COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **6**, Number **2**, June **2007**

Website: http://AIMsciences.org

pp. **X–XX**

TIME PERIODIC VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

Mihai Bostan

Laboratoire de Mathématiques de Besançon
Université de Franche-Comté
16 route de Gray, Besançon, 25030 Cedex, FRANCE

Gawtum Namah

Laboratoire de Mathématiques de Besançon
Université de Franche-Comté
16 route de Gray, Besançon, 25030 Cedex, FRANCE

(Communicated by Martino Bardi)

Abstract. This paper discusses viscosity solutions of general Hamilton-Jacobi equations in the time periodic case. Existence results are presented under usual hypotheses. The main idea is to reduce the study of time periodic problems to the study of stationary problems obtained by averaging the source term over a period. These results hold also for almost-periodic viscosity solutions.

1. **Introduction.** In this paper, we will be interested in time periodic solutions of first order Hamilton-Jacobi equations of the form

$$\partial_t u + H(x, u, Du) = f(t), \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1}$$

i.e., we will look for viscosity solutions of (1) which satisfy $u(x, t) = u(x, t + T), \ (x, t) \in \mathbb{R}^N \times \mathbb{R}$, where the hamiltonian H and f are continuous functions, f is T periodic in t and $Du = (\partial_{x_1} u, \partial_{x_2} u, ..., \partial_{x_N} u)$ denotes the gradient of u.

Clearly the conditions ensuring the existence and uniqueness of such solutions will be closely related to those giving the existence and uniqueness results of the initial value problem

$$\begin{cases} \partial_t u + H(x, u, Du) = f(t), & (x, t) \in \mathbb{R}^N \times]0, T[, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{2}$$

For this latter purpose, one may refer to the series of papers by Crandall and Lions where the notion of viscosity solution was introduced, cf. [8], [9], [10], [13]. They proved the uniqueness and stability of this type of solutions for a large class of equations, in particular for the initial value problem

$$\begin{cases} \partial_t u + H(x, t, u, Du) = 0, & (x, t) \in \mathbb{R}^N \times]0, T[, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3}$$

2000 *Mathematics Subject Classification.* Primary: 35F20; Secondary: 35B45.

Key words and phrases. Hamilton-Jacobi equations, time periodic viscosity solutions, almost-periodic viscosity solutions.

and also for the stationary problem

$$H(x,u,Du)=0,\ \ x\in\mathbb{R}^N. \tag{4}$$

These results were extended by several papers. We just mention the one by Souganidis [19] where general existence results are discussed and the book by Barles [1] for a clear presentation of viscosity solutions.

Let us start by listing the usual hypotheses used for the existence and uniqueness results. We formulate them for time dependent hamiltonians, whereas when dealing with hamiltonians not depending on time, stationary variants have to be considered.

$$\forall\, 0<R<+\infty,\ \ \exists\gamma_R>0\ :H(x,t,u,p)-H(x,t,v,p)\geq\gamma_R(u-v), \tag{5}$$

for all $x\in\mathbb{R}^N,\ 0\leq t\leq T,\ -R\leq v\leq u\leq R,\ p\in\mathbb{R}^N$;

$$\forall R>0,\ \ \exists m_R\ :|H(x,t,u,p)-H(y,t,u,p)|\leq m_R(\ |x-y|\cdot(1+|p|)\ ), \tag{6}$$

for all $x,y\in\mathbb{R}^N,\ t\in[0,T],\ -R\leq u\leq R,\ p\in\mathbb{R}^N$, where $\lim_{z\to 0}m_R(z)=0$;

$$\forall R>0,\ \lim_{|p|\to\infty}H(x,t,u,p)=\infty,\ \text{uniformly for } (x,t,u)\in\mathbb{R}^N\times[0,T]\times[-R,R]\ ; \tag{7}$$

$$\forall\, 0<R<+\infty,\ \ H \text{ is uniformly continuous on } \mathbb{R}^N\times[0,T]\times[-R,R]\times\overline{B}_R\ ; \tag{8}$$

$$\exists M>0\ :H(x,t,-M,0)\leq 0\leq H(x,t,M,0),\ \ \forall\, x\in\mathbb{R}^N,\ t\in[0,T]. \tag{9}$$

Recall that hypotheses (5), (6 or 7), (8), (9) ensure the existence of a unique solution for the stationary equation (4). It is well known that the condition (5) is crucial for the uniqueness result. For example if (5) is replaced by

$$H(x,u,p)-H(x,v,p)\geq 0,\ \forall x\in\mathbb{R}^N,\ v\leq u,\ p\in\mathbb{R}^N, \tag{10}$$

(which comes to taking $\gamma_R=0$ in (5)), then uniqueness fails even if under the hypotheses (10), (7), (8) and (9) one still has the existence of a solution $u\in W^{1,\infty}(\mathbb{R}^N)$. The regularity of the solution is in fact a consequence of the coercivity condition (7). Thus without (7), for example under the hypotheses (10), (6), (8) and (9), one can just ensure the existence of a bounded semi continuous viscosity solution of (4). And finally note that a further weakening of the above hypotheses, for example that of (9), may not guarantee even the existence of a viscosity solution. The hypotheses (5) (with $\gamma_R\in\mathbb{R},\ \forall R>0$), (6), (8), (9) ensure existence and uniqueness results for the Cauchy problem (3). An easy consequence is the existence of a unique periodic solution for

$$\partial_t u+H(x,t,u,Du)=0,\ \ (x,t)\in\mathbb{R}^N\times\mathbb{R}, \tag{11}$$

(start with an arbitrary data u_0 and use the fixed point method, see Section 3).

As mentioned before, in the above results, the monotonicity of H with respect to u happens to be crucial notably for the uniqueness results. A much more difficult situation arises when the hamiltonian is just nondecreasing with respect to u. In this case we consider the particular problem (1) where f is a periodic continuous function. The main idea is to observe that there is a close relation between the existence of time periodic solutions of (1) and that of stationary solutions for the time averaged problem

$$H(x,u,Du)=\langle f\rangle:=\frac{1}{T}\int_0^T f(t)\ dt,\ \ x\in\mathbb{R}^N. \tag{12}$$

A very easy example is given by the following ode

$$x'(t)+g(x(t))=f(t),\ \ t\in\mathbb{R}, \tag{13}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and for which we have the following result (see [6])

Proposition 1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous, T periodic and $g : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing. Then* (13) *admits a T periodic (classical) solution iff there exists x_0 which solves $g(x) = \langle f \rangle$.*

We will now give our main results for Hamilton-Jacobi equations in the case where the hamiltonian is nondecreasing in u, *i.e.,* satisfies (10).
Theorem 4.1 *Let $H = H(x, z, p)$ be a hamiltonian verifying* (10), (7), (8) *and* $\sup\{|H(x, 0, 0)| \ : x \in \mathbb{R}^N\} = C < +\infty$ *and $f \in C(\mathbb{R})$ be a continuous time periodic function. Then there is a bounded lipschitz time periodic viscosity solution of* (1) *iff there is a bounded continuous viscosity solution of* (12).

Another interesting problem is the long time behaviour for the viscosity solution of the Cauchy problem (2). For some classes of initial conditions we show the convergence towards a time periodic viscosity solution of (1), see Proposition 7. The proof relies on monotonicity and stability arguments. At this stage let us point out that asymptotic behaviours of Hamilton-Jacobi equations in periodic settings have been studied essentially for hamiltonians independent on u and which are periodic in the x variable, cf. [3], [11], [15], [16], notably since the paper of Lions, Papanicolaou, Varadhan [14] concerning the homogenization of such equations. These space periodic solutions are generally shown to converge, as $t \to +\infty$, to steady solutions or to travelling waves. We note here the relationship with ergodic problems as the speed of the underlying waves appears as the ergodic constant related to the solvability of

$$H(x, Du) = \lambda, \ \ x \in \mathbb{R}^N.$$

In the space-time periodic case, one may expect the existence of space-time periodic solutions φ of (1) and then the convergence of the solution of the initial value problem towards φ when $t \to +\infty$. We mention here the paper by Roquejoffre [17] where this programme is carried out under appropriate assumptions including the strict convexity of the hamiltonian. His results, whose proofs essentially come from the dynamical system theory and which call for the Aubry-Mather set, may be viewed as an extension, to the time dependent case, of Fathi's result [11], where convergence to travelling fronts are proved for strictly convex hamiltonians.

However it is worth noting that in general, convergence to space-time periodic solutions fails *i.e.,* the results of [17] cannot be extended to more general hamiltonians, see counterexamples in [2], [12]. Coming back to this work, in this setting, our results roughly say that for some classes of initial conditions, we have convergence towards periodic solutions iff $\lambda = \langle f \rangle$. Moreover no convexity argument is required. We study also the asymptotic behaviour of time periodic viscosity solutions for high frequencies. Let us analyze our model equation (13), with f a T periodic function. Introduce also $f_n(t) = f(nt), \forall t \in \mathbb{R}, \ n \geq 1$, which is $\frac{T}{n}$ periodic and has the same average as f. Suppose that $\langle f \rangle \in g(\mathbb{R})$ and let x_n be a $\frac{T}{n}$ periodic solution of

$$x_n'(t) + g(x_n(t)) = f_n(t), \ t \in \mathbb{R},$$

such that $\sup_n \|x_n\|_\infty < +\infty$. We are interested in the limit of $(x_n)_n$ when the period goes to 0.

After the change of variable $y_n(t) = x_n(\frac{t}{n})$ we deduce that y_n are T periodic and solves $n \cdot y_n'(t) + g(y_n(t)) = f(t), \ t \in \mathbb{R}, \ n \geq 1$. We can guess that $(y_n)_n$ converges uniformly to a constant y_0 and since $\int_0^T g(y_n(t))\,dt = \int_0^T f(t)\,dt$, $\forall\, n \geq 1$ we deduce that $g(y_0) = \langle f\rangle$. Thus we obtain that $(x_n)_n$ converges towards a solution of $g(x) = \langle f\rangle$. The same result holds in the context of minimal l.s.c. viscosity supersolutions, resp. maximal u.s.c. viscosity subsolutions for equations (1), (12) (see Section 5 for the definitions of minimal l.s.c. supersolution, resp. maximal u.s.c. subsolution). We have the theorem

Theorem 5.1 *Let $H = H(x,z,p)$ satisfy* (10), (6), (8), $H(x,-M,0) \leq f(t)$, $\forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}$ *for some $M > 0$, where f is a T periodic continuous function. Suppose also that there is a bounded l.s.c. viscosity supersolution $\tilde{V} \geq -M$ of* (12) *and denote by V, v_n the minimal l.s.c. viscosity supersolutions of* (12), *resp.* $\partial_t v_n + H(x, v_n, Dv_n) = f_n(t)$, *in $\mathbb{R}^N \times \mathbb{R}$. Then the sequence $(v_n)_n$ converges uniformly on $\mathbb{R}^N \times \mathbb{R}$ towards V and* $\|v_n - V\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \leq \frac{1}{n}\|f - \langle f\rangle\|_{L^1(0,T)}$, $\forall n \geq 1$.

The above result may be assimilated to a homogenization process where the period goes to zero. Note also that we have a 'more regular' version of this theorem when the hamiltonian satisfies the coercivity condition (7), see Remark 7.

The paper is organized as follows. In Section 2, after recalling a few basic results on viscosity solutions we give a comparison result which will be one of the key points in our proofs. Section 3 is devoted to the case where the hamiltonian satisfies the strong monotonicity condition (5). We show the existence of a unique time periodic solution which is the limit of any corresponding initial value problem's solution. In Section 4 we deal with the case where the hamiltonian satisfies the weaker condition (10) and prove Theorem 5 for coercif hamiltonians. Moreover we analyze the long time behaviour of the solutions as well as the relation with ergodic problems. In the next section we look at the asymptotic behaviour of time periodic solutions for large frequencies. We end up with some generalizations for the case of time almost-periodic solutions.

2. **Preliminaries.** In this section we recall some basic properties of viscosity solutions. We present also a slightly improved version of comparison result. Let H be a T periodic continuous function and u a viscosity subsolution (resp. supersolution) of the equation

$$\partial_t u + H(x,t,u,Du) = 0, \ \ (x,t) \in \mathbb{R}^N \times]0,T[. \tag{14}$$

Note that if u is a T periodic viscosity subsolution (resp. supersolution) of (14), then u is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x,t,u,Du) = 0$ in $\mathbb{R}^N \times \mathbb{R}$. This is essentially due to the following classical result (see [1]).

Lemma 1. *Assume that $H \in C(\mathbb{R}^N \times]0,T] \times \mathbb{R} \times \mathbb{R}^N)$ and $u \in C(\mathbb{R}^N \times]0,T])$ is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x,t,u,Du) = 0, (x,t) \in \mathbb{R}^N \times]0,T[$. Then u is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x,t,u,Du) = 0, (x,t) \in \mathbb{R}^N \times]0,T]$.*

Now by time periodicity one gets

Proposition 2. *Assume that $H \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N)$ and $u \in C(\mathbb{R}^N \times \mathbb{R})$ are T periodic such that u is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x,t,u,Du) = 0$, $(x,t) \in \mathbb{R}^N \times]0,T[$. Then u is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x,t,u,Du) = 0$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.*

Let us recall a few results concerning the stationary equation (4). We have the following comparison result (see [1]).

Theorem 1. *Let u, v be bounded u.s.c. (upper semi continuous) subsolution, resp. l.s.c. (lower semi continuous) supersolution of* (4). *We assume that* (5), (6), (8) *hold. Then we have $u(x) \leq v(x), \quad \forall\ x \in \mathbb{R}^N$. Moreover the hypothesis* (6) *can be replaced by $u \in W^{1,\infty}(\mathbb{R}^N)$ or $v \in W^{1,\infty}(\mathbb{R}^N)$.*

The main existence result for (4) is given by the following theorem (see [1]). We use the notation $BUC(X) = \{v \in C(X) : v$ is bounded, uniformly continuous on $X\}$.

Theorem 2. *Assume that* (5), (6), (8), (9) *hold. Then there is a unique viscosity solution $u \in BUC(\mathbb{R}^N)$ of* (4).

For the time dependent case we have the following comparison result (see [1]).

Theorem 3. *Let u, v be bounded u.s.c. subsolution, resp. l.s.c. supersolution of* (14). *We assume that* (5), (6), (8) *hold (with $\gamma_R \in \mathbb{R}$ not necessarily positive),*

$$\lim_{t\searrow 0}(u(x,t) - u(x,0))_+ = \lim_{t\searrow 0}(v(x,t) - v(x,0))_- = 0, \text{ uniformly for } x \in \mathbb{R}^N, \quad (15)$$

(here $(\cdot)_\pm$ denotes the positive/negative part $a_\pm = \max(\pm a, 0)$, $\forall\ a \in \mathbb{R}$) and

$$u(\cdot,0) \in BUC(\mathbb{R}^N) \text{ or } v(\cdot,0) \in BUC(\mathbb{R}^N). \quad (16)$$

Then we have $e^{\gamma t}\sup_{x\in\mathbb{R}^N}(u(x,t) - v(x,t))_+ \leq \sup_{x\in\mathbb{R}^N}(u(x,0) - v(x,0))_+$, $\forall\ t \in [0,T]$, where $\gamma = \gamma_{R_0}$, $R_0 = \max(\sup_{\mathbb{R}^N\times[0,T]} u, -\inf_{\mathbb{R}^N\times[0,T]} v)$. If the hypotheses (15), (16) *are not verified, we have*

$$e^{\gamma t}\sup_{x\in\mathbb{R}^N}(u(x,t) - v(x,t))_+ \leq (\sup_{\mathbb{R}^N\times[0,T]} u - \inf_{\mathbb{R}^N\times[0,T]} v)_+ \leq 2R_0.$$

Moreover, the hypothesis (6) *can be replaced by $u \in W^{1,\infty}(\mathbb{R}^N\times]0,T[)$ or $v \in W^{1,\infty}(\mathbb{R}^N\times]0,T[)$.*

Corollary 1. *Let u, v be bounded u.s.c. subsolution of $\partial_t u + H(x,t,u,Du) = f(x,t)$ in $\mathbb{R}^N\times]0,T[$, resp. l.s.c. supersolution of $\partial_t v + H(x,t,v,Dv) = g(x,t)$ in $\mathbb{R}^N\times]0,T[$ where $f, g \in BUC(\mathbb{R}^N\times[0,T])$. Then under the assumptions of Theorem* 3 *we have for all $t \in [0,T]$*

$$e^{\gamma t}\|(u(\cdot,t) - v(\cdot,t))_+\|_{L^\infty(\mathbb{R}^N)} \leq \|(u(\cdot,0) - v(\cdot,0))_+\|_{L^\infty(\mathbb{R}^N)} + \int_0^t e^{\gamma s}\|(f(\cdot,s) - g(\cdot,s))_+\|_{L^\infty(\mathbb{R}^N)}ds, \quad (17)$$

where $\gamma = \gamma_{R_0}$, $R_0 = \max(\sup_{\mathbb{R}^N\times[0,T]} u, -\inf_{\mathbb{R}^N\times[0,T]} v)$.

The main existence result for (3) is given by the following theorem (see [19], [1]).

Theorem 4. *Assume that* (5), (6), (8), (9) *hold (with $\gamma_R \in \mathbb{R}$, $\forall\ R > 0$). Then for every $u_0 \in BUC(\mathbb{R}^N)$ there is a unique viscosity solution $u \in BUC(\mathbb{R}^N\times[0,T])$ of* (3), $\forall\ T > 0$.

We end this section with the following comparison result for semi continuous viscosity solutions.

Proposition 3. *Let u a bounded viscosity u.s.c subsolution of $\partial_t u + H(x,t,u,Du) = f(x,t)$ in $\mathbb{R}^N\times]0,T[$ and v a bounded viscosity l.s.c. supersolution of $\partial_t v + H(x,t,v,Dv) = g(x,t)$ in $\mathbb{R}^N\times]0,T[$, where $f, g \in BUC(\mathbb{R}^N\times[0,T])$. We assume that* (5), (6), (8), (15), (16) *hold (with $\gamma_R \geq 0, \forall\ 0 < R < +\infty$). Then we have for all $t \in [0,T]$*

$$\sup_{x\in\mathbb{R}^N}(u(x,t) - v(x,t)) \leq e^{-\gamma t}\sup_{x\in\mathbb{R}^N}(u(x,0) - v(x,0))_+ + \sup_{0\leq s\leq t}\int_s^t \sup_{x\in\mathbb{R}^N}(f(x,\sigma) - g(x,\sigma))\, d\sigma, \quad (18)$$

where $\gamma = \gamma_{R_0}$, $R_0 = \max(\sup_{\mathbb{R}^N\times[0,T]} u, -\inf_{\mathbb{R}^N\times[0,T]} v)$. *If the hypotheses* (15), (16) *are not verified, we have*

$$\sup_{x\in\mathbb{R}^N}(u(x,t)-v(x,t)) \le 2R_0\cdot e^{-\gamma t} + \sup_{0\le s\le t}\int_s^t \sup_{x\in\mathbb{R}^N}(f(x,\sigma)-g(x,\sigma))\,d\sigma.$$

Moreover, the hypothesis (6) *can be replaced by* $u \in W^{1,\infty}(\mathbb{R}^N\times]0,T[)$ *or* $v \in W^{1,\infty}(\mathbb{R}^N\times]0,T[)$.

Remark 1. Note that the main difference between (17) and (18) is that in the right hand side term of (18) we have now $\sup_{\mathbb{R}^N}(f(\cdot,\sigma)-g(\cdot,\sigma))$ and not $\sup_{\mathbb{R}^N}(f(\cdot,\sigma)-g(\cdot,\sigma))_+$.

Proof. Let us fix $t\in[0,T]$. We denote by $h:[0,T]\to\mathbb{R}$ the function $h(s) = \sup_{x\in\mathbb{R}^N}(f(x,s)-g(x,s))$, $s\in[0,T]$. Consider the function $w:\mathbb{R}^N\times[0,t]\to\mathbb{R}$ given by

$$w(x,s) = v(x,s) + \int_0^s h(\sigma)\,d\sigma + \sup_{0\le\tau\le t}\left(-\int_0^\tau h(\sigma)\,d\sigma\right),\ \ (x,s)\in\mathbb{R}^N\times[0,t].$$

It is easily seen that w is a bounded viscosity l.s.c. supersolution of $\partial_s w + H = f$, $(x,s)\in\mathbb{R}^N\times]0,t[$, since $w\ge v$ on $\mathbb{R}^N\times[0,t]$ and H is nondecreasing with respect to the third variable (we use here $\gamma_R\ge 0, \forall\, R>0$). We have

$$R_0 \ge \max(\sup_{\mathbb{R}^N\times[0,t]} u, -\inf_{\mathbb{R}^N\times[0,t]} v) \ge \max(\sup_{\mathbb{R}^N\times[0,t]} u, -\inf_{\mathbb{R}^N\times[0,t]} w),$$

and by Theorem 3 we deduce that for any $(x,s)\in\mathbb{R}^N\times[0,t]$

$$e^{\gamma s}(u(x,s)-w(x,s)) \le \sup_{y\in\mathbb{R}^N}(u(y,0)-w(y,0))_+ \le \sup_{y\in\mathbb{R}^N}(u(y,0)-v(y,0))_+,$$

implying that

$$u(x,s)-v(x,s) \le e^{-\gamma s}\sup_{y\in\mathbb{R}^N}(u(y,0)-v(y,0))_+ + \int_0^s h(\sigma)\,d\sigma + \sup_{0\le\tau\le t}\left(-\int_0^\tau h(\sigma)\,d\sigma\right).$$

In particular for $s=t$ one gets

$$u(x,t)-v(x,t) \le e^{-\gamma t}\sup_{y\in\mathbb{R}^N}(u(y,0)-v(y,0))_+ + \sup_{0\le\tau\le t}\left(\int_\tau^t h(\sigma)\,d\sigma\right),\ \forall\, x\in\mathbb{R}^N.$$

If (15), (16) are not verified we have by Theorem 3

$$e^{\gamma s}(u(x,s)-w(x,s)) \le (\sup_{\mathbb{R}^N\times[0,t]} u - \inf_{\mathbb{R}^N\times[0,t]} w)_+ \le (\sup_{\mathbb{R}^N\times[0,t]} u - \inf_{\mathbb{R}^N\times[0,t]} v)_+ \le 2R_0,$$

and therefore

$$u(x,s)-v(x,s) \le 2R_0 e^{-\gamma s} + \int_0^s h(\sigma)\,d\sigma + \sup_{0\le\tau\le t}\left(-\int_0^\tau h(\sigma)\,d\sigma\right),\ \forall\, x\in\mathbb{R}^N.$$

Our conclusion follows by taking $s=t$. □

Corollary 2. *Let* u *be a bounded time periodic viscosity u.s.c subsolution of* $\partial_t u + H(x,t,u,Du) = f(x,t)$ *in* $\mathbb{R}^N\times\mathbb{R}$ *and* v *a bounded time periodic viscosity l.s.c. supersolution of* $\partial_t v + H(x,t,v,Dv) = g(x,t)$ *in* $\mathbb{R}^N\times\mathbb{R}$, *where* $f,g\in BUC(\mathbb{R}^N\times\mathbb{R})$ *and* H *are* T *periodic such that* (5), (6), (8) *hold (with* $\gamma_R>0,\ \forall\, R>0$*). Then we have*

$$\sup_{x\in\mathbb{R}^N}(u(x,t)-v(x,t)) \le \sup_{s\le t}\int_s^t \sup_{x\in\mathbb{R}^N}(f(x,\sigma)-g(x,\sigma))\,d\sigma.$$

Moreover, the hypothesis (6) *can be replaced by* $u \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$ *or* $v \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$.

Proof. By using the time periodicity and Proposition 3 we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^N}(u(x,t)-v(x,t)) &= \sup_{x\in\mathbb{R}^N}(u(x,t+nT)-v(x,t+nT))\\ &\leq e^{-\gamma(t+nT)}(\|u\|_{L^\infty}+\|v\|_{L^\infty})\\ &\quad+\sup_{0\leq s\leq t+nT}\int_s^{t+nT}\sup_{x\in\mathbb{R}^N}(f(x,\sigma)-g(x,\sigma))\,d\sigma. \quad (19)\end{aligned}$$

Observe that

$$\begin{aligned}\int_s^{t+nT}\sup_{x\in\mathbb{R}^N}(f(x,\sigma)-g(x,\sigma))\,d\sigma &= \int_s^{t+nT}\sup_{x\in\mathbb{R}^N}(f(x,\sigma-nT)-g(x,\sigma-nT))\,d\sigma\\ &= \int_{s-nT}^{t}\sup_{x\in\mathbb{R}^N}(f(x,\sigma)-g(x,\sigma))\,d\sigma\\ &\leq \sup_{r\leq t}\int_r^t\sup_{x\in\mathbb{R}^N}(f(x,\sigma)-g(x,\sigma))\,d\sigma.\end{aligned}$$

The conclusion follows by letting $n \to +\infty$ in (19). □

Remark 2. Assume that the hypotheses of Corollary 2 hold. If u, v are bounded time periodic viscosity solutions of $\partial_t u + H(x,t,u,Du) = f(x,t)$ in $\mathbb{R}^N \times \mathbb{R}$, resp. $\partial_t v + H(x,t,v,Dv) = g(x,t)$ in $\mathbb{R}^N \times \mathbb{R}$, then we have for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$

$$\inf_{s\leq t}\int_s^t\inf_{y\in\mathbb{R}^N}(f(y,\sigma)-g(y,\sigma))d\sigma \leq u(x,t)-v(x,t) \leq \sup_{s\leq t}\int_s^t\sup_{y\in\mathbb{R}^N}(f(y,\sigma)-g(y,\sigma))d\sigma.$$

Remark 3. Note that under the hypotheses of Corollary 2 we have a strong comparison result for discontinuous time periodic sub/supersolutions, *i.e.*, if u, v are bounded time periodic viscosity u.s.c. subsolution, resp. l.s.c. supersolution of (11), then we have $u(x,t) \leq v(x,t)$, $\forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}$.

3. **Time periodic viscosity solution when $\gamma_R > 0$.** In this section we study the existence and uniqueness of time periodic viscosity solution of (11) when hypothesis (5) holds. This is a direct consequence of the results given in Section 2.

Proposition 4. *Let* $u, v \in BUC(\mathbb{R}^N \times \mathbb{R})$ *be bounded time periodic viscosity subsolution, resp. supersolution of* (11) *where* $H \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N)$ *is time periodic. We assume that* (5), (6), (8) *hold (with* $\gamma_R > 0$, $\forall\ R > 0$*). Then we have*

$$u(x,t) \leq v(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Moreover, the hypothesis (6) *can be replaced by* $u \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$ *or* $v \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$.

Proof. By Theorem 3 we have

$$e^{\gamma T}\|(u(\cdot,T)-v(\cdot,T))_+\|_{L^\infty(\mathbb{R}^N)} \leq \|(u(\cdot,0)-v(\cdot,0))_+\|_{L^\infty(\mathbb{R}^N)},$$

where $\gamma = \gamma_{R_0} > 0$, $R_0 = \max(\|u\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})}, \|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})})$. By periodicity we have $u(\cdot,T) - v(\cdot,T) = u(\cdot,0) - v(\cdot,0)$ and since $e^{\gamma T} > 1$ we deduce that $(u(x,0) - v(x,0))_+ \leq 0$, $\forall x \in \mathbb{R}^N$. The conclusion follows by Theorem 3. □

Corollary 3. *Under the assumptions of Proposition* 4 *there is at most one time periodic viscosity solution of* (11).

For the existence part we solve the problem (3) with arbitrary initial condition $u_0 \in BUC(\mathbb{R}^N)$ and we pass to the limit for $t \to +\infty$.

Proposition 5. *Let $u_0 \in BUC(\mathbb{R}^N)$ and assume that* (5), (6), (8), (9) *hold (with $\gamma_R > 0$, $\forall\, R > 0$). Denote by $u \in C(\mathbb{R}^N \times [0, +\infty[)$ the unique viscosity solution of* (3) *and let $u_n(x,t) = u(x, t + nT)$, $(x,t) \in \mathbb{R}^N \times [0,T]$ for $n \geq 0$. Then $(u_n)_n$ converges uniformly on $\mathbb{R}^N \times [0,T]$ to a time periodic viscosity solution of* (11).

Proof. By hypothesis (9) we deduce that $-M$ (resp. M) is subsolution (resp. supersolution) of (3) and by Theorem 3 we have

$$-M - \|(-u_0 - M)_+\|_\infty \leq u(x,t) \leq \|(u_0 - M)_+\|_\infty + M, \ \ \forall (x,t) \in \mathbb{R}^N \times [0, +\infty[.$$

Let $\gamma = \gamma_{R_0} > 0$ where $R_0 = \|u\|_{L^\infty(\mathbb{R}^N \times]0,+\infty[)}$. Consider $v(x,t) = u(x, t+T)$, $\forall (x,t) \in \mathbb{R}^N \times [0, +\infty[$. By the periodicity of H we deduce that v is viscosity solution of $\partial_t v + H(x,t,v,Dv) = 0$, $(x,t) \in \mathbb{R}^N \times [0,+\infty[$. By using Theorem 3 we have

$$\begin{aligned}
\|u(\cdot, t+T) - u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} &= \|v(\cdot,t) - u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \\
&\leq e^{-\gamma t}\|v(\cdot,0) - u(\cdot,0)\|_{L^\infty(\mathbb{R}^N)} \\
&\leq 2e^{-\gamma t}\|u\|_{L^\infty(\mathbb{R}^N \times]0,+\infty[)}.
\end{aligned}$$

In particular, by taking $t = s + nT$, $s \in [0,T]$ we deduce that

$$\|u_{n+1}(\cdot, s) - u_n(\cdot, s)\|_{L^\infty(\mathbb{R}^N)} \leq 2e^{-nT\gamma}\|u\|_{L^\infty(\mathbb{R}^N \times]0,+\infty[)},$$

and therefore there is $w \in C(\mathbb{R}^N \times \mathbb{R})$, T periodic such that $u_n \to w|_{\mathbb{R}^N \times [0,T]}$ uniformly on $\mathbb{R}^N \times [0,T]$. Now by using the stability result for continuous viscosity solutions (see [9], [1]) we deduce that w is viscosity solution of (11) in $\mathbb{R}^N \times]0,T[$. Therefore by Proposition 2 w is periodic viscosity solution of (11) in $\mathbb{R}^N \times \mathbb{R}$. Note also that $w \in BUC(\mathbb{R}^N \times \mathbb{R})$, since $u_n \in BUC(\mathbb{R}^N \times [0,T])$, $\forall\, n$. □

Since we have uniqueness of the time periodic viscosity solution of (11), the solution constructed above does not depend on the initial condition u_0.

4. **Time periodic viscosity solution when $\gamma_R = 0$.** In this section we study the time periodic viscosity solutions of

$$\partial_t u + H(x, u, Du) = f(t), \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{20}$$

for hamiltonians satisfying (10). We introduce also the stationary equation

$$H(x,u,Du) = \langle f \rangle := \frac{1}{T}\int_0^T f(t)\, dt, \ \ x \in \mathbb{R}^N. \tag{21}$$

4.1. **Existence of time periodic viscosity solution.** For coercif hamiltonians *i.e.,* hamiltonians verifying (7), we have the following necessary and sufficient condition for the existence of time periodic viscosity solution.

Theorem 5. *Let $H = H(x,z,p)$ be a hamiltonian verifying* (10), (7), (8) *and* $\sup\{|H(x,0,0)| \ : x \in \mathbb{R}^N\} = C < +\infty$ *and $f \in C(\mathbb{R})$ be a continuous time periodic function. Then there is a bounded lipschitz time periodic viscosity solution of* (20) *iff there is a bounded continuous viscosity solution of* (21).

Proof. Assume that there is a bounded viscosity solution V of (21). Since the hamiltonian satisfies (7) we can prove as usual that V is a lipschitz function. For any $\alpha > 0$ take $M_\alpha = \|V\|_{L^\infty(\mathbb{R}^N)} + \frac{1}{\alpha}(C + \|f\|_{L^\infty(\mathbb{R})})$ and observe that

$$\alpha(-M_\alpha - V(x)) + H(x, -M_\alpha, 0) \le f(t) \le \alpha(M_\alpha - V(x)) + H(x, M_\alpha, 0), \ \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Therefore we can construct the stationary viscosity solution V_α of

$$\alpha(V_\alpha - V(x)) + H(x, V_\alpha, DV_\alpha) = \langle f \rangle, \ \ x \in \mathbb{R}^N,$$

and the time periodic viscosity solution v_α of

$$\alpha(v_\alpha - V(x)) + \partial_t v_\alpha + H(x, v_\alpha, Dv_\alpha) = f(t), \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{22}$$

In fact we have $V_\alpha = V$, $\forall \alpha > 0$ and by using Corollary 2 we obtain

$$|v_\alpha(x,t) - V(x)| = |v_\alpha(x,t) - V_\alpha(x)| \le \|f - \langle f \rangle\|_{L^1(0,T)}, \ \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \forall \alpha > 0, \tag{23}$$

which implies that $(v_\alpha)_\alpha$ is uniformly bounded

$$\|v_\alpha\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \le \|V\|_{L^\infty(\mathbb{R}^N)} + \|f - \langle f \rangle\|_{L^1(0,T)}, \ \ \forall \alpha > 0.$$

In order to extract a subsequence which converges uniformly on compact sets we prove that $(v_\alpha)_\alpha$ are uniformly lipschitz on $\mathbb{R}^N \times \mathbb{R}$. For this note that $w_\alpha(x,t) = v_\alpha(x, t+h)$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ is time periodic viscosity solution of

$$\alpha \cdot (w_\alpha - V(x)) + \partial_t w_\alpha + H(x, w_\alpha, Dw_\alpha) = f(t+h), \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{24}$$

By using Corollary 2 we have

$$\begin{aligned} v_\alpha(x,t+h) - v_\alpha(x,t) &\le \sup_{s \le t} \int_s^t (f(\sigma + h) - f(\sigma))\, d\sigma \\ &= \sup_{s \le t} \left\{ \int_t^{t+h} f(\sigma)\, d\sigma - \int_s^{s+h} f(\sigma)\, d\sigma \right\} \\ &\le 2|h| \cdot \|f\|_{L^\infty(\mathbb{R})}, \ \ \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}, h \in \mathbb{R}. \end{aligned} \tag{25}$$

Let us prove now that v_α are uniformly lipschitz with respect to x. Take $K > 0$ such that $H(x,z,p) \ge 3 \cdot \|f\|_{L^\infty(\mathbb{R})} + 1$, for $x \in \mathbb{R}^N, z \in \mathbb{R}, |z| \le \sup_{\alpha > 0} \|v_\alpha\|_{L^\infty}$, $p \in \mathbb{R}^N, |p| \ge K$. For $\varepsilon > 0$, $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$ fixed we consider the function $\psi(x,t) = v_\alpha(x,t) - K \cdot |x-y| - \frac{|t-\tau|^2}{\varepsilon^2}$ and let $(x_0, t_0) = (x_0(\alpha, \varepsilon, y, \tau), t_0(\alpha, \varepsilon, y, \tau))$ a maximum point of ψ. The inequality $\psi(x_0, t) \le \psi(x_0, t_0)$, $\forall t \in \mathbb{R}$ implies

$$v_\alpha(x_0, t) - \frac{|t-\tau|^2}{\varepsilon^2} \le v_\alpha(x_0, t_0) - \frac{|t_0 - \tau|^2}{\varepsilon^2}, \ \ \forall t \in \mathbb{R},$$

and therefore by (25) we find $2 \cdot \frac{|t_0 - \tau|}{\varepsilon^2} \le 2 \cdot \|f\|_{L^\infty(\mathbb{R})}$. Suppose now that $x_0 \neq y$ and thus we have the viscosity inequality

$$\alpha \cdot (v_\alpha(x_0,t_0) - V(x_0)) + 2 \cdot \frac{t_0 - \tau}{\varepsilon^2} + H\left(x_0, v_\alpha(x_0,t_0), K \cdot \frac{x_0 - y}{|x_0 - y|}\right) \le f(t_0),$$

which implies

$$1 + 3 \cdot \|f\|_{L^\infty(\mathbb{R})} \le H\left(x_0, v_\alpha(x_0,t_0), K \cdot \frac{x_0 - y}{|x_0 - y|}\right) \le 3 \cdot \|f\|_{L^\infty(\mathbb{R})} + \alpha(\|v_\alpha\|_{L^\infty} + \|V\|_{L^\infty}).$$

This clearly gives a contradiction for α small and therefore we deduce that

$$\psi(x,t_0) = v_\alpha(x,t_0) - K \cdot |x-y| - \frac{|t_0 - \tau|^2}{\varepsilon^2} \le v_\alpha(y,t_0) - \frac{|t_0 - \tau|^2}{\varepsilon^2} = \psi(y,t_0),$$

which implies $v_\alpha(x,t_0) \le v_\alpha(y,t_0) + K\cdot|x-y|$ for all $x,y\in\mathbb{R}^N$, $\tau\in\mathbb{R}$, $\alpha,\varepsilon>0$. By passing to the limit for $\varepsilon \searrow 0$ (note that $|t_0-\tau| \le \varepsilon^2\cdot\|f\|_{L^\infty(\mathbb{R})}$ and thus $t_0\to\tau$) we obtain $v_\alpha(x,\tau) \le v_\alpha(y,\tau)+K\cdot|x-y|$, $\forall x,y\in\mathbb{R}^N, \tau\in\mathbb{R}$. Therefore the functions $(v_\alpha)_{\alpha>0}$ are uniformly lipschitz and we can extract a subsequence which converges uniformly on compact sets of $\mathbb{R}^N\times\mathbb{R}$ to a bounded lipschitz function v. By using the stability result for continuous viscosity solutions we deduce that v is a viscosity solution of (20).

Assume now that there is a bounded lipschitz time periodic viscosity solution v of (20). For any $\alpha>0$ take $\tilde{M}_\alpha = \frac{1}{\alpha}(C+\|f\|_{L^\infty(\mathbb{R})})$ and observe that

$$-\alpha\tilde{M}_\alpha + H(x,-\tilde{M}_\alpha,0) \le f(t) \le \alpha\tilde{M}_\alpha + H(x,\tilde{M}_\alpha,0),\ \forall\ (x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

Therefore we can construct the time periodic viscosity solution v_α of

$$\alpha v_\alpha + \partial_t v_\alpha + H(x,v_\alpha,Dv_\alpha) = f(t),\ \ (x,t)\in\mathbb{R}^N\times\mathbb{R},$$

and the stationary viscosity solution V_α of

$$\alpha V_\alpha + H(x,V_\alpha,DV_\alpha) = \langle f\rangle,\ \ x\in\mathbb{R}^N.$$

Since v is bounded, lipschitz and time periodic, it is also time periodic viscosity solution of $\alpha v + \partial_t v + H(x,v,Dv) = f(t) + \alpha\ v(x,t)$, $(x,t)\in\mathbb{R}^N\times\mathbb{R}$. By using Corollary 1 we deduce that for all $t\ge t_0$, $\alpha>0$ we have

$$\begin{aligned}\|v_\alpha(\cdot,t)-v(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} &\le e^{-\alpha(t-t_0)}\|v_\alpha(\cdot,t_0)-v(\cdot,t_0)\|_{L^\infty(\mathbb{R}^N)}\\ &\quad + e^{-\alpha t}\int_{t_0}^t e^{\alpha s}\alpha\ \|v(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}\ ds\\ &\le e^{-\alpha(t-t_0)}(\|v_\alpha\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})}+\|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})})\\ &\quad +\|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})}.\end{aligned}$$

By passing to the limit for $t_0\to-\infty$ we obtain

$$\|v_\alpha(\cdot,t)-v(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})},\ \ \forall t\in\mathbb{R},\alpha>0,$$

and therefore $\sup_{\alpha>0}\|v_\alpha\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \le 2\|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})}$. By using Corollary 2 one gets

$$|V_\alpha(x)-v_\alpha(x,t)| \le \|f-\langle f\rangle\|_{L^1(0,T)},\ \ \forall (x,t)\in\mathbb{R}^N\times\mathbb{R},\forall\alpha>0,$$

and therefore $(V_\alpha)_\alpha$ is uniformly bounded

$$\sup_{\alpha>0}\|V_\alpha\|_{L^\infty(\mathbb{R}^N)} \le 2\|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} + \|f-\langle f\rangle\|_{L^1(0,T)}.$$

The conclusion follows easily by observing that $(V_\alpha)_\alpha$ are uniformly lipschitz (use hypothesis (7)) and by extracting a subsequence convergent on compact sets of $\mathbb{R}^N$ (use also the stability result for continuous viscosity solutions). □

Remark 4. Note that in the above result we do not need hypothesis (6) which is replaced by (7) just as in the stationary case. In fact, the uniform bound on $\partial_t u$ comes from the autonomous character of the hamiltonian H and of the use of Corollary 2. The coercivity condition (7) then allows to obtain the lipschitz estimate in x.

In the above computations we have used several times Corollary 2. Actually, similar conclusions can be obtained by using standard comparison results. For example, we check easily that $V(x) + \int_0^t \{f(s) - \langle f\rangle\}\ ds \mp \|f - \langle f\rangle\|_{L^1(0,T)}$ is a sub/supersolution of (22) and therefore we deduce for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $\alpha > 0$

$$\int_0^t \{f(s)-\langle f\rangle\}\ ds - \|f-\langle f\rangle\|_{L^1(0,T)} \leq v_\alpha(x,t) - V(x) \leq \int_0^t \{f(s)-\langle f\rangle\}\ ds + \|f-\langle f\rangle\|_{L^1(0,T)}.$$

We obtain the inequality

$$|v_\alpha(x,t) - V(x)| \leq 2\ \|f - \langle f\rangle\|_{L^1(0,T)},\ \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R},\ \forall\ \alpha > 0,$$

which is similar to (23). In the same manner, by observing that $v_\alpha + \int_0^t \{f(s+h) - f(s)\}\ ds \mp 2\ |h|\ \|f\|_{L^\infty(\mathbb{R})}$ is a sub/supersolution of (24), we deduce that

$$\begin{aligned}\int_0^t \{f(s+h) - f(s)\}ds - 2|h|\|f\|_{L^\infty} &\leq v_\alpha(x,t+h) - v_\alpha(x,t)\\ &\leq \int_0^t \{f(s+h) - f(s)\}ds + 2|h|\|f\|_{L^\infty}.\end{aligned}$$

One gets the inequality

$$|v_\alpha(x,t+h) - v_\alpha(x,t)| \leq 4\ |h|\ \|f\|_{L^\infty(\mathbb{R})},\ \ (x,t) \in \mathbb{R}^N \times \mathbb{R},\ h \in \mathbb{R},\ \alpha > 0,$$

which is similar to (25). Let us mention that the above alternative proof was pointed to us by the referee to whom we are thankful.

The Theorem 5 establishes the equivalence between the solvability of (20) and (21). Therefore one gets existence of time periodic viscosity solution for (20) by imposing sufficient conditions for the existence of stationary viscosity solution for (21).

Theorem 6. *Let $H = H(x,z,p)$ be a hamiltonian verifying* (10), (7), (8) *and f be a continuous time periodic function. Assume that there is $M > 0$ such that*

$$H(x, -M, 0) \leq \langle f\rangle \leq H(x, M, 0),\ \ \forall x \in \mathbb{R}^N. \tag{26}$$

Then there is a bounded lipschitz time periodic viscosity solution v of (20) *satisfying*

$$-M - \|f - \langle f\rangle\|_{L^1(0,T)} \leq v(x,t) \leq M + \|f - \langle f\rangle\|_{L^1(0,T)},\ \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Proof. For any $\alpha > 0$ consider the stationary viscosity solution of

$$\alpha(V_\alpha + M) + H(x, V_\alpha, DV_\alpha) = \langle f\rangle,\ \ x \in \mathbb{R}^N.$$

By (26) we have $-M \leq V_\alpha(x) \leq M,\ \ x \in \mathbb{R}^N,\ \alpha > 0$. For any $\alpha > 0$ take $M_\alpha = M + \frac{2}{\alpha}\|f\|_{L^\infty(\mathbb{R})}$ and observe that

$$\alpha(-M_\alpha + M) + H(x, -M_\alpha, 0) \leq f(t) \leq \alpha(M_\alpha + M) + H(x, M_\alpha, 0),\ \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Therefore we can construct for all $\alpha > 0$ the time periodic viscosity solution of

$$\alpha(v_\alpha + M) + \partial_t v_\alpha + H(x, v_\alpha, Dv_\alpha) = f(t),\ \ (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

By using Corollary 2 we obtain

$$|v_\alpha(x,t)| \leq |V_\alpha(x)| + \|f - \langle f\rangle\|_{L^1(0,T)},\ \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R},\ \forall\ \alpha > 0.$$

As before we check that $(v_\alpha)_\alpha$ are uniformly lipschitz and we can extract a subsequence which converges uniformly on compact sets of $\mathbb{R}^N \times \mathbb{R}$ towards a lipschitz time periodic viscosity solution v of (20) satisfying

$$-M - \|f - \langle f\rangle\|_{L^1(0,T)} \leq v(x,t) \leq M + \|f - \langle f\rangle\|_{L^1(0,T)},\ \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

□

4.2. **Relation with ergodic problems and long time behaviour.** The Theorem 5 makes clear the central role played by the solvability of $H(x,u,Du) = \langle f\rangle$. We may in fact reformulate the problem in terms of ergodic constants. Indeed let us suppose that

$$\exists\,!\,\lambda \in \mathbb{R}\;:\; H(x,u,Du) = \lambda,\;\; x \in \mathbb{R}^N \quad \text{is solvable.} \tag{27}$$

In this context, under hypothesis (27) the Theorem 5 says that (20) admits a time periodic solution iff the ergodic constant is $\lambda = \langle f\rangle$. Let us mention a rather widely studied case where (27) is known to hold. It concerns coercif hamiltonians of the form $H = H(x,p)$, periodic in the x variable. The first classical result in this direction is due to Lions, Papanicolaou, Varadhan [14], where (27) appears as the cell problem in a homogenization process. Of course, our result remains valid in this case *i.e.,* a time periodic solution exists iff $\lambda = \langle f\rangle$. This space periodic setting, which ensures a compactness property of the domain, has been a privileged ground for the steady of long time behaviour of solutions of Hamilton-Jacobi equations, whether H depends on t or not. In this context, the ergodic constant often appears as the speed of the underlying travelling wave solution or of the periodic front, see for example [15], [3], [16], [11] for H independent of t and the recent papers by Roquejoffre [18], [17], Fathi and Mather [12] for hamiltonians which are also time periodic. We mention also the paper by Barles and Souganidis [4] where a similar analysis is carried out for quasi-linear parabolic equations.

Globally speaking, the results for time periodic hamiltonians state that there exists a unique $\mu \in \mathbb{R}$ such that

$$\partial_t\phi + H(x,t,D\phi) + \mu = 0,\;\; (x,t) \in \mathbb{R}^N\times]0,+\infty[\tag{28}$$

has space-time periodic solutions. Then under some "appropriate hypotheses" on the hamiltonian H notably its convexity with respect to p, they show convergence results of the type $\lim_{t\to+\infty}\{u(x,t) - \mu t - \phi(x,t)\} = 0$, uniformly for $x \in \mathbb{R}^N$, where ϕ is a space-time periodic solution of (28) and u is the solution of the initial value problem

$$\begin{cases} \partial_t u + H(x,t,Du) = 0, & (x,t) \in \mathbb{R}^N\times]0,+\infty[,\\ u(x,0) = u_0(x), & x \in \mathbb{R}^N.\end{cases}$$

It is worth mentioning here that generally there is no convergence. For example consider the problem (borrowed from [2])

$$\begin{cases} \partial_t u + |1 - \partial_x u| = 1, & (x,t) \in \mathbb{R}\times]0,+\infty[,\\ u(x,0) = \sin x, & x \in \mathbb{R},\end{cases} \tag{29}$$

whose solution is $u(x,t) = \sin(x+t)$, $\forall\,(x,t) \in \mathbb{R}\times[0,+\infty[$. Observe that for any $x \in \mathbb{R}$ there is no limit of $u(x,t)$ as t goes to infinity. We have the following result for hamiltonians non depending on u.

Proposition 6. *Let $H = H(x,p)$ be a hamiltonian which belongs to $BUC(\mathbb{R}^N \times \overline{B}_R)$ $\forall\; R > 0$ and satisfies $\lim_{|p|\to+\infty} H(x,p) = +\infty$ uniformly with respect to $x \in \mathbb{R}^N$, f a time periodic continuous function and u a viscosity solution of*

$$\partial_t u + H(x,Du) = f(t),\;\; (x,t) \in \mathbb{R}^N\times]0,+\infty[,$$

with a bounded lipschitz initial condition u_0. If there is a bounded viscosity solution U for $H(x,DU) = \lambda$, $x \in \mathbb{R}^N$ for some $\lambda \in \mathbb{R}$ then we have

$$|u(x,t) - (\langle f\rangle - \lambda)t| \le \|u_0\|_{L^\infty(\mathbb{R}^N)} + 2\|U\|_{L^\infty(\mathbb{R}^N)} + \|f - \langle f\rangle\|_{L^1(0,T)},\; (x,t) \in \mathbb{R}^N\times[0,\infty[.$$

In particular if $\lambda < \langle f \rangle$ *then* $\lim_{t\to+\infty} u(x,t) = +\infty$*, uniformly with respect to* $x \in \mathbb{R}^N$ *and if* $\lambda > \langle f \rangle$ *then* $\lim_{t\to+\infty} u(x,t) = -\infty$*, uniformly with respect to* $x \in \mathbb{R}^N$*.*

Proof. By the comparison result we obtain for any $(x,t) \in \mathbb{R}^N \times [0,+\infty[$

$$U(x) + \int_0^t \{f(s) - \lambda\}\, ds - M \leq u(x,t) \leq U(x) + \int_0^t \{f(s) - \lambda\}\, ds + M,$$

with $M = \|U - u_0\|_{L^\infty(\mathbb{R}^N)}$. We deduce that

$$\begin{aligned} |u(x,t) - (\langle f \rangle - \lambda)t| &\leq \|U\|_{L^\infty(\mathbb{R}^N)} + M + \left| \int_0^t \{f(s) - \langle f \rangle\}\, ds \right| \\ &\leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + 2\|U\|_{L^\infty(\mathbb{R}^N)} + \|f - \langle f \rangle\|_{L^1(0,T)}. \end{aligned}$$

□

We examine now the situation $\lambda = \langle f \rangle$. In this case we can consider hamiltonians which depend on u. We investigate the long time behaviour of viscosity solutions for the initial value problem

$$\begin{cases} \partial_t u + H(x,u,Du) = f(t), & (x,t) \in \mathbb{R}^N \times]0,+\infty[, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{30}$$

when the hamiltonian H satisfies (10), (7), (8) and f is a continuous time periodic function. The case when the hamiltonian verifies (5) instead of (10) is much easier : as shown in Proposition 5 we have convergence towards the unique time periodic viscosity solution of (20), with exponential decay.

Proposition 7. *Let* $H = H(x,z,p)$ *be a hamiltonian verifying* (10)*,* (7)*,* (8)*,* f *a continuous time periodic function and* $u_0 \in W^{1,\infty}(\mathbb{R}^N)$*. We assume that there is a bounded viscosity solution* U *for* $H(x,U,DU) = \langle f \rangle, \ x \in \mathbb{R}^N$*.*
1) Then there is a unique viscosity solution u *of* (30) *which belongs to* $W^{1,\infty}(\mathbb{R}^N \times]0,+\infty[)$ *;*
2) If the initial condition is such that $u_0(x) \leq u(x,T), \ \forall\, x \in \mathbb{R}^N$ *then we have*

$$\lim_{k\to+\infty} u(x,t+kT) = \varphi(x,t), \textit{uniformly for } x \textit{ in compact sets of } \mathbb{R}^N, t \geq 0,$$

where φ *is the minimal time periodic viscosity solution of* (20) *verifying* $\varphi(x,0) \geq u_0(x), x \in \mathbb{R}^N$ *;*
3) If the initial condition is such that $u_0(x) \geq u(x,T), \ \forall\, x \in \mathbb{R}^N$ *then we have*

$$\lim_{k\to+\infty} u(x,t+kT) = \Phi(x,t), \textit{uniformly for } x \textit{ in compact sets of } \mathbb{R}^N, t \geq 0,$$

where Φ *is the maximal time periodic viscosity solution of* (20) *verifying* $\Phi(x,0) \leq u_0(x), x \in \mathbb{R}^N$*.*

Before giving the proof, let us illustrate the previous results by the following example

$$\begin{cases} \partial_t u + |1 + \partial_x u| - 1 = f(t), & (x,t) \in \mathbb{R} \times]0,+\infty[, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{31}$$

where $f \in C(\mathbb{R})$ is T periodic and $u_0 \in W^{1,\infty}(\mathbb{R})$, such that $\|u_0'\|_{L^\infty(\mathbb{R})} \leq 1$. Then the solution of (31) is given by $u(x,t) = u_0(x-t) + \int_0^t f(s)\, ds, \ (x,t) \in \mathbb{R} \times [0,+\infty[$. Observe that the equation $|1 + U'(x)| - 1 = \lambda, x \in \mathbb{R}$ has bounded solutions for $\lambda = 0$. We can easily check that if $\langle f \rangle > 0$ we have $\lim_{t\to+\infty} u(x,t) = +\infty$ for any $x \in \mathbb{R}$ and if $\langle f \rangle < 0$ we have $\lim_{t\to+\infty} u(x,t) = -\infty$ for any $x \in \mathbb{R}$.

Assume now that $\langle f\rangle = 0$. Take as initial condition $u_0(x) = -\arctan x, x \in \mathbb{R}$ and observe that $u_0(x) \le u(x,T),\ \forall\, x \in \mathbb{R}$. In this case we have $\lim_{k\to+\infty} u(x,t+kT) = \frac{\pi}{2} + \int_0^t f(s)\ ds,\ \forall\,(x,t) \in \mathbb{R}\times[0,T]$. Take now as initial condition $u_0(x) = \sin x, x \in \mathbb{R}$ and assume that $\frac{T}{2\pi} \notin \mathbb{Z}$, implying that u_0 doesn't meet the hypotheses of statements $2), 3)$ of Proposition 7. We check easily that in this case there is no $\lim_{k\to+\infty} u(x,t+kT)$.

Proof. (of Proposition 7)
1) Take $M = \max\{\|u_0 - U\|_{L^\infty(\mathbb{R}^N)}, \|f - \langle f\rangle\|_{L^1(0,T)}\}$. We check easily that $U(x) + \int_0^t\{f(s) - \langle f\rangle\}\ ds \mp M$ are viscosity sub/supersolutions of (30) and therefore we deduce that there is a unique viscosity solution of (30) satisfying

$$U(x) + \int_0^t \{f(s) - \langle f\rangle\}\ ds - M \le u(x,t) \le U(x) + \int_0^t \{f(s) - \langle f\rangle\}\ ds + M.$$

Finally one gets for any $(x,t) \in \mathbb{R}^N \times [0,+\infty[$

$$|u(x,t)| \le \|u_0\|_{L^\infty(\mathbb{R}^N)} + 2\|U\|_{L^\infty(\mathbb{R}^N)} + 2\|f - \langle f\rangle\|_{L^1(0,T)} =: C_2. \tag{32}$$

By using the results in [19] we have

$$|u(x,h) - u_0(x)| \le C_1 h,\ \forall\,(x,h) \in \mathbb{R}^N \times [0,+\infty[,$$

where $C_1 = \sup\{|f(s) - H(y,z,p)| \ :\ (y,s,z,p) \in \mathbb{R}^N \times \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N, |z| \le \|u_0\|_{L^\infty(\mathbb{R}^N)}, |p| \le \|Du_0\|_{L^\infty(\mathbb{R}^N)}\}$. We obtain also for any $(x,t,h) \in \mathbb{R}^N \times [0,+\infty[^{\,2}$ by comparison results

$$-2h\|f\|_{L^\infty(\mathbb{R})} - C_1 h \le u(x,t+h) - u(x,t) - \int_0^t \{f(s+h) - f(s)\}ds \le 2h\|f\|_{L^\infty(\mathbb{R})} + C_1 h,$$

and therefore

$$|u(x,t+h) - u(x,t)| \le (4\|f\|_{L^\infty(\mathbb{R})} + C_1)h,\ \forall\,(x,t,h) \in \mathbb{R}^N \times [0,+\infty[^{\,2}. \tag{33}$$

By the hypothesis (7) and using similar arguments as those in the proof of Theorem 5 we can prove that u is lipschitz with respect to x, uniformly for $t \in [0,+\infty[$

$$|u(x,t) - u(y,t)| \le K \cdot |x-y|,\ \forall\, x,y \in \mathbb{R}^N, t \in [0,+\infty[. \tag{34}$$

We prove now the second statement, the last one following in a similar way.
2) Assume that

$$u_0(x) \le u(x,T),\ \forall\, x \in \mathbb{R}^N. \tag{35}$$

Consider the sequence of functions $(u^k)_{k\ge0}$ given by $u^k(x,t) = u(x,t+kT),\ \forall\,(x,t) \in \mathbb{R}^N \times [0,T],\ k \ge 0$. Since f is T periodic in time we have for any $k \ge 0$

$$\partial_t u^k + H(x,u^k,Du^k) = f(t),\ \ (x,t) \in \mathbb{R}^N \times [0,T]. \tag{36}$$

Observe that the sequence $(u^k)_{k\ge0}$ is nondecreasing. Indeed, by (36) it is sufficient to check that

$$u^k(x,0) \le u^{k+1}(x,0),\ \ x \in \mathbb{R}^N,\ k \ge 0. \tag{37}$$

By (35) observe that the above inequality holds true for $k = 0$. Assume now that (37) holds for some $k \ge 0$ and let us prove that the same inequality is valid for $k+1$. By the comparison result we have $u^k(x,t) \le u^{k+1}(x,t),\ \forall\,(x,t) \in \mathbb{R}^N \times [0,T]$. In particular, for $t = T$ one gets for any $x \in \mathbb{R}^N$

$$u^{k+1}(x,0) = u(x,(k+1)T) = u^k(x,T) \le u^{k+1}(x,T) = u(x,(k+2)T) = u^{k+2}(x,0).$$

Using the estimates (32), (33), (34) we can extract a subsequence $(u^{k_r})_r$ which converges uniformly on compact sets of $\mathbb{R}^N \times [0,T]$. And by monotonicity we deduce that all the sequence $(u^k)_{k\geq 0}$ converges uniformly on compact sets of $\mathbb{R}^N \times [0,T]$ towards a bounded lipschitz function φ

$$\lim_{k\to+\infty} u^k(x,t) = \lim_{k\to+\infty} u(x,t+kT) = \varphi(x,t),\ \forall\ (x,t) \in \mathbb{R}^N \times [0,T].$$

Observe that we have for any $x \in \mathbb{R}^N$

$$\begin{aligned}\varphi(x,0) &= \lim_{k\to+\infty} u^k(x,0) = \lim_{k\to+\infty} u(x,kT)\\ &= \lim_{k\to+\infty} u(x,(k+1)T) = \lim_{k\to+\infty} u^k(x,T) = \varphi(x,T),\end{aligned}$$

and therefore we can extend φ by periodicity over $\mathbb{R}^N \times \mathbb{R}$. By passing to the limit with respect to k in (36) we deduce by the stability result for continuous viscosity solutions that $\partial_t\varphi + H(x,\varphi,D\varphi) = f(t),\ \ (x,t) \in \mathbb{R}^N\times]0,T[$, and thus, by Proposition 2, φ is a time periodic viscosity solution of (20). It remains to check the minimality of the solution φ. Notice that $u^k(x,0) \geq u_0(x),\ \forall\ x \in \mathbb{R}^N, k \geq 0$, and therefore we have $\varphi(x,0) \geq u_0(x),\ \forall\ x \in \mathbb{R}^N$. Consider now another time periodic viscosity solution ψ of (20) such that $\psi(x,0) \geq u_0(x),\ \forall\ x \in \mathbb{R}^N$. By the comparison result one then has $\psi(x,t) \geq u(x,t),\ \forall\ (x,t) \in \mathbb{R}^N \times [0,+\infty[$, implying that

$$\psi(x,t) = \psi(x,t+kT) \geq u(x,t+kT) = u^k(x,t),\ \forall\ (x,t) \in \mathbb{R}^N \times [0,T],\ \forall\ k \geq 0.$$

By passing to the limit with respect to k we obtain

$$\psi(x,t) \geq \lim_{k\to+\infty} u^k(x,t) = \varphi(x,t),\ \forall\ (x,t) \in \mathbb{R}^N \times [0,T].$$

□

In the particular case of hamiltonians non depending on u the previous theorem ensures the convergence towards periodic fronts. Indeed, let $H = H(x,p)$ be a hamiltonian which belongs to $BUC(\mathbb{R}^N \times \overline{B}_R)\ \forall\ R > 0$ and satisfies $\lim_{|p|\to+\infty} H(x,p) = +\infty$ uniformly with respect to $x \in \mathbb{R}^N$, f a time periodic continuous function and u a viscosity solution of

$$\partial_t u + H(x,Du) = f(t),\ \ (x,t) \in \mathbb{R}^N\times]0,+\infty[,$$

with a bounded lipschitz initial condition u_0. Assume also that there is a bounded viscosity solution of $H(x,DU) = \lambda,\ x \in \mathbb{R}^N$ for some $\lambda \in \mathbb{R}$. Take $\mu = \langle f\rangle - \lambda$ and consider $u_\mu(x,t) = u(x,t) - \mu t,\ (x,t) \in \mathbb{R}^N \times [0,+\infty[,\ f_\mu(t) = f(t) - \mu,\ t \in \mathbb{R}$. Obviously we have

$$\begin{cases}\partial_t u_\mu + H(x,Du_\mu) = f_\mu(t), & (x,t) \in \mathbb{R}^N\times]0,+\infty[,\\ u_\mu(x,0) = u_0(x), & x \in \mathbb{R}^N.\end{cases}$$

By Proposition 7 we deduce that $u(x,t) - \mu t - \varphi(x,t)$ converges to 0 when $t \to +\infty$ if the initial condition is such that $u_0(x) \leq u(x,T) - \mu T,\ x \in \mathbb{R}^N$ (resp. $u_0(x) \geq u(x,T) - \mu T,\ x \in \mathbb{R}^N$) and φ is the minimal (resp. maximal) time periodic viscosity solution of $\partial_t\varphi + H(x,D\varphi) = f(t) - \mu,\ (x,t) \in \mathbb{R}^N \times \mathbb{R}$, satisfying $\varphi(x,0) \geq u_0(x), x \in \mathbb{R}^N$ (resp. $\varphi(x,0) \leq u_0(x), x \in \mathbb{R}^N$).

We have the following analogous result concerning the convergence towards steady states. The proof is left to the reader.

Proposition 8. *Let $H = H(x,z,p)$ be a hamiltonian verifying* (10), (7), (8) *and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. We assume that there is a bounded viscosity solution U for*

$$H(x,U,DU) = \lambda,\ \ x \in \mathbb{R}^N, \tag{38}$$

and we denote by u the unique viscosity solution of

$$\begin{cases} \partial_t u + H(x,u,Du) = \lambda, & (x,t) \in \mathbb{R}^N \times]0,+\infty[, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{39}$$

1) If the initial condition u_0 is a viscosity subsolution of (38) *then we have*

$$\lim_{t\to+\infty} u(x,t) = \varphi(x), \text{ uniformly for } x \text{ in compact sets of } \mathbb{R}^N,$$

where φ is the minimal viscosity solution of (38) *verifying $\varphi(x) \geq u_0(x), x \in \mathbb{R}^N$;*
2) If the initial condition $u_0(x)$ is a viscosity supersolution of (38) *then we have*

$$\lim_{t\to+\infty} u(x,t) = \Phi(x), \text{ uniformly for } x \text{ in compact sets of } \mathbb{R}^N,$$

where Φ is the maximal viscosity solution of (38) *verifying $\Phi(x) \leq u_0(x), x \in \mathbb{R}^N$.*

Remark 5. Observe that the initial condition of the counterexample (29) doesn't meet the hypotheses of the statements 1), 2) in Proposition 8.

5. **Asymptotic behaviour for large frequencies.** In this section we study the asymptotic behaviour of time periodic viscosity solutions for high frequencies. Our convergence result is a direct consequence of Corollary 2. We consider $f : \mathbb{R} \to \mathbb{R}$ a T periodic continuous function and denote by f_n the $\frac{T}{n}$ periodic functions given by $f_n(t) = f(nt)$, $\forall t \in \mathbb{R}$. Suppose that for all $n \geq 1$ there is a $\frac{T}{n}$ periodic viscosity solution of

$$\partial_t u_n + H(x,u_n,Du_n) = f_n(t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{40}$$

There are several natural questions arising in this context. Does the sequence $(u_n)_n$ converge? What are the limits in the convergence case? An easy example is the following. Consider $H(u) = \gamma \cdot u$, $\gamma \geq 0$, $f(t) = \cos t + \sin t$. Therefore the $\frac{2\pi}{n}$ periodic solution of $u_n' + \gamma \cdot u_n = f_n(t)$, $t \in \mathbb{R}$ are given by

$$u_n(t) = \frac{\gamma - n}{\gamma^2 + n^2}\cos(nt) + \frac{\gamma + n}{\gamma^2 + n^2}\sin(nt), \quad t \in \mathbb{R}.$$

Observe that $\lim_{n\to+\infty} u_n(t) = 0$ uniformly with respect to $t \in \mathbb{R}$ for all $\gamma \geq 0$. Generally we will see that, under appropriate hypotheses the sequence $(u_n)_n$ converges towards a viscosity solution of (21). This can be justified at least formally by introducing the fast oscillating variable $s = nt$ and by using the asymptotic expansion

$$u_n(x,t) = u^0(x) + \frac{1}{n}u^1(x,nt) + ... \tag{41}$$

which is one of the standard tools in homogenization problems, see [5]. Plugging the ansatz (41) into (40) we obtain

$$\partial_s u^1(x,s) + ... + H(x, u^0(x) + \frac{1}{n}u^1(x,s) + ..., Du^0(x) + \frac{1}{n}Du^1(x,s) + ...) = f(s),$$
$$(x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

Since u_n is $\frac{T}{n}$ periodic with respect to t, we are looking for a T periodic function $u^1(x,s)$ with respect to s. After integration over $[0,T]$ one gets

$$\frac{1}{T}\int_0^T H(x, u^0(x) + \frac{1}{n}u^1(x,s) + ..., Du^0(x) + \frac{1}{n}Du^1(x,s) + ...)\, ds = \langle f \rangle, \quad x \in \mathbb{R}^N.$$

After passing to the limit for $n \to +\infty$ we deduce formally that $\lim_{n\to+\infty} u_n = u^0$, where u^0 solves the cell problem (see [14])

$$H(x, u^0, Du^0) = \langle f \rangle, \ \ x \in \mathbb{R}^N.$$

Before formulating our statements let us introduce some notations. Assume that the hamiltonian H satisfies (10), (6), (8) and that the following condition holds

$$\exists M > 0 \text{ such that } H(x, -M, 0) \le f(t), \ \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{42}$$

Suppose also that there is a bounded l.s.c. viscosity supersolution $\tilde{V} \ge -M$ of (21). We construct the viscosity solutions $(V_\alpha)_{\alpha>0}$ of

$$\alpha(V_\alpha + M) + H(x, V_\alpha, DV_\alpha) = \langle f \rangle, \quad x \in \mathbb{R}^N.$$

By the comparison result we obtain easily that $-M \le V_\beta \le V_\alpha \le \tilde{V}, 0 < \alpha \le \beta$, and therefore $V = \sup_{\alpha>0} V_\alpha$ is a bounded l.s.c. function. We introduce the semi limits

$$\liminf{}_\star V_\alpha(x) := \liminf_{y\to x, \alpha\searrow 0} V_\alpha(y), \ \ \limsup{}^\star V_\alpha(x) := \limsup_{y\to x,\alpha\searrow 0} V_\alpha(y), \ \forall\, x \in \mathbb{R}^N.$$

Notice that we have $\liminf_\star V_\alpha(x) = V(x), \ \ \limsup^\star V_\alpha(x) = V^\star(x), \ \forall\, x \in \mathbb{R}^N$, and by using the stability result for semi continuous viscosity solutions (see [1], p. 85) we deduce that V is a bounded l.s.c. viscosity supersolution of (21) and $V^\star$ is a bounded u.s.c. viscosity subsolution of (21) (we say that V is a discontinuous viscosity solution of (21)). Observe also that $V = \sup_{\alpha>0} V_\alpha$ is the minimal bounded l.s.c. viscosity supersolution of (21) satisfying $V \ge -M$. Indeed, if W is a bounded l.s.c. viscosity supersolution of (21) such that $W \ge -M$ then by the comparison result we have $W \ge V_\alpha$ for any $\alpha > 0$, implying that $W \ge \sup_{\alpha>0} V_\alpha = V$. Similarly we construct the time periodic viscosity solutions $(v_\alpha)_{\alpha>0}$ of

$$\alpha(v_\alpha + M) + \partial_t v_\alpha + H(x, v_\alpha, Dv_\alpha) = f(t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

As before we obtain $-M \le v_\beta \le v_\alpha, \ \forall\, 0 < \alpha \le \beta$. By Corollary 2 we have

$$|v_\alpha(x,t) - V_\alpha(x)| \le \|f - \langle f \rangle\|_{L^1(0,T)}, \ \forall\, (x,t) \in \mathbb{R}^N \times \mathbb{R}, \ \forall\, \alpha > 0$$

and we deduce that $v = \sup_{\alpha>0} v_\alpha$ is a bounded time periodic l.s.c. function. Using the stability result for semi continuous viscosity solutions yields that $\liminf_\star v_\alpha = v$ is a bounded time periodic l.s.c. viscosity supersolution of (20) and $\limsup^\star v_\alpha = v^\star$ is a bounded time periodic u.s.c. viscosity subsolution of (20). Actually v is the minimal bounded time periodic l.s.c. viscosity supersolution of (20) satisfying $v \ge -M$.

Theorem 7. *Let $H = H(x, z, p)$ be a hamiltonian satisfying* (10)*,* (6)*,* (8)*,* (42)*, where f is a T periodic continuous function. Suppose also that there is a bounded l.s.c. viscosity supersolution $\tilde{V} \ge -M$ of* (21) *and denote by V, v_n the minimal stationary, resp. time periodic l.s.c. viscosity supersolution of* (21)*, resp.* (40)*. Then the sequence $(v_n)_n$ converges uniformly on $\mathbb{R}^N \times \mathbb{R}$ towards V and we have $\|v_n - V\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \le \frac{1}{n}\|f - \langle f \rangle\|_{L^1(0,T)}, \ \ \forall n \ge 1$.*

Proof. Note that $v_n = \sup_{\alpha>0} v_{n,\alpha}$ is $\frac{T}{n}$ periodic. We introduce also $w_{n,\alpha}(x,t) = v_{n,\alpha}(x, \frac{t}{n})$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, which is T periodic. As $v_{n,\alpha}$ satisfies in the viscosity sense $\alpha(v_{n,\alpha} + M) + \partial_t v_{n,\alpha} + H(x, v_{n,\alpha}, Dv_{n,\alpha}) = f_n(t), \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R}$, we deduce that $w_{n,\alpha}$ satisfies in the viscosity sense

$$\alpha(w_{n,\alpha} + M) + n\, \partial_t w_{n,\alpha} + H(x, w_{n,\alpha}, Dw_{n,\alpha}) = f(t), \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

which we can rewrite as

$$\partial_t w_{n,\alpha} + \frac{1}{n}(\alpha w_{n,\alpha} + H(x, w_{n,\alpha}, Dw_{n,\alpha})) = \frac{1}{n}(f(t) - \alpha M), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{43}$$

Recall also that we have in the viscosity sense

$$\frac{1}{n}(\alpha V_\alpha + H(x, V_\alpha, DV_\alpha)) = \frac{1}{n}(\langle f \rangle - \alpha M), \quad x \in \mathbb{R}^N. \tag{44}$$

By using Corollary 2 we deduce that

$$w_{n,\alpha}(x,t) - V_\alpha(x) \quad \leq \quad \sup_{s \leq t} \frac{1}{n} \int_s^t (f(\sigma) - \langle f \rangle)\, d\sigma \leq \frac{1}{n} \|f - \langle f \rangle\|_{L^1(0,T)},$$

and similarly $V_\alpha(x) - w_{n,\alpha}(x,t) \leq \frac{1}{n}\|f - \langle f \rangle\|_{L^1(0,T)}$, $\forall n \geq 1$. We obtain for all $n \geq 1$

$$\left| v_{n,\alpha}\left(x, \frac{t}{n}\right) - V_\alpha(x) \right| \leq \frac{1}{n}\|f - \langle f \rangle\|_{L^1(0,T)},$$

and after passing to the limit for $\alpha \searrow 0$ one gets for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$

$$\left| v_n\left(x, \frac{t}{n}\right) - V(x) \right| \leq \frac{1}{n}\|f - \langle f \rangle\|_{L^1(0,T)}.$$

Finally we deduce that $\|v_n - V\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq \frac{1}{n}\|f - \langle f \rangle\|_{L^1(0,T)}$ for all $n \geq 1$. □

Remark 6. With the above notations we have for all $n \geq 1$

$$|v_n(x, t+h) - v_n(x,t)| \leq 2 \cdot |h| \cdot \|f\|_{L^\infty(\mathbb{R})}, \ \forall\, (x,t) \in \mathbb{R}^N \times \mathbb{R}, \ \forall\, h \in \mathbb{R}.$$

Proof. Note that $z_{n,\alpha}(x,t) = w_{n,\alpha}(x, t+h)$ is T periodic and satisfies in the viscosity sense

$$\partial_t z_{n,\alpha} + \frac{1}{n}(\alpha z_{n,\alpha} + H(x, z_{n,\alpha}, Dz_{n,\alpha})) = \frac{1}{n}(f(t+h) - \alpha M), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{45}$$

Now by using (43), (45) and Corollary 2 we obtain

$$w_{n,\alpha}(x, t+h) - w_{n,\alpha}(x,t) \leq \sup_{s \leq t} \frac{1}{n} \int_s^t (f(\sigma + h) - f(\sigma))\, d\sigma \leq \frac{2}{n} \cdot |h| \cdot \|f\|_{L^\infty(\mathbb{R})}.$$

We deduce that $v_{n,\alpha}\left(x, \frac{t+h}{n}\right) - v_{n,\alpha}\left(x, \frac{t}{n}\right) \leq \frac{2}{n} \cdot |h| \cdot \|f\|_{L^\infty(\mathbb{R})}$, $\forall\, n \geq 1$, $x \in \mathbb{R}^N$, $t, h \in \mathbb{R}$. Our conclusion follows easily after passing to the limit for $\alpha \searrow 0$. □

Remark 7. If hypothesis (6) is replaced by (7) we obtain that for $\alpha > 0$, $n \geq 1$ the functions $V_\alpha, V, v_{n,\alpha}, v_n$ are uniformly lipschitz with respect to $x \in \mathbb{R}^N$, resp. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. In particular $V = V^\star$ and $v_n = v_n^\star$ are continuous viscosity solutions of (21), resp. (40).

6. **Almost periodic viscosity solutions.** In this section we generalize the results obtained for time periodic viscosity solutions to the class of almost-periodic viscosity solutions. We recall briefly the notion of almost-periodic function and some basic properties. For more details on the theory of almost-periodic functions the reader can refer to [7].

Proposition 9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The following conditions are equivalent*
1) $\forall\, \varepsilon > 0$, $\exists\, l(\varepsilon) > 0$ *such that* $\forall\, a \in \mathbb{R}$, $\exists\, \tau \in [a, a + l(\varepsilon)[$ *satisfying*

$$|f(t+\tau) - f(t)| < \varepsilon, \quad \forall t \in \mathbb{R} \ ; \tag{46}$$

2) $\forall \varepsilon > 0$, *there is a trigonometric polynomial* $T_\varepsilon(t) = \sum_{k=1}^{n} \{a_k \cdot \cos(\lambda_k t) + b_k \cdot \sin(\lambda_k t)\}$ *where* $a_k, b_k, \lambda_k \in \mathbb{R}$, $1 \le k \le n$ *such that* $|f(t) - T_\varepsilon(t)| < \varepsilon$, $\forall t \in \mathbb{R}$ *;*
3) for all real sequence $(h_n)_n$ *there is a subsequence* $(h_{n_k})_k$ *such that* $(f(\cdot + h_{n_k}))_k$ *converges uniformly on* $\mathbb{R}$.

Definition 1. We say that a continuous function f is almost-periodic iff f satisfies one of the previous three conditions.

A number τ verifying (46) is called ε almost-period. By using Proposition 9 we obtain easily the following properties of almost-periodic functions.

Proposition 10. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is almost-periodic. Then* f *is bounded uniformly continuous function.*

Another important property is the following.

Proposition 11. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is almost-periodic. Then* $\frac{1}{T}\int_a^{a+T} f(t)\,dt$ *converges as* $T \to +\infty$ *uniformly with respect to* $a \in \mathbb{R}$. *Moreover the limit does not depend on* a *and it is called the average of* f

$$\exists \langle f \rangle := \lim_{T \to +\infty} \frac{1}{T} \int_a^{a+T} f(t)\,dt, \quad \textit{uniformly with respect to } a \in \mathbb{R}.$$

Note that if f is periodic then $\langle f \rangle$ coincides with the usual definition of the mean of f over one period. We finish this brief introduction on the notion of almost-periodicity with the following result concerning primitives of almost-periodic functions.

Proposition 12. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is almost-periodic and denote by* F *a primitive of* f. *Then* F *is almost-periodic iff* F *is bounded.*

In the following we adapt the results of previous sections for almost-periodic viscosity solutions.

Definition 2. We say that $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is almost-periodic in t uniformly with respect to x if u is continuous in t uniformly with respect to x and $\forall\, \varepsilon > 0$, $\exists\, l(\varepsilon) > 0$ such that all interval of length $l(\varepsilon)$ contains a number τ which is ε almost-period for $u(x, \cdot)$, $\forall x \in \mathbb{R}^N$

$$|u(x, t + \tau) - u(x, t)| < \varepsilon, \ \ \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}. \tag{47}$$

We establish existence and uniqueness results for hamiltonians $H = H(x, z, p)$ satisfying (5). For the uniqueness we have the more general result.

Proposition 13. *Let* u *a bounded u.s.c. viscosity subsolution of* $\partial_t u + H(x, t, u, Du) = f(x, t)$, *in* $\mathbb{R}^N \times \mathbb{R}$ *and* v *a bounded l.s.c. viscosity supersolution of* $\partial_t v + H(x, t, v, Dv) = g(x, t)$, *in* $\mathbb{R}^N \times \mathbb{R}$ *where* $f, g \in BUC(\mathbb{R}^N \times \mathbb{R})$ *and* (5), (6), (8) *hold uniformly for* $t \in \mathbb{R}$. *Then we have for all* $t \in \mathbb{R}$

$$\sup_{x \in \mathbb{R}^N} (u(x, t) - v(x, t))_+ \le e^{-\gamma t} \int_{-\infty}^{t} e^{\gamma \sigma} \sup_{x \in \mathbb{R}^N} (f(x, \sigma) - g(x, \sigma))_+ \, d\sigma.$$

Moreover, the hypothesis (6) *can be replaced by* $u \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$ *or* $v \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$.

Proof. Take $t_0, t \in \mathbb{R}$, $t_0 \le t$ and by using Corollary 1 write for all $x \in \mathbb{R}^N$

$$u(x, t) - v(x, t) \le e^{-\gamma(t - t_0)} \cdot (\|u\|_\infty + \|v\|_\infty) + e^{-\gamma t} \int_{t_0}^{t} e^{\gamma \sigma} \sup_{y \in \mathbb{R}^N} (f(y, \sigma) - g(y, \sigma))_+ \, d\sigma,$$

where $\gamma = \gamma_{R_0}$, $R_0 = \max(\|u\|_\infty, \|v\|_\infty)$. The conclusion follows by passing $t_0 \to -\infty$. □

We concentrate now on the existence part. We will see that the proof is a little more complicated than in the periodic case.

Proposition 14. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is almost-periodic and that the hamiltonian $H = H(x, z, p)$ satisfies the hypotheses* (5), (6), (8) *and $\exists M > 0$ such that $H(x, -M, 0) \le f(t) \le H(x, M, 0)$, $\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}$. Then there is a time almost-periodic viscosity solution in $BUC(\mathbb{R}^N \times \mathbb{R})$ of $\partial_t u + H(x, u, Du) = f(t)$, in $\mathbb{R}^N \times \mathbb{R}$.*

Proof. For all $n \ge 1$ we consider the unique viscosity solution of the problem

$$\begin{cases} \partial_t u_n + H(x, u_n, Du_n) = f(t), & (x,t) \in \mathbb{R}^N \times]-n, +\infty[, \\ u_n(x, -n) = 0, & x \in \mathbb{R}^N. \end{cases} \tag{48}$$

Such a solution exists, cf. Theorem 4. We will prove that for all $t \in \mathbb{R}$, $(u_n(t))_{n \ge -t}$ converges to an almost-periodic viscosity solution of $\partial_t u + H(x, u, Du) = f(t)$, in $\mathbb{R}^N \times \mathbb{R}$. Since $\mp M$ is sub/supersolution of $\partial_t u + H = f$ we deduce by Theorem 3 that $-M \le u_n(x,t) \le M$, $\forall (x,t) \in \mathbb{R}^N \times [-n, +\infty[$. Consider $\gamma = \gamma_M > 0$. Take $t \in \mathbb{R}$ and for $m \ge n$ large enough, by Proposition 3 we can write for all $x \in \mathbb{R}^N$, $\forall t \ge t_0 \ge -n$

$$|u_n(x,t) - u_m(x,t)| \le e^{-\gamma(t - t_0)} \cdot (\|u_n\|_\infty + \|u_m\|_\infty) \le e^{-\gamma(t-t_0)} \cdot 2M.$$

For $t_0 = -n$ we deduce that $|u_n(x,t) - u_m(x,t)| \le 2M \cdot e^{-\gamma t} \cdot e^{-\gamma n}$ and thus there is $\lim_{n \to +\infty} u_n(x,t) = u(x,t)$, $\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$. Moreover $(u_n)_n$ converges uniformly on $\mathbb{R}^N \times [a, +\infty[$, $\forall a \in \mathbb{R}$. In particular we obtain that $u \in BUC(\mathbb{R}^N \times [a,b])$, $\forall a, b \in \mathbb{R}$, $a \le b$. By using the stability result for continuous viscosity solutions we deduce that u verifies in the viscosity sense $\partial_t u + H(x, u, Du) = f(t)$, in $\mathbb{R}^N \times \mathbb{R}$. We have to prove that u is almost-periodic. For all $\varepsilon > 0$ consider $l(\gamma \cdot \varepsilon)$ such that any interval of length $l(\gamma \cdot \varepsilon)$ contains a $\gamma \cdot \varepsilon$ almost-period of f. We will show that any interval of length $l(\gamma \cdot \varepsilon)$ contains a number τ which is an ε almost-period for $u(x, \cdot)$, $\forall x \in \mathbb{R}^N$. Indeed, consider an interval of length $l(\gamma \cdot \varepsilon)$, take τ a $\gamma \cdot \varepsilon$ almost-period of f and let us fix $\tilde{t} \in \mathbb{R}$. Observe that the function $v_n : \mathbb{R}^N \times [-n - \tau, +\infty[\to \mathbb{R}$, $v_n(x,t) = u_n(x, t + \tau)$ solves in the viscosity sense

$$\partial_t v_n + H(x, v_n, Dv_n) = f(t + \tau), \quad (x,t) \in \mathbb{R}^N \times]-n-\tau, +\infty[.$$

By Corollary 1 we have for all $t \ge t_n = \max\{-n, -n-\tau\}$

$$e^{\gamma t} |u_n(x,t) - v_n(x,t)| \le e^{\gamma t_n} \cdot (\|u_n\|_\infty + \|v_n\|_\infty) + \int_{t_n}^{t} e^{\gamma \sigma} |f(\sigma + \tau) - f(\sigma)| \, d\sigma.$$

In particular for $t = \tilde{t}$ and n large enough we deduce

$$|u_n(x, \tilde{t}) - u_n(x, \tilde{t} + \tau)| \le 2M \cdot e^{-\gamma(\tilde{t} - t_n)} + e^{-\gamma \tilde{t}} \int_{t_n}^{\tilde{t}} e^{\gamma \sigma} \gamma \varepsilon \, d\sigma \le 2M \cdot e^{-\gamma(\tilde{t} - t_n)} + \varepsilon.$$

By passing $n \to +\infty$ we have $t_n \to -\infty$ and therefore

$$|u(x, \tilde{t}) - u(x, \tilde{t} + \tau)| \le \varepsilon, \quad (x, \tilde{t}) \in \mathbb{R}^N \times \mathbb{R}.$$

Since we already know that $u \in BUC(\mathbb{R}^N \times [a,b])$, $\forall a, b \in \mathbb{R}$, $a \le b$, by time almost-periodicity we deduce also that $u \in BUC(\mathbb{R}^N \times \mathbb{R})$. □

Now we are ready to study the case of hamiltonians satisfying only (10). We have the following theorem analogous to Theorem 5.

Theorem 8. *Let $H = H(x, z, p)$ be a hamiltonian verifying the hypotheses* (10), (7), (8), $\sup\{|H(x,0,0)| \; : x \in \mathbb{R}^N\} = C < +\infty$ *and f be a time almost-periodic function such that $t \to F(t) = \int_0^t \{f(\sigma) - \langle f \rangle\} \, d\sigma$ is bounded on $\mathbb{R}$. Then there is a bounded lipschitz time almost-periodic viscosity solution of* (20) *iff there is a bounded viscosity solution of* (21).

Proof. Assume that there is a bounded viscosity solution V of (21). Since the hamiltonian satisfies (7) we deduce that V is a lipschitz function. For any $\alpha > 0$ take $M_\alpha = \|V\|_{L^\infty(\mathbb{R}^N)} + \frac{1}{\alpha}(C + \|f\|_{L^\infty(\mathbb{R})})$ and observe that

$$\alpha(-M_\alpha - V(x)) + H(x, -M_\alpha, 0) \le f(t) \le \alpha(M_\alpha - V(x)) + H(x, M_\alpha, 0), \forall \, (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Therefore we can construct the family of solutions V_α for

$$\alpha(V_\alpha - V(x)) + H(x, V_\alpha, DV_\alpha) = \langle f \rangle, \;\; x \in \mathbb{R}^N,$$

and, cf. Propositions 13, 14, the family of time almost-periodic solutions v_α for

$$\alpha(v_\alpha - V(x)) + \partial_t v_\alpha + H(x, v_\alpha, Dv_\alpha) = f(t), \;\; (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

In fact we have $V_\alpha = V$ for any $\alpha > 0$ and

$$v_\alpha(x,t) - V(x) = v_\alpha(x,t) - V_\alpha(x) \le \sup_{s \le t} \int_s^t \{f(\sigma) - \langle f \rangle\} d\sigma = \sup_{s \le t} \{F(t) - F(s)\} \le 2\|F\|_\infty.$$

Similarly one gets $V(x) - v_\alpha(x,t) = V_\alpha(x) - v_\alpha(x,t) \le 2\|F\|_\infty$, which implies that the family $(v_\alpha)_\alpha$ is also bounded. As in the periodic case we deduce that $(v_\alpha)_{\alpha>0}$ are uniformly lipschitz and we can extract a sequence which converges uniformly on compact sets of $\mathbb{R}^N \times \mathbb{R}$ towards a bounded lipschitz solution v of (20). The difficult thing to do is to check that v is almost-periodic. By the hypotheses and Proposition 12 we deduce that F is almost-periodic and thus, for all $\varepsilon > 0$ there is $l(\frac{\varepsilon}{2})$ such that any interval of length $l(\frac{\varepsilon}{2})$ contains an $\frac{\varepsilon}{2}$ almost-period of F. Take an interval of length $l(\frac{\varepsilon}{2})$ and τ an $\frac{\varepsilon}{2}$ almost-period of F in this interval. We have for all $\alpha > 0$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$

$$\begin{aligned} v_\alpha(x, t+\tau) - v_\alpha(x,t) &\le \sup_{s \le t} \int_s^t \{f(\sigma + \tau) - f(\sigma)\} \, d\sigma \\ &= \sup_{s \le t} \left\{ \int_{s+\tau}^{t+\tau} (f(\sigma) - \langle f \rangle) \, d\sigma - \int_s^t (f(\sigma) - \langle f \rangle) \, d\sigma \right\} \\ &= \sup_{s \le t} \{(F(t+\tau) - F(t)) - (F(s+\tau) - F(s))\} \\ &\le \varepsilon. \end{aligned} \tag{49}$$

After passing to the limit for $\alpha \searrow 0$ one gets $v(x, t+\tau) - v(x,t) \le \varepsilon$ and similarly $v(x,t) - v(x, t+\tau) \le \varepsilon$, $\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$. By using the uniform continuity of F, we can prove exactly in the same manner that v is continuous in t uniformly with respect to x. The converse implication follows easily. □

We focus now our attention to the asymptotic behaviour of the almost-periodic solutions of (40) when f is almost-periodic function. Notice that for all $n \ge 1$ the function f_n given by $f_n(t) = f(nt)$ is almost-periodic and has the same average as f. The reader can easily adapt the proof of Theorem 7 to the case of almost-periodic functions. We have the following result.

Theorem 9. *Let $H = H(x, z, p)$ be a hamiltonian satisfying* (10)*,* (6)*,* (8)*,* (42) *where f is almost-periodic function. Suppose also that there is a bounded l.s.c. viscosity supersolution $\tilde{V} \geq -M$ of* (21)*, that $t \to F(t) = \int_0^t \{f(s) - \langle f \rangle\} ds$ is bounded and denote by V, v_n the minimal stationary, resp. time almost-periodic l.s.c. viscosity supersolution of* (21)*, resp.* (40)*. Then the sequence $(v_n)_n$ converges uniformly on $\mathbb{R}^N \times \mathbb{R}$ towards V and $\|v_n - V\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq \frac{2}{n} \|F\|_{L^\infty(\mathbb{R})}$, $\forall n \geq 1$.*

Acknowledgements. The authors are thankful to Prof. J.-M. Roquejoffre and to the referees for helpful remarks and advices.

REFERENCES

[1] G. Barles, "Solutions de viscosité des équations de Hamilton-Jacobi," Springer-Verlag, Berlin, 1994.

[2] G. Barles and P.E. Souganidis, *Some counterexamples on the asymptotic behavior of the solutions of the Hamilton-Jacobi equations*, C. R. Acad. Sci. Paris, Sér. I Math., **330** (2000), 963–968.

[3] G. Barles and P.E. Souganidis, *On the large time behavior of solutions of Hamilton-Jacobi equations*, SIAM J. Math. Anal., **31** (2001), 925–939.

[4] G. Barles and P.E. Souganidis, *Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations*, SIAM J. Math. Anal., **32** (2001), 1311–1323.

[5] A. Bensoussan, J.-L. Lions and G. Papanicolaou, "Asymptotic Analysis for Periodic Structures," North-Holland, Amsterdam, 1987.

[6] M. Bostan, *Periodic solutions for evolution equations*, Electronic J. Differential Equations, Monograph **3** (2002), 41 pp.

[7] C. Corduneanu, "Almost Periodic Functions," Chelsea, New York, 1989.

[8] M.G. Crandall and P.-L. Lions, *Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre*, C. R. Acad. Sci. Paris, Sér. I Math., **292** (1981), 183–186.

[9] M.G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **277** (1983), 1–42.

[10] M.G. Crandall, L.C. Evans and P.-L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **282** (1984), 487–502.

[11] A. Fathi, *Sur la convergence du semi-groupe de Lax-Oleinik,* C. R. Acad. Sci. Paris, Sér. I Math., **327** (1998), 267–270.

[12] A. Fathi and J. N. Mather, *Failure of convergence of the Lax-Oleinik Semi-group in the Time-Periodic case,* Bull. Soc. Math. France, **128** (2000), 473–483.

[13] P.-L. Lions, "Generalized solutions of Hamilton-Jacobi Equations," Research Notes in Mathematics, Pitman, 1982.

[14] P.-L. Lions, G. Papanicolaou and S.R.S. Varadhan, *Homogeneization of Hamilton-Jacobi equations*, Preprint.

[15] G. Namah and J.-M. Roquejoffre, *Remarks on the long time behaviour of the solutions of Hamilton-Jacobi equations,* Comm. Partial Differential Equations, **24** (1999), 883–893.

[16] J.-M. Roquejoffre, *Comportement asymptotique des solutions d'équations de Hamilton-Jacobi monodimensionnelles*, C. R. Acad. Sci. Paris Sér. I Math., **326** (1998), 185–189.

[17] J.-M. Roquejoffre, *Convergence to steady states or periodic solutions in a class of Hamilton-Jacobi equations*, J. Math. Pures Appl., **80** (2001), 85–104.

[18] J.-M. Roquejoffre, *Large time convergence in Hamilton-Jacobi equations*, Proceedings in Control Systems : Theory, Numerics and Applications, Rome (2005), to appear.

[19] P.E. Souganidis, *Existence of viscosity solutions of Hamilton-Jacobi equations*, J. Differential Equations, **56** (1985), 345–390.

Received November 2005; revised September 2006.

E-mail address: mbostan@math.univ-fcomte.fr; gawtum.namah@math.univ-fcomte.fr