Analysis of a particle method for the one dimensional Vlasov-Maxwell system

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Abstract

In this article we present a particle method for solving numerically the one dimensional Vlasov-Maxwell equations. This method is based on the formulation by characteristics. We perform the error analysis and we investigate the properties of this scheme.

Keywords: Maxwell equations, Vlasov equation, particle method, error analysis.

AMS classification: 35A05, 35Q75, 65M15, 65M25, 82D10.

1 Introduction

There are now several methods for the numerical resolution of the Vlasov-Maxwell equations. We can use Lagrangian methods like particle-in-cell methods (PIC), relying on the approximation of the population of charged particles by a finite number

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of macro-particles. At each time iteration we compute the new positions and impulsions of the macro-particles by using the characteristics of the Vlasov equation whereas the new values of the electro-magnetic field are calculated by using the Maxwell equations. For updating the electro-magnetic field we need to approximate the charge and current densities (which are the source terms of the Maxwell equations) on a mesh of the physical space, see [4]. For the convergence analysis point of view, the approximation of the charge and current densities requires the smoothness of the particle distribution.

Eulerian methods have been proposed as well. Finite volume schemes were constructed in [5], [8], [14], [17]. Other methods are the semi-Lagrangian ones, which combine the advantages of both Eulerian and Lagrangian schemes. At each time step the particle distribution function is computed on a fixed Cartesian mesh in the phase space by backward integration along the characteristics and interpolation with respect to the particle distribution values at the previous time step, cf. [1], [2], [25], [12]. When performing the error analysis, in order to handle the interpolation on the phase space mesh, we need to assume that the particle distribution is smooth.

The aim of this paper is to construct a new numerical scheme for solving the one dimensional Vlasov-Maxwell equations, based on the formulation by characteristics. Consider a population of charged particles with mass m and charge q. We assume that the collisions between particles are so rare such that we can neglect them. The particle distribution function f, depending on the time $t \in [0, T]$, position $x \in \mathbb{R}$ and impulsion $p \in \mathbb{R}$ satisfies the Vlasov equation

$$\partial_t f + v(p)\partial_x f + q\, E(t,x)\partial_p f = 0, \quad (t,x,p) \in]0,T[\times\mathbb{R}\times\mathbb{R}, \tag{1}$$

where E represents the self-consistent electric field, verifying the Maxwell equations

$$\partial_t E = -\frac{j(t,x)}{\varepsilon_0}, \quad \partial_x E = \frac{\rho(t,x)}{\varepsilon_0}, \quad (t,x) \in]0,T[\times\mathbb{R}. \tag{2}$$

Here ε_0 is the dielectric permittivity of the vacuum, ρ is the charge density and j is the current density, given by

$$\rho(t,x) = q \int_{\mathbb{R}} f(t,x,p)\, dp, \quad j(t,x) = q \int_{\mathbb{R}} v(p) f(t,x,p)\, dp, \ \forall\ (t,x) \in [0,T] \times \mathbb{R}.$$

The notation $v(p)$ stands for the velocity associated to a given impulsion $p \in \mathbb{R}$. In the non relativistic case we have $v(p) = \frac{p}{m}$ and in the relativistic case we have $v(p) = \frac{p}{m}\left(1 + \frac{p^2}{m^2c^2}\right)^{-1/2}$ where c represents the light speed in the vacuum. Integrating the Vlasov equation with respect to $p \in \mathbb{R}$ leads to the continuity equation

$$\partial_t \rho + \partial_x j = 0, \quad (t,x) \in]0,T[\times\mathbb{R}, \tag{3}$$

and we deduce as usual that if $\partial_x E(0,\cdot) = \frac{\rho(0,\cdot)}{\varepsilon_0}$ then $\partial_x E(t,\cdot) = \frac{\rho(t,\cdot)}{\varepsilon_0}$ holds true for any $t \in]0,T]$. We prescribe initial conditions for the particle distribution function and the electric field

$$f(t=0,x,p) = f_0(x,p), \quad (x,p) \in \mathbb{R}^2, \tag{4}$$

$$E(t=0,x) = E_0(x), \quad x \in \mathbb{R}. \tag{5}$$

We assume that the initial conditions satisfy the constraint

$$\frac{d}{dx} E_0 = \frac{\rho_0(x)}{\varepsilon_0} := \frac{q}{\varepsilon_0} \int_{\mathbb{R}} f_0(x,p)\, dp, \ \forall\ x \in \mathbb{R}. \tag{6}$$

Obviously, in one space dimension and one impulsion dimension the Maxwell equations (2) degenerate to the Poisson equation

$$-\partial_x^2 \Phi = \frac{\rho(t,x)}{\varepsilon_0}, \quad E(t,x) = -\partial_x \Phi, \quad (t,x) \in]0,T[\times\mathbb{R}.$$

We consider physical units such that $m = 1$, $q = 1$, $\varepsilon_0 = 1$. Assume for the moment that the electric field is a smooth given function and let us introduce the system of characteristics for (1)

$$\frac{dX}{ds} = v(P(s)), \quad \frac{dP}{ds} = E(s, X(s)), \quad 0 < s < T, \tag{7}$$

with the conditions

$$X(s=t) = x, \quad P(s=t) = p. \tag{8}$$

Notice that if $E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))$ then for any $(t,x,p) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ there is a unique solution for (7), (8) denoted

$$(X(s), P(s)) = (X(s;t,x,p), P(s;t,x,p)), \ \ 0 \le s \le T.$$

Observe that (1) can be written formally

$$\frac{d}{ds} f(s, X(s;t,x,p), P(s;t,x,p)) = 0, \ \ (s,t,x,p) \in [0,T]^2 \times \mathbb{R}^2, \tag{9}$$

and therefore we define the solution by characteristics (or mild solution) of (1), (4) by

$$f(t,x,p) = f_0(X(0;t,x,p), P(0;t,x,p)), \ \ (t,x,p) \in [0,T] \times \mathbb{R}^2.$$

Since $\operatorname{div}_{(x,p)}(v(p), E(t,x)) = 0$ we have

$$\det\left(\frac{\partial(X(s;t,x,p), P(s;t,x,p))}{\partial(x,p)}\right) = 1, \ \forall \ (s,t,x,p) \in [0,T]^2 \times \mathbb{R}^2. \tag{10}$$

Assume that f_0 is a non negative function which belongs to $L^1(\mathbb{R}^2)$. By using the change of variables $(x,p) \to (X(s;0,x,p), P(s;0,x,p)), \quad (x,p) \in \mathbb{R}^2, \quad s \in [0,T]$, we obtain easily that the mild solution satisfies the following formulation by characteristics for any continuous bounded function ψ

$$\int_0^T\!\!\int_{\mathbb{R}}\int_{\mathbb{R}} f\psi \ dp\ dx\ dt = \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^T \psi(s, X(s;0,x,p), P(s;0,x,p))\ ds\ dp\ dx. \tag{11}$$

Assume now that (f,E) is a solution of the one dimensional Vlasov-Maxwell problem (1), (2), (4), (5) and let us see how the electric field can be expressed in terms of characteristics. For any smooth function φ we have for $t \in [0,T]$

$$\int_{\mathbb{R}} E(t,x)\varphi(x)\ dx \ \ = \ \ \int_{\mathbb{R}} E_0(x)\varphi(x)\ dx - \int_0^t\!\!\int_{\mathbb{R}}\int_{\mathbb{R}} f(s,x,p)v(p)\varphi(x)\ dp\ dx\ ds. \tag{12}$$

By using the formulation (11) with the test function $\psi(t,x,p) = v(p)\varphi(x)$ we obtain

$$\int_0^t\!\!\int_{\mathbb{R}}\int_{\mathbb{R}} f(s,x,p)v(p)\varphi(x)\ dp\ dx\ ds = \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_x^{X(t;0,x,p)} \varphi(u)\ du\ dp\ dx. \tag{13}$$

Combining (12), (13) yields for any smooth function φ and $t \in [0,T]$

$$\int_{\mathbb{R}} E(t,x)\varphi(x)\ dx = \int_{\mathbb{R}} E_0(x)\varphi(x)\ dx - \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_x^{X(t;0,x,p)} \varphi(u)\ du\ dp\ dx. \tag{14}$$

Actually, when $f_0 \in L^1(\mathbb{R}^2)$, $E_0 \in L^\infty(\mathbb{R})$ the above formula holds for any function $\varphi \in L^1(\mathbb{R})$. We intend to construct a numerical scheme based on formula (14). The main point here is that the computation of the electric field do not require neither the calculation of the charge and current densities nor the explicit resolution of the Maxwell equations. In fact such schemes rely only on the approximation of the characteristics and the electric field, which are generally more regular than the particle distribution (think that f can be a L^1 function or even a measure). We do not need to ask for the smoothness of the particle distribution of the exact solution since no interpolation is performed. We will see that this method allows us to approximate solutions by characteristics, launched by initial particle densities $f_0 \in L^1(\mathbb{R}^2)$ verifying $\sup_{x\in\mathbb{R}} f_0(x,\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. At this stage let us mention the recent analysis of the relativistic Vlasov-Maxwell equations in [16], where the authors approximate particle distributions of bounded variation.

The convergence of the particle method for solving the one dimensional Vlasov-Poisson system was analyzed in [10], [11], [29], [30]. Results for the multi dimensional Vlasov-Poisson system were obtained in [24], [18], [26], [27], [28]. The convergence of a finite volume scheme for the one dimensional Vlasov-Poisson system is done in [15] and the convergence of a semi-Lagrangian scheme for the one dimensional Vlasov-Poisson system is performed in [3]. The analysis of the particle method for the relativistic Vlasov-Maxwell system can be found in [22].

This paper is organized as follows. In Section 2 we recall the existence and uniqueness result of the solution by characteristics for the one dimensional Vlasov-Maxwell equations. In Section 3 we introduce our numerical scheme. We perform the error analysis and we indicate how this scheme can be localized in space, due to the finite speed propagation property. The last section is devoted to numerical simulations based on our numerical scheme and we compare these results with respect to that obtained by standard particle methods.

2 The one dimensional Vlasov-Maxwell system

The Cauchy problem of the Vlasov-Maxwell equations was studied by many authors, cf. [9], [13], [19], [20], [21], [23], [7]. In this section we recall the existence and uniqueness results for the one dimensional Vlasov-Maxwell equations. We present them in the relativistic setting (*i.e.*, $v(p) = p/\sqrt{1+p^2/c^2}$), but analogous results hold in the classical case (*i.e.*, $v(p) = p$). We assume that the initial conditions satisfy

H1) there is a function $g_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ non decreasing on $\mathbb{R}^-$ and non increasing on $\mathbb{R}^+$ such that $0 \le f_0(x,p) \le g_0(p)$, $\forall\ (x,p) \in \mathbb{R}^2$;

H2) f_0 belongs to $L^1(\mathbb{R}^2)$;

H3) E_0 belongs to $L^\infty(\mathbb{R})$ such that $E_0' = \rho_0 := \int_{\mathbb{R}} f_0\ dp$.

Following the arguments in [6] we obtain by standard successive approximations

Theorem 2.1 *Assume that* (f_0, E_0) *satisfy the hypotheses H1, H2, H3. Then there is a global unique (mild) solution* $(f, E) \in L^\infty(]0;+\infty[; L^1(\mathbb{R}^2)) \times L^\infty(]0,+\infty[\times\mathbb{R})$ *of* (1), (2), (4), (5) *satisfying*

$$\|E\|_{L^\infty(]0,+\infty[\times\mathbb{R})} \le a,$$

$$\|\partial_x E\|_{L^\infty(]0,T[\times\mathbb{R})} = \left\| \int_{\mathbb{R}} f(\cdot,\cdot,p)\ dp \right\|_{L^\infty(]0,T[\times\mathbb{R})} \le b(T),\ \forall\ T > 0,$$

$$\|\partial_t E\|_{L^\infty(]0,T[\times\mathbb{R})} = \left\| \int_{\mathbb{R}} v(p) f(\cdot,\cdot,p)\ dp \right\|_{L^\infty(]0,T[\times\mathbb{R})} \le c\, b(T),\ \forall\ T > 0,$$

where $a = \|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)}$ *and* $b(T) = \|g_0\|_{L^1(\mathbb{R})} + 2\|g_0\|_{L^\infty(\mathbb{R})}\ T\ a$.

The next result emphasizes the finite propagation speed of the solution constructed above (in the relativistic case). The proof follows very similar lines to those presented in [6].

Theorem 2.2 *Assume that* $(f_0^k, E_0^k)_{k\in\{1,2\}}$ *satisfy the hypotheses H1, H2, H3 and denote by* $(f^k, E^k)_{k\in\{1,2\}}$ *the global unique solutions of the relativistic Vlasov-Maxwell system in one dimension corresponding to the initial conditions* $(f_0^k, E_0^k)_{k\in\{1,2\}}$*. Then for any* $R > 0$ *there is a constant* $C(R/c)$ *such that for all* $t \in [0, R/c]$ *we have*

$$\|E^1(t)-E^2(t)\|_{L^\infty(]-(R-ct),R-ct[)} \le C(\frac{R}{c})\left(\|f_0^1 - f_0^2\|_{L^1(]-R,R[\times\mathbb{R})} + \|E_0^1 - E_0^2\|_{L^\infty(]-R,R[)}\right).$$

In particular if $f_0^1(x,p) = f_0^2(x,p),\ \forall\ (x,p) \in]-R,R[\times\mathbb{R}$ *and* $E_0^1(x) = E_0^2(x),\ \forall\ x \in]-R,R[$ *for some* $R > 0$ *then, for any* $t \in]0, R/c[$ *we have*

$$f^1(t,x,p) = f^2(t,x,p),\ \forall\ (x,p) \in]-(R-ct), R-ct[\times\mathbb{R},$$

$$E^1(t,x) = E^2(t,x),\ \forall\ x \in]-(R-ct), R-ct[.$$

Corollary 2.1 *Assume that* $(f_0^k, E_0^k)_{k\in\{1,2\}}$ *satisfy the hypotheses H1, H2, H3 and denote by* $(f^k, E^k)_{k\in\{1,2\}}$ *the global unique solutions of the relativistic Vlasov-Maxwell system in one dimension corresponding to the initial conditions* $(f_0^k, E_0^k)_{k\in\{1,2\}}$*. Then for any* $T > 0$ *there is a constant* C_T *depending on* T *and the initial conditions such that*

$$\begin{aligned}&\|(X^1 - X^2)(t;0,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^2)} + \|(P^1 - P^2)(t;0,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^2)} + \|E^1(t) - E^2(t)\|_{L^\infty(\mathbb{R})}\\ &\quad\le\ C_T\left(\|f_0^1 - f_0^2\|_{L^1(\mathbb{R}^2)} + \|E_0^1 - E_0^2\|_{L^\infty(\mathbb{R})}\right).\end{aligned}$$

Proof. For any $(x,t) \in \mathbb{R}\times]0,T[$ we deduce by Theorem 2.2 applied with $R = cT$ that

$$\begin{aligned}\|E^1(t) - E^2(t)\|_{L^\infty(]x-c(T-t),x+c(T-t)[)} &\le C_T\|f_0^1 - f_0^2\|_{L^1(]x-cT,x+cT[\times\mathbb{R})}\\ &+ C_T\|E_0^1 - E_0^2\|_{L^\infty(]x-cT,x+cT[)}\\ &\le C_T\left(\|f_0^1 - f_0^2\|_{L^1(\mathbb{R}^2)} + \|E_0^1 - E_0^2\|_{L^\infty(\mathbb{R})}\right),\end{aligned}$$

and our conclusion comes by the continuous dependence of the characteristics upon the electric field

$$\begin{aligned}|X^1(t;0,x,p) - X^2(t;0,x,p)| &+ |P^1(t;0,x,p) - P^2(t;0,x,p)| \le \exp(t(1 + b(t)))\\ &\times \int_0^t \|E^1(s) - E^2(s)\|_{L^\infty(\mathbb{R})} ds,\end{aligned}$$

where $b(t) = \max\{b^1(t), b^2(t)\}$. □

3 Numerical approximation of the one dimensional Vlasov-Maxwell system

Assume that the initial conditions (f_0, E_0) satisfy the hypotheses H1, H2, H3 and denote by (f, E) the global unique solution of the relativistic Vlasov-Maxwell system (1), (2), (4), (5). In this section we construct a scheme for the numerical computation of the solution (f, E). We perform the error analysis and study the properties of this scheme.

Consider $(x_i, p_j) = (i\Delta x, j\Delta p)$, $(i, j) \in \mathbb{Z}^2$ the mesh points of the phase space, where $\Delta x, \Delta p > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a compactly supported non negative function satisfying $\int_{\mathbb{R}} \varphi(u)\, du = 1$. The time step is denoted by Δt and we consider $t^n = n\Delta t$, $n \in \mathbb{N}$.

3.1 Approximation of the initial conditions

For any $(i, j) \in \mathbb{Z}^2$ let $C_i^x = [(i - 1/2)\Delta x, (i + 1/2)\Delta x[$, $C_j^p = [(j - 1/2)\Delta p, (j + 1/2)\Delta p[$ and consider

$$f_{ij}^0 = \frac{1}{\Delta x \Delta p} \int_{C_i^x} \int_{C_j^p} f_0(x, p)\, dp\, dx,\ \forall\ (i, j) \in \mathbb{Z}^2,\ \ \rho_i^0 = \sum_{j\in\mathbb{Z}} \Delta p f_{ij}^0,\ \forall\ i \in \mathbb{Z}.$$

Obviously we have

$$\sum_{(i,j)\in\mathbb{Z}^2} \Delta x \Delta p f_{ij}^0 = \sum_{i\in\mathbb{Z}} \Delta x \rho_i^0 = \|f_0\|_{L^1(\mathbb{R}^2)}.$$

In order to approximate the initial electric field consider the function $\tilde{\rho}_0 : \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{\rho}_0(x) = \sum_{i\in\mathbb{Z}} \rho_i^0 \varphi\left(\frac{x_i - x}{\Delta x}\right),\ \ x \in \mathbb{R}. \tag{15}$$

Since φ has compact support the above function is well defined. Moreover $\tilde{\rho}_0$ belongs to $L^1(\mathbb{R})$ and

$$\|\tilde{\rho}_0\|_{L^1(\mathbb{R})} = \sum_{i\in\mathbb{Z}} \Delta x \rho_i^0 = \|\rho_0\|_{L^1(\mathbb{R})}.$$

We approximate the initial electric field E_0 by $\tilde{E}_0 : \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{E}_0(x) = E_0(x) + \int_{-\infty}^{x} \{\tilde{\rho}_0(y) - \rho_0(y)\}\ dy, \quad x \in \mathbb{R}. \tag{16}$$

Since $\frac{d}{dx}E_0 = \rho_0 \geq 0$ and E_0 is bounded, we deduce that $\lim_{y\to\pm\infty} E_0(y) \in \mathbb{R}$ and therefore the above formula can be written

$$\tilde{E}_0(x) = \lim_{y\to-\infty} E_0(y) + \int_{-\infty}^{x} \tilde{\rho}_0(y)\ dy = \lim_{y\to-\infty} E_0(y) + \sum_{i\in\mathbb{Z}} \rho_i^0 \int_{-\infty}^{x} \varphi\left(\frac{x_i - y}{\Delta x}\right)\ dy. \tag{17}$$

This choice will be motivated by further computations. Let us estimate the error $\|\tilde{E}_0 - E_0\|_{L^\infty(\mathbb{R})}$.

Proposition 3.1 *Assume that (f_0, E_0) satisfy H1, H2, H3 and consider φ a non negative compactly supported function verifying $\int_{\mathbb{R}} \varphi(u)\ du = 1$. Then $\tilde{E}_0$ is bounded, non decreasing and we have*

$$\|\tilde{E}_0 - E_0\|_{L^\infty(\mathbb{R})} \leq C\Delta x \|g_0\|_{L^1(\mathbb{R})}, \quad \|\tilde{E}_0'\|_{L^\infty(\mathbb{R})} \leq C\|g_0\|_{L^1(\mathbb{R})}, \tag{18}$$

for some constant C depending on φ.

Proof. Since $\frac{d}{dx}\tilde{E}_0 = \tilde{\rho}_0 \geq 0$ we deduce that $\tilde{E}_0$ is non decreasing. Take $R > 0$ such that $\operatorname{supp} \varphi \subset [-R, R]$ and for any $x \in \mathbb{R}$ consider the sets $\tilde{I}_1(x) = \{i \in \mathbb{Z} : x_i + R\Delta x \leq x\}$, $\tilde{I}_2(x) = \{i \in \mathbb{Z} : x_i - R\Delta x \leq x < x_i + R\Delta x\}$, $\tilde{I}_3(x) = \{i \in \mathbb{Z} : x_i - R\Delta x > x\}$. Observe that we have $\int_{-\infty}^{x} \varphi\left(\frac{x_i - y}{\Delta x}\right)\ dy = \Delta x,\ \forall\, i \in \tilde{I}_1(x)$ and $\int_{-\infty}^{x} \varphi\left(\frac{x_i - y}{\Delta x}\right)\ dy = 0,\ \forall\, i \in \tilde{I}_3(x)$. Therefore we obtain

$$\int_{-\infty}^{x} \tilde{\rho}_0(y)\ dy = \sum_{i\in\tilde{I}_1(x)} \rho_i^0 \Delta x + \sum_{i\in\tilde{I}_2(x)} \rho_i^0 \int_{-\infty}^{x} \varphi\left(\frac{x_i - y}{\Delta x}\right)\ dy. \tag{19}$$

Notice that

$$\operatorname{card} \tilde{I}_2(x) \leq 1 + 2R. \tag{20}$$

By using H1 we have for any $i \in \mathbb{Z}$

$$\rho_i^0 \Delta x = \int_{C_i^x} \int_{\mathbb{R}} f_0(x, p)\ dp\ dx \leq \Delta x \|g_0\|_{L^1(\mathbb{R})}. \tag{21}$$

Combining (20), (21) yields

$$\sum_{i\in\tilde{I}_2(x)} \rho_i^0 \int_{-\infty}^{x} \varphi\left(\frac{x_i - y}{\Delta x}\right)\, dy \le (1+2R)\Delta x \|g_0\|_{L^1(\mathbb{R})}. \tag{22}$$

Consider now the subsets of $\mathbb{Z}$

$$I_1(x) = \{i \in \mathbb{Z}\ :\ x_i + \frac{\Delta x}{2} \le x\} = \left\{i\in\mathbb{Z}\ :\ i < \left[\frac{x}{\Delta x} + \frac{1}{2}\right]\right\},$$

$$I_2(x) = \{i \in \mathbb{Z}\ :\ x_i - \frac{\Delta x}{2} \le x < x_i + \frac{\Delta x}{2}\} = \left\{\left[\frac{x}{\Delta x} + \frac{1}{2}\right]\right\},$$

$$I_3(x) = \{i \in \mathbb{Z}\ :\ x_i - \frac{\Delta x}{2} > x\} = \left\{i\in\mathbb{Z}\ :\ i > \left[\frac{x}{\Delta x} + \frac{1}{2}\right]\right\}.$$

We can write

$$\int_{-\infty}^{x} \rho_0(y)\, dy = \sum_{i\in I_1(x)} \rho_i^0 \Delta x + \sum_{i\in I_2(x)} \int_{x_i-\Delta x/2}^{x} \rho_0(y)\, dy. \tag{23}$$

Observe also that we have

$$\sum_{i\in I_2(x)} \int_{x_i-\Delta x/2}^{x} \rho_0(y)\, dy \le \Delta x \|\rho_0\|_{L^\infty(\mathbb{R})} \le \Delta x \|g_0\|_{L^1(\mathbb{R})}. \tag{24}$$

We deduce from (19), (22), (23), (24) that

$$\begin{aligned} \left|\int_{-\infty}^{x} (\tilde{\rho}_0(y) - \rho_0(y))\, dy\right| &\le 2(R+1)\Delta x \|g_0\|_{L^1(\mathbb{R})} \\ &+ \sum_{i\in I_1(x)-\tilde{I}_1(x)} \rho_i^0 \Delta x + \sum_{i\in \tilde{I}_1(x)-I_1(x)} \rho_i^0 \Delta x. \end{aligned} \tag{25}$$

We check easily that $\tilde{I}_1(x) \subset I_1(x)$ and card $(I_1(x) \cap \complement \tilde{I}_1(x)) \le R + 1/2$ if $R > 1/2$, $I_1(x) \subset \tilde{I}_1(x)$ and card $(\tilde{I}_1(x) \cap \complement I_1(x)) \le 3/2 - R$ if $R < 1/2$ and $I_1(x) = \tilde{I}_1(x)$ if $R = 1/2$. The inequality (25) implies immediately that

$$\left|\int_{-\infty}^{x} (\tilde{\rho}_0(y) - \rho_0(y))\, dy\right| \le C(R)\Delta x \|g_0\|_{L^1(\mathbb{R})},$$

for some constant $C(R)$ depending on the support of φ. Finally one gets from (16)

$$|\tilde{E}_0(x) - E_0(x)| = \left|\int_{-\infty}^{x} (\tilde{\rho}_0(y) - \rho_0(y))\, dy\right| \le C(R)\Delta x \|g_0\|_{L^1(\mathbb{R})},\ \forall\, x\in\mathbb{R}.$$

By (21) we have $\rho_i^0 \leq \|g_0\|_{L^1(\mathbb{R})}$ for any $i \in \mathbb{Z}$ and therefore we obtain

$$\begin{aligned} |\tilde{E}_0'(x)| &= \tilde{\rho}_0(x) \leq \|g_0\|_{L^1(\mathbb{R})} \|\varphi\|_{L^\infty(\mathbb{R})} \, \mathrm{card}\, \{i \in \mathbb{Z} \, : \, |x - x_i| < R\Delta x\} \\ &\leq (2R+1)\|\varphi\|_{L^\infty(\mathbb{R})} \|g_0\|_{L^1(\mathbb{R})}, \; \forall \, x \in \mathbb{R}. \end{aligned}$$

□

3.2 Numerical scheme

Consider $(X_{ij}^0, P_{ij}^0) = (x_i, p_j)$ for any $(i,j) \in \mathbb{Z}^2$ and $\tilde{E}_0$ the electric field given by formula (16). We define our numerical scheme as follows

$$(X_{ij}^n, P_{ij}^n)_{(i,j)\in\mathbb{Z}^2} \to (X_{ij}^{n+1}, P_{ij}^{n+1})_{(i,j)\in\mathbb{Z}^2},$$

where for any $n \in \mathbb{N}$ we put

$$X_{ij}^{n+1} = X_{ij}^n + \Delta t \; v(P_{ij}^n), \;\; (i,j) \in \mathbb{Z}^2, \tag{26}$$

$$E_{kl}^n = \tilde{E}_0(X_{kl}^n) - \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u - X_{kl}^n}{\Delta x}\right) \, du, \;\; (k,l) \in \mathbb{Z}^2, \tag{27}$$

$$P_{ij}^{n+1} = P_{ij}^n + \Delta t \; E_{ij}^n, \;\; (i,j) \in \mathbb{Z}^2. \tag{28}$$

For any $n \geq 1$ let us introduce the electric field

$$\tilde{E}_n(x) = \tilde{E}_0(x) - \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u - x}{\Delta x}\right) \, du, \; \forall \, x \in \mathbb{R}. \tag{29}$$

By using the above definition, which is motivated by (14), the scheme (26), (27), (28) can be written

$$X_{ij}^{n+1} = X_{ij}^n + \Delta t \; v(P_{ij}^n), \;\; (i,j) \in \mathbb{Z}^2, \tag{30}$$

$$P_{ij}^{n+1} = P_{ij}^n + \Delta t \; \tilde{E}_n(X_{ij}^n), \;\; (i,j) \in \mathbb{Z}^2. \tag{31}$$

Surely, for practical calculations one can only use a large but finite number of $(f_{ij}^0, X_{ij}^n, P_{ij}^n)_{(i,j)\in\mathbb{Z}^2}$. However we analyze first the mathematical properties of the theoretical scheme (29), (30), (31). Later on we will see how to implement such schemes. This will be achieved by neglecting the initial density $f_0(x,p)$ for x or p large enough (for example by taking $f^0_{ij} = 0$ for any $(i,j) \in \mathbb{Z}^2 \cap \complement(\{i_1, i_1+1, ..., i_2 - 1\} \times \{j_1, j_1+1, ..., j_2-1\})$ with i_1, j_1 small enough and i_2, j_2 large enough). Obviously, more accurate scheme can be used for the numerical approximation of the characteristics. For example start with $P^{-1/2}_{ij} = p_j - \frac{\Delta t}{2}\ \tilde{E}_0(x_i),\ X^0_{ij} = x_i$ and use the leap-frog scheme

$$P^{n+1/2}_{ij} = P^{n-1/2}_{ij} + \Delta t\ \tilde{E}_n(X^n_{ij}),\ \ (i,j) \in \mathbb{Z}^2,$$

$$X^{n+1}_{ij} = X^n_{ij} + \Delta t\ v(P^{n+1/2}_{ij}),\ \ (i,j) \in \mathbb{Z}^2.$$

The reader can easily adapt the error analysis of our method in this case. Nevertheless we study here only the simpler scheme (29), (30), (31).

Proposition 3.2 *Assume that (f_0, E_0) satisfy H1, H2, H3 and consider $\varphi : \mathbb{R} \to \mathbb{R}$ a non negative compactly supported function verifying $\int_{\mathbb{R}} \varphi(u)\ du = 1$. Then for any $n \in \mathbb{N}$ $(X^n_{ij}, P^n_{ij})_{(i,j)\in\mathbb{Z}^2}, \tilde{E}_n$ are well defined and we have for any $(i,j) \in \mathbb{Z}^2, n \in \mathbb{N}, 0 \le m \le n$*

$$|X^n_{ij} - X^m_{ij}| \le c(n-m)\Delta t, \tag{32}$$

$$\|\tilde{E}_n\|_{L^\infty(\mathbb{R})} \le \|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)} + C\Delta x \|g_0\|_{L^1(\mathbb{R})}, \tag{33}$$

$$|P^n_{ij} - P^m_{ij}| \le (n-m)\Delta t \left(\|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)} + C\Delta x \|g_0\|_{L^1(\mathbb{R})}\right). \tag{34}$$

Proof. Assume that $(X^n_{ij}, P^n_{ij})_{(i,j)\in\mathbb{Z}^2}, \tilde{E}_n$ are well defined and that (32), (33), (34) hold. Obviously, by (30) X^{n+1}_{ij} is well defined for any $(i,j) \in \mathbb{Z}^2$ and we have $|X^{n+1}_{ij} - X^n_{ij}| \le c\Delta t$ implying that

$$|X^{n+1}_{ij} - X^m_{ij}| \le |X^{n+1}_{ij} - X^n_{ij}| + |X^n_{ij} - X^m_{ij}| \le (n+1-m)c\Delta t,\ \ 0 \le m \le n.$$

We analyze now the field $\tilde{E}_{n+1}$ given by

$$\tilde{E}_{n+1}(x) = \tilde{E}_0(x) - \sum_{(i,j)\in\mathbb{Z}^2} f^0_{ij} \Delta p \int_{x_i}^{X^{n+1}_{ij}} \varphi\left(\frac{u-x}{\Delta x}\right)\ du,\ \ x \in \mathbb{R}. \tag{35}$$

Observe that for any $x \in \mathbb{R}$ the sum in the above formula is well defined since we have

$$\sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta p \left| \int_{x_i}^{X_{ij}^{n+1}} \varphi\left(\frac{u-x}{\Delta x}\right)\, du \right| \le \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta x \Delta p = \|f_0\|_{L^1(\mathbb{R}^2)},$$

and therefore, by Proposition 3.1 we have

$$\|\tilde{E}_{n+1}\|_{L^\infty(\mathbb{R})} \le \|\tilde{E}_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)} \le \|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)} + C\Delta x \|g_0\|_{L^1(\mathbb{R})}.$$

By formula (31) one gets easily that

$$\begin{aligned} |P_{ij}^{n+1} - P_{ij}^m| &\le |P_{ij}^{n+1} - P_{ij}^n| + |P_{ij}^n - P_{ij}^m| \\ &\le (n+1-m)\Delta t (\|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)} + C\Delta x \|g_0\|_{L^1(\mathbb{R})}). \end{aligned}$$

□

Recall that the exact solution (f, E) satisfies $\partial_x E = \rho \ge 0$ and therefore $E(t, \cdot)$ is non decreasing for any $t \ge 0$. In the following proposition we prove that $\tilde{E}_n$ is non decreasing for any $n \in \mathbb{N}$.

Proposition 3.3 *Assume that* (f_0, E_0) *satisfy H1, H2, H3 and consider* $\varphi : \mathbb{R} \to \mathbb{R}$ *a non negative compactly supported function verifying* $\int_{\mathbb{R}} \varphi(u)\, du = 1$*. Then for any* $n \in \mathbb{N}$ *the electric field* $\tilde{E}_n$ *is non decreasing and we have*

$$\frac{d}{dx}\tilde{E}_n = \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta p\, \varphi\left(\frac{X_{ij}^n - x}{\Delta x}\right), \quad x \in \mathbb{R}.$$

Proof. By formula (29) we can write

$$\begin{aligned} \frac{d}{dx}\tilde{E}_n &= \frac{d}{dx}\tilde{E}_0 - \frac{d}{dx} \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta x \Delta p \int_{(x_i - x)/\Delta x}^{(X_{ij}^n - x)/\Delta x} \varphi(y)\, dy \\ &= \tilde{\rho}_0(x) + \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta p \left(\varphi\left(\frac{X_{ij}^n - x}{\Delta x}\right) - \varphi\left(\frac{x_i - x}{\Delta x}\right) \right) \\ &= \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta p\, \varphi\left(\frac{X_{ij}^n - x}{\Delta x}\right) \in [0, +\infty[, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \end{aligned} \tag{36}$$

□

Remark 3.1 *The above computation justifies our choice for the charge density* $\tilde{\rho}_0$ *in formula* (15).

3.3 Convergence of the numerical scheme

We estimate now the error of the numerical scheme (29), (30), (31). We split our computations in several steps.

Lemma 3.1 *Under the hypotheses of Proposition* 3.1 *we have*

$$\left\| \tilde{E}_0(\cdot) - \int_{\mathbb{R}} E_0(u) \frac{1}{\Delta x} \varphi\left(\frac{u-\cdot}{\Delta x}\right)\, du \right\|_{L^\infty(\mathbb{R})} \leq C\Delta x\, \|g_0\|_{L^1(\mathbb{R})},$$

for some constant C depending on φ.

Proof. For any $x \in \mathbb{R}$ we can write

$$\begin{aligned} \left| \int_{\mathbb{R}} E_0(u) \frac{1}{\Delta x} \varphi\left(\frac{u-x}{\Delta x}\right)\, du - E_0(x) \right| &= \left| \int_{\mathbb{R}} (E_0(u) - E_0(x)) \frac{1}{\Delta x} \varphi\left(\frac{u-x}{\Delta x}\right)\, du \right| \\ &\leq \|E_0'\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u-x| \frac{1}{\Delta x} \varphi\left(\frac{u-x}{\Delta x}\right)\, du \\ &\leq \|g_0\|_{L^1(\mathbb{R})} \Delta x \int_{\mathbb{R}} |y| \varphi(y)\, dy. \end{aligned}$$

Combining with Proposition 3.1 we obtain

$$\left\| \tilde{E}_0(\cdot) - \int_{\mathbb{R}} E_0(u) \frac{1}{\Delta x} \varphi\left(\frac{u-\cdot}{\Delta x}\right)\, du \right\|_{L^\infty(\mathbb{R})} \leq C\Delta x\, \|g_0\|_{L^1(\mathbb{R})},$$

where C depends on φ. □

Lemma 3.2 *Under the hypotheses of Proposition* 3.1 *denote by* (f, E) *the unique global solution of the relativistic Vlasov-Maxwell system* (1), (2), (4), (5) *and by* (X, P) *the characteristics of the Vlasov equation* (1) *associated to the electric field* E*. Then for any* $(i, j) \in \mathbb{Z}^2$, $0 \leq n \leq N$ *we have*

$$|X_{ij}^n - X(t^n; 0, x_i, p_j)| + |P_{ij}^n - P(t^n; 0, x_i, p_j)| \leq C(T) \left[\Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty} + \Delta t \right]$$

with $C(T) = (1 + (a/2 + cb(T))\, T)\, \exp((1 + b(T))T)$ *and* $T = N\Delta t$.

Proof. We introduce the notations $DX_{ij}^n = |X_{ij}^n - X_{ij}(t^n)|$, $DP_{ij}^n = |P_{ij}^n - P_{ij}(t^n)|$ for any $(i,j) \in \mathbb{Z}^2, 0 \le n \le N$, where $(X_{ij}(t), P_{ij}(t)) = (X(t;0,x_i,p_j), P(t;0,x_i,p_j))$. We can write

$$
\begin{aligned}
DX_{ij}^{n+1} &= \left| X_{ij}^n + \Delta t\, v(P_{ij}^n) - X_{ij}(t^n) - \int_{t^n}^{t^{n+1}} v(P_{ij}(s))\, ds \right| \\
&\le DX_{ij}^n + \left| \int_{t^n}^{t^{n+1}} \{ v(P_{ij}^n) - v(P_{ij}(t^n)) + v(P_{ij}(t^n)) - v(P_{ij}(s)) \}\, ds \right| \\
&\le DX_{ij}^n + \|v\,'\|_{L^\infty} DP_{ij}^n\, \Delta t + \|v\,'\|_{L^\infty} \int_{t^n}^{t^{n+1}} |P_{ij}(s) - P_{ij}(t^n)|\, ds. \quad (37)
\end{aligned}
$$

Observe that $\|v\,'\|_{L^\infty} \le 1$ and that

$$
|P_{ij}(s) - P_{ij}(t^n)| = \left| \int_{t^n}^{s} E(\tau, X_{ij}(\tau))\, d\tau \right| \le (s - t^n) a, \quad s \ge t^n, \quad (38)
$$

where $a = \|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)}$. Combining (37), (38) yields

$$
DX_{ij}^{n+1} \le DX_{ij}^n + \Delta t\; DP_{ij}^n + \frac{\Delta t^2}{2} a. \quad (39)
$$

We have also

$$
\begin{aligned}
DP_{ij}^{n+1} &= \left| P_{ij}^n + \Delta t\; \tilde{E}_n(X_{ij}^n) - P_{ij}(t^n) - \int_{t^n}^{t^{n+1}} E(s, X_{ij}(s))\, ds \right| \quad (40) \\
&\le DP_{ij}^n + \Delta t\, |\tilde{E}_n(X_{ij}^n) - E(t^n, X_{ij}^n)| + \int_{t^n}^{t^{n+1}} |E(s, X_{ij}(s)) - E(t^n, X_{ij}^n)|\, ds.
\end{aligned}
$$

By using the estimates $\|\partial_x E(s)\|_{L^\infty(\mathbb{R})} \le \|g_0\|_{L^1(\mathbb{R})} + 2\|g_0\|_{L^\infty(\mathbb{R})}\; sa = b(s)$ and $\|\partial_t E(s)\|_{L^\infty(\mathbb{R})} \le cb(s)$ for any $s \ge 0$ one gets

$$
\begin{aligned}
|E(s, X_{ij}(s)) - E(t^n, X_{ij}^n)| &\le |E(s, X_{ij}(s)) - E(t^n, X_{ij}(t^n))| + |E(t^n, X_{ij}(t^n)) - E(t^n, X_{ij}^n)| \\
&\le |E(s, X_{ij}(s)) - E(s, X_{ij}(t^n))| + |E(s, X_{ij}(t^n)) - E(t^n, X_{ij}(t^n))| \\
&+ |E(t^n, X_{ij}(t^n)) - E(t^n, X_{ij}^n)| \\
&\le \|\partial_x E(s)\|_{L^\infty} |X_{ij}(s) - X_{ij}(t^n)| + \|\partial_t E\|_{L^\infty} (s - t^n) + \|\partial_x E(t^n)\|_{L^\infty} DX_{ij}^n \\
&\le 2\, cb(s)(s - t^n) + b(t^n) DX_{ij}^n, \ \forall\, s \ge t^n. \quad (41)
\end{aligned}
$$

Combining (40), (41) yields

$$
DP_{ij}^{n+1} \le DP_{ij}^n + \Delta t\, \|\tilde{E}_n - E(t^n)\|_{L^\infty(\mathbb{R})} + \Delta t\; b(t^n) DX_{ij}^n + cb(t^{n+1})\, \Delta t^2. \quad (42)
$$

Finally one gets from (39), (42)

$$\begin{aligned} DX_{ij}^{n+1} + DP_{ij}^{n+1} &\leq DX_{ij}^n + DP_{ij}^n + \Delta t\ (b(t^n) DX_{ij}^n + DP_{ij}^n) + \Delta t\ \|\tilde{E}_n - E(t^n)\|_{L^\infty} \\ &+ \Delta t^2 \left(\frac{a}{2} + cb(t^{n+1})\right), \quad (i,j) \in \mathbb{Z}^2,\ n \in \mathbb{N}, \end{aligned} \tag{43}$$

and therefore we obtain

$$\begin{aligned} DX_{ij}^n + DP_{ij}^n &\leq \left[\Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty} + \Delta t^2 \left(\frac{na}{2} + ncb(t^n)\right)\right] \exp(t^n(1 + b(t^n))) \\ &= \left[\Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty} + \Delta t \left(\frac{a}{2} + cb(t^n)\right) t^n\right] \exp(t^n(1 + b(t^n))) \\ &\leq C(T) \left[\Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty} + \Delta t\right]. \end{aligned}$$

□

We can prove the following error estimates.

Proposition 3.4 *Under the hypotheses of Proposition* 3.1 *denote by* (f, E) *the unique global solution of* (1), (2), (4), (5), *by* (X, P) *the characteristics associated to the electric field* E *and by* $(X_{ij}^n, P_{ij}^n)_{(n,i,j) \in \mathbb{N} \times \mathbb{Z}^2}$, $(\tilde{E}_n)_{n \in \mathbb{N}}$ *the numerical solution given by* (29), (30), (31). *Then there is a constant* C *depending on the initial conditions and* $T = N\Delta t$ *such that we have for any* $0 \leq n \leq N$

$$\begin{aligned} \sup_{(i,j) \in \mathbb{Z}^2} \{|X_{ij}^n - X(t^n; 0, x_i, p_j)| + |P_{ij}^n - P(t^n; 0, x_i, p_j)|\} &+ \|\tilde{E}_n(\cdot) - E(t^n, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\leq C(\Delta t + \Delta x + \Delta p). \end{aligned}$$

Proof. We check easily that for any $n \in \mathbb{N}$ we have

$$\begin{aligned} \left\| E(t^n, \cdot) - \int_{\mathbb{R}} E(t^n, u) \frac{1}{\Delta x} \varphi\left(\frac{u - \cdot}{\Delta x}\right)\ du \right\|_{L^\infty(\mathbb{R})} &\leq \|\partial_x E(t^n)\|_{L^\infty(\mathbb{R})} \Delta x \int_{\mathbb{R}} |y| \varphi(y)\ dy \\ &\leq b(t^n) \Delta x \int_{\mathbb{R}} |y| \varphi(y)\ dy. \end{aligned} \tag{44}$$

By using (14), (29) and Lemma 3.1 one gets for any $y \in \mathbb{R}, n \in \mathbb{N}$

$$
\begin{aligned}
&\left|\tilde{E}_n(y) - \int_{\mathbb{R}} E(t^n,u)\frac{1}{\Delta x}\varphi\left(\frac{u-y}{\Delta x}\right)\, du\right| \le \left|\tilde{E}_0(y) - \int_{\mathbb{R}} E_0(u)\frac{1}{\Delta x}\varphi\left(\frac{u-y}{\Delta x}\right)\, du\right| \\
+ &\left|\sum_{(i,j)\in\mathbb{Z}^2} f^0_{ij}\Delta p \int_{x_i}^{X^n_{ij}} \varphi\left(\frac{u-y}{\Delta x}\right)\, du - \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \int_x^{X(t^n;0,x,p)} \frac{1}{\Delta x}\varphi\left(\frac{u-y}{\Delta x}\right)\, du\, dp\, dx\right| \\
\le\ & C\Delta x\, \|g\|_{L^1(\mathbb{R})} + T_1 + T_2, \qquad (45)
\end{aligned}
$$

where

$$
T_1 = \left|\sum_{(i,j)\in\mathbb{Z}^2} \int_{C^x_i}\int_{C^p_j} \frac{f_0}{\Delta x}\left(\int_x^{X(t^n;0,x,p)} \varphi\left(\frac{u-y}{\Delta x}\right)\, du - \int_{x_i}^{X(t^n;0,x_i,p_j)} \varphi\left(\frac{u-y}{\Delta x}\right) du\right) dp dx\right|,
$$

and

$$
T_2 = \left|\sum_{(i,j)\in\mathbb{Z}^2} f^0_{ij}\Delta p \int_{X(t^n;0,x_i,p_j)}^{X^n_{ij}} \varphi\left(\frac{u-y}{\Delta x}\right)\, du\right|. \qquad (46)
$$

For any $x \in \mathbb{R}$ we denote by $x_i(x) = i(x)\Delta x$ the point x_i such that $x_i - \frac{\Delta x}{2} \le x < x_i + \frac{\Delta x}{2}$ and similarly, for any $p \in \mathbb{R}$, the notation $p_j(p)$ stands for the point p_j such that $p_j - \frac{\Delta p}{2} \le p < p_j + \frac{\Delta p}{2}$. Observe that the term T_1 can be written

$$
\begin{aligned}
T_1 \ &= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f_0}{\Delta x}\left(\int_x^{X(t^n;0,x,p)} \varphi\left(\frac{u-y}{\Delta x}\right)\, du - \int_{x_i(x)}^{X(t^n;0,x_i(x),p_j(p))} \varphi\left(\frac{u-y}{\Delta x}\right) du\right)\, dp\, dx\right| \\
&\le T_{11} + T_{12}, \qquad (47)
\end{aligned}
$$

where

$$
T_{11} = \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f_0}{\Delta x}\int_x^{x_i(x)} \varphi\left(\frac{u-y}{\Delta x}\right)\, du\, dp\, dx\right|,
$$

and

$$
T_{12} = \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f_0}{\Delta x}\int_{X(t^n;0,x,p)}^{X(t^n;0,x_i(x),p_j(p))} \varphi\left(\frac{u-y}{\Delta x}\right)\, du\, dp\, dx\right|.
$$

Let us estimate

$$
T_{12}(s) := \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f_0}{\Delta x}\int_{X(s;0,x,p)}^{X(s;0,x_i(x),p_j(p))} \varphi\left(\frac{u-y}{\Delta x}\right)\, du\, dp\, dx\right|, \quad s \ge 0.
$$

We have

$$T_{12}(s) = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,p) \int_{(X(s;0,x,p)-y)/\Delta x}^{(X(s;0,x_i(x),p_j(p))-y)/\Delta x} \varphi(z)\ dz\ dp\ dx \right|$$

$$\leq \int_{\mathbb{R}} \varphi(z) \int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,p) \mathbf{1}_{\{\left|z - \frac{X(s;0,x,p)-y}{\Delta x}\right| \leq \frac{X(s;0,x_i(x),p_j(p))-X(s;0,x,p)}{\Delta x}\}}\ dp\ dx\ dz$$

$$= \int_{\mathbb{R}} \varphi(z) \int_{\mathbb{R}} \int_{\mathbb{R}} f(s, X(s;0,x,p), P(s;0,x,p))$$

$$\times\ \mathbf{1}_{\{|X(s;0,x,p)-y-z\Delta x| \leq |X(s;0,x,p)-X(s;0,x_i(x),p_j(p))|\}}\ dp\ dx\ dz. \tag{48}$$

We check easily by using the characteristic equations that

$$|X(s;0,x,p) - X(s;0,x_i(x),p_j(p))| + |P(s;0,x,p) - P(s;0,x_i(x),p_j(p))|$$

$$\leq (|x - x_i(x)| + |p - p_j(p)|) \exp(s(1+b(s)))$$

$$\leq \left(\frac{\Delta x}{2} + \frac{\Delta p}{2}\right) \exp(s(1+b(s))) =: \tilde{\tilde{R}}(s). \tag{49}$$

Therefore we obtain the following estimate for the term $T_{12}(s)$

$$T_{12}(s) \leq \int_{\mathbb{R}} \varphi(z) \int_{\mathbb{R}} \int_{\mathbb{R}} f(s, X(s;0,x,p), P(s;0,x,p)) \mathbf{1}_{\{|X(s;0,x,p)-y-z\Delta x| \leq \tilde{\tilde{R}}(s)\}}\ dp\ dx\ dz$$

$$= \int_{\mathbb{R}} \varphi(z) \int_{\mathbb{R}} \int_{\mathbb{R}} f(s,X,P) \mathbf{1}_{\{|X-y-z\Delta x| \leq \tilde{\tilde{R}}(s)\}}\ dP\ dX\ dz$$

$$\leq \int_{\mathbb{R}} \varphi(z) \|\rho(s)\|_{L^\infty(\mathbb{R})} 2\ \tilde{\tilde{R}}(s)\ dz$$

$$= (\Delta x + \Delta p) \exp((1+b(s))s)\ b(s). \tag{50}$$

Taking $s = 0$, $s = t^n$ we deduce that $T_{11} \leq (\Delta x + \Delta p) b(0)$, respectively $T_{12} \leq (\Delta x + \Delta p) \exp((1+b(t^n))t^n)\ b(t^n)$ and thus we have

$$T_1 \leq 2\ (\Delta x + \Delta p) \exp((1+b(t^n))t^n)\ b(t^n). \tag{51}$$

It remains to estimate the term T_2. As before, observe that

$$T_2 = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f_0(x,p)}{\Delta x} \int_{X(t^n;0,x_i(x),p_j(p))}^{X^n_{i(x)j(p)}} \varphi\left(\frac{u-y}{\Delta x}\right)\ du\ dp\ dx \right| \tag{52}$$

$$\leq \int_{\mathbb{R}} \varphi(z) \int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,p) \mathbf{1}_{\{|X(t^n;0,x_i(x),p_j(p))-y-z\Delta x| \leq |X^n_{i(x)j(p)} - X(t^n;0,x_i(x),p_j(p))|\}}\ dp\ dx\ dz.$$

By Lemma 3.2 we know that for any $0 \le n \le N, (x,p) \in \mathbb{R}^2$ we have

$$\begin{aligned} |X^n_{i(x)j(p)} - X(t^n;0,x_i(x),p_j(p))| &\le C(T)\left[\Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty(\mathbb{R})} + \Delta t\right] \\ &=: R_n. \end{aligned} \tag{53}$$

Since $|X(t^n;0,x_i(x),p_j(p)) - X(t^n;0,x,p)| \le \frac{\Delta x + \Delta p}{2} \exp((1+b(t^n))t^n)$ we deduce that the following inclusion holds for any n

$$\begin{aligned} &\{(x,p) : |X(t^n;0,x_i(x),p_j(p)) - y - z\Delta x| \le |X^n_{i(x)j(p)} - X(t^n;0,x_i(x),p_j(p))|\} \\ \subset\ &\{(x,p) : |X(t^n;0,x,p) - y - z\Delta x| \le \tilde{R}_n\}, \end{aligned} \tag{54}$$

where $\tilde{R}_n = R_n + \frac{\Delta x + \Delta p}{2} \exp((1+b(t^n))t^n)$. From (52), (54) one gets

$$\begin{aligned} T_2 &\le \int_{\mathbb{R}} \varphi(z) \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \mathbf{1}_{\{|X(t^n;0,x,p) - y - z\Delta x| \le \tilde{R}_n\}}\, dp\, dx\, dz \\ &= \int_{\mathbb{R}} \varphi(z) \int_{\mathbb{R}}\int_{\mathbb{R}} f(t^n, X(t^n;0,x,p), P(t^n;0,x,p)) \mathbf{1}_{\{|X(t^n;0,x,p) - y - z\Delta x| \le \tilde{R}_n\}}\, dp\, dx\, dz \\ &= \int_{\mathbb{R}} \varphi(z) \int_{\mathbb{R}}\int_{\mathbb{R}} f(t^n, X, P) \mathbf{1}_{\{|X - y - z\Delta x| \le \tilde{R}_n\}}\, dP\, dX\, dz \\ &\le \|\rho(t^n)\|_{L^\infty(\mathbb{R})}\, 2\, \tilde{R}_n \\ &\le b(t^n)\left(2R_n + (\Delta x + \Delta p)\exp((1+b(t^n))t^n)\right). \end{aligned} \tag{55}$$

Collecting the partial computations (44), (45), (51), (55) one gets for any $y \in \mathbb{R}$, $0 \le n \le N$

$$\begin{aligned} |\tilde{E}_n(y) - E(t^n,y)| &\le b(t^n) \int_{\mathbb{R}} |z|\varphi(z)\, dz\, \Delta x + C\Delta x \|g\|_{L^1(\mathbb{R})} \\ &+ 3(\Delta x + \Delta p)\exp((1+b(t^n))t^n)\, b(t^n) \\ &+ 2b(t^n)C(T)\left[\Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty(\mathbb{R})} + \Delta t\right] \\ &\le C(\Delta t + \Delta x + \Delta p) + C\Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

for some constant C depending on the initial conditions and $T = N\Delta t$, but not on $y \in \mathbb{R}$. Therefore we deduce that

$$\|\tilde{E}_n - E(t^n)\|_{L^\infty(\mathbb{R})} \le C(\Delta t + \Delta x + \Delta p) + C\Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty(\mathbb{R})}. \tag{56}$$

We consider $S_n = \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty(\mathbb{R})}$ for any $n \geq 1$ and $S_0 = 0$. From the above inequality we deduce that

$$S_{n+1} \leq (1 + C\Delta t)S_n + C(\Delta t + \Delta x + \Delta p), \;\; 0 \leq n \leq N,$$

and therefore one gets easily

$$\Delta t S_n \;\; = \;\; \Delta t \sum_{m=0}^{n-1} \|\tilde{E}_m - E(t^m)\|_{L^\infty(\mathbb{R})} \leq C(\Delta t + \Delta x + \Delta p), \;\; 0 \leq n \leq N. \quad (57)$$

Combining (56), (57) we obtain

$$\|\tilde{E}_n - E(t^n)\|_{L^\infty(\mathbb{R})} \leq C(\Delta t + \Delta x + \Delta p), \;\; 0 \leq n \leq N,$$

for some constant depending on the initial conditions and $T = N\Delta t$. By Lemma 3.2 we deduce also that

$$\sup_{(i,j)\in\mathbb{Z}^2} \{|X_{ij}^n - X(t^n; 0, x_i, p_j)| + |P_{ij}^n - P(t^n; 0, x_i, p_j)|\} \leq C(\Delta t + \Delta x + \Delta p), \;\; 0 \leq n \leq N.$$

$\square$

For practical implementation of the scheme (29), (30), (31) we approximate the initial particle density by

$$\overline{f}_0(x,p) = f_0(x,p)\mathbf{1}_{\{(i_1-1/2)\Delta x \leq x < (i_2-1/2)\Delta x\}} \times \mathbf{1}_{\{(j_1-1/2)\Delta p \leq p < (j_2-1/2)\Delta p\}}, \quad (58)$$

where i_1, j_1 are small enough and i_2, j_2 are large enough such that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \{f_0(x,p) - \overline{f}_0(x,p)\}\; dp\; dx < \varepsilon,$$

with ε a small parameter. We consider the charge density $\overline{\rho}_0 = \int_{\mathbb{R}} \overline{f}_0 \; dp$ and the electric field

$$\overline{E}_0(x) = E_0(x) + \int_{-\infty}^{x} \{\overline{\rho}_0(y) - \rho_0(y)\}\; dy, \;\; x \in \mathbb{R}. \quad (59)$$

We have $\overline{E}_0^{\,\prime} = \overline{\rho}_0$ and

$$|\overline{E}_0(x) - E_0(x)| \leq \int_{\mathbb{R}} \{\rho(y) - \overline{\rho}_0(y)\}\; dy \leq \int_{\mathbb{R}}\int_{\mathbb{R}} \{f_0(y,p) - \overline{f}_0(y,p)\}\; dp\; dy < \varepsilon, \;\; x \in \mathbb{R}.$$

Notice that by (58) we have $\overline{f}_{ij}^0 = 0$ for any $(i,j) \in \mathbb{Z}^2 \cap \complement(\{i_1, i_1+1, ..., i_2-1\} \times \{j_1, j_1+1, ..., j_2-1\})$ and therefore, by using the numerical scheme (29), (30), (31) associated to the initial conditions $(\overline{f}_0, \overline{E}_0)$, we can compute $X_{ij}^n, P_{ij}^n, \tilde{E}_n$ for any $(i,j) \in \{i_1, i_1+1, ..., i_2-1\} \times \{j_1, j_1+1, ..., j_2-1\}$ and $n \in \mathbb{N}$. We obtain the following result

Theorem 3.1 *Assume that (f_0, E_0) satisfy H1, H2, H3 and consider φ a non negative compactly supported function verifying $\int_{\mathbb{R}} \varphi(u)\ du = 1$. For any $\varepsilon > 0$ let $(\overline{f}_0, \overline{E}_0)$ given by* (58), (59)*. Denote by (f, E) the unique global solution of* (1), (2) *with the initial conditions (f_0, E_0), by (X, P) the characteristics associated to the electric field E and by $X_{ij}^n, P_{ij}^n, \tilde{E}_n, (i,j) \in \{i_1, i_1+1, ..., i_2-1\} \times \{j_1, j_1+1, ..., j_2-1\}, n \in \{0, 1, ..., N\}$ the numerical solution given by* (29), (30), (31) *and the initial conditions $(\overline{f}_0, \overline{E}_0)$. Then we have*

$$\begin{aligned}
\sup_{(i,j)\in\{i_1,i_1+1,...,i_2-1\}\times\{j_1,j_1+1,...,j_2-1\}} & \{|X_{ij}^n - X(t^n; 0, x_i, p_j)| + |P_{ij}^n - P(t^n; 0, x_i, p_j)|\} \\
+ & \quad \|\tilde{E}_n - E(t^n)\|_{L^\infty(\mathbb{R})} \\
\leq & \quad C(\Delta t + \Delta x + \Delta p + \varepsilon), \quad 0 \leq n \leq N,
\end{aligned} \tag{60}$$

for some constant depending on the initial conditions and $T = N\Delta t$.

Proof. Observe that since (f_0, E_0) satisfy H1, H2, H3 therefore $(\overline{f}_0, \overline{E}_0)$ satisfy H1, H2, H3 as well. We denote by $(\overline{f}, \overline{E})$ the unique global solution of (1), (2) with the initial conditions $(\overline{f}_0, \overline{E}_0)$. By Corollary 2.1 we have

$$\begin{aligned}
\|E(t) - \overline{E}(t)\|_{L^\infty(\mathbb{R})} \quad + & \quad \|(X - \overline{X})(t; 0, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^2)} + \|(P - \overline{P})(t; 0, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^2)} \\
\leq & \quad C(T) \left(\|f_0 - \overline{f}_0\|_{L^1(\mathbb{R}^2)} + \|E_0 - \overline{E}_0\|_{L^\infty(\mathbb{R})} \right) \\
\leq & \quad 2\varepsilon C(T),\ \forall\, t \in [0, T],
\end{aligned} \tag{61}$$

where $(\overline{X}, \overline{P})$ are the characteristics associated with the electric field $\overline{E}$. By Proposition 3.4 we obtain

$$\begin{aligned} \sup_{(i,j)\in\{i_1,i_1+1,...,i_2-1\}\times\{j_1,j_1+1,...,j_2-1\}} & \{|X_{ij}^n - \overline{X}(t^n;0,x_i,p_j)| + |P_{ij}^n - \overline{P}(t^n;0,x_i,p_j)|\} \\ + & \ \|\tilde{E}_n - \overline{E}(t^n)\|_{L^\infty(\mathbb{R})} \\ \leq & \ \overline{C}(T)(\Delta t + \Delta x + \Delta p), \ \ 0 \leq n \leq N. \end{aligned} \tag{62}$$

Our conclusion follows easily by combining (61), (62). $\square$

We proved that the solution of the relativistic one dimensional Vlasov-Maxwell system propagates with finite speed, cf. Theorem 2.2. We use this property for localizing the numerical scheme (29), (30), (31) in space. Assume that we want to approximate the solution of (1), (2), (4), (5) for $(t, x, p) \in [0, T] \times [a, b] \times \mathbb{R}$. Take $i_1, i_2 \in \mathbb{Z}, N \in \mathbb{N}$ such that $N\Delta t \geq T, x_{i_1} + cN\Delta t \leq a < b \leq x_{i_2-1} - cN\Delta t$. For simplifying suppose also that $f_{ij}^0 = 0, \ \ (i,j) \in \mathbb{Z} \times (\mathbb{Z} \cap \complement\{j_1, ..., j_2 - 1\})$. For any $0 \leq n \leq N$ we consider

$$T^n = \{(i,j) \in \mathbb{Z} \times \{j_1, ..., j_2 - 1\} \ : \ x_{i_1} + nc\Delta t \leq X_{ij}^n \leq x_{i_2-1} - nc\Delta t\}.$$

From (32) one gets immediately that

$$\{i_1, ..., i_2 - 1\} \times \{j_1, ..., j_2 - 1\} = T^0 \supset T^1 \supset ... \supset T^n \supset T^{n+1} \supset ... \supset T^N.$$

Proposition 3.5 *Denote by* $(X_{ij}^n, P_{ij}^n), \tilde{E}_n$, $(i,j) \in \mathbb{Z}\times\{j_1, ..., j_2-1\}, n \in \{0, 1, ..., N\}$ *the numerical solution given by* (29), (30), (31)*. Then for any* $n \in \{0, 1, ..., N\}$ *we have*

$$\begin{aligned} \tilde{E}_n(x) &= \tilde{E}_0(x) - \sum_{m=1}^{n} \sum_{(i,j)\in T^{m-1}\cap\complement T^m} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^m} \varphi\left(\frac{u-x}{\Delta x}\right) du \\ &- \sum_{(i,j)\in T^n} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right) du, \end{aligned} \tag{63}$$

for any $x \in [x_{i_1} + nc\Delta t + R\Delta x, x_{i_2-1} - nc\Delta t - R\Delta x]$*, where* $[-R, R] \supset \operatorname{supp} \varphi$*.*

Proof. Take $x \in [x_{i_1} + nc\Delta t + R\Delta x, x_{i_2-1} - nc\Delta t - R\Delta x]$. For any $(i,j) \in \mathbb{Z}^2$ such that $i < i_1$ we have $\max\{x_i, X_{ij}^n\} \le x_{i_1} + nc\Delta t \le x - R\Delta x$. We deduce that $\int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du = 0$ for any $i < i_1$. Similarly, for any $(i,j) \in \mathbb{Z}^2$ such that $i > i_2 - 1$ we have $\min\{x_i, X_{ij}^n\} \ge x_{i_2-1} - nc\Delta t \ge x + R\Delta x$, and therefore $\int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du = 0$ for any $i > i_2 - 1$. By the definition of $\tilde{E}_n$ (see (29)) one gets

$$
\begin{aligned}
\tilde{E}_n(x) &= \tilde{E}_0(x) - \sum_{(i,j)\in T^0} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du \\
&= \tilde{E}_0(x) - \sum_{m=1}^{n} \sum_{(i,j)\in T^{m-1}\cap \complement T^m} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du \\
&- \sum_{(i,j)\in T^n} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du. \qquad (64)
\end{aligned}
$$

We are done if we prove that $\int_{X_{ij}^m}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du = 0$ for any $m \in \{1, ..., n\}, (i,j) \in T^{m-1} \cap \complement T^m$. Indeed, if $(i,j) \in T^{m-1} \cap \complement T^m$ we have $X_{ij}^m < x_{i_1} + mc\Delta t$ or $X_{ij}^m > x_{i_2-1} - mc\Delta t$. In the first case we deduce that $\max\{X_{ij}^m, X_{ij}^n\} \le X_{ij}^m + (n-m)c\Delta t < x_{i_1} + nc\Delta t < x - R\Delta x$, and in the second case one gets $\min\{X_{ij}^m, X_{ij}^n\} \ge X_{ij}^m - (n-m)c\Delta t > x_{i_2-1} - nc\Delta t > x + R\Delta x$. Therefore in both cases we have $\int_{X_{ij}^m}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du = 0$. □

By using the previous proposition it is possible to construct a local numerical scheme by slightly modifying the scheme (29), (30), (31). We start with $(X_{ij}^0, P_{ij}^0) = (x_i, p_j)$, $(i,j) \in T^0$ and we take as initial field $\tilde{\tilde{E}}_0$ on $[x_{i_1}, x_{i_2-1}]$ the restriction of $\tilde{E}_0$ on $[x_{i_1}, x_{i_2-1}]$. Assume that for some $n \ge 0$ we know (X_{ij}^m, P_{ij}^m) for any $(i,j) \in T^m$, $m \in \{0, 1, ..., n\}$ and a field $\tilde{\tilde{E}}_n$ on $[x_{i_1} + nc\Delta t, x_{i_2-1} - nc\Delta t]$. We have to define $(X_{ij}^{n+1}, P_{ij}^{n+1})$ for any $(i,j) \in T^{n+1}$ and a field $\tilde{\tilde{E}}_{n+1}$ on $[x_{i_1} + (n+1)c\Delta t, x_{i_2-1} - (n+1)c\Delta t]$. We can take $X_{ij}^{n+1} = X_{ij}^n + \Delta t v(P_{ij}^n)$ for any (i,j) in $T^n \supset T^{n+1}$. Note that for any $(i,j) \in T^{n+1}$ we have $X_{ij}^n \in [x_{i_1} + nc\Delta t, x_{i_2-1} - nc\Delta t]$ and therefore we can define

$$
P_{ij}^{n+1} = P_{ij}^n + \Delta t \tilde{\tilde{E}}_n(X_{ij}^n), \ \ (i,j) \in T^{n+1}.
$$

It remains to define the field $\tilde{\tilde{E}}_{n+1}$ on $[x_{i_1} + (n+1)c\Delta t, x_{i_2-1} - (n+1)c\Delta t]$. By Proposition 3.5 we can compute the field $\tilde{E}_{n+1}$ on $[a_{n+1}, b_{n+1}] = [x_{i_1} + (n+1)c\Delta t +$

$R\Delta x, x_{i_2-1} - (n+1)c\Delta t - R\Delta x]$ by using the values $X_{ij}^{m+1} = X_{ij}^m + \Delta t v(P_{ij}^m)$, $(i,j) \in T^m$, $m \in \{0,1,...,n\}$. We can take $\tilde{\tilde{E}}_{n+1}(x) = \tilde{E}_{n+1}(a_{n+1})$ for $x \in [x_{i_1} + (n+1)c\Delta t, a_{n+1}[$, $\tilde{\tilde{E}}_{n+1}(x) = \tilde{E}_n(x)$ for $x \in [a_{n+1}, b_{n+1}]$ and $\tilde{\tilde{E}}_{n+1}(x) = \tilde{E}_n(b_{n+1})$ for $x \in]b_{n+1}, x_{i_2-1} - (n+1)c\Delta t]$. After N time steps we obtain a field $\tilde{\tilde{E}}_N$ defined on $[x_{i_1} + Nc\Delta t, x_{i_2-1} - Nc\Delta t] \supset [a,b]$. We expect that the above scheme has the same properties as the numerical scheme (29), (30), (31).

Finally, for any $(n,i) \in \mathbb{N}\times\mathbb{Z}$ we introduce the charge and current approximations

$$\tilde{\rho}_i^n = \frac{\tilde{E}_n(x_{i+1}) - \tilde{E}_n(x_i)}{\Delta x}, \quad \tilde{j}_i^n = -\frac{\tilde{E}_{n+1}(x_i) - \tilde{E}_n(x_i)}{\Delta t}.$$

The quantities $(\tilde{\rho}_i^n, \tilde{j}_i^n)_{(n,i)\in\mathbb{N}\times\mathbb{Z}}$ verify the properties

Proposition 3.6 *Under the hypotheses of Theorem* 3.1 *we have*

$$\tilde{\rho}_i^n \geq 0, \quad \frac{\tilde{\rho}_i^{n+1} - \tilde{\rho}_i^n}{\Delta t} + \frac{\tilde{j}_{i+1}^n - \tilde{j}_i^n}{\Delta x} = 0, \quad (n,i) \in \mathbb{N}\times\mathbb{Z}, \tag{65}$$

and

$$\sum_{i\in\mathbb{Z}} \tilde{\rho}_i^n \Delta x = \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\, dp\, dx, \quad n \in \mathbb{N}.$$

Proof. By Proposition 3.3 the electric field $\tilde{E}_n$ is non decreasing and therefore $\tilde{\rho}_i^n \geq 0$ for any $(n,i) \in \mathbb{N}\times\mathbb{Z}$. Obviously, the discrete continuity equation (65) holds true. It remains to compute the total charge $Q^n = \sum_{i\in\mathbb{Z}} \tilde{\rho}_i^n \Delta x$. Since $\tilde{E}_n$ is non decreasing and bounded we have $Q^n = \lim_{x\to+\infty} \tilde{E}_n(x) - \lim_{x\to-\infty} \tilde{E}_n(x)$. By (29) we have $\tilde{E}_n = \tilde{E}_0 - \tilde{F}_n$ where

$$\tilde{F}_n(x) = \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right) du, \quad x \in \mathbb{R}.$$

We claim that $\lim_{x\to\pm\infty} \tilde{F}_n(x) = 0$ for any $n \in \mathbb{N}$. Indeed, for any $\varepsilon > 0$ take i_ε large enough such that

$$\sum_{(i,j)\in\mathbb{Z}^2, |i|>i_\varepsilon} f_{ij}^0 \Delta x \Delta p < \varepsilon.$$

Take $R > 0$ such that supp $\varphi \subset [-R,R]$ and x such that $|x| \geq R\Delta x + i_\varepsilon \Delta x + nc\Delta t$. Observe that for any $(i,j) \in \mathbb{Z}^2$, $|i| \leq i_\varepsilon$ and u between x_i and X_{ij}^n we have $\frac{|u-x|}{\Delta x} \geq R$

saying that

$$\sum_{(i,j)\in\mathbb{Z}^2, |i|\le i_\varepsilon} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du = 0, \quad |x| \ge R\Delta x + i_\varepsilon \Delta x + nc\Delta t.$$

We deduce that

$$|\tilde{F}_n(x)| \le \sum_{(i,j)\in\mathbb{Z}^2, |i|>i_\varepsilon} f_{ij}^0 \Delta x \Delta p < \varepsilon, \quad |x| \ge R\Delta x + i_\varepsilon \Delta x + nc\Delta t,$$

and therefore we have

$$\lim_{x\to\pm\infty} \tilde{F}_n(x) = 0. \tag{66}$$

By (16) one gets easily that

$$\lim_{x\to-\infty} \tilde{E}_0(x) = \lim_{x\to-\infty} E_0(x), \quad \lim_{x\to+\infty} \tilde{E}_0(x) = \lim_{x\to+\infty} E_0(x). \tag{67}$$

Combining (66), (67) we obtain

$$Q^n = \lim_{x\to+\infty} E_0(x) - \lim_{x\to-\infty} E_0(x) = \int_{\mathbb{R}} E_0'(x)\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\, dp\, dx, \ \forall\, n \in \mathbb{N}.$$

□

4 Numerical simulations

In this section we intend to compare the numerical scheme (29), (30), (31) with respect to classical particle methods for solving the Vlasov-Maxwell equations. As noticed before, the main advantage of (29), (30), (31) is that the convergence analysis permits very rough particle densities (typically integrable densities) while the electric field and the characteristics remain smooth (Lipschitz continuous functions at least). We have also seen that the total charge is conserved (cf. Proposition 3.6) and, in the relativistic case, we have proved that the numerical solution propagates with finite speed (cf. Proposition 3.5). We wish to perform some numerical computations. For this we will introduce also a standard particle method and we will compare the results obtained by using both methods. We work in the setting of spatial L periodic functions, *i.e.,* we are looking for solutions (f, E) of (1), (2) satisfying

$$f(t, x + L, p) = f(t, x, p), \quad E(t, x + L) = E(t, x), \quad (t, x, p) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}.$$

At this stage let us point out that the electric potential $\Phi(t, x)$ (*i.e.,* $E(t, x) = -\partial_x \Phi(t, x)$) is not necessarily L periodic, saying that, generally (2) can not be replaced by the Poisson problem

$$-\partial_x^2 \Phi = \rho(t, x), \quad \Phi(t, x + L) = \Phi(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Indeed, the electric potential is L periodic iff the average of the electric field over a period vanishes at any time $t \in \mathbb{R}_+$. Anyway, the system (1), (2) is equivalent to the Vlasov equation (1) and the Ampère law $\partial_t E = -j$ in (2) since the Gauss law $\partial_x E = \rho$ is a consequence of the continuity equation (3), when the initial conditions in (4), (5) satisfy the constraint $\frac{d}{dx} E_0 = \int_{\mathbb{R}} f_0 dp$.

A standard particle method for approximating the Vlasov equation combined to the Ampère law would be, with the notations at the beginning of Section 3

- Consider $(X_{ij}^0, P_{ij}^0) = (x_i, p_j)$ for any $(i, j) \in \mathbb{Z}^2$ and $E_k^0 = E_0(x_k)$ for any $k \in \mathbb{Z}$.
- Given $(X_{ij}^n, P_{ij}^n)_{(i,j) \in \mathbb{Z}^2}$ and $(E_k^n)_{k \in \mathbb{Z}}$ compute $(X_{ij}^{n+1}, P_{ij}^{n+1})_{(i,j) \in \mathbb{Z}^2}$ and $(E_k^{n+1})_{k \in \mathbb{Z}}$ by using

$$X_{ij}^{n+1} = X_{ij}^n + \Delta t \; v(P_{ij}^n), \quad (i, j) \in \mathbb{Z}^2 \tag{68}$$

$$P_{ij}^{n+1} = P_{ij}^n + \Delta t \; \mathcal{I}(E^n)(X_{ij}^n), \quad (i, j) \in \mathbb{Z}^2 \tag{69}$$

$$E_k^{n+1} = E_k^n - \Delta t \sum_{(i,j) \in \mathbb{Z}^2} \Delta p f_{ij}^0 v(P_{ij}^n) \varphi\left(\frac{X_{ij}^n - x_k}{\Delta x}\right), \quad k \in \mathbb{Z} \tag{70}$$

where the notation $\mathcal{I}$ in (69) stands for an interpolation operator let say $\mathcal{I} : l^\infty(\mathbb{Z}) \to L^\infty(\mathbb{R})$. The formula (70) comes from the Ampère law, by observing that

$$\begin{aligned}\int_{t^n}^{t^{n+1}} j(s, x_k)\, ds &\approx \Delta t \int_{\mathbb{R}} j(t^n, x) \frac{1}{\Delta x} \varphi\left(\frac{x - x_k}{\Delta x}\right)\, dx \\ &= \Delta t \int_{\mathbb{R}}\int_{\mathbb{R}} v(p) f(t^n, x, p) \frac{1}{\Delta x} \varphi\left(\frac{x - x_k}{\Delta x}\right)\, dpdx \\ &\approx \Delta t \sum_{(i,j)\in\mathbb{Z}^2} \Delta p f^0_{ij} v(P^n_{ij}) \varphi\left(\frac{X^n_{ij} - x_k}{\Delta x}\right). \end{aligned} \tag{71}$$

Surely, for more accuracy we can start with

$$(X^0_{ij}, P^{-1/2}_{ij}) = (x_i, p_j - \frac{\Delta t}{2} E_0(x_i)), \quad (i, j) \in \mathbb{Z}^2$$

and replace (68), (69), (70) by

$$P^{n+1/2}_{ij} = P^{n-1/2}_{ij} + \Delta t\, \mathcal{I}(E^n)(X^n_{ij}), \quad (i, j) \in \mathbb{Z}^2 \tag{72}$$

$$X^{n+1}_{ij} = X^n_{ij} + \Delta t\, v(P^{n+1/2}_{ij}), \quad (i, j) \in \mathbb{Z}^2 \tag{73}$$

$$E^{n+1}_k = E^n_k - \Delta t \sum_{(i,j)\in\mathbb{Z}^2} \Delta p f^0_{ij} v(P^{n+1/2}_{ij}) \varphi\left(\frac{X^{n+1/2}_{ij} - x_k}{\Delta x}\right), \quad k \in \mathbb{Z} \tag{74}$$

with $X^{n+1/2}_{ij} = X^n_{ij} + \frac{\Delta t}{2} v(P^{n+1/2}_{ij})$.

Obviously, the main difference between the schemes (29), (30), (31) and (68), (69), (70) comes from the expressions for the electric field in (29) and (70) respectively. Let us denote by ψ the function given by $\psi(x) = \int_{-\infty}^{x} \frac{1}{\Delta x} \varphi\left(\frac{y}{\Delta x}\right)\, dy$. If $[-R, R]$ contains the support of φ then it is easily seen that $\psi(x) = 0$ for any $x < -R\Delta x$, $\psi(x) = 1$ for any $x > R\Delta x$ and $\psi(x) \in [0, 1]$ for any $x \in [-R\Delta x, R\Delta x]$. Estimating the difference between the electric fields in (29) and (70) leads, up to other error terms, to terms like

$$
\begin{aligned}
T_k^n &= \sum_{(i,j)\in\mathbb{Z}^2} \Delta x \Delta p f_{ij}^0 \int_{x_i}^{X_{ij}^n} \frac{1}{\Delta x}\varphi\left(\frac{u-x_k}{\Delta x}\right)\,du \\
&- \sum_{m=0}^{n-1}\sum_{(i,j)\in\mathbb{Z}^2} \Delta t \Delta x \Delta p f_{ij}^0 v(P_{ij}^n)\frac{1}{\Delta x}\varphi\left(\frac{X_{ij}^n-x_k}{\Delta x}\right) \\
&= \sum_{(i,j)\in\mathbb{Z}^2} \Delta x \Delta p f_{ij}^0 \left[\psi(X_{ij}^n - x_k) - \psi(x_i - x_k)\right] \\
&- \sum_{(i,j)\in\mathbb{Z}^2} \Delta x \Delta p f_{ij}^0 \left[\sum_{m=0}^{n-1}(X_{ij}^{m+1} - X_{ij}^m)\psi'(X_{ij}^m - x_k)\right] \\
&= \sum_{(i,j)\in\mathbb{Z}^2} \Delta x \Delta p f_{ij}^0 \sum_{m=0}^{n-1} R_{ijk}^m
\end{aligned}
$$

where the notation R_{ijk}^m stands for

$$R_{ijk}^m = \psi(X_{ij}^{m+1} - x_k) - \psi(X_{ij}^m - x_k) - (X_{ij}^{m+1} - X_{ij}^m)\psi'(X_{ij}^m - x_k).$$

Therefore, the convergence analysis of (68), (69), (70) reduces to the convergence towards zero for the error terms T_k^n when $\Delta t, \Delta x, \Delta p$ tend to 0. But estimating terms like R_{ijk}^m is not an easy task, since the function ψ converges to the unitary step as Δx tends to 0. Indeed, by Taylor expansion we are led to

$$|R_{ijk}^m| = \frac{1}{2}(X_{ij}^{m+1} - X_{ij}^m)^2 |\psi''(\xi_{ijk}^m)| = \mathcal{O}\left(\frac{V\Delta t}{\Delta x}\right)^2 \tag{75}$$

where ξ_{ijk}^m are intermediate points and V is a bound for the maximal velocity (take $V = 1$ in the relativistic case or assume that f_0 has compact support in the non relativistic case). A better idea is to observe that $R_{ijk}^m = 0$ if

$$\max\{X_{ij}^{m+1} - x_k, X_{ij}^m - x_k\} < -R\Delta x \ \text{ or } \ \min\{X_{ij}^{m+1} - x_k, X_{ij}^m - x_k\} > R\Delta x. \tag{76}$$

Obviously, in the other cases R_{ijk}^m remain uniformly bounded (cf. (75)) under a CFL condition. Nevertheless, the difficult job is to estimate for any fixed k the cardinal of the set

$$\mathcal{A}_k = \{(i,j,m) \ : \ R_{ijk}^m \neq 0\}.$$

Based on (76) this reduces to finding an upper-bound for card$\{(i,j,m) : X_{ij}^m, X_{ij}^{m+1} \in [-R\Delta x - V\Delta t + x_k, R\Delta x + V\Delta t + x_k]\}$. And this is not of all evident, at least when the particle density is only an integrable function. Actually we will see that in certain cases the numerical schemes considered above behave differently . Therefore they are not equivalent.

The first numerical computation we address here concerns the oscillations of a spatial homogeneous plasma in the non relativistic case ($v(p) = p$). We consider the initial conditions

$$f_0 = \frac{n}{\sqrt{2\pi\theta}} \exp\left(-\frac{p^2}{2\theta}\right), \quad E_0 = \sqrt{n\theta}$$

and we check immediately that the exact solution is given by

$$f(t,p) = \frac{n}{\sqrt{2\pi\theta}} \exp\left(-\frac{(p-\sqrt{\theta}\sin(\sqrt{n}t))^2}{2\theta}\right), \quad E(t) = \sqrt{n\theta}\cos(\sqrt{n}t). \qquad (77)$$

The following figures illustrate the behavior of the numerical approximations for (f, E) obtained by using both new and standard particle method. These numerical results are compared to the analytical expressions (77). The parameter values are $n = 25, \theta = 0.1, \Delta t = 0.019$ whereas the phase space domain $[0, L] \times [-p_{\max}, p_{\max}]$ (here $L = 0.5$, $p_{\max} = 2$) is discretized by using $N_x = 20$ points along the space direction and $N_p = 40$ points along the momentum direction. On Figure 1 we plot the time evolution of the electric field. The Figures 2, 3, 4 illustrate the time variation of the total current $\int_0^L \int_{\mathbb{R}} pf \; dpdx$, kinetic energy $\int_0^L \int_{\mathbb{R}} \frac{p^2}{2} f \; dpdx$ and electric energy $\int_0^L \frac{1}{2} E^2 \; dx$. We observe that the curves are in very good agreement for both methods. Actually in this case the numerical schemes have similar behaviors.

We consider now the spatial periodic initial conditions

$$f_0(x,p) = \frac{n}{\sqrt{2\pi\theta}} \exp\left(-\frac{p^2}{2\theta}\right)\left(1 + \cos\left(\frac{2\pi x}{L}\right)\right)$$

$$E_0(x) = \sqrt{n\theta}\left(1 + \frac{L\sqrt{n}}{2\pi\sqrt{\theta}}\sin\left(\frac{2\pi x}{L}\right)\right).$$

The values of the parameters are the same. The Figure 5 represents the time evolution of the total current $\int_0^L \int_{\mathbb{R}} pf(t,x,p) \; dpdx$ and kinetic energy $\int_0^L \int_{\mathbb{R}} \frac{p^2}{2} f(t,x,p) \; dpdx$.

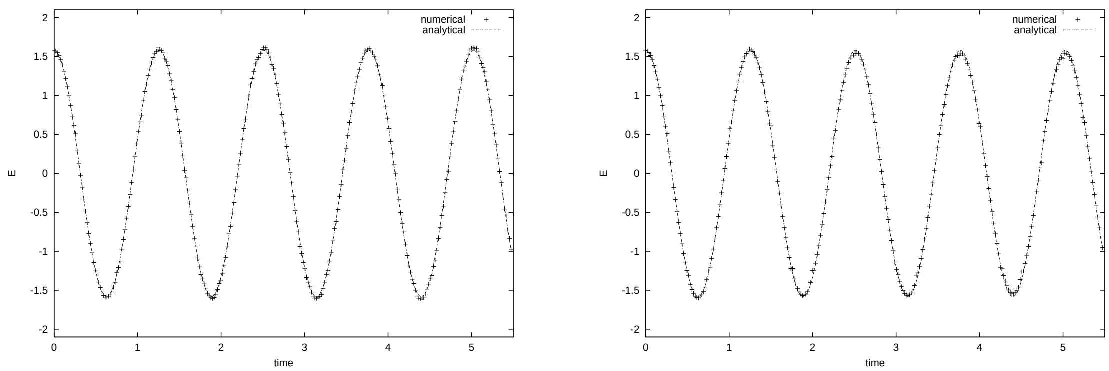

Figure 1: Time evolution of the electric field (left : new scheme/right : standard scheme)

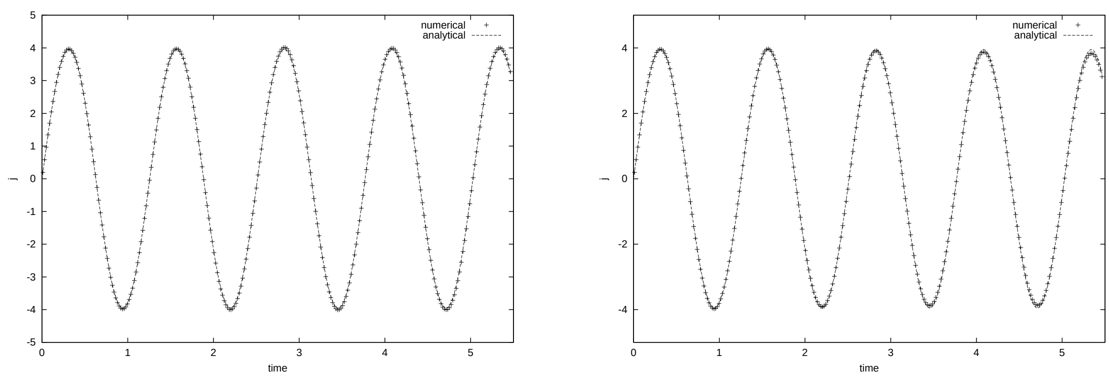

Figure 2: Time evolution of the current (left : new scheme/right : standard scheme)

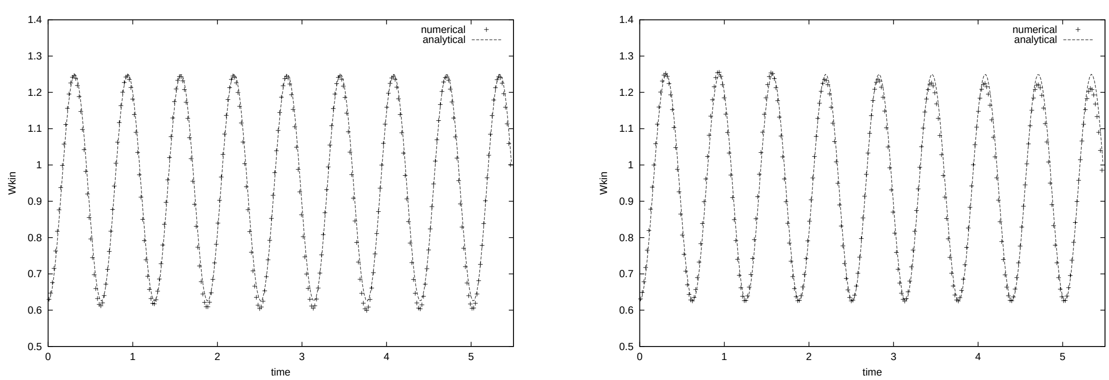

Figure 3: Time evolution of the kinetic energy (left : new scheme/right : standard scheme)

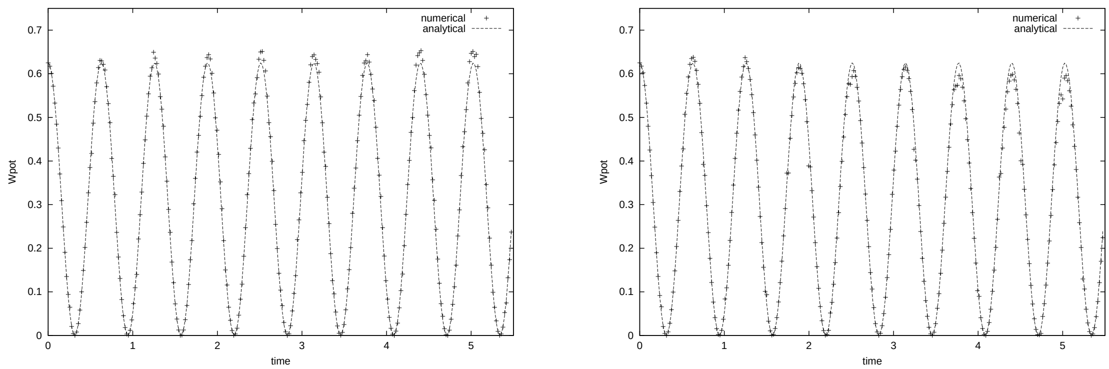

Figure 4: Time evolution of the electric energy (left : new scheme/right : standard scheme)

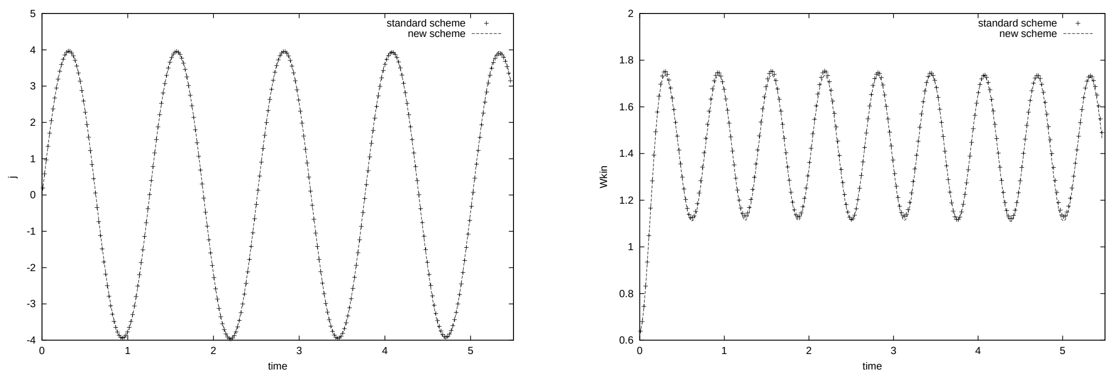

Figure 5: Time evolution of the current and kinetic energy (new scheme and standard scheme)

The left curves in Figure 6 show the variation of the total electric energy $\int_0^L \frac{1}{2}|E(t,x)|^2 \, dx$. We observe that the curves are in very good agreement up to the time $t = 5.5$. The curves in the right part of Figure 6 illustrate the long time evolution of the total energy (kinetic and electric). They show that the total energy is better preserved when using the new particle method. The total energy is conserved with a precision of 1% when using the new particle method and with a precision of 14% when using the standard particle method.

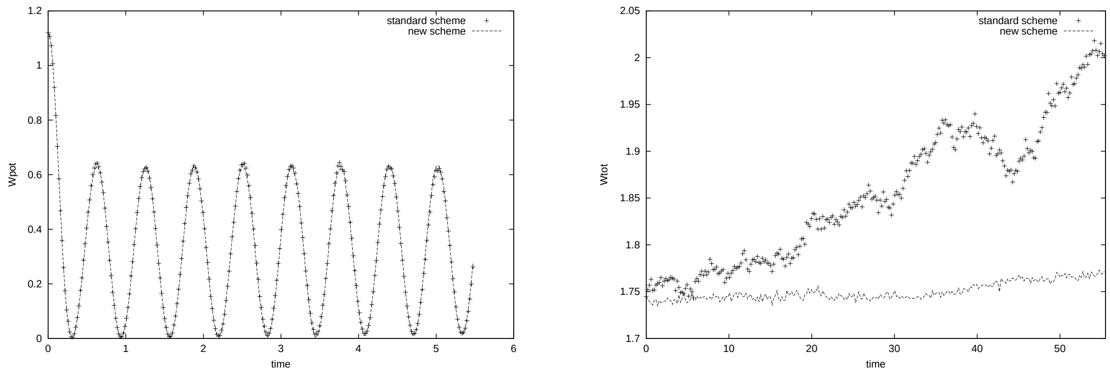

Figure 6: Time evolution of the electric and total energy (new scheme and standard scheme)

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