrowed from [3], Propositions 2.1, 2.2.

Proposition 2.1. The average operator is linear continuous. Moreover it coincides with the orthogonal projection on the kernel of $\mathcal{T}$ *i.e.,*

$$\langle u\rangle \in \ker\mathcal{T} \ \text{ and } \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(u - \langle u\rangle)\varphi\,\mathrm{d}v\mathrm{d}x = 0,\ \ \forall\,\varphi\in\ker\mathcal{T}. \tag{2.2}$$

REMARK 2.1. Notice that $(\overline{X}, \overline{V})$ depends only on s and $(\overline{x}, \overline{v})$ and thus the variational characterization in (2.2) holds true at any fixed $(x_3, v_3) \in \mathbb{R}^2$. Indeed, for any $\varphi \in \ker \mathcal{T}$, $(x_3, v_3) \in \mathbb{R}^2$ we have

$$\begin{aligned}
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (u\varphi)(x,v)\;\mathrm{d}\overline{v}\mathrm{d}\overline{x} &= \frac{1}{T_c}\int_0^{T_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,v)\varphi(\overline{X}(-s;x,v),x_3,\overline{V}(-s;x,v),v_3)\;\mathrm{d}\overline{v}\mathrm{d}\overline{x}\mathrm{d}s\\
&= \frac{1}{T_c}\int_0^{T_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(\overline{X}(s;x,v),x_3,\overline{V}(s;x,v),v_3)\varphi(x,v)\;\mathrm{d}\overline{v}\mathrm{d}\overline{x}\mathrm{d}s\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \langle u\rangle\,(x,v)\varphi(x,v)\;\mathrm{d}\overline{v}\mathrm{d}\overline{x}.
\end{aligned}$$

We have the orthogonal decomposition of $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ into invariant functions along the characteristics (1.7) and zero average functions

$$u = \langle u\rangle + (u - \langle u\rangle),\quad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (u - \langle u\rangle)\,\langle u\rangle\;\mathrm{d}v\mathrm{d}x = 0.$$

Since $\mathcal{T}^\star = -\mathcal{T}$ we have

$$\ker\langle\cdot\rangle = \ker \mathrm{Proj}_{\ker\mathcal{T}} = (\ker\mathcal{T})^\perp = (\ker\mathcal{T}^\star)^\perp = \overline{\mathrm{Range}\,\mathcal{T}}$$

implying $\mathrm{Range}\,\mathcal{T} \subset \ker\langle\cdot\rangle$. Actually we show that $\mathrm{Range}\,\mathcal{T}$ is closed, which will give a solvability condition for $\mathcal{T}u = w$ (cf. [3], Propositions 2.2).

PROPOSITION 2.2. The restriction of $\mathcal{T}$ to $\ker\langle\cdot\rangle$ is one to one map onto $\ker\langle\cdot\rangle$. Its inverse belongs to $\mathcal{L}(\ker\langle\cdot\rangle, \ker\langle\cdot\rangle)$ and we have the Poincaré inequality

$$\|u\| \le \frac{2\pi}{|\omega_c|}\|\mathcal{T}u\|,\quad \omega_c = \frac{qB}{m} \neq 0$$

for any $u \in \mathrm{D}(\mathcal{T}) \cap \ker\langle\cdot\rangle$.

3. Average and first order differential operators. In order to average transport operators, see (1.8), and the Fokker-Planck-Landau kernel, it is convenient to identify derivations which leave invariant $\ker\mathcal{T}$. It turns out that these derivations are those along the invariants

$$\psi_1 = x_1 + \frac{v_2}{\omega_c},\;\; \psi_2 = x_2 - \frac{v_1}{\omega_c},\;\; \psi_3 = x_3,\;\; \psi_4 = \sqrt{(v_1)^2 + (v_2)^2},\;\; \psi_5 = v_3.$$

We introduce also $\psi_0 = -\frac{\alpha}{\omega_c}$, with $\overline{v} = |\overline{v}|e^{i\alpha}$, $\alpha \in [0, 2\pi[$. Notice that ψ_0 has a jump of $\frac{2\pi}{\omega_c}$ across $\overline{v} \in \mathbb{R}_+ \times \{0\}$ but not its gradient with respect to $\overline{v}$

$$\nabla_{\overline{v}}\alpha = -\frac{{}^\perp\overline{v}}{|\overline{v}|^2}, \;\; \nabla_{\overline{v}}\psi_0 = \frac{{}^\perp\overline{v}}{\omega_c|\overline{v}|^2}, \;\; \mathcal{T}\psi_0 = 1.$$

The idea is to consider the fields $(b^i)_{0\leq i\leq 5}$ such that

$$b^i \cdot \nabla_{x,v}\psi_j = \delta^i_j, \;\; 0 \leq i, j \leq 5.$$

Indeed, the map $(x, v) \to (\psi_i(x, v))_{0\leq i\leq 5}$ defines a change of coordinates

$$x_1 = \psi_1 + \frac{\psi_4}{\omega_c}\sin(\omega_c\psi_0), \;\; x_2 = \psi_2 + \frac{\psi_4}{\omega_c}\cos(\omega_c\psi_0), \;\; x_3 = \psi_3$$

$$v_1 = \psi_4\cos(\omega_c\psi_0), \;\; v_2 = -\psi_4\sin(\omega_c\psi_0), \;\; v_3 = \psi_5.$$

Therefore any function $u = u(x, v)$ can be written $u(x, v) = U(\psi(x, v))$, $\psi = (\psi_i)_{0\leq i\leq 5}$ and thus, for any $i \in \{0, 1, ..., 5\}$ we have

$$b^i \cdot \nabla_{x,v}u = b^i \cdot \sum_{j=0}^{5} \frac{\partial U}{\partial\psi_j}(\psi(x, v))\nabla_{x,v}\psi_j = \frac{\partial U}{\partial\psi_i}(\psi(x, v)).$$

In other words the derivations $b^i \cdot \nabla_{x,v}$ act like $\partial_{\psi_i}, 0 \leq i \leq 5$. In particular if $u \in \ker \mathcal{T}$, meaning that U does not depend on ψ_0, then $b^i \cdot \nabla_{x,v}u = \partial_{\psi_i}U(\psi(x, v))$ does not depend on ψ_0, saying that $\ker \mathcal{T}$ is left invariant by $b^i \cdot \nabla_{x,v}$, $0 \leq i \leq 5$. The following result comes by direct computation and is left to the reader. For any smooth vector fields ξ, η on $\mathbb{R}^6$, the notation $[\xi, \eta]$ stands for their Poisson bracket *i.e.,*

$$[\xi, \eta] = (\xi \cdot \nabla_{x,v})\eta - (\eta \cdot \nabla_{x,v})\xi.$$

PROPOSITION 3.1. The fields $(b^i)_{0\leq i\leq 5}$ satisfying $b^i \cdot \nabla_{x,v}\psi_j = \delta^i_j$, $0 \leq i, j \leq 5$ are given by

$$b^0 \cdot \nabla_{x,v} = \overline{v} \cdot \nabla_{\overline{x}} + \omega_c\, {}^\perp\overline{v} \cdot \nabla_{\overline{v}}, \;\; b^1 \cdot \nabla_{x,v} = \partial_{x_1}, \;\; b^2 \cdot \nabla_{x,v} = \partial_{x_2}, \;\; b^3 \cdot \nabla_{x,v} = \partial_{x_3}$$

$$b^4 \cdot \nabla_{x,v} = -\frac{{}^\perp\overline{v}}{\omega_c|\overline{v}|} \cdot \nabla_{\overline{x}} + \frac{\overline{v}}{|\overline{v}|} \cdot \nabla_{\overline{v}}, \;\; b^5 \cdot \nabla_{x,v} = \partial_{v_3}.$$

Moreover the Poisson brackets between $(b^i)_{0\leq i\leq 5}$ vanishes or equivalently the derivations $b^i \cdot \nabla_{x,v}$, $0 \leq i \leq 5$ are commuting.

REMARK 3.1. Notice that $(b^i)_{i\neq 4}$ are divergence free and $\mathrm{div}_{x,v}b^4 = \frac{1}{|\overline{v}|}$.

We claim that the operators $u \to \mathrm{div}_{x,v}(ub^i)$, with domain

$$\mathrm{D}(\mathrm{div}_{x,v}(\cdot\ b^i)) = \{u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\ :\ \mathrm{div}_{x,v}(ub^i) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\},\ \ 0 \le i \le 5$$

are commuting with the average operator. More generally we establish the following result cf. [5].

Proposition 3.2. Assume that the field $c \cdot \nabla_{x,v}$ is in involution with $b \cdot \nabla_{x,v} = \overline{v} \cdot \nabla_{\overline{x}} + \omega_c\, {}^\perp\overline{v} \cdot \nabla_{\overline{v}}$ *i.e.,* $[c, b] = 0$. Then the operator $\mathrm{div}_{x,v}(\cdot\ c)$ is commuting with the average operator associated to the flow of $b \cdot \nabla_{x,v}$ that is, for any function $u \in \mathrm{D}(\mathrm{div}_{x,v}(\cdot\ c))$ its average $\langle u \rangle$ belongs to $\mathrm{D}(\mathrm{div}_{x,v}(\cdot\ c))$ and

$$\mathrm{div}_{x,v}(\langle u \rangle\ c) = \langle \mathrm{div}_{x,v}(uc) \rangle\,.$$

Another useful tool is the following commutation formula between average and divergence cf. [5].

Proposition 3.3. For any smooth field $\xi = (\xi_x, \xi_v) \in \mathbb{R}^6$ we have the equality

$$\begin{aligned} \langle \mathrm{div}_{x,v}\xi \rangle &= \mathrm{div}_{\overline{x}} \left\{ \left\langle \xi_{\overline{x}} + \frac{{}^\perp\xi_{\overline{v}}}{\omega_c} \right\rangle + \left\langle \xi_{\overline{v}} \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{\omega_c|\overline{v}|} - \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{{}^\perp\overline{v}}{\omega_c|\overline{v}|} \right\} + \partial_{x_3} \langle \xi_{x_3} \rangle \\ &+ \mathrm{div}_{\overline{v}} \left\{ \left\langle \xi_{\overline{v}} \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|} \right\rangle \frac{{}^\perp\overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{|\overline{v}|} \right\} + \partial_{v_3} \langle \xi_{v_3} \rangle\,. \end{aligned}$$

In particular we have for any smooth field $\xi_x \in \mathbb{R}^3$

$$\langle \mathrm{div}_x \xi_x \rangle = \mathrm{div}_x \langle \xi_x \rangle$$

and for any smooth field $\xi_v \in \mathbb{R}^3$

$$\begin{aligned} \langle \mathrm{div}_v \xi_v \rangle &= \mathrm{div}_{\overline{x}} \left\{ \left\langle \frac{{}^\perp\xi_{\overline{v}}}{\omega_c} \right\rangle + \left\langle \xi_{\overline{v}} \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{\omega_c|\overline{v}|} - \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{{}^\perp\overline{v}}{\omega_c|\overline{v}|} \right\} \\ &+ \mathrm{div}_{\overline{v}} \left\{ \left\langle \xi_{\overline{v}} \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|} \right\rangle \frac{{}^\perp\overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{|\overline{v}|} \right\} + \partial_{v_3} \langle \xi_{v_3} \rangle\,. \end{aligned}$$

A direct consequence of Proposition 3.3 is the computation of the average for the transport operator in (1.6).

PROPOSITION 3.4. Assume that the electric field derives from a smooth potential *i.e.,* $E = -\nabla_x \phi$. Then for any $f \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}$ we have

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 \right\rangle = \partial_t f + \frac{\langle {}^\perp \overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f.$$

4. The Fokker-Planck-Landau operator. In this section we focus on the Fokker-Planck-Landau equation [6, 7, 8, 9]. The rate of change of the density f_s, corresponding to a population of charged particles of specie s, due to collisions with charge particles of specie s' writes

$$Q_{FPL}(f_s, f_{s'}) = \frac{1}{m_s} \mathrm{div}_v \int_{\mathbb{R}^3} \mu_{ss'}^2 \sigma_{ss'}(|v - v'|)|v - v'|^3$$
$$\times S(v - v') \left(\frac{1}{m_s} f_{s'}(v')(\nabla_v f_s)(v) - \frac{1}{m_{s'}} f_s(v)(\nabla_{v'} f_{s'})(v') \right) \mathrm{d}v'$$

where $\mu_{ss'} = \frac{m_s m_{s'}}{m_s + m_{s'}}$ is the reduced mass of the pair $\{m_s, m_{s'}\}$, $\sigma_{ss'} = \sigma_{s's} > 0$ is the scattering cross section between species $\{s, s'\}$ and the matrix $S(w) = \left(I - \frac{w \otimes w}{|w|^2} \right)$ corresponds to the orthogonal projection on the plane orthogonal to w. As the electron mass is much smaller than the ion mass, we consider only the collisions between ions, whose distribution function is denoted by f^ε and satisfies

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(\overline{v} \cdot \nabla_{\overline{x}} + \omega_c \, {}^\perp \overline{v} \cdot \nabla_{\overline{v}}) f^\varepsilon + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon = Q_{FPL}(f^\varepsilon, f^\varepsilon), \; (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \tag{4.1}$$

where $q > 0$ is the ion charge and m is the ion mass. The limit model comes by averaging the collision kernel Q_{FPL}. The treatment of the Fokker-Planck-Landau kernel is very elaborated. Therefore we content ourselves of formal computations. The main properties of the Fokker-Planck-Landau operator are summarized below

PROPOSITION 4.1. Consider the Fokker-Planck-Landau kernel between ions

$$Q_{FPL}(f, f) = \frac{1}{4} \mathrm{div}_v \int_{\mathbb{R}^3} \sigma_{ii}(|v - v'|)|v - v'|^3 S(v - v') \left(f(v')(\nabla_v f)(v) - f(v)(\nabla_{v'} f)(v') \right) \mathrm{d}v'.$$

Then the mass, momentum and kinetic energy balances hold true

$$\int_{\mathbb{R}^3} m Q_{FPL}(f, f) \,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} m v Q_{FPL}(f, f) \,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} m \frac{|v|^2}{2} Q_{FPL}(f, f) \,\mathrm{d}v = 0.$$

Moreover the entropy production $D := -\int_{\mathbb{R}^3}(1+\ln f)Q_{FPL}(f,f)\,\mathrm{d}v$ is non negative

$$D = \frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sigma(|v-v'|)|v-v'|f(v)f(v')|(v-v')\wedge(\nabla_v \ln f(v) - \nabla_{v'}\ln f(v'))|^2\,\mathrm{d}v'\mathrm{d}v \geq 0.$$

Proof. All statements come easily by integration by parts, observing that $A_{ii}(v,v') + A_{ii}(v',v) = 0$, where $A_{ii}(v,v') = \sigma_{ii}(|v-v'|)|v-v'|^3 S(v-v')(f(v')\nabla_v f(v) - f(v)\nabla_{v'}f(v'))$ and $S(v-v')(v-v') = 0$. □

With the notation $\sigma(|v-v'|) = \frac{1}{4}\sigma_{ii}(|v-v'|)|v-v'|^3$ the collision kernel becomes

$$Q_{FPL}(f,f)(v) = \mathrm{div}_v \int_{\mathbb{R}^3}\sigma(|v-v'|)S(v-v')\left(f(v')(\nabla_v f)(v) - f(v)(\nabla_{v'}f)(v')\right)\,\mathrm{d}v'.$$

4.1. *Preliminary computations.* The Fokker-Planck-Landau operator combines convolution and differential operators in v. Therefore its average can be determined by studying the commutation properties between convolution and derivation with respect to the average. First we apply the commutation formula between divergence and average. Next we are looking for commutation between convolution and average. It is convenient to split Q_{FPL} into its gain and loss parts $Q^{\pm}_{FPL}$. We introduce the following notations, for any function g and vectors w_1, w_2

$$\langle g\rangle_{\sigma S} := \left\langle \int_{\mathbb{R}^3} g(x,v')\sigma(|v-v'|)S(v-v')\,\mathrm{d}v'\right\rangle$$

$$\langle g, w_1\rangle_{\sigma S} := \left\langle \int_{\mathbb{R}^3} g(x,v')\sigma(|v-v'|)S(v-v')w_1\,\mathrm{d}v'\right\rangle$$

$$\langle g, w_1, w_2\rangle_{\sigma S} := \left\langle \int_{\mathbb{R}^3} g(x,v')\sigma(|v-v'|)S(v-v') : w_1\otimes w_2\,\mathrm{d}v'\right\rangle.$$

Let us establish some useful formulae based on [5] Proposition 4.2. We consider functions C which are left invariant by any rotation around $e_3 = (0,0,1)$. Therefore we assume that for any orthogonal matrix $\mathcal{O} \in \mathcal{M}_3(\mathbb{R})$ such that $\mathcal{O}e_3 = e_3$ we have

$$C({}^t\mathcal{O}v, {}^t\mathcal{O}v') = C(v,v'),\quad v, v' \in \mathbb{R}^3. \tag{4.2}$$

These functions are precisely those depending only on $|\overline{v}|, v_3, |\overline{v'}|, v'_3$ and the angle between $\overline{v}$ and $\overline{v'}$

$$C(v,v') = \tilde{C}(|\overline{v}|, v_3, |\overline{v'}|, v'_3, \varphi),\quad \varphi = \arg\overline{v'} - \arg\overline{v}.$$

PROPOSITION 4.2. (4.2, [5]) Assume that the function $C(v,v')$ satisfies (4.2) and belongs to the space $L^2(\mathrm{d}v\mathrm{d}v')$. Then for any function $f \in \ker \mathcal{T}$ we have

$$\left\langle \int_{\mathbb{R}^3} C(v,v') f(x,v')\,\mathrm{d}v' \right\rangle (x,v) = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) f(\overline{x'}, x_3, v')\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2' \tag{4.3}$$

where $z = \omega_c \overline{x} + {}^\perp\overline{v} - (\omega_c \overline{x'} + {}^\perp\overline{v'})$

$$\mathcal{C}(r, v_3, r', v_3', z) = \frac{\tilde{C}(r, v_3, r', v_3', \varphi) + \tilde{C}(r, v_3, r', v_3', -\varphi)}{2}\,\chi(r, r', z)$$

$$\chi(r, r', z) = \frac{\mathbf{1}_{\{|r-r'|<|z|<r+r'\}}}{\pi^2\sqrt{|z|^2 - (r-r')^2}\sqrt{(r+r')^2 - |z|^2}}$$

and for any $|z| \in (|r-r'|, r+r')$, $\varphi \in (0,\pi)$ is the unique angle such that

$$|z|^2 = r^2 + (r')^2 - 2rr'\cos\varphi.$$

For any orthogonal matrix $\mathcal{O} \in \mathcal{M}_3(\mathbb{R})$ we consider the application $(v,v') \to S({}^t\mathcal{O}v - {}^t\mathcal{O}v')$. It is easily seen that

$$S({}^t\mathcal{O}v - {}^t\mathcal{O}v') = {}^t\mathcal{O}S(v-v')\mathcal{O}.$$

Notice also that for any orthogonal matrix $\mathcal{O} \in \mathcal{M}_3(\mathbb{R})$ such that $\mathcal{O}e_3 = e_3$ we have $(\overline{{}^t\mathcal{O}v}, 0) = {}^t\mathcal{O}(\overline{v}, 0)$ and $({}^\perp\overline{{}^t\mathcal{O}v}, 0) = {}^t\mathcal{O}({}^\perp\overline{v}, 0)$ for any $v \in \mathbb{R}^3$.

LEMMA 4.1. The following applications are left invariant by any rotation around e_3, that is they satisfy (4.2)

$$S(v-v') : (\overline{v},0)\otimes(\overline{v},0),\;\; S(v-v') : (\overline{v},0)\otimes({}^\perp\overline{v},0),\;\; S(v-v') : ({}^\perp\overline{v},0)\otimes({}^\perp\overline{v},0)$$

$$S : (\overline{v'},0)\otimes(\overline{v},0),\; S : (\overline{v'},0)\otimes({}^\perp\overline{v},0),\; S : ({}^\perp\overline{v'},0)\otimes(\overline{v},0),\; S : ({}^\perp\overline{v'},0)\otimes({}^\perp\overline{v},0).$$

Proof. For any rotation $\mathcal{O}$ around e_3 we have

$$\begin{aligned} S({}^t\mathcal{O}v - {}^t\mathcal{O}v') : (\overline{{}^t\mathcal{O}v},0)\otimes(\overline{{}^t\mathcal{O}v},0) &= {}^t\mathcal{O}S(v-v')\mathcal{O} : {}^t\mathcal{O}(\overline{v},0)\otimes{}^t\mathcal{O}(\overline{v},0) \\ &= S(v-v') : (\overline{v},0)\otimes(\overline{v},0). \end{aligned}$$

The other invariances follow similarly. □

Therefore the formula in Proposition 4.2 applies to all previous functions and we obtain

PROPOSITION 4.3. For any $z \in \mathbb{R}^2$ such that $|r-r'| < |z| < r+r'$ we denote by $\varphi \in (0, 2\pi)$ the angle satisfying $r^2 + (r')^2 - 2rr' \cos\varphi = |z|^2$. Then for any function $f \in \ker \mathcal{T}$ we have, with the notation $z = (\omega_c \overline{x} + {}^\perp\overline{v}) - (\omega_c \overline{x'} + {}^\perp\overline{v'})$

(1)

$$\langle f, (\overline{v}, 0), (\overline{v}, 0)\rangle_{\sigma S} = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(\overline{x'}, x_3, v')\chi(|\overline{v}|, |\overline{v'}|, z) \times \left\{ r^2 - \frac{r^2(r - r'\cos\varphi)^2}{|z|^2 + (v_3 - v_3')^2} \right\} \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

(2)

$$\left\langle f, (\overline{v}, 0), ({}^\perp\overline{v}, 0)\right\rangle_{\sigma S} = \left\langle f, ({}^\perp\overline{v}, 0), (\overline{v}, 0)\right\rangle_{\sigma S} = 0$$

(3)

$$\left\langle f, ({}^\perp\overline{v}, 0), ({}^\perp\overline{v}, 0)\right\rangle_{\sigma S} = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(\overline{x'}, x_3, v')\chi(|\overline{v}|, |\overline{v'}|, z) \times \left\{ r^2 - \frac{r^2(r')^2\sin^2\varphi}{|z|^2 + (v_3 - v_3')^2} \right\} \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

(4)

$$\left\langle f, (\overline{v'}, 0), (\overline{v}, 0)\right\rangle_{\sigma S} = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(\overline{x'}, x_3, v')\chi(|\overline{v}|, |\overline{v'}|, z) \times \left\{ rr'\cos\varphi - \frac{rr'(r\cos\varphi - r')(r - r'\cos\varphi)}{|z|^2 + (v_3 - v_3')^2} \right\} \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

(5)

$$\left\langle f, (\overline{v'}, 0), ({}^\perp\overline{v}, 0)\right\rangle_{\sigma S} = \left\langle f, ({}^\perp\overline{v'}, 0), (\overline{v}, 0)\right\rangle_{\sigma S} = 0$$

(6)

$$\left\langle f, ({}^\perp\overline{v'}, 0), ({}^\perp\overline{v}, 0)\right\rangle_{\sigma S} = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(\overline{x'}, x_3, v')\chi(|\overline{v}|, |\overline{v'}|, z) \times \left\{ rr'\cos\varphi - \frac{r^2(r')^2\sin^2\varphi}{|z|^2 + (v_3 - v_3')^2} \right\} \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.$$

Proof. We need to compute the functions $\tilde{C}, \mathcal{C}$ defined in Proposition 4.2. In each case we have

1.

$$C(v,v') = \sigma(|v-v'|)S(v-v') : (\overline{v},0)\otimes(\overline{v},0) = \sigma(|v-v'|)\left\{|\overline{v}|^2 - \frac{[(\overline{v}-\overline{v'})\cdot\overline{v}]^2}{|v-v'|^2}\right\}$$

$$\tilde{C}(r,v_3,r',v_3',\varphi) = \sigma\left(\sqrt{r^2+(r')^2-2rr'\cos\varphi+(v_3-v_3')^2}\ \right)\left\{r^2 - \frac{(r^2-rr'\cos\varphi)^2}{|z|^2+(v_3-v_3')^2}\right\}$$

$$\mathcal{C}(r,v_3,r',v_3',z) = \chi(r,r',z)\sigma\left(\sqrt{|z|^2+(v_3-v_3')^2}\ \right)\left\{r^2 - \frac{r^2(r-r'\cos\varphi)^2}{|z|^2+(v_3-v_3')^2}\right\}.$$

2.

$$C(v,v') = \sigma(|v-v'|)S(v-v') : (\overline{v},0)\otimes({}^\perp\overline{v},0)$$

$$= -\sigma(|v-v'|)\frac{[(\overline{v}-\overline{v'})\cdot\overline{v}]\ [(\overline{v}-\overline{v'})\cdot{}^\perp\overline{v}]}{|v-v'|^2}$$

$$\tilde{C}(r,v_3,r',v_3',\varphi) = -\sigma\left(\sqrt{|z|^2+(v_3-v_3')^2}\ \right)\frac{(r^2-rr'\cos\varphi)rr'\sin\varphi}{|z|^2+(v_3-v_3')^2}$$

$$\mathcal{C} = 0.$$

3.

$$C(v,v') = \sigma(|v-v'|)S(v-v') : ({}^\perp\overline{v},0)\otimes({}^\perp\overline{v},0) = \sigma(|v-v'|)\left\{|\overline{v}|^2 - \frac{(\overline{v'}\cdot{}^\perp\overline{v})^2}{|v-v'|^2}\right\}$$

$$\tilde{C}(r,v_3,r',v_3',\varphi) = \sigma\left(\sqrt{|z|^2+(v_3-v_3')^2}\ \right)\left\{r^2 - \frac{r^2(r')^2\sin^2\varphi}{|z|^2+(v_3-v_3')^2}\right\}$$

$$\mathcal{C}(r,v_3,r',v_3',z) = \chi(r,r',z)\sigma\left(\sqrt{|z|^2+(v_3-v_3')^2}\ \right)\left\{r^2 - \frac{r^2(r')^2\sin^2\varphi}{|z|^2+(v_3-v_3')^2}\right\}.$$

4.

$$C(v,v') = \sigma(|v-v'|)S(v-v') : (\overline{v'},0)\otimes(\overline{v},0)$$

$$= \sigma(|v-v'|)\left\{\overline{v'}\cdot\overline{v} - \frac{[(\overline{v}-\overline{v'})\cdot\overline{v'}]\ [(\overline{v}-\overline{v'})\cdot\overline{v}]}{|v-v'|^2}\right\}$$

$$\tilde{C}(r,v_3,r',v_3',\varphi) = \sigma\left\{rr'\cos\varphi - \frac{(rr'\cos\varphi-(r')^2)(r^2-rr'\cos\varphi)}{|z|^2+(v_3-v_3')^2}\right\}$$

$$\mathcal{C}(r,v_3,r',v_3',z) = \chi(r,r',z)\sigma\left\{rr'\cos\varphi - \frac{rr'(r\cos\varphi-r')(r-r'\cos\varphi)}{|z|^2+(v_3-v_3')^2}\right\}.$$

5.

$$C(v,v') = \sigma(|v-v'|)S(v-v') : (\overline{v'},0)\otimes({}^\perp\overline{v},0)$$

$$= \sigma(|v-v'|)\left\{\overline{v'}\cdot{}^\perp\overline{v} - \frac{[(\overline{v}-\overline{v'})\cdot\overline{v'}]\ [(\overline{v}-\overline{v'})\cdot{}^\perp\overline{v}]}{|v-v'|^2}\right\}$$

$$\tilde{C}(r, v_3, r', v_3', \varphi) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) \left\{ -rr' \sin\varphi - \frac{(rr'\cos\varphi - (r')^2) rr' \sin\varphi}{|z|^2 + (v_3 - v_3')^2} \right\}$$

$$\mathcal{C}(r, v_3, r', v_3', z) = 0.$$

6.

$$C(v, v') = \sigma(|v - v'|) S(v - v') : ({}^\perp\overline{v'}, 0) \otimes ({}^\perp\overline{v}, 0)$$

$$= \sigma(|v - v'|) \left\{ \overline{v} \cdot \overline{v'} - \frac{[(\overline{v} - \overline{v'}) \cdot {}^\perp\overline{v'}]\,[(\overline{v} - \overline{v'}) \cdot {}^\perp\overline{v}]}{|v - v'|^2} \right\}$$

$$\tilde{C}(r, v_3, r', v_3', \varphi) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) \left\{ rr' \cos\varphi - \frac{r^2 (r')^2 \sin^2\varphi}{|z|^2 + (v_3 - v_3')^2} \right\}$$

$$\mathcal{C}(r, v_3, r', v_3', z) = \chi(r, r', z) \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) \left\{ rr' \cos\varphi - \frac{r^2 (r')^2 \sin^2\varphi}{|z|^2 + (v_3 - v_3')^2} \right\}.$$

□

We also need to compute the averages

$$\langle f, (\overline{v}, 0) \rangle_{\sigma S}, \quad \langle f, ({}^\perp\overline{v}, 0) \rangle_{\sigma S}, \quad \langle f, (\overline{v'}, 0) \rangle_{\sigma S}, \quad \langle f, ({}^\perp\overline{v'}, 0) \rangle_{\sigma S}. \tag{4.4}$$

Notice that the functions $\sigma S(v - v')(\overline{v}, 0), \sigma S(v - v')({}^\perp\overline{v}, 0), \sigma S(v - v')(\overline{v'}, 0), \sigma S(v - v')({}^\perp\overline{v'}, 0)$ writes

$$D(v, v') = (\tilde{D}\,\overline{v} + \tilde{D}'\,\overline{v'}, \tilde{D}_3\, v_3 + \tilde{D}_3'\, v_3')$$

for some scalar functions $\tilde{D}, \tilde{D}', \tilde{D}_3, \tilde{D}_3'$ depending on $|\overline{v}|, v_3, |\overline{v'}|, v_3'$ and φ, the angle between $(\overline{v}, 0), (\overline{v'}, 0)$. Performing the same steps as in the proof of Proposition 4.2 we obtain (see Appendix A for proof details)

PROPOSITION 4.4. Consider $\tilde{D} = \tilde{D}(|\overline{v}|, v_3, |\overline{v'}|, v_3', \varphi), \tilde{D}' = \tilde{D}'(|\overline{v}|, v_3, |\overline{v'}|, v_3', \varphi)$ two functions and

$$D(v, v') = \tilde{D}\,\overline{v} + \tilde{D}'\,\overline{v'}, \;\; D_3(v, v') = \tilde{D}\, v_3 + \tilde{D}'\, v_3'.$$

Then for any $f \in \ker \mathcal{T}$ we have

(1)

$$\left\langle \int_{\mathbb{R}^3} D(v, v') f(x, v')\, \mathrm{d}v' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{D}(|\overline{v}|, v_3, |\overline{v'}|, v_3', (\omega_c \overline{x} + {}^\perp\overline{v}) - (\omega_c \overline{x'} + {}^\perp\overline{v'})) \times f(\overline{x'}, x_3, v')\, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

where

$$\mathcal{D}(r, v_3, r', v_3', z) = \frac{\begin{pmatrix} z_2 & z_1 \\ -z_1 & z_2 \end{pmatrix}}{2|z|}[\tilde{D}(r, v_3, r', v_3', \varphi) re^{-i\psi} + \tilde{D}(r, v_3, r', v_3', -\varphi) re^{i\psi}$$
$$+ \tilde{D}'(r, v_3, r', v_3', \varphi) r' e^{i(\varphi-\psi)} + \tilde{D}'(r, v_3, r', v_3', -\varphi) r' e^{-i(\varphi-\psi)}]$$
$$\times \chi(r, r', z).$$

(2)

$$\left\langle \int_{\mathbb{R}^3} D_3(v, v') f(x, v') \, \mathrm{d}v' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{D}_3(|\overline{v}|, v_3, |\overline{v'}|, v_3', (\omega_c \overline{x} + {}^\perp \overline{v}) - (\omega_c \overline{x'} + {}^\perp \overline{v'}))$$
$$\times f(\overline{x'}, x_3, v') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

where

$$\mathcal{D}_3(r, v_3, r', v_3', z) = \frac{1}{2}[(\tilde{D}(r, v_3, r', v_3', \varphi) + \tilde{D}(r, v_3, r', v_3', -\varphi)) v_3$$
$$+ (\tilde{D}'(r, v_3, r', v_3', \varphi) + \tilde{D}'(r, v_3, r', v_3', -\varphi)) v_3'] \chi(r, r', z).$$

The angles $\varphi, \psi \in (0, \pi)$ are such that $|z|^2 = r^2 + (r')^2 - 2rr' \cos\varphi$, $(r')^2 = r^2 + |z|^2 + 2r|z|\cos\psi$.

It is worth analyzing the case of even/odd coefficients $\tilde{D}, \tilde{D}'$.

PROPOSITION 4.5. With the same notations as in Proposition 4.4 assume that the functions $\tilde{D}, \tilde{D}'$ are even with respect to φ. Then we have

(1)

$$\mathcal{D}(r, v_3, r', v_3', z) = [(\tilde{D}(\varphi) + \tilde{D}'(\varphi)) \, r \cos\psi + |z| \tilde{D}'(\varphi)] \chi(r, r', z) \frac{{}^\perp z}{|z|}$$

(2)

$$\mathcal{D}_3(r, v_3, r', v_3', z) = [(\tilde{D}(\varphi) + \tilde{D}'(\varphi)) \, v_3 + \tilde{D}'(\varphi)(v_3' - v_3)] \chi(r, r', z).$$

Proof. 1. Clearly we have

$$\frac{1}{2}[\tilde{D}(r, v_3, r', v_3', \varphi) \, re^{-i\psi} + \tilde{D}(r, v_3, r', v_3', -\varphi) \, re^{i\psi}] = \tilde{D}(r, v_3, r', v_3', \varphi) \, r(\cos\psi, 0)$$

and

$$\frac{1}{2}[\tilde{D}'(r,v_3,r',v_3',\varphi)\; r'e^{i(\varphi-\psi)} + \tilde{D}'(r,v_3,r',v_3',-\varphi)\; r'e^{-i(\varphi-\psi)}] = \tilde{D}'\; r'(\cos(\psi-\varphi),0).$$

Consider now the triangle of vertices $O=(0,0), A=(r,0), A'=r'e^{i\varphi}$ in $\mathbb{R}^2$. The definitions for φ,ψ assure that $|z|=|AA'|$ and that ψ is the supplement of the angle opposite to OA'. Applying the cosine theorem with respect to the angle opposite to OA one gets

$$r^2 = (r')^2 + |z|^2 - 2r'|z|\cos(\psi-\varphi). \tag{4.5}$$

Combining with the definition of ψ yields

$$0 = 2|z|^2 - 2r'|z|\cos(\psi-\varphi) + 2r|z|\cos\psi$$

implying

$$r\cos\psi - r'\cos(\psi-\varphi) + |z| = 0. \tag{4.6}$$

Finally one gets

$$\begin{aligned}\mathcal{D}(r,v_3,r',v_3',z) &= [\tilde{D}(\varphi)\; r\cos\psi + \tilde{D}'(\varphi)\; r'\cos(\psi-\varphi)]\chi(r,r',z)\frac{{}^\perp z}{|z|}\\ &= [(\tilde{D}(\varphi)+\tilde{D}'(\varphi))\; r\cos\psi + \tilde{D}'(\varphi)(r'\cos(\psi-\varphi) - r\cos\psi)]\chi(r,r',z)\frac{{}^\perp z}{|z|}\\ &= [(\tilde{D}(\varphi)+\tilde{D}'(\varphi))\; r\cos\psi + |z|\tilde{D}'(\varphi)]\chi(r,r',z)\frac{{}^\perp z}{|z|}.\end{aligned}$$

2. It follows immediately observing that

$$\begin{aligned}\mathcal{D}_3(r,v_3,r',v_3',z) &= (\tilde{D}(\varphi)\; v_3 + \tilde{D}'(\varphi)\; v_3')\chi(r,r',z)\\ &= [(\tilde{D}(\varphi)+\tilde{D}'(\varphi))\; v_3 + \tilde{D}'(\varphi)(v_3'-v_3)]\chi(r,r',z).\end{aligned}$$

□

PROPOSITION 4.6. With the same notations as in Proposition 4.4 assume that the functions $\tilde{D},\tilde{D}'$ are odd with respect to φ. Then we have

(1)

$$\mathcal{D}(r,v_3,r',v_3',z) = -[\tilde{D}(\varphi)+\tilde{D}'(\varphi)]\; r\sin\psi\; \chi(r,r',z)\; \frac{z}{|z|}$$

(2)

$$\mathcal{D}_3(r, v_3, r', v_3', z) = 0.$$

Proof. 1. We have

$$\frac{1}{2}[\tilde{D}(r, v_3, r', v_3', \varphi)\ re^{-i\psi} + \tilde{D}(r, v_3, r', v_3', -\varphi)\ re^{i\psi}] = -(0, \tilde{D}(r, v_3, r', v_3', \varphi)\ r \sin\psi)$$

and

$$\frac{1}{2}[\tilde{D}'(r, v_3, r', v_3', \varphi)\ r'e^{i(\varphi-\psi)} + \tilde{D}'(r, v_3, r', v_3', -\varphi)\ r'e^{-i(\varphi-\psi)}] = -(0, \tilde{D}'\ r' \sin(\psi - \varphi)).$$

The sine theorem applied in the triangle of vertices $O = (0,0), A = (r,0), A' = r'e^{i\varphi}$ implies

$$r \sin\psi = r' \sin(\psi - \varphi).$$

We deduce that

$$\begin{aligned}\mathcal{D}(r, v_3, r', v_3', z) &= -[\tilde{D}(\varphi)\ r \sin\psi + \tilde{D}'(\varphi)\ r' \sin(\psi - \varphi)]\ \chi(r, r', z) \frac{z}{|z|} \\ &= -[\tilde{D}(\varphi) + \tilde{D}'(\varphi)]\ r \sin\psi\ \chi(r, r', z)\ \frac{z}{|z|}.\end{aligned}$$

2. Clearly we have $\tilde{D}(\varphi) + \tilde{D}(-\varphi) = \tilde{D}'(\varphi) + \tilde{D}'(-\varphi) = 0$ and therefore $\mathcal{D}_3 = 0$. □

The averages in (4.4) come immediately appealing to Propositions 4.5, 4.6.

COROLLARY 4.1. With the notations in Proposition 4.4 we have for any function $f \in \ker \mathcal{T}$

(1)

$$\langle f, (\overline{v}, 0)\rangle_{\sigma S} = -\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi \frac{r^2 - rr' \cos\varphi}{|z|^2 + (v_3 - v_3')^2} \left(\frac{(v_3 - v_3')^2}{|z|^2}\ {}^\perp z, v_3 - v_3' \right)\ \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

(2)

$$\left\langle f, (\overline{v'}, 0)\right\rangle_{\sigma S} = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi \frac{(r')^2 - rr' \cos\varphi}{|z|^2 + (v_3 - v_3')^2} \left(\frac{(v_3 - v_3')^2}{|z|^2}\ {}^\perp z, v_3 - v_3' \right)\ \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

(3)

$$\left\langle f, ({}^\perp\overline{v}, 0)\right\rangle_{\sigma S} = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi \frac{r^2 - rr' \cos\varphi}{|z|^2}\ (z, 0)\ \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

(4)

$$\left\langle f, ({}^\perp\overline{v'}, 0)\right\rangle_{\sigma S} = -\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi \frac{(r')^2 - rr'\cos\varphi}{|z|^2}\,(z,0)\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.$$

Proof. 1. We consider the function $D(v,v') = \sigma(|v-v'|)S(v-v')(\overline{v},0) = (\tilde{D}\ \overline{v} + \tilde{D}'\ \overline{v'}, \tilde{D}'(v_3'-v_3))$ where

$$\tilde{D}(r,v_3,r',v_3',\varphi) = \sigma\left(1 - \frac{r^2 - rr'\cos\varphi}{r^2 + (r')^2 - 2rr'\cos\varphi + (v_3 - v_3')^2}\right)$$

$$\tilde{D}'(r,v_3,r',v_3',\varphi) = \sigma\frac{r^2 - rr'\cos\varphi}{r^2 + (r')^2 - 2rr'\cos\varphi + (v_3 - v_3')^2}.$$

Thanks to Proposition 4.5 and the identity $r\cos\psi = -\frac{r^2 - rr'\cos\varphi}{|z|}$ we obtain

$$\begin{aligned}\left\langle \int_{\mathbb{R}^3} (\tilde{D}\ \overline{v} + \tilde{D}'\ \overline{v'}) f(x,v')\ \mathrm{d}v'\right\rangle &= \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f(x_1',x_2',x_3,v')\chi \\ &\quad\times\left(r\cos\psi + |z|\frac{r^2 - rr'\cos\varphi}{|z|^2 + (v_3 - v_3')^2}\right)\frac{{}^\perp z}{|z|}\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2' \\ &= -\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f(x_1',x_2',x_3,v')\chi \\ &\quad\times \frac{r^2 - rr'\cos\varphi}{|z|^2 + (v_3 - v_3')^2}\,\frac{(v_3 - v_3')^2}{|z|}\,\frac{{}^\perp z}{|z|}\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\end{aligned}$$

and also

$$\begin{aligned}\left\langle \int_{\mathbb{R}^3} \tilde{D}'(v_3' - v_3) f(x,v')\ \mathrm{d}v'\right\rangle &= \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f(x_1',x_2',x_3,v')\chi \\ &\quad\times \frac{r^2 - rr'\cos\varphi}{|z|^2 + (v_3 - v_3')^2}(v_3' - v_3)\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\end{aligned}$$

which justifies the first statement.

2. We take $D(v,v') = \sigma(|v-v'|)S(v-v')(\overline{v'},0) = (\tilde{D}\ \overline{v} + \tilde{D}'\ \overline{v'}, \tilde{D}(v_3 - v_3'))$ where

$$\tilde{D}(r,v_3,r',v_3',\varphi) = \sigma\frac{(r')^2 - rr'\cos\varphi}{r^2 + (r')^2 - 2rr'\cos\varphi + (v_3 - v_3')^2}$$

$$\tilde{D}'(r,v_3,r',v_3',\varphi) = \sigma\left(1 - \frac{(r')^2 - rr'\cos\varphi}{r^2 + (r')^2 - 2rr'\cos\varphi + (v_3 - v_3')^2}\right).$$

Notice that by (4.5), (4.6) we have

$$r\cos\psi + |z| = r'\cos(\psi - \varphi) = \frac{(r')^2 + |z|^2 - r^2}{2|z|} = \frac{(r')^2 - rr'\cos\varphi}{|z|}$$

and in this case we obtain

$$\left\langle \int_{\mathbb{R}^3} (\tilde{D}\ \overline{v} + \tilde{D}'\ \overline{v'}) f(x,v')\ \mathrm{d}v' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f(x_1', x_2', x_3, v')\chi$$
$$\times \left[r\cos\psi + |z| \left(1 - \frac{(r')^2 - rr'\cos\varphi}{|z|^2 + (v_3 - v_3')^2} \right) \right] \frac{{}^\perp z}{|z|}\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$
$$= \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f(x_1', x_2', x_3, v')\chi$$
$$\times \frac{(r')^2 - rr'\cos\varphi}{|z|^2 + (v_3 - v_3')^2} \frac{(v_3 - v_3')^2}{|z|} \frac{{}^\perp z}{|z|}\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

and

$$\left\langle \int_{\mathbb{R}^3} \tilde{D}(v_3 - v_3') f(x,v')\ \mathrm{d}v' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f(x_1', x_2', x_3, v')\chi$$
$$\times \frac{(r')^2 - rr'\cos\varphi}{|z|^2 + (v_3 - v_3')^2} (v_3 - v_3')\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

justifying the second statement.

3. We take $D(v,v') = -\sigma(|v-v'|)\frac{(v-v')\otimes(v-v')}{|v-v'|^2}({}^\perp\overline{v}, 0) = (\tilde{D}\ \overline{v} + \tilde{D}'\ \overline{v'}, \tilde{D}v_3 + \tilde{D}'v_3')$ where

$$\tilde{D}(r, v_3, r', v_3', \varphi) = -\sigma \frac{rr'\sin\varphi}{r^2 + (r')^2 - 2rr'\cos\varphi + (v_3 - v_3')^2} = -\tilde{D}'.$$

By Proposition 4.6 we deduce that $\left\langle \int_{\mathbb{R}^3} D(v,v') f(x,v')\ \mathrm{d}v' \right\rangle = 0$. Therefore

$$\left\langle f, ({}^\perp\overline{v}, 0) \right\rangle_{\sigma S} = \left\langle \int_{\mathbb{R}^3} \sigma(|v-v'|)({}^\perp\overline{v}, 0) f(x,v')\ \mathrm{d}v' \right\rangle.$$

Applying now Proposition 4.5 with $\tilde{D} = \sigma, \tilde{D}' = 0$ we obtain

$$\left\langle \int_{\mathbb{R}^3} \sigma(|v-v'|) f(x,v')\overline{v}\ \mathrm{d}v' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi\ r\cos\psi \frac{{}^\perp z}{|z|}\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

and finally

$$\left\langle f, ({}^\perp\overline{v}, 0) \right\rangle_{\sigma S} = -\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi\ r\cos\psi \frac{(z,0)}{|z|}\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$
$$= \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi \frac{r^2 - rr'\cos\varphi}{|z|^2} (z,0)\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.$$

4. As before, by Proposition 4.6 we have

$$\left\langle \int_{\mathbb{R}^3} \sigma(|v-v'|) f(x,v') \frac{(v-v')\otimes(v-v')}{|v-v'|^2} ({}^\perp\overline{v'}, 0)\ \mathrm{d}v' \right\rangle = 0$$

and by Proposition 4.5 we obtain

$$\left\langle \int_{\mathbb{R}^3} \sigma(|v-v'|) f(x,v') \overline{v'} \,\mathrm{d}v' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi \; (r\cos\psi + |z|) \frac{{}^\perp z}{|z|} \,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

$$= \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi \; \frac{(r')^2 - rr'\cos\varphi}{|z|^2} \;{}^\perp z \,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.$$

At the end one gets

$$\left\langle f, ({}^\perp\overline{v'}, 0)\right\rangle_{\sigma S} = -\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma f \chi \frac{(r')^2 - rr'\cos\varphi}{|z|^2} \,(z,0)\; \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.$$

□

The last average we will need is $\langle f\rangle_{\sigma S} = \left\langle \int_{\mathbb{R}^3} f(x,v')\sigma(|v-v'|)S(v-v')\,\mathrm{d}v'\right\rangle$. By similar computations as those in the proofs of Propositions 4.2, 4.4 we obtain (see Appendix A for details)

Proposition 4.7. For any function $f \in \ker \mathcal{T}$ we have

$$\langle f\rangle_{\sigma S} = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma\left(\sqrt{|z|^2 + (v_3 - v_3')^2}\,\right) f(\overline{x'}, x_3, v') \chi(|\overline{v}|, |\overline{v'}|, z) S\left(\,({}^\perp z, v_3' - v_3)\,\right) \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.$$

4.2. *The averaged Fokker-Planck-Landau operator.* We are ready to determine the average of the Fokker-Planck-Landau kernel. For the sake of presentation we treat separately the gain and loss parts. Recall that the Fokker-Planck-Landau gain part appears as a velocity diffusion, where the diffusion matrix is a convolution in velocity

$$Q^+_{FPL}(f,f) = \mathrm{div}_v \left\{ \int_{\mathbb{R}^3} \sigma(|v-v'|) S(v-v') f(v')\,\mathrm{d}v' \;\nabla_v f(v) \right\}.$$

The averaged Fokker-Planck-Landau kernel will keep the same structure, nevertheless diffusion and convolution have to be considered both in velocity and space perpendicular directions. The proof is postponed to Appendix B.

Proposition 4.8. For any function $f = f(x,v)$ satisfying the constraint $\mathcal{T} f = 0$ we have

$$\left\langle Q^+_{FPL}(f,f)\right\rangle = \mathrm{div}_{\omega_c x, v} \{ \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma\left(\sqrt{|z|^2 + (v_3 - v_3')^2}\,\right) f(x_1', x_2', x_3, v')\; \chi(|\overline{v}|, |\overline{v'}|, z)$$
$$\times A^+ \nabla_{\omega_c x, v} f(x,v)\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2' \} \tag{4.7}$$

with $\mathrm{div}_{\omega_c x} = \frac{1}{\omega_c}\mathrm{div}_x$, $\nabla_{\omega_c x} = \frac{1}{\omega_c}\nabla_x$

$$A^+(r, v_3, r', v_3', z) = \frac{(r')^2 \sin^2\varphi\,(v_3 - v_3')^2}{|z|^2[|z|^2 + (v_3 - v_3')^2]}\left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{(\,^\perp\overline{v},0)}{|\overline{v}|}\right)^{\otimes 2}$$
$$+ \left[\frac{r - r'\cos\varphi}{|z|}\left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{(\,^\perp\overline{v},0)}{|\overline{v}|}\right) + \left(\frac{(^\perp z,0)}{|z|}, 0\right)\right]^{\otimes 2} + \frac{(r')^2\sin^2\varphi}{|z|^2}\left(\frac{(\,^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right)^{\otimes 2}$$
$$+ \left[\frac{(r'\cos\varphi - r)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\left(\frac{(\,^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right) + \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right]^{\otimes 2}$$

where $z = (\omega_c \overline{x} + \,^\perp\overline{v}) - (\omega_c\overline{x'} + \,^\perp\overline{v'})$ and for any $r, r' \in \mathbb{R}_+, z \in \mathbb{R}^2$ such that $|r - r'| < |z| < r + r'$, the angle $\varphi \in (0, \pi)$ is given by $|z|^2 = r^2 + (r')^2 - 2rr'\cos\varphi$.

Remark 4.2. Clearly A^+ is symmetric and positive. Notice also that the vectors $(e_3, 0)$ and $(\,(z,0), (-^\perp z, v_3 - v_3')\,)$ are orthogonal on

$$\left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{(\,^\perp\overline{v},0)}{|\overline{v}|}\right),\ \left(\frac{(\,^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right),\ \left(\frac{(^\perp z,0)}{|z|}, 0\right),\ \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}$$

saying that

$$A^+(e_3, 0) = A^+(\,(z,0), (-^\perp z, v_3 - v_3')\,) = 0.$$

Actually we have for any $z \neq 0$

$$\ker A^+(r, v_3, r', v_3', z) = \mathrm{span}\{(e_3, 0),\ \ (\,(z,0), (-^\perp z, v_3 - v_3')\,)\}.$$

A similar result can be carried out for the loss part Q^-_{FPL} (see Appendix B for the proof).

Proposition 4.9. For any function $f = f(x, v)$ satisfying the constraint $\mathcal{T}f = 0$ we have

$$\langle Q^-_{FPL}(f,f)\rangle = \mathrm{div}_{\omega_c x, v}\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(x,v)\,\chi(|\overline{v}|, |\overline{v'}|, z)$$
$$\times A^-\nabla_{\omega_c x', v'} f(x_1', x_2', x_3, v')\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\} \tag{4.8}$$

with

$$
\begin{aligned}
A^-(r, v_3, r', v_3', z) &= \frac{rr' \sin^2\varphi\,(v_3 - v_3')^2}{|z|^2[|z|^2 + (v_3 - v_3')^2]} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}\right) \otimes \left(\frac{(\overline{v'}, 0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'}, 0)}{|\overline{v'}|}\right) \\
&+ \left[\frac{r - r'\cos\varphi}{|z|}\left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}\right) + \left(\frac{({}^\perp z, 0)}{|z|}, 0\right)\right] \\
&\otimes \left[\frac{r\cos\varphi - r'}{|z|}\left(\frac{(\overline{v'}, 0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'}, 0)}{|\overline{v'}|}\right) + \left(\frac{({}^\perp z, 0)}{|z|}, 0\right)\right] \\
&+ \frac{rr'\sin^2\varphi}{|z|^2}\left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) \otimes \left(\frac{({}^\perp\overline{v'}, 0)}{|\overline{v'}|}, -\frac{(\overline{v'}, 0)}{|\overline{v'}|}\right) \\
&+ \left[\frac{(r'\cos\varphi - r)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) + \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right] \\
&\otimes \left[\frac{(r' - r\cos\varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\left(\frac{({}^\perp\overline{v'}, 0)}{|\overline{v'}|}, -\frac{(\overline{v'}, 0)}{|\overline{v'}|}\right) + \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right]
\end{aligned}
$$

where $z = (\omega_c \overline{x} + {}^\perp\overline{v}) - (\omega_c \overline{x'} + {}^\perp\overline{v'})$ and for any $r, r' \in \mathbb{R}_+, z \in \mathbb{R}^2$ such that $|r - r'| < |z| < r + r'$, the angle $\varphi \in (0, \pi)$ is given by $|z|^2 = r^2 + (r')^2 - 2rr'\cos\varphi$.

For any $\overline{x} = (x_1, x_2), \overline{x'} = (x_1', x_2') \in \mathbb{R}^2$, $v, v' \in \mathbb{R}^3$ we introduce the fields

$$
\begin{aligned}
\xi^1(\overline{x}, v, \overline{x'}, v') &= \{\sigma\chi\}^{1/2} \frac{r'\sin\varphi\,(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}\right) \\
\xi^2(\overline{x}, v, \overline{x'}, v') &= \{\sigma\chi\}^{1/2} \left[\frac{r - r'\cos\varphi}{|z|}\left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}\right) + \left(\frac{({}^\perp z, 0)}{|z|}, 0\right)\right] \\
\xi^3(\overline{x}, v, \overline{x'}, v') &= \{\sigma\chi\}^{1/2} \frac{r'\sin\varphi}{|z|}\left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) \\
\frac{\xi^4(\overline{x}, v, \overline{x'}, v')}{\{\sigma\chi\}^{1/2}} &= \frac{(r'\cos\varphi - r)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) + \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}
\end{aligned}
$$

where $r = |\overline{v}|, r' = |\overline{v'}|, z = (\omega_c \overline{x} + {}^\perp\overline{v}) - (\omega_c \overline{x'} + {}^\perp\overline{v'}), \sigma = \sigma\sqrt{|z|^2 + (v_3 - v_3')^2}, \chi = \chi(r, r', z)$ and $\varphi \in (0, \pi)$ is given by $|z|^2 = r^2 + (r')^2 - 2rr'\cos\varphi$, $|r - r'| < |z| < r + r'$. Thanks to Propositions 4.8, 4.9 we obtain the representation formula

PROPOSITION 4.10. Consider a function $f = f(x, v)$ satisfying the constraint $\mathcal{T} f = 0$. Then

1. The averaged Fokker-Planck-Landau operator writes

$$
\begin{aligned}
&\omega_c^{-2} \langle Q_{FPL}(f,f)\rangle (x,v) = \\
&\mathrm{div}_{\omega_c x, v} \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sum_{i=1}^{4} f(\overline{x'}, x_3, v')\xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x}, v, \overline{x'}, v')\nabla_{\omega_c x, v} f(x,v)\, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2' \right\} \\
&-\mathrm{div}_{\omega_c x, v} \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sum_{i=1}^{4} f(x,v)\xi^i(\overline{x}, v, \overline{x'}, v') \otimes \varepsilon_i \xi^i(\overline{x'}, v', \overline{x}, v)\nabla_{\omega_c x', v'} f(\overline{x'}, x_3, v')\, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2' \right\}
\end{aligned}
\tag{4.9}
$$

where $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1$.

2. The following properties hold true for any fixed $x_3 \in \mathbb{R}$

$$
\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q_{FPL}(f,f)\rangle (x,v)\, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = 0, \ \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v\, \langle Q_{FPL}(f,f)\rangle (x,v)\, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = 0
$$

$$
\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|v|^2}{2} \langle Q_{FPL}(f,f)\rangle (x,v)\, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = 0, \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f\, \langle Q_{FPL}(f,f)\rangle (x,v)\, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \le 0.
$$

Proof. 1. By Proposition 4.8 we know that

$$
\begin{aligned}
&\omega_c^{-2} \left\langle Q^+_{FPL}(f,f)\right\rangle (x,v) = \\
&\quad \mathrm{div}_{\omega_c x, v} \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sum_{i=1}^{4} f(\overline{x'}, x_3, v')(\xi^i)^{\otimes 2}(\overline{x}, v, \overline{x'}, v')\nabla_{\omega_c x, v} f(x,v)\, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2' \right\}.
\end{aligned}
\tag{4.10}
$$

Observe that we have $\chi(r', r, -z) = \chi(r, r', z)$. Therefore the permutation $(\overline{x}, v) \longleftrightarrow (\overline{x'}, v')$ leads to

$$
\xi^1(\overline{x'}, v', \overline{x}, v) = \{\sigma\chi\}^{1/2} \frac{r\sin\varphi\, (v_3' - v_3)}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{(\overline{v'}, 0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'}, 0)}{|\overline{v'}|} \right)
$$

$$
\xi^2(\overline{x'}, v', \overline{x}, v) = \{\sigma\chi\}^{1/2} \left[\frac{r' - r\cos\varphi}{|z|} \left(\frac{(\overline{v'}, 0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'}, 0)}{|\overline{v'}|} \right) - \left(\frac{({}^\perp z, 0)}{|z|}, 0 \right) \right]
$$

$$
\xi^3(\overline{x'}, v', \overline{x}, v) = \{\sigma\chi\}^{1/2} \frac{r\sin\varphi}{|z|} \left(\frac{({}^\perp\overline{v'}, 0)}{|\overline{v'}|}, -\frac{(\overline{v'}, 0)}{|\overline{v'}|} \right)
$$

$$
\frac{\xi^4(\overline{x'}, v', \overline{x}, v)}{\{\sigma\chi\}^{1/2}} = \frac{(r\cos\varphi - r')(v_3' - v_3)}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{({}^\perp\overline{v'}, 0)}{|\overline{v'}|}, -\frac{(\overline{v'}, 0)}{|\overline{v'}|} \right) + \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3 \right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}
$$

where $z = (\omega_c \overline{x} + {}^\perp \overline{v}) - (\omega_c \overline{x'} + {}^\perp \overline{v'})$. By Proposition 4.9 one gets

$$\omega_c^{-2} \left\langle Q^-_{FPL}(f,f)\right\rangle (x,v) = \tag{4.11}$$

$$\operatorname{div}_{\omega_c x, v} \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sum_{i=1}^{4} f(x,v)\xi^i(\overline{x}, v, \overline{x'}, v') \otimes \varepsilon_i \xi^i(\overline{x'}, v', \overline{x}, v) \nabla_{\omega_c x', v'} f(\overline{x'}, x_3, v') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \right\}$$

and the first statement follows combining (4.10) and (4.11).

2. The mass, third momentum component and kinetic energy balances for the averaged Fokker-Planck-Landau operator come by the corresponding properties of the Fokker-Planck-Landau kernel. Indeed, since $1, v_3, \frac{|v|^2}{2}$ belong to $\ker \mathcal{T}$, we can write for any $x_3 \in \mathbb{R}$, thanks to Remark 2.1

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \left\{1, v_3, \frac{|v|^2}{2}\right\} \langle Q_{FPL}(f,f)\rangle \, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \left\{1, v_3, \frac{|v|^2}{2}\right\} Q_{FPL}(f,f) \, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2$$
$$= (0,0,0).$$

Similarly since $\ln f \in \ker \mathcal{T}$ one gets

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f \, \langle Q_{FPL}(f,f)\rangle \, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f \; Q_{FPL}(f,f) \, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \tag{4.12}$$
$$= -\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sigma(|v-v'|) f(x,v) f(x,v') \frac{|(v-v') \wedge (\nabla_v \ln f - \nabla_{v'} \ln f)|^2}{|v-v'|^2} \, \mathrm{d}v' \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \leq 0.$$

Finally observe that $\langle v_1 \rangle = \langle v_2 \rangle = 0$, $\langle Q_{FPL}(f,f)\rangle \in \ker \mathcal{T}$ and thus trivially

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_1, v_2) \, \langle Q_{FPL}(f,f)\rangle \, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = (0,0).$$

□

We establish formally the limit model stated in Theorem 1.1.

Proof. (of Theorem 1.1) Plugging the Ansatz $f^\varepsilon = f + \varepsilon f^1 + \varepsilon^2 f^2 + ...$ into (4.1) we obtain

$$\left(\partial_t + v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v + \frac{1}{\varepsilon}\mathcal{T}\right)(f + \varepsilon f^1 + ...) = Q_{FPL}(f,f)$$
$$+ \varepsilon(Q_{FPL}(f,f^1) + Q_{FPL}(f^1,f)) + ...$$

implying that

$$\mathcal{T} f = 0, \;\; \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 = Q_{FPL}(f,f).$$

Applying the average operator, we deduce by Propositions 3.4, 4.10 that f satisfies

$$\partial_t f + \frac{\left\langle {}^\perp \overline{E} \right\rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f = \langle Q_{FPL} \rangle (f,f).$$

□

We are searching for extensions of the averaged Fokker-Planck-Landau operator to the whole space of densities $f = f(x,v)$, not necessarily in the kernel of $\mathcal{T}$. One possibility is to consider the extension $\langle Q_{FPL} \rangle$ obtained thanks to (4.9), that is for any f

$$\begin{aligned}
&\omega_c^{-2} \langle Q_{FPL} \rangle (f,f) = \\
&\mathrm{div}_{\omega_c x, v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^4 f(\overline{x'}, x_3, v') \xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x}, v, \overline{x'}, v') \nabla_{\omega_c x, v} f(x,v) \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \right\} \\
&- \mathrm{div}_{\omega_c x, v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^4 f(x,v) \xi^i(\overline{x}, v, \overline{x'}, v') \otimes \varepsilon_i \xi^i(\overline{x'}, v', \overline{x}, v) \nabla_{\omega_c x', v'} f(\overline{x'}, x_3, v') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \right\}.
\end{aligned} \tag{4.13}$$

What is remarkable is that this extension still satisfies the mass, third momentum component, kinetic energy balances and decreases the entropy $f \ln f$, globally in $(\overline{x}, v)$.

Proposition 4.11. Consider two functions $f = f(x,v), \varphi = \varphi(x,v)$. For any $x_3 \in \mathbb{R}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_{FPL} \rangle (f,f) \varphi \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = -\frac{\omega_c^2}{2} \times \\
&\sum_{i=1}^4 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f f' (\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \nabla' \ln f')(\xi^i \cdot \nabla \varphi - \varepsilon_i (\xi^i)' \nabla' \varphi') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2
\end{aligned} \tag{4.14}$$

where

$$f = f(x,v), \; f' = f'(x_1', x_2', x_3, v'), \; \nabla \varphi = \nabla_{\omega_c x, v} \varphi(x,v), \; \nabla' \varphi' = \nabla_{\omega_c x', v'} \varphi(x_1', x_2', x_3, v')$$

$$\xi^i = \xi^i(x_1, x_2, v, x_1', x_2', v'), \;\; (\xi^i)' = \xi^i(x_1', x_2', v', x_1, x_2, v).$$

In particular the averaged Fokker-Planck-Landau operator satisfies the mass, third momentum component, kinetic energy balances (globally in (x_1, x_2, v))

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \left(1, v_3, \frac{|v|^2}{2} \right) \langle Q_{FPL} \rangle (f,f) \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = (0,0,0)$$

and decreases the entropy $f \ln f$ (globally in (x_1, x_2, v))

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f \, \langle Q_{FPL}\rangle (f,f) \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 \le 0.$$

Proof. Notice that for any $1 \le i \le 4$ we have $\xi^i \cdot (e_3, 0) = 0$ and therefore the operator $\mathrm{div}_{\omega_c x, v}$ acts only in (x_1, x_2, v). Thus, for any fixed $x_3 \in \mathbb{R}$ we can perform integration by parts with respect to (x_1, x_2, v).

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q_{FPL}\rangle (f,f)\varphi \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = -\sum_{i=1}^{4} \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f f' \tag{4.15}$$
$$\times \left\{ (\xi^i \cdot \nabla\varphi)(\xi^i \cdot \nabla \ln f) - \varepsilon_i (\xi^i \cdot \nabla\varphi)((\xi^i)' \cdot \nabla' \ln f') \right\} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2.$$

Performing the change of variables $(x_1', x_2', v') \leftrightarrow (x_1, x_2, v)$ yields

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q_{FPL}\rangle (f,f)\varphi \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = -\sum_{i=1}^{4} \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f f' \tag{4.16}$$
$$\times \left\{ ((\xi^i)' \cdot \nabla'\varphi')((\xi^i)' \cdot \nabla' \ln f') - \varepsilon_i ((\xi^i)' \cdot \nabla'\varphi')(\xi^i \cdot \nabla \ln f) \right\} \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'.$$

Combining (4.15), (4.16) one gets by Fubini theorem

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q_{FPL}\rangle (f,f)\varphi \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = -\frac{\omega_c^2}{2} \sum_{i=1}^{4} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f f' T^i \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2$$

where

$$T^i = \left(\xi^i \cdot \nabla\varphi - \varepsilon_i (\xi^i)' \cdot \nabla'\varphi' \right) \, \left(\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln f' \right), \ \ 1 \le i \le 4.$$

Clearly, the divergence form of $\langle Q_{FPL}\rangle$ guarantees the mass conservation and (4.14) applied with $\varphi = \ln f$ ensures that the entropy $f \ln f$ is decreasing

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f \, \langle Q_{FPL}\rangle (f,f) \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = -\frac{\omega_c^2}{2} \sum_{i=1}^{4} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f f'$$
$$\times \left(\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln f' \right)^2 \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 \le 0, \ x_3 \in \mathbb{R}.$$

It remains to show the kinetic energy and third momentum component balances. Thanks to formula (4.14) it is sufficient to check that

$$\xi^i \cdot \nabla \frac{|v|^2}{2} - \varepsilon_i (\xi^i)' \cdot \nabla' \frac{|v'|^2}{2} = 0, \ \ 1 \le i \le 4.$$

The above condition is trivially satisfied for $i \in \{1,2\}$. For $i = 3$ we have

$$\xi^3 \cdot \nabla \frac{|v|^2}{2} - \varepsilon_3 (\xi^3)' \cdot \nabla' \frac{|v'|^2}{2} = -\{\sigma\chi\}^{1/2} \frac{r' \sin\varphi}{|z|} r + \{\sigma\chi\}^{1/2} \frac{r \sin\varphi}{|z|} r' = 0.$$

Finally, when $i = 4$ we obtain

$$\begin{aligned}
&\xi^4 \cdot \nabla \frac{|v|^2}{2} - \varepsilon_4 (\xi^4)' \cdot \nabla' \frac{|v'|^2}{2} \\
&= \{\sigma\chi\}^{1/2} \left\{ - \frac{(r'\cos\varphi - r)(v_3 - v_3')r + |z|^2 v_3}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} + \frac{(r\cos\varphi - r')(v_3' - v_3)r' + |z|^2 v_3'}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \right\} \\
&= \{\sigma\chi\}^{1/2} \frac{v_3 - v_3'}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \left[r^2 + (r')^2 - 2rr'\cos\varphi - |z|^2 \right] = 0.
\end{aligned}$$

Notice also that for any $i \in \{1,2,3,4\}$ we have

$$\xi^i \cdot \nabla v_3 - \varepsilon_i (\xi^i)' \cdot \nabla' v_3' = 0$$

saying that $\langle Q_{FPL} \rangle$ satisfies

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} v_3 \langle Q_{FPL} \rangle (f,f) \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = 0.$$

□

REMARK 4.3. By the formula (4.14) we deduce that the positive smooth functions f satisfying $\langle Q_{FPL} \rangle (f,f) = 0$ are those verifying

$$\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln f' = 0, \;\; i \in \{1,2,3,4\}. \tag{4.17}$$

In particular (4.17) holds true for any Maxwellian f which belongs to $\ker \mathcal{T}$, since in that case

$$\langle Q_{FPL} \rangle (f,f) = \langle Q_{FPL}(f,f) \rangle = \langle 0 \rangle = 0.$$

We deduce that

$$\xi^i \cdot \nabla \varphi - \varepsilon_i (\xi^i)' \cdot \nabla' \varphi' = 0, \;\; i \in \{1,2,3,4\} \tag{4.18}$$

for any function $\varphi(x,v) = \alpha(x)|v|^2 + \beta(x) \cdot v + \gamma(x)$ satisfying $\mathcal{T}\varphi = 0$, and in particular for the functions

$$x_1 + \frac{v_2}{\omega_c}, \; x_2 - \frac{v_1}{\omega_c}, \; x_3, \; v_3, \; |\overline{x}|^2 + 2\overline{x} \cdot \frac{{}^\perp\overline{v}}{\omega_c} = \left| \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} \right|^2 - \frac{|\overline{v}|^2}{\omega_c^2}.$$

Notice that $\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}$ is the center of the circle obtained by projecting the Larmor circle $C_{x,v}$ onto (x_1', x_2') and $|\overline{x}|^2 + 2\overline{x} \cdot \frac{{}^\perp\overline{v}}{\omega_c}$ is the power of the origin $(0,0)$ with respect to the same circle. We obtain, thanks to (4.14) and (4.18), the balances

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right)\langle Q_{FPL}\rangle(f,f)\ \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = (0,0)$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\left(|\overline{x}|^2 + 2\overline{x}\cdot\frac{{}^\perp\overline{v}}{\omega_c}\right)\langle Q_{FPL}\rangle(f,f)\ \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = 0$$

for any smooth function f.

The previous identities allow us to establish the mass, momentum, total energy, Larmor circle center and power conservations for smooth solutions of (1.10), (1.11) coupled with the Poisson equation for the electric field

$$E = -\nabla_x\varphi,\ \ \varepsilon_0\ \mathrm{div}_x E(t,x) = q\int_{\mathbb{R}^3} f(t,x,v)\ \mathrm{d}v =: \rho(t,x),\ \ (t,x)\in\mathbb{R}_+\times\mathbb{R}^3. \quad (4.19)$$

THEOREM 4.4. Assume that (f,E) is a smooth solution for

$$\partial_t f + \frac{\left\langle {}^\perp\overline{E}\right\rangle}{B}\cdot\nabla_{\overline{x}} f + v_3\partial_{x_3} f + \frac{q}{m}\langle E_3\rangle\partial_{v_3} f = \langle Q_{FPL}\rangle(f,f),\ \ \mathcal{T}f = 0,\ \ (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$$

$$\varepsilon_0\mathrm{div}_x E(t,x) = q\int_{\mathbb{R}^3} f(t,x,v)\ \mathrm{d}v,\ \ E = -\nabla_x\phi,\ \ (t,x)\in\mathbb{R}_+\times\mathbb{R}^3.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\ \mathrm{d}v\mathrm{d}x = 0,\ \ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\overline{x}f\ \mathrm{d}v\mathrm{d}x = 0,\ \ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} vf\ \mathrm{d}v\mathrm{d}x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\varepsilon_0}{2}\int_{\mathbb{R}^3}|E|^2\ \mathrm{d}x + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} m\frac{|v|^2}{2}f\ \mathrm{d}v\mathrm{d}x\right\} = 0,\ \ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\frac{|\overline{x}|^2}{2} + \overline{x}\cdot\frac{{}^\perp\overline{v}}{\omega_c}\right) f\ \mathrm{d}v\mathrm{d}x = 0.$$

Proof. The mass conservation comes by the conservative formulation of the Vlasov equation, which writes

$$\partial_t f + \mathrm{div}_{\overline{x}}\left\{f\frac{\left\langle {}^\perp\overline{E}\right\rangle}{B}\right\} + \partial_{x_3}\{fv_3\} + \frac{q}{m}\partial_{v_3}\{f\langle E_3\rangle\} = \langle Q_{FPL}\rangle(f,f). \quad (4.20)$$

Multiplying (4.20) by v and integrating with respect to (x,v) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} vf\ \mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{q}{m}f\langle E_3\rangle e_3\ \mathrm{d}v\mathrm{d}x = 0$$

since for functions in $\ker \mathcal{T}$ we can write

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v \left\langle Q_{FPL}\right\rangle(f,f)\;\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v \left\langle Q_{FPL}(f,f)\right\rangle\;\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle v\rangle\, Q_{FPL}(f,f)\;\mathrm{d}v\mathrm{d}x$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(0,0,v_3)Q_{FPL}(f,f)\;\mathrm{d}v\mathrm{d}x = 0.$$

Using the Poisson equation and the identity

$$\mathrm{div}_x E\; E = \mathrm{div}_x(E\otimes E) - \frac{1}{2}\nabla_x |E|^2 \tag{4.21}$$

we obtain, taking into account that $f \in \ker \mathcal{T}$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{q}{m} f\,\langle E_3\rangle\;\mathrm{d}v\mathrm{d}x = \frac{q}{m}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f E_3\;\mathrm{d}v\mathrm{d}x = \frac{\varepsilon_0}{m}\int_{\mathbb{R}^3} E_3\;\mathrm{div}_x E\;\mathrm{d}x$$
$$= \frac{\varepsilon_0}{m}\int_{\mathbb{R}^3}\left\{\mathrm{div}_x(E_3 E) - \frac{1}{2}\partial_{x_3}|E|^2\right\}\;\mathrm{d}x = 0$$

and thus $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v f\;\mathrm{d}v\mathrm{d}x = 0$. Multiplying (4.20) by $\frac{m|v|^2}{2}$ and integrating with respect to (x,v) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{m|v|^2}{2} f\;\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} q f v_3\,\langle E_3\rangle\;\mathrm{d}v\mathrm{d}x = 0.$$

Thanks to the continuity equation

$$\partial_t \int_{\mathbb{R}^3} f\;\mathrm{d}v + \mathrm{div}_{\overline{x}}\int_{\mathbb{R}^3} f\frac{\left\langle {}^\perp\overline{E}\right\rangle}{B}\;\mathrm{d}v + \partial_{x_3}\int_{\mathbb{R}^3} f v_3\;\mathrm{d}v = 0$$

we have

$$- q\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f v_3\,\langle E_3\rangle\;\mathrm{d}v\mathrm{d}x = -q\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f v_3 E_3\;\mathrm{d}v\mathrm{d}x = q\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f v_3 \partial_{x_3}\phi\;\mathrm{d}v\mathrm{d}x$$
$$= -q\int_{\mathbb{R}^3}\phi\;\partial_{x_3}\int_{\mathbb{R}^3} f v_3\;\mathrm{d}v\;\mathrm{d}x = \int_{\mathbb{R}^3}\phi\left\{\partial_t\rho + q\;\mathrm{div}_{\overline{x}}\int_{\mathbb{R}^3} f\frac{\left\langle {}^\perp\overline{E}\right\rangle}{B}\;\mathrm{d}v\right\}\;\mathrm{d}x$$
$$= \int_{\mathbb{R}^3}\phi\;\partial_t(\varepsilon_0 \mathrm{div}_x E)\;\mathrm{d}x - q\;\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nabla_x\phi\cdot\frac{\left\langle {}^\perp\overline{E}\right\rangle}{B} f\;\mathrm{d}v\mathrm{d}x$$
$$= -\int_{\mathbb{R}^3}\nabla_x\phi\cdot\partial_t(\varepsilon_0 E)\;\mathrm{d}x + q\;\int_{\mathbb{R}^3}\langle E\rangle\cdot\frac{\left\langle {}^\perp\overline{E}\right\rangle}{B} f\;\mathrm{d}x$$
$$= \frac{\varepsilon_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|E|^2\;\mathrm{d}x$$

implying

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\varepsilon_0}{2}\int_{\mathbb{R}^3}|E|^2\;\mathrm{d}x + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} m\frac{|v|^2}{2} f\;\mathrm{d}v\mathrm{d}x\right\} = 0.$$

By Remark 4.3 we know that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \overline{x} \left\langle Q_{FPL}\right\rangle (f,f)\, \mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right) \left\langle Q_{FPL}\right\rangle (f,f)\, \mathrm{d}v\mathrm{d}x \\ - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{{}^\perp\overline{v}}{\omega_c} \left\langle Q_{FPL}\right\rangle (f,f)\, \mathrm{d}v\mathrm{d}x = 0$$

and therefore, multiplying (4.20) by $\overline{x}$ one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\overline{x}\, \mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f \frac{\left\langle {}^\perp\overline{E}\right\rangle}{B}\, \mathrm{d}v\mathrm{d}x = 0.$$

Appealing to the Poisson equation we have

$$q\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f \left\langle \overline{E}\right\rangle\, \mathrm{d}v\mathrm{d}x = q\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\overline{E}\, \mathrm{d}v\mathrm{d}x = \int \rho\overline{E} = \varepsilon_0 \int_{\mathbb{R}^3} \mathrm{div}_x E\overline{E}\, \mathrm{d}x = \\ = \varepsilon_0 \int_{\mathbb{R}^3} \left(\mathrm{div}_x(\overline{E}\otimes E) - \frac{1}{2}\nabla_{\overline{x}}|E|^2\right)\, \mathrm{d}x = 0$$

and therefore $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\overline{x}\, \mathrm{d}v\mathrm{d}x = 0$. In particular the mean Larmor circle center is left invariant

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{ \frac{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f \left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right)\, \mathrm{d}v\mathrm{d}x}{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\, \mathrm{d}v\mathrm{d}x} \right\} = 0.$$

By Remark 4.3 we know that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\omega_c^2|\overline{x}|^2 + 2\omega_c \overline{x}\cdot {}^\perp\overline{v}) \left\langle Q_{FPL}\right\rangle (f,f)\, \mathrm{d}v\mathrm{d}x = 0$$

and for any $\psi(x) \in C_c^1(\mathbb{R}^3)$ we can write

$$\int_{\mathbb{R}^3} \psi(x)\, \mathrm{div}_{\overline{x}} \int_{\mathbb{R}^3} f\overline{v}\, \mathrm{d}v\, \mathrm{d}x = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\, \overline{v}\cdot\nabla_{\overline{x}}\psi\, \mathrm{d}v\mathrm{d}x \\ = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\, \mathcal{T}\psi\, \mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi\, \mathcal{T} f\, \mathrm{d}v\mathrm{d}x = 0$$

saying that $\mathrm{div}_{\overline{x}} \int_{\mathbb{R}^3} f\overline{v}\ \mathrm{d}v = 0$. Therefore we deduce

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\frac{|\overline{x}|^2}{2} + \overline{x}\cdot\frac{{}^\perp\overline{v}}{\omega_c}\right) f\ \mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\frac{\left\langle {}^\perp\overline{E}\right\rangle}{B}\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right)\ \mathrm{d}v\mathrm{d}x \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\frac{{}^\perp\overline{E}}{B}\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right)\ \mathrm{d}v\mathrm{d}x = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\frac{\overline{E}}{B}\cdot {}^\perp\overline{x}\ \mathrm{d}v\mathrm{d}x + \int_{\mathbb{R}^3}\frac{\phi(x)}{B\omega_c}\mathrm{div}_{\overline{x}}\int_{\mathbb{R}^3} f\overline{v}\ \mathrm{d}v\ \mathrm{d}x \\
&= -\frac{\varepsilon_0}{qB}\int_{\mathbb{R}^3} {}^\perp\overline{x}\cdot\overline{E}\ \mathrm{div}_x E\ \mathrm{d}x = -\frac{\varepsilon_0}{qB}\int_{\mathbb{R}^3} {}^\perp\overline{x}\cdot\left(\mathrm{div}_x(\overline{E}\otimes E) - \frac{1}{2}\nabla_{\overline{x}}|E|^2\right)\ \mathrm{d}x \\
&= -\frac{\varepsilon_0}{qB}\int_{\mathbb{R}^3}\left\{x_2\sum_{j=1}^3 \partial_{x_j}(E_1E_j) - x_1\sum_{j=1}^3\partial_{x_j}(E_2E_j)\right\}\ \mathrm{d}x \\
&= \frac{\varepsilon_0}{qB}\int_{\mathbb{R}^3}\sum_{j=1}^3\left\{\frac{\partial x_2}{\partial x_j}E_1E_j - \frac{\partial x_1}{\partial x_j}E_2E_j\right\}\ \mathrm{d}x \\
&= 0
\end{aligned}
$$

implying that the mean Larmor circle power (with respect to the origin) is left invariant. □

Appendix A. Proofs of Propositions 4.4, 4.7.

Proof. (of Proposition 4.4) We follow the same arguments as those in the proof of Proposition 4.2. The details are left to the reader.

1. Using cylindrical coordinates we obtain

$$
\begin{aligned}
J &:= \left\langle\int_{\mathbb{R}^3} D(v,v')f(x,v')\ \mathrm{d}v'\right\rangle = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}\int_{-\pi}^{\pi}\int_{\mathbb{R}_+} D(|\overline{v}|e^{i\alpha}, v_3, r'e^{i(\varphi+\alpha)}, v_3') \\
&\times f\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\{|\overline{v}|e^{i\alpha}\}}{\omega_c}, x_3, r'e^{i(\varphi+\alpha)}, v_3'\right) r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}\int_{-\pi}^{\pi}\int_{\mathbb{R}_+} D(|\overline{v}|e^{i\alpha}, v_3, r'e^{i(\varphi+\alpha)}, v_3') \\
&\times g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\{|\overline{v}|e^{i\alpha}\}}{\omega_c} + \frac{{}^\perp\{r'e^{i(\varphi+\alpha)}\}}{\omega_c}, x_3, r', v_3'\right) r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha \qquad \text{(A.1)}
\end{aligned}
$$

where in the last equality we have used the constraint $f \in \ker \mathcal{T}$ *i.e.*, there is g such that

$$f(x,v) = g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}, x_3, |\overline{v}|, v_3\right),\ \ (x,v)\in\mathbb{R}^3\times\mathbb{R}^3.$$

We have $r'e^{i(\varphi+\alpha)} - |\overline{v}|e^{i\alpha} = le^{i(\psi+\alpha)}$ where $l^2 = r^2 + (r')^2 - 2rr'\cos\varphi$, $r = |\overline{v}|$ and $(r')^2 = r^2 + l^2 + 2rl\cos\psi$. Notice that $\psi \in (0,\pi)$ if $\varphi\in(0,\pi)$ and $\psi\in(-\pi,0)$ if

$\varphi \in (-\pi, 0)$. Also $\psi = \psi(\varphi)$ is odd with respect to φ that is $\psi(-\varphi) = -\psi(\varphi)$. By hypothesis we deduce that

$$D(re^{i\alpha}, v_3, r'e^{i(\varphi+\alpha)}, v_3') = \tilde{D}(r, v_3, r', v_3', \varphi)\; re^{i\alpha} + \tilde{D}'(r, v_3, r', v_3', \varphi)\; r'e^{i(\varphi+\alpha)}$$

and thus, if we denote by $R(\alpha)$ the rotation of angle α in $\mathbb{R}^2$ one gets

$$\begin{aligned}
J = &\frac{1}{2\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{-\pi}^{\pi}\int_{\mathbb{R}_+} R(\alpha+\psi)[\tilde{D}(r, v_3, r', v_3', \varphi)\; re^{-i\psi} + \tilde{D}'(r, v_3, r', v_3', \varphi)\; r'e^{i(\varphi-\psi)}] \\
&\times g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} + \frac{{}^\perp\{le^{i(\alpha+\psi)}\}}{\omega_c}, x_3, r', v_3'\right) r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha \\
= &\frac{1}{2\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{-\pi}^{\pi}\int_{\mathbb{R}_+} R(\alpha)[\tilde{D}(r, v_3, r', v_3', \varphi)\; re^{-i\psi} + \tilde{D}'(r, v_3, r', v_3', \varphi)\; r'e^{i(\varphi-\psi)}] \\
&\times g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} + \frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha.
\end{aligned}$$

Using the symmetry of ψ with respect to φ and changing φ against l yield

$$\begin{aligned}
J = &\frac{1}{2\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_0^{\pi}\int_{\mathbb{R}_+} R(\alpha)[\tilde{D}(r, v_3, r', v_3', \varphi)\; re^{-i\psi} + \tilde{D}(r, v_3, r', v_3', -\varphi)\; re^{i\psi} \\
&+ \tilde{D}'(r, v_3, r', v_3', \varphi)\; r'e^{i(\varphi-\psi)} + \tilde{D}'(r, v_3, r', v_3', -\varphi)\; r'e^{-i(\varphi-\psi)}] \\
&\times g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} + \frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha \\
= &\frac{1}{\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{|r-r'|}^{(r+r')}\int_{\mathbb{R}_+} R(\alpha)[\tilde{D}(r, v_3, r', v_3', \varphi)\; re^{-i\psi} + \tilde{D}(r, v_3, r', v_3', -\varphi)\; re^{i\psi} \\
&+ \tilde{D}'(r, v_3, r', v_3', \varphi)\; r'e^{i(\varphi-\psi)} + \tilde{D}'(r, v_3, r', v_3', -\varphi)\; r'e^{-i(\varphi-\psi)}] \\
&\times g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} + \frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) \frac{r'\mathrm{d}r'\; l\mathrm{d}l\; \mathrm{d}v_3'\mathrm{d}\alpha}{\sqrt{l^2-(r-r')^2}\,\sqrt{(r+r')^2-l^2}}.
\end{aligned}$$

Notice that $R(\alpha) = e_1 \otimes e^{-i\alpha} - e_2 \otimes {}^\perp e^{-i\alpha}$ and for any $\alpha' \in [0, 2\pi)$ we have

$$g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} + \frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) = f\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} + \frac{{}^\perp\{le^{i\alpha}\}}{\omega_c} - \frac{{}^\perp\{r'e^{i\alpha'}\}}{\omega_c}, x_3, r'e^{i\alpha'}, v_3'\right).$$

Performing the change of coordinates $v' = (r'e^{i\alpha'}, v_3')$ and $-z = {}^\perp\{le^{i\alpha}\}$ leads to

$$J = \frac{1}{2\pi^2}\int_0^{2\pi}\!\!\int_{\mathbb{R}_+}\int_{\mathbb{R}}\int_0^{2\pi}\!\!\int_{\mathbb{R}_+}\{e_1\otimes e^{-i\alpha} - e_2 \otimes {}^\perp e^{-i\alpha}\}[\tilde{D}(\varphi)\, re^{-i\psi} + \tilde{D}(-\varphi)\, re^{i\psi}$$
$$+ \tilde{D}'(\varphi)\, r'e^{i(\varphi-\psi)} + \tilde{D}'(-\varphi)\, r'e^{-i(\varphi-\psi)}]$$
$$\times f\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} + \frac{{}^\perp\{le^{i\alpha}\}}{\omega_c} - \frac{{}^\perp\{r'e^{i\alpha'}\}}{\omega_c}, x_3, r'e^{i\alpha'}, v_3'\right)\chi(r, r', -{}^\perp\{le^{i\alpha}\})r'\mathrm{d}r'\mathrm{d}\alpha'\mathrm{d}v_3' l\mathrm{d}l\mathrm{d}\alpha$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{D}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) f\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{z}{\omega_c} - \frac{{}^\perp\overline{v'}}{\omega_c}, x_3, v'\right)\,\mathrm{d}v'\mathrm{d}z$$
$$= \omega_c^2\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^3}\mathcal{D}(|\overline{v}|, v_3, |\overline{v'}|, v_3', (\omega_c\overline{x} + {}^\perp\overline{v}) - (\omega_c\overline{x'} + {}^\perp\overline{v'}))f(x_1', x_2', x_3, v')\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.$$

The statement in 2. follows similarly. □

Proof. (of Proposition 4.7) Observe that

$$\langle f\rangle_{\sigma S} = \left\langle\int_{\mathbb{R}^3} f(x,v')\sigma(|v-v'|)\,\mathrm{d}v'\right\rangle\, I - \left\langle\int_{\mathbb{R}^3} f(x,v')\sigma(|v-v'|)\frac{(v-v')\otimes(v-v')}{|v-v'|^2}\,\mathrm{d}v'\right\rangle.$$

Applying Proposition 4.2 with $C = 1$ one gets $\left\langle\int_{\mathbb{R}^3} f\sigma\,\mathrm{d}v'\right\rangle = \omega_c^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\sigma f\chi\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$. It remains to compute the second average. Using cylindrical coordinates and the constraint $\mathcal{T}f = 0$ *i.e.,* $f(x,v) = g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}, x_3, |\overline{v}|, v_3\right)$ we obtain

$$K := \left\langle\int_{\mathbb{R}^3} f(x,v')\sigma(|v-v'|)\frac{(v-v')\otimes(v-v')}{|v-v'|^2}\,\mathrm{d}v'\right\rangle$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{-\pi}^{\pi}\int_{\mathbb{R}_+}\frac{\sigma(\,|\,(|\overline{v}|e^{i\alpha} - r'e^{i(\varphi+\alpha)}, v_3 - v_3')\,|\,)}{|\,(|\overline{v}|e^{i\alpha} - r'e^{i(\varphi+\alpha)}, v_3 - v_3')\,|^2}\left(|\overline{v}|e^{i\alpha} - r'e^{i(\varphi+\alpha)}, v_3 - v_3'\right)^{\otimes 2}$$
$$\times f\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\{|\overline{v}|e^{i\alpha}\}}{\omega_c}, x_3, r'e^{i(\varphi+\alpha)}, v_3'\right)r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{-\pi}^{\pi}\int_{\mathbb{R}_+}\frac{\sigma(\,|\,(|\overline{v}|e^{i\alpha} - r'e^{i(\varphi+\alpha)}, v_3 - v_3')\,|\,)}{|\,(|\overline{v}|e^{i\alpha} - r'e^{i(\varphi+\alpha)}, v_3 - v_3')\,|^2}\left(|\overline{v}|e^{i\alpha} - r'e^{i(\varphi+\alpha)}, v_3 - v_3'\right)^{\otimes 2}$$
$$\times g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\{|\overline{v}|e^{i\alpha}\}}{\omega_c} + \frac{{}^\perp\{r'e^{i(\varphi+\alpha)}\}}{\omega_c}, x_3, r', v_3'\right)r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha.$$

We introduce l, ψ such that $r'e^{i(\varphi+\alpha)} - |\overline{v}|e^{i\alpha} = le^{i(\psi+\alpha)}$. We have the relations $l^2 = r^2 + (r')^2 - 2rr'\cos\varphi$, $r = |\overline{v}|$ and $(r')^2 = r^2 + l^2 + 2rl\cos\psi$. Since l, ψ are not depending on α one gets

$$
\begin{aligned}
K = & \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{-\pi}^{\pi}\int_{\mathbb{R}_+} \frac{\sigma(\sqrt{l^2+(v_3-v_3')^2}\,)}{l^2+(v_3-v_3')^2}\left(le^{i(\alpha+\psi)}, v_3'-v_3\right)^{\otimes 2}\\
& \times g\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}+\frac{{}^\perp\{le^{i(\alpha+\psi)}\}}{\omega_c}, x_3, r', v_3'\right) r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha\\
= & \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{-\pi}^{\pi}\int_{\mathbb{R}_+} \frac{\sigma(\sqrt{l^2+(v_3-v_3')^2}\,)}{l^2+(v_3-v_3')^2}\left(le^{i\alpha}, v_3'-v_3\right)^{\otimes 2}\\
& \times g\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}+\frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha.
\end{aligned}
$$

Since l is even with respect to φ we obtain, after changing φ against l

$$
\begin{aligned}
K = & \frac{1}{\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{0}^{\pi}\int_{\mathbb{R}_+} \frac{\sigma(\sqrt{l^2+(v_3-v_3')^2}\,)}{l^2+(v_3-v_3')^2}\left(le^{i\alpha}, v_3'-v_3\right)^{\otimes 2}\\
& \times g\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}+\frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) r'\mathrm{d}r'\mathrm{d}\varphi\mathrm{d}v_3'\mathrm{d}\alpha\\
= & \frac{2}{\pi}\int_0^{2\pi}\!\!\int_{\mathbb{R}}\int_{|r-r'|}^{(r+r')}\int_{\mathbb{R}_+} \frac{\sigma(\sqrt{l^2+(v_3-v_3')^2}\,)}{l^2+(v_3-v_3')^2}\left(le^{i\alpha}, v_3'-v_3\right)^{\otimes 2}\\
& \times g\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}+\frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) \frac{r'\mathrm{d}r'\ l\mathrm{d}l\ \mathrm{d}v_3'\mathrm{d}\alpha}{\sqrt{l^2-(r-r')^2}\sqrt{(r+r')^2-l^2}}.
\end{aligned}
$$

For any $\alpha' \in [0, 2\pi)$ we have

$$
g\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}+\frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) = f\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}+\frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}-\frac{{}^\perp\{r'e^{i\alpha'}\}}{\omega_c}, x_3, r'e^{i\alpha'}, v_3'\right).
$$

Performing the change of coordinates $v' = (r'e^{i\alpha'}, v_3')$ and $z = -{}^\perp\{le^{i\alpha}\}$ leads to

$$
\begin{aligned}
K = & \int_0^{2\pi}\!\!\int_{\mathbb{R}_+}\int_{\mathbb{R}}\int_0^{2\pi}\!\!\int_{\mathbb{R}_+} \frac{\sigma(\sqrt{l^2+(v_3-v_3')^2}\,)}{l^2+(v_3-v_3')^2}\left(le^{i\alpha}, v_3'-v_3\right)^{\otimes 2}\\
& \times f\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}+\frac{{}^\perp\{le^{i\alpha}\}}{\omega_c}-\frac{{}^\perp\{r'e^{i\alpha'}\}}{\omega_c}, x_3, r'e^{i\alpha'}, v_3'\right)\chi(r, r', -{}^\perp\{le^{i\alpha}\})r'\mathrm{d}r'\mathrm{d}\alpha'\mathrm{d}v_3' l\mathrm{d}l\mathrm{d}\alpha\\
= & \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{\sigma(\sqrt{|z|^2+(v_3-v_3')^2}\,)}{|z|^2+(v_3-v_3')^2}(\,{}^\perp z, v_3'-v_3)^{\otimes 2} f\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}-\frac{z}{\omega_c}-\frac{{}^\perp\overline{v'}}{\omega_c}, x_3, v'\right)\chi\ \mathrm{d}v'\mathrm{d}z\\
= & \,\omega_c^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\sigma(\sqrt{|z|^2+(v_3-v_3')^2}\,)f(x_1', x_2', x_3, v')\ \chi\ \frac{(\,{}^\perp z, v_3'-v_3)^{\otimes 2}}{|z|^2+(v_3-v_3')^2}\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.
\end{aligned}
$$

Finally we obtain

$$
\langle f\rangle_{\sigma S} = \omega_c^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\sigma(\sqrt{|z|^2+(v_3-v_3')^2}\,)f(x_1', x_2', x_3, v')\ \chi\ S(\,({}^\perp z, v_3'-v_3)\,)\ \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.
$$

□

Appendix B. Proofs of Propositions 4.8, 4.9.

Proof. (of Proposition 4.8) Let us introduce the notation

$$\xi_v(x,v) = \int_{\mathbb{R}^3} \sigma(|v-v'|)S(v-v')f(x,v')\nabla_v f(x,v)\,\mathrm{d}v'.$$

Thanks to Proposition 3.3 we have

$$\left\langle Q^+_{FPL}(f,f)\right\rangle = \langle \mathrm{div}_v\ \xi_v\rangle = \frac{1}{\omega_c}\mathrm{div}_{\overline{x}}\left\{\left\langle {}^\perp\xi_{\overline{v}}\right\rangle + \left\langle \xi_{\overline{v}}\cdot\frac{{}^\perp\overline{v}}{|\overline{v}|}\right\rangle\frac{\overline{v}}{|\overline{v}|} - \left\langle \xi_{\overline{v}}\cdot\frac{\overline{v}}{|\overline{v}|}\right\rangle\frac{{}^\perp\overline{v}}{|\overline{v}|}\right\}$$
$$+\,\mathrm{div}_{\overline{v}}\left\{\left\langle \xi_{\overline{v}}\cdot\frac{{}^\perp\overline{v}}{|\overline{v}|}\right\rangle\frac{{}^\perp\overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}}\cdot\frac{\overline{v}}{|\overline{v}|}\right\rangle\frac{\overline{v}}{|\overline{v}|}\right\} + \partial_{v_3}\left\langle \xi_{v_3}\right\rangle.$$

We need to compute $\langle \xi_{\overline{v}}\rangle, \langle \xi_{\overline{v}}\cdot\overline{v}\rangle, \langle \xi_{\overline{v}}\cdot {}^\perp\overline{v}\rangle$. By Proposition 3.1 we know that $\sum_{i=0}^5 b^i\otimes \nabla_{x,v}\psi_i = I$ and thus

$$\partial_{v_1}f = \sum_{i=0}^5 \partial_{v_1}\psi_i\ b^i\cdot\nabla_{x,v}f = \frac{v_2}{\omega_c|\overline{v}|^2}\ b^0\cdot\nabla_{x,v}f - \frac{1}{\omega_c}b^2\cdot\nabla_{x,v}f + \frac{v_1}{|\overline{v}|}\ b^4\cdot\nabla_{x,v}f.$$

Similarly we have

$$\partial_{v_2}f = \sum_{i=0}^5 \partial_{v_2}\psi_i\ b^i\cdot\nabla_{x,v}f = -\frac{v_1}{\omega_c|\overline{v}|^2}\ b^0\cdot\nabla_{x,v}f + \frac{1}{\omega_c}b^1\cdot\nabla_{x,v}f + \frac{v_2}{|\overline{v}|}\ b^4\cdot\nabla_{x,v}f$$

leading to

$$\nabla_v f = b^0\cdot\nabla_{x,v}f\ \frac{({}^\perp\overline{v},0)}{\omega_c|\overline{v}|^2} + \left(-\frac{{}^\perp\nabla_{\overline{x}}f}{\omega_c},\partial_{v_3}f\right) + b^4\cdot\nabla_{x,v}f\ \frac{(\overline{v},0)}{|\overline{v}|}.$$

Taking into account that all derivations $b^i\cdot\nabla_{x,v}, 0\le i\le 5$ leave invariant $\ker\mathcal{T}$, cf. Proposition 3.1, we obtain

$$\langle\xi_v\rangle = \left\langle f,({}^\perp\overline{v},0)\right\rangle_{\sigma S}\frac{b^0\cdot\nabla_{x,v}f}{\omega_c|\overline{v}|^2} + \langle f\rangle_{\sigma S}\left(-\frac{{}^\perp\nabla_{\overline{x}}f}{\omega_c},\partial_{v_3}f\right) + \langle f,(\overline{v},0)\rangle_{\sigma S}\frac{b^4\cdot\nabla_{x,v}f}{|\overline{v}|}$$
$$= \left\{\frac{\left\langle f,({}^\perp\overline{v},0)\right\rangle_{\sigma S}}{|\overline{v}|}\otimes\left(\frac{(\overline{v},0)}{|\overline{v}|},\frac{({}^\perp\overline{v},0)}{|\overline{v}|}\right) - \frac{\langle f,(\overline{v},0)\rangle_{\sigma S}}{|\overline{v}|}\otimes\left(\frac{({}^\perp\overline{v},0)}{|\overline{v}|},-\frac{(\overline{v},0)}{|\overline{v}|}\right)\right\}\nabla_{\omega_c x,v}f$$
$$-\langle f\rangle_{\sigma S}(E,-e_3\otimes e_3)\nabla_{\omega_c x,v}f$$

where the lines of the matrix $E \in \mathcal{M}_3(\mathbb{R})$ are $e_2, -e_1, 0$. Similarly, thanks to the identities $\big\langle f, (\overline{v}, 0), ({}^\perp\overline{v}, 0)\big\rangle_{\sigma S} = \big\langle f, ({}^\perp\overline{v}, 0), (\overline{v}, 0)\big\rangle_{\sigma S} = 0$ we obtain

$$\begin{aligned}
&\left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle = \left\langle \xi_v \cdot \frac{(\overline{v}, 0)}{|\overline{v}|} \right\rangle \\
&= -\frac{\langle f, (\overline{v}, 0)\rangle_{\sigma S}}{|\overline{v}|} \cdot \left(\frac{{}^\perp\nabla_{\overline{x}} f}{\omega_c}, -\partial_{v_3} f \right) + \frac{\langle f, (\overline{v}, 0), (\overline{v}, 0)\rangle_{\sigma S}}{|\overline{v}|^2}\ b^4 \cdot \nabla_{x,v} f \\
&= \left\{ -\frac{\langle f, (\overline{v}, 0), (\overline{v}, 0)\rangle_{\sigma S}}{|\overline{v}|^2} \left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|} \right) + \left(E \frac{\langle f, (\overline{v}, 0)\rangle_{\sigma S}}{|\overline{v}|}, e_3 \otimes e_3 \frac{\langle f, (\overline{v}, 0)\rangle_{\sigma S}}{|\overline{v}|} \right) \right\} \\
&\cdot \nabla_{\omega_c x, v} f
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle \xi_{\overline{v}} \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|} \right\rangle = \left\langle \xi_v \cdot \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|} \right\rangle \\
&= \frac{\big\langle f, ({}^\perp\overline{v}, 0), ({}^\perp\overline{v}, 0)\big\rangle_{\sigma S}}{|\overline{v}|^2}\ \frac{b^0 \cdot \nabla_{x,v} f}{\omega_c |\overline{v}|} - \frac{\big\langle f, ({}^\perp\overline{v}, 0)\big\rangle_{\sigma S}}{|\overline{v}|} \cdot \left(\frac{{}^\perp\nabla_{\overline{x}} f}{\omega_c}, -\partial_{v_3} f \right) \\
&= \left\{ \frac{\big\langle f, ({}^\perp\overline{v}, 0), ({}^\perp\overline{v}, 0)\big\rangle_{\sigma S}}{|\overline{v}|^2} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|} \right) + \left(E \frac{\big\langle f, ({}^\perp\overline{v}, 0)\big\rangle_{\sigma S}}{|\overline{v}|}, 0 \right) \right\} \cdot \nabla_{\omega_c x, v} f.
\end{aligned}$$

In the last equality we have taken into account that

$$e_3 \otimes e_3 \frac{\big\langle f, ({}^\perp\overline{v}, 0)\big\rangle_{\sigma S}}{|\overline{v}|} = 0.$$

Clearly $\left\langle Q^+_{FPL}(f,f)\right\rangle$ has the form in (4.7) with $A^+ = \begin{pmatrix} A^+_{xx} & A^+_{xv} \\ A^+_{vx} & A^+_{vv} \end{pmatrix}$ where

$$
\begin{aligned}
(A^+_{xx}, A^+_{xv}) = {} & \frac{r - r'\cos\varphi}{|z|}\ \frac{({}^\perp z, 0)}{|z|} \otimes \left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{({}^\perp\overline{v},0)}{|\overline{v}|}\right) \\
& - \frac{(r - r'\cos\varphi)(v_3 - v_3')^2}{|z|[\ |z|^2 + (v_3 - v_3')^2\ ]}\ \frac{(z,0)}{|z|} \otimes \left(\frac{({}^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right) \\
& + \left(1 - \frac{(r')^2\sin^2\varphi}{|z|^2 + (v_3 - v_3')^2}\right) \frac{(\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{({}^\perp\overline{v},0)}{|\overline{v}|}\right) \\
& + \frac{r - r'\cos\varphi}{|z|}\ \frac{(\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{({}^\perp z,0)}{|z|}, 0\right) \\
& + \left(1 - \frac{(r - r'\cos\varphi)^2}{|z|^2 + (v_3 - v_3')^2}\right) \frac{({}^\perp\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{({}^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right) \\
& - \frac{(r - r'\cos\varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\ \frac{({}^\perp\overline{v},0)}{|\overline{v}|} \otimes \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \\
& + ({}^t ES(\ ({}^\perp z, v_3' - v_3)\ )E, ES(\ ({}^\perp z, v_3' - v_3)\ )e_3 \otimes e_3)
\end{aligned} \tag{B.1}
$$

and

$$
\begin{aligned}
(A^+_{vx}, A^+_{vv}) = {} & \left(1 - \frac{(r')^2\sin^2\varphi}{|z|^2 + (v_3 - v_3')^2}\right) \frac{({}^\perp\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{({}^\perp\overline{v},0)}{|\overline{v}|}\right) \\
& + \frac{r - r'\cos\varphi}{|z|}\ \frac{({}^\perp\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{({}^\perp z,0)}{|z|}, 0\right) \\
& - \left(1 - \frac{(r - r'\cos\varphi)^2}{|z|^2 + (v_3 - v_3')^2}\right) \frac{(\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{({}^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right) \\
& + \frac{(r - r'\cos\varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\ \frac{(\overline{v},0)}{|\overline{v}|} \otimes \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \\
& + \frac{(r - r'\cos\varphi)(v_3 - v_3')}{|z|^2 + (v_3 - v_3')^2}\ e_3 \otimes \left(\frac{({}^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right) \\
& + (-e_3 \otimes e_3 S(\ ({}^\perp z, v_3' - v_3)\ )E, e_3 \otimes e_3 S(\ ({}^\perp z, v_3' - v_3)\ )e_3 \otimes e_3).
\end{aligned} \tag{B.2}
$$

It is easily seen that the matrix A^+ writes

$$
\begin{aligned}
A^+ &= \left(1 - \frac{(r')^2 \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2}\right) \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}\right)^{\otimes 2} \\
&+ \left(1 - \frac{(r - r' \cos \varphi)^2}{|z|^2 + (v_3 - v_3')^2}\right) \left(\frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right)^{\otimes 2} \\
&+ \frac{r - r' \cos \varphi}{|z|} \left(\frac{(^\perp z, 0)}{|z|}, 0\right) \otimes \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}\right) \\
&+ \frac{r - r' \cos \varphi}{|z|} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}\right) \otimes \left(\frac{(^\perp z, 0)}{|z|}, 0\right) \\
&- \frac{(r - r' \cos \varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z| e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \otimes \left(\frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) \\
&- \frac{(r - r' \cos \varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) \otimes \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z| e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \\
&+ B^+ = A_1^+ + A_2^+ + A_3^+ + A_4^+ + A_5^+ + A_6^+ + B^+ \qquad \text{(B.3)}
\end{aligned}
$$

where

$$
B^+ = \begin{pmatrix} {}^t E S(\,(^\perp z, v_3' - v_3)\,) E & E S(\,(^\perp z, v_3' - v_3)\,) e_3 \otimes e_3 \\ -e_3 \otimes e_3 S(\,(^\perp z, v_3' - v_3)\,) E & e_3 \otimes e_3 S(\,(^\perp z, v_3' - v_3)\,) e_3 \otimes e_3 \end{pmatrix}.
$$

Observe that for any $z \in \mathbb{R}^2, v_3, v_3' \in \mathbb{R}$ the family

$$
\frac{(z,0)}{|z|}, \quad \left(\frac{v_3 - v_3'}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \frac{^\perp z}{|z|}, \frac{|z|}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right), \quad \frac{(^\perp z, v_3' - v_3)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}
$$

is a orthonormal basis of $\mathbb{R}^3$. Therefore we have

$$
S(\,(^\perp z, v_3' - v_3)\,) = \frac{(z,0)}{|z|} \otimes \frac{(z,0)}{|z|} + \left(\frac{v_3 - v_3'}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \frac{^\perp z}{|z|}, \frac{|z|}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right)^{\otimes 2}
$$

implying that for any $\xi = (\xi_x, \xi_v) \in \mathbb{R}^6$

$$
\begin{aligned}
&(B^+ \xi, \xi) = S(\,(^\perp z, v_3' - v_3)\,) : (^\perp \xi_{\overline{x}}, -\xi_{v_3}) \otimes (^\perp \xi_{\overline{x}}, -\xi_{v_3}) \\
&= \left[\left(\frac{(^\perp z, 0)}{|z|}, 0\right)^{\otimes 2} + \left(\frac{v_3 - v_3'}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \frac{(z,0)}{|z|}, -\frac{|z| e_3}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right)^{\otimes 2}\right] : \xi \otimes \xi
\end{aligned}
$$

and thus

$$B^+ = \left(\frac{(^\perp z, 0)}{|z|}, 0\right)^{\otimes 2} + \left(\frac{v_3 - v_3'}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \frac{(z,0)}{|z|}, -\frac{|z| e_3}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right)^{\otimes 2} =: B_1^+ + B_2^+. \tag{B.4}$$

Observe that

$$A_1^+ + A_3^+ + A_4^+ + B_1^+ = \frac{(r')^2 \sin^2 \varphi \, (v_3 - v_3')^2}{|z|^2 [\, |z|^2 + (v_3 - v_3')^2 \,]} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}\right)^{\otimes 2}$$
$$+ \left[\frac{r - r' \cos\varphi}{|z|} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}\right) + \left(\frac{(^\perp z, 0)}{|z|}, 0\right)\right]^{\otimes 2} \tag{B.5}$$

since $(\frac{r - r'\cos\varphi}{|z|})^2 = 1 - \frac{(r')^2 \sin^2 \varphi}{|z|^2}$ and

$$A_2^+ + A_5^+ + A_6^+ + B_2^+ = \frac{(r')^2 \sin^2 \varphi}{|z|^2} \left(\frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right)^{\otimes 2} +$$
$$\left[\frac{(r - r'\cos\varphi)(v_3 - v_3')}{|z| \sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) - \frac{\left((v_3 - v_3') \frac{(z,0)}{|z|}, -|z| e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right]^{\otimes 2}. \tag{B.6}$$

Our conclusion follows by combining (B.3), (B.4), (B.5), (B.6). □

Proof. (of Proposition 4.9) We consider

$$\xi_v(x, v) = \int_{\mathbb{R}^3} \sigma(|v - v'|) S(v - v') f(x, v) \nabla_{v'} f(x, v') \, \mathrm{d}v'.$$

By Proposition 3.3 we have

$$\left\langle Q^-_{FPL}(f, f)\right\rangle = \langle \mathrm{div}_v \, \xi_v \rangle = \frac{1}{\omega_c} \mathrm{div}_{\overline{x}} \left\{\langle ^\perp \xi_{\overline{v}} \rangle + \left\langle \xi_{\overline{v}} \cdot \frac{^\perp \overline{v}}{|\overline{v}|}\right\rangle \frac{\overline{v}}{|\overline{v}|} - \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|}\right\rangle \frac{^\perp \overline{v}}{|\overline{v}|}\right\}$$
$$+ \mathrm{div}_{\overline{v}} \left\{\left\langle \xi_{\overline{v}} \cdot \frac{^\perp \overline{v}}{|\overline{v}|}\right\rangle \frac{^\perp \overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|}\right\rangle \frac{\overline{v}}{|\overline{v}|}\right\} + \partial_{v_3} \langle \xi_{v_3} \rangle$$
$$= \mathrm{div}_{\omega_c x} \left\{\langle E \xi_v \rangle + \left\langle \xi_v \cdot \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}\right\rangle \frac{(\overline{v}, 0)}{|\overline{v}|} - \left\langle \xi_v \cdot \frac{(\overline{v}, 0)}{|\overline{v}|}\right\rangle \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}\right\}$$
$$+ \mathrm{div}_v \left\{\left\langle \xi_v \cdot \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|}\right\rangle \frac{(\,^\perp \overline{v}, 0)}{|\overline{v}|} + \left\langle \xi_v \cdot \frac{(\overline{v}, 0)}{|\overline{v}|}\right\rangle \frac{(\overline{v}, 0)}{|\overline{v}|} + \langle e_3 \otimes e_3 \xi_v \rangle\right\}.$$

As in the proof of Proposition 4.8 we obtain

$$\nabla_{v'} f(x, v') = b^0 \cdot \nabla_{x,v'} f \, \frac{(\,^\perp \overline{v'}, 0)}{\omega_c |\overline{v'}|^2} + \left(-\frac{^\perp \nabla_{\overline{x}} f}{\omega_c}, \partial_{v_3'} f\right) + b^4 \cdot \nabla_{x,v'} f \, \frac{(\overline{v'}, 0)}{|\overline{v'}|}$$

and therefore

$$\langle \xi_v \rangle = f(x,v) \left\langle \frac{b^0 \cdot \nabla_{x,v'} f}{\omega_c |\overline{v'}|^2}, ({}^\perp \overline{v'}, 0) \right\rangle_{\sigma S} - f(x,v) \left\langle \frac{\partial_{x_2} f}{\omega_c}, e_1 \right\rangle_{\sigma S} + f(x,v) \left\langle \frac{\partial_{x_1} f}{\omega_c}, e_2 \right\rangle_{\sigma S}$$

$$+ f(x,v) \left\langle \partial_{v'_3} f, e_3 \right\rangle_{\sigma S} + f(x,v) \left\langle \frac{b^4 \cdot \nabla_{x,v'} f}{|\overline{v'}|}, (\overline{v'}, 0) \right\rangle_{\sigma S}$$

$$= -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x,v) \chi \frac{r' - r\cos\varphi}{|z|} \frac{(z,0)}{|z|} \otimes \left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp \overline{v'}, 0)}{|\overline{v'}|} \right) \nabla_{\omega_c x, v'} f \, \mathrm{d}v' \mathrm{d}x'_1 \mathrm{d}x'_2$$

$$- \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x,v) \chi \frac{(r' - r\cos\varphi)(v_3 - v'_3)}{|z|^2 + (v_3 - v'_3)^2} \left(\frac{v_3 - v'_3}{|z|^2} \, {}^\perp z, 1 \right)$$

$$\otimes \left(\frac{({}^\perp \overline{v'}, 0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|} \right) \nabla_{\omega_c x, v'} f \, \mathrm{d}v' \mathrm{d}x'_1 \mathrm{d}x'_2$$

$$- \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x,v) \chi S(\, ({}^\perp z, v'_3 - v_3)\,)(E, -e_3 \otimes e_3) \nabla_{\omega_c x, v'} f \, \mathrm{d}v' \mathrm{d}x'_1 \mathrm{d}x'_2.$$

Similarly, thanks to the identities

$$\left\langle \frac{b^0 \cdot \nabla_{x,v'} f}{\omega_c |\overline{v'}|^2}, ({}^\perp \overline{v'}, 0), (\overline{v}, 0) \right\rangle_{\sigma S} = \left\langle \frac{b^4 \cdot \nabla_{x,v'} f}{|\overline{v'}|}, (\overline{v'}, 0), ({}^\perp \overline{v}, 0) \right\rangle_{\sigma S} = 0$$

we obtain

$$\left\langle \xi_v \cdot \frac{(\overline{v},0)}{|\overline{v}|} \right\rangle = -f(x,v) \left\langle \frac{\partial_{x_2} f(x,v')}{\omega_c |\overline{v}|}, (\overline{v}, 0) \right\rangle_{\sigma S} \cdot e_1 + f(x,v) \left\langle \frac{\partial_{x_1} f(x,v')}{\omega_c |\overline{v}|}, (\overline{v}, 0) \right\rangle_{\sigma S} \cdot e_2$$

$$+ f(x,v) \left\langle \frac{\partial_{v'_3} f(x,v')}{|\overline{v}|}, (\overline{v}, 0) \right\rangle_{\sigma S} \cdot e_3 + f(x,v) \left\langle \frac{b^4 \cdot \nabla_{x,v'} f(x,v')}{|\overline{v}| \, |\overline{v'}|}, (\overline{v'}, 0), (\overline{v}, 0) \right\rangle_{\sigma S}$$

$$= -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f \chi \frac{(r'\cos\varphi - r)(v_3 - v'_3)}{|z|^2 + (v_3 - v'_3)^2} \left(\frac{v_3 - v'_3}{|z|} \frac{(z,0)}{|z|}, -e_3 \right) \cdot \nabla_{\omega_c x, v'} f \, \mathrm{d}v' \mathrm{d}x'_1 \mathrm{d}x'_2$$

$$- \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f \chi \left(\cos\varphi + \frac{(r - r'\cos\varphi)(r' - r\cos\varphi)}{|z|^2 + (v_3 - v'_3)^2} \right) \left(\frac{({}^\perp \overline{v'}, 0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|} \right)$$

$$\cdot \nabla_{\omega_c x, v'} f \, \mathrm{d}v' \mathrm{d}x'_1 \mathrm{d}x'_2$$

and

$$\left\langle \xi_v \cdot \frac{({}^\perp \overline{v},0)}{|\overline{v}|} \right\rangle = -f(x,v) \left\langle \frac{\partial_{x_2} f(x,v')}{\omega_c |\overline{v}|}, ({}^\perp \overline{v}, 0) \right\rangle_{\sigma S} \cdot e_1 + f(x,v) \left\langle \frac{\partial_{x_1} f(x,v')}{\omega_c |\overline{v}|}, ({}^\perp \overline{v}, 0) \right\rangle_{\sigma S} \cdot e_2$$

$$+ f(x,v) \left\langle \frac{\partial_{v'_3} f(x,v')}{|\overline{v}|}, ({}^\perp \overline{v}, 0) \right\rangle_{\sigma S} \cdot e_3 + f(x,v) \left\langle \frac{b^0 \cdot \nabla_{x,v'} f(x,v')}{\omega_c |\overline{v}| \, |\overline{v'}|^2}, ({}^\perp \overline{v'}, 0), ({}^\perp \overline{v}, 0) \right\rangle_{\sigma S}$$

$$= -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f \chi \frac{r'\cos\varphi - r}{|z|} \left(\frac{({}^\perp z, 0)}{|z|}, 0 \right) \cdot \nabla_{\omega_c x, v'} f \, \mathrm{d}v' \mathrm{d}x'_1 \mathrm{d}x'_2$$

$$+ \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f \chi \left(\cos\varphi - \frac{r r' \sin^2\varphi}{|z|^2 + (v_3 - v'_3)^2} \right) \left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp \overline{v'}, 0)}{|\overline{v'}|} \right) \cdot \nabla_{\omega_c x, v'} f \, \mathrm{d}v' \mathrm{d}x'_1 \mathrm{d}x'_2.$$

Obviously $\left\langle Q^-_{FPL}(f,f)\right\rangle$ has the form in (4.8) with $A^- = \begin{pmatrix} A^-_{xx} & A^-_{xv} \\ A^-_{vx} & A^-_{vv} \end{pmatrix}$ where

$$
\begin{aligned}
(A^-_{xx}, A^-_{xv}) = & -\frac{r' - r\cos\varphi}{|z|}\ \frac{({}^\perp z, 0)}{|z|} \otimes \left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}\right) \\
& + \frac{(r' - r\cos\varphi)(v_3 - v_3')^2}{|z|[\ |z|^2 + (v_3 - v_3')^2\ ]}\ \frac{(z,0)}{|z|} \otimes \left(\frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|}\right) \\
& + \left(\cos\varphi - \frac{rr'\sin^2\varphi}{|z|^2 + (v_3 - v_3')^2}\right) \frac{(\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}\right) \\
& + \frac{r - r'\cos\varphi}{|z|}\ \frac{(\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{({}^\perp z,0)}{|z|}, 0\right) \\
& + \left(\cos\varphi + \frac{(r - r'\cos\varphi)(r' - r\cos\varphi)}{|z|^2 + (v_3 - v_3')^2}\right) \frac{({}^\perp\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|}\right) \\
& - \frac{(r - r'\cos\varphi)(v_3 - v_3')}{|z|^2 + (v_3 - v_3')^2}\ \frac{({}^\perp\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{v_3 - v_3'}{|z|}\ \frac{(z,0)}{|z|}, -e_3\right) \\
& - ES(\ ({}^\perp z, v_3' - v_3)\ )(E, -e_3 \otimes e_3)
\end{aligned} \tag{B.7}
$$

and

$$
\begin{aligned}
(A^-_{vx}, A^-_{vv}) = & \left(\cos\varphi - \frac{rr'\sin^2\varphi}{|z|^2 + (v_3 - v_3')^2}\right) \frac{({}^\perp\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}\right) \\
& + \frac{r - r'\cos\varphi}{|z|}\ \frac{({}^\perp\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{({}^\perp z,0)}{|z|}, 0\right) \\
& - \left(\cos\varphi + \frac{(r - r'\cos\varphi)(r' - r\cos\varphi)}{|z|^2 + (v_3 - v_3')^2}\right) \frac{(\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|}\right) \\
& + \frac{(r - r'\cos\varphi)(v_3 - v_3')}{|z|^2 + (v_3 - v_3')^2}\ \frac{(\overline{v},0)}{|\overline{v}|} \otimes \left(\frac{v_3 - v_3'}{|z|}\ \frac{(z,0)}{|z|}, -e_3\right) \\
& - \frac{(r' - r\cos\varphi)(v_3 - v_3')}{|z|^2 + (v_3 - v_3')^2}\ e_3 \otimes \left(\frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|}\right) \\
& - e_3 \otimes e_3\ S(\ ({}^\perp z, v_3' - v_3)\ )(E, -e_3 \otimes e_3).
\end{aligned} \tag{B.8}
$$

It is easily seen that the matrix A^- writes

$$\begin{aligned}
A^- = &\left(\cos\varphi - \frac{rr'\sin^2\varphi}{|z|^2+(v_3-v_3')^2}\right)\left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{({}^\perp\overline{v},0)}{|\overline{v}|}\right)\otimes\left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}\right)\\
&+\left(\cos\varphi + \frac{(r-r'\cos\varphi)(r'-r\cos\varphi)}{|z|^2+(v_3-v_3')^2}\right)\left(\frac{({}^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right)\otimes\left(\frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|}\right)\\
&-\frac{r'-r\cos\varphi}{|z|}\left(\frac{({}^\perp z,0)}{|z|},0\right)\otimes\left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}\right)\\
&+\frac{r-r'\cos\varphi}{|z|}\left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{({}^\perp\overline{v},0)}{|\overline{v}|}\right)\otimes\left(\frac{({}^\perp z,0)}{|z|},0\right)\\
&+\frac{(r'-r\cos\varphi)(v_3-v_3')}{|z|\sqrt{|z|^2+(v_3-v_3')^2}}\frac{\left((v_3-v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2+(v_3-v_3')^2}}\otimes\left(\frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|}\right)\\
&-\frac{(r-r'\cos\varphi)(v_3-v_3')}{|z|\sqrt{|z|^2+(v_3-v_3')^2}}\left(\frac{({}^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right)\otimes\frac{\left((v_3-v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2+(v_3-v_3')^2}}\\
&+B^- = A_1^- + A_2^- + A_3^- + A_4^- + A_5^- + A_6^- + B^-
\end{aligned} \tag{B.9}$$

where, cf. (B.4)

$$B^- = \begin{pmatrix} {}^tES(\ ({}^\perp z, v_3'-v_3)\ )E & ES(\ ({}^\perp z, v_3'-v_3)\ )e_3\otimes e_3 \\ -e_3\otimes e_3 S(\ ({}^\perp z, v_3'-v_3)\ )E & e_3\otimes e_3 S(\ ({}^\perp z, v_3'-v_3)\ )e_3\otimes e_3 \end{pmatrix} \tag{B.10}$$

$$= \left(\frac{({}^\perp z,0)}{|z|},0\right)^{\otimes 2} + \left(\frac{v_3-v_3'}{\sqrt{|z|^2+(v_3-v_3')^2}}\frac{(z,0)}{|z|}, -\frac{|z|e_3}{\sqrt{|z|^2+(v_3-v_3')^2}}\right)^{\otimes 2} =: B_1^- + B_2^-.$$

Observe that

$$\begin{aligned}
A_1^- + A_3^- + A_4^- + B_1^- = &\frac{rr'\sin^2\varphi\,(v_3-v_3')^2}{|z|^2[\ |z|^2+(v_3-v_3')^2\ ]}\left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{({}^\perp\overline{v},0)}{|\overline{v}|}\right)\otimes\left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}\right)\\
&+\left[\frac{r-r'\cos\varphi}{|z|}\left(\frac{(\overline{v},0)}{|\overline{v}|}, \frac{({}^\perp\overline{v},0)}{|\overline{v}|}\right) + \left(\frac{({}^\perp z,0)}{|z|},0\right)\right]\\
&\otimes\left[\frac{r\cos\varphi - r'}{|z|}\left(\frac{(\overline{v'},0)}{|\overline{v'}|}, \frac{({}^\perp\overline{v'},0)}{|\overline{v'}|}\right) + \left(\frac{({}^\perp z,0)}{|z|},0\right)\right]
\end{aligned} \tag{B.11}$$

since

$$\frac{rr'\sin^2\varphi\,(v_3-v_3')^2}{|z|^2[\ |z|^2+(v_3-v_3')^2\ ]} + \frac{(r-r'\cos\varphi)(r\cos\varphi-r')}{|z|^2} = \cos\varphi - \frac{rr'\sin^2\varphi}{|z|^2+(v_3-v_3')^2}$$

and

$$A_2^- + A_5^- + A_6^- + B_2^- = \frac{rr'\sin^2\varphi}{|z|^2}\left(\frac{(\,^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right) \otimes \left(\frac{(\,^\perp\overline{v'},0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|}\right)$$

$$+ \left[\frac{(r - r'\cos\varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\left(\frac{(\,^\perp\overline{v},0)}{|\overline{v}|}, -\frac{(\overline{v},0)}{|\overline{v}|}\right) - \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right]$$

$$\otimes \left[\frac{(r\cos\varphi - r')(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\left(\frac{(\,^\perp\overline{v'},0)}{|\overline{v'}|}, -\frac{(\overline{v'},0)}{|\overline{v'}|}\right) - \frac{\left((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3\right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\right] \tag{B.12}$$

since

$$\frac{rr'\sin^2\varphi}{|z|^2} + \frac{(r'\cos\varphi - r)(r' - r\cos\varphi)(v_3 - v_3')^2}{|z|^2\,[\,|z|^2 + (v_3 - v_3')^2\,]} = \cos\varphi + \frac{(r - r'\cos\varphi)(r' - r\cos\varphi)}{|z|^2 + (v_3 - v_3')^2}$$

Our conclusion follows by combining (B.9), (B.10), (B.11), (B.12). □

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