Low field regime for the relativistic Vlasov-Maxwell-Fokker-Planck system; the one and one half dimensional case

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Abstract

We study the asymptotic regime for the relativistic Vlasov-Maxwell-Fokker-Planck system which corresponds to a small mean free path compared to the Debye length, chosen as an observation length scale, combined to a large thermal velocity assumption. We are led to a convection-diffusion equation. The analysis is performed in the one and one half dimensional case.

Keywords: Vlasov-Maxwell-Fokker-Planck system, Asymptotic behavior, Diffusion approximation.

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1 Introduction

We consider a population of charged particles interacting both through collisions and the action of their self-consistent electro-magnetic field. The evolution of such a system is governed by the relativistic Vlasov-Maxwell-Fokker-Planck (VMFP) equations. After a dimensional analysis (see the Appendix) we obtain the following equations

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v(p)\cdot\nabla_x f^\varepsilon + \left(\frac{1}{\varepsilon}E^\varepsilon(t,x) + \delta^2 v(p)\wedge B^\varepsilon(t,x)\right)\cdot\nabla_p f^\varepsilon = \frac{\theta}{\varepsilon^2}L(f^\varepsilon)$$
$$:= \frac{\theta}{\varepsilon^2}\mathrm{div}_p(\nabla_p f^\varepsilon + v(p) f^\varepsilon),\quad (t,x,p)\in]0,T[\times\mathbb{R}^3\times\mathbb{R}^3, \quad (1)$$

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$$\partial_t E^\varepsilon - \mathrm{curl}_x B^\varepsilon = -\frac{j^\varepsilon}{\varepsilon} + J, \;\; \varepsilon^2\delta^2\partial_t B^\varepsilon + \mathrm{curl}_x E^\varepsilon = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3, \tag{2}$$

$$\mathrm{div}_x E^\varepsilon = \rho^\varepsilon - D, \;\; \mathrm{div}_x B^\varepsilon = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3, \tag{3}$$

where $\varepsilon, \delta, \theta$ are dimensionless parameters and $v(p) = \nabla_p \; \mathcal{E}(p)$ is the scaled relativistic velocity (see (81)). Here $\rho^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon$ and $j^\varepsilon = \int_{\mathbb{R}^3} v(p) f^\varepsilon$ are respectively the charge and current densities of the distribution f^ε and D, J are the charge and current densities of a background particle distribution of opposite sign, ensuring the global neutrality condition

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^\varepsilon(t,x,p)\,dp\,dx = \int_{\mathbb{R}^3} D(t,x)\,dx, \;\; t \in [0,T], \; \varepsilon > 0.$$

The dimensionless parameter $\varepsilon > 0$ is proportional with the scaled thermal mean free path and also with the scaled macroscopic velocity. We are interested in the asymptotic regime $0 < \varepsilon << 1$, $\delta = \mathcal{O}(1), \theta = \mathcal{O}(1)$.
By neglecting the magnetic field we obtain the Vlasov-Poisson-Fokker-Planck (VPFP) system

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v(p)\cdot\nabla_x f^\varepsilon - \frac{1}{\varepsilon}\nabla_x \Phi^\varepsilon \cdot \nabla_p f^\varepsilon = \frac{\theta}{\varepsilon^2}\mathrm{div}_p(\nabla_p f^\varepsilon + v(p) f^\varepsilon), \tag{4}$$

$$-\Delta_x \Phi^\varepsilon = \rho^\varepsilon - D. \tag{5}$$

The asymptotic behavior of the non relativistic system (4), (5) when ε goes to 0 was studied in [31], [23]. It was shown that the limit $(\rho, \Phi) := \lim_{\varepsilon \searrow 0}(\rho^\varepsilon, \Phi^\varepsilon)$ solves the following drift-diffusion system

$$\partial_t \rho - \frac{1}{\theta}\mathrm{div}_x(\nabla_x \rho + \rho\nabla_x \Phi) = 0, \;\; -\Delta_x \Phi = \rho(t,x) - D(t,x), \;\; (t,x) \in]0,T[\times\mathbb{R}^3. \tag{6}$$

Another interesting regime is obtained by taking as small parameter ε the square of the ratio of the thermal mean free path with respect to the Debye length and by assuming that the distance travelled by the light during the relaxation time is of order of the Debye length. In this case we obtain the equations

$$\partial_t f^\varepsilon + v(p)\cdot\nabla_x f^\varepsilon + \left(\frac{1}{\varepsilon}E^\varepsilon(t,x) + v(p)\wedge B^\varepsilon(t,x)\right)\cdot\nabla_p f^\varepsilon = \frac{1}{\varepsilon}\mathrm{div}_p(v(p) f^\varepsilon + \nabla_p f^\varepsilon), \tag{7}$$

$$\partial_t E^\varepsilon - \mathrm{curl}_x B^\varepsilon = J(t,x) - j^\varepsilon(t,x), \;\; \varepsilon\partial_t B^\varepsilon + \mathrm{curl}_x E^\varepsilon = 0, (t,x) \in]0,T[\times\mathbb{R}^3, \tag{8}$$

$$\mathrm{div}_x E^\varepsilon = \rho^\varepsilon(t,x) - D(t,x), \;\; \mathrm{div}_x B^\varepsilon = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3. \tag{9}$$

Notice that in (7) the non linear term $E^\varepsilon \cdot \nabla_p f^\varepsilon$ is of the same order of magnitude that the diffusion Fokker-Planck term. This asymptotic regime is called the high-electric field limit and the non relativistic case was studied recently in [7]. The following limit system was obtained

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho E) = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3, \\ \mathrm{div}_x E = \rho(t,x) - D(t,x), \;\; \mathrm{curl}_x E = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3, \\ \partial_t E - \mathrm{curl}_x B = J(t,x) - \rho(t,x)E(t,x), \;\; \mathrm{div}_x B = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3. \end{cases} \tag{10}$$

The high-field limit of the VPFP system was studied in [28], [22].

We analyze here the parabolic limit of the relativistic one and one half dimensional VMFP system, *i.e.*, $f = f(t, x, p_1, p_2), \mathbf{E} = (E_1(t,x), E_2(t,x), 0), \mathbf{B} = (0, 0, B(t,x))$ for any $(t, x, p_1, p_2) \in [0,T] \times \mathbb{R}^3$. We derive a limit system very similar to (6), which was obtained when analyzing the VPFP system. Our proofs rely on compactness arguments. One of the crucial point is to obtain L^∞ bounds for the electro-magnetic field, uniformly with respect to the small parameter $\varepsilon > 0$. This is why we restrict our analysis to solutions depending on only one spatial variable. We obtain the equations

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v_1(p) \partial_x f^\varepsilon + \left(\frac{1}{\varepsilon} E_1^\varepsilon + \delta^2 v_2(p) B^\varepsilon \right) \partial_{p_1} f^\varepsilon + \left(\frac{1}{\varepsilon} E_2^\varepsilon - \delta^2 v_1(p) B^\varepsilon \right) \partial_{p_2} f^\varepsilon$$
$$= \frac{\theta}{\varepsilon^2} \mathrm{div}_p (\nabla_p f^\varepsilon + v(p) f^\varepsilon), \quad (t,x,p) \in]0,T[\times \mathbb{R} \times \mathbb{R}^2, \qquad (11)$$

$$\partial_t E_1^\varepsilon = -\frac{1}{\varepsilon} j_1^\varepsilon(t,x) + J(t,x), \quad (t,x) \in]0,T[\times\mathbb{R}, \qquad (12)$$

$$\partial_t E_2^\varepsilon + \partial_x B^\varepsilon = -\frac{1}{\varepsilon} j_2^\varepsilon(t,x), \quad (t,x) \in]0,T[\times\mathbb{R}, \qquad (13)$$

$$\varepsilon^2 \delta^2 \partial_t B^\varepsilon + \partial_x E_2^\varepsilon = 0, \quad (t,x) \in]0,T[\times\mathbb{R}, \qquad (14)$$

$$\partial_x E_1^\varepsilon = \rho^\varepsilon(t,x) - D(t,x), \quad (t,x) \in [0,T]\times\mathbb{R}, \qquad (15)$$

where $D, J : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given functions satisfying $D \geq 0$ and the continuity equation

$$\partial_t D + \partial_x J = 0, \quad (t,x) \in]0,T[\times\mathbb{R}.$$

We prescribe initial conditions for the particle distribution and the electro-magnetic field

$$f^\varepsilon(0,x,p) = f_0^\varepsilon(x,p), \quad (x,p) \in \mathbb{R} \times \mathbb{R}^2, \qquad (16)$$

$$E^\varepsilon(0,x) = E_0^\varepsilon(x), \quad B^\varepsilon(0,x) = B_0^\varepsilon(x), \quad x \in \mathbb{R}, \qquad (17)$$

satisfying

$$\frac{d}{dx} E_{0,1}^\varepsilon = \int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\, dp - D(0,x), \quad x \in \mathbb{R}. \qquad (18)$$

After integration of (11) with respect to $p \in \mathbb{R}^2$ we deduce that the charge and the current densities verify the continuity equation

$$\partial_t \rho^\varepsilon + \frac{1}{\varepsilon} \partial_x j_1^\varepsilon = 0, \quad (t,x) \in]0,T[\times\mathbb{R}.$$

By using the continuity equations for positive/negative charges and by taking the derivative of (12) with respect to x we deduce that (15) is a consequence of (18). Notice that if initially the neutrality condition is satisfied *i.e.*, $\int_{\mathbb{R}} \int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\, dp\, dx = \int_{\mathbb{R}} D(0,x)\, dx$, then we have $\int_{\mathbb{R}} \int_{\mathbb{R}^2} f^\varepsilon(t,x,p)\, dp\, dx = \int_{\mathbb{R}} D(t,x)\, dx$ for any $t \in]0,T]$. We consider only smooth solutions. Unfortunately, to our knowledge, there are no mathematical results concerning the existence and uniqueness of strong solution for the VMFP system. For the VPFP system the situation is better : results concerning the existence of weak solutions can be found in [13], [34] while for existence and uniqueness results of strong solution we refer to [8], [9], [17], [29]. The existence of classical solutions in the collisionless case has been investigated by different approaches, see [20], [10], [25]. Recently global existence and uniqueness results have been obtained for reduced model for laser-plasma interaction, cf. [14], [6].

The analysis of such asymptotic regimes is motivated by applications in the theory of semiconductors, the evolution of laser-produced plasmas or description of tokamaks. High-field asymptotics for the kinetic theory of semiconductors have been analyzed in [30], [15]. Results for different physical models have been obtained in [1], [4], [18], [27]. Generally we appeal to usual compactness methods. Another approach uses the modulated energy method, as introduced in [35]. This method has been used for studying various asymptotic problems in plasma physics [11], [12], [21], [33], [5], [24].

The paper is organized as follows. In Section 2 we establish a priori estimates, uniformly with respect to the small parameter $\varepsilon > 0$. These bounds are obtained by performing classical computations involving the energy and the entropy of the VMFP system and by using also the hyperbolic structure of the Maxwell equations. In Section 3 we detail the passage to the limit. The dimensional analysis can be found in the Appendix.

2 A priori estimates

In this section we establish a priori estimates for smooth solutions $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ of the relativistic VMFP system in one and one half dimension. We will use the hypotheses

H1) $f_0^\varepsilon \geq 0,\;\; D \geq 0,\;\; \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\,dp\,dx = \int_{\mathbb{R}} D(0,x)\,dx,\; \forall\, \varepsilon > 0$;

H2) $\sup_{\varepsilon>0}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^2}(1 + |\ln f_0^\varepsilon| + |x| + \mathcal{E}(p))f_0^\varepsilon dp\,dx + \frac{1}{2}\int_{\mathbb{R}}(|E_0^\varepsilon|^2 + \varepsilon^2\delta^2|B_0^\varepsilon|^2)\,dx\right) < \infty$;

H3) D, J are given smooth functions satisfying $\partial_t D + \partial_x J = 0,\;\; (t,x) \in]0,T[\times\mathbb{R}$;

H4) $J \in L^1(]0,T[; L^2(\mathbb{R}))$;

H5) $\sup_{\varepsilon>0}\left(\|E_0^\varepsilon\|_{L^\infty(\mathbb{R})} + \varepsilon\delta\|B_0^\varepsilon\|_{L^\infty(\mathbb{R})}\right) < +\infty$;

H6) $J \in L^1(]0,T[; L^\infty(\mathbb{R}))$;

H7) there is $r > 1$ such that $\sup_{\varepsilon>0}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f_0^\varepsilon(x,p))^r e^{(r-1)\mathcal{E}(p)}\,dp\,dx < +\infty$.

We introduce the notations

$$M_0^\varepsilon := \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\;dp\,dx,$$

$$W_0^\varepsilon := \int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{E}(p) f_0^\varepsilon(x,p)\;dp\,dx + \frac{1}{2}\int_{\mathbb{R}}\{|E_0^\varepsilon(x)|^2 + \varepsilon^2\delta^2|B_0^\varepsilon(x)|^2\}\;dx,$$

$$L_0^\varepsilon := \int_{\mathbb{R}}\int_{\mathbb{R}^2} |x| f_0^\varepsilon(x,p)\;dp\,dx,$$

$$R_0^\varepsilon := \|E_0^\varepsilon\|_{L^\infty(\mathbb{R})} + \varepsilon\delta\|B_0^\varepsilon\|_{L^\infty(\mathbb{R})}.$$

The following proposition states the usual bounds for the mass, energy and entropy (see Lemma 2.1 for the definition of the constant $C_{1/4}$).

Proposition 2.1 *Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ be a smooth solution of the problem* (11) – (17). *Assume that the initial conditions satisfy H1, H2 and that H3, H4 hold. Then we have for any $t \in [0,T]$*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon(t,x,p)\;dp\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\;dp\,dx < +\infty,$$

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^2} (|\ln f^\varepsilon| + |x| + \mathcal{E}(p)) f^\varepsilon(t,x,p)\;dp\,dx \;&+\; \frac{1}{2}\int_{\mathbb{R}}(|E^\varepsilon(t,x)|^2 + \varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\;dx \\ &\leq\; 4\left(\frac{T}{2\theta}M_0^\varepsilon + L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon + C_{1/4}\right) \\ &+\; 2\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))},\end{aligned}$$

$$\begin{aligned}\theta\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} |h^\varepsilon(t,x,p)|^2\;dp\,dx\,dt \;&\leq\; 4\left(\frac{T}{2\theta}M_0^\varepsilon + L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon + C_{1/4}\right) \\ &+\; 2\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))},\end{aligned}$$

$$\left\|\frac{j^\varepsilon}{\varepsilon}\right\|^2_{L^2(]0,T[;L^1(\mathbb{R}))} \leq M_0^\varepsilon\;\|h^\varepsilon\|^2_{L^2(]0,T[\times\mathbb{R}\times\mathbb{R}^2)}. \tag{19}$$

The above estimates come by standard computations involving the energy conservation and the entropy dissipation. We use the following lemma, based on classical arguments due to Carleman.

Lemma 2.1 *Assume that* $f = f(x,p)$ *satisfies* $f \geq 0$, $(|x| + \mathcal{E}(p) + |\ln f|)f \in L^1(\mathbb{R} \times \mathbb{R}^2)$, *where* $\mathcal{E}(\cdot)$ *is the scaled relativistic energy given by* (81). *Then for all* $k > 0$ *we have*

$$f|\ln f| \leq f \ln f + 2k(|x| + \mathcal{E}(p))f + 2Ce^{-\frac{k}{2}(|x|+\mathcal{E}(p))}, \quad \text{with } C = \sup_{0<y<1}\{-\sqrt{y}\ln y\},$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} f|\ln f| \; dp\, dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}^2} f \ln f \; dp\, dx + 2k \int_{\mathbb{R}}\int_{\mathbb{R}^2} (|x| + \mathcal{E}(p))f \; dp\, dx + C_k,$$

with $C_k = 2C \int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{-\frac{k}{2}(|x|+\mathcal{E}(p))} \; dp\, dx$.

Proof. Since $f|\ln f| = f \ln f + 2f(\ln f)_-$, it is sufficient to estimate $f(\ln f)_-$. Take $k > 0$ and let $C = \sup_{0<y<1}\{-\sqrt{y}\ln y\} < +\infty$. We have

$$\begin{aligned} f(\ln f)_- &= -f \ln f \cdot \mathbf{1}_{\{0<f<e^{-k(|x|+\mathcal{E}(p))}\}} - f \ln f \cdot \mathbf{1}_{\{e^{-k(|x|+\mathcal{E}(p))} \leq f < 1\}} \\ &\leq Ce^{-\frac{k}{2}(|x|+\mathcal{E}(p))} + k(|x| + \mathcal{E}(p))f, \;\; \forall\, (x,p) \in \mathbb{R} \times \mathbb{R}^2. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(\ln f)_- \; dp\, dx \leq k \int_{\mathbb{R}}\int_{\mathbb{R}^2} (|x| + \mathcal{E}(p))f \; dp\, dx + C \int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{-\frac{k}{2}(|x|+\mathcal{E}(p))} \; dp\, dx,$$

and the conclusion follows easily. $\square$

Proof. (of Proposition 2.1)
Integrating (11) with respect to $(x,p) \in \mathbb{R} \times \mathbb{R}^2$ yields the charge conservation

$$\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon(t,x,p) \; dp\, dx = 0, \;\; t \in]0,T[,$$

which implies that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon(t,x,p) \; dp\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p) \; dp\, dx = M_0^\varepsilon, \;\; t \in [0,T].$$

Similarly by H3 one gets $\int_{\mathbb{R}} D(t,x) \; dx = \int_{\mathbb{R}} D(0,x) \; dx, \; t \in [0,T]$. We multiply now the Vlasov equation by $(1+\ln f^\varepsilon+\mathcal{E}(p))$ and integrate with respect to $(x,p) \in \mathbb{R}\times\mathbb{R}^2$. We obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2} (\ln f^\varepsilon + \mathcal{E}(p))f^\varepsilon(t,x,p) \; dp\, dx - \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2} v(p) \cdot E^\varepsilon(t,x) f^\varepsilon(t,x,p) \; dp\, dx \\ = -\frac{\theta}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \frac{|\nabla_p f^\varepsilon + v(p) f^\varepsilon|^2}{f^\varepsilon} \; dp\, dx. \quad (20) \end{aligned}$$

Multiplying (12) by E_1^ε, (13) by E_2^ε and (14) by B^ε yields after integration with respect to $x \in \mathbb{R}$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(|E^\varepsilon(t,x)|^2+\varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\;dx=-\frac{1}{\varepsilon}\int_{\mathbb{R}}(j^\varepsilon(t,x)\cdot E^\varepsilon(t,x)+JE_1^\varepsilon(t,x))\;dx. \tag{21}$$

By combining (20), (21) we deduce

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(\ln f^\varepsilon+\mathcal{E}(p))f^\varepsilon\;dp\,dx &+ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(|E^\varepsilon(t,x)|^2+\varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\;dx\\ &= -\theta\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\;dp\,dx+\int_{\mathbb{R}}JE_1^\varepsilon\;dx,\end{aligned} \tag{22}$$

where $h^\varepsilon(t,x,p)=\frac{1}{\varepsilon}(v(p)\sqrt{f^\varepsilon}+2\nabla_p\sqrt{f^\varepsilon}),\ (t,x,p)\in[0,T]\times\mathbb{R}\times\mathbb{R}^2$. In order to apply Lemma 2.1 let us multiply the Vlasov equation by $|x|$ and integrate with respect to $(x,p)\in\mathbb{R}\times\mathbb{R}^2$. We deduce that

$$\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2}|x|f^\varepsilon(t,x,p)\;dp\,dx-\frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\frac{x}{|x|}v_1(p)f^\varepsilon(t,x,p)\;dp\,dx=0,\quad t\in]0,T[, \tag{23}$$

implying that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^2}|x|f^\varepsilon\;dp\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}^2}|x|f_0^\varepsilon\;dp\,dx+\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}\frac{x}{|x|}(v_1(p)f^\varepsilon+\partial_{p_1}f^\varepsilon)\;dp\,dx\,ds\\ &\leq L_0^\varepsilon+\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}\sqrt{f^\varepsilon(s,x,p)}\;\;|h_1^\varepsilon(s,x,p)|\;dp\,dx\,ds\\ &\leq L_0^\varepsilon+\left(\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}f^\varepsilon\;dp\,dx\,ds\right)^{1/2}\left(\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\;dp\,dx\,ds\right)^{1/2}\\ &\leq L_0^\varepsilon+\frac{t}{2\theta}M_0^\varepsilon+\frac{\theta}{2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\;dp\,dx\,ds,\quad t\in[0,T].\end{aligned} \tag{24}$$

Combining (22), (24) and Lemma 2.1 with $k=\frac{1}{4}$ yields for any $t\in[0,T]$

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}^2}(|\ln f^\varepsilon|+|x|+\mathcal{E}(p))f^\varepsilon(t,x,p)\;dp\,dx+\frac{1}{2}\int_{\mathbb{R}}(|E^\varepsilon(t,x)|^2+\varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\;dx\\ &\quad+\ \theta\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\;dp\,dx\,ds\\ &\quad\leq\ \int_{\mathbb{R}}\int_{\mathbb{R}^2}(\ln f_0^\varepsilon+|x|+\mathcal{E}(p))f_0^\varepsilon\;dp\,dx+\frac{1}{2}\int_{\mathbb{R}}(|E_0^\varepsilon|^2+\varepsilon^2\delta^2|B_0^\varepsilon|^2)\;dx\\ &\quad+\ \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(|x|+\mathcal{E}(p))f^\varepsilon(t,x,p)\;dp\,dx+\frac{\theta}{2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}|h^\varepsilon|^2\;dp\,dx\,ds\\ &\quad+\ \frac{t}{2\theta}M_0^\varepsilon+C_{1/4}+\int_0^t\|J(s)\|_{L^2(\mathbb{R})}\|E^\varepsilon(s)\|_{L^2(\mathbb{R})}ds,\end{aligned} \tag{25}$$

which implies for any $t \in [0,T]$

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} (|\ln f^\varepsilon| + \frac{1}{2}(|x| + \mathcal{E}(p)))f^\varepsilon(t,x,p)\, dp\, dx + \frac{1}{2}\int_{\mathbb{R}}(|E^\varepsilon(t,x)|^2 + \varepsilon^2\delta^2|B^\varepsilon(t,x)|^2)\, dx$$
$$+ \quad \frac{\theta}{2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} |h^\varepsilon|^2\, dp\, dx\, ds \tag{26}$$
$$\leq \quad \frac{T}{2\theta}M_0^\varepsilon + L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon + C_{1/4} + \int_0^t \|J(s)\|_{L^2(\mathbb{R})}\|E^\varepsilon(s)\|_{L^2(\mathbb{R})}ds,$$

and the first two statements follow easily by using Bellman lemma. For the last one write

$$\int_0^T \left\|\frac{j^\varepsilon(t)}{\varepsilon}\right\|^2_{L^1(\mathbb{R})} dt \quad = \quad \int_0^T \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}^2} \sqrt{f^\varepsilon(t,x,p)}h^\varepsilon(t,x,p)\, dp\right|\, dx\right)^2 dt,$$

and we apply the Cauchy-Schwartz inequality. □

Since we intend to use compactness arguments we need to estimate $\int_{\mathbb{R}} \rho^\varepsilon |\ln \rho^\varepsilon|\, dx$. This can be done by using the standard result

Lemma 2.2 *Assume that f is a non negative function satisfying*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} (1 + |\ln f(x,p)| + |x| + \mathcal{E}(p))f(x,p)\, dp\, dx < +\infty,$$

and denote by $\rho(x) = \int_{\mathbb{R}^2} f(x,p)\, dp,\ x \in \mathbb{R}$. Then we have

$$\int_{\mathbb{R}} |\ln \rho(x)|\, \rho(x)\, dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}^2} (\ln K + |\ln f| + |x| + \mathcal{E}(p))f(x,p)\, dp\, dx + 2C\int_{\mathbb{R}} e^{-\frac{|x|}{4}}\, dx,$$

where $C = \sup\{-\sqrt{y}\ln y\ :\ 0 < y < 1\}$ and $K = \int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\, dp$.

Proof. Consider the convex function $\varphi : [0,+\infty[\to \mathbb{R}$, $\varphi(s) = s\ln s, s > 0$, $\varphi(0) = 0$ and the measure $d\nu = \frac{e^{-\mathcal{E}(p)}}{K}dp$. By applying Jensen inequality

$$\varphi\left(\int_{\mathbb{R}^2} g(p)d\nu\right) \leq \int_{\mathbb{R}^2} \varphi(g(p))d\nu,$$

with the function $g(\cdot) = Kf(x,\cdot)e^{\mathcal{E}(\cdot)}$ one gets

$$\rho(x)\ln\rho(x) \leq \int_{\mathbb{R}^2} (\ln K + \ln f(x,p) + \mathcal{E}(p))f(x,p)\, dp.$$

As in the proof of Lemma 2.1 we have

$$\rho(x)|\ln\rho(x)| \leq \rho(x)\ln\rho(x) + 2k|x|\rho(x) + 2Ce^{-\frac{k|x|}{2}},$$

with $C = \sup\{-\sqrt{y}\ln y\ :\ 0 < y < 1\}$ and therefore, by taking $k = 1/2$ we deduce

$$\int_{\mathbb{R}} \rho(x)|\ln\rho(x)|\, dx \quad \leq \quad \int_{\mathbb{R}} (\ln\rho(x) + |x|)\rho(x)\, dx + 2C\int_{\mathbb{R}} e^{-\frac{|x|}{4}}\, dx$$
$$\leq \quad \int_{\mathbb{R}}\int_{\mathbb{R}^2} (\ln K + |\ln f| + |x| + \mathcal{E}(p))f\, dp\, dx + 2C\int_{\mathbb{R}} e^{-\frac{|x|}{4}}\, dx.$$

□

Corollary 2.1 *Under the hypotheses of Proposition* 2.1 *we have for any* $t \in [0,T]$

$$\int_{\mathbb{R}} \rho^\varepsilon(t,x)|\ln \rho^\varepsilon|\, dx \le C_T(1 + M_0^\varepsilon + L_0^\varepsilon + W_0^\varepsilon + H_0^\varepsilon + \|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}),$$

for some constant C_T *depending on* T *but not on* ε.

Another way of estimating the solutions of the Fokker-Planck equation comes by multiplication with $H'(fe^{\mathcal{E}(p)})$, where H is a convex function, cf. [31].

Proposition 2.2 *Assume that* $E^\varepsilon, B^\varepsilon$ *are bounded smooth functions and that* f^ε *is a smooth solution of* (11), (16) *with a non negative initial condition* f_0^ε *satisfying*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f_0^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx < +\infty,$$

for some convex non negative function H. *Then we have for any* $t \in [0,T]$

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f^\varepsilon(t)e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx \;&+\; \frac{\theta}{2\varepsilon^2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})|\nabla_p f^\varepsilon + vf^\varepsilon|^2\, dp\, dx\, ds\\
&\le \int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f_0^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx\\
&+ \frac{1}{2\theta}(\|E_1^\varepsilon\|_{L^\infty} + \|E_2^\varepsilon\|_{L^\infty} + 2\varepsilon\delta\|B^\varepsilon\|_{L^\infty})^2\\
&\times \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f^\varepsilon)^2 e^{\mathcal{E}(p)}H''(f^\varepsilon(s)e^{\mathcal{E}(p)})\, dp\, dx\, ds.
\end{aligned}$$

Proof. After multiplication of (11) by $H'(f^\varepsilon e^{\mathcal{E}(p)})$ we obtain

$$\begin{aligned}
\partial_t H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)} &+ \frac{v_1}{\varepsilon}\partial_x H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)} + \partial_{p_1}\left\{\left(\frac{E_1^\varepsilon}{\varepsilon} + \delta^2 v_2 B^\varepsilon\right)f^\varepsilon\right\}H'(f^\varepsilon e^{\mathcal{E}(p)})\\
&+ \partial_{p_2}\left\{\left(\frac{E_2^\varepsilon}{\varepsilon} - \delta^2 v_1 B^\varepsilon\right)f^\varepsilon\right\}H'(f^\varepsilon e^{\mathcal{E}(p)})\\
&= \frac{\theta}{\varepsilon^2}\mathrm{div}_p(\nabla_p f^\varepsilon + v(p)f^\varepsilon)H'(f^\varepsilon e^{\mathcal{E}(p)}).
\end{aligned} \tag{27}$$

After integration with respect to $(x,p) \in \mathbb{R}\times\mathbb{R}^2$ one gets

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx &- \int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon H''(f^\varepsilon e^{\mathcal{E}(p)})\left(\frac{E_1^\varepsilon}{\varepsilon} + \delta^2 v_2 B^\varepsilon\right)\partial_{p_1}(e^{\mathcal{E}(p)}f^\varepsilon)\, dp\, dx\\
&- \int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon H''(f^\varepsilon e^{\mathcal{E}(p)})\left(\frac{E_2^\varepsilon}{\varepsilon} - \delta^2 v_1 B^\varepsilon\right)\partial_{p_2}(e^{\mathcal{E}(p)}f^\varepsilon)\, dp\, dx\\
&= -\frac{\theta}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})|\nabla_p f^\varepsilon + vf^\varepsilon|^2\, dp\, dx.
\end{aligned} \tag{28}$$

We introduce the notation $R^\varepsilon(t) = \|E_1^\varepsilon(t)\|_{L^\infty(\mathbb{R})} + \|E_2^\varepsilon(t)\|_{L^\infty(\mathbb{R})} + 2\varepsilon\delta\|B^\varepsilon(t)\|_{L^\infty(\mathbb{R})}$ and

$$q_H^\varepsilon(t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})|\nabla_p f^\varepsilon + v(p)f^\varepsilon|^2\, dp\, dx.$$

By Cauchy-Schwartz inequality and by taking into account that $|v(p)| < 1/\delta$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon H''(f^\varepsilon e^{\mathcal{E}(p)})\left\{\left(\frac{E_1^\varepsilon}{\varepsilon}+\delta^2 v_2 B^\varepsilon\right)\partial_{p_1}(e^{\mathcal{E}(p)}f^\varepsilon)+\left(\frac{E_2^\varepsilon}{\varepsilon}-\delta^2 v_1 B^\varepsilon\right)\partial_{p_2}(e^{\mathcal{E}(p)}f^\varepsilon)\right\}dp\, dx\\
&\quad\le\; R^\varepsilon(t)\frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})|\nabla_p f^\varepsilon + v(p)f^\varepsilon|\, dp\, dx\\
&\quad\le\; R^\varepsilon(t)\,(q_H^\varepsilon(t))^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon)^2 e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})\, dp\, dx\right)^{1/2}.
\end{aligned}\tag{29}$$

Combining (28), (29) yields

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx + \theta\, q_H^\varepsilon(t) &\le R^\varepsilon(t)\,(q_H^\varepsilon(t))^{1/2}\\
&\quad\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon)^2 e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})\, dp\, dx\right)^{1/2}\\
&\le \theta\frac{q_H^\varepsilon(t)}{2} + \frac{R^\varepsilon(t)^2}{2\theta}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon)^2 e^{\mathcal{E}(p)}H''(f^\varepsilon e^{\mathcal{E}(p)})\, dp\, dx.
\end{aligned}\tag{30}$$

Finally one gets for any $t \in [0, T]$

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx + \frac{\theta}{2}\int_0^t q_H^\varepsilon(s)\, ds &\le \int_{\mathbb{R}}\int_{\mathbb{R}^2} H(f_0^\varepsilon e^{\mathcal{E}(p)})e^{-\mathcal{E}(p)}\, dp\, dx\\
&+ \frac{\|R^\varepsilon\|_{L^\infty}^2}{2\theta}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon(s))^2 e^{\mathcal{E}(p)}H''(f^\varepsilon(s)e^{\mathcal{E}(p)})\, dp\, dx\, ds.
\end{aligned}$$

$\square$

Corollary 2.2 *Assume that $E^\varepsilon, B^\varepsilon$ are bounded smooth functions and that f^ε is a smooth solution of* (11), (16) *with a non negative initial condition f_0^ε satisfying*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f_0^\varepsilon)^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx < +\infty,$$

for some $r > 1$. Then for any $t \in [0, T]$ we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon(t))^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx \le e^{C^\varepsilon(t)}\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f_0^\varepsilon)^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx,$$

and

$$\begin{aligned}
\frac{\theta}{\varepsilon^2}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f^\varepsilon)^{r-2}e^{(r-1)\mathcal{E}(p)}|\nabla_p f^\varepsilon + v(p)f^\varepsilon|^2\, dp\, dx\, dt \;&\le\; \frac{2e^{C^\varepsilon(T)}}{r(r-1)}\\
&\times\int_{\mathbb{R}}\int_{\mathbb{R}^2}(f_0^\varepsilon)^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx,
\end{aligned}$$

where $C^\varepsilon(t) = \frac{t\, r(r-1)}{2\theta}(\|E_1^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} + \|E_2^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} + 2\varepsilon\delta\|B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})})^2$.

Proof. By applying the previous proposition with the convex function $H(s) = s^r$, $s \geq 0$ we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f^\varepsilon(t))^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx \;+\; \frac{\theta}{2}\int_0^t q_H^\varepsilon(s)\, ds \leq \int_{\mathbb{R}}\int_{\mathbb{R}^2} (f_0^\varepsilon)^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx$$
$$+\; r(r-1)\frac{\|R^\varepsilon\|_{L^\infty}^2}{2\theta}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f^\varepsilon(s))^r e^{(r-1)\mathcal{E}(p)}\, dp\, dx\, ds.$$

We conclude by Gronwall lemma. □

We are looking now for L^∞ bounds of the electro-magnetic field. We exploit the hyperbolic structure of the Maxwell equations and the entropy dissipation of the Fokker-Planck collision operator. We adapt the method used in [19], where L^∞ bounds of the electro-magnetic field have been obtained for the collisionless relativistic Vlasov-Maxwell system in one and one half dimension. Notice that the Maxwell equations (12), (13), (14) can be written

$$\partial_t E_1^\varepsilon = -\frac{j_1^\varepsilon(t,x)}{\varepsilon} + J(t,x), \;\; (t,x) \in]0,T[\times\mathbb{R}, \tag{31}$$

$$\partial_t(E_2^\varepsilon + \varepsilon\delta B^\varepsilon) + \frac{1}{\varepsilon\delta}\partial_x(E_2^\varepsilon + \varepsilon\delta B^\varepsilon) = -\frac{j_2^\varepsilon(t,x)}{\varepsilon}, \;\; (t,x) \in]0,T[\times\mathbb{R}, \tag{32}$$

$$\partial_t(E_2^\varepsilon - \varepsilon\delta B^\varepsilon) - \frac{1}{\varepsilon\delta}\partial_x(E_2^\varepsilon - \varepsilon\delta B^\varepsilon) = -\frac{j_2^\varepsilon(t,x)}{\varepsilon}, \;\; (t,x) \in]0,T[\times\mathbb{R}. \tag{33}$$

Therefore the electro-magnetic field is given by

$$E_1^\varepsilon(t,x) = E_{0,1}^\varepsilon(x) - U^\varepsilon(t,x) + \int_0^t J(s,x)\, ds, \;\; (t,x) \in [0,T]\times\mathbb{R}, \tag{34}$$

$$\begin{aligned} E_2^\varepsilon(t,x) &= \frac{1}{2}(E_{0,2}^\varepsilon + \varepsilon\delta B_0^\varepsilon)(x - \frac{t}{\varepsilon\delta}) + \frac{1}{2}(E_{0,2}^\varepsilon - \varepsilon\delta B_0^\varepsilon)(x + \frac{t}{\varepsilon\delta}) \\ &- \frac{1}{2}V_+^\varepsilon(t,x) - \frac{1}{2}V_-^\varepsilon(t,x), \;\; (t,x) \in [0,T]\times\mathbb{R}, \end{aligned} \tag{35}$$

$$\begin{aligned} \varepsilon\delta B^\varepsilon(t,x) &= \frac{1}{2}(E_{0,2}^\varepsilon + \varepsilon\delta B_0^\varepsilon)(x - \frac{t}{\varepsilon\delta}) - \frac{1}{2}(E_{0,2}^\varepsilon - \varepsilon\delta B_0^\varepsilon)(x + \frac{t}{\varepsilon\delta}) \\ &- \frac{1}{2}V_+^\varepsilon(t,x) + \frac{1}{2}V_-^\varepsilon(t,x), \;\; (t,x) \in [0,T]\times\mathbb{R}, \end{aligned} \tag{36}$$

where

$$U^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_0^t j_1^\varepsilon(s,x)\, ds, \;\; V_\pm^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_0^t j_2^\varepsilon(s, x \mp \frac{t-s}{\varepsilon\delta})\, ds.$$

We need to find L^∞ bounds for the functions $U^\varepsilon, V_\pm^\varepsilon$. This can be done by using the local energy conservation and entropy dissipation.

Proposition 2.3 *Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ be a smooth solution of the problem* $(11)-(17)$. *Then we have*

$$\partial_t e^\varepsilon + \frac{1}{\varepsilon}\partial_x \pi^\varepsilon + s^\varepsilon(t,x) = r^\varepsilon(t,x), \quad (t,x) \in]0,T[\times\mathbb{R},$$

where

$$e^\varepsilon(t,x) = \int_{\mathbb{R}^2}\left(\frac{mc_0^2}{\mu} + \ln f^\varepsilon + |x| + \mathcal{E}(p)\right) f^\varepsilon(t,x,p)dp + \frac{1}{2}\left(|E^\varepsilon(t,x)|^2 + \varepsilon^2\delta^2|B^\varepsilon(t,x)|^2\right),$$

$$\pi^\varepsilon(t,x) = \int_{\mathbb{R}^2} v_1(p)\left(\frac{mc_0^2}{\mu} + \ln f^\varepsilon + |x| + \mathcal{E}(p)\right) f^\varepsilon(t,x,p)\, dp + \varepsilon E_2^\varepsilon(t,x)B^\varepsilon(t,x),$$

$$s^\varepsilon(t,x) = \theta\int_{\mathbb{R}^2} |h^\varepsilon(t,x,p)|^2\, dp,$$

$$r^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^2}\frac{x}{|x|}v_1(p)f^\varepsilon\, dp + E_1^\varepsilon(t,x)J(t,x) = \frac{x}{|x|}\int_{\mathbb{R}^2}\sqrt{f^\varepsilon}h_1^\varepsilon\, dp + E_1^\varepsilon(t,x)J(t,x).$$

Proof. Multiplying the Vlasov equation by $1 + \ln f^\varepsilon + |x| + \mathcal{E}(p)$ and integrating with respect to $p \in \mathbb{R}^2$ yields for any $(t,x) \in]0,T[\times\mathbb{R}$

$$\begin{aligned}\partial_t\int_{\mathbb{R}^2}(\ln f^\varepsilon + |x| + \mathcal{E}(p))f^\varepsilon dp \;+\; &\frac{1}{\varepsilon}\partial_x\int_{\mathbb{R}^2} v_1(p)(\ln f^\varepsilon + |x| + \mathcal{E}(p))f^\varepsilon dp + \theta\int_{\mathbb{R}^2}|h^\varepsilon|^2 dp \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^2} E^\varepsilon\cdot v(p)f^\varepsilon\, dp + \frac{1}{\varepsilon}\int_{\mathbb{R}^2}\frac{x}{|x|}v_1(p)f^\varepsilon\, dp.\end{aligned} \tag{37}$$

Multiplying the Maxwell equations (12), (13), (14) by $E_1^\varepsilon, E_2^\varepsilon, B^\varepsilon$ respectively implies

$$\frac{1}{2}\partial_t(|E^\varepsilon|^2 + \varepsilon^2\delta^2|B^\varepsilon|^2) + \partial_x(E_2^\varepsilon B^\varepsilon) = -\frac{1}{\varepsilon}E^\varepsilon(t,x)\cdot j^\varepsilon(t,x) + E_1^\varepsilon(t,x)J(t,x). \tag{38}$$

By the continuity equation we have also

$$\partial_t\int_{\mathbb{R}^2}\frac{mc_0^2}{\mu}f^\varepsilon\, dp + \frac{1}{\varepsilon}\partial_x\int_{\mathbb{R}^2}\frac{mc_0^2}{\mu}v_1(p)f^\varepsilon\, dp = 0, \tag{39}$$

and our conclusion follows by (37), (38), (39). □

For our further computations we will use the elementary results

Lemma 2.3 *Let $\mathcal{E}(p) = \frac{mc_0^2}{\mu}(\sqrt{1+|p|^2p_{th}^2/(mc_0)^2} - 1)$ be the scaled relativistic energy, $v(p) = \nabla_p\mathcal{E}(p)$ and $\delta = v_{th}/(\theta c_0)$ (see the Appendix for the definitions of p_{th}, v_{th}, θ). Then we have the inequality*

$$\left(\mathcal{E}(p) + \frac{mc_0^2}{\mu}\right)\left(\frac{1}{\delta} - |v_1(p)|\right) \geq \frac{mc_0^2}{\mu}|v_2(p)|, \quad p \in \mathbb{R}^2.$$

In particular we have $|v_1(p)| \leq \frac{1}{\delta}$, $\forall\, p \in \mathbb{R}^2$.

Proof. For any $p \in \mathbb{R}^2$ we obtain

$$Q(p) \;:=\; \left(\mathcal{E}(p) + \frac{mc_0^2}{\mu}\right)\left(\frac{1}{\delta} - |v_1(p)|\right) = \frac{mc_0^2}{\mu}\frac{c_0 p_{th}}{\mu}\left(\sqrt{1+|q|^2} - |q_1|\right), \quad (40)$$

where $q = \frac{p_{th}}{mc_0}p$. Obviously we have

$$\sqrt{1+|q|^2} - |q_1| = \frac{1+|q_2|^2}{\sqrt{1+|q|^2}+|q_1|} \geq \frac{|q_2|}{\sqrt{1+|q|^2}}. \quad (41)$$

Combining (40), (41) yields

$$Q(p) \geq \frac{mc_0^2}{\mu}\frac{p_{th}^2}{m\mu}\frac{|p_2|}{\sqrt{1+\left(\frac{p_{th}}{mc}\right)^2|p|^2}} = \frac{mc_0^2}{\mu}|v_2(p)|.$$

$\square$

Lemma 2.4 *Let $u, z, w : [0,T] \times \mathbb{R} \to \mathbb{R}$ be smooth functions satisfying*

$$\partial_t u + \frac{1}{\varepsilon}\partial_x z = w(t,x), \quad (t,x) \in]0,T[\times\mathbb{R}. \quad (42)$$

Then for any $(t,x) \in [0,T] \times \mathbb{R}$ we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (\delta^{-1}u \mp z)\left(s, x \mp \frac{t-s}{\varepsilon\delta}\right)\,ds \pm \frac{1}{\varepsilon}\int_0^t z(s,x)\,ds \;&=\; \pm\int_{x\mp\frac{t}{\varepsilon\delta}}^x u(0,y)\,dy \quad (43)\\ &\pm \int_0^t\int_{x\mp\frac{t-s}{\varepsilon\delta}}^x w(s,y)\,dy\,ds,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (\delta^{-1}u - z)\left(s, x - \frac{t-s}{\varepsilon\delta}\right)\,ds \;+\; \frac{1}{\varepsilon}\int_0^t (\delta^{-1}u + z)\left(s, x + \frac{t-s}{\varepsilon\delta}\right)\,ds&\\ = \int_{x-\frac{t}{\varepsilon\delta}}^{x+\frac{t}{\varepsilon\delta}} u(0,y)\,dy + \int_0^t\int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}} w(s,y)\,dy\,ds.& \quad (44)\end{aligned}$$

Proof. For any $(t,x) \in [0,T] \times \mathbb{R}$ consider the sets $\Delta_\pm^\varepsilon$ given by

$$\Delta_+^\varepsilon = \{(s,y) \in]0,T[\times\mathbb{R} \;:\; x - \frac{t-s}{\varepsilon\delta} < y < x\},$$

$$\Delta_-^\varepsilon = \{(s,y) \in]0,T[\times\mathbb{R} \;:\; x < y < x + \frac{t-s}{\varepsilon\delta}\}.$$

Integrating (42) with respect to $(s,y) \in \Delta_+^\varepsilon$ yields

$$\begin{aligned}\int_{x-\frac{t}{\varepsilon\delta}}^{x} \{u(t-\varepsilon\delta(x-y),y)-u(0,y)\}\,dy \quad &+ \quad \frac{1}{\varepsilon}\int_0^t \{z(s,x)-z(s,x-\frac{t-s}{\varepsilon\delta})\}\,ds \\ &= \quad \int_0^t\int_{x-\frac{t-s}{\varepsilon\delta}}^{x} w(s,y)\,dy\,ds,\end{aligned}$$

and therefore we obtain

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (\delta^{-1}u-z)\left(s,x-\frac{t-s}{\varepsilon\delta}\right)\,ds+\frac{1}{\varepsilon}\int_0^t z(s,x)\,ds \quad &= \quad \int_{x-\frac{t}{\varepsilon\delta}}^{x} u(0,y)\,dy \\ &+ \quad \int_0^t\int_{x-\frac{t-s}{\varepsilon\delta}}^{x} w(s,y)\,dy\,ds.\end{aligned} \tag{45}$$

Similarly, integrating (42) over Δ_-^ε implies

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (\delta^{-1}u+z)\left(s,x+\frac{t-s}{\varepsilon\delta}\right)\,ds-\frac{1}{\varepsilon}\int_0^t z(s,x)\,ds \quad &= \quad \int_{x}^{x+\frac{t}{\varepsilon\delta}} u(0,y)\,dy \\ &+ \quad \int_0^t\int_{x}^{x+\frac{t-s}{\varepsilon\delta}} w(s,y)\,dy\,ds.\end{aligned} \tag{46}$$

The equality (44) follows by adding (45), (46). □

Proposition 2.4 *Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ be a smooth solution of the problem (11) – (17), assume that the initial conditions satisfy H1, H2 and that H3, H4 hold. Then we have*

$$|U^\varepsilon(t,x)| \le M_0^\varepsilon, \;\; (t,x)\in[0,T]\times\mathbb{R}, \tag{47}$$

$$|V_+^\varepsilon|+|V_-^\varepsilon| \le \frac{2\mu}{mc_0^2}\left(\left(\frac{mc_0^2}{\mu}+\frac{5T}{2\theta}\right)M_0^\varepsilon+5(W_0^\varepsilon+L_0^\varepsilon+H_0^\varepsilon)+\frac{5}{2}\|J\|^2_{L^1(]0,T[;L^2)}+6C_{1/4}\right). \tag{48}$$

Proof. Combining Proposition 2.3, and Lemma 2.4, (44) we obtain

$$
\begin{aligned}
& \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}^2} (\delta^{-1} - v_1)\left(\ln f^\varepsilon + \left|x - \frac{t-s}{\varepsilon\delta}\right| + \mathcal{E}(p) + \frac{mc_0^2}{\mu}\right) f^\varepsilon(s, x - \frac{t-s}{\varepsilon\delta}, p)\, dp\, ds \\
+ \quad & \frac{1}{2\varepsilon\delta}\int_0^t (|E^\varepsilon|^2 + \varepsilon^2\delta^2|B^\varepsilon|^2 - 2\varepsilon\delta E_2^\varepsilon B^\varepsilon)(s, x - \frac{t-s}{\varepsilon\delta})\, ds \\
+ \quad & \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}^2} (\delta^{-1} + v_1)\left(\ln f^\varepsilon + \left|x + \frac{t-s}{\varepsilon\delta}\right| + \mathcal{E}(p) + \frac{mc_0^2}{\mu}\right) f^\varepsilon(s, x + \frac{t-s}{\varepsilon\delta}, p)\, dp\, ds \\
+ \quad & \frac{1}{2\varepsilon\delta}\int_0^t (|E^\varepsilon|^2 + \varepsilon^2\delta^2|B^\varepsilon|^2 + 2\varepsilon\delta E_2^\varepsilon B^\varepsilon)(s, x + \frac{t-s}{\varepsilon\delta})\, ds \\
+ \quad & \theta\int_0^t \int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}} \int_{\mathbb{R}^2} |h^\varepsilon(s,y,p)|^2\, dp\, dy\, ds \\
= \quad & \int_{x-\frac{t}{\varepsilon\delta}}^{x+\frac{t}{\varepsilon\delta}}\!\!\int_{\mathbb{R}^2}\left(\ln f_0^\varepsilon + |y| + \mathcal{E}(p) + \frac{mc_0^2}{\mu}\right) f_0^\varepsilon\, dp\, dy + \frac{1}{2}\int_{x-\frac{t}{\varepsilon\delta}}^{x+\frac{t}{\varepsilon\delta}} (|E_0^\varepsilon|^2 + \varepsilon^2\delta^2|B_0^\varepsilon|^2)\, dy \\
+ \quad & \int_0^t \int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}} \left(\int_{\mathbb{R}^2} \frac{y}{|y|}\sqrt{f^\varepsilon(s,y,p)}\; h_1^\varepsilon(s,y,p)\, dp + E_1^\varepsilon(s,y)J(s,y)\right) dy\, ds \\
\leq \quad & \frac{mc_0^2}{\mu}M_0^\varepsilon + W_0^\varepsilon + L_0^\varepsilon + H_0^\varepsilon + \frac{t}{2\theta}M_0^\varepsilon + \frac{\theta}{2}\int_0^t \int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}} \int_{\mathbb{R}^2} |h^\varepsilon(s,y,p)|^2\, dp\, dy\, ds \\
+ \quad & \|E^\varepsilon\|_{L^\infty(]0,T[;L^2(\mathbb{R}))}\|J\|_{L^1(]0,T[;L^2(\mathbb{R}))}. \qquad (49)
\end{aligned}
$$

For any fixed $(t,x) \in [0,T] \times \mathbb{R}$ and $s \in [0,t]$ we prove exactly as in Lemma 2.1 that

$$
\begin{aligned}
f^\varepsilon|\ln f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p)| \quad \leq \quad & \left(\ln f^\varepsilon + \frac{1}{2}\left(|x \pm \frac{t-s}{\varepsilon\delta}| + \mathcal{E}(p)\right)\right) f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p) \\
+ \quad & 2Ce^{-\frac{1}{8}(|x\pm\frac{t-s}{\varepsilon\delta}| + \mathcal{E}(p))}, \qquad (50)
\end{aligned}
$$

where $C = \sup\{-\sqrt{y}\ln y \;:\; 0 < y < 1\}$. We obtain the inequalities

$$
\begin{aligned}
& \frac{1}{\varepsilon}\int_0^t \int_{\mathbb{R}^2} (\delta^{-1} \pm v_1(p))|\ln f^\varepsilon| f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p)\, dp\, ds \qquad (51) \\
\leq \quad & \frac{1}{\varepsilon}\int_0^t \int_{\mathbb{R}^2} (\delta^{-1} \pm v_1(p))(\ln f^\varepsilon) f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p)\, dp\, ds \\
+ \quad & \frac{1}{2\varepsilon}\int_0^t \int_{\mathbb{R}^2} (\delta^{-1} \pm v_1(p))(|x \pm \frac{t-s}{\varepsilon\delta}| + \mathcal{E}(p)) f^\varepsilon(s, x \pm \frac{t-s}{\varepsilon\delta}, p)\, dp\, ds + C_{\varepsilon\delta},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{\varepsilon\delta} \quad := \quad & \frac{2C}{\varepsilon\delta}\int_0^t \int_{\mathbb{R}^2} e^{-\frac{1}{8}(|x\pm\frac{t-s}{\varepsilon\delta}| + \mathcal{E}(p))}\, dp\, ds \\
\leq \quad & 2C\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{-\frac{1}{8}(|y| + \mathcal{E}(p))}\, dp\, dy \\
= \quad & C_{1/4}, \quad \varepsilon > 0, \delta > 0.
\end{aligned}
$$

Combining (49), (51) yields

$$
\begin{aligned}
& \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}^2}(\delta^{-1}-v_1)(|\ln f^\varepsilon|+\frac{1}{2}|x-\frac{t-s}{\varepsilon\delta}|+\frac{1}{2}\mathcal{E}+\frac{mc_0^2}{\mu})f^\varepsilon(s,x-\frac{t-s}{\varepsilon\delta},p)\,dp\,ds\\
+ \ & \frac{1}{2\varepsilon\delta}\int_0^t(|E_1^\varepsilon|^2+|E_2^\varepsilon-\varepsilon\delta B^\varepsilon|^2)(s,x-\frac{t-s}{\varepsilon\delta})\,ds\\
+ \ & \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}^2}(\delta^{-1}+v_1)(|\ln f^\varepsilon|+\frac{1}{2}|x+\frac{t-s}{\varepsilon\delta}|+\frac{1}{2}\mathcal{E}+\frac{mc_0^2}{\mu})f^\varepsilon(s,x+\frac{t-s}{\varepsilon\delta},p)\,dp\,ds\\
+ \ & \frac{1}{2\varepsilon\delta}\int_0^t(|E_1^\varepsilon|^2+|E_2^\varepsilon+\varepsilon\delta B^\varepsilon|^2)(s,x+\frac{t-s}{\varepsilon\delta})\,ds\\
+ \ & \frac{\theta}{2}\int_0^t\int_{x-\frac{t-s}{\varepsilon\delta}}^{x+\frac{t-s}{\varepsilon\delta}}\int_{\mathbb{R}^2}|h^\varepsilon(s,y,p)|^2\,dp\,dy\,ds\\
\leq \ & \frac{mc_0^2}{\mu}M_0^\varepsilon+W_0^\varepsilon+L_0^\varepsilon+H_0^\varepsilon+\frac{t}{2\theta}M_0^\varepsilon+2C_{1/4}+\frac{1}{2}\|E^\varepsilon\|^2_{L^\infty(]0,T[;L^2)}+\frac{1}{2}\|J\|^2_{L^1(]0,T[;L^2)}\\
\leq \ & \left(\frac{mc_0^2}{\mu}+\frac{5T}{2\theta}\right)M_0^\varepsilon+5(L_0^\varepsilon+W_0^\varepsilon+H_0^\varepsilon)+6C_{1/4}+\frac{5}{2}\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}=:C_0^\varepsilon. \quad (52)
\end{aligned}
$$

We deduce that

$$
\sum_{k=0}^{1}\frac{1}{2\varepsilon}\int_0^t\int_{\mathbb{R}^2}(\delta^{-1}+(-1)^k v_1(p))(\mathcal{E}(p)+\frac{mc_0^2}{\mu})f^\varepsilon(s,x+(-1)^k\frac{t-s}{\varepsilon\delta},p)\,dp\,ds\leq C_0^\varepsilon,
$$

and finally by Lemma 2.3 one gets

$$
\begin{aligned}
\frac{mc_0^2}{\mu}(|V_+^\varepsilon|+|V_-^\varepsilon|) \ &\leq \ \frac{mc_0^2}{\varepsilon\mu}\int_0^t\!\!\int_{\mathbb{R}^2}|v_2|(f^\varepsilon(s,x-\frac{t-s}{\varepsilon\delta},p)+f^\varepsilon(s,x+\frac{t-s}{\varepsilon\delta},p))\,dp\,ds\\
&\leq \ \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}^2}(\delta^{-1}-v_1(p))(\mathcal{E}(p)+\frac{mc_0^2}{\mu})f^\varepsilon(s,x-\frac{t-s}{\varepsilon\delta},p)\,dp\,ds\\
&+ \ \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}^2}(\delta^{-1}+v_1(p))(\mathcal{E}(p)+\frac{mc_0^2}{\mu})f^\varepsilon(s,x+\frac{t-s}{\varepsilon\delta},p)\,dp\,ds\\
&\leq \ 2C_0^\varepsilon.
\end{aligned}
$$

The estimate of U^ε follows by applying Lemma 2.4 to the continuity equation

$$
\partial_t\int_{\mathbb{R}^2}f^\varepsilon\,dp+\frac{1}{\varepsilon}\partial_x\int_{\mathbb{R}^2}v_1(p)f^\varepsilon\,dp=0,\quad (t,x)\in]0,T[\times\mathbb{R}.
$$

Indeed, by (43) we have for any $(t,x)\in[0,T]\times\mathbb{R}$

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}(\delta^{-1}\mp v_1(p))f^\varepsilon(s,x \ &\mp \ \frac{t-s}{\varepsilon\delta},p)\,dp\,ds\pm\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}v_1(p)f^\varepsilon(s,x,p)\,dp\,ds\\
&= \ \pm\int_{x\mp\frac{t}{\varepsilon\delta}}^{x}\int_{\mathbb{R}^2}f_0^\varepsilon(y,p)\,dp\,dy,
\end{aligned}
$$

and thus we deduce that $\pm U^\varepsilon(t,x)\leq M_0^\varepsilon$. $\square$

Corollary 2.3 *Let* $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ *be a smooth solution of the problem* $(11)-(17)$ *and assume that H1-H6 hold. Then we have*

$$\|E_1^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} \le \|E_{0,1}^\varepsilon\|_{L^\infty(\mathbb{R})} + M_0^\varepsilon + \|J\|_{L^1(]0,T[;L^\infty(\mathbb{R}))},$$

and

$$\begin{aligned}\max(\|E_2^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})}, \varepsilon\delta\|B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})}) &\le R_0^\varepsilon + \frac{\mu}{mc_0^2}((\frac{mc_0^2}{\mu} + \frac{5T}{2\theta})M_0^\varepsilon + 6C_{1/4} \\ &+ 5(W_0^\varepsilon + L_0^\varepsilon + H_0^\varepsilon) + \frac{5}{2}\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}).\end{aligned}$$

Proof. By (34) and Proposition 2.4 one gets

$$\|E_1^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} \le \|E_{0,1}^\varepsilon\|_{L^\infty(\mathbb{R})} + M_0^\varepsilon + \|J\|_{L^1(]0,T[;L^\infty(\mathbb{R}))}.$$

Similarly, combining (35), (36), Proposition 2.4 implies

$$\begin{aligned}\max(\|E_2^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})}, \varepsilon\delta\|B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})}) &\le \|E_{0,2}^\varepsilon\|_{L^\infty(\mathbb{R})} + \varepsilon\delta\|B_0^\varepsilon\|_{L^\infty(\mathbb{R})} \\ &+ \frac{\mu}{mc_0^2}((\frac{mc_0^2}{\mu} + \frac{5T}{2\theta})M_0^\varepsilon + 6C_{1/4} \\ &+ 5(W_0^\varepsilon + L_0^\varepsilon + H_0^\varepsilon) + \frac{5}{2}\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}))}).\end{aligned}$$

$\square$

3 Asymptotic analysis

We are now in position to perform the asymptotic analysis when ε goes to zero. The uniform estimates obtained in the previous section allow us to extract sequences as follows.

Proposition 3.1 *Assume that the hypotheses H1-H6 hold. Suppose that for any* $\varepsilon > 0$ $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ *is a smooth solution of* $(11)-(17)$. *Then there is a sequence* $(\varepsilon_k)_k$ *decreasing to zero such that the sequences*

$$(f_0^k, E_0^k, B_0^k)_k := (f_0^{\varepsilon_k}, E_0^{\varepsilon_k}, B_0^{\varepsilon_k})_k, \;\; (f^k, E^k, B^k)_k := (f^{\varepsilon_k}, E^{\varepsilon_k}, B^{\varepsilon_k})_k,$$

satisfy the following convergences

$$f_0^k \rightharpoonup f_0 \text{ weakly in } L^1(\mathbb{R}^2), \tag{53}$$

$$\rho_0^k := \int_{\mathbb{R}^2} f_0^k \, dp \rightharpoonup \rho_0 := \int_{\mathbb{R}^2} f_0 \, dp \text{ weakly in } L^1(\mathbb{R}), \tag{54}$$

$$E^k_{0,1} \to E_{0,1} \text{ uniformly on compact sets of } \mathbb{R}, \tag{55}$$

$$E^k_{0,2} \rightharpoonup E_{0,2} \text{ weakly in } L^2(\mathbb{R}), \tag{56}$$

$$f^k \rightharpoonup f \text{ weakly in } L^1(]0,T[\times\mathbb{R}\times\mathbb{R}^2), \tag{57}$$

$$\rho^k := \int_{\mathbb{R}^2} f^k \, dp \rightharpoonup \rho := \int_{\mathbb{R}^2} f \, dp \text{ weakly in } L^1(]0,T[\times\mathbb{R}), \tag{58}$$

$$E^k_1 \to E_1 \text{ strongly in } L^1_{\text{loc}}([0,T]\times\mathbb{R}), \text{ weakly in } L^2(]0,T[\times\mathbb{R}), \text{ weakly} \star \text{ in } L^\infty, \tag{59}$$

$$(E^k_2, \varepsilon_k \delta B^k) \rightharpoonup (0,0) \text{ weakly in } L^2(]0,T[\times\mathbb{R})^2, \text{ weakly } \star \text{ in } L^\infty(]0,T[\times\mathbb{R})^2. \tag{60}$$

Proof. By Proposition 2.1 we have

$$\sup_{\varepsilon>0, t\in[0,T]} \left\{ \int_{\mathbb{R}}\int_{\mathbb{R}^2} (1+|x|+\mathcal{E}(p)+|\ln f^\varepsilon|) f^\varepsilon \, dp\, dx + \int_{\mathbb{R}} (|E^\varepsilon|^2 + \varepsilon^2\delta^2|B^\varepsilon|^2)\, dx \right\} < \infty,$$

and therefore (53), (57) follow by classical arguments. Moreover the function f is non negative and satisfies

$$\left\| \int_{\mathbb{R}}\int_{\mathbb{R}^2} (1+|x|+\mathcal{E}(p)) f(\cdot,x,p)\, dp\, dx \right\|_{L^\infty(]0,T[)} < +\infty.$$

Similarly by Corollary 2.1 we deduce

$$\sup_{\varepsilon>0, t\in[0,T]} \int_{\mathbb{R}} (1+|x|+|\ln\rho^\varepsilon|)\rho^\varepsilon(t,x)\, dx < +\infty,$$

which implies (54), (58). Since $\sup_{\varepsilon>0}\int_{\mathbb{R}} |\ln\rho^\varepsilon_0|\rho^\varepsilon_0(x)\, dx < +\infty$ we deduce that for any $\eta > 0$ there is $h = h(\eta) > 0$ such that $\int_x^{x+h} \rho^\varepsilon_0(y)\, dy < \eta$ for any $\varepsilon > 0$ and $x \in \mathbb{R}$. Taking h small enough, since $D(0,\cdot)$ belongs to $L^1(\mathbb{R})$ we have also $\int_x^{x+h} D(0,y)\, dy < \eta$ for any $x \in \mathbb{R}$. Therefore we have for any $\varepsilon > 0$ and $x \in \mathbb{R}$

$$|E^\varepsilon_{0,1}(x+h) - E^\varepsilon_{0,1}(x)| = \left| \int_x^{x+h} \{\rho^\varepsilon_0(y) - D(0,y)\}\, dy \right| < 2\eta,$$

and since $(E^\varepsilon_{0,1})_\varepsilon$ is bounded in $L^\infty(\mathbb{R})$, by using the Arzela-Ascoli theorem we deduce (55). The convergence (56) is obvious since $(E^\varepsilon_{0,2})_{\varepsilon>0}$ is bounded in $L^2(\mathbb{R})$. Moreover we check easily that $E_{0,1}, E_{0,2} \in L^\infty(\mathbb{R})$ and $\frac{d}{dx}E_{0,1} = \rho_0 - D(0,\cdot)$. We claim that

$(E_1^\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,1}_{\mathrm{loc}}([0,T]\times\mathbb{R})$. Indeed, $(E_1^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$ and thus in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$. Obviously $(\partial_x E_1^\varepsilon)_{\varepsilon>0} = (\rho^\varepsilon - D)_{\varepsilon>0}$ is bounded in $L^1(]0,T[\times\mathbb{R})$ and $(\partial_t E_1^\varepsilon)_{\varepsilon>0} = (-\frac{1}{\varepsilon} j_1^\varepsilon + J)_{\varepsilon>0}$ is bounded in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$ since by Proposition 2.1 we have $\sup_{\varepsilon>0}\|\varepsilon^{-1} j^\varepsilon\|_{L^2(]0,T[;L^1(\mathbb{R}))} < +\infty$ and $J \in L^1(]0,T[;L^2(\mathbb{R}))$. We deduce that $(E_1^\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$. Observe also that $(E_1^\varepsilon, E_2^\varepsilon, \varepsilon\delta B^\varepsilon)_{\varepsilon>0}$ is weakly relatively compact in $(L^2(]0,T[\times\mathbb{R}))^3$ and weakly $\star$ relatively compact in $L^\infty(]0,T[\times\mathbb{R})^3$. Thus we obtain (59) and that $(E_2^k, \varepsilon_k \delta B^k) \rightharpoonup (E_2, B)$ weakly in $L^2(]0,T[\times\mathbb{R})^2$, weakly $\star$ in $L^\infty(]0,T[\times\mathbb{R})^2$. Moreover the limits E_1, E_2, B belong to $L^\infty(]0,T[;L^2(\mathbb{R}))$ and we have $\partial_x E_1 = \rho - D$. Notice also that by (14) we have for any $\varphi \in C^1_c(]0,T[\times\mathbb{R})$

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}} E_2^\varepsilon \partial_x\varphi \,dx\,dt\right| &= \varepsilon^2\delta^2\left|\int_0^T\int_{\mathbb{R}} B^\varepsilon \partial_t\varphi\,dx\,dt\right| \\ &\le \varepsilon\delta\|\varepsilon\delta B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})}\int_0^T\int_{\mathbb{R}}|\partial_t\varphi|\,dx\,dt.\end{aligned}$$

Since $\sup_{\varepsilon>0}\|\varepsilon\delta B^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} < +\infty$ we obtain $\partial_x E_2 = 0$. Taking into account that $E_2 \in L^\infty(]0,T[;L^2(\mathbb{R}))$ we deduce that $E_2 = 0$. Similarly for any $\varphi \in C^1_c(]0,T[\times\mathbb{R})$ we have by (13)

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}} \varepsilon\delta B^\varepsilon \partial_x\varphi\,dx\,dt\right| &= \varepsilon\delta\left|\int_0^T\int_{\mathbb{R}}\left(\frac{j_2^\varepsilon}{\varepsilon}\varphi - E_2^\varepsilon\partial_t\varphi\right)\,dx\,dt\right| \\ &\le \varepsilon\delta\left(\|\varphi\|_{C^0}\left\|\frac{j_2^\varepsilon}{\varepsilon}\right\|_{L^1(]0,T[\times\mathbb{R})} + \|E_2^\varepsilon\|_{L^\infty}\int_0^T\int_{\mathbb{R}}|\partial_t\varphi|\,dx\,dt\right).\end{aligned}$$

By using the uniform bounds $\sup_{\varepsilon>0}\|E_2^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{R})} < \infty$, $\sup_{\varepsilon>0}\left\|\frac{j^\varepsilon}{\varepsilon}\right\|_{L^2(]0,T[;L^1(\mathbb{R}))} < \infty$ we obtain $\partial_x B = 0$ and since we know that $B \in L^\infty(]0,T[;L^2(\mathbb{R}))$ finally one gets $B = 0$. $\square$

We focus our attention to the moment equations of (11). Integrating (11) with respect to $p \in \mathbb{R}^2$ yields the continuity equation $\partial_t \int_{\mathbb{R}^2} f^\varepsilon\,dp + \frac{1}{\varepsilon}\partial_x\int_{\mathbb{R}^2} v_1(p) f^\varepsilon\,dp = 0$. Let us multiply now by p_1 and integrate with respect to $p \in \mathbb{R}^2$.

$$\varepsilon\partial_t\int_{\mathbb{R}^2} p_1 f^\varepsilon\,dp + \partial_x\int_{\mathbb{R}^2} v_1(p)p_1 f^\varepsilon\,dp - E_1^\varepsilon\rho^\varepsilon - \varepsilon\delta^2 B^\varepsilon j_2^\varepsilon = \theta(\partial_t E_1^\varepsilon - J). \tag{61}$$

We need to examine the limit as ε goes to zero of each term in the above equation. We identify easily the limits of all these terms, except for the term $\partial_x\int_{\mathbb{R}^2} v_1(p)p_1 f^\varepsilon\,dp$. In order to analyze formally this term, observe that by Proposition 2.1 we have

$$\sup_{\varepsilon>0}\left(\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2}\frac{|\nabla_p f^\varepsilon + v(p)f^\varepsilon|^2}{f^\varepsilon}\,dp\,dx\,dt\right) < +\infty,$$

and therefore $f^\varepsilon(t,x,p) \approx \rho(t,x)\frac{e^{-\mathcal{E}(p)}}{K}$ where $K = \int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\,dp$. In this case we obtain

$$\int_{\mathbb{R}^2} v_1(p)p_1 f^\varepsilon\,dp \approx -\frac{\rho(t,x)}{K}\int_{\mathbb{R}^2} p_1\partial_{p_1}(e^{-\mathcal{E}(p)})\,dp = \rho(t,x),$$

and thus we can guess that $\lim_{\varepsilon \searrow 0} \partial_x \int_{\mathbb{R}^2} v_1(p) p_1 f^\varepsilon \, dp = \partial_x \rho$. Multiplying now (11) by p_2 and integrating with respect to $p \in \mathbb{R}^2$ yields

$$\varepsilon \partial_t \int_{\mathbb{R}^2} p_2 f^\varepsilon \, dp + \partial_x \int_{\mathbb{R}^2} v_1(p) p_2 f^\varepsilon \, dp - E_2^\varepsilon \rho^\varepsilon + \varepsilon \delta^2 B^\varepsilon j_1^\varepsilon = -\theta \, \frac{j_2^\varepsilon}{\varepsilon}. \tag{62}$$

Similarly one gets at least formally

$$\int_{\mathbb{R}^2} v_1(p) p_2 f^\varepsilon \, dp \approx -\frac{\rho(t,x)}{K} \int_{\mathbb{R}^2} p_2 \partial_{p_1} (e^{-\mathcal{E}(p)}) \, dp = 0,$$

and therefore $\lim_{\varepsilon \searrow 0} \partial_x \int_{\mathbb{R}^2} v_1(p) p_2 f^\varepsilon \, dp = 0$.

Proposition 3.2 *Assume that the hypotheses H1-H7 hold. Suppose that* $(f^\varepsilon, E^\varepsilon, B^\varepsilon)_{\varepsilon > 0}$ *are smooth solutions of* $(11) - (17)$ *and consider* $(\varepsilon_k)_k$ *the sequence constructed in Proposition* 3.1. *Then we have*

$$\lim_{k \to +\infty} \varepsilon_k \left(\partial_t \int_{\mathbb{R}^2} p_1 f^k \, dp, \partial_t \int_{\mathbb{R}^2} p_2 f^k \, dp \right) = (0,0) \text{ in } \mathcal{D}'([0,T] \times \mathbb{R})^2, \tag{63}$$

$$\lim_{k \to +\infty} \left(\partial_x \int_{\mathbb{R}^2} v_1(p) p_1 f^k \, dp, \partial_x \int_{\mathbb{R}^2} v_1(p) p_2 f^k \, dp \right) = (\partial_x \rho, 0) \text{ in } \mathcal{D}'([0,T] \times \mathbb{R})^2, \tag{64}$$

$$\lim_{k \to +\infty} (E_1^k \rho^k, E_2^k \rho^k) = (E_1 \rho, 0) \text{ in } \mathcal{D}'([0,T] \times \mathbb{R})^2, \tag{65}$$

$$\lim_{k \to +\infty} (\varepsilon_k \delta^2 B^k j_1^k, \varepsilon_k \delta^2 B^k j_2^k) = (0,0) \text{ in } L^1(]0,T[\times \mathbb{R})^2. \tag{66}$$

Proof. By Proposition 2.1, Corollary 2.2 and Corollary 2.3 we have

$$\sup_{k, t \in [0,T]} \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^2} (1 + |x| + \mathcal{E}(p) + |\ln f^k|) f^k \, dp \, dx + \int_{\mathbb{R}} (|E^k|^2 + \varepsilon_k^2 \delta^2 |B^k|^2) \, dx \right\} < \infty,$$

$$\sup_{k \in \mathbb{N}} \left(\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} |h^k(t,x,p)|^2 \, dp \, dx \, dt + \left\| \frac{j^k}{\varepsilon_k} \right\|^2_{L^2(]0,T[;L^1(\mathbb{R}))} \right) < +\infty,$$

$$\sup_{k \in \mathbb{N}} \left(\|E^k\|_{L^\infty(]0,T[\times \mathbb{R})} + \varepsilon_k \delta \|B^k\|_{L^\infty(]0,T[\times \mathbb{R})} \right) < +\infty,$$

$$\sup_{k \in \mathbb{N}, t \in [0,T]} \int_{\mathbb{R}} \int_{\mathbb{R}^2} (f^k(t,x,p))^r e^{(r-1)\mathcal{E}(p)} \, dp \, dx < +\infty.$$

For any function $\varphi \in C_c^1([0,T] \times \mathbb{R})$, $l \in \{1,2\}$ we have

$$\sup_{k \in \mathbb{N}} \left| \langle \partial_t \int_{\mathbb{R}^2} p_l f^k \, dp, \varphi \rangle \right| \leq \|\varphi\|_{C^1} \sup_{k \in \mathbb{N}} \left(2 \sup_{t \in [0,T]} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |p| f^k \, dp \, dx + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} |p| f^k \, dp \, dx \, dt \right),$$

and therefore $\lim_{k\to+\infty} \varepsilon_k \partial_t \int_{\mathbb{R}^2} p_l f^k\, dp = 0$ in $\mathcal{D}'([0,T]\times\mathbb{R})$. Observe that

$$\langle \partial_x \int_{\mathbb{R}^2} v_1(p) p_l f^k \, dp, \varphi\rangle \;=\; -\varepsilon_k \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} \partial_x \varphi \; p_l \sqrt{f^k}\; h_1^k \; dp\, dx\, dt - \int_0^T \int_{\mathbb{R}} \partial_x \varphi \delta_{1l} \rho^k \; dx\, dt,$$

and therefore (64) holds provided that for any $R > 0$ we have

$$\sup_{k\in\mathbb{N}} \int_0^T \int_{-R}^R \int_{\mathbb{R}^2} |p| \sqrt{f^k}\; |h^k| \; dp \; dx \; dt < +\infty. \tag{67}$$

By Cauchy-Schwartz inequality we deduce that

$$\begin{aligned}\int_0^T \int_{-R}^R \int_{\mathbb{R}^2} |p|\sqrt{f^k}\;|h^k| \; dp\; dx\; dt \;&\leq\; \left(\int_0^T \int_{-R}^R \int_{\mathbb{R}^2} |p|^2 f^k \; dp\; dx\; dt\right)^{1/2} \\ &\times\; \left(\int_0^T \int_{-R}^R \int_{\mathbb{R}^2} |h_k|^2 \; dp\; dx\; dt\right)^{1/2},\end{aligned}$$

and therefore we are done if we prove that $\sup_{k\in\mathbb{N}, t\in[0,T]} \int_{-R}^R \int_{\mathbb{R}^2} |p|^2 f^k(t,x,p)\; dp\; dx < +\infty$. Indeed, we obtain by Hölder inequality

$$\begin{aligned}\sup_{k\in\mathbb{N}, t\in[0,T]} \int_{-R}^R \int_{\mathbb{R}^2} |p|^2 f^k(t,x,p)\; dp \;&\leq\; \sup_{k\in\mathbb{N}, t\in[0,T]} \left(\int_{-R}^R \int_{\mathbb{R}^2} (f^k(t))^r e^{(r-1)\mathcal{E}(p)} \; dp\; dx\right)^{1/r} \\ &\times\; \left(\int_{-R}^R \int_{\mathbb{R}^2} |p|^{2r'} e^{-\mathcal{E}(p)}\; dp\right)^{1/r'} < +\infty\end{aligned}$$

where r' is the conjugate exponent of r, *i.e.*, $1/r + 1/r' = 1$. Consider now the term $E_1^k \rho^k = E_1^k(\partial_x E_1^k + D)$. Since $(E_1^k)_k$ converges towards E_1 weakly $\star$ in $L^\infty(]0,T[\times\mathbb{R})$ we have for any $\varphi \in C_c^1([0,T]\times\mathbb{R})$

$$\lim_{k\to+\infty} \int_0^T \int_{\mathbb{R}} E_1^k(t,x) D(t,x)\varphi(t,x)\; dx\, dt = \int_0^T \int_{\mathbb{R}} E_1(t,x) D(t,x) \varphi(t,x)\; dx\, dt.$$

It remains to analyze the term $E_1^k \partial_x E_1^k$

$$\begin{aligned}|\langle E_1^k \partial_x E_1^k - E_1 \partial_x E_1, \varphi\rangle| &= \left| \int_0^T \int_{\mathbb{R}} \frac{1}{2} \partial_x |E_1^k|^2 \varphi \; dx\, dt - \frac{1}{2}\langle \partial_x |E_1|^2, \varphi\rangle\right| \\ &= \left| -\frac{1}{2} \int_0^T \int_{\mathbb{R}} (E_1^k - E_1) E_1^k \partial_x \varphi \; dx\, dt + \frac{1}{2}\int_0^T \int_{\mathbb{R}} E_1(E_1 - E_1^k)\partial_x \varphi \; dx\, dt\right| \\ &\leq \|\varphi\|_{C^1} \sup_{k'\in\mathbb{N}} \|E_1^{k'}\|_{L^\infty(]0,T[\times\mathbb{R})} \|E_1^k - E_1\|_{L^1(\mathrm{supp}\varphi)},\end{aligned}$$

and therefore $\lim_{k\to+\infty} E_1^k \rho^k = E_1 \rho$ in $\mathcal{D}'([0,T[\times\mathbb{R})$. Consider now the term $E_2^k \rho^k$. Since $(E_2^k)_k$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$ and $(\rho^k)_k$ is bounded in $L^\infty(]0,T[; L^1(\mathbb{R}))$

it is sufficient to prove that $E_2^k \rho^k = E_2^k(\partial_x E_1^k + D) \to 0$ in $\mathcal{D}'(]0,T[\times\mathbb{R})$. Take $\varphi \in C_c^1(]0,T[\times\mathbb{R})$. As before we have

$$\lim_{k\to+\infty} \int_0^T \int_{\mathbb{R}} E_2^k(t,x)D(t,x)\varphi(t,x)\,dx\,dt = \int_0^T \int_{\mathbb{R}} E_2(t,x)D(t,x)\varphi(t,x)\,dx\,dt,$$

and for the term $E_2^k \partial_x E_1^k$ we write

$$\begin{aligned}
|\langle E_2^k \partial_x E_1^k, \varphi\rangle| &= \left| \int_0^T \int_{\mathbb{R}} (\partial_x(E_2^k E_1^k) - \partial_x E_2^k E_1^k)\varphi\,dx\,dt \right| \\
&\leq Q_1^k + Q_2^k, \qquad (68)
\end{aligned}$$

where $Q_1^k := \left|\int_0^T\int_{\mathbb{R}} E_2^k E_1^k \partial_x\varphi\,dx\,dt\right|$ and $Q_2^k := \left|\int_0^T\int_{\mathbb{R}} \partial_x E_2^k E_1^k \varphi\,dx\,dt\right|$. Observe that

$$Q_1^k \leq \left| \int_0^T \int_{\mathbb{R}} E_2^k(E_1^k - E_1)\partial_x\varphi\,dx\,dt \right| + \left| \int_0^T \int_{\mathbb{R}} E_1 E_2^k \partial_x\varphi\,dx\,dt \right|, \qquad (69)$$

and therefore, by using the strong convergence of $(E_1^k)_k$ in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$ and the weak convergence of $(E_2^k)_k$ in $L^2(]0,T[\times\mathbb{R})$ we deduce that $\lim_{k\to+\infty} Q_1^k = 0$. By using (14), (12) we have

$$\begin{aligned}
Q_2^k &= \varepsilon_k\delta \left| \int_0^T \int_{\mathbb{R}} \varepsilon_k\delta B^k E_1^k \partial_t\varphi\,dx\,dt \right| + \varepsilon_k\delta \left| \int_0^T \int_{\mathbb{R}} \varepsilon_k\delta B^k \partial_t E_1^k \varphi\,dx\,dt \right| \\
&\leq \varepsilon_k\delta \|\varepsilon_k\delta B^k\|_{L^\infty(]0,T[\times\mathbb{R})} \|E_1^k\|_{L^\infty(]0,T[\times\mathbb{R})} \int_0^T \int_{\mathbb{R}} |\partial_t\varphi|\,dx\,dt \\
&+ \varepsilon_k\delta \|\varepsilon_k\delta B^k\|_{L^\infty(]0,T[\times\mathbb{R})} \left| \int_0^T \int_{\mathbb{R}} \left(J - \frac{j_1^k}{\varepsilon_k} \right) \varphi\,dx\,dt \right|.
\end{aligned}$$

Since $(E_1^k)_k$, $(\varepsilon_k\delta B^k)_k$ are bounded in $L^\infty(]0,T[\times\mathbb{R})$, $(\frac{j_1^k}{\varepsilon_k})_k$ is bounded in $L^2(]0,T[;L^1(\mathbb{R}))$ and J belongs to $L^1(]0,T[;L^\infty(\mathbb{R}))$ we deduce that $\lim_{k\to+\infty} Q_2^k = 0$. Thus we proved that $\lim_{k\to+\infty} E_2^k \partial_x E_1^k = 0$ in $\mathcal{D}'(]0,T[\times\mathbb{R})$ and therefore the second convergence in (65) holds. The convergence (66) follows easily since $(\varepsilon_k\delta B^k)_k$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$ and $(\frac{j^k}{\varepsilon_k})_k$ is bounded in $L^2(]0,T[;L^1(\mathbb{R}))$. We have

$$\|\varepsilon_k\delta^2 B^k j^k\|_{L^1(]0,T[\times\mathbb{R})} \leq \varepsilon_k\delta \sup_{k'\in\mathbb{N}} \left(\|\varepsilon_{k'}\delta B^{k'}\|_{L^\infty(]0,T[\times\mathbb{R})} \left\| \frac{j^{k'}}{\varepsilon_{k'}} \right\|_{L^1(]0,T[\times\mathbb{R})} \right).$$

□

Remark 3.1 *By easy density arguments we deduce that* $\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}} E_2^k\rho^k\varphi\,dx\,dt = 0$ *for any continuous bounded function* $\varphi \in C^0([0,T]\times\mathbb{R})$ *(use the uniform bounds* $\sup_{k\in\mathbb{N}}\|E_2^k\|_{L^\infty(]0,T[\times\mathbb{R})} < +\infty$ *and* $\sup_{k\in\mathbb{N},t\in[0,T]}\int_{\mathbb{R}}(1+|x|)\rho^k(t,x)\,dx < +\infty$*).*

Notice that under the hypotheses of Proposition 3.2 with $r = 2$ we have the uniform bound $\sup_{\varepsilon>0}\|\frac{j^\varepsilon}{\varepsilon}\|_{L^2(]0,T[\times\mathbb{R})} < +\infty$. Indeed, by Corollary 2.2 we know that

$$\sup_{\varepsilon>0}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon e^{\mathcal{E}(p)}|h^\varepsilon|^2\,dp\,dx\,dt = \sup_{\varepsilon>0}\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\mathcal{E}(p)}|\nabla_p f^\varepsilon + v(p) f^\varepsilon|^2\,dp\,dx\,dt < \infty,$$

and thus we obtain for any $(t,x)\in[0,T]\times\mathbb{R}$

$$\left|\frac{j^\varepsilon(t,x)}{\varepsilon}\right|^2 = \left|\int_{\mathbb{R}^2}\sqrt{f^\varepsilon}h^\varepsilon(t,x,p)\,dp\right|^2 \le \int_{\mathbb{R}^2} f^\varepsilon e^{\mathcal{E}(p)}|h^\varepsilon|^2\,dp\int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\,dp,$$

implying that

$$\sup_{\varepsilon>0}\left\|\frac{j^\varepsilon}{\varepsilon}\right\|^2_{L^2(]0,T[\times\mathbb{R})} \le \left(\int_{\mathbb{R}^2} e^{-\mathcal{E}(p)}\,dp\right)\sup_{\varepsilon>0}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^2} f^\varepsilon e^{\mathcal{E}(p)}|h^\varepsilon|^2\,dp\,dx\,dt.$$

The convergences of Proposition 3.2 are sufficient for passing to the limit with respect to k in (61). We obtain the equations

$$\theta\partial_t E_1 + \rho E_1 - \partial_x^2 E_1 = \partial_x D + \theta J,\quad (t,x)\in]0,T[\times\mathbb{R},$$

$$\partial_x E_1 = \rho - D,\quad (t,x)\in[0,T]\times\mathbb{R},$$

$$E_1(0,x) = E_{0,1}(x),\quad x\in\mathbb{R}.$$

We have already proved that $\lim_{k\to+\infty} E_2^k = 0$ weakly in $L^2(]0,T[\times\mathbb{R})$. Under the hypothesis

H8) $\lim_{R\to+\infty}\sup_{\varepsilon>0}\int_{|x|>R}\left(\frac{|E^\varepsilon_{0,2}(x)|}{\varepsilon} + \delta|B_0^\varepsilon(x)|\right)\,dx = 0,$

we claim that $\lim_{k\to+\infty}\left(\frac{E_2^k}{\varepsilon_k},\delta B^k\right) = 0$ in $\mathcal{D}'(]0,T[\times\mathbb{R})^2$. For any $\varphi\in C_c^1(]0,T[\times\mathbb{R})$ we have by (36)

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}}\delta B^k\varphi\,dx\,dt\right| &\le \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}}\left(\frac{E^k_{0,2}}{\varepsilon_k} + \delta B_0^k\right)(x)\varphi\left(t,x+\frac{t}{\varepsilon_k\delta}\right)\,dx\,dt\right|\\ &+ \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}}\left(\frac{E^k_{0,2}}{\varepsilon_k} - \delta B_0^k\right)(x)\varphi\left(t,x-\frac{t}{\varepsilon_k\delta}\right)\,dx\,dt\right|\\ &+ \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}}V_+^k\frac{\varphi}{\varepsilon_k}\,dx\,dt\right| + \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}}V_-^k\frac{\varphi}{\varepsilon_k}\,dx\,dt\right|. \end{aligned}\tag{70}$$

Take R large enough such that $\int_{|x|>R}\left(\frac{E^k_{0,2}(x)}{\varepsilon_k} + \delta|B_0^k(x)|\right)\,dx \le \eta$ for any k. Take $d>0$ large enough such that $\operatorname{supp}\varphi\subset[\frac{1}{d},T-\frac{1}{d}]\times[-d,d]$. Then for any $(t,x)\in[\frac{1}{d},T]\times[-R,R]$ and k satisfying $\varepsilon_k < \frac{1}{\delta d(R+d)}$ we have $|x\pm\frac{t}{\varepsilon_k\delta}| \ge \frac{1}{d\varepsilon_k\delta} - R > d$ saying that

$$\int_0^T\int_{-R}^R (E^k_{0,2}\pm\varepsilon_k\delta B_0^k)(x)\varphi\left(t,x\pm\frac{t}{\varepsilon_k\delta}\right)\,dx\,dt = 0,$$

and thus we have

$$\left|\int_0^T\int_{\mathbb{R}}(\frac{E^k_{0,2}}{\varepsilon_k}\pm\delta B^k_0)(x)\varphi(t,x\pm\frac{t}{\varepsilon_k\delta})\;dx\,dt\right| \begin{array}{l}\leq \|\varphi\|_{L^\infty}\int_0^T\int_{|x|>R}(\frac{|E^k_{0,2}|}{\varepsilon_k}+\delta|B^k_0|)\;dxdt\\ \leq \eta\|\varphi\|_{L^\infty}.\end{array}$$

Therefore the first and second term in the right hand side of (70) vanish as $k\to+\infty$. For the last two terms observe that we have

$$\int_0^T\int_{\mathbb{R}} V^k_{\pm}(t,x)\frac{\varphi(t,x)}{\varepsilon_k}\;dx\,dt = \int_0^T\int_{\mathbb{R}}\frac{j^k_2(t,x)}{\varepsilon_k}\psi^k_{\pm}(t,x)\;dx\,dt, \tag{71}$$

where for any $(t,x)\in[0,T]\times\mathbb{R}$

$$\psi^k_{\pm}(t,x)=\frac{1}{\varepsilon_k}\int_t^T\varphi(s,x\pm\frac{s-t}{\varepsilon_k\delta})\;ds=\pm\delta\int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}}\varphi(t\pm\varepsilon_k\delta(y-x),y)\;dy.$$

By using (62) we can write

$$\begin{aligned}-\theta\int_0^T\int_{\mathbb{R}}V^k_{\pm}\frac{\varphi}{\varepsilon_k}\;dx\,dt &= \int_0^T\int_{\mathbb{R}}\varepsilon_k\partial_t\left(\int_{\mathbb{R}^2}p_2f^k\;dp\right)\psi^k_{\pm}\;dx\,dt\\ &+ \int_0^T\int_{\mathbb{R}}\partial_x\left(\int_{\mathbb{R}^2}v_1(p)p_2f^k\;dp\right)\psi^k_{\pm}\;dx\,dt\\ &- \int_0^T\int_{\mathbb{R}}E^k_2\rho^k\psi^k_{\pm}\;dx\,dt+\int_0^T\int_{\mathbb{R}}\varepsilon_k\delta^2B^kj^k_1\psi^k_{\pm}\;dx\,dt\\ &= T^k_{\pm,1}+T^k_{\pm,2}+T^k_{\pm,3}+T^k_{\pm,4}.\end{aligned}\tag{72}$$

We are done if we prove that $\lim_{k\to+\infty}T^k_{\pm,l}=0$, $l\in\{1,2,3,4\}$. Observe that

$$\partial_t\psi^k_{\pm}(t,x)=\pm\delta\int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}}\partial_t\varphi(t\pm\varepsilon_k\delta(y-x),y)\;dy, \tag{73}$$

and

$$\begin{aligned}\partial_x\psi^k_{\pm}(t,x) &= -\varepsilon_k\delta^2\int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}}\partial_t\varphi(t\pm\varepsilon_k\delta(y-x),y)\;dy\mp\delta\varphi(t,x)\\ &= \mp\varepsilon_k\delta\partial_t\psi^k_{\pm}\mp\delta\varphi(t,x).\end{aligned}\tag{74}$$

Notice that $\psi^k_{\pm}$, $\partial_t\psi^k_{\pm}$, $\partial_x\psi^k_{\pm}$ are uniformly bounded for $k\geq 1$

$$\|\psi^k_{\pm}\|_{L^\infty(]0,T[\times\mathbb{R})}\leq\delta\int_{\mathbb{R}}\sup_{t\in[0,T]}|\varphi(t,x)|\;dx,$$

$$\|\partial_t\psi^k_{\pm}\|_{L^\infty(]0,T[\times\mathbb{R})}\leq\delta\int_{\mathbb{R}}\sup_{t\in[0,T]}|\partial_t\varphi(t,x)|\;dx,$$

$$\|\partial_x \psi_\pm^k\|_{L^\infty(]0,T[\times\mathbb{R})} \leq \delta \|\varphi\|_{L^\infty(]0,T[\times\mathbb{R})} + \varepsilon_1 \delta^2 \int_{\mathbb{R}} \sup_{t\in[0,T]} |\partial_t \varphi(t,x)| \, dx.$$

After integration by parts, by taking into account that $\psi_\pm^k(T,\cdot) = 0$ we find

$$\begin{aligned}
|T_{\pm,1}^k| &\leq \left| \varepsilon_k \int_{\mathbb{R}} \int_{\mathbb{R}^2} p_2 f_0^k(x,p) \psi_\pm^k(0,x) \, dp \, dx \right| \\
&+ \left| \varepsilon_k \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} p_2 f^k(t,x,p) \partial_t \psi_\pm^k \, dp \, dx \, dt \right| \\
&\leq C\varepsilon_k \left(\sup_{k'} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |p| f_0^{k'} \, dp \, dx + \sup_{k'} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} |p| f^{k'} \, dp \, dx \, dt \right),
\end{aligned}$$

implying that $\lim_{k\to+\infty} T_{\pm,1}^k = 0$. Similarly one gets by using (74)

$$\begin{aligned}
|T_{\pm,2}^k| &\leq \left| \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} v_1(p) p_2 f^k \varepsilon_k \delta \partial_t \psi_\pm^k \, dp \, dx \, dt \right| \\
&+ \left| \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} v_1(p) p_2 f^k \delta \varphi \, dp \, dx \, dt \right| \\
&\leq \varepsilon_k \|\partial_t \psi_\pm^k\|_{L^\infty} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} |p| f^k \, dp \, dx \, dt + \varepsilon_k \delta \left| \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} p_2 \sqrt{f^k} \, h_1^k \varphi \, dp \, dx \, dt \right|.
\end{aligned}$$

Notice that φ has compact support and then we deduce by (67) that $\lim_{k\to+\infty} T_{\pm,2}^k = 0$. The convergence $\lim_{k\to+\infty} T_{\pm,4}^k = 0$ follows by (66). Let us concentrate our attention on the convergence of $(T_{\pm,3}^k)_k$. Consider the functions

$$\tilde{\psi}_\pm(t,x) = \pm\delta \int_x^{\pm\infty} \varphi(t,y) \, dy, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

Here $\varphi \in C_c^1(]0,T[\times\mathbb{R})$ with supp $\subset [\frac{1}{d}, T - \frac{1}{d}] \times [-d,d]$ with $d > 0$ large enough. By Remark 3.1 we know that

$$\lim_{k\to+\infty} \int_0^T \int_{\mathbb{R}} E_2^k \rho^k \tilde{\psi}_\pm \, dx \, dt = 0.$$

Since $(E_2^k)_k$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$, $\sup_{k\in\mathbb{N}, t\in[0,T]} \int_{\mathbb{R}} (1+|x|) \rho^k(t,x) \, dx < +\infty$ and $\sup_{k\in\mathbb{N}} \|\psi_\pm^k\|_{L^\infty(]0,T[\times\mathbb{R})} < +\infty$ for any $\eta > 0$ there is $R = R(\eta)$ large enough such that

$$\left| \int_0^T \int_{|x|>R} E_2^k \rho^k \psi_\pm^k \, dx \, dt \right| < \eta, \quad \left| \int_0^T \int_{|x|>R} E_2^k \rho^k \tilde{\psi}_\pm \, dx \, dt \right| < \eta, \quad k \geq 1.$$

Take $k_1(\eta)$ such that

$$\left| \int_0^T \int_{\mathbb{R}} E_2^k \rho^k \tilde{\psi}_\pm \, dx \, dt \right| < \eta, \quad k \geq k_1(\eta).$$

Therefore we can write for any $k \geq k_1(\eta)$

$$\begin{aligned}
\left|\int_0^T\int_{\mathbb{R}} E_2^k \rho^k \psi_\pm^k \; dx\, dt\right| &< \eta + \left|\int_0^T\int_{\mathbb{R}} E_2^k \rho^k (\psi_\pm^k - \tilde{\psi}_\pm) \; dx\, dt\right| \\
&< 3\eta + \left|\int_0^T\int_{-R}^{R} E_2^k \rho^k (\psi_\pm^k - \tilde{\psi}_\pm) \; dx\; dt\right| \\
&= 3\eta + \left|\int_0^{T-\frac{1}{d}}\int_{-R}^{R} E_2^k \rho^k (\psi_\pm^k - \tilde{\psi}_\pm) \; dx\; dt\right|. \qquad (75)
\end{aligned}$$

Take now k_2 large enough such that $\frac{1}{d\varepsilon_k\delta} > R + d$ for any $k \geq k_2$. Observe that for all $(t,x) \in [0, T - \frac{1}{d}] \times [-R, R]$ and $k \geq k_2$ we have

$$x + \frac{T-t}{\varepsilon_k\delta} \geq \frac{1}{d\varepsilon_k\delta} - R > d, \;\; x - \frac{T-t}{\varepsilon_k\delta} \leq R - \frac{1}{d\varepsilon_k\delta} < -d,$$

saying that for any $(t,x) \in [0, T - \frac{1}{d}] \times [-R, R]$ and $k \geq k_2$ we have

$$\tilde{\psi}_\pm(t,x) = \pm\delta\int_x^{\pm\infty}\varphi(t,y)\; dy = \pm\delta\int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}}\varphi(t,y)\; dy.$$

Thus for any $(t,x) \in [0, T - \frac{1}{d}] \times [-R, R]$ and $k \geq k_2$ we have

$$\begin{aligned}
|\psi_\pm^k(t,x) - \tilde{\psi}_\pm(t,x)| &= \delta\left|\int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}}\{\varphi(t \pm \varepsilon_k\delta(y-x), y) - \varphi(t,y)\}\; dy\right| \\
&\leq \delta\left|\int_x^{x\pm\frac{T-t}{\varepsilon_k\delta}} \|\partial_t\varphi\|_{L^\infty}\; \varepsilon_k\delta\; |y-x|\mathbf{1}_{\{y\in[-d,d]\}}\; dy\right| \\
&\leq 2\|\partial_t\varphi\|_{L^\infty}\; \varepsilon_k\delta^2(d+R)d. \qquad (76)
\end{aligned}$$

Combining (75), (76) yields for any $k \geq \max\{k_1(\eta), k_2\}$

$$\left|\int_0^T\int_{\mathbb{R}} E_2^k \rho^k \psi_\pm^k \; dx\, dt\right| < 3\eta + 2\|\partial_t\varphi\|_{L^\infty}\varepsilon_k\delta^2(d+R)d\; \sup_{k'}\|E_2^{k'}\|_{L^\infty} T \sup_{k'}\|\rho_0^{k'}\|_{L^1},$$

and we deduce that $\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}} E_2^k\rho^k\psi_\pm^k \;\; dx\, dt = 0$. The above computations show also that $(\frac{E_2^k}{\varepsilon_k})_k$ converges to 0 in $\mathcal{D}'(]0,T[\times\mathbb{R})$ (use (35)). We have proved the theorem

Theorem 3.1 *Let* $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ *be smooth solutions of the problem* (11) − (17)*. Assume that H1-H7 hold and consider* $(\varepsilon_k)_k$ *the sequence constructed in Proposition* 3.1*. Then we have*

$$\rho^k \rightharpoonup \rho \geq 0 \;\; \textit{weakly in} \;\; L^1(]0,T[\times\mathbb{R}),$$

$$(E_1^k, E_2^k, \delta\varepsilon_k B^k) \rightharpoonup (E_1, 0, 0) \;\; \textit{weakly in} \;\; L^2(]0,T[\times\mathbb{R})^3, \;\; \textit{weakly} \; \star \;\; \textit{in} \; L^\infty(]0,T[\times\mathbb{R})^3,$$

$$E_1^k \to E_1, \ \ \textit{strongly in} \ \ L^1_{\text{loc}}([0,T]\times\mathbb{R}).$$

The limits ρ, E_1 satisfy in distribution sense

$$\theta\partial_t E_1 + \rho(t,x)E_1(t,x) - \partial_x^2 E_1 = \partial_x D + \theta J(t,x), \ \ (t,x) \in]0,T[\times\mathbb{R},$$

$$\partial_x E_1 = \rho(t,x) - D(t,x), \ \ (t,x) \in [0,T]\times\mathbb{R},$$

$$E_1(0,x) = \lim_{k\to+\infty} E_{0,1}^k(x), \ \ \textit{uniformly on compact sets of} \ \mathbb{R}.$$

Moreover if H8 holds we have

$$\lim_{k\to+\infty}\left(\frac{E_2^k}{\varepsilon_k}, \delta B^k\right) = 0, \ \ \textit{in} \ \mathcal{D}'(]0,T[\times\mathbb{R})^2.$$

It is possible to show also that $(f^k)_k$ converges towards $\rho(t,x)\frac{e^{-\mathcal{E}(p)}}{\int_{\mathbb{R}^2} e^{-\mathcal{E}(q)}dq}$ in some sense. We need to establish first that $(\rho^k)_k$ converges towards ρ in $C^0([0,T];\text{w}-L^1(\mathbb{R}))$. As in [22] we can prove

Lemma 3.1 *Assume that $(\rho^\varepsilon)_{\varepsilon>0}$, $(j_1^\varepsilon)_{\varepsilon>0}$ satisfy $\rho^\varepsilon \geq 0$,*

$$\partial_t \rho^\varepsilon + \partial_x \frac{j_1^\varepsilon}{\varepsilon} = 0, \ \ \text{in} \ \mathcal{D}'(]0,T[\times\mathbb{R}),$$

$$\sup_{\varepsilon>0, t\in[0,T]} \int_{\mathbb{R}} (1+|x|+|\ln\rho^\varepsilon|)\rho^\varepsilon(t,x)\,dx < +\infty,$$

and

$$\sup_{\varepsilon>0}\int_0^T \left(\int_{\mathbb{R}} \frac{|j_1^\varepsilon(t,x)|}{\varepsilon}\,dx\right)^2 dt < +\infty.$$

Then $(\rho^\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T];\text{w}-L^1(\mathbb{R}))$.

Proof. Following the ideas in [22] we can extract a sequence $(\varepsilon_k)_k$ decreasing towards 0 such that for any $\varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we have

$$\lim_{k\to+\infty}\int_{\mathbb{R}} \rho^k(t,x)\varphi(x)\,dx = \int_{\mathbb{R}} \rho(t,x)\varphi(x)\,dx, \ \ \text{uniformly in} \ \ t\in[0,T]. \tag{77}$$

Actually (77) holds for any $\varphi \in L^\infty(\mathbb{R})$. Indeed, for any $\eta > 0$ take $R > 0$ large enough such that

$$\sup_{k\in\mathbb{N}, t\in[0,T]} \int_{|x|>R} \rho^k(t,x)\,dx < \eta, \ \ \sup_{t\in[0,T]}\int_{|x|>R}\rho(t,x)\,dx < \eta. \tag{78}$$

By the hypotheses we can find $\mu = \mu(\eta) > 0$ such that for any $t \in [0,T]$ and measurable set A satisfying $\text{meas}(A) < \mu$ we have

$$\sup_{\varepsilon>0, t\in[0,T]}\int_A \rho^\varepsilon(t,x)\,dx < \eta, \ \ \sup_{t\in[0,T]}\int_A \rho(t,x)\,dx < \eta. \tag{79}$$

By Lusin theorem (cf. [32], p. 52) there is a function $\varphi_\eta \in C_c^0(\mathbb{R})$, $\|\varphi_\eta\|_{L^\infty} \leq \|\varphi\|_{L^\infty}$ such that

$$\text{meas}\,(\{x \in [-R, R] \ : \ \varphi_\eta(x) \neq \varphi(x)\}) < \mu. \tag{80}$$

Combining (78), (79), (80) yields

$$\begin{aligned}
\left|\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))\varphi(x)\,dx\right| &\leq \left|\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))\varphi_\eta(x)\,dx\right| \\
&+ \left|\int_{-R}^{R}(\rho^k(t,x) - \rho(t,x))(\varphi(x) - \varphi_\eta(x))\,dx\right| \\
&+ \left|\int_{|x|>R}(\rho^k(t,x) - \rho(t,x))(\varphi(x) - \varphi_\eta(x))\,dx\right| \\
&\leq \left|\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))\varphi_\eta(x)\,dx\right| + 8\eta\|\varphi\|_{L^\infty}.
\end{aligned}$$

Since we know that $\lim_{k\to+\infty}\int_{\mathbb{R}} \rho^k(t,x)\varphi_\eta(x)\,dx = \int_{\mathbb{R}} \rho(t,x)\varphi_\eta(x)\,dx$ uniformly in $t \in [0,T]$ we conclude that $\lim_{k\to+\infty}\int_{\mathbb{R}} \rho^k(t,x)\varphi(x)\,dx = \int_{\mathbb{R}} \rho(t,x)\varphi(x)\,dx$ uniformly in $t \in [0,T]$. $\square$

Corollary 3.1 *Let us set* $M(p) = \frac{e^{-\mathcal{E}(p)}}{\int_{\mathbb{R}^2} e^{-\mathcal{E}(q)}dq}$ *for any* $p \in \mathbb{R}^2$*. Under the assumptions of Theorem* 3.1*,* $(f^k)_k$ *converges towards* $\rho(t,x)M(p)$ *in the following sense*

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}}(f^k(t,x,p) - \rho(t,x)M(p))\varphi(x)\,dx\right|\,dp\,dt = 0,$$

for any test function $\varphi \in L^\infty(\mathbb{R})$.

Proof. We write $f^k - \rho(t,x)M(p) = f^k - \rho^k(t,x)M(p) + (\rho^k(t,x) - \rho(t,x))M(p)$. Consider now $\varphi \in L^\infty(\mathbb{R})$. By Lemma 3.1 we have

$$\lim_{k\to+\infty}\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))\varphi(x)\,dx = 0, \ \text{ uniformly in } \ t \in [0,T].$$

By using the dominated convergence theorem we have

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}}(\rho^k(t,x) - \rho(t,x))M(p)\varphi(x)\,dx\right|\,dp\,dt = 0.$$

It remains to discuss $f^k - \rho^k M$. By logarithmic Sobolev inequality (see [3], [2]) we obtain

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^2} \left\{ \frac{f^k}{\rho^k M(p)} \ln\left(\frac{f^k}{\rho^k M(p)}\right) - \frac{f^k}{\rho^k M(p)} + 1 \right\} \rho^k M(p)\, dp \\
&= \int_{\mathbb{R}^2} f^k \ln\left(\frac{f^k}{\rho^k M(p)}\right)\, dp \\
&\leq \lambda \int_{\mathbb{R}^2} \left| \nabla_p \sqrt{\frac{f^k}{M(p)}} \right|^2 M(p)\, dp \\
&= \frac{\lambda \varepsilon_k^2}{4} \int_{\mathbb{R}^2} |h^k(t,x,p)|^2\, dp,
\end{aligned}
$$

for some $\lambda > 0$. We conclude by using the Csiszar-Kullback-Pinsker inequality, see [16], [26]

$$
\left(\int_{\mathbb{R}} \int_{\mathbb{R}^2} |f^k - \rho^k M(p)|\, dp\, dx \right)^2 \leq \mu \int_{\mathbb{R}} \int_{\mathbb{R}^2} f^k \ln\left(\frac{f^k}{\rho^k M(p)}\right)\, dp\, dx,
$$

for some $\mu > 0$ which implies that

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} (f^k - \rho^k M)\varphi\, dx \right| dpdt &\leq \|\varphi\|_{L^\infty} \sqrt{T} \left(\int_0^T \left(\int_{\mathbb{R}} \int_{\mathbb{R}^2} |f^k - \rho^k M|\, dp\, dx \right)^2 dt \right)^{1/2} \\
&\leq |\|\varphi\|_{L^\infty} \sqrt{\mu T} \left(\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} f^k \ln\left(\frac{f^k}{\rho^k M}\right)\, dp\, dx\, dt \right)^{1/2} \\
&\leq \|\varphi\|_{L^\infty} \sqrt{\lambda \mu T} \frac{\varepsilon_k}{2} \left(\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^2} |h^k|^2\, dp\, dx\, dt \right)^{1/2} \to 0.
\end{aligned}
$$

□

4 Appendix

We detail here the dimensional analysis of the equations and the physical meaning of the different parameters. We introduce the following physical constants
- ε_0 the vacuum permittivity ;
- c_0 the vacuum light speed ;
- q the charge of particles ;
- m the mass of particles ;
- τ the relaxation time which characterizes the interactions of the particles with the thermal bath ;
- K_B the Boltzmann constant ;
- T_{th} the temperature of the thermal bath.

Let $f(t,x,p)$ denote the particle distribution function, which depends on the time $t>0$, space coordinates $x\in\mathbb{R}^3$ and impulsion coordinates $p\in\mathbb{R}^3$. The evolution of f is described by the Fokker-Planck equation

$$\partial_t f+v(p)\cdot\nabla_x f+q(E(t,x)+v(p)\wedge B(t,x))\cdot\nabla_p f = L_{FP}(f),\ \ (t,x,p)\in]0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3,$$

where the relativistic Fokker-Planck collision operator is given by

$$L_{FP}(f)=\frac{p_{\mathrm{th}}^2}{\tau}\mathrm{div}_p\left(\frac{v(p)}{\mu}f+\nabla_p f\right)=\frac{p_{\mathrm{th}}^2}{\tau}\mathrm{div}_p\left(\mathcal{M}(p)\nabla_p\left(\frac{f}{\mathcal{M}(p)}\right)\right).$$

Here $\mathcal{M}(p)=e^{-\mathcal{E}(p)/\mu}$ is the relativistic maxwellian where $\mathcal{E}(p)=mc_0^2(\sqrt{1+|p|^2/(m^2c_0^2)}-1)$ is the relativistic energy, $\mu=KT_{\mathrm{th}}$ and the thermal impulsion $p_{\mathrm{th}}>0$ is given by

$$mc_0^2\left(\sqrt{1+\frac{p_{\mathrm{th}}^2}{m^2c_0^2}}-1\right)=\mu,$$

which is equivalent to $p_{\mathrm{th}}=\sqrt{\mu^2/c_0^2+2\mu m}$. The evolution of the electro-magnetic field (E,B) is given by the Maxwell equations

$$\partial_t E-c_0^2\,\mathrm{curl}_x B=-\frac{j(t,x)}{\varepsilon_0},\ \ \partial_t B+\mathrm{curl}_x E=0,\ \ (t,x)\in]0,+\infty[\times\mathbb{R}^3,$$

$$\mathrm{div}_x E=\frac{\rho(t,x)}{\varepsilon_0},\ \ \mathrm{div}_x B=0,\ \ (t,x)\in]0,+\infty[\times\mathbb{R}^3,$$

where $\rho=q\int_{\mathbb{R}^3}f\ dp$, $j=q\int_{\mathbb{R}^3}v(p)f\ dp$ are respectively the charge and current densities. We denote by $v_{\mathrm{th}}>0$ the thermal velocity given by

$$v_{\mathrm{th}}=\frac{p_{\mathrm{th}}}{m\sqrt{1+\frac{p_{\mathrm{th}}^2}{m^2c_0^2}}}=c_0\sqrt{1-\frac{1}{\left(1+\frac{\mu}{mc_0^2}\right)^2}}.$$

By direct computation we check that $\frac{p_{\mathrm{th}}v_{\mathrm{th}}}{\mu}=\frac{\mu+2mc_0^2}{\mu+mc_0^2}=:\theta\in]1,2[$. We introduce a length unit L, a time unit T and the parameters $\alpha=\frac{Tv_{\mathrm{th}}}{L}$, $\beta=\frac{\tau v_{\mathrm{th}}}{L}$. As impulsion unit we take $P=p_{\mathrm{th}}$. We define dimensionless variables and unknowns by the relations

$$t=Tt',\ \ x=Lx',\ \ p=p_{\mathrm{th}}p',$$

$$f(t,x,p)=\frac{\mathcal{N}}{L^3p_{\mathrm{th}}^3}f'(\frac{t}{T},\frac{x}{L},\frac{p}{P}),\ \ E(t,x)=\frac{U_{\mathrm{th}}}{L}E'(\frac{t}{T},\frac{x}{L}),\ \ B(t,x)=\frac{U_{\mathrm{th}}}{Tc_0^2}B'(\frac{t}{T},\frac{x}{L}),$$

where $\mathcal{N}$ is the total number of particles, U_{th} is the thermal potential given by $qU_{\mathrm{th}}=\mu$. After changing variables and unknowns, we obtain dropping the primes

$$\partial_t f+\frac{\alpha}{\theta}v(p)\cdot\nabla_x f+\left(\frac{\alpha}{\theta}\ E(t,x)+\frac{v_{\mathrm{th}}^2}{\theta^2c_0^2}\ v(p)\wedge B(t,x)\right)\cdot\nabla_p f=\frac{\alpha}{\beta}L(f),$$

$$\partial_t E - \mathrm{curl}_x B = -\frac{\alpha}{\beta}\frac{L^2}{\Lambda(L)^2}\, j(t,x), \quad \frac{v_{\mathrm{th}}^2}{\alpha^2 c_0^2}\partial_t B + \mathrm{curl}_x E = 0,$$

$$\mathrm{div}_x E = \frac{L^2}{\Lambda(L)^2}\, \rho(t,x), \quad \mathrm{div}_x B = 0,$$

where $L(f) = \mathrm{div}_p \left(\mathcal{M}(p)\nabla_p \left(\frac{f}{\mathcal{M}(p)}\right)\right)$, $\mathcal{M}(p) = e^{-\mathcal{E}(p)}$

$$\mathcal{E}(p) = \frac{mc_0^2}{\mu}\left(\sqrt{1+\frac{p_{\mathrm{th}}^2}{m^2c_0^2}|p|^2} - 1\right), \quad v(p) = \nabla_p \mathcal{E}(p) = \frac{p_{\mathrm{th}}^2}{m\mu}\frac{p}{\sqrt{1+\frac{p_{\mathrm{th}}^2}{m^2c_0^2}|p|^2}}, \tag{81}$$

$$\rho = \int_{\mathbb{R}^3} f\, dp, \quad j = \int_{\mathbb{R}^3} v(p) f\, dp, \quad \Lambda(L) = \sqrt{\frac{\varepsilon_0 \mu L^3}{q^2\mathcal{N}}},$$

is the Debye length. We take $L(\mu) = \frac{q^2\mathcal{N}}{\varepsilon_0\mu}$ which means $L = \Lambda(L)$ and $T = \frac{q^4\mathcal{N}^2}{\varepsilon_0^2\tau}\frac{1}{\mu^2 v_{\mathrm{th}}^2}$ which is equivalent to $\alpha\,\beta = \theta$ saying that the scaled thermal velocity $\frac{v_{\mathrm{th}}}{L/T}$ is proportional with the inverse of the scaled thermal mean free path. We are interested on the asymptotic behavior when the scaled thermal mean free path goes to 0. Finally, by introducing the small parameter $\varepsilon > 0$ such that $\frac{\alpha}{\theta} = \frac{1}{\varepsilon}$, $\beta = \varepsilon$, one gets the system

$$\partial_t f + \frac{1}{\varepsilon} v(p)\cdot\nabla_x f + \left(\frac{1}{\varepsilon}E(t,x) + \delta^2 v(p)\wedge B(t,x)\right)\cdot\nabla_p f = \frac{\theta}{\varepsilon^2}L(f),$$

$$\partial_t E - \mathrm{curl}_x B = -\frac{1}{\varepsilon} j(t,x), \quad \varepsilon^2\delta^2\partial_t B + \mathrm{curl}_x E = 0,$$

$$\mathrm{div}_x E = \rho(t,x), \quad \mathrm{div}_x B = 0,$$

where $\delta = \frac{v_{\mathrm{th}}}{\theta c_0}$.

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