Periodic Solutions for Evolution Equations

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Abstract. *We study here the existence and uniqueness of periodic solutions for evolution equations. First of all we analyse the one-dimensional case. In arbitrary dimension (finite or not) the case of linear symmetric operators is considered. Finally we prove that the same result applies for non-linear sub-differential operators $A = \partial\varphi$ where φ is convex i.s.c.*

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1 Introduction

Many theoretical and numerical studies in applied mathematics focus on permanent regimes for ordinary or partial differential equations. The main purpose of this paper is to establish existence and uniqueness results for periodic solutions in the general framework of evolution equations :

$$x'(t) + Ax(t) = f(t),\ t \in I\!R, \tag{1}$$

by using the penalization method. Note that in the linear case a necessary condition for the existence is given by :

$$< f >:= \frac{1}{T}\int_0^T f(t)dt \in Range(A). \tag{2}$$

Unfortunately this condition is not always sufficient for the existence (see the example of the orthogonal rotation of $I\!R^2$). Nevertheless the condition (2) is sufficient in the symmetric case. The key point consists of considering first perturbed equations :

$$\alpha x_\alpha(t) + x'_\alpha(t) + Ax_\alpha(t) = f(t), \quad t \in I\!R, \tag{3}$$

where $\alpha > 0$. By using the Banach's fixed point theorem we deduce the existence and uniqueness of the periodic solutions $x_\alpha,\ \alpha > 0$. Under the assumption (2), in the linear symmetric case we show that $(x_\alpha)_{\alpha>0}$ is a Cauchy sequence in C^1 and by passing to the limit for $\alpha \to 0$ it follows that the limit function is a periodic solution for (1).

These results have been announced by [4]. The same approach applies for the study of almost periodic solutions (see [5]).

Results concerning this topic have been obtained previously by other authors by using different methods. A similar condition (2) has been investigated in [3] by studying the range of sums of monotone operators. A different way consists of applying fixed point techniques, see for example [2], [7].

The article is organized as follows. First we analyse the one dimensional case. Necessary and sufficient conditions for the existence and uniqueness of periodic solutions are shown. Results for sub(super)-periodic solutions are proved as well in this case. In the next section we show that the same existence result holds for linear symmetric maximal monotone operators on Hilbert spaces. In the last section the case of non-linear sub-differential operators is considered.

2 Periodic solutions for one dimensional evolution equations

In order to study the periodic solutions for evolution equations it is convenient to consider first the one dimensional case :

$$x'(t) + g(x(t)) = f(t), \quad t \in \mathbb{R}, \tag{4}$$

where $g : \mathbb{R} \to \mathbb{R}$ is increasing Lipschitz continuous in x and $f : \mathbb{R} \to \mathbb{R}$ is T-periodic and continuous in t. By Picard's theorem it follows that for each initial data $x(0) = x_0 \in \mathbb{R}$ there is an unique solution $x \in C^1(\mathbb{R}; \mathbb{R})$ for (4). We are looking for T-periodic solutions. Let us start by the uniqueness study :

2.1 Uniqueness

Proposition 1 *Assume that g is strictly increasing and f is periodic. Then there is at most one periodic solution for (4).*

Proof : Let x_1, x_2 be two periodic solutions for (4). By taking the difference between the two equations and multiplying by $x_1(t) - x_2(t)$ we get :

$$\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + [g(x_1(t)) - g(x_2(t))][x_1(t) - x_2(t)] = 0, \quad t \in \mathbb{R}. \tag{5}$$

Since g is increasing we have $(g(x_1) - g(x_2))(x_1 - x_2) \geq 0 \ \forall x_1, x_2 \in \mathbb{R}$ and therefore we deduce that $|x_1(t) - x_2(t)|$ is decreasing. Moreover as x_1 and x_2 are periodic it follows that $|x_1(t) - x_2(t)|$ doesn't depend on $t \in \mathbb{R}$ and therefore, from (5) we get :

$$[g(x_1(t)) - g(x_2(t))][x_1(t) - x_2(t)] = 0, \ t \in \mathbb{R}.$$

Finally, the strictly monotony of g implies that $x_1 = x_2$.

Remark 1 *If g is only increasing, it is possible that (4) has several periodic solutions. Let us consider the function :*

$$g(x) = \begin{cases} x+\varepsilon, & x < -\varepsilon, \\ 0, & x \in [-\varepsilon, \varepsilon], \\ x-\varepsilon, & x > \varepsilon, \end{cases} \tag{6}$$

and $f(t) = \frac{\varepsilon}{2}\cos t$. We can easily check that $x_\lambda(t) = \lambda + \frac{\varepsilon}{2}\sin t$ are periodic solutions for (4) for $\lambda \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$.

Generally we can prove that every two periodic solutions differ by a constant :

Proposition 2 *Let g be an increasing function and x_1, x_2 two periodic solutions of (4). Then there is a constant $C \in I\!R$ such that :*

$$x_1(t) - x_2(t) = C, \quad \forall t \in I\!R.$$

Proof : As shown before there is a constant $C \in I\!R$ such that $|x_1(t) - x_2(t)| = C,\ t \in I\!R$. Moreover $x_1(t) - x_2(t)$ has constant sign, otherwise $x_1(t_0) = x_2(t_0)$ for some $t_0 \in I\!R$ and it follows that $|x_1(t) - x_2(t)| = |x_1(t_0) - x_2(t_0)| = 0,\ t \in I\!R$ or $x_1 = x_2$. Finally we find that :

$$x_1(t) - x_2(t) = sign(x_1(0) - x_2(0))C,\ t \in I\!R.$$

Before analysing in detail the uniqueness for increasing functions, let us denote by $\mathcal{O}(y),\ y \in I\!R$ the set :

$$\mathcal{O}(y) = \begin{cases} \{x \in I\!R \,|\, x + \int_0^t (f(s) - y)ds \in g^{-1}(y) \quad \forall t \in I\!R\} \subset g^{-1}(y), & y \in g(I\!R), \\ \emptyset, & y \notin g(I\!R). \end{cases}$$

Proposition 3 *Let g be an increasing function and f periodic. Then equation (4) has different periodic solutions iff $Int(\mathcal{O} < f >) \neq \emptyset$.*

Proof : Assume that (4) has two periodic solutions $x_1 \neq x_2$. By the previous proposition we have $x_2 - x_1 = C > 0$. By integration on $[0, T]$ one gets :

$$\int_0^T g(x_1(t))dt = \int_0^T f(t)dt = \int_0^T g(x_2(t))dt. \tag{7}$$

Since g is increasing we have $g(x_1(t)) \leq g(x_2(t)),\ t \in I\!R$ and therefore :

$$\int_0^T g(x_1(t))dt \leq \int_0^T g(x_2(t))dt. \tag{8}$$

From (7) and (8) we deduce that $g(x_1(t)) = g(x_2(t))$, $t \in I\!R$ and thus g is constant on each interval $[x_1(t), x_2(t)] = [x_1(t), x_1(t)+C]$, $t \in I\!R$. Finally it implies that g is constant on $Range(x_1)+[0, C] = \{x_1(t) + y \mid t \in [0, T], y \in [0, C]\}$ and this constant is exactly the time average of f :

$$g(x_1(t)) = g(x_2(t)) =< f >, \quad t \in [0, T]. \tag{9}$$

Let x be an arbitrary real number in $]x_1(0), x_1(0)+C[$. Then :

$$\begin{aligned} x + \int_0^t \{f(s) - <f>\}ds &= x - x_1(0) + x_1(0) + \int_0^t \{f(s) - g(x_1(s))\}ds \\ &= x - x_1(0) + x_1(t) \\ &> x_1(t), \quad t \in I\!\!R. \end{aligned}$$

Similarly :

$$\begin{aligned} x + \int_0^t \{f(s) - <f>\}ds &= x - x_2(0) + x_2(0) + \int_0^t \{f(s) - g(x_2(s))\}ds \\ &= x - x_2(0) + x_2(t) \\ &< x_2(t), \quad t \in I\!\!R. \end{aligned}$$

Therefore $x + \int_0^t \{f(s) - <f>\}ds \in]x_1(t), x_2(t)[\subset g^{-1}(<f>)$, $t \in I\!\!R$ which implies that $x \in \mathcal{O} <f>$ and hence $]x_1(0), x_2(0)[\subset \mathcal{O} <f>$.

Conversely, suppose that there is x and $C > 0$ small enough such that $x, x+C \in \mathcal{O} <f>$. It is easy to check that x_1, x_2 given below are different periodic solutions for (4) :

$$x_1(t) = x + \int_0^t \{f(s) - <f>\}ds, \quad t \in I\!\!R,$$

$$x_2(t) = x + C + \int_0^t \{f(s) - <f>\}ds = x_1(t) + C, \quad t \in I\!\!R.$$

Remark 2 *The condition $Int(\mathcal{O} <f>) \neq \emptyset$ is equivalent to :*

$$diam(g^{-1} <f>) > diam(Range \int \{f(t) - <f>\}dt).$$

Example : Consider the equation $x'(t) + g(x(t)) = \eta \cos t, t \in I\!\!R$ with g given in *Remark 1.* We have $<\eta \cos t> = 0 \in g(I\!\!R)$ and :

$$\begin{aligned} \mathcal{O}(0) &= \{x \in I\!\!R \,|\, x + \int_0^t \eta \cos s \; ds \in g^{-1}(0), \quad t \in I\!\!R\} && (10) \\ &= \{x \in I\!\!R \,|\, x + \eta \sin t \in g^{-1}(0), \; t \in I\!\!R\} \\ &= \{x \in I\!\!R \,|\, -\varepsilon \le x + \eta \sin t \le \varepsilon, t \in I\!\!R\} \\ &= \begin{cases} \emptyset, & |\eta| > \varepsilon, \\ \{0\}, & |\eta| = \varepsilon, \\ [|\eta| - \varepsilon, \varepsilon - |\eta|] & |\eta| < \varepsilon. \end{cases} && (11) \end{aligned}$$

Therefore uniqueness doesn't occur if $|\eta| < \varepsilon$, for example if $\eta = \varepsilon/2$, as seen before in the *Remark 1.* If $|\eta| \ge \varepsilon$ there is an unique periodic solution.

In the following we suppose that g is increasing and we establish the existence result.

2.2 Existence

In order to study the existence, note that a necessary condition is given by :

Proposition 4 *Assume that equation (4) has* T*-periodic solutions. Then there is* $x_0 \in I\!\!R$ *such that* $< f >:= \frac{1}{T}\int_0^T f(t)dt = g(x_0)$.

Proof : Indeed, after integration on a period interval $[0, T]$ we obtain :

$$x(T) - x(0) + \int_0^T g(x(t))dt = \int_0^T f(t)dt. \tag{12}$$

Since x is periodic and $g \circ x$ is continuous we get :

$$Tg(x(\tau)) = \int_0^T f(t)dt, \quad \tau \in]0, T[, \tag{13}$$

and hence :

$$< f >:= \frac{1}{T}\int_0^T f(t)dt \in Range(g). \tag{14}$$

In the following we will show that this condition is also sufficient for the existence of periodic solutions. We will prove this result in several steps. First we establish the existence for the equation :

$$\alpha x_\alpha(t) + x'_\alpha(t) + g(x_\alpha(t)) = f(t), \quad t \in I\!\!R, \ \alpha > 0. \tag{15}$$

Proposition 5 *Suppose that* g *is increasing Lipschitz continuous and* f *is* T*-periodic and continuous. Then for every* $\alpha > 0$ *the equation (15) has exactly one periodic solution.*

Remark 3 *Before starting the proof let us observe that (15) reduces to an equation of type (4) with* $g_\alpha = \alpha 1_{I\!\!R} + g$*. Since* g *is increasing, is clear that* g_α *is strictly increasing and by the Proposition 1 we deduce that the uniqueness holds. Moreover since* $Range(g_\alpha) = I\!\!R$*, the necessary condition (14) is trivially verified and therefore, in this case we can expect to prove existence.*

Proof : First of all remark that the existence of periodic solutions reduces to finding $x_0 \in I\!\!R$ such that the solution of the evolution problem :

$$\begin{cases} \alpha x_\alpha(t) + x'_\alpha(t) + g(x_\alpha(t)) = f(t), & t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{16}$$

verifies $x(T\,; 0, x_0) = x_0$. Here we denote by $x(\cdot\,; 0, x_0)$ the solution of (16) (existence and uniqueness assured by Picard's theorem). We define the map $S : I\!\!R \to I\!\!R$ given by :

$$S(x_0) = x(T\,; 0, x_0), \quad x_0 \in I\!\!R. \tag{17}$$

We demonstrate the existence and uniqueness of the periodic solution of (16) by showing that the Banach's fixed point theorem applies. Let us consider two solutions of (16) corresponding to the initial datas x_0^1 and x_0^2. Using the monotony of g we can write :

$$\alpha|x(t\,;0,x_0^1)-x(t\,;0,x_0^2)|^2+\frac{1}{2}\frac{d}{dt}|x(t\,;0,x_0^1)-x(t\,;0,x_0^2)|^2\leq 0, \tag{18}$$

which implies :

$$\frac{1}{2}\frac{d}{dt}\{e^{2\alpha t}|x(t\,;0,x_0^1)-x(t\,;0,x_0^2)|^2\}\leq 0, \tag{19}$$

and therefore :

$$|S(x_0^1)-S(x_0^2)|=|x(T\,;0,x_0^1)-x(T\,;0,x_0^2)|\leq e^{-\alpha T}|x_0^1-x_0^2|. \tag{20}$$

For $\alpha>0$ S is a contraction and the Banach's fixed point theorem applies. Therefore $S(x_0)=x_0$ for an unique $x_0\in I\!\!R$ and hence $x(\cdot\,;0,x_0)$ is a periodic solution of (4).

Naturally, in the following proposition we inquire about the convergence of $(x_\alpha)_{\alpha>0}$ to a periodic solution of (4) as $\alpha\to 0$. In view of the *Proposition 4* this convergence doesn't hold if (14) is not verified. Assume for the moment that (4) has at least one periodic solution. In this case convergence holds :

Proposition 6 *If equation (4) has at least one periodic solution, then $(x_\alpha)_{\alpha>0}$ is convergent in $C^0(I\!\!R;I\!\!R)$ and the limit is also a periodic solution of (4).*

Proof : Denote by x a periodic solution of (4). By elementary calculations we find :

$$\alpha|x_\alpha(t)-x(t)|^2+\frac{1}{2}\frac{d}{dt}|x_\alpha(t)-x(t)|^2\leq-\alpha x(t)(x_\alpha(t)-x(t)),\ t\in I\!\!R, \tag{21}$$

which can be also written as :

$$\frac{1}{2}\frac{d}{dt}\{e^{2\alpha t}|x_\alpha(t)-x(t)|^2\}\leq\alpha e^{\alpha t}|x(t)|\cdot e^{\alpha t}|x_\alpha(t)-x(t)|,\ t\in I\!\!R. \tag{22}$$

Therefore, by integration on $[0,t]$ we deduce :

$$\frac{1}{2}\{e^{\alpha t}|x_\alpha(t)-x(t)|\}^2\leq\frac{1}{2}|x_\alpha(0)-x(0)|^2+\int_0^t\alpha e^{\alpha s}|x(s)|\cdot e^{\alpha s}|x_\alpha(s)-x(s)|ds. \tag{23}$$

Using Bellman's lemma, formula (23) gives :

$$e^{\alpha t}|x_\alpha(t)-x(t)|\leq|x_\alpha(0)-x(0)|+\int_0^t\alpha e^{\alpha s}|x(s)|ds,\ t\in I\!\!R. \tag{24}$$

Let us consider $\alpha>0$ fixed for the moment. Since x is periodic and continuous, it is also bounded and therefore from (24) we get :

$$|x_\alpha(t)-x(t)|\leq e^{-\alpha t}|x_\alpha(0)-x(0)|+(1-e^{-\alpha t})\|x\|_{L^\infty(I\!\!R)},\quad t\in I\!\!R. \tag{25}$$

By periodicity we have :

$$\begin{aligned}|x_\alpha(t)-x(t)| &= |x_\alpha(nT+t)-x(nT+t)|\\ &\leq e^{-\alpha(nT+t)}|x_\alpha(0)-x(0)|+(1-e^{-\alpha(nT+t)})\|x\|_{L^\infty(I\!R)}\\ &\leq e^{-\alpha(nT+t)}|x_\alpha(0)-x(0)|+\|x\|_{L^\infty(I\!R)},\ t\in I\!R, n\geq 0.\end{aligned}$$

By passing to the limit for $n\to\infty$ we deduce that $(x_\alpha)_{\alpha>0}$ is uniformly bounded in $L^\infty(I\!R)$:

$$|x_\alpha(t)|\leq|x_\alpha(t)-x(t)|+|x(t)|\leq 2\|x\|_{L^\infty(I\!R)},\ t\in I\!R,\ \alpha>0.$$

The derivatives x'_α are also uniformly bounded in $L^\infty(I\!R)$ for $\alpha\to 0$:

$$\begin{aligned}|x'_\alpha(t)| &= |f(t)-\alpha x_\alpha(t)-g(x_\alpha(t))|\\ &\leq \|f\|_{L^\infty(I\!R)}+2\alpha\|x\|_{L^\infty(I\!R)}+\max\{g(2\|x\|_{L^\infty(I\!R)}),-g(-2\|x\|_{L^\infty(I\!R)})\}.\end{aligned}$$

The uniform convergence of $(x_\alpha)_{\alpha>0}$ follows now from the Arzela-Ascoli's theorem. Denote by u the limit of $(x_\alpha)_{\alpha>0}$ as $\alpha\to 0$. Obviously u is also periodic :

$$u(0)=\lim_{\alpha\to 0}x_\alpha(0)=\lim_{\alpha\to 0}x_\alpha(T)=u(T).$$

In order to prove that u verify equation (4), we write :

$$x_\alpha(t)=x_\alpha(0)+\int_0^t\{f(s)-g(x_\alpha(s))-\alpha x_\alpha(s)\}ds,\ t\in I\!R.$$

Since the convergence is uniform, by passing to the limit for $\alpha\to 0$ we obtain :

$$u(t)=u(0)+\int_0^t\{f(s)-g(u(s))\}ds,$$

and hence $u\in C^1(I\!R;I\!R)$ and :

$$u'(t)+g(u(t))=f(t),\quad t\in I\!R.$$

From the previous proposition we conclude that the existence of periodic solutions for (4) reduces to uniform estimates in $L^\infty(I\!R)$ for $(x_\alpha)_{\alpha>0}$:

Proposition 7 *Assume that g is increasing Lipschitz continuous and f is T-periodic and continuous. Then the following statements are equivalent :*
(i) equation (4) has periodic solutions;
(ii) the sequence $(x_\alpha)_{\alpha>0}$ is uniformly bounded in $L^\infty(I\!R)$. Moreover, in this case $(x_\alpha)_{\alpha>0}$ is convergent in $C^0(I\!R;I\!R)$ and the limit is a periodic solution for (4).

Note that generally we can not estimate $(x_\alpha)_{\alpha>0}$ uniformly in $L^\infty(I\!R)$. Indeed, by standard computations we obtain :

$$\alpha(x_\alpha(t)-u)^2+\frac{1}{2}\frac{d}{dt}(x_\alpha(t)-u)^2\leq|f(t)-\alpha u-g(u)|\cdot|x_\alpha(t)-u|,\ t,u\in I\!R$$

and therefore :

$$\frac{1}{2}\frac{d}{dt}\{e^{2\alpha t}(x_\alpha(t)-u)^2\} \leq e^{\alpha t}|f(t)-\alpha u - g(u)| \cdot e^{\alpha t}|x_\alpha(t)-u|,\ t,u \in I\!R.$$

By integration on $[t, t+h]$ we get :

$$\begin{aligned}\frac{1}{2}e^{2\alpha(t+h)}(x_\alpha(t+h)-u)^2 &\leq \int_t^{t+h} e^{2\alpha s}|f(s)-\alpha u - g(u)| \cdot |x_\alpha(s)-u|ds \\ &+ \frac{1}{2}e^{2\alpha t}(x_\alpha(t)-u)^2,\ t < t+h,\ u \in I\!R.\end{aligned}$$

Now by using Bellman's lemma we deduce :

$$|x_\alpha(t+h)-u| \leq e^{-\alpha h}|x_\alpha(t)-u| + \int_t^{t+h} e^{-\alpha(t+h-s)}|f(s)-\alpha u - g(u)|ds,\ t < t+h.$$

Since x_α is T-periodic, by taking $h = T$ we can write :

$$|x_\alpha(t)-u| \leq \frac{1}{1-e^{-\alpha T}}\int_0^T e^{-\alpha(T-s)}|f(s)-\alpha u - g(u)|ds,\ t \in I\!R,$$

and thus for $u = 0$ we obtain :

$$\|x_\alpha\|_{L^\infty(I\!R)} \leq \frac{1}{1-e^{-\alpha T}}\int_0^T |f(s)-g(0)|ds \sim \mathcal{O}\left(\frac{1}{\alpha}\right),\ \alpha > 0.$$

Now we can state our main existence result :

Theorem 1 *Assume that g is increasing Lipschitz continuous, and f is T-periodic and continuous. Then equation (4) has periodic solutions iff $< f >:= \frac{1}{T}\int_0^T f(t)dt \in Range(g)$ (there is $x_0 \in I\!R$ such that $< f >= g(x_0)$). Moreover in this case we have the estimate :*

$$\|x\|_{L^\infty(I\!R)} \leq |x_0| + \int_0^T |f(t)- < f > |dt,\ \forall\ x_0 \in g^{-1} < f >,$$

and the solution is unique iff $Int(\mathcal{O} < f >) = \emptyset$ or :

$$diam(g^{-1} < f >) \leq diam(Range \int \{f(t)- < f >\}dt).$$

Proof : The condition is necessary (see *Proposition 4*). We will prove now that it is also sufficient. Let us consider the sequence of periodic solutions $(x_\alpha)_{\alpha>0}$ of (15). Accordingly to the *Proposition 7* we need to prove uniform estimates in $L^\infty(I\!R)$ for $(x_\alpha)_{\alpha>0}$. Since x_α is T-periodic by integration on $[0, T]$ we get :

$$\int_0^T \{\alpha x_\alpha(t) + g(x_\alpha(t))\}dt = T < f >,\ \alpha > 0.$$

Using the average formula for continuous functions we have :

$$\int_0^T \{\alpha x_\alpha(t) + g(x_\alpha(t))\}dt = T\{\alpha x_\alpha(t_\alpha) + g(x_\alpha(t_\alpha))\},\ t_\alpha \in]0, T[,\ \alpha > 0.$$

By the hypothesis there is $x_0 \in I\!R$ such that $< f >= g(x_0)$ and thus :

$$\alpha x_\alpha(t_\alpha) + g(x_\alpha(t_\alpha)) = g(x_0),\ \alpha > 0. \tag{26}$$

Since g is increasing, we deduce :

$$\alpha x_\alpha(t_\alpha)[x_0 - x_\alpha(t_\alpha)] = [g(x_0) - g(x_\alpha(t_\alpha))][x_0 - x_\alpha(t_\alpha)] \geq 0,\ \alpha > 0,$$

and thus :

$$|x_\alpha(t_\alpha)|^2 \leq x_\alpha(t_\alpha)x_0 \leq |x_\alpha(t_\alpha)||x_0|.$$

Finally we deduce that $x_\alpha(t_\alpha)$ is uniformly bounded in $I\!R$:

$$|x_\alpha(t_\alpha)| \leq |x_0|, \quad \forall\, \alpha > 0.$$

Now we can easily find uniform estimates in $L^\infty(I\!R)$ for $(x_\alpha)_{\alpha>0}$. Let us take in the previous calculus $u = x_\alpha(t_\alpha)$and integrate on $[t_\alpha, t]$:

$$\frac{1}{2}e^{2\alpha t}(x_\alpha(t) - x_\alpha(t_\alpha))^2 \leq \int_{t_\alpha}^t e^{2\alpha s}|f(s) - \alpha x_\alpha(t_\alpha) - g(x_\alpha(t_\alpha))| \cdot |x_\alpha(s) - x_\alpha(t_\alpha)| ds.$$

By using Bellman's lemma we get :

$$|x_\alpha(t) - x_\alpha(t_\alpha)| \leq \int_{t_\alpha}^t e^{-\alpha(t-s)}|f(s) - \alpha x_\alpha(t_\alpha) - g(x_\alpha(t_\alpha))|ds,\ t > t_\alpha,$$

and hence by (26) we deduce :

$$\begin{aligned} |x_\alpha(t)| &\leq |x_0| + \int_0^T |f(t) - \alpha x_\alpha(t_\alpha) - g(x_\alpha(t_\alpha))|dt \\ &= |x_0| + \int_0^T |f(t) - <f>|dt,\ t \in I\!R, \alpha > 0. \end{aligned} \tag{27}$$

Now by passing to the limit in (27) we get :

$$|x(t)| \leq |x_0| + \int_0^T |f(t) - <f>|dt,\ t \in I\!R,\ \forall\, x_0 \in g^{-1} <f>.$$

2.3 Sub(super)-periodic solutions

In this part we generalize the previous existence results for sub(super)-periodic solutions. We will see that similar results hold. Let us introduce the concept of sub(super)-periodic solutions :

Definition 1 *We say that $x \in C^1([0,T]; I\!R)$ is a sub-periodic solution for (4) if :*

$$x'(t) + g(x(t)) = f(t), \quad t \in [0,T],$$

and $x(0) \leq x(T)$.

Note that a necessary condition for the existence is given by :

Proposition 8 *If equation (4) has sub-periodic solutions, then there is $x_0 \in I\!R$ such that $g(x_0) \leq$ $<f>$.*

Proof : Let x be a sub-periodic solution of (4) . By integration on $[0,T]$ we find :

$$x(T) - x(0) + \int_0^T g(x(t))dt = T < f > .$$

Since $g \circ x$ is continuous, there is $\tau \in]0,T[$ such that :

$$g(x(\tau)) =< f > -\frac{1}{T}(x(T) - x(0)) \leq < f > .$$

Similarly we define the notion of super-periodic solution :

Definition 2 *We say that $y \in C^1([0,T]; I\!R)$ is a super-periodic solution for (4) if :*

$$y'(t) + g(y(t)) = f(t), \quad t[0,T],$$

and $y(0) \geq y(T)$.

The analogous necessary condition holds :

Proposition 9 *If equation (4) has super-periodic solutions, then there is $y_0 \in I\!R$ such that $g(y_0) \geq$ $< f >$.*

Remark 4 *It is clear that x is periodic solution for (4) iff is in the same time sub-periodic and super-periodic solution. Therefore there are $x_0, y_0 \in I\!R$ such that :*

$$g(x_0) \leq < f > \leq g(y_0).$$

Since g is continuous, we deduce that $< f >\in Range(g)$ which is exactly the necessary condition given by the Proposition 4.

As before we will prove that the necessary condition of *Proposition 8* is also sufficient for the existence of sub-periodic solutions. We have the theorem :

Theorem 2 *Assume that g is increasing Lipschitz continuous and f is T-periodic continuous. Then equation (4) has sub-periodic solutions iff there is $x_0 \in I\!R$ such that $g(x_0) \leq < f >$.*

Proof : The condition is necessary (see *Proposition 8*). Let us prove now that it is also sufficient. Consider z_0 an arbitrary initial data and denote by $x : [0, \infty[\to I\!R$ the solution for (4) with the initial condition $x(0) = z_0$. If there is $t_0 \geq 0$ such that $x(t_0) \leq x(t_0+T)$, thus $x_{t_0}(t) := x(t_0+t)$, $t \in [0,T]$ is a sub-periodic solution. Suppose now that $x(t) > x(t+T)$, $\forall t \in I\!R$. By integration on $[nT, (n+1)T]$, $n \geq 0$ we get :

$$x((n+1)T) - x(nT) + \int_0^T g(x(nT+t))dt = T < f >, n \geq 0.$$

Using the hypothesis and the average formula we have :

$$g(x(nT+\tau_n)) =< f > +\frac{1}{T}\{x(nT) - x((n+1)T)\} > g(x_0), \, \tau_n \in]0,T[, \, n \geq 0.$$

Since g is increasing we deduce that $x(nT+\tau_n) > x_0$, $n \geq 0$. We have also $x(nT+\tau_n) \leq x((n-1)T+\tau_n) \leq \cdots \leq x(\tau_n) \leq \sup_{t\in[0,T]} |x(t)|$ and thus we deduce that $(x(nT+\tau_n))_{n\geq 0}$ is bounded :

$$|x(nT+\tau_n)| \leq K,\ n \geq 0.$$

Consider now the functions $x_n : [0,T] \to I\!\!R$ given by :

$$x_n(t) = x(nT+t),\ t \in [0,T].$$

By standard computation we get :

$$\frac{1}{2}\frac{d}{dt}|x_n(t)|^2 + [g(x_n(t)) - g(0)]x_n(t) = [f(t) - g(0)]x_n(t),\ t \in [0,T].$$

Using the monotony of g we obtain :

$$|x_n(t)| \leq |x_n(s)| + \int_s^t |f(u) - g(0)|du,\ 0 \leq s \leq t \leq T.$$

Taking $s = \tau_n \in]0,T[$ we can write :

$$|x_n(t)| \leq |x_n(\tau_n)| + \int_{\tau_n}^t |f(u) - g(0)|du \leq K + \int_0^T |f(u) - g(0)|du,\ t \in [\tau_n, T].$$

For $t \in [0,\tau_n]$, $n \geq 1$ we have :

$$\begin{aligned} |x_n(t)| = |x(nT+t)| &\leq |x((n-1)T+\tau_{n-1})| + \int_{(n-1)T+\tau_{n-1}}^{nT+t} |f(u) - g(0)|du \\ &\leq K + \int_{(n-1)T}^{(n+1)T} |f(u) - g(0)|du \\ &\leq K + 2\int_0^T |f(u) - g(0)|du. \end{aligned}$$

Therefore the sequence $(x_n)_{n\geq 0}$ is uniformly bounded in $L^\infty(I\!\!R)$ and :

$$\|x_n\|_{L^\infty(I\!\!R)} \leq K + 2\int_0^T |f(t) - g(0)|dt := M.$$

Moreover, $(x_n')_{n\geq 0}$ is also uniformly bounded in $L^\infty(I\!\!R)$. Indeed we have :

$$|x_n'(t)| = |f(t) - g(x_n(t))| \leq \|f\|_{L^\infty(I\!\!R)} + \max\{g(M), -g(-M)\},$$

and hence, by Arzela-Ascoli's theorem we deduce that $(x_n)_{n\geq 0}$ converges in $C^0([0,T], I\!\!R)$:

$$\lim_{n\to\infty} x_n(t) = u(t), \text{ uniformly for } t \in [0,T].$$

As usual, by passing to the limit for $n \to \infty$ we find that u is also solution for (4). Moreover since $(x(nT))_{n\geq 0}$ is decreasing and bounded, it is convergent and we can prove that u is periodic :

$$u(0) = \lim_{n\to\infty} x_n(0) = \lim_{n\to\infty} x(nT) = \lim_{n\to\infty} x((n+1)T) = \lim_{n\to\infty} x_n(T) = u(T).$$

Therefore u is a sub-periodic solution for (4). The analogous result holds for super-periodic solutions :

Proposition 10 *Under the same assumptions as in Theorem 2 the equation (4) has super-periodic solutions iff there is* $y_0 \in I\!R$ *such that* $g(y_0) \geq < f >$.

We state now a comparaison result between sub-periodic and super-periodic solutions :

Proposition 11 *If* g *is increasing,* x *is a sub-periodic solution and* y *is a super-periodic solution we have :*

$$x(t) \leq y(t), \, \forall t \in [0, T],$$

provided that x *and* y *are not both periodic.*

Proof : Both x and y verify (4), thus :

$$(x - y)'(t) + g(x(t)) - g(y(t)) = 0, \, t \in [0, T].$$

With the notation :

$$r(t) = \begin{cases} \frac{g(x(t)) - g(y(t))}{x(t) - y(t)} & , t \in [0, T], \, x(t) \neq y(t) \\ 0 & , t \in [0, T], \, x(t) = y(t), \end{cases} \tag{28}$$

we can write $g(x(t)) - g(y(t)) = r(t)(x(t) - y(t)), \; t \in [0, T]$ and therefore :

$$(x - y)'(t) + r(t)(x(t) - y(t)) = 0, \, t \in [0, T],$$

which implies that :

$$x(t) - y(t) = (x(0) - y(0))e^{-\int_0^t r(s) ds}. \tag{29}$$

Now it is clear that if $x(0) \leq y(0)$ we also have $x(t) \leq y(t), \, t \in [0, T]$. Suppose now that $x(0) > y(0)$. Taking $t = T$ in (29) we obtain :

$$x(T) - y(T) = (x(0) - y(0))e^{-\int_0^T r(t) dt}. \tag{30}$$

Since g is increasing, by the definition of the function r we have $r \geq 0$. Two cases are possible :(i) either $\int_0^T r(t)dt > 0$, (ii) either $\int_0^T r(t)dt = 0$ in wich case $r(t) = 0, \, t \in [0, T]$ (r vanisches in all points of continuity t such that $x(t) \neq y(t)$ and also in all points t with $x(t) = y(t)$ by the definition). Let us analyse the first case (i). By (30) we deduce that $x(T) - y(T) < x(0) - y(0)$ or $x(T) - x(0) < y(T) - y(0)$. Since x is sub-periodic we have $x(0) \leq x(T)$ which implies that $y(T) > y(0)$ which is in contradiction with the super-periodicity of y ($y(T) \leq y(0)$).
In the second case (ii) we have $g(x(t)) = g(y(t)), \, t \in [0, T]$ so $(x - y)' = 0$ and therefore there is a constant $C \in I\!R$ such that $x(t) = y(t) + C, \, t \in [0, T]$. Taking $t = 0$ and $t = T$ we obtain :

$$0 \geq x(0) - x(T) = y(0) - y(T) \geq 0,$$

and thus x and y are both periodic which is in contradiction with the hypothesis.

In the following we will see how it is possible to retrieve the existence result for periodic solutions by using the method of sub(super)-periodic solutions. Suppose that $< f >\in Range(g)$. Obviously both sufficient conditions for existence of sub(super)-periodic solutions are satisfied and thus there are $x_0(y_0)$ sub(super)-periodic solutions. If y_0 is even periodic the proof is finisched. Assume that y_0 is not periodic ($y_0(0) > y_0(T)$). Denote by $\mathcal{M}$ the set of sub-periodic solutions for (4) :

$$\mathcal{M} = \{x : [0,T] \to I\!R \,|\, x \text{ sub-periodic solution }, \; x_0(t) \le x(t), \; t \in [0,T]\}.$$

Since $x_0 \in \mathcal{M}$ we have $\mathcal{M} \ne \emptyset$. Moreover, from the comparaison result since y_0 is super-periodic but not periodic we have $x \le y_0, \forall x \in \mathcal{M}$. We prove that $\mathcal{M}$ contains a maximal element in respect to the order :

$$x_1 \prec x_2 \text{ (iff) } x_1(t) \le x_2(t), \; t \in [0,T].$$

Finally we show that this maximal element is even a periodic solution for (4) since otherwise it would be possible to construct a sub-periodic solution greater than the maximal element.

We state now the following generalization :

Theorem 3 *Assume that $g : I\!R \times I\!R \to I\!R$ is increasing Lipschitz continuous function in x, T-periodic and continuous in t and $f : I\!R \to I\!R$ is T-periodic and continuous in t. Then the equation :*

$$x'(t) + g(t, x(t)) = f(t), \, t \in I\!R, \tag{31}$$

has periodic solutions iff there is $x_0 \in I\!R$ such that :

$$< f >:= \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^T g(t, x_0)dt = G(x_0). \tag{32}$$

Moreover, in this case we have the estimate :

$$\|x\|_{L^\infty(I\!R)} \le |x_0| + \int_0^T |f(t) - g(t, x_0)|dt, \; \forall \, x_0 \in G^{-1} < f > .$$

Proof : Consider the average function $G : I\!R \to I\!R$ given by :

$$G(x) = \frac{1}{T}\int_0^T g(t, x)dt, \, x \in I\!R.$$

It is easy to check that G is also increasing and Lipschitz continuous with the same constant. Let us prove that the condition (32) is necessary. Suppose that x is a periodic solution for (31). By integration on $[0,T]$ we get :

$$\frac{1}{T}\int_0^T g(t, x(t))dt =< f > . \tag{33}$$

We can write :

$$m \le x(t) \le M, \, t \in [0,T],$$

and thus :

$$g(t,m) \leq g(t,x(t)) \leq g(t,M),\, t \in [0,T],$$

which implies :

$$G(m) = \frac{1}{T}\int_0^T g(t,m)dt \leq \frac{1}{T}\int_0^T g(t,x(t))dt \leq \frac{1}{T}\int_0^T g(t,M)dt = G(M).$$

Since G is continuous it follows that there is $x_0 \in [m,M]$ such that $G(x_0) = \frac{1}{T}\int_0^T g(t,x(t))dt$ and from (33) we deduce that $< f >= G(x_0)$.

Let us show that the condition (32) is also sufficient. As before let us consider the unique periodic solution for :

$$\alpha x_\alpha(t) + x'_\alpha(t) + g(t,x_\alpha(t)) = f(t),\, t \in [0,T],\ \alpha > 0,$$

(existence and uniqueness follow by the Banach's fixed point theorem exactly as before). All we need to prove is that $(x_\alpha)_{\alpha>0}$ is uniformly bounded in $L^\infty(I\!R)$ (then $(x'_\alpha)_{\alpha>0}$ is also uniformly bounded in $L^\infty(I\!R)$ and by Arzela-Ascoli's theorem we deduce that x_α converges to a periodic solution for (31)). Taking the average on $[0,T]$ we get :

$$\frac{1}{T}\int_0^T \{\alpha x_\alpha(t) + g(t,x_\alpha(t))\}dt =< f >= G(x_0),\ \alpha > 0.$$

As before we can write :

$$\alpha m_\alpha + g(t,m_\alpha) \leq \alpha x_\alpha(t) + g(t,x_\alpha(t)) \leq \alpha M_\alpha + g(t,M_\alpha),\, t \in [0,T],\ \alpha > 0,$$

where :

$$m_\alpha \leq x_\alpha(t) \leq M_\alpha,\, t \in [0,T],\ \alpha > 0,$$

and hence :

$$\alpha m_\alpha + G(m_\alpha) \leq \frac{1}{T}\int_0^T \{\alpha x_\alpha(t) + g(t,x_\alpha(t))\}dt \leq \alpha M_\alpha + G(M_\alpha),\ \alpha > 0.$$

Finally we get :

$$G(x_0) = \frac{1}{T}\int_0^T \{\alpha x_\alpha(t) + g(t,x_\alpha(t))\}dt = \alpha u_\alpha + G(u_\alpha),\ u_\alpha \in]m_\alpha, M_\alpha[,\ \alpha > 0. \qquad (34)$$

Multiplying by $u_\alpha - x_0$ we obtain :

$$\alpha u_\alpha(u_\alpha - x_0) = -(G(x_0) - G(u_\alpha))(x_0 - u_\alpha),\ \alpha > 0.$$

Since G is increasing we deduce that $|u_\alpha|^2 \leq u_\alpha x_0 \leq |u_\alpha| \cdot |x_0|,\ \alpha > 0$ and hence $(u_\alpha)_{\alpha>0}$ is bounded :

$$|u_\alpha| \leq |x_0|,\ \alpha > 0.$$

Now using (34) it follows :

$$\frac{1}{T}\int_0^T \{\alpha x_\alpha(t) + g(t,x_\alpha(t))\}dt = \frac{1}{T}\int_0^T \{\alpha u_\alpha + g(t,u_\alpha)\}dt,$$

and thus there is $t_\alpha \in]0, T[$ such that :

$$\alpha x_\alpha(t_\alpha) + g(t_\alpha, x_\alpha(t_\alpha)) = \alpha u_\alpha + g(t_\alpha, u_\alpha), \ \alpha > 0.$$

Since $\alpha(x_\alpha(t_\alpha) - u_\alpha)^2 = -[g(t_\alpha, x_\alpha(t_\alpha)) - g(t_\alpha, u_\alpha)][x_\alpha(t_\alpha) - u_\alpha] \leq 0$ we find that $x_\alpha(t_\alpha) = u_\alpha, \ \alpha > 0$ and thus $(x_\alpha(t_\alpha))_{\alpha>0}$ is also bounded :

$$|x_\alpha(t_\alpha)| \leq |x_0|, \ \alpha > 0.$$

Now by standard calculations we can write :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|x_\alpha(t) - x_\alpha(t_\alpha)|^2 &+ [g(t, x_\alpha(t)) - g(t, x_\alpha(t_\alpha))][x_\alpha(t) - x_\alpha(t_\alpha)] \\ &\leq [f(t) - \alpha x_\alpha(t_\alpha) - g(t, x_\alpha(t_\alpha))][x_\alpha(t) - x_\alpha(t_\alpha)], \ t \in I\!\!R, \end{aligned}$$

and thus :

$$|x_\alpha(t) - x_\alpha(t_\alpha)| \leq \int_{t_\alpha}^{t} |f(s) - \alpha x_\alpha(t_\alpha) - g(s, x_\alpha(t_\alpha))| ds, \ t > t_\alpha, \ \alpha > 0,$$

which implies :

$$|x_\alpha(t)| \leq |x_0| + \int_0^T |f(t) - \alpha x_\alpha(t_\alpha) - g(t, x_\alpha(t_\alpha))| dt, \ t \in [0, T], \ \alpha > 0. \tag{35}$$

Since $(x_\alpha(t_\alpha))_{\alpha>0}$ is bounded we have :

$$u_\alpha = x_\alpha(t_\alpha) \to x_1,$$

such that :

$$G(x_0) = \lim_{\alpha \to 0} \{\alpha u_\alpha + G(u_\alpha)\} = G(x_1).$$

Moreover, if $x_0 \leq x_1$ we have :

$$0 \leq \frac{1}{T}\int_0^T [g(t, x_1) - g(t, x_0)] dt = G(x_1) - G(x_0) = 0,$$

and hence $g(t, x_1) = g(t, x_0), \ \forall \ t \in [0, T]$. Obviously the same equalities hold if $x_0 > x_1$. Now by passing to the limit in (35) we find :

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^T |f(t) - g(t, x_1)| dt \qquad (36) \\ &= |x_0| + \int_0^T |f(t) - g(t, x_0)| dt, \ t \in [0, T], \ \forall \ x_0 \in G^{-1} < f >, \end{aligned}$$

and therefore $(x_\alpha)_{\alpha>0}$ is uniformly bounded in $L^\infty(I\!\!R)$.

3 Periodic solutions for evolution equations on Hilbert spaces

In this section we analyse the existence and uniqueness of periodic solutions for general evolution equations on Hilbert spaces :

$$x'(t) + Ax(t) = f(t),\ t > 0, \tag{37}$$

where $A : D(A) \subset H \to H$ is a maximal monotone operator on a Hilbert space H and $f \in C^1(I\!R; H)$ is a T-periodic function. As known by the theory of Hille-Yosida, for every initial data $x_0 \in D(A)$ there is an unique solution $x \in C^1([0, +\infty[\ ; H) \cap C([0, +\infty[\ ; D(A))$ for (37), see [6], pp. 101. Obviously, the periodic problem reduces to find $x_0 \in D(A)$ such that $x(T) = x_0$. As in the one dimensional case we demonstrate uniqueness for strictly monotone operators. We state also necessary and sufficient condition for the existence in the linear symmetric case. Finally the case of non-linear sub-differential operators is considered. Let us start with the definition of periodic solutions for (37) :

Definition 3 *Let $A : D(A) \subset H \to H$ be a maximal monotone operator on a Hilbert space H and $f \in C^1(I\!R\ ; H)$ a T-periodic function. We say that $x \in C^1([0,T]\ ; H) \cap C([0,T]\ ; D(A))$ is a periodic solution for (37) iff :*

$$x'(t) + Ax(t) = f(t),\ t \in [0, T],$$

and $x(0) = x(T)$.

3.1 Uniqueness

Generally the uniqueness doesn't hold (see the example in the following paragraph). However it occurs under the hypothesis of strictly monotony :

Proposition 12 *Assume that A is strictly monotone ($(Ax_1 - Ax_2, x_1 - x_2) = 0$ implies $x_1 = x_2$). Then (37) has at most one periodic solution.*

Proof : Let x_1, x_2 be two different periodic solutions. By taking the difference of (37) and multiplying both sides by $x_1(t) - x_2(t)$ we find :

$$\frac{1}{2}\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 + (Ax_1(t) - Ax_2(t),\ x_1(t) - x_2(t)) = 0,\ t \in [0, T].$$

By the monotony of A we deduce that $\|x_1 - x_2\|^2$ is decreasing and therefore we have :

$$\|x_1(0) - x_2(0)\| \geq \|x_1(t) - x_2(t)\| \geq \|x_1(T) - x_2(T)\|,\ t \in [0, T].$$

Since x_1 and x_2 are T-periodic we have :

$$\|x_1(0) - x_2(0)\| = \|x_1(T) - x_2(T)\|,$$

which implies that $\|x_1(t) - x_2(t)\|$ is constant for $t \in [0, T]$ and thus :

$$(Ax_1(t) - Ax_2(t),\ x_1(t) - x_2(t)) = 0,\ t \in [0, T].$$

Now uniqueness follows by the strictly monotony of A.

3.2 Existence

In this section we establish existence results. In the linear case we state the following necessary condition :

Proposition 13 *Let $A : D(A) \subset H \to H$ be a linear maximal monotone operator and $f \in L^1(]0, T[;\ H)$ a T-periodic function. If (37) has T-periodic solutions, then the following necessary condition holds :*

$$< f >:= \frac{1}{T}\int_0^T f(t)dt \in Range(A),$$

(there is $x_0 \in D(A)$ such that $< f >= Ax_0$).

Proof: Suppose that $x \in C^1([0, T];\ H) \cap C([0, T];\ D(A))$ is a T-periodic solution for (37). Let us consider the divisions $\Delta_n : 0 = t_0^n < t_1^n < ... < t_n^n = T$ such that :

$$\lim_{n\to\infty} \max_{1\le i\le n} |t_i^n - t_{i-1}^n| = 0. \tag{38}$$

We can write :

$$(t_i^n - t_{i-1}^n)x'(t_{i-1}^n) + (t_i^n - t_{i-1}^n)Ax(t_{i-1}^n) = (t_i^n - t_{i-1}^n)f(t_{i-1}^n),\ 1 \le i \le n.$$

Since A is linear we deduce :

$$\frac{1}{T}\sum_{i=1}^n (t_i^n - t_{i-1}^n)x'(t_{i-1}^n) + A\left(\frac{1}{T}\sum_{i=1}^n (t_i^n - t_{i-1}^n)x(t_{i-1}^n)\right) = \frac{1}{T}\sum_{i=1}^n (t_i^n - t_{i-1}^n)f(t_{i-1}^n),$$

and hence :

$$\left[\frac{1}{T}\sum_{i=1}^n (t_i^n - t_{i-1}^n)x(t_{i-1}^n)), \frac{1}{T}\sum_{i=1}^n (t_i^n - t_{i-1}^n)[f(t_{i-1}^n) - x'(t_{i-1}^n)]\right] \in A.$$

By (38) we deduce that :

$$\frac{1}{T}\sum_{i=1}^n (t_i^n - t_{i-1}^n)x(t_{i-1}^n)) \to \frac{1}{T}\int_0^T x(t)dt,$$

and :

$$\begin{aligned}
\frac{1}{T}\sum_{i=1}^n (t_i^n - t_{i-1}^n)[f(t_{i-1}^n) - x'(t_{i-1}^n)] \quad &\to \quad \frac{1}{T}\int_0^T [f(t) - x'(t)]dt \\
&= \quad \frac{1}{T}\int_0^T f(t)dt - \frac{1}{T}x(t)|_0^T \\
&= \quad \frac{1}{T}\int_0^T f(t)dt.
\end{aligned}$$

Since A is maximal monotone $Graph(A)$ is closed and therefore :

$$\left[\frac{1}{T}\int_0^T x(t)dt,\ \frac{1}{T}\int_0^T f(t)dt\right] \in A.$$

Thus $\frac{1}{T}\int_0^T x(t)dt \in D(A)$ and $< f >= A\left(\frac{1}{T}\int_0^T x(t)dt\right)$.

Generally the previous condition is not sufficient for the existence of periodic solutions. For example let us analyse the periodic solutions $x = (x_1, x_2) \in C^1([0,T];\ I\!R^2)$ for :

$$x'(t) + Ax(t) = f(t),\ t \in [0,T], \tag{39}$$

where $A : I\!R^2 \to I\!R^2$ is the orthogonal rotation :

$$A(x_1,x_2) = (-x_2, x_1),\ (x_1,x_2) \in I\!R^2,$$

and $f = (f_1, f_2) \in L^1(]0,T[;\ I\!R^2)$ is T-periodic. For a given initial data $x(0) = x_0 \in I\!R^2$ the solution writes :

$$x(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A} f(s)ds,\ t > 0, \tag{40}$$

where the semigroup e^{-tA} is given by :

$$e^{-tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, t \in I\!R. \tag{41}$$

Since $e^{-2\pi A} = 1$ we deduce that the equation (39) has 2π-periodic solutions iff :

$$\int_0^{2\pi} e^{tA} f(t)dt = 0. \tag{42}$$

Thus if $\int_0^{2\pi}\{f_1(t)\cos t - f_2(t)\sin t\}dt \neq 0$ or $\int_0^{2\pi}\{f_1(t)\sin t + f_2(t)\cos t\}dt \neq 0$ equation (39) doesn't have any 2π-periodic solution and the necessary condition still holds because $Range(A) = I\!R^2$. Moreover if (42) is satisfied then every solution of (39) is periodic and therefore uniqueness doesn't occur (the operator A is not strictly monotone).

Let us analyse now the existence. As in the one dimensional case we have :

Proposition 14 *Suppose that $A : D(A) \subset H \to H$ is maximal monotone and $f \in C^1(I\!R;\ H)$ is T-periodic. Then for every $\alpha > 0$ the equation :*

$$\alpha x(t) + x'(t) + Ax(t) = f(t),\ t \in I\!R, \tag{43}$$

has an unique T-periodic solution in $C^1(I\!R; H) \cap C(I\!R; D(A))$.

Proof : Since $\alpha + A$ is strictly monotone the uniqueness follows from *Proposition 12*. Indeed :

$$\alpha\|x-y\|^2 + (Ax - Ay, x-y) = 0,\ x, y \in D(A),$$

implies $\alpha \|x-y\|^2 = 0$ and hence $x = y$.

Consider now an arbitrary initial data $x_0 \in D(A)$. By the Hille-Yosida's theorem, there is $x \in C^1([0,+\infty[; H) \cap C([0,+\infty[; D(A))$ solution for (43). Denote by $(x_n)_{n\geq 0}$ the functions :

$$x_n(t) = x(nT+t),\ t \in [0,T],\ n \geq 0.$$

We have :

$$\alpha x_{n+1}(t) + x'_{n+1}(t) + Ax_{n+1}(t) = f((n+1)T+t),\ t \in [0,T],$$

and :

$$\alpha x_n(t) + x'_n(t) + Ax_n(t) = f(nT+t),\ t \in [0,T].$$

Since f is T-periodic, after usual computations we get :

$$\begin{aligned}\alpha\|x_{n+1}(t)-x_n(t)\|^2 &+ \frac{1}{2}\frac{d}{dt}\|x_{n+1}(t)-x_n(t)\|^2 \\ &+ (Ax_{n+1}(t)-Ax_n(t),\ x_{n+1}(t)-x_n(t)) = 0,\ t \in [0,T].\end{aligned}$$

Taking into account that A is monotone we deduce :

$$\|x_{n+1}(t)-x_n(t)\| \leq e^{-\alpha t}\|x_{n+1}(0)-x_n(0)\|,\ t \in [0,T],$$

and hence :

$$\begin{aligned}\|x_{n+1}(0)-x_n(0)\| &= \|x_n(T)-x_{n-1}(T)\| \\ &\leq e^{-\alpha T}\|x_n(0)-x_{n-1}(0)\| \\ &\leq e^{-2\alpha T}\|x_{n-1}(0)-x_{n-2}(0)\| \\ &\leq \ldots \\ &\leq e^{-n\alpha T}\|x_1(0)-x_0(0)\|,\ n \geq 0. \qquad (44)\end{aligned}$$

Finally we get the estimate :

$$\|x_{n+1}(t)-x_n(t)\| \leq e^{-\alpha(nT+t)}\|S_\alpha(T;0,x_0)-x_0\|,\ t \in [0,T],\ n \geq 0.$$

Here $S_\alpha(t;0,x_0)$ represents the solution of (43) for the initial data x_0. From the previous estimate it is clear that $(x_n)_{n\geq 0}$ is convergent in $C^0([0,T];H)$:

$$x_n(t) = x_0(t) + \sum_{k=0}^{n-1}(x_{k+1}(t)-x_k(t)),\ t \in [0,T],$$

where :

$$\begin{aligned}\|\sum_{k=0}^{n-1}(x_{k+1}(t)-x_k(t))\| &\leq \sum_{k=0}^{n-1}\|x_{k+1}(t)-x_k(t)\| \\ &\leq \sum_{k=0}^{n-1} e^{-\alpha(kT+t)}\|S_\alpha(T;0,x_0)-x_0\| \\ &\leq \frac{e^{-\alpha t}}{1-e^{-\alpha T}}\|S_\alpha(T;0,x_0)-x_0\|.\end{aligned}$$

Moreover $\|x_n(t)\| \leq \|S_\alpha(t;0,x_0)\| + \frac{1}{1-e^{-\alpha T}}\|S_\alpha(T;0,x_0)-x_0\|$. Denote by x_α the limit of $(x_n)_{n\geq 0}$ as $n\to\infty$. We should note that without any other hypothesis $(x_\alpha)_{\alpha>0}$ is not uniformly bounded in $L^\infty(]0,T[;H)$. We have only estimate in $\mathcal{O}(1+\frac{1}{\alpha})$:

$$\|x_\alpha\|_{L^\infty([0,T];H)} \leq C\left(1+\frac{1}{1-e^{-\alpha T}}\right) \sim \mathcal{O}\left(1+\frac{1}{\alpha}\right).$$

The above estimate leads immediately to the following :

Remark 5 *The sequence $(\alpha x_\alpha)_{\alpha>0}$ is uniformly bounded in $L^\infty([0,T];H)$.*

Let us demonstrate that x_α is T-periodic and solution for (43). Indeed :

$$x_\alpha(0) = \lim_{n\to\infty} x_n(0) = \lim_{n\to\infty} x_{n-1}(T) = x_\alpha(T).$$

Now let us show that $(x_n')_{n\geq 0}$ is also uniformly bounded in $L^\infty(]0,T[;H)$. By taking the difference between the equations (43) at the moments t and $t+h$ we have :

$$\alpha(x(t+h)-x(t)) + x'(t+h) - x'(t) + Ax(t+h) - Ax(t) = f(t+h) - f(t),\ t<t+h.$$

After multiplication by $x(t+h)-x(t)$ we obtain :

$$\alpha\|x(t+h)-x(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|x(t+h)-x(t)\|^2 \leq \|f(t+h)-f(t)\|\cdot\|x(t+h)-x(t)\|,$$

which can be also rewritten as :

$$\begin{aligned}\frac{1}{2}e^{2\alpha t}\|x(t+h)-x(t)\|^2 &\leq \int_0^t e^{\alpha s}\|f(s+h)-f(s)\|\cdot e^{\alpha s}\|x(s+h)-x(s)\|ds \\ &+ \frac{1}{2}\|x(h)-x(0)\|^2,\ t<t+h.\end{aligned}$$

By using Bellman's lemma we conclude that :

$$\begin{aligned}\frac{1}{h}\|x(t+h)-x(t)\| &\leq \int_0^t e^{-\alpha(t-s)}\frac{1}{h}\|f(s+h)-f(s)\|ds \\ &+ e^{-\alpha t}\frac{1}{h}\|x(h)-x(0)\|,\ 0\leq t<t+h.\end{aligned} \tag{45}$$

By passing to the limit for $h\to 0$ the previous formula yields :

$$\begin{aligned}\|x'(t)\| &\leq e^{-\alpha t}\|x'(0)\| + \int_0^t e^{-\alpha(t-s)}\|f'(s)\|ds \\ &\leq e^{-\alpha t}\|f(0)-\alpha x_0 - Ax_0\| + \frac{1}{\alpha}(1-e^{-\alpha t})\|f'\|_{L^\infty(]0,T[;H)} \\ &\leq \|f(0)-\alpha x_0 - Ax_0\| + \frac{1}{\alpha}\|f'\|_{L^\infty(]0,T[;H)} < +\infty.\end{aligned}$$

Therefore $(x_n')_{n\geq 0}$ is uniformly bounded in $L^\infty(]0,T[;H)$ since :

$$\|x_n'\|_{L^\infty(]0,T[;H)} = \|x'(nT+(\cdot))\|_{L^\infty(]0,T[;H)} \leq \|x'\|_{L^\infty([0,+\infty[;H)},$$

and thus we have $x'_n(t) \rightharpoonup y_\alpha(t),\ t \in [0,T]$. We can write :

$$(x_n(t), z) = (x_n(0), z) + \int_0^t (x'_n(s), z)ds,\ z \in H,\ t \in [0,T],\ n \geq 0,$$

and by passing to the limit for $n \to \infty$ we deduce :

$$(x_\alpha(t), z) = (x_\alpha(0), z) + \int_0^t (y_\alpha(s), z)ds,\ z \in H,\ t \in [0,T],$$

wich is equivalent to :

$$x_\alpha(t) = x_\alpha(0) + \int_0^t y_\alpha(s)ds,\ t \in [0,T].$$

Therefore x_α is differentiable and $x'_\alpha = y_\alpha$. Finally we can write $x'_n(t) \rightharpoonup x'_\alpha(t),\ t \in [0,T]$. Let us show that x_α is also solution for (43). We have :

$$[x_n(t), f(t) - \alpha x_n(t) - x'_n(t)] \in A,\ n \geq 0,\ t \in [0,T].$$

Since $x_n(t) \to x_\alpha(t)$, $x'_n(t) \rightharpoonup x'_\alpha(t)$ and A is maximal monotone we conclude that :

$$[x_\alpha(t), f(t) - x'_\alpha(t)] \in A,\ t \in [0,T],\ \alpha > 0,$$

which means that $x_\alpha(t) \in D(A)$ and $Ax_\alpha(t) = f(t) - x'_\alpha(t),\ t \in [0,T]$.

Now we establish for the linear case the similar result stated in *Proposition 7*. Before let us recall a standard result concerning maximal monotone operators on Hilbert spaces :

Proposition 15 *Assume that A is a maximal monotone operator (linear or not) and $\alpha u_\alpha + Au_\alpha = f,\ u_\alpha \in D(A),\ f \in H,\ \alpha > 0$. Then the following statements are equivalent :*
(i) $f \in Range(A)$;
(ii) $(u_\alpha)_{\alpha>0}$ is bounded in H. Moreover, in this case $(u_\alpha)_{\alpha>0}$ is convergent in H to the element of minimal norm in $A^{-1}f$.

Proof : *(i) $\to$ (ii)* By the hypothesis there is $u \in D(A)$ such that $f = Au$. After multiplication by $u_\alpha - u$ we get :

$$\alpha(u_\alpha, u_\alpha - u) + (Au_\alpha - Au, u_\alpha - u) = 0,\ \alpha > 0.$$

Taking into account that A is monotone we deduce :

$$\|u_\alpha\|^2 \leq (u_\alpha, u) \leq \|u_\alpha\| \cdot \|u\|,\ \alpha > 0,$$

and hence $\|u_\alpha\| \leq \|u\|,\ \alpha > 0,\ u \in A^{-1}f$ which implies that $u_\alpha \rightharpoonup u_0$. We have $[u_\alpha, f - \alpha u_\alpha] \in A,\ \alpha > 0$ and since A is maximal monotone, by passing to the limit for $\alpha \to 0$ we deduce that $[u_0, f] \in A$, or $u_0 \in A^{-1}f$. Moreover :

$$\|u_0\| = \|w - \lim_{\alpha \to 0} u_\alpha\| \leq \liminf_{\alpha \to 0} \|u_\alpha\| \leq \limsup_{\alpha \to 0} \|u_\alpha\| \leq \|u\|,\ \forall u \in A^{-1}f.$$

In particulat taking $u = u_0 \in A^{-1}f$ we get :

$$\|w - \lim_{\alpha\to 0} u_\alpha\| = \lim_{\alpha\to 0} \|u_\alpha\|,$$

and hence, since any Hilbert space is strictly convex, by Mazur's theorem we deduce that the convergence is strong :

$$u_\alpha \to u_0 \in A^{-1}f,\ \alpha \to 0,$$

where $\|u_0\| = \inf_{u\in A^{-1}f} \|u\| = \min_{u\in A^{-1}f} \|u\|$.

(ii) → (i) Conversely, suppose that $(u_\alpha)_{\alpha>0}$ is bounded in H. Therefore $u_\alpha \rightharpoonup u$ in H. We have $[u_\alpha, f - \alpha u_\alpha] \in A,\ \alpha > 0$ and since A is maximal monotone by passing to the limit for $\alpha \to 0$ we deduce that $[u, f] \in A$ or $u \in D(A)$ and $f = Au$.

Theorem 4 *Assume that $A : D(A) \subset H \to H$ is a linear maximal monotone operator on a compact Hilbert space H and $f \in C^1(\mathbb{R}; H)$ is a T-periodic function. Then the following statements are equivalent :*

(i) equation (37) has periodic solutions ;

(ii) the sequence of periodic solutions for (43) is bounded in $C^1(\mathbb{R}; H)$. Moreover in this case $(x_\alpha)_{\alpha>0}$ is convergent in $C^0(\mathbb{R}; H)$ and the limit is also a T-periodic solution for (37).

Proof : *(i) → (ii)* Denote by x, x_α the periodic solutions for (37) and (43). By taking the difference and after multiplication by $x_\alpha(t) - x(t)$ we get :

$$\alpha\|x_\alpha(t) - x(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|x_\alpha(t) - x(t)\|^2 \le \alpha\|x(t)\| \cdot \|x_\alpha(t) - x(t)\|,\ t \in \mathbb{R}. \tag{46}$$

Finally, after integration and by using Bellman's lemma, formula (46) yields :

$$\begin{aligned} \|x_\alpha(t) - x(t)\| &\le e^{-\alpha t}\|x_\alpha(0) - x(0)\| + \int_0^t \alpha e^{-\alpha(t-s)}\|x(s)\|ds \\ &\le e^{-\alpha t}\|x_\alpha(0) - x(0)\| + (1 - e^{-\alpha t})\|x\|_{L^\infty},\ t \in \mathbb{R}. \end{aligned}$$

Since x_α and x are T-periodic we can also write :

$$\begin{aligned} \|x_\alpha(t) - x(t)\| &= \|x_\alpha(nT + t) - x(nT + t)\| \\ &\le e^{-\alpha(nT+t)}\|x_\alpha(0) - x(0)\| + (1 - e^{-\alpha(nT+t)})\|x\|_{L^\infty}. \end{aligned}$$

By passing to the limit for $n \to \infty$ we obtain :

$$\|x_\alpha - x\|_{L^\infty} \le \|x\|_{L^\infty},\ \alpha > 0,$$

and hence :

$$\|x_\alpha\|_{L^\infty} \le 2\|x\|_{L^\infty},\ \alpha > 0.$$

Since A is linear we can write :

$$\begin{aligned}\frac{\alpha}{h}(x_\alpha(t+h)-x_\alpha(t)) &+ \frac{1}{h}(x'_\alpha(t+h)-x'_\alpha(t))+\frac{1}{h}A(x_\alpha(t+h)-x_\alpha(t))\\ &= \frac{1}{h}(f(t+h)-f(t)),\ t<t+h,\ \alpha>0,\end{aligned} \tag{47}$$

and :

$$\frac{1}{h}(x'(t+h)-x'(t))+\frac{1}{h}A(x(t+h)-x(t))=\frac{1}{h}(f(t+h)-f(t)),\ t<t+h.$$

For every $h>0$ denote by $y_{\alpha,h}$, y_h and g_h the periodic functions :

$$y_{\alpha,h}(t)=\frac{1}{h}(x_\alpha(t+h)-x_\alpha(t)),\ t\in I\!R,\ \alpha>0,$$

$$y_h(t)=\frac{1}{h}(x(t+h)-x(t)),\ t\in I\!R,$$

$$g_h(t)=\frac{1}{h}(f(t+h)-f(t)),\ t\in I\!R,$$

and hence we have :

$$\alpha y_{\alpha,h}(t)+y'_{\alpha,h}(t)+Ay_{\alpha,h}(t)=g_h(t),\ t\in I\!R,$$

and :

$$y'_h(t)+Ay_h(t)=g_h(t),\ t\in I\!R.$$

By the same computations we get :

$$\|y_{\alpha,h}(t)-y_h(t)\|\le e^{-\alpha t}\|y_{\alpha,h}(0)-y_h(0)\|+\int_0^t \alpha e^{-\alpha(t-s)}\|y_h(s)\|ds.$$

Now by passing to the limit for $h\to 0$ we deduce :

$$\begin{aligned}\|x'_\alpha(t)-x'(t)\| &\le e^{-\alpha t}\|x'_\alpha(0)-x'(0)\|+\int_0^t \alpha e^{-\alpha(t-s)}\|x'(s)\|ds\\ &\le e^{-\alpha t}\|x'_\alpha(0)-x'(0)\|+(1-e^{-\alpha t})\|x'\|_{L^\infty},\ t\in[0,T].\end{aligned}$$

By the periodicity we obtain as before that :

$$\begin{aligned}\|x'_\alpha(t)-x'(t)\| &= \|x'_\alpha(nT+t)-x'(nT+t)\|\\ &\le e^{-\alpha(nT+t)}\|x'_\alpha(0)-x'(0)\|+(1-e^{-\alpha(nT+t)})\|x'\|_{L^\infty},\end{aligned}$$

and hence by passing to the limit for $n\to\infty$ we conclude that :

$$\|x'_\alpha-x'\|_{L^\infty}\le\|x'\|_{L^\infty},\ \alpha>0.$$

Therefore $(x'_\alpha)_{\alpha>0}$ is also uniformly bounded in L^∞ :

$$\|x'_\alpha\|_{L^\infty}\le 2\|x'\|_{L^\infty},\ \alpha>0.$$

Conversely, the implication *(ii)* $\to$ *(i)* follows by using Arzela-Ascoli's theorem and by passing to the limit for $\alpha\to 0$ in (43).

Let us continue the analysis of the previous example. The semigroup associated to the equation (43) is given by :

$$e^{-t(\alpha+A)} = e^{-\alpha t}e^{-tA} = e^{-\alpha t}\begin{pmatrix} \cos t, & \sin t \\ -\sin t, & \cos t \end{pmatrix}, \; t \in I\!\!R, \; \alpha > 0,$$

and the periodic solution for equation (43) writes :

$$\begin{aligned} x_\alpha(t) &= (1 - e^{-T(\alpha+A)})^{-1} \int_0^T e^{-(T-s)(\alpha+A)} f(s)ds \\ &+ \int_0^t e^{-(t-s)(\alpha+A)} f(s)ds \\ &= \frac{1 - e^{-T(\alpha-A)}}{(1 - e^{-\alpha T}\cos T)^2 + (e^{-\alpha T}\sin T)^2} \int_0^T e^{-(T-s)(\alpha+A)} f(s)ds \\ &+ \int_0^t e^{-(t-s)(\alpha+A)} f(s)ds, \; t > 0, \; \alpha > 0. \end{aligned}$$

As we have seen, the existence of periodic solutions reduces to uniform $L^\infty(]0,T[;H)$ estimates for $(x_\alpha)_{\alpha>0}$ and $(x'_\alpha)_{\alpha>0}$. Since A is linear bounded operator ($\|A\|_{\mathcal{L}(H;\,H)} = 1$) we have :

$$\begin{aligned} \|x'_\alpha\|_{L^\infty(]0,T[;H)} &= \|f - \alpha x_\alpha - Ax_\alpha\|_{L^\infty(]0,T[;H)} \\ &\leq \|f\|_{L^\infty(]0,T[;H)} + (\alpha + \|A\|_{\mathcal{L}(H;\,H)})\|x_\alpha\|_{L^\infty(]0,T[;H)}, \; \alpha > 0, \end{aligned}$$

and hence in this case it is sufficient to find only uniform $L^\infty(]0,T[;H)$ estimates for $(x_\alpha)_{\alpha>0}$ or uniform estimates for $(x_\alpha(0))_{\alpha>0}$ in H.

Case 1 : $T = 2n\pi, \; n \geq 0$. We have :

$$\lim_{\alpha\to 0} x_\alpha(0) = \lim_{\alpha\to 0} \frac{1}{1 - e^{-\alpha T}} \int_0^T e^{-(T-s)(\alpha+A)} f(s)ds.$$

If $\int_0^T e^{-(T-s)A} f(s)ds \neq 0$, then $(x_\alpha(0))_{\alpha>0}$ is not bounded. In fact since $e^{-2n\pi A} = 1$ it is easy to check that equation (39) doesn't have any periodic solution. If $\int_0^T e^{-(T-s)A} f(s)ds = 0$ then every solution of (39) is T-periodic and $(x_\alpha(0))_{\alpha>0}$ is convergent for $\alpha \to 0$:

$$\begin{aligned} \lim_{\alpha\to 0} x_\alpha(0) &= \lim_{\alpha\to 0} \frac{\int_0^T (e^{-\alpha(T-s)} - 1)e^{-(T-s)A} f(s)ds}{1 - e^{-\alpha T}} \\ &= -\int_0^T \frac{T-s}{T} e^{-(T-s)A} f(s) \\ &= \frac{1}{T}\int_0^T s e^{-(T-s)A} f(s). \end{aligned}$$

Case 2 : $T \neq 2n\pi \; \forall n \geq 0$. In this case $(1 - e^{-TA})$ is invertible and $(x_\alpha(0))_{\alpha>0}$ converges to $x(0)$ where x is the unique T-periodic solution of (39) :

$$\lim_{\alpha\to 0} x_\alpha(0) = \lim_{\alpha\to 0} (1 - e^{-T(\alpha+A)})^{-1} \int_0^T e^{-(T-s)(\alpha+A)} f(s)ds$$

$$
\begin{aligned}
&= (1-e^{-TA})^{-1}\int_0^T e^{-(T-s)A} f(s)ds \\
&= \frac{1}{2\sin(\frac{T}{2})}\int_0^T e^{-(\frac{T+\pi}{2}-s)A} f(s)ds.
\end{aligned}
$$

We state now our main result of existence in the linear and symmetric case :

Theorem 5 *Assume that $A : D(A) \subset H \to H$ is a linear maximal monotone and symmetric operator and $f \in C^1([0,T];H)$ is a T-periodic function. Then the necessary and sufficient condition for the existence of periodic solutions for (37) is given by :*

$$
< f >:= \frac{1}{T}\int_0^T f(t)dt \in Range(A).
$$

In this case we have the estimates :

$$
\|x\|_{L^\infty(]0,T[;H)} \le \|A^{-1} < f > \| + \frac{\sqrt{T}}{2}\|f\|_{L^2(]0,T[;H)} + \frac{T}{2}\|f'\|_{L^1(]0,T[;H)},
$$

and :

$$
\|x'\|_{L^\infty(]0,T[;H)} \le \frac{1}{\sqrt{T}}\|f\|_{L^2(]0,T[;H)} + \|f'\|_{L^1(]0,T[;H)},
$$

and the solution is unique up to a constant in $A^{-1}(0)$.

Proof : The condition is necessary (see *Proposition 13*). Let us show now that it is also sufficient. Consider the T-periodic solutions $(x_\alpha)_{\alpha>0}$ for :

$$
\alpha x_\alpha(t) + x'_\alpha(t) + Ax_\alpha(t) = f(t),\ t \in [0,T],\ \alpha > 0.
$$

First we prove that $(x_\alpha)_{\alpha>0}$ is uniformly bounded in $C^1([0,T];H)$. Let us multiply by $x'_\alpha(t)$ and integrate on a period :

$$
\int_0^T \|x'_\alpha(t)\|^2 dt + \int_0^T \alpha(x_\alpha(t), x'_\alpha(t)) + (Ax_\alpha(t), x'_\alpha(t))dt = \int_0^T (f(t), x'_\alpha(t))dt.
$$

Since A is symmetric and x_α is T-periodic we have :

$$
\begin{aligned}
\int_0^T \alpha(x_\alpha(t), x'_\alpha(t)) + (Ax_\alpha(t), x'_\alpha(t))dt &= \int_0^T \frac{\alpha}{2}\frac{d}{dt}\|x_\alpha(t)\|^2 dt \\
&+ \int_0^T \frac{1}{2}\frac{d}{dt}(Ax_\alpha(t), x_\alpha(t))dt \\
&= \frac{1}{2}\left\{\alpha\|x_\alpha(t)\|^2 + (Ax_\alpha(t), x_\alpha(t))\right\}\Big|_0^T \\
&= 0.
\end{aligned}
$$

Finally we get :

$$
\|x'_\alpha\|^2_{L^2(]0,T[;H)} \le (f, x'_\alpha)_{L^2(]0,T[;H)} \le \|f\|_{L^2(]0,T[;H)} \cdot \|x'_\alpha\|_{L^2(]0,T[;H)},
$$

and hence :

$$\|x'_\alpha\|_{L^2(]0,T[;H)} \le \|f\|_{L^2(]0,T[;H)},\ \alpha > 0.$$

Therefore we can write :

$$\min_{t\in[0,T]} \|x'_\alpha(t)\| \le \frac{1}{\sqrt{T}}\|x'_\alpha\|_{L^2(]0,T[;H)} \le \frac{1}{\sqrt{T}}\|f\|_{L^2(]0,T[;H)}. \tag{48}$$

As seen before, since A is linear we can write :

$$\frac{\alpha}{h}(x_\alpha(t+h)-x_\alpha(t)) + \frac{1}{h}(x'_\alpha(t+h)-x'_\alpha(t)) + \frac{1}{h}A(x_\alpha(t+h)-x_\alpha(t)) = \frac{1}{h}(f(t+h)-f(t)),$$

and by standard calculations we get :

$$\begin{aligned}\frac{1}{h}\|x_\alpha(t+h)-x_\alpha(t)\| &\le e^{-\alpha(t-s)}\frac{1}{h}\|x_\alpha(s+h)-x_\alpha(s)\| \\ &+ \int_s^t e^{-\alpha(t-\tau)}\frac{1}{h}\|f(\tau+h)-f(t)\|d\tau,\ s<t,\ h>0.\end{aligned}$$

By passing to the limit for $h \to 0$ we deduce :

$$\begin{aligned}\|x'_\alpha(t)\| &\le e^{-\alpha(t-s)}\|x'_\alpha(s)\| + \int_s^t e^{-\alpha(t-\tau)}\|f'(\tau)\|d\tau \\ &\le \|x'_\alpha(s)\| + \int_s^t \|f'(\tau)\|d\tau,\ s\le t,\ \alpha>0. \end{aligned} \tag{49}$$

From (48) and (49) we conclude that $(x'_\alpha)_{\alpha>0}$ is uniformly bounded in $L^\infty(]0,T[;H)$:

$$\|x'_\alpha\|_{L^\infty(]0,T[;H)} \le \frac{1}{\sqrt{T}}\|f\|_{L^2(]0,T[;H)} + \|f'\|_{L^1(]0,T[;H)},\ \alpha>0.$$

As shown before, since A is linear and x_α is T-periodic we have also :

$$\alpha < x_\alpha > + A < x_\alpha > = < f > . \tag{50}$$

By the hypothesis there is $x_0 \in D(A)$ such that $< f >= Ax_0$ and hence :

$$\| < x_\alpha > \| = \|(\alpha+A)^{-1} < f > \| = \|(\alpha+A)^{-1}Ax_0\| \le \|x_0\|,\ \alpha>0.$$

Now it is easy to check that $(x_\alpha)_{\alpha>0}$ is uniformly bounded in $L^\infty(]0,T[;H)$:

$$\begin{aligned}\|x_\alpha(t) - < x_\alpha > \| &= \left\|\frac{1}{T}\int_0^T (x_\alpha(t)-x_\alpha(s))ds\right\| \\ &= \left\|\frac{1}{T}\int_0^T\int_s^t x'_\alpha(\tau)d\tau ds\right\| \\ &\le \frac{\sqrt{T}}{2}\|f\|_{L^2(]0,T[;H)} + \frac{T}{2}\|f'\|_{L^1(]0,T[;H)},\end{aligned}$$

and thus :

$$\begin{aligned}\|x_\alpha\|_{L^\infty(]0,T[;H)} &\le \| < x_\alpha > \| + \frac{\sqrt{T}}{2}\|f\|_{L^2(]0,T[;H)} + \frac{T}{2}\|f'\|_{L^1(]0,T[;H)} \\ &\le \|x_0\| + \frac{\sqrt{T}}{2}\|f\|_{L^2(]0,T[;H)} + \frac{T}{2}\|f'\|_{L^1(]0,T[;H)}.\end{aligned}$$

Now we can prove that $(x_\alpha)_{\alpha>0}$ is convergent in $C^1([0,T];H)$. Indeed, by taking the difference between the equations (43) written for $\alpha, \beta > 0$, after multiplication by $x'_\alpha(t)-x'_\beta(t)$ and integration on $[0,T]$ we get :

$$\begin{aligned}\int_0^T \{\alpha(x_\alpha(t)-x_\beta(t), x'_\alpha(t)-x'_\beta(t)) &+ \|x'_\alpha(t)-x'_\beta(t)\|^2 + (A(x_\alpha(t)-x_\beta(t)), x'_\alpha(t)-x'_\beta(t))\}dt \\ &= -(\alpha-\beta)\int_0^T (x_\beta(t), x'_\alpha(t)-x'_\beta(t))dt,\ \alpha,\beta>0.\end{aligned}$$

Since A is symmetric, x_α, x_β are T-periodic and uniformly bounded in $L^\infty(]0,T[;H)$ we deduce that :

$$\|x'_\alpha - x'_\beta\|_{L^2(]0,T[;H)} \leq |\alpha-\beta| \cdot \sup_{\gamma>0} \|x_\gamma\|_{L^2(]0,T[;H)},\ \alpha,\beta>0,$$

or :

$$\|x'_\alpha - x'_\beta\|_{L^\infty(]0,T[;H)} \leq \frac{|\alpha-\beta|}{\sqrt{T}} \cdot \sup_{\gamma>0}\|x_\gamma\|_{L^2(]0,T[;H)} + |\alpha-\beta|\cdot \sup_{\gamma>0}\|x'_\gamma\|_{L^1(]0,T[;H)},\ \alpha,\beta>0,$$

and therefore $(x'_\alpha)_{\alpha>0}$ converges in $C([0,T];H)$.
We already know that $(< x_\alpha >)_{\alpha>0} = ((\alpha+A)^{-1} < f >)_{\alpha>0}$ is bounded in H and by the *Proposition 15* it follows that $(< x_\alpha >)_{\alpha>0}$ is convergent to the element of minimal norm in $A^{-1} < f >$. We have :

$$x_\alpha(t) = x_\alpha(0) + \int_0^t x'_\alpha(s)ds,\ t \in I\!R, \alpha>0.$$

By taking the average we deduce that $x_\alpha(0) =< x_\alpha > - < \int_0^t x'_\alpha(s)ds >$ and therefore, since $(x'_\alpha)_{\alpha>0}$ is uniformly convergent, it follows that $(x_\alpha(0))_{\alpha>0}$ is also convergent. Finally we conclude that $(x_\alpha)_{\alpha>0}$ is convergent in $C^1([0,T];H)$ to the periodic solution x for (37) such that $< x >$ is the element of minimal norm in $A^{-1} < f >$.
Before analysing the periodic solution for the heat equation, following an idea of [7], let us state the proposition :

Proposition 16 *Assume that $A : D(A) \subset H \to H$ is a linear maximal monotone and symmetric operator and $f \in C^1([0,T];H)$ is a T-periodic function. Then for every $x_0 \in D(A)$ we have :*

$$\lim_{t\to\infty} \frac{1}{T}(x(t+T;0,x_0) - x(t;0,x_0)) =< f > -Proj_{\overline{R(A)}} < f >, \tag{51}$$

where $x(\cdot;0,x_0)$ represents the solution of (37) with the initial data x_0 and $R(A)$ is the range of A.

Remark 6 *A being maximal monotone, A^{-1} is also maximal monotone and therefore $\overline{D(A^{-1})} = \overline{R(A)}$ is convex.*

Proof : Consider $x_0 \in D(A)$ and denote by $x(\cdot)$ the corresponding solution. By integration on $[t, t+T]$ we get :

$$\frac{1}{T}(x(t+T)-x(t)) + A\left(\frac{1}{T}\int_t^{t+T} x(s)ds\right) =< f > . \tag{52}$$

For each $\alpha > 0$ consider $x_\alpha \in D(A)$ such that $\alpha x_\alpha + Ax_\alpha =< f >$. Denoting by $y(\cdot)$ the function $y(t) = \frac{1}{T}\int_t^{t+T} x(s)ds,\ t \geq 0$, equation (52) writes :

$$y'(t) + Ay(t) = \alpha x_\alpha + Ax_\alpha,\ t \geq 0,\ \alpha > 0.$$

Let us search for y of the form $y_1 + y_2$ where :

$$y_1'(t) + Ay_1(t) = \alpha x_\alpha,\ t \geq 0,$$

with the initial condition $y_1(0) = 0$ and :

$$y_2'(t) + Ay_2(t) = Ax_\alpha,\ t \geq 0, \tag{53}$$

with the initial condition $y_2(0) = y(0) = \frac{1}{T}\int_0^T x(t)dt$. We are interested on the asymptotic behaviour of $Ay(t) = Ay_1(t) + Ay_2(t)$ for large t. We have :

$$\begin{aligned} y_1(t) &= e^{-tA}y_1(0) + \int_0^t e^{-(t-s)A}\alpha x_\alpha ds \\ &= \int_0^t e^{-(t-s)A}\alpha x_\alpha ds, \end{aligned}$$

and therefore :

$$\begin{aligned} Ay_1(t) &= \int_0^t Ae^{-(t-s)A}\alpha x_\alpha ds \\ &= e^{-(t-s)A}\alpha x_\alpha|_0^t \\ &= (1 - e^{-tA})\alpha x_\alpha,\ t \geq 0. \end{aligned}$$

By the other hand, after multiplication of (53) by $y_2'(t) = (y_2(t) - x_\alpha)'$ we get :

$$\|y_2'(t)\|^2 + (A(y_2(t) - x_\alpha), (y_2(t) - x_\alpha)') = 0,\ t \geq 0.$$

Since A is symmetric, after integration on $[0,t]$ we obtain :

$$\int_0^t \|y_2'(s)\|^2 ds + \frac{1}{2}(A(y_2(t) - x_\alpha), y_2(t) - x_\alpha) = \frac{1}{2}(A(y_2(0) - x_\alpha), y_2(0) - x_\alpha),$$

and therefore, by the monotony of A it follows that :

$$\int_0^\infty \|y_2'(t)\|^2 dt \leq \frac{1}{2}(A(y_2(0) - x_\alpha), y_2(0) - x_\alpha).$$

Thus $\lim_{t\to\infty} y_2'(t) = 0$ and by passing to the limit in (53) we deduce that $\lim_{t\to\infty} Ay_2(t) = \lim_{t\to\infty}(Ax_\alpha - y_2'(t)) = Ax_\alpha$. Finally we find that :

$$\begin{aligned} \lim_{t\to\infty}\{\frac{1}{T}(x(t+T) - x(t)) - e^{-tA}\alpha x_\alpha\} &= \lim_{t\to\infty}\{y'(t) - e^{-tA}\alpha x_\alpha\} \\ &= \lim_{t\to\infty}\{< f > -Ay(t) - e^{-tA}\alpha x_\alpha\} \\ &= \lim_{t\to\infty}\{< f > -Ay_1(t) - Ay_2(t) - e^{-tA}\alpha x_\alpha\} \\ &= < f > -\alpha x_\alpha - Ax_\alpha \\ &= 0,\ \alpha > 0. \end{aligned} \tag{54}$$

Now let us put $y_\alpha = Ax_\alpha$ and observe that $y_\alpha + \alpha A^{-1} y_\alpha = Ax_\alpha + \alpha x_\alpha =< f >,\ \alpha > 0$. Therefore :

$$\begin{aligned}
\lim_{\alpha \searrow 0} y_\alpha &= \lim_{\alpha \searrow 0} (1 + \alpha A^{-1})^{-1} < f > \\
&= \lim_{\alpha \searrow 0} J_\alpha^{A^{-1}} < f > \\
&= Proj_{\overline{D(A^{-1})}} < f > \\
&= Proj_{\overline{R(A)}} < f >,
\end{aligned}$$

and it follows that :

$$\begin{aligned}
\lim_{\alpha \searrow 0} \alpha x_\alpha &= \lim_{\alpha \searrow 0} (< f > - Ax_\alpha) \\
&= \lim_{\alpha \searrow 0} (< f > - y_\alpha) \\
&= < f > - Proj_{\overline{R(A)}} < f > .
\end{aligned}$$

Since $Graph(A)$ is closed and $[\alpha x_\alpha, \alpha y_\alpha] = [\alpha x_\alpha, A(\alpha x_\alpha)] \in A,\ \alpha > 0$, by passing to the limit for $\alpha \searrow 0$ we deduce that $< f > - Proj_{\overline{R(A)}} < f > \in D(A)$ and $A(< f > - Proj_{\overline{R(A)}} < f >) = 0$. It is easy to see that we can pass to the limit for $\alpha \searrow 0$ in (54). Indeed, for $\varepsilon > 0$ let us consider $\alpha_\varepsilon > 0$ such that $\| \lim_{\alpha \searrow 0} \alpha x_\alpha - \alpha_\varepsilon x_{\alpha_\varepsilon} \| < \frac{\varepsilon}{2}$. We have :

$$\begin{aligned}
\left\| \frac{1}{T}(x(t+T) - x(t)) - e^{-tA} \lim_{\alpha \searrow 0} \alpha x_\alpha \right\| &\leq \left\| \frac{1}{T}(x(t+T) - x(t)) - e^{-tA} \alpha_\varepsilon x_{\alpha_\varepsilon} \right\| \\
&+ \left\| e^{-tA} \alpha_\varepsilon x_{\alpha_\varepsilon} - e^{-tA} \lim_{\alpha \searrow 0} \alpha x_\alpha \right\| \\
&\leq \left\| \frac{1}{T}(x(t+T) - x(t)) - e^{-tA} \alpha_\varepsilon x_{\alpha_\varepsilon} \right\| \\
&+ \| \alpha_\varepsilon x_{\alpha_\varepsilon} - \lim_{\alpha \searrow 0} \alpha x_\alpha \| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon, \qquad t \geq t(\alpha_\varepsilon, \frac{\varepsilon}{2}) = t(\varepsilon),
\end{aligned}$$

and thus :

$$\lim_{t \to \infty} \{ \frac{1}{T}(x(t+T) - x(t)) - e^{-tA}(< f > - Proj_{\overline{R(A)}} < f >) \} = 0.$$

But $e^{-tA}(< f > - Proj_{\overline{R(A)}} < f >)$ doesn't depend on $t \geq 0$:

$$\begin{aligned}
\frac{d}{dt} e^{-tA}(< f > - Proj_{\overline{R(A)}} < f >) &= -Ae^{-tA}(< f > - Proj_{\overline{R(A)}} < f >) \\
&= -e^{-tA} A(< f > - Proj_{\overline{R(A)}} < f >) \\
&= 0,
\end{aligned}$$

and thus the previous formula writes :

$$\lim_{t \to \infty} \frac{1}{T}(x(t+T) - x(t)) =< f > - Proj_{\overline{R(A)}} < f > .$$

Remark 7 *Under the same hypothesis as above we can easily check that :*

$$\inf_{x_0 \in D(A)} \frac{\|x(T; 0, x_0) - x_0\|}{T} = \| < f > - Proj_{\overline{R(A)}} < f > \| = dist(< f >, \overline{R(A)}).$$

3.3 Periodic solutions for the heat equation

Consider $\Omega \subset I\!R^d,\ d \geq 1$ an open bounded set with $\partial\Omega \in C^2$. The heat equation writes :

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = f(t,x),\ (t,x) \in I\!R \times \Omega, \tag{55}$$

with the Dirichlet boundary condition :

$$u(t,x) = g(t,x),\ (t,x) \in I\!R \times \partial\Omega, \tag{56}$$

or the Neumann boundary condition :

$$\frac{\partial u}{\partial n}(t,x) = g(t,x),\ (t,x) \in I\!R \times \partial\Omega, \tag{57}$$

where we denote by $n(x)$ the outward normal in $x \in \partial\Omega$.

Theorem 6 *Assume that $f \in C^1(I\!R; L^2(\Omega))$ is T-periodic and $g(t,x) = \frac{\partial u_0}{\partial n}(t,x),\ (t,x) \in I\!R \times \partial\Omega$ where $u_0 \in C^1(I\!R; H^2(\Omega)) \cap C^2(I\!R; L^2(\Omega))$ is T-periodic. Then the heat problem (55), (57) has T-periodic solutions $u \in C(I\!R; H^2(\Omega)) \cap C^1(I\!R; L^2(\Omega))$ iff :*

$$\int_{\partial\Omega}\int_0^T g(t,x)dtd\sigma + \int_\Omega \int_0^T f(t,x)dtdx = 0.$$

In this case the periodic solutions verifie the estimates :

$$\begin{aligned} \|u' - u_0'\|_{L^\infty([0,T];L^2(\Omega))} &\leq \frac{1}{\sqrt{T}}\|f - u_0' + \Delta u_0\|_{L^2(]0,T[;L^2(\Omega))} \\ &+ \|f' - u_0'' + \Delta u_0'\|_{L^1(]0,T[;L^2(\Omega))}, \end{aligned} \tag{58}$$

and the solution is unique up to a constant.

Proof : Let us search for solutions $u = u_0 + v$ where :

$$\frac{\partial v}{\partial t}(t,x) - \Delta v(t,x) = f(t,x) - \frac{\partial u_0}{\partial t}(t,x) + \Delta u_0(t,x),\ (t,x) \in I\!R \times \Omega, \tag{59}$$

and :

$$\frac{\partial v}{\partial n}(t,x) = g(t,x) - \frac{\partial u_0}{\partial n}(t,x) = 0,\ (t,x) \in I\!R \times \partial\Omega. \tag{60}$$

Consider the operator $A_N : D(A_N) \subset L^2(\Omega) \to L^2(\Omega)$ given by :

$$D(A_N) = \{v \in H^2(\Omega) \mid \frac{\partial v}{\partial n}(x) = 0,\ \forall\, x \in \partial\Omega\},$$

and :

$$A_N v = -\Delta v,\ \forall\, v \in D(A_N).$$

The operator A_N is linear monotone :

$$\begin{aligned} (A_N v, v) &= -\int_\Omega \Delta v(x) v(x) dx \\ &= -\int_{\partial\Omega} \frac{\partial v}{\partial n}(x) v(x) d\sigma + \int_\Omega \|\nabla v(x)\|^2 dx \\ &= \int_\Omega \|\nabla v(x)\|^2 dx \\ &\geq 0,\ \forall\, v \in D(A_N). \end{aligned} \tag{61}$$

Since the equation $\lambda v - \Delta v = f$ has unique solution in $D(A_N)$ for every $f \in L^2(\Omega)$, $\lambda > 0$ it follows that A_N is maximal (see [6]). Moreover, it is symmetric :

$$(A_N v_1, v_2) = \int_\Omega \nabla v_1(x) \cdot \nabla v_2(x) dx = (v_1, A_N v_2), \ \forall \, v_1, v_2 \in D(A_N).$$

Note that by the hypothesis the second member in (59) $f - u_0' + \Delta u_0$ belongs to $C^1(I\!R; L^2(\Omega))$. Therefore the *Theorem 5* applies and hence the problem (59), (60) has periodic solutions iff there is $w \in D(A_N)$ such that :

$$-\Delta w = \frac{1}{T} \int_0^T \{f(t) - \frac{du_0}{dt}(t) + \Delta u_0(t)\} dt.$$

Since u_0 is T-periodic we have $\int_0^T \frac{du_0}{dt}(t) dt = 0$ and thus $w + \frac{1}{T}\int_0^T u_0(t) dt$ is solution for the elliptic problem :

$$-\Delta \left(w + \frac{1}{T} \int_0^T u_0(t) dt \right) = \frac{1}{T} \int_0^T f(t) dt = F,$$

with the boundary condition :

$$\begin{aligned} \frac{\partial}{\partial n} \left(w + \frac{1}{T} \int_0^T u_0(t) dt \right) &= \frac{\partial w}{\partial n} + \frac{1}{T} \int_0^T \frac{\partial u_0}{\partial n}(t) dt \\ &= \frac{1}{T} \int_0^T g(t) dt = G. \end{aligned}$$

As known from the general theory of partial differential equations (see [6]) this problem has solution iff $\int_{\partial\Omega} G(x) d\sigma + \int_\Omega F(x) dx = 0$ or :

$$\int_{\partial\Omega} \int_0^T g(t,x) dt d\sigma + \int_\Omega \int_0^T f(t,x) dt dx = 0.$$

The estimate (58) follows from the *Theorem 5*.

For the heat equation with Dirichlet boundary condition we have the existence result :

Theorem 7 *Assume that $f \in C^1(I\!R; L^2(\Omega))$ is T-periodic and $g(t,x) = u_0(t,x)$, $(t,x) \in I\!R \times \partial\Omega$ where $u_0 \in C^1(I\!R; H^2(\Omega)) \cap C^2(I\!R; L^2(\Omega))$ is T-periodic. Then the heat problem (55), (56) has an unique T-periodic solution $u \in C(I\!R; H^2(\Omega)) \cap C^1(I\!R; L^2(\Omega))$ and there is a constant $C(\Omega)$ such that :*

$$\begin{aligned} \|u - u_0\|_{L^\infty([0,T];L^2(\Omega))} &\leq C(\Omega) \|f + \Delta u_0\|_{L^\infty([0,T];L^2(\Omega))} \\ &+ \frac{\sqrt{T}}{2} \|f - u_0' + \Delta u_0\|_{L^2(]0,T[;L^2(\Omega))} \\ &+ \frac{T}{2} \|f' - u_0'' + \Delta u_0'\|_{L^1(]0,T[;L^2(\Omega))}, \end{aligned} \tag{62}$$

and :

$$\begin{aligned} \|u' - u_0'\|_{L^\infty([0,T];L^2(\Omega))} &\leq \frac{1}{\sqrt{T}} \|f - u_0' + \Delta u_0\|_{L^2(]0,T[;L^2(\Omega))} \\ &+ \|f' - u_0'' + \Delta u_0'\|_{L^1(]0,T[;L^2(\Omega))}. \end{aligned} \tag{63}$$

Proof : This time we consider the operator $A_D : D(A_D) \subset L^2(\Omega) \to L^2(\Omega)$ given by :

$$D(A_D) = \{v \in H^2(\Omega) \mid v(x) = 0,\ \forall x \in \partial\Omega\},$$

and :

$$A_D v = -\Delta v,\ \forall v \in D(A_D).$$

As before A_D is linear, monotone and symmetric and thus our problem reduces to the existence for an elliptic equation :

$$-\Delta w = \frac{1}{T}\int_0^T \{f(t) + \Delta u_0(t)\}dt,$$

with homogenous Dirichlet boundary condition $w = 0$ on $\partial\Omega$. Since the previous problem has all the time unique solution verifying :

$$\|w\|_{L^2(\Omega)} \leq C(\Omega)\|\frac{1}{T}\int_0^T \{f(t) + \Delta u_0(t)\}dt\|_{L^2(\Omega)} \leq C(\Omega)\|f + \Delta u_0\|_{L^\infty([0,T];L^2(\Omega))}, \tag{64}$$

we prove the existence for (55), (56). Here we denote by $C(\Omega)$ the Poincaré's constant :

$$\left(\int_\Omega |w(x)|^2 dx\right)^{1/2} \leq C(\Omega)\left(\int_\Omega \|\nabla w(x)\|^2 dx\right)^{1/2},\ \forall w \in H_0^1(\Omega).$$

Moreover in this case the operator A_D is strictly monotone. Indeed, by using the Poincaré's inequality we have :

$$\left(\int_\Omega |v(x)|^2 dx\right)^{1/2} \leq C(\Omega)\left(\int_\Omega \|\nabla v(x)\|^2 dx\right)^{1/2} = C(\Omega)(A_D v, v)^{1/2},\ \forall v \in D(A_D),$$

and hence if $(A_D v, v) = 0$ we deduce that $v = 0$. Therefore, by the *Proposition 12* we deduce the uniqueness of the periodic solution for (55), (56). The estimates of the solution follow immediately from (64) and *Theorem 5*.

3.4 Non-linear case

Throughout this section we will consider evolution equations associated to subdifferential operators. Let $\varphi : H \to]-\infty, +\infty]$ be a lower-semicontinuous proper convex function on a real Hilbert space H. Denote by $\partial\varphi \subset H \times H$ the subdifferential of φ :

$$\partial\varphi(x) = \{y \in H;\ \varphi(x) - \varphi(u) \leq (y, x - u),\ \forall u \in H\}, \tag{65}$$

and denote by $D(\varphi)$ the effective domain of φ :

$$D(\varphi) = \{x \in H;\ \varphi(x) < +\infty\}. \tag{66}$$

Under the previous assumptions on φ we recall that $A = \partial\varphi$ is maximal monotone in $H \times H$ and $\overline{D(A)} = \overline{D(\varphi)}$. Consider the equation :

$$x'(t) + \partial\varphi x(t) \ni f(t),\ 0 < t < T. \tag{67}$$

We say that x is solution for (67) if $x \in C([0,T];H)$, x is absolutely continuous on every compact of $]0,T[$ (and therefore a.e. differentiable on $]0,T[$) and satisfies $x(t) \in D(\partial\varphi)$ a.e. on $]0,T[$ and $x'(t) + \partial\varphi x(t) \ni f(t)$ a.e. on $]0,T[$. We have the following main result (see [1]) :

Theorem 8 *Let f be given in $L^2(]0,T[;H)$ and $x_0 \in \overline{D(\partial\varphi)}$. Then the Cauchy problem (67) with the initial condition $x(0) = x_0$ has a unique solution $x \in C([0,T];H)$ which satisfies :*

$$x \in W^{1,2}(]\delta,T[;H)\ \forall\, 0<\delta<T, \quad \sqrt{t}\cdot x' \in L^2(]0,T[;H), \quad \varphi\circ x \in L^1(0,T).$$

Moreover, if $x_0 \in D(\varphi)$ then :

$$x' \in L^2(]0,T[;H), \quad \varphi\circ x \in L^\infty(0,T).$$

We are interested in finding sufficient conditions on $A = \partial\varphi$ and f such that equation (67) has unique T-periodic solution, i.e. $x(0) = x(T)$. Obviously, if such a solution exists, by periodicity we deduce that it is absolutely continuous on $[0,T]$ and belongs to $W^{1,2}(]0,T[;H)$.
It is well known that if φ is strictly convex then $\partial\varphi$ is strictly monotone and therefore the uniqueness holds :

Proposition 17 *Assume that $\varphi : H \to]-\infty,+\infty]$ is a lower-semicontinuous proper, strictly convex function. Then equation (67) has at most one periodic solution.*

Proof : By using the *Proposition 12* it is sufficient to prove that $\partial\varphi$ is strictly monotone. Suppose that there are $u_1, u_2 \in D(\partial\varphi)$, $u_1 \neq u_2$ such that :

$$(\partial\varphi(u_1) - \partial\varphi(u_2), u_1 - u_2) = 0.$$

We have :

$$\begin{aligned}
\varphi(u_2) - \varphi(u_1) &\geq (\partial\varphi(u_1), u_2 - u_1)\\
&= -(\partial\varphi(u_2), u_1 - u_2)\\
&\geq \varphi(u_2) - \varphi(u_1),
\end{aligned}$$

and hence :

$$\varphi(u_2) - \varphi(u_1) = (\partial\varphi(u_1), u_2 - u_1).$$

We can also write for $\lambda \in]0,1[$:

$$\begin{aligned}
\varphi((1-\lambda)u_1 + \lambda u_2) &= \varphi(u_1 + \lambda(u_2 - u_1))\\
&\geq \varphi(u_1) + (\partial\varphi(u_1), \lambda(u_2 - u_1))\\
&= \varphi(u_1) + \lambda(\partial\varphi(u_1), u_2 - u_1)\\
&= \varphi(u_1) + \lambda(\varphi(u_2) - \varphi(u_1))\\
&= (1-\lambda)\varphi(u_1) + \lambda\varphi(u_2).
\end{aligned}$$

Since φ is strictly convex we have also :

$$\varphi((1-\lambda)u_1 + \lambda u_2) < (1-\lambda)\varphi(u_1) + \lambda\varphi(u_2),$$

which is in contradiction with the previous inequality. Thus $u_1 = u_2$ and hence $\partial\varphi$ is strictly monotone.

We state now the result concerning the existence of periodic solutions :

Theorem 9 *Suppose that $\varphi : H \to]-\infty, +\infty]$ is a lower-semicontinuous proper convex function and $f \in L^2(]0,T[;H)$ such that :*

$$\lim_{\|x\|\to\infty} \{\varphi(x) - (x, < f >)\} = +\infty, \tag{68}$$

and every level subset $\{x \in H;\ \varphi(x) + \|x\|^2 \le M\}$ is compact. Then equation (67) has T-periodic solutions $x \in C([0,T];H) \cap W^{1,2}(]0,T[;H)$ which satisfy :

$$\|x'\|_{L^2(]0,T[;H)} \le \|f\|_{L^2(]0,T[;H)}, \quad x(t) \in D(\varphi)\ \forall\, t \in [0,T], \quad \varphi \circ x \in L^\infty(0,T).$$

Before showing this result, notice that the condition (68) implies that the lower-semicontinuous proper convex function $\psi : H \to]-\infty, +\infty]$ given by $\psi(x) = \varphi(x) - (x, < f >)$ has a minimum point $x_0 \in H$ and therefore $< f > \in Range(\partial\varphi)$ since $0 = \partial\psi(x_0) = \partial\varphi(x_0) - < f >$.

Proof : As previous for every $\alpha > 0$ we consider the unique periodic solution x_α for :

$$\alpha x_\alpha(t) + x'_\alpha(t) + \partial\varphi x_\alpha(t) = f(t),\ 0 < t < T. \tag{69}$$

(In order to prove the existence and uniqueness of the periodic solution for (69) consider the application $S_\alpha : \overline{D(\partial\varphi)} \to \overline{D(\partial\varphi)}$ defined by $S_\alpha(x_0) = x(T;0,x_0)$, where $x(\cdot;0,x_0)$ denote the unique solution of (69) with the initial condition x_0 and apply the Banach's fixed point theorem. By the previous theorem it follows that the periodic solution x_α is absolutely continuous on $[0,T]$ and belongs to $C([0,T];H) \cap W^{1,2}(]0,T[;H)$).

First of all we will show that $(x'_\alpha)_{\alpha>0}$ is uniformly bounded in $L^2(]0,T[;H)$. Indeed, after multiplication by $x'_\alpha(t)$ we obtain :

$$\int_0^T \|x'_\alpha(t)\|^2 dt + \int_0^T \{\alpha(x_\alpha(t), x'_\alpha(t)) + (\partial\varphi x_\alpha(t), x'_\alpha(t))\} dt = \int_0^T (f(t), x'_\alpha(t)) dt.$$

Since x_α is T-periodic we deduce that :

$$\begin{aligned} \int_0^T \{\alpha(x_\alpha(t), x'_\alpha(t)) + (\partial\varphi x_\alpha(t), x'_\alpha(t))\} dt &= \int_0^T \frac{d}{dt}\{\frac{\alpha}{2}\|x_\alpha(t)\|^2 + \varphi(x_\alpha(t))\} dt \\ &= \frac{\alpha}{2}\|x_\alpha(t)\|^2 + \varphi(x_\alpha(t))|_0^T \\ &= 0. \end{aligned} \tag{70}$$

Therefore $\|x'_\alpha\|^2_{L^2(]0,T[;H)} \leq (f, x'_\alpha)_{L^2(]0,T[;H)}$ and thus :

$$\|x'_\alpha\|_{L^2(]0,T[;H)} \leq \|f\|_{L^2(]0,T[;H)}, \ \alpha > 0.$$

Before estimate $(x_\alpha)_{\alpha>0}$, let us check that $(\alpha x_\alpha)_{\alpha>0}$ is bounded. By taking $x_0 \in D(\partial\varphi)$, after standard calculation we find that :

$$\|x_\alpha(t) - x_0\| \leq e^{-\alpha t}\|x_\alpha(0) - x_0\| + \int_0^t e^{-\alpha(t-s)}\|f(s) - \alpha x_0 - \partial\varphi(x_0)\| \, ds, \ t > 0, \ \alpha > 0. \quad (71)$$

Since x_α is T-periodic we can write :

$$\begin{aligned}
\|x_\alpha(t) - x_0\| &= \lim_{n\to\infty} \|x_\alpha(nT+t) - x_0\| \\
&\leq \lim_{n\to\infty} \left\{ e^{-\alpha(nT+t)}\|x_\alpha(0) - x_0\| + \int_0^{nT+t} e^{-\alpha(nT+t-s)}\|f(s) - \alpha x_0 - \partial\varphi(x_0)\| \, ds \right\} \\
&\leq \frac{1}{\alpha}\|\alpha x_0 + \partial\varphi(x_0)\| + \lim_{n\to\infty} \int_0^{nT+t} e^{-\alpha(nT+t-s)}\|f(s)\| \, ds \\
&\leq \frac{1}{\alpha}\|\alpha x_0 + \partial\varphi(x_0)\| + \lim_{n\to\infty} \left\{ \left[1 + e^{-\alpha t}(e^{-\alpha(n-1)T} + ... + e^{-\alpha T} + 1)\right] \cdot \|f\|_{L^1} \right\} \\
&= \frac{1}{\alpha}\|\alpha x_0 + \partial\varphi(x_0)\| + \left(1 + \frac{e^{-\alpha t}}{1 - e^{-\alpha T}}\right) \cdot \|f\|_{L^1(]0,T[;H)} \\
&\leq C_1(x_0, T, \|f\|_{L^2(]0,T[;H)}) \left(1 + \frac{1}{\alpha}\right), \ 0 \leq t \leq T, \ \alpha > 0.
\end{aligned}$$

It follows that $\alpha\|x_\alpha(t)\| \leq C_2(x_0, T, \|f\|_{L^2(]0,T[;H)}), \ 0 \leq t \leq T, \ 0 < \alpha < 1$. Now we can estimate $x_\alpha, \ \alpha > 0$. After multiplication by $x_\alpha(t)$ and integration on $[0, T]$ we obtain :

$$\int_0^T \alpha\|x_\alpha(t)\|^2 dt + \int_0^T (\partial\varphi(x_\alpha(t)), x_\alpha(t)) dt = \int_0^T (f(t), x_\alpha(t)) dt. \quad (72)$$

We have :

$$\varphi(x_0) \geq \varphi(x_\alpha(t)) + (\partial\varphi(x_\alpha(t)), x_0 - x_\alpha(t)), \ t \in [0, T], \ \alpha > 0,$$

and thus we deduce that :

$$\int_0^T (\partial\varphi(x_\alpha(t)), x_\alpha(t)) dt \geq \int_0^T \varphi(x_\alpha(t)) dt + \int_0^T \{(\partial\varphi(x_\alpha(t)), x_0) - \varphi(x_0)\} dt, \ \alpha > 0.$$

On the other hand :

$$\begin{aligned}
\int_0^T (\partial\varphi(x_\alpha(t)), x_0) \, dt &= \int_0^T (f(t) - \alpha x_\alpha(t) - x'_\alpha(t), x_0) \, dt \\
&= \left(\int_0^T f(t) \, dt, x_0\right) - \int_0^T (\alpha x_\alpha(t), x_0) \, dt \\
&\geq -C_3(x_0, T, \|f\|_{L^2(]0,T[;H)}), \ 0 < \alpha < 1,
\end{aligned}$$

and therefore :

$$\int_0^T (\partial\varphi(x_\alpha(t)), x_\alpha(t)) dt \geq \int_0^T \varphi(x_\alpha(t)) dt - C_4(x_0, T, \|f\|_{L^2(]0,T[;H)}), \ 0 < \alpha < 1. \quad (73)$$

Now by combining (72) and (73) we deduce that :

$$\begin{aligned}\int_0^T \varphi(x_\alpha(t))dt &\leq C_4 + \int_0^T (\partial\varphi(x_\alpha(t)), x_\alpha(t))dt \\ &= C_4 + \int_0^T (f(t), x_\alpha(t))dt - \int_0^T \alpha\|x_\alpha(t)\|^2 dt \\ &\leq C_4 + \int_0^T (f(t), x_\alpha(t))dt,\ 0 < \alpha < 1. \end{aligned} \tag{74}$$

On the other hand we have :

$$\begin{aligned}\int_0^T (f(t), x_\alpha(t))dt &= \int_0^T (f(t) - < f >, x_\alpha(t))dt + \left(\int_0^T x_\alpha(t)dt, < f >\right) \\ &= \int_0^T (f(t) - < f >, x_\alpha(0) + \int_0^t x'_\alpha(s)\ ds)dt + T(< x_\alpha >, < f >) \\ &= \int_0^T (f(t) - < f >, \int_0^t x'_\alpha(s)\ ds)dt + T(< x_\alpha >, < f >) \\ &\leq \int_0^T \|f(t) - < f > \| \cdot \left(\int_0^t \|x'_\alpha(s)\|^2\ ds\right)^{1/2} \cdot t^{1/2}\ dt + T(< x_\alpha >, < f >) \\ &\leq \|f - < f > \|_{L^2(]0,T[;H)} \cdot \|f\|_{L^2(]0,T[;H)} \cdot \frac{T}{\sqrt{2}} + T(< x_\alpha >, < f >). \end{aligned}$$

Finally we deduce that :

$$\int_0^T \{\varphi(x_\alpha(t)) - (x_\alpha(t), < f >)\}\ dt \leq C_5(x_0, T, \|f\|_{L^2(]0,T[;H)}),\ 0 < \alpha < 1, \tag{75}$$

and thus there is $t_\alpha \in [0,T]$ such that :

$$\varphi(x_\alpha(t)) - (x_\alpha(t), < f >) \leq \frac{C_5}{T},\ 0 < \alpha < 1. \tag{76}$$

By the hypothesis (68) we get that $(x_\alpha(t_\alpha))_{0<\alpha<1}$ is bounded and therefore from (71) :

$$\|x_\alpha(t) - x_0\| \leq e^{-\alpha(t-t_\alpha)}\|x_\alpha(t_\alpha) - x_0\| + \int_{t_\alpha}^t e^{-\alpha(t-s)}\|f(s) - \alpha x_0 - \partial\varphi(x_0)\|\ ds,\ t \in [t_\alpha, t_\alpha + T],$$

we deduce that $(x_\alpha)_{0<\alpha<1}$ is bounded in $L^\infty(]0,T[;H)$ and that there is $x \in L^\infty(]0,T[;H)$ such that $x_\alpha(t) \rightharpoonup x(t)$ when α goes to 0 for $t \in [0,T]$. Moreover, from (76) it follows that $(\varphi(x_\alpha(t_\alpha)))_{0<\alpha<1}$ is bounded from above and we deduce that :

$$\begin{aligned}\varphi(x_\alpha(t)) &= \varphi(x_\alpha(t_\alpha)) + \int_{t_\alpha}^t (\partial\varphi(x_\alpha(s)), x'_\alpha(s))\ ds \\ &\leq \varphi(x_\alpha(t_\alpha)) + \int_{t_\alpha}^t (f(s) - \alpha x_\alpha(s) - x'_\alpha(s), x'_\alpha(s))\ ds \\ &\leq C_6(x_0, T, \|f\|_{L^2(]0,T[;H)}),\ 0 < \alpha < 1. \end{aligned}$$

On the other hand, by writting $\varphi(x_\alpha(t)) \geq \varphi(x_0) + (\partial\varphi(x_0), x_\alpha(t) - x_0),\ 0 \leq t \leq T,\ \alpha > 0$ we deduce that $\varphi(x_\alpha(t))$ is also bounded from below so that finally $(\varphi \circ x_\alpha)_{0<\alpha<1}$ is bounded in $L^\infty(]0,T[;H)$.

Now, by using the second hypothesis of the theorem (every level subset is compact) we deduce that $x_\alpha(0) \to x(0)$ when α goes to 0 (at least for a subsequence $\alpha_n \searrow 0$). In fact we can easily check that x_α converges uniformly to x on $[0,T]$ since :

$$\|x_\alpha(t) - x_\beta(t)\| \le \|x_\alpha(0) - x_\beta(0)\| + |\alpha - \beta| \cdot T \cdot \sup_{0<\gamma<1} \|x_\gamma\|_{L^\infty(]0,T[;H)},\ 0 \le t \le T,\ 0 < \alpha, \beta < 1.$$

Now, since $\lim_{\alpha \searrow 0} dx_\alpha/dt = dx/dt$ in the sense of H-valued vectorial distribution on $]0,T[$ and $(x'_\alpha)_{\alpha>0}$ is bounded in $L^2(]0,T[;H)$ it follows that x' belongs to $L^2(]0,T[;H)$ and in particular x is absolutely continuous on every compact of $]0,T[$ and therefore a.e. differentiable on $]0,T[$. In order to complete the proof we need to show that $x(t) \in D(\varphi)$ a.e. on $]0,T[$ and $x'(t) + \partial\varphi x(t) \ni f(t)$ a.e. on $]0,T[$. For arbitrarly $[u,v] \in \partial\varphi$ we have :

$$\frac{1}{2}e^{2\alpha t}\|x_\alpha(t) - u\|^2 \le \frac{1}{2}e^{2\alpha s}\|x_\alpha(s) - u\|^2 + \int_s^t e^{2\alpha\tau}(f(\tau) - \alpha u - v, x_\alpha(\tau) - u)\, d\tau,\ 0 \le s \le t \le T,\ \alpha > 0.$$

By passing to the limit for $\alpha \searrow 0$ we get :

$$\frac{1}{2}\|x(t) - u\|^2 \le \frac{1}{2}\|x(s) - u\|^2 + \int_s^t (f(\tau) - v, x(\tau) - u)\, d\tau,\ 0 \le s \le t \le T,$$

and thus :

$$(x(t) - x(s), x(s) - u) \le \frac{1}{2}\|x(t) - u\|^2 - \frac{1}{2}\|x(s) - u\|^2 \le \int_s^t (f(\tau) - v, x(\tau) - u)\, d\tau,\ 0 \le s \le t \le T.$$

Since x is a.e. differentiable on $]0,T[$ we find that :

$$\begin{aligned}(x'(t), x(t) - u) &= \lim_{s \nearrow t} \frac{1}{t-s}(x(t) - x(s), x(s) - u) \\ &\le \lim_{s \nearrow t} \frac{1}{t-s}\int_s^t (f(\tau) - v, x(\tau) - u)\, d\tau \\ &= (f(t) - v, x(t) - u),\ a.e.\ t \in]0,T[,\ \forall\ [u,v] \in \partial\varphi. \qquad (77)\end{aligned}$$

Finally, since $\partial\varphi$ is maximal monotone and $(f(t) - x'(t) - v, x(t) - u) \ge 0\ \forall\ [u,v] \in \partial\varphi$ we deduce that $x(t) \in D(\partial\varphi)$ a.e. on $]0,T[$ and $x'(t) + \partial\varphi x(t) \ni f(t)$ a.e. on $]0,T[$. Since φ is lower-semicontinuous we also have :

$$\varphi(x(t)) \le \liminf_{\alpha \searrow 0} \varphi(x_\alpha(t)) \le \liminf_{\alpha \searrow 0} \|\varphi \circ x_\alpha\|_{L^\infty} \le \sup_{0<\gamma<1} \|\varphi \circ x_\gamma\|_{L^\infty}.$$

As previous, by writting :

$$\begin{aligned}\varphi(x(t)) &\ge \varphi(x_0) + (\partial\varphi(x_0), x(t) - x_0) \\ &\ge \varphi(x_0) - \|\partial\varphi(x_0)\| \cdot (\|x_0\| + \liminf_{\alpha \searrow 0} \|x_\alpha(t)\|) \\ &\ge \varphi(x_0) - \|\partial\varphi(x_0)\| \cdot (\|x_0\| + \sup_{0<\gamma<1} \|x_\gamma\|_{L^\infty}),\ 0 \le t \le T,\end{aligned}$$

we deduce finally that $\varphi \circ x \in L^\infty(0,T)$.

Remark 8 *If* $\dim H < +\infty$ *then the level subsets* $\{x \in H\ ;\ \varphi(x) + \|x\|^2 \leq M\}$ *are compact as bounded sets.*

Remark 9 *Assume that* $\varphi : H \to]-\infty, +\infty]$ *is a lower-semicontinuous proper convex function such that Range*$(\partial\varphi) = H$ *which is equivalent to :*

$$\lim_{\|x\|\to\infty} \{\varphi(x) - (x, y)\} = +\infty,\ \forall y \in H,$$

see [2], pp.41. In particular, by taking $y =< f >$ *we deduce that the hypothesis (68) is verified.*

Remark 10 *Assume that* φ *is coercive :*

$$\lim_{\|x\|\to\infty} \frac{(\partial\varphi(x), x - x_0)}{\|x\|} = +\infty,\ \forall x_0 \in D(\varphi),$$

which is equivalent to $\lim_{\|x\|\to\infty} \frac{\varphi(x)}{\|x\|} = +\infty$ *(see [2], pp.42). Then Range*$(\varphi) = H$ *because the previous condition is satisfied :* $\lim_{\|x\|\to\infty}\{\varphi(x) - (x, y)\} = +\infty,\ \forall\ y \in H$ *and therefore (68) is verified.*

We state also the following result :

Theorem 10 *Suppose that* $\varphi : H \to]-\infty, +\infty]$ *is a lower-semicontinuous proper convex function and* $f \in W^{1,1}(]0, T[; H)$ *such that :*

$$\lim_{\|x\|\to\infty} \{\varphi(x) - (x, < f >)\} = +\infty, \tag{78}$$

and every level subset $\{x \in H;\ \varphi(x) + \|x\|^2 \leq M\}$ *is compact. Then equation (67) has* T*-periodic solutions* $x \in C([0, T]; H) \cap W^{1,\infty}(]0, T[; H)$ *which satisfy :*

$$x(t) \in D(\partial\varphi), \forall\, t \in [0, T], \quad \frac{d^+}{dt}x(t) + (\partial\varphi x(t) - f(t))^\circ = 0,\ \forall\ t \in [0, T],$$

where $(\partial\varphi - f)^\circ$ *denote the minimal section of* $\partial\varphi - f$.

Proof : Since $W^{1,1}(]0, T[; H) \subset L^2(]0, T[; H)$ the previous theorem applies. Consider $x \in C([0, T]; H) \cap W^{1,2}(]0, T[; H)$ a T-periodic solution for (67). Since $\|x'\|_{L^2(]0,T[;H)} \leq \|f\|_{L^2(]0,T[;H)}$ it follows that there is $t^\star \in]0, T[$ such that x is differentiable in $t^\star$ and $\|x'(t^\star)\| \leq \frac{1}{\sqrt{T}}\|f\|_{L^2(]0,T[;H)}$. By standard calculation we find that :

$$\|\frac{1}{h}(x(t + h) - x(t))\| \leq \|\frac{1}{h}(x(t^\star + h) - x(t^\star))\| + \int_{t^\star}^{t} \|\frac{1}{h}(f(\tau + h) - f(\tau)\|\, d\tau,$$

and therefore $\sup_{0\leq t\leq T,\ h>0} \|\frac{1}{h}(x(t + h) - x(t))\| \leq C$ which implies that $x \in W^{1,\infty}(]0, T[; H)$. Making use of the inequality :

$$\frac{1}{h}(x(t + h) - x(t), x(t) - u) \leq \frac{1}{h}\int_{t}^{t+h} (f(\tau) - v, x(\tau) - u)\, d\tau,\ 0 \leq t < t + h \leq T,$$

which holds for every $[u,v] \in \partial\varphi$ we deduce that $x(t) \in D(\partial\varphi),\ \forall t \in [0,T]$ and the weak closure of the set $\{\frac{1}{h}(x(t+h) - x(t)),\ h > 0\}$ belongs to $f(t) - \partial\varphi x(t),\ \forall t \in [0,T]$. On the other hand by writting :

$$\|x(t+h) - u\| \leq \|x(t) - u\| + \int_t^{t+h} \|f(\tau) - v\|\, d\tau,\ 0 \leq t < t+h \leq T,$$

for $u = x(t)$ and $v \in \partial\varphi x(t)$ we find that :

$$\|(\partial\varphi x(t) - f(t))^\circ\| \leq \|w - \lim_{h \searrow 0} \frac{1}{h}(x(t+h) - x(t))\ \| \leq \limsup_{h \searrow 0} \|\frac{1}{h}(x(t+h) - x(t))\ \| \leq \|(\partial\varphi x(t) - f(t))^\circ\|.$$

This shows that $\lim_{h \searrow 0} \frac{1}{h}(x(t+h) - x(t)) = \frac{d^+}{dt}x(t)$ exists for every $t \in [0,T]$ and coincides with $-(\partial\varphi x(t) - f(t))^\circ$.

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