Asymptotic analysis of parabolic equations with stiff transport terms by a multi-scale approach

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Abstract

We perform the asymptotic analysis of parabolic equations with stiff transport terms. This kind of problem occurs, for example, in collisional gyrokinetic theory for tokamak plasmas, where the velocity diffusion of the collision mechanism is dominated by the velocity advection along the Laplace force corresponding to a strong magnetic field. This work appeal to the filtering techniques. Removing the fast oscillations associated to the singular transport operator, leads to a stable family of profiles. The limit profile comes by averaging with respect to the fast time variable, and still satisfies a parabolic model, whose diffusion matrix is completely characterized in terms of the original diffusion matrix and the stiff transport operator. Introducing first order correctors allows us to obtain strong convergence results, for general initial conditions (not necessarily well prepared).

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1 Introduction

In many applications we deal with disparate scales. The solutions of the problems in hand fluctuate at very different scales and for the moment, solving numerically for both slow and fast scales seems out of reach. Depending on the particular regimes we are interested on, it could be worth to solve with respect to the slow variable, after smoothing out the fast oscillations. In this work we focus on parabolic models perturbed by stiff transport operators

$$\begin{cases} \partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) + \dfrac{1}{\varepsilon}\, b(y)\cdot\nabla_y u^\varepsilon = 0, & (t,y)\in\mathbb{R}_+\times\mathbb{R}^m \\ u^\varepsilon(0,y) = u^{\mathrm{in}}(y), & y\in\mathbb{R}^m. \end{cases} \tag{1}$$

Here $b:\mathbb{R}^m\to\mathbb{R}^m$ and $D:\mathbb{R}^m\to\mathcal{M}_m(\mathbb{R})$ are given fields of vectors and symmetric matrices, and $\varepsilon>0$ is a small parameter destinated to converge to 0. If the vector field b is divergence free, the energy balance writes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m}(u^\varepsilon(t,y))^2\,\mathrm{d}y + \int_{\mathbb{R}^m} D(y)\nabla_y u^\varepsilon\cdot\nabla_y u^\varepsilon\,\mathrm{d}y = 0.$$

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Therefore, when the matrices $D(y)$ are positive, the L^2 norms of the solutions $(u^\varepsilon)_{\varepsilon>0}$ decrease in time, and we expect that the limit model still behaves like a parabolic one, whose diffusion matrix field is to be determined. This work is motivated by the study of collisional models for the gyrokinetic theory in tokamak plasmas. The fluctuations of the presence density of charged particles are due to the transport in space and velocity (under the action of electromagnetic fields), but also to the collision mechanisms. In the framework of the magnetic confinement fusion, the external magnetic fields are very large, leading to a stiff velocity advection, due to the magnetic force $qv \wedge B^\varepsilon = qv \wedge \frac{B}{\varepsilon}$. Here q stands for the particle charge and $B^\varepsilon = \frac{B}{\varepsilon}$ represents a strong magnetic field, when ε goes to 0. Using a Fokker-Planck operator for taking into account the collisions between particles, we are led to the Fokker-Planck equation

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m}\left(E + v \wedge \frac{B}{\varepsilon}\right) \cdot \nabla_v f^\varepsilon = \nu \,\mathrm{div}_v(\Theta \nabla_v f^\varepsilon + v f^\varepsilon), \;\; (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (2)$$

where E is the electric field, m is the particle mass, ν is the collision frequency and Θ is the temperature. The asymptotic analysis of (2), when neglecting the collisions is now well understood [4, 15, 16, 17]. It can be handled by averaging the perturbed model along the characteristic flow associated to the dominant transport operator. Recently, models including collisions have been analyzed formally by using the averaging method [5, 6]. In particular, it was emphasized that, averaging with respect to the fast cyclotronic motion leads to diffusion not only in velocity, but also with respect to the perpendicular space directions, see (27). The study of the averaged diffusion matrix field is crucial when determining the equilibria of the limit Fokker-Planck equation (2), when ε goes to zero. Numerical results concerning strongly anisotropic elliptic and parabolic problems were obtained in [13, 14, 10].

This work concentrates on the asymptotic analysis for the parabolic models in (1). We expect that part of these arguments applies to other perturbed models, for example in the framework of strongly anisotropic parabolic models, which will be studied in future works. Our paper is organized as follows. The main results are introduced in Section 2. We indicate the main lines of our arguments, performing formal computations. In Section 3 we present a brief overview on the construction of the average operators for matrix fields. Section 4 is devoted to uniform estimates, in view of convergence results. In Section 5 we establish two-scale convergence results, in the ergodic setting, which allows us to handle situations with non periodic fast variables. Up to our knowledge, these results have not been reported yet. The proofs of the main theorems are detailed in Section 6. Some technical arguments are presented in Appendix A.

2 Presentation of the main results and formal approach

The subject matter of this paper concentrates on the asymptotic analysis of (1), when ε becomes small. Obviously, the fast time oscillations come through the large advection field $\frac{b(y)}{\varepsilon} \cdot \nabla_y$. Indeed, think that when neglecting the diffusion operator, the problem (1) reduces to a transport model, whose solution writes

$$u^\varepsilon(t,y) = u^{\mathrm{in}}(Y(-t/\varepsilon; y)), \;\; (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m. \quad (3)$$

Here $(s,y) \in \mathbb{R} \times \mathbb{R}^m \to Y(s;y) \in \mathbb{R}^m$ stands for the characteristic flow of $b \cdot \nabla_y$

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = b(Y(s;y)), \;\; (s,y) \in \mathbb{R} \times \mathbb{R}^m, \;\; Y(0;y) = y, \;\; y \in \mathbb{R}^m.$$

This flow is well defined under standard smoothness assumptions

$$b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m), \;\; \mathrm{div}_y b = 0 \quad (4)$$

and

$$\exists C > 0 \text{ such that } |b(y)| \le C(1+|y|), \quad y \in \mathbb{R}^m. \tag{5}$$

Under the above hypotheses the flow Y is global and smooth, $Y \in W^{1,\infty}_{\text{loc}}(\mathbb{R}\times\mathbb{R}^m)$. Moreover, since the field b is divergence free, the transformation $y \in \mathbb{R}^m \to Y(s;y) \in \mathbb{R}^m$ is measure preserving for any $s \in \mathbb{R}$. Motivated by (3), we introduce the new unknowns

$$v^\varepsilon(t,z) = u^\varepsilon(t, Y(t/\varepsilon; z)), \quad (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m, \quad \varepsilon > 0 \tag{6}$$

and we expect to get stability for the family $(v^\varepsilon)_{\varepsilon>0}$, when ε goes to 0. In that case we will deduce that, for small $\varepsilon > 0$, u^ε behaves like $v(t, Y(-t/\varepsilon; y))$, for some profile $v = \lim_{\varepsilon\searrow 0} v^\varepsilon$, that is, u^ε appears as the composition product between a stable profile and the fast oscillating flow $Y(-t/\varepsilon; y)$. We prove mainly two strong convergence results for general initial conditions (not necessarily well prepared), whose simplified versions are stated below. For detailed assertions see Theorems 2.2, 2.3.

Theorem *We denote by $(u^\varepsilon)_{\varepsilon>0}$ the variational solutions of* (1) *and by $(v^\varepsilon)_{\varepsilon>0}$ the functions*

$$v^\varepsilon(t,z) = u^\varepsilon(t, Y(t/\varepsilon; z)), \quad (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m, \quad \varepsilon > 0.$$

1. *Under suitable hypotheses on the vector field b, the matrix field D and the initial condition u^{in}, the family $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$ to the unique variational solution $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of* (15), *whose diffusion matrix field $\langle D\rangle$ comes by averaging the matrix field D along the flow of the vector field b (cf. Theorem 2.1).*

2. *Under more regularity hypotheses, we have*

$$u^\varepsilon(t,\cdot) = v(t, Y(-t/\varepsilon;\cdot)) + \mathcal{O}(\varepsilon) \ \text{ in } L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$$

that is, for any $T \in \mathbb{R}_+$, there is a constant C_T such that

$$\sup_{t\in[0,T]} \|u^\varepsilon(t,\cdot) - v(t, Y(-t/\varepsilon;\cdot))\|_{L^2(\mathbb{R}^m)} \le C_T \varepsilon.$$

The problem satisfied by v^ε is obtained by performing the change of variable $y = Y(t/\varepsilon; z)$ in (1). A straightforward computation based on the chain rule leads to (see Remark 6.2)

$$\partial_t v^\varepsilon(t,z) = \partial_t u^\varepsilon(t, Y(t/\varepsilon;z)) + \frac{1}{\varepsilon}\, b(Y(t/\varepsilon;z)) \cdot (\nabla_y u^\varepsilon)(t, Y(t/\varepsilon;z))$$

and

$$\operatorname{div}_z\{\partial Y^{-1}(t/\varepsilon;z) D(Y(t/\varepsilon;z))\, {}^t\partial Y^{-1}(t/\varepsilon;z)\nabla_z v^\varepsilon\} = \{\operatorname{div}_y(D(y)\nabla_y u^\varepsilon)\}(t, Y(t/\varepsilon;z))$$

and therefore (1) becomes

$$\begin{cases} \partial_t v^\varepsilon - \operatorname{div}_z\left((G(t/\varepsilon)D)\nabla_z v^\varepsilon\right) = 0, & (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v^\varepsilon(0,z) = u^\varepsilon(0,z) = u^{\text{in}}(z), & z \in \mathbb{R}^m,\ \varepsilon > 0 \end{cases} \tag{7}$$

where $(G(s)D)_{s\in\mathbb{R}}$ is the family of matrix fields given by

$$\begin{aligned} (G(s)D)(z) &= \partial Y^{-1}(s;z) D(Y(s;z))\, {}^t\partial Y^{-1}(s;z) \\ &= \partial Y(-s; Y(s;z))\, D(Y(s;z))\, {}^t\partial Y(-s; Y(s;z)), \quad (s,z) \in \mathbb{R}\times\mathbb{R}^m. \end{aligned} \tag{8}$$

The new diffusion problem (7) seems simpler than the original problem (1), because the singular term $\frac{1}{\varepsilon}\, b \cdot \nabla_y$ has disappeared. Nevertheless, the new model depends on a fast time variable $s = t/\varepsilon$, through the diffusion matrix field $G(s = t/\varepsilon)D$, and a slow time variable t. We deal with a two-scale problem in time. As often in asymptotic analysis of multiple scale problems, a way to understand the behavior of the solutions $(v^\varepsilon)_{\varepsilon>0}$ when ε goes to 0 and to identify the limit problem is to use a formal development whose terms depend both on the slow and fast time variables

$$v^\varepsilon(t, z) = v(t, t/\varepsilon, z) + \varepsilon v^1(t, t/\varepsilon, z) + ... \tag{9}$$

This method is used in many frameworks such as periodic homogenization for elliptic and parabolic systems [1, 18], transport equations [9, 11] or kinetic equations [7]. Plugging the Ansatz (9) in (7) and identifying the terms of the same order with respect to ε, lead to the hierarchy of equations

$$\partial_s v = 0 \tag{10}$$

$$\partial_t v - \mathrm{div}_z(G(s)D\nabla_z v) + \partial_s v^1 = 0 \tag{11}$$

$$\vdots$$

Equation (10) says that the first profile v does not depend on the fast time variable s, that is $v = v(t, z)$. We expect that v is the limit of the family $(v^\varepsilon)_{\varepsilon>0}$, when ε goes to 0. The slow time evolution of v is given by (11), but we need to eliminate the second profile v^1. Actually v^1 appears as a Lagrange multiplier which guarantees that at any time t, the profile v satisfies the constraint $\partial_s v = 0$. In the periodic case, we eliminate v^1 by taking the average over one period. More general, we appeal to ergodic average and we write

$$\begin{cases} \partial_t v - \mathrm{div}_z \left\{ \left(\lim_{S\to+\infty} \dfrac{1}{S} \displaystyle\int_0^S G(s)D \,\mathrm{d}s \right) \nabla_z v \right\} = 0, & (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v(0, z) = u^{\mathrm{in}}(z), & z \in \mathbb{R}^m. \end{cases} \tag{12}$$

The key point is that $(G(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators (on some Hilbert space to be determined), and thanks to von Neumann's ergodic mean theorem [20], the limit $\langle D\rangle = \lim_{S\to+\infty} \frac{1}{S}\int_0^S G(s)D \,\mathrm{d}s$ makes sense. The Hilbert space which realizes $(G(s))_{s\in\mathbb{R}}$ as a C^0-group of unitary operators appears as a L^2 weighted space, with respect to some field of symmetric definite positive matrices. We assume that there is a matrix field P such that

$${}^tP = P,\;\; P(y)\xi\cdot\xi > 0,\;\; \xi \in \mathbb{R}^m \setminus \{0\},\;\; y \in \mathbb{R}^m,\;\; P^{-1}, P \in L^2_{\mathrm{loc}}(\mathbb{R}^m) \tag{13}$$

$$[b, P] := (b\cdot\nabla_y)P - \partial_y bP - P^t\partial_y b = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^m). \tag{14}$$

For example, when the vector field b is uniform, we can take $P = I_m$. Notice that we have the following characterization for (14) cf. Proposition 3.8 [8]

Proposition 2.1 *Consider $b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)$ (not necessarily divergence free) with at most linear growth at infinity and $A(y) \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Then $[b, A] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ iff*

$$A(Y(s; y)) = \partial Y(s; y)A(y)\, {}^t\partial Y(s; y),\;\; s \in \mathbb{R},\;\; y \in \mathbb{R}^m.$$

Given a matrix field P satisfying (13), (14), we consider the set of matrix fields

$$H_Q = \left\{ A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R}) \text{ measurable } : \; Q^{1/2}AQ^{1/2} \in L^2 \right\}$$

where $Q = P^{-1}$, and the scalar product on H_Q

$$(A, B)_Q = \int_{\mathbb{R}^m} Q^{1/2}(y)\, A(y)\, Q^{1/2}(y) : Q^{1/2}(y)\, B(y)\, Q^{1/2}(y) \,\mathrm{d}y = \int_{\mathbb{R}^m} QA : BQ \,\mathrm{d}y$$

for any $A, B \in H_Q$. For any two matrices in $\mathcal{M}_m(\mathbb{R})$, the notation $A : B$ stands for $\mathrm{tr}({}^t AB)$. Notice that the application $J : H_Q \to L^2(\mathbb{R}^m; \mathcal{M}_m(\mathbb{R}))$, given by $J(A) = Q^{1/2}\, A\, Q^{1/2}, A \in H_Q$ is an isometry, implying that $(H_Q, (\cdot,\cdot))$ is a Hilbert space. We prove that the family of applications $G(s) : H_Q \to H_Q, s \in \mathbb{R}$, is a C^0-group of unitary operators on H_Q cf. Proposition 3.1. Thanks to Theorem 3.1 (see [20] for more details), the average of a matrix field $\langle A \rangle := \lim_{S\to+\infty} \frac{1}{S}\int_0^S G(s)A \,\mathrm{d}s$ is well defined and coincides with the orthogonal projection on $\{B \in H_Q \,:\, G(s)B = B \text{ for any } s \in \mathbb{R}\}$.

Theorem 2.1 *Assume that* (4), (5), (13), (14) *hold true. We denote by L the infinitesimal generator of the group* $(G(s))_{s\in\mathbb{R}}$.

1. *For any matrix field* $A \in H_Q$ *we have the strong convergence in* H_Q

$$\langle A \rangle := \lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} \partial Y(-s; Y(s;\cdot))A(Y(s;\cdot))\, {}^t\partial Y(-s; Y(s;\cdot)) \,\mathrm{d}s = \mathrm{Proj}_{\ker L} A$$

uniformly with respect to $r \in \mathbb{R}$.

2. *If* $A \in H_Q$ *is a field of symmetric positive matrices, then so is* $\langle A \rangle$.

3. *If* $A \in H_Q$ *and there is* $\alpha > 0$ *such that*

$$Q^{1/2}(y)A(y)Q^{1/2}(y) \geq \alpha I_m, \;\; y \in \mathbb{R}^m$$

therefore we have

$$Q^{1/2}(y)\,\langle A \rangle\,(y)Q^{1/2}(y) \geq \alpha I_m, \;\; y \in \mathbb{R}^m$$

and in particular, $\langle A \rangle\,(y)$ *is definite positive for* $y \in \mathbb{R}^m$.

4. *If* $A \in H_Q \cap H_Q^\infty$ *(see* (18) *for the definition of the Banach space* H_Q^∞*), then* $\langle A \rangle \in H_Q \cap H_Q^\infty$ *and*

$$|\,\langle A \rangle\,|_Q \leq |A|_Q, \;\; |\,\langle A \rangle\,|_{H_Q^\infty} \leq |A|_{H_Q^\infty}.$$

In view of Theorem 2.1, the limit of the parabolic problems (7) becomes, accordingly to (12)

$$\begin{cases} \partial_t v - \mathrm{div}_z(\langle D \rangle\, \nabla_z v) = 0, & (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v(0,z) = u^{\mathrm{in}}(z), & z \in \mathbb{R}^m. \end{cases} \tag{15}$$

Under some regularity assumptions (see Section 6), we obtain a strong convergence result for the family $(v^\varepsilon)_{\varepsilon>0}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, toward the solution v of the problem (15). Coming back to the family $(u^\varepsilon)_{\varepsilon>0}$, through the variable change in (6), and thanks to the fact that for any $s \in \mathbb{R}$, $Y(s;\cdot)$ is measure preserving, we justify that at any time $t \in \mathbb{R}_+, \varepsilon > 0$, $u^\varepsilon(t,\cdot)$ behaves (in $L^2(\mathbb{R}^m)$) like the composition product between $v(t,\cdot)$ and $Y(-t/\varepsilon;\cdot)$, that is

$$\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^m} (\; u^\varepsilon(t,y) - v(t, Y(-t/\varepsilon; y))\;)^2 \,\mathrm{d}y = \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^m} (\; v^\varepsilon(t,z) - v(t,z)\;)^2 \,\mathrm{d}z = 0$$

uniformly with respect to $t \in [0,T]$, for any $T \in \mathbb{R}_+$.

Theorem 2.2 *Assume that the hypotheses* (4), (5), (22), (23), (28), (29), (32) *hold true together with all the regularity conditions in Proposition 4.4. We suppose that* $u^{\mathrm{in}} \in H_R^2$ *(see* (24), (26) *for the definitions of* H_R^k *and* ∇^R*) and we denote by* $(u^\varepsilon)_{\varepsilon>0}$ *the variational solutions of* (1) *and by* $(v^\varepsilon)_{\varepsilon>0}$ *the functions*

$$v^\varepsilon(t,z) = u^\varepsilon(t, Y(t/\varepsilon; z)), \;\; (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m, \;\; \varepsilon > 0.$$

Then the family $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$ to the unique variational solution $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of (15)*. The function v has the regularity*

$$\partial_t v,\ \nabla_z^R v,\ \nabla_z^R \otimes \nabla_z^R v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)),\ \ \partial_t \nabla_z^R v \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$$

and $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ converges toward $\nabla_z v$ in $L^2_{\mathrm{loc}}(\mathbb{R}_+; X_P)$ when ε goes to 0 *(see Section 3 for the definition of the Hilbert space X_P).*

Under additional hypotheses we can justify that $v^\varepsilon = v + \mathcal{O}(\varepsilon)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as suggested by the formal Ansatz (9).

Theorem 2.3 *Assume that the hypotheses* (4)*,* (5)*,* (22)*,* (23)*,* (28)*,* (29) *hold true. Moreover, we assume that the solution v of the limit model* (15) *is smooth enough, that is*

$$\nabla_z^R v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)),\quad \nabla_z^R \otimes \nabla_z^R v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$$

$$\nabla_z^R \partial_t v \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)),\quad \nabla_z^R \otimes \nabla_z^R \partial_t v \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$$

$$\nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)),\quad \nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$$

and that there is a smooth matrix field C, that is

$$\mathrm{div}_y(RC),\ \ b_k \cdot \nabla_y \mathrm{div}_y(RC),\ \ b_l \cdot \nabla_y (b_k \cdot \nabla_y \mathrm{div}_y(RC)) \in L^\infty(\mathbb{R}^m),\ \ k, l \in \{1, ..., m\}$$

$$RC\,{}^tR,\ \ b_k \cdot \nabla_y (RC\,{}^tR),\ \ b_l \cdot \nabla_y (b_k \cdot \nabla_y (RC\,{}^tR)) \in L^\infty(\mathbb{R}^m),\ \ k, l \in \{1, ..., m\}$$

such that the following decomposition holds true

$$D = \langle D \rangle + L(C),\ \ C \in (\ker L)^\perp.$$

We denote by $(u^\varepsilon)_{\varepsilon>0}$ the variational solutions of (1)*. Then for any $T \in \mathbb{R}_+$, there is a constant C_T such that*

$$\sup_{t\in[0,T]} \|u^\varepsilon(t,\cdot) - v(t, Y(-t/\varepsilon;\cdot))\|_{L^2(\mathbb{R}^m)} \le C_T \varepsilon$$

$$\left(\int_0^T |\nabla_y u^\varepsilon(t,\cdot) - \nabla_y v(t, Y(-t/\varepsilon;\cdot))|_P^2 \,\mathrm{d}t \right)^{1/2} \le C_T \varepsilon.$$

3 The average of a matrix field

Consider a matrix field P satisfying the hypotheses (13), (14) and the inverse matrix field $Q = P^{-1}$. We introduce the set

$$H_Q = \left\{ A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R})\ :\ \int_{\mathbb{R}^m} Q(y)A(y) : A(y)Q(y)\,\mathrm{d}y < +\infty \right\}$$

and the application

$$(\cdot,\cdot)_Q : H_Q \times H_Q \to \mathbb{R},\ \ (A, B)_Q = \int_{\mathbb{R}^m} Q(y)A(y) : B(y)Q(y)\,\mathrm{d}y,\ A, B \in H_Q.$$

It is easily seen that the bilinear application $(\cdot,\cdot)_Q$ is symmetric and positive definite and that the set H_Q endowed with the scalar product $(\cdot,\cdot)_Q$ is a Hilbert space, whose norm is denoted by $|A|_Q = (A, A)_Q^{1/2}, A \in H_Q$. Clearly $C^0_c(\mathbb{R}^m, \mathcal{M}_m(\mathbb{R})) \subset H_Q$. Observe that

$H_Q \subset L^1_{\mathrm{loc}}(\mathbb{R}^m; \mathcal{M}_m(\mathbb{R}))$. Indeed, if for any matrix M the notation $|M|$ stands for the norm subordonated to the euclidian norm of $\mathbb{R}^m$

$$|M| = \sup_{\xi \in \mathbb{R}^m \setminus \{0\}} \frac{|M\xi|}{|\xi|} \leq (M : M)^{1/2}$$

we have

$$\begin{aligned}
|A| &= \sup_{\xi,\eta \neq 0} \frac{A\xi \cdot \eta}{|\xi|\ |\eta|} \\
&= \sup_{\xi,\eta \neq 0} \frac{Q^{1/2} A Q^{1/2} P^{1/2}\xi \cdot P^{1/2}\eta}{|P^{1/2}\xi|\ |P^{1/2}\eta|} \frac{|P^{1/2}\xi|}{|\xi|} \frac{|P^{1/2}\eta|}{|\eta|} \\
&\leq |Q^{1/2} A Q^{1/2}|\ |P^{1/2}|^2 \\
&\leq (Q^{1/2} A Q^{1/2} : Q^{1/2} A Q^{1/2})^{1/2}\ |P|.
\end{aligned}$$

We deduce that for any $R > 0$

$$\int_{B_R} |A(y)|\ \mathrm{d}y \leq \int_{B_R} (Q^{1/2} A Q^{1/2} : Q^{1/2} A Q^{1/2})^{1/2}\ |P|\ \mathrm{d}y \leq (A, A)_Q^{1/2} \left(\int_{B_R} |P(y)|^2\ \mathrm{d}y \right)^{1/2}.$$

When replacing the matrix field Q by the matrix field P, we obtain the Hilbert space

$$H_P = \left\{ A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R})\ :\ \int_{\mathbb{R}^m} P(y)A(y) : A(y)P(y)\ \mathrm{d}y < +\infty \right\}$$

endowed with the scalar product

$$(\cdot,\cdot)_P : H_P \times H_P \to \mathbb{R},\ \ (A, B)_P = \int_{\mathbb{R}^m} P(y)A(y) : B(y)P(y)\ \mathrm{d}y,\ A, B \in H_P.$$

Motivated by the computations leading to (8), we consider the family of linear transformations $(G(s))_{s\in\mathbb{R}}$, acting on matrix fields. It happens that $(G(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators on H_Q (see [8] Proposition 3.12 for details). For any function $f = f(y), y \in \mathbb{R}^m$, the notation $f_s = f_s(z)$ stands for the composition product $f_s = f \circ Y(s; \cdot)$.

Proposition 3.1 *Assume that the hypotheses* (4), (5), (13), (14) *hold true.*

1. *The family of applications*

$$A \to G(s)A := \partial Y^{-1}(s; \cdot) A_s\ {}^t\partial Y^{-1}(s; \cdot) = \partial Y(-s; Y(s; \cdot)) A_s\ {}^t\partial Y(-s; Y(s; \cdot))$$

   is a C^0-group of unitary operators on H_Q.

2. *If A is a field of symmetric matrices, then so is $G(s)A$, for any $s \in \mathbb{R}$.*

3. *If A is a field of positive matrices, then so is $G(s)A$, for any $s \in \mathbb{R}$.*

4. *If there is $\alpha > 0$ such that $Q^{1/2}(y)A(y)Q^{1/2}(y) \geq \alpha I_m, y \in \mathbb{R}^m$, then for any $s \in \mathbb{R}$ we have $Q^{1/2}(y)(G(s)A)(y)Q^{1/2}(y) \geq \alpha I_m, y \in \mathbb{R}^m$.*

Proof.
1. Thanks to the characterization in Proposition 2.1 we know that

$$P_s = \partial Y(s; \cdot) P\ {}^t\partial Y(s; \cdot),\quad s \in \mathbb{R}. \tag{16}$$

For any $s \in \mathbb{R}$ we consider the matrix field $\mathcal{O}(s;\cdot) = Q_s^{1/2}\partial Y(s;\cdot)Q^{-1/2}$. Observe that $\mathcal{O}(s;\cdot)$ is a field of orthogonal matrices, for any $s \in \mathbb{R}$. Indeed we have, thanks to (16)

$$\begin{aligned} {}^t\mathcal{O}(s;\cdot)\mathcal{O}(s;\cdot) &= Q^{-1/2}\,{}^t\partial Y(s;\cdot)Q_s^{1/2}Q_s^{1/2}\partial Y(s;\cdot)Q^{-1/2} \\ &= Q^{-1/2}\left(\partial Y^{-1}(s;\cdot)P_s\,{}^t\partial Y^{-1}(s;\cdot)\right)^{-1}Q^{-1/2} \\ &= Q^{-1/2}P^{-1}Q^{-1/2} \\ &= I_m \end{aligned}$$

implying that for any matrix field A we have

$$Q^{1/2}G(s)AQ^{1/2} = Q^{1/2}\partial Y^{-1}(s;\cdot)A_s\,{}^t\partial Y^{-1}(s;\cdot)Q^{1/2} = {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot). \quad (17)$$

It is easily seen that if $A \in H_Q$, then for any $s \in \mathbb{R}$

$$\begin{aligned} |G(s)A|_Q^2 &= \int_{\mathbb{R}^m} Q^{1/2}G(s)AQ^{1/2} : Q^{1/2}G(s)AQ^{1/2}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^m} {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot) : {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^m} Q_s^{1/2}A_sQ_s^{1/2} : Q_s^{1/2}A_sQ_s^{1/2}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^m} Q^{1/2}AQ^{1/2} : Q^{1/2}AQ^{1/2}\,\mathrm{d}y = |A|_Q^2 \end{aligned}$$

saying that $G(s)$ is a unitary transformation for any $s \in \mathbb{R}$. The group property of the family $(G(s))_{s\in\mathbb{R}}$ follows easily from the group property of the flow $(Y(s;\cdot))_{s\in\mathbb{R}}$

$$\begin{aligned} G(s)G(t)A &= \partial Y^{-1}(s;\cdot)(G(t)A)_s\,{}^t\partial Y^{-1}(s;\cdot) \\ &= \partial Y^{-1}(s;\cdot)\partial Y^{-1}(t;Y(s;\cdot))(A_t)_s\,{}^t\partial Y^{-1}(t;Y(s;\cdot))\,{}^t\partial Y^{-1}(s;\cdot) \\ &= \partial Y^{-1}(t+s;\cdot)A_{t+s}\,{}^t\partial Y^{-1}(t+s;\cdot) = G(t+s)A, \quad A \in H_Q. \end{aligned}$$

The continuity of the group, *i.e.*, $\lim_{s\to 0} G(s)A = A$ strongly in H_Q, is left to the reader.
2. Notice that $G(s)$ commutes with transposition

$$\begin{aligned} {}^t(G(s)A) &= {}^t\left(\partial Y^{-1}(s;\cdot)A_s\,{}^t\partial Y^{-1}(s;\cdot)\right) \\ &= \partial Y^{-1}(s;\cdot)\,{}^tA_s\,{}^t\partial Y^{-1}(s;\cdot) \\ &= G(s)\,{}^tA. \end{aligned}$$

In particular, if ${}^tA = A$, then ${}^t(G(s)A) = G(s)A$.
3. We use the formula (17). For any $\xi, \eta \in \mathbb{R}^m$, the notation $\xi \otimes \eta$ stands for the matrix whose (i,j) entry is $\xi_i\eta_j$. For any $\xi \in \mathbb{R}^m$ we have

$$\begin{aligned} G(s)A : Q^{1/2}\xi \otimes Q^{1/2}\xi &= Q^{1/2}G(s)AQ^{1/2} : \xi \otimes \xi \\ &= {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot) : \xi \otimes \xi \\ &= Q_s^{1/2}A_sQ_s^{1/2} : \mathcal{O}(s;\cdot)(\xi \otimes \xi)\,{}^t\mathcal{O}(s;\cdot) \\ &= Q_s^{1/2}A_sQ_s^{1/2} : (\mathcal{O}(s;\cdot)\xi) \otimes (\mathcal{O}(s;\cdot)\xi) \\ &= A_s : (Q_s^{1/2}\mathcal{O}(s;\cdot)\xi) \otimes (Q_s^{1/2}\mathcal{O}(s;\cdot)\xi). \end{aligned}$$

As A is a field of positive matrices, therefore $G(s)A$ is a field of positive matrices as well.
4. Assume that there is $\alpha > 0$ such that $Q^{1/2}AQ^{1/2} \geq \alpha I_m$. As before we write for any $\xi \in \mathbb{R}^m$

$$Q^{1/2}G(s)AQ^{1/2} : \xi \otimes \xi = (Q^{1/2}AQ^{1/2})_s : (\mathcal{O}(s;\cdot)\xi) \otimes (\mathcal{O}(s;\cdot)\xi) \geq \alpha|\mathcal{O}(s;\cdot)\xi|^2 = \alpha|\xi|^2$$

saying that $Q^{1/2}G(s)AQ^{1/2} \geq \alpha I_m$. □

We denote by L the infinitesimal generator of the group G

$$L : \mathrm{dom}(L) \subset H_Q \to H_Q, \ \ \mathrm{dom}L = \{A \in H_Q \ : \ \exists \lim_{s\to 0} \frac{G(s)A - A}{s} \ \text{ in } \ H_Q\}$$

and $L(A) = \lim_{s\to 0} \frac{G(s)A-A}{s}$ for any $A \in \mathrm{dom}(L)$. Notice that $C^1_c(\mathbb{R}^m) \subset \mathrm{dom}(L)$ and $L(A) = (b \cdot \nabla_y)A - \partial_y b A - A\, {}^t\partial_y b$, $A \in C^1_c(\mathbb{R}^m)$ (use the hypothesis $Q \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ and the dominated convergence theorem). The main properties of the operator L are summarized below (see [8] Proposition 3.13 for details)

Proposition 3.2 *Assume that the hypotheses* (4), (5), (13), (14) *hold true.*

1. *The domain of L is dense in H_Q and L is closed.*
2. *The matrix field $A \in H_Q$ belongs to* $\mathrm{dom}(L)$ *iff there is a constant $C > 0$ such that*

$$|G(s)A - A|_Q \leq C|s|, \ \ s \in \mathbb{R}.$$

3. *The operator L is skew-adjoint and we have the orthogonal decomposition $H_Q = \ker L \overset{\perp}{\oplus} \overline{\mathrm{Range}\, L}$.*

The transformations $(G(s))_{s\in\mathbb{R}}$ behave nicely also when applied on weighted L^∞ spaces. We introduce the set

$$H^\infty_Q = \{A(y) \text{ measurable } : \ Q^{1/2}AQ^{1/2} : Q^{1/2}AQ^{1/2} = QA : AQ \in L^\infty(\mathbb{R}^m)\}. \tag{18}$$

It is a Banach space with respect to the norm

$$|A|_{H^\infty_Q} = \mathrm{ess\ sup}_{y\in\mathbb{R}^m}(Q(y)A(y) : A(y)Q(y))^{1/2}.$$

This space is left invariant by $(G(s))_{s\in\mathbb{R}}$. Indeed, let us consider $A \in H^\infty_Q$ and, thanks to (17) and to the orthogonality of $\mathcal{O}(s;\cdot)$, observe that

$$\begin{aligned} Q^{1/2}G(s)AQ^{1/2} : Q^{1/2}G(s)AQ^{1/2} &= {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot) : {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot) \\ &= (Q^{1/2}\,A\,Q^{1/2} : Q^{1/2}\,A\,Q^{1/2})_s, \quad s \in \mathbb{R}. \end{aligned}$$

We deduce that for any $s \in \mathbb{R}$ we have $G(s)A \in H^\infty_Q$ and $|G(s)A|_{H^\infty_Q} = |A|_{H^\infty_Q}$.
We are now in position to apply the von Neumann's ergodic mean theorem.

Theorem 3.1 *(von Neumann's ergodic mean theorem)*
Let $(\mathcal{G}(s))_{s\in\mathbb{R}}$ be a C^0-group of unitary operators on an Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ and $\mathcal{L}$ be its infinitesimal generator. Then for any $x \in \mathcal{H}$, we have the strong convergence in $\mathcal{H}$

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} \mathcal{G}(s)x \ \mathrm{d}s = \mathrm{Proj}_{Ker\mathcal{L}}x, \quad \textit{uniformly with respect to } r \in \mathbb{R}.$$

The proof of Theorem 2.1 comes immediately, by applying Theorem 3.1 to the group in Proposition 3.1.

Proof. (of Theorem 2.1) The first and second statements are obvious.
3. For any $\xi \in \mathbb{R}^m, \psi \in C^0_c(\mathbb{R}^m), \psi \geq 0$ we have $\psi(\cdot)P^{1/2}\xi \otimes P^{1/2}\xi \in H_Q$ and we can write,

thanks to (17)

$$
\begin{aligned}
(G(s)A, \psi(\cdot)P^{1/2}\xi \otimes P^{1/2}\xi)_Q &= \int_{\mathbb{R}^m} Q^{1/2}G(s)AQ^{1/2} : \psi(y)\xi \otimes \xi \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} \psi(y) \, {}^t\mathcal{O}(s;y)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;y)\xi \cdot \xi \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} \psi(y)Q_s^{1/2}A_sQ_s^{1/2} : \mathcal{O}(s;y)\xi \otimes \mathcal{O}(s;y)\xi \, \mathrm{d}y \\
&\geq \alpha \int_{\mathbb{R}^m} |\mathcal{O}(s;y)\xi|^2\psi(y) \, \mathrm{d}y \\
&= \alpha|\xi|^2 \int_{\mathbb{R}^m} \psi(y) \, \mathrm{d}y.
\end{aligned}
$$

Taking the average over $[0, S]$ and letting $S \to +\infty$ yield

$$
\begin{aligned}
\int_{\mathbb{R}^m} Q^{1/2} \langle A \rangle Q^{1/2} : \xi \otimes \xi\psi(y) \, \mathrm{d}y &= (\langle A \rangle, \psi P^{1/2}\xi \otimes P^{1/2}\xi)_Q \\
&= \lim_{S\to+\infty} \frac{1}{S} \int_0^S (G(s)A, \psi P^{1/2}\xi \otimes P^{1/2}\xi)_Q \, \mathrm{d}s \\
&\geq \int_{\mathbb{R}^m} \alpha|\xi|^2\psi(y) \, \mathrm{d}y
\end{aligned}
$$

implying that

$$
Q^{1/2}(y) \langle A \rangle (y)Q^{1/2}(y) \geq \alpha I_m, \ \ y \in \mathbb{R}^m.
$$

4. Obviously, for any $A \in H_Q$, we have by the properties of the orthogonal projection on $\ker L$ that $| \langle A \rangle |_Q = |\mathrm{Proj}_{\ker L} A|_Q \leq |A|_Q$. For the last inequality, consider $M \in \mathcal{M}_m(\mathbb{R})$ a fixed matrix, $\psi \in C_c^0(\mathbb{R}^m), \psi \geq 0$ and, as before, observe that $\psi P^{1/2}MP^{1/2} \in H_Q$, which allows us to write

$$
\begin{aligned}
(G(s)A, &\psi P^{1/2}MP^{1/2})_Q = \int_{\mathbb{R}^m} Q^{1/2}G(s)AQ^{1/2} : \psi M \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} {}^t\mathcal{O}(s;y)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;y) : \psi M \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} Q_s^{1/2}A_sQ_s^{1/2} : \mathcal{O}(s;y)M \, {}^t\mathcal{O}(s;y) \, \mathrm{d}y \\
&\leq \int_{\mathbb{R}^m} \sqrt{Q_s^{1/2}A_sQ_s^{1/2} : Q_s^{1/2}A_sQ_s^{1/2}} \sqrt{\mathcal{O}(s;y)M \, {}^t\mathcal{O}(s;y) : \mathcal{O}(s;y)M \, {}^t\mathcal{O}(s;y)}\psi \, \mathrm{d}y \\
&\leq |A|_{H_Q^\infty}(M : M)^{1/2} \int_{\mathbb{R}^m} \psi(y) \, \mathrm{d}y.
\end{aligned}
$$

Taking the average over $[0, S]$ and letting $S \to +\infty$, lead to

$$
\int_{\mathbb{R}^m} Q^{1/2} \langle A \rangle Q^{1/2} : M\psi(y) \, \mathrm{d}y = (\langle A \rangle, \psi P^{1/2}MP^{1/2})_Q \leq |A|_{H_Q^\infty}(M : M)^{1/2} \int_{\mathbb{R}^m} \psi(y) \, \mathrm{d}y.
$$

We deduce that

$$
Q^{1/2}(y) \langle A \rangle (y)Q^{1/2}(y) : M \leq |A|_{H_Q^\infty}(M : M)^{1/2}, \ \ y \in \mathbb{R}^m, \ \ M \in \mathcal{M}_m(\mathbb{R})
$$

saying that

$$
| \langle A \rangle |_{H_Q^\infty} = \mathrm{ess\ sup}_{y\in\mathbb{R}^m} \sqrt{Q^{1/2}(y) \langle A \rangle (y)Q^{1/2}(y) : Q^{1/2}(y) \langle A \rangle (y)Q^{1/2}(y)} \leq |A|_{H_Q^\infty}.
$$

□

We introduce also the sets of vector fields

$$X_Q = \{c : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable } : \int_{\mathbb{R}^m} Q(y) : c(y) \otimes c(y) \,\mathrm{d}y < +\infty\}$$

$$X_Q^\infty = \{c : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable } : |Q^{1/2}c| \in L^\infty(\mathbb{R}^m)\}.$$

The vector space X_Q, endowed with the scalar product

$$(\cdot,\cdot)_Q : X_Q \times X_Q \to \mathbb{R}, \ \ (c,d)_Q = \int_{\mathbb{R}^m} Q(y) : c(y) \otimes d(y) \,\mathrm{d}y, \ \ c, d \in X_Q$$

becomes a Hilbert space, whose norm is denoted by $|c|_Q = (c,c)_Q^{1/2}, c \in X_Q$. We use the same notation for the scalar product and norm of H_Q, resp. X_Q. Obviously, it should be understood in the right framework, depending on the arguments being matrix fields, resp. vector fields.
The vector space X_Q^∞ is a Banach space with respect to the norm

$$|c|_{X_Q^\infty} = \text{ess sup}_{y \in \mathbb{R}^m} |Q^{1/2}(y)c(y)|.$$

We end this section by indicating a sufficient condition for (13), (14). Assume that there is a matrix field $R(y)$ such that

$$\det R(y) \neq 0, \ y \in \mathbb{R}^m, \ \ R \in L^1_{\text{loc}}(\mathbb{R}^m) \tag{19}$$

$$(b \cdot \nabla_y) R + R \partial_y b = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m). \tag{20}$$

The hypothesis (20) is equivalent to $R(Y(s;y))\partial Y(s;y) = R(y), \ \ (s,y) \in \mathbb{R} \times \mathbb{R}^m$, which also writes

$$\partial Y(s;y) R^{-1}(y) = R^{-1}(Y(s;y)), \ \ (s,y) \in \mathbb{R} \times \mathbb{R}^m. \tag{21}$$

We deduce that (20) is equivalent to $(b \cdot \nabla_y) R^{-1} = \partial_y b R^{-1}$, saying that the columns of R^{-1} are vector fields in involution with b. The vector fields in the columns of R^{-1} are denoted $b_i, 1 \le i \le m$. At any point $y \in \mathbb{R}^m$ they form a basis for $\mathbb{R}^m$ cf. (19) and are supposed smooth

$$b_i \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^m), \ \ \text{div}_y b_i \in L^\infty(\mathbb{R}^m), \ \ 1 \le i \le m. \tag{22}$$

We assume that any field b_i satisfies the growth condition

$$\forall i \in \{1, ..., m\}, \ \exists C_i > 0 \text{ such that } |b_i(y)| \le C_i(1 + |y|), \ \ y \in \mathbb{R}^m \tag{23}$$

which guarantees the existence of the global flows $Y_i(s;y) \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^m), i \in \{1, ..., m\}$. Clearly $R^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^m)$, since b_i, which are the columns of R^{-1}, are supposed locally bounded on $\mathbb{R}^m$. Since $y \to R^{-1}(y)$ is continuous, the function $y \to \det R^{-1}(y)$ remains away from 0 on any compact set of $\mathbb{R}^m$, implying that $R = (R^{-1})^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^m)$. In particular ${}^tRR, ({}^tRR)^{-1}$ are locally bounded, and therefore locally square integrable on $\mathbb{R}^m$. We define $Q = {}^tRR, P = Q^{-1} = R^{-1}\,{}^tR^{-1}$ and observe that (13), (14) are satisfied. Indeed, $P(y)$ is symmetric, definite positive, locally square integrable, together with its inverse $Q = P^{-1}$ and, thanks to (21), we have

$$\begin{aligned} P(Y(s;y)) &= R^{-1}(Y(s;y))\,{}^tR^{-1}(Y(s;y)) \\ &= \partial Y(s;y) R^{-1}(y)\,{}^tR^{-1}(y)\,{}^t\partial Y(s;y) \\ &= \partial Y(s;y) P(y)\,{}^t\partial Y(s;y) \end{aligned}$$

saying that $[b, P] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ cf. Proposition 2.1. Under the hypotheses (22), (23), the space H_Q, H_Q^∞ also write

$$H_Q = \left\{ A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R}) \text{ measurable } : \int_{\mathbb{R}^m} R(y)A(y)\,{}^tR(y) : R(y)A(y)\,{}^tR(y)\ \mathrm{d}y < +\infty \right\}$$

and

$$H_Q^\infty = \{A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R}) \text{ measurable } : R(y)A(y)\,{}^tR(y) : R(y)A(y)\,{}^tR(y) \in L^\infty(\mathbb{R}^m)\}.$$

Given the family $(b_i)_{1\le i\le m}$ of vector fields in involution with respect to b, we construct the following H^1 type space on $\mathbb{R}^m$

$$H_R^1 = \bigcap_{i=1}^m \mathrm{dom}(b_i \cdot \nabla_y) = \{u \in L^2(\mathbb{R}^m) \ : \ {}^tR^{-1}\nabla_y u := {}^t(b_1 \cdot \nabla_y u, ..., b_m \cdot \nabla_y u) \in L^2(\mathbb{R}^m)^m\} \tag{24}$$

endowed with the scalar product

$$(u, v)_R = \int_{\mathbb{R}^m} u(y)v(y)\ \mathrm{d}y + \sum_{i=1}^m \int_{\mathbb{R}^m} (b_i \cdot \nabla_y u)(b_i \cdot \nabla_y v)\ \mathrm{d}y,\ u, v \in H_R^1.$$

It is a Hilbert space, whose norm is denoted by $|\cdot|_R$. The operators $b_i \cdot \nabla_y$ are the infinitesimal generators of the C^0-groups of linear transformations on $L^2(\mathbb{R}^m)$ given by

$$\tau_i(s)u = u \circ Y_i(s;\cdot),\ \ u \in L^2(\mathbb{R}^m),\ \ s \in \mathbb{R},\ \ i \in \{1, ..., m\}.$$

The hypothesis $\mathrm{div}_y b_i \in L^\infty(\mathbb{R}^m)$ plays a crucial role when looking for a bound for the Jacobian determinant of ∂Y_i.

Remark 3.1 *Notice that every element of H_R^1 has a weak gradient in $L^2_{loc}(\mathbb{R}^m)$. Indeed, if $u \in H_R^1$, we define $v_i = b_i \cdot \nabla_y u$, $i \in \{1, ..., m\}$ and consider $V(y) = {}^tR(y)\,{}^t(v_1(y), ..., v_m(y))$, $y \in \mathbb{R}^m$. The field V is locally square integrable on $\mathbb{R}^m$ since R is locally bounded on $\mathbb{R}^m$ and $(v_i)_{1\le i\le m}$ are square integrable on $\mathbb{R}^m$. Using the dual basis $\{c_1, ..., c_m\}$ of $\{b_1, ..., b_m\}$ we write for any $\xi \in (C_c^1(\mathbb{R}^m))^m$*

$$\begin{aligned}\int_{\mathbb{R}^m} V(y) \cdot \xi(y)\ \mathrm{d}y &= \int_{\mathbb{R}^m} \sum_{i=1}^m (V(y) \cdot b_i(y))\ (c_i(y) \cdot \xi(y))\ \mathrm{d}y \\ &= -\int_{\mathbb{R}^m} u(y)\mathrm{div}_y \left(\sum_{i=1}^m (c_i(y) \cdot \xi(y)) b_i(y) \right)\ \mathrm{d}y = -\int_{\mathbb{R}^m} u(y)\mathrm{div}_y \xi\ \mathrm{d}y\end{aligned}$$

which shows that V is the weak gradient of u. The H_R^1 norm of u can be written using the X_P norm of the gradient

$$\begin{aligned}|u|_R^2 &= \|u\|^2_{L^2(\mathbb{R}^m)} + \|{}^tR^{-1}\nabla_y u\|^2_{L^2(\mathbb{R}^m)} \\ &= \int_{\mathbb{R}^m} \{(u(y))^2 + (R^{-1}\,{}^tR^{-1} V \cdot V)\}\ \mathrm{d}y \\ &= \int_{\mathbb{R}^m} \{(u(y))^2 + (P(y)V(y) \cdot V(y))\}\ \mathrm{d}y \\ &= \|u\|^2_{L^2(\mathbb{R}^m)} + |V|_P^2.\end{aligned} \tag{25}$$

In the sequel, for any $u \in H^1_R$, the notation $\nabla_y u$ stands for the weak gradient of u, and we have $\nabla_y u = {}^t R\, {}^t(b_1 \cdot \nabla_y u, ..., b_m \cdot \nabla_y u)$. We introduce the differential operator

$$\nabla^R_y := {}^t R^{-1} \nabla_y = {}^t(b_1 \cdot \nabla_y, ..., b_m \cdot \nabla_y).$$

We will also use the space

$$H^2_R = \{u \in H^1_R \;:\; \nabla^R_y u \in (H^1_R)^m\} = \{u \in H^1_R \;:\; b_j \cdot \nabla_y(b_i \cdot \nabla_y u) \in L^2(\mathbb{R}^m), 1 \le i, j \le m\} \tag{26}$$

and the differential operator $(\nabla^R_y)^2 := \nabla^R_y \otimes \nabla^R_y$ given by

$$(\nabla^R_y \otimes \nabla^R_y)_{ij} = b_j \cdot \nabla_y(b_i \cdot \nabla_y), \;\; i, j \in \{1, ..., m\}.$$

3.1 Examples

In this paragraph we compute explicitly the average matrix field in two cases. Both of them deal with periodic flows. Consider the vector field $b(y) = (\gamma y_2, -\beta y_1)$, for any $y = (y_1, y_2) \in \mathbb{R}^2$, with $\beta, \gamma \in \mathbb{R}^\star_+$. We denote by $Y(s; y)$ the flow of the vector field b. We intend to determine the average along the flow Y of the matrix field

$$D(y) = \begin{pmatrix} \lambda_1(y) & 0 \\ 0 & \lambda_2(y) \end{pmatrix}, \;\; y \in \mathbb{R}^2$$

where λ_1, λ_2 are two given functions. It is easily seen that the flow is $2\pi/\sqrt{\beta\gamma}$-periodic and writes $Y(s; y) = \mathcal{R}(-s; \beta, \gamma)y, \; (s, y) \in \mathbb{R} \times \mathbb{R}^2$, with

$$\mathcal{R}(s; \beta, \gamma) = \begin{pmatrix} \cos(\sqrt{\beta\gamma}\, s) & -\sqrt{\frac{\gamma}{\beta}} \sin(\sqrt{\beta\gamma}\, s) \\ \sqrt{\frac{\beta}{\gamma}} \sin(\sqrt{\beta\gamma}\, s) & \cos(\sqrt{\beta\gamma}\, s) \end{pmatrix}.$$

By Theorem 2.1 we deduce that

$$\begin{aligned} \langle D \rangle &= \frac{\sqrt{\beta\gamma}}{2\pi} \int_0^{2\pi/\sqrt{\beta\gamma}} \partial Y(-s; Y(s; \cdot)) D(Y(s; \cdot))\, {}^t\partial Y(-s; Y(s; \cdot))\, \mathrm{d}s \\ &= \frac{\sqrt{\beta\gamma}}{2\pi} \int_0^{2\pi/\sqrt{\beta\gamma}} \mathcal{R}(s; \beta, \gamma) D(Y(s; \cdot)) \mathcal{R}(-s; \gamma, \beta)\, \mathrm{d}s \\ &= \begin{pmatrix} \langle D \rangle_{11} & \langle D \rangle_{12} \\ \langle D \rangle_{21} & \langle D \rangle_{22} \end{pmatrix} \end{aligned}$$

where

$$\langle D \rangle_{11} = \frac{1}{2} \left\langle \lambda_1 [1 + \cos(2\sqrt{\beta\gamma}\, \cdot)] \right\rangle + \frac{\gamma}{2\beta} \left\langle \lambda_2 [1 - \cos(2\sqrt{\beta\gamma}\, \cdot)] \right\rangle$$

$$\langle D \rangle_{12} = \langle D \rangle_{21} = \frac{\sqrt{\beta}}{2\sqrt{\gamma}} \left\langle \lambda_1 \sin(2\sqrt{\beta\gamma}\, \cdot) \right\rangle - \frac{\sqrt{\gamma}}{2\sqrt{\beta}} \left\langle \lambda_2 \sin(2\sqrt{\beta\gamma}\, \cdot) \right\rangle$$

$$\langle D \rangle_{22} = \frac{\beta}{2\gamma} \left\langle \lambda_1 [1 - \cos(2\sqrt{\beta\gamma}\, \cdot)] \right\rangle + \frac{1}{2} \left\langle \lambda_2 [1 - \cos(2\sqrt{\beta\gamma}\, \cdot)] \right\rangle$$

and for any function h the notation $\langle \lambda_i h(\cdot) \rangle$ stands for $\frac{\sqrt{\beta\gamma}}{2\pi} \int_0^{2\pi/\sqrt{\beta\gamma}} \lambda_i(Y(s; y)) h(s)\, \mathrm{d}s, \; i \in \{1, 2\}$. Notice that when λ_1, λ_2 are constant functions along the flow $Y(s; \cdot)$ (that is, when λ_1, λ_2 depend only on $\beta(y_1)^2 + \gamma(y_2)^2$), the expression for $\langle D \rangle$ reduces to

$$\langle D \rangle = \begin{pmatrix} \frac{1}{2}\lambda_1 + \frac{\gamma}{2\beta}\lambda_2 & 0 \\ 0 & \frac{\beta}{2\gamma}\lambda_1 + \frac{1}{2}\lambda_2 \end{pmatrix}.$$

We inquire now about the Fokker-Planck equation. The external electro-magnetic field is given by

$$E = -\nabla_x \Phi, \;\; B^\varepsilon = (0, 0, B/\varepsilon), \;\; x \in \mathbb{R}^3$$

where, for simplicity, we assume that the magnetic field is uniform. In the finite Larmor radius regime (*i.e.,* the typical length in the orthogonal directions is much smaller then the typical length in the parallel direction), the presence density f^ε satisfies

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v_1 \partial_{x_1} + v_2 \partial_{x_2}) f^\varepsilon + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon + \frac{qB}{m\varepsilon}(v_2 \partial_{v_1} - v_1 \partial_{v_2}) f^\varepsilon = \nu \mathrm{div}_v \{\Theta \nabla_v f^\varepsilon + v f^\varepsilon\}.$$

Here m is the particle mass, q is the particle charge, ν is the collision frequency and Θ is the temperature. In this case, the flow $Y(s; x, v)$ to be considered corresponds to the vector field

$$b(x, v) \cdot \nabla_{x,v} = v_1 \partial_{x_1} + v_2 \partial_{x_2} + \omega_c (v_2 \partial_{v_1} - v_1 \partial_{v_2}), \;\; \omega_c = \frac{qB}{m}, \;\; (x, v) \in \mathbb{R}^6.$$

It is easily seen that

$$\overline{X}(s; \overline{x}, \overline{v}) = \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{\mathcal{R}(-\omega_c s)}{\omega_c} {}^\perp\overline{v}, \;\; X_3(s; x_3) = x_3, \;\; \overline{V}(s; \overline{v}) = \mathcal{R}(-\omega_c s)\overline{v}, \;\; V_3(s; v_3) = v_3$$

where we have used the notations $\overline{x} = (x_1, x_2), \overline{v} = (v_1, v_2), {}^\perp\overline{v} = (v_2, -v_1)$ and $\mathcal{R}(\theta)$ stands for the rotation of angle $\theta \in \mathbb{R}$. The Jacobian matrix writes

$$\partial_{x,v} Y(s; x, v) = \begin{pmatrix} I_2 & O_{2\times1} & \frac{I_2 - \mathcal{R}(-\omega_c s)}{\omega_c}\mathcal{E} & O_{2\times1} \\ O_{1\times2} & 1 & O_{1\times2} & 0 \\ O_{2\times2} & O_{2\times1} & \mathcal{R}(-\omega_c s) & O_{2\times1} \\ O_{1\times2} & 0 & O_{1\times2} & 1 \end{pmatrix}$$

where $O_{m\times n}$ stands for the null matrix with m lines and n columns, and $\mathcal{E} = \mathcal{R}(-\pi/2)$. The diffusion matrix field to be averaged is

$$D = \sum_{i=1}^{3} e_{v_i} \otimes e_{v_i} = \begin{pmatrix} O_3 & O_3 \\ O_3 & I_3 \end{pmatrix}$$

and by Theorem 2.1 we obtain after direct computations

$$\begin{aligned} \langle D \rangle = \left\langle\!\!\left\langle \begin{pmatrix} O_3 & O_3 \\ O_3 & I_3 \end{pmatrix} \right\rangle\!\!\right\rangle &= \frac{\omega_c}{2\pi} \int_0^{2\pi/\omega_c} \partial Y(-s; Y(s; \cdot)) \begin{pmatrix} O_3 & O_3 \\ O_3 & I_3 \end{pmatrix} {}^t\partial Y(-s; Y(s; \cdot)) \,\mathrm{d}s \\ &= \begin{pmatrix} \frac{2I_2}{\omega_c^2} & O_{2\times1} & -\frac{\mathcal{E}}{\omega_c} & O_{2\times1} \\ O_{1\times2} & 0 & O_{1\times2} & 0 \\ \frac{\mathcal{E}}{\omega_c} & O_{2\times1} & I_2 & O_{2\times1} \\ O_{1\times2} & 0 & O_{1\times2} & 1 \end{pmatrix}. \end{aligned} \tag{27}$$

Notice that the average Fokker-Planck kernel contains diffusion terms not only in velocity variables (as in the Fokker-Planck kernel) but also in space variables (orthogonal to the magnetic lines), as observed in gyrokinetic experiments and numerical simulations.

4 Well posedness for the perturbed problem and uniform estimates

For solving (1), we appeal to variational methods. We use the continuous embedding $H^1_R \hookrightarrow L^2(\mathbb{R}^m)$, with dense image (since $C^1_c(\mathbb{R}^m) \subset H^1_R$). We work under the hypotheses (4), (5), (22), (23). Moreover we assume that

$${}^tD = D, \;\; D \in H_Q \cap H_Q^\infty, \;\; b \in X_Q^\infty \tag{28}$$

and

$$\exists\, \alpha > 0 \ \text{ such that } Q^{1/2}(y)D(y)Q^{1/2}(y) \geq \alpha I_m, \ \ y \in \mathbb{R}^m \tag{29}$$

where $Q = {}^t RR$ and the columns of R^{-1} are given by the vector fields $b_1, ..., b_m$.

Proposition 4.1 *Assume that the hypotheses* (4), (5), (22), (23), (28), (29) *hold true.*

1. *Let us consider the application* $a^\varepsilon : H^1_R \times H^1_R \to \mathbb{R}$

$$a^\varepsilon(u,v) = \int_{\mathbb{R}^m} D(y)\nabla_y u \cdot \nabla_y v \; \mathrm{d}y + \frac{1}{\varepsilon}\int_{\mathbb{R}^m} (b\cdot\nabla_y u)v(y)\; \mathrm{d}y, \ \ u, v \in H^1_R.$$

The bilinear form a^ε *is well defined, continuous and coercive on* H^1_R *with respect to* $L^2(\mathbb{R}^m)$.

2. *Let us consider the application* $\langle a\rangle : H^1_R \times H^1_R \to \mathbb{R}$

$$\langle a\rangle(u,v) = \int_{\mathbb{R}^m} \langle D\rangle(y)\nabla_y u \cdot \nabla_y v \; \mathrm{d}y, \ \ u, v \in H^1_R.$$

The bilinear form $\langle a\rangle$ *is well defined, continuous and coercive on* H^1_R *with respect to* $L^2(\mathbb{R}^m)$.

Proof.

1. For any $u, v \in H^1_R$ we have

$$\begin{aligned} |D(y)\nabla_y u\cdot\nabla_y v| &= |Q^{1/2}(y)D(y)Q^{1/2}(y) : (P^{1/2}\nabla_y v)\otimes(P^{1/2}\nabla_y u)| \\ &\leq |D|_{H^\infty_Q}|P^{1/2}(y)\nabla_y v|\,|P^{1/2}(y)\nabla_y u|, \ \ y\in\mathbb{R}^m \end{aligned}$$

and

$$|b(y)\cdot\nabla_y u\; v(y)| = |Q^{1/2}(y)b(y)\cdot P^{1/2}(y)\nabla_y u\; v(y)| \leq |b|_{X^\infty_Q}|P^{1/2}(y)\nabla_y u|\,|v(y)|, \ \ y\in\mathbb{R}^m.$$

Therefore, it is easily seen, thanks to (25), that

$$\begin{aligned} |a^\varepsilon(u,v)| &\leq |D|_{H^\infty_Q}|\nabla_y u|_P|\nabla_y v|_P + \frac{1}{\varepsilon}|b|_{X^\infty_Q}|\nabla_y u|_P\|v\|_{L^2(\mathbb{R}^m)} \\ &\leq \left(|D|_{H^\infty_Q} + \frac{1}{\varepsilon}|b|_{X^\infty_Q}\right)|u|_R|v|_R \end{aligned}$$

saying that the bilinear application $a^\varepsilon(\cdot,\cdot)$ is well defined and continuous. We inquire now about the coercivity of a^ε on H^1_R, with respect to $L^2(\mathbb{R}^m)$. For any $u \in H^1_R$ we have, thanks to the anti-symmetry of $b\cdot\nabla_y$

$$\begin{aligned} a^\varepsilon(u,u) + \alpha\|u\|^2_{L^2(\mathbb{R}^m)} &= \int_{\mathbb{R}^m} D(y)\nabla_y u\cdot\nabla_y u\;\mathrm{d}y + \alpha\|u\|^2_{L^2(\mathbb{R}^m)} \\ &= \int_{\mathbb{R}^m} Q^{1/2}(y)D(y)Q^{1/2}(y) : (P^{1/2}(y)\nabla_y u)\otimes(P^{1/2}(y)\nabla_y u)\;\mathrm{d}y + \alpha\|u\|^2_{L^2(\mathbb{R}^m)} \\ &\geq \alpha\int_{\mathbb{R}^m}|P^{1/2}(y)\nabla_y u|^2\;\mathrm{d}y + \alpha\|u\|^2_{L^2(\mathbb{R}^m)} \\ &= \alpha\left(|\nabla_y u|^2_P + \|u\|^2_{L^2(\mathbb{R}^m)}\right) = \alpha|u|^2_R. \end{aligned}$$

We emphasize the following inequality, which will be used several times in the sequel

$$D(y)\nabla_y u\cdot\nabla_y u \geq \alpha|\,\nabla^R_y u|^2, \ \ u\in H^1_R. \tag{30}$$

2. Follow the same lines as before by using the third and fourth assertions of Theorem 2.1, that is

$$Q^{1/2}(y)\langle D\rangle(y)Q^{1/2}(y) \geq \alpha I_m, \ \ y\in\mathbb{R}^m$$

and $|\langle D\rangle|_{H^\infty_Q} \leq |D|_{H^\infty_Q}$. □

Proposition 4.2 *Assume that the hypotheses* (4), (5), (22), (23), (28), (29) *hold true. There exists* u^ε *(resp. v) a unique variational solution of* (1) *(resp.* (15)*). Moreover, we have*

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le \|u^{\rm in}\|_{L^2(\mathbb{R}^m)}, \quad \|\nabla_y u^\varepsilon\|_{L^2(\mathbb{R}_+;X_P)} \le \frac{\|u^{\rm in}\|_{L^2(\mathbb{R}^m)}}{\sqrt{2\alpha}}, \quad \varepsilon > 0$$

and

$$\|v\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le \|u^{\rm in}\|_{L^2(\mathbb{R}^m)}, \quad \|\nabla_z v\|_{L^2(\mathbb{R}_+;X_P)} \le \frac{\|u^{\rm in}\|_{L^2(\mathbb{R}^m)}}{\sqrt{2\alpha}}.$$

Proof. By Theorems 1, 2 of [12] p. 513, see also [19], we deduce that for any $u^{\rm in} \in L^2(\mathbb{R}^m)$, there is a unique variational solution u^ε for the problem (1), that is $u^\varepsilon \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2(\mathbb{R}_+; H^1_R)$, $\partial_t u^\varepsilon \in L^2(\mathbb{R}_+; (H^1_R)')$ and

$$u^\varepsilon(0) = u^{\rm in}, \quad \frac{\rm d}{{\rm d}t}\int_{\mathbb{R}^m} u^\varepsilon(t,y)\varphi(y)\;{\rm d}y + a^\varepsilon(u^\varepsilon(t),\varphi) = 0, \;\; \text{in } \mathcal{D}'(\mathbb{R}_+), \;\text{for any } \varphi \in H^1_R.$$

Similarly, there is a unique variational solution v for the limit model (15), that is $v \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2(\mathbb{R}_+; H^1_R)$, $\partial_t v \in L^2(\mathbb{R}_+; (H^1_R)')$ and

$$v(0) = u^{\rm in}, \quad \frac{\rm d}{{\rm d}t}\int_{\mathbb{R}^m} v(t,z)\psi(z)\;{\rm d}y + \langle a\rangle\,(v(t),\psi) = 0, \;\; \text{in } \mathcal{D}'(\mathbb{R}_+), \;\text{for any } \psi \in H^1_R.$$

The above estimates come immediately by the energy balance

$$\frac{1}{2}\frac{\rm d}{{\rm d}t}\|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} + a^\varepsilon(u^\varepsilon(t),u^\varepsilon(t)) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}_+)$$

which implies

$$\frac{1}{2}\|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} + \int_0^t a^\varepsilon(u^\varepsilon(\tau),u^\varepsilon(\tau))\;{\rm d}\tau = \frac{1}{2}\|u^{\rm in}\|^2_{L^2(\mathbb{R}^m)}, \quad t \in \mathbb{R}_+.$$

In particular we deduce $\|u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \le \|u^{\rm in}\|_{L^2(\mathbb{R}^m)}$, for any $t \in \mathbb{R}_+, \varepsilon > 0$, and

$$2\alpha\int_0^t |\nabla_y u^\varepsilon(\tau)|^2_P\;{\rm d}\tau \le 2\int_0^t a^\varepsilon(u^\varepsilon(\tau),u^\varepsilon(\tau))\;{\rm d}\tau \le \|u^{\rm in}\|^2_{L^2(\mathbb{R}^m)}, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0$$

saying that $\|\nabla_y u^\varepsilon\|_{L^2(\mathbb{R}_+;X_P)} \le \frac{\|u^{\rm in}\|_{L^2(\mathbb{R}^m)}}{\sqrt{2\alpha}}$, for any $\varepsilon > 0$. The estimates for v follow similarly, using the energy balance

$$\frac{1}{2}\frac{\rm d}{{\rm d}t}\|v(t)\|^2_{L^2(\mathbb{R}^m)} + \langle a\rangle\,(v(t),v(t)) = 0, \;\; \text{in } \mathcal{D}'(\mathbb{R}_+)$$

and the inequality $\langle a\rangle\,(v(t),v(t)) \ge \alpha|\nabla_z v(t)|^2_P$. □

Remark 4.1 *The family* $(v^\varepsilon(t,\cdot))_{\varepsilon>0} = (u^\varepsilon(t, Y(t/\varepsilon;\cdot)))_{\varepsilon>0}$ *satisfies the same estimates as the family* $(u^\varepsilon)_{\varepsilon>0}$*. Indeed, performing the change of variable* $y = Y(t/\varepsilon; z)$*, which is measure preserving, one gets*

$$\|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} = \int_{\mathbb{R}^m}(u^\varepsilon(t,Y(t/\varepsilon;z)))^2\;{\rm d}z = \int_{\mathbb{R}^m}(u^\varepsilon(t,y))^2\;{\rm d}y \le \|u^{\rm in}\|^2_{L^2(\mathbb{R}^m)}, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

Notice that we have, thanks to (21)

$$\begin{aligned}\nabla_z^R v^\varepsilon(t) &= {}^t R^{-1}(z)\nabla_z v^\varepsilon(t) = {}^t R^{-1}(z)\, {}^t\partial Y(t/\varepsilon; z)\, \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z))\\ &= {}^t(\partial Y(t/\varepsilon; z)\, R^{-1}(z))\nabla_y u^\varepsilon(t, Y(t/\varepsilon; z))\\ &= {}^t R^{-1}(Y(t/\varepsilon; z))\nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) = \left(\nabla_y^R u^\varepsilon(t)\right)(Y(t/\varepsilon; z))\end{aligned}$$

and therefore

$$\begin{aligned}\|\nabla_z v^\varepsilon\|^2_{L^2(\mathbb{R}_+;X_P)} &= \int_0^{+\infty} |\nabla_z v^\varepsilon(\tau)|_P^2 \,\mathrm{d}\tau = \int_0^{+\infty} \|\nabla_z^R v^\varepsilon(\tau)\|^2_{L^2(\mathbb{R}^m)} \,\mathrm{d}\tau\\ &= \int_0^{+\infty} \|\nabla_y^R u^\varepsilon(\tau)\|^2_{L^2(\mathbb{R}^m)} \,\mathrm{d}\tau = \int_0^{+\infty} |\nabla_y u^\varepsilon(\tau)|_P^2 \,\mathrm{d}\tau = \|\nabla_y u^\varepsilon\|^2_{L^2(\mathbb{R}_+;X_P)}\\ &\leq \frac{\|u^{\mathrm{in}}\|^2_{L^2(\mathbb{R}^m)}}{2\alpha}, \quad \varepsilon > 0.\end{aligned}$$

Using twice the formula

$$b_i \cdot \nabla_z v^\varepsilon(t) = b_i \cdot \nabla_z(u^\varepsilon(t) \circ Y(t/\varepsilon; \cdot)) = (b_i \cdot \nabla_y u^\varepsilon(t)) \circ Y(t/\varepsilon; \cdot)$$

we deduce that

$$b_j \cdot \nabla_z(b_i \cdot \nabla_z v^\varepsilon(t)) = b_j \cdot \nabla_z\left[(b_i \cdot \nabla_y u^\varepsilon(t)) \circ Y(t/\varepsilon; \cdot)\right] = [b_j \cdot \nabla_y(b_i \cdot \nabla_y u^\varepsilon(t))] \circ Y(t/\varepsilon; \cdot)\,.$$

Therefore $\|b_j \cdot \nabla_z(b_i \cdot \nabla_z v^\varepsilon(t))\|_{L^2(\mathbb{R}^m)} = \|b_j \cdot \nabla_y(b_i \cdot \nabla_y u^\varepsilon(t))\|_{L^2(\mathbb{R}^m)}$, $i, j \in \{1, ..., m\}$ *anytime that* $u^\varepsilon(t) \in H^2_R$.

Up to now, we have considered solutions with initial condition $u^{\mathrm{in}} \in L^2(\mathbb{R}^m)$. In order to study the stability of the family $(v^\varepsilon)_{\varepsilon>0}$ when ε goes to 0, we need more regularity. This will be the object of the next propositions, in which we analyze how the regularity of the initial condition propagates in time. The idea is to take the directional derivative $b_i \cdot \nabla_y$ of (1), leading to

$$\partial_t(b_i \cdot \nabla_y u^\varepsilon) - \mathrm{div}_y(D(y)\nabla_y(b_i \cdot \nabla_y u^\varepsilon)) + \frac{1}{\varepsilon} b \cdot \nabla_y(b_i \cdot \nabla_y u^\varepsilon) = [b_i \cdot \nabla_y, \mathrm{div}_y(D\nabla_y)]u^\varepsilon. \quad (31)$$

Notice that the key point was to take advantage of the involution between b_i and b, for any $i \in \{1, ..., m\}$, which guarantees that there is no commutator between the first order operators $b_i \cdot \nabla_y$ and $b \cdot \nabla_y$. More generally, if we apply the directional derivative $c \cdot \nabla_y$ in (1), the right hand side of the corresponding equation in (31) will contain the extra term $\frac{1}{\varepsilon}[b \cdot \nabla_y, c \cdot \nabla_y]u^\varepsilon$, which is clearly unstable, when ε goes to 0, if b and c are not in involution. The estimate for $b_i \cdot \nabla_y u^\varepsilon$ follows by using the energy balance of (31), observing that, thanks to the anti-symmetry of $b \cdot \nabla_y$, we get rid of the term of order $1/\varepsilon$. We assume that for any $i, j \in \{1, ..., m\}$, the coordinates of the Poisson bracket $[b_i, b_j]$ in the basis $(b_k)_{1\leq k\leq m}$ are bounded

$$[b_i, b_j] = \sum_{k=1}^m \alpha_{ij}^k b_k, \;\; \alpha_{ij}^k \in L^\infty(\mathbb{R}^m), \; i, j, k \in \{1, ..., m\}. \quad (32)$$

Proposition 4.3 *Assume that the hypotheses* (4), (5), (22), (23), (28), (29), (32) *hold true. Moreover we assume that for any* $i, j \in \{1, ..., m\}$

$$b_i \cdot \nabla_y \mathrm{div}_y b_j \in L^\infty(\mathbb{R}^m), \;\; \mathrm{div}_y(RD) \in L^\infty(\mathbb{R}^m)$$

$$R[b_i, D]\, {}^t R \in L^\infty(\mathbb{R}^m), \;\; \sum_{i=1}^m b_i \cdot \nabla_y(R[b_i, D]\, {}^t R) \in L^\infty(\mathbb{R}^m).$$

If the initial condition belongs to H^1_R, then we have for any $T \in \mathbb{R}_+$

$$\sup_{\varepsilon>0} \| \nabla^R_y u^\varepsilon \|_{L^\infty([0,T];L^2(\mathbb{R}^m))} = \sup_{\varepsilon>0} \| \nabla^R_z v^\varepsilon \|_{L^\infty([0,T];L^2(\mathbb{R}^m))} < +\infty$$

$$\sup_{\varepsilon>0} \| \nabla^R_y \otimes \nabla^R_y u^\varepsilon \|_{L^2([0,T];L^2(\mathbb{R}^m))} = \sup_{\varepsilon>0} \| \nabla^R_z \otimes \nabla^R_z v^\varepsilon \|_{L^2([0,T];L^2(\mathbb{R}^m))} < +\infty$$

$$\sup_{\varepsilon>0} \|\partial_t v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} < +\infty.$$

Here the notation $\nabla^R \otimes \nabla^R w$ stands for the matrix whose entry (i,j) is $b_j \cdot \nabla(b_i \cdot \nabla w)$, $i,j \in \{1,...,m\}$.

Proof. We want to estimate the L^2 norms of $b_i \cdot \nabla_y u^\varepsilon$, $i \in \{1,...,m\}, \varepsilon > 0$. This can be done by analyzing the translations along the flows Y_i and estimating the L^2 norms of $(u^\varepsilon(t, Y_i(h;y)) - u^\varepsilon(t,y))/h$ uniformly with respect to $h \in \mathbb{R}^\star$ and $\varepsilon > 0$. For simplicity, we justify the estimates only for smooth solutions and coefficients (and therefore we use clasical derivatives). The general case is left to the reader. We need to compute the commutator between a first order operator $c \cdot \nabla_y$ and the diffusion operator $\mathrm{div}_y(D\nabla_y)$. Here $c(y)$ is a vector field, not necessarily in involution with the vector field $b(y)$ (the involution does not play any role when computing $[c \cdot \nabla_y, \mathrm{div}_y(D\nabla_y)]$). A straightforward computation shows that the commutator between $c \cdot \nabla_y$ and div_y is given by

$$[c \cdot \nabla_y, \mathrm{div}_y]\xi = \xi \cdot \nabla_y \mathrm{div}_y c - \mathrm{div}_y(\partial_y c\, \xi), \;\; \xi \in (C^2(\mathbb{R}^m))^m.$$

Using the above formula with $\xi = D(y)\nabla_y u^\varepsilon$, one gets

$$c \cdot \nabla_y(\mathrm{div}_y(D(y)\nabla_y u^\varepsilon)) - \mathrm{div}_y(c \cdot \nabla_y(D(y)\nabla_y u^\varepsilon)) = D(y)\nabla_y u^\varepsilon \cdot \nabla_y \mathrm{div}_y c - \mathrm{div}_y(\partial_y c\, D(y)\nabla_y u^\varepsilon). \quad (33)$$

Taking into account that

$$\begin{aligned} c \cdot \nabla_y(D(y)\nabla_y u^\varepsilon) &= (c \cdot \nabla_y D)\nabla_y u^\varepsilon + D(y)(\partial^2 u^\varepsilon)c(y) \\ &= (c \cdot \nabla_y D)\nabla_y u^\varepsilon + D(y)\nabla_y(c \cdot \nabla_y u^\varepsilon) - D(y)\, {}^t\partial_y c \nabla_y u^\varepsilon \end{aligned}$$

we deduce by (33)

$$\begin{aligned} c \cdot \nabla_y(\mathrm{div}_y(D(y)\nabla_y u^\varepsilon)) - \mathrm{div}_y(D(y)\nabla_y(c \cdot \nabla_y u^\varepsilon)) &= D(y)\nabla_y u^\varepsilon \cdot \nabla_y \mathrm{div}_y c \\ &+ \mathrm{div}_y((c \cdot \nabla_y D - \partial_y c D(y) - D(y)\, {}^t\partial_y c)\nabla_y u^\varepsilon). \end{aligned}$$

Finally the commutator between $c \cdot \nabla_y$ and $\mathrm{div}_y(D(y)\nabla_y)$ writes

$$[c \cdot \nabla_y, \mathrm{div}_y(D(y)\nabla_y)]u^\varepsilon = \mathrm{div}_y([c, D]\nabla_y u^\varepsilon) + D(y)\nabla_y u^\varepsilon \cdot \nabla_y \mathrm{div}_y c. \quad (34)$$

Multiplying (31) by $b_i \cdot \nabla_y u^\varepsilon$, integrating with respect to y over $\mathbb{R}^m$ and observing that the contribution of the singular term $\frac{1}{\varepsilon} b \cdot \nabla_y(b_i \cdot \nabla_y u^\varepsilon)$ cancels by the anti-symmetry of the operator $b \cdot \nabla_y$, yield

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m}(b_i \cdot \nabla_y u^\varepsilon(t))^2 \;\mathrm{d}y + \int_{\mathbb{R}^m} D(y)\nabla_y(b_i \cdot \nabla_y u^\varepsilon(t)) \cdot \nabla_y(b_i \cdot \nabla_y u^\varepsilon(t)) \;\mathrm{d}y \quad (35) \\ = -\int_{\mathbb{R}^m}[b_i, D]\nabla_y u^\varepsilon(t) \cdot \nabla_y(b_i \cdot \nabla_y u^\varepsilon(t)) \;\mathrm{d}y \\ + \int_{\mathbb{R}^m} D(y)\nabla_y u^\varepsilon(t) \cdot \nabla_y \mathrm{div}_y b_i \;(b_i \cdot \nabla_y u^\varepsilon(t)) \;\mathrm{d}y. \end{aligned}$$

By hypothesis (29) we have

$$D\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t))\cdot\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t)) = Q^{1/2}DQ^{1/2} \tag{36}$$
$$: P^{1/2}\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t))\otimes P^{1/2}\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t))$$
$$\geq \alpha|P^{1/2}\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t))|^2 = \alpha|\nabla_y^R(b_i\cdot\nabla_y u^\varepsilon(t))|^2$$
$$= \alpha\sum_{j=1}^m (b_j\cdot\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t)))^2.$$

Combining (35), (36) leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla_y^R u^\varepsilon(t)\right\|^2_{L^2(\mathbb{R}^m)} + \alpha\left\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon(t)\right\|^2_{L^2(\mathbb{R}^m)} \tag{37}$$
$$\leq -\sum_{i=1}^m \underbrace{\int_{\mathbb{R}^m}[b_i,\, D]\nabla_y u^\varepsilon(t)\cdot\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t))\,\mathrm{d}y}_{:=I_i^1}$$
$$+\sum_{i=1}^m \underbrace{\int_{\mathbb{R}^m} D\nabla_y u^\varepsilon(t)\cdot\nabla_y(\mathrm{div}_y b_i)\,(b_i\cdot\nabla_y u^\varepsilon(t))\,\mathrm{d}y}_{:=I_i^2}.$$

In order to upper bound the term I_i^1 in the right hand side of (37) we write

$$I_i^1 = \int_{\mathbb{R}^m} R\,[b_i,\, D]\,{}^tR : {}^tR^{-1}\nabla_y u^\varepsilon(t)\otimes{}^tR^{-1}\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t))\,\mathrm{d}y \tag{38}$$
$$= \int_{\mathbb{R}^m} R\,[b_i,\, D]\,{}^tR :\,\nabla_y^R u^\varepsilon(t)\otimes\,\nabla_y^R(b_i\cdot\nabla_y u^\varepsilon(t))\,\mathrm{d}y.$$

Notice that for any $j\in\{1,...,m\}$ we have

$$b_j\cdot\nabla_y(b_i\cdot\nabla_y u^\varepsilon(t)) = b_i\cdot\nabla_y(b_j\cdot\nabla_y u^\varepsilon(t)) - [b_i, b_j]\cdot\nabla_y u^\varepsilon(t)$$
$$= b_i\cdot\nabla_y(b_j\cdot\nabla_y u^\varepsilon(t)) - \sum_{k=1}^m \alpha_{ij}^k b_k\cdot\nabla_y u^\varepsilon(t)$$

and therefore we obtain

$$\nabla_y^R(b_i\cdot\nabla_y u^\varepsilon(t)) = b_i\cdot\nabla_y(\,\nabla_y^R u^\varepsilon(t)) - \mathcal{A}_i\,\nabla_y^R u^\varepsilon(t) \tag{39}$$

where the entry (j,k) of the matrix $\mathcal{A}_i$ is given by α_{ij}^k. Combining (38), (39) yields

$$I_i^1 = \int_{\mathbb{R}^m} R\,[b_i,\, D]\,{}^tR :\,\nabla_y^R u^\varepsilon(t)\otimes b_i\cdot\nabla_y(\,\nabla_y^R u^\varepsilon(t))\,\mathrm{d}y$$
$$-\int_{\mathbb{R}^m} R\,[b_i,\, D]\,{}^tR :\,\nabla_y^R u^\varepsilon(t)\otimes\mathcal{A}_i\,\nabla_y^R u^\varepsilon(t)\,\mathrm{d}y$$
$$=: J_i^1 + J_i^2.$$

For estimating the term J_i^1, we use the symmetry of the matrix field D and the formula

$$b_i\cdot\nabla_y\big(R\,[b_i,\, D]\,{}^tR :\,\nabla_y^R u^\varepsilon(t)\otimes\,\nabla_y^R u^\varepsilon(t)\big) = b_i\cdot\nabla_y(R\,[b_i,\, D]\,{}^tR) :\,\nabla_y^R u^\varepsilon(t)\otimes\,\nabla_y^R u^\varepsilon(t)$$
$$+ 2R[b_i, D]\,{}^tR :\,\nabla_y^R u^\varepsilon(t)\otimes b_i\cdot\nabla_y(\,\nabla_y^R u^\varepsilon(t)).$$

Integrating by parts leads to

$$
\begin{aligned}
2\left|\sum_{i=1}^m J_i^1\right| &= \left| - \sum_{i=1}^m \int_{\mathbb{R}^m} \mathrm{div}_y b_i \, R\,[b_i,\, D]\,{}^t R :\, \nabla_y^R u^\varepsilon(t) \otimes \nabla_y^R u^\varepsilon(t)\,\mathrm{d}y \right.\\
&\quad \left. - \int_{\mathbb{R}^m} \sum_{i=1}^m b_i \cdot \nabla_y (R\,[b_i,\, D]\,{}^t R) :\, \nabla_y^R u^\varepsilon(t) \otimes \nabla_y^R u^\varepsilon(t)\,\mathrm{d}y \right|\\
&\leq \left(\sum_{i=1}^m \|\mathrm{div}_y b_i\|_{L^\infty} \|R[b_i, D]\,{}^t R\|_{L^\infty} + \left\| \sum_{i=1}^m b_i \cdot \nabla_y (R[b_i, D]\,{}^t R) \right\|_{L^\infty} \right)\\
&\quad \times \| \nabla_y^R u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}.
\end{aligned}
$$

The estimate for the term J_i^2 follows immediately, thanks to the hypothesis (32)

$$
|J_i^2| \leq \|R[b_i, D]\,{}^t R\|_{L^\infty(\mathbb{R}^m)} \|\mathcal{A}_i\|_{L^\infty(\mathbb{R}^m)} \| \nabla_y^R u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}
$$

and finally there is a constant C_1 depending on $\max_{1\leq i \leq m} \|\mathrm{div}_y b_i\|_{L^\infty}$, $\max_{1\leq i\leq m} \|\mathcal{A}_i\|_{L^\infty}$, $\max_{1\leq i\leq m} \|R[b_i, D]\,{}^t R\|_{L^\infty}$, $\|\sum_{i=1}^m b_i \cdot \nabla_y (R[b_i, D]\,{}^t R)\|_{L^\infty}$ such that

$$
\left| \sum_{i=1}^m I_i^1 \right| \leq C_1 \| \nabla_y^R u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)},\;\; t \in \mathbb{R}_+,\;\; \varepsilon > 0. \tag{40}
$$

For the term I_i^2 we can write, using the inequality $(R(y)D(y)\,{}^t R(y) : R(y)D(y)\,{}^t R(y))^{1/2} \leq |D|_{H_Q^\infty}, y \in \mathbb{R}^m$

$$
\begin{aligned}
|I_i^2| &= \left| \int_{\mathbb{R}^m} RD\,{}^t R\,{}^t R^{-1} \nabla_y u^\varepsilon(t) \cdot {}^t R^{-1} \nabla_y (\mathrm{div}_y b_i)\,(b_i \cdot \nabla_y u^\varepsilon(t))\,\mathrm{d}y \right|\\
&\leq |D|_{H_Q^\infty} \| \nabla_y^R \mathrm{div}_y b_i\|_{L^\infty(\mathbb{R}^m)} \int_{\mathbb{R}^m} | \nabla_y^R u^\varepsilon(t)|\, |b_i \cdot \nabla_y u^\varepsilon(t)|\,\mathrm{d}y\\
&\leq |D|_{H_Q^\infty} \| \nabla_y^R \mathrm{div}_y b_i\|_{L^\infty(\mathbb{R}^m)} \| \nabla_y^R u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}
\end{aligned}
$$

implying that there is a constant C_2 depending on $|D|_{H_Q^\infty}, \max_{1\leq i\leq m} \| \nabla_y^R \mathrm{div}_y b_i\|_{L^\infty(\mathbb{R}^m)}$ such that

$$
\left| \sum_{i=1}^m I_i^2 \right| \leq C_2 \| \nabla_y^R u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)},\;\; t \in \mathbb{R}_+,\;\; \varepsilon > 0. \tag{41}
$$

Combining (37), (40), (41) and applying Gronwall's lemma imply

$$
\| \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \leq e^{(C_1+C_2)t} \| \nabla_y^R u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)},\;\; t \in \mathbb{R}_+,\;\; \varepsilon > 0
$$

and

$$
\| \nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} \leq \frac{e^{(C_1+C_2)t}}{\sqrt{2\alpha}} \| \nabla_y^R u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)},\;\; \varepsilon > 0.
$$

The first and second conclusions follow thanks to Remark 4.1. For the last one, notice that

$$
\begin{aligned}
\partial_t v^\varepsilon(t,z) &= \partial_t u^\varepsilon(t, Y(t/\varepsilon; z)) + \frac{1}{\varepsilon} b(Y(t/\varepsilon; z)) \cdot \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z))\\
&= \mathrm{div}_y (D \nabla_y u^\varepsilon(t))(Y(t/\varepsilon; z))
\end{aligned} \tag{42}
$$

which implies

$$
\|\partial_t v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} = \|\mathrm{div}_y (D \nabla_y u^\varepsilon(t))\|_{L^2(\mathbb{R}^m)}.
$$

By direct computation we obtain

$$\operatorname{div}_y(D\nabla_y u^\varepsilon) = \operatorname{div}_y(D\,{}^tR\,\nabla_y^R u^\varepsilon) = \operatorname{div}_y(RD)\cdot \nabla_y^R u^\varepsilon + RD\,{}^tR : \partial_y(\nabla_y^R u^\varepsilon)R^{-1}$$
$$= \operatorname{div}_y(RD)\cdot \nabla_y^R u^\varepsilon + RD\,{}^tR : \nabla_y^R \otimes \nabla_y^R u^\varepsilon$$

and therefore

$$\sup_{\varepsilon>0}\|\partial_t v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} = \sup_{\varepsilon>0}\|\operatorname{div}_y(D\nabla_y u^\varepsilon)\|_{L^2([0,T];L^2(\mathbb{R}^m))}$$
$$\leq \sqrt{T}\|\operatorname{div}_y(RD)\|_{L^\infty(\mathbb{R}^m)}\sup_{\varepsilon>0}\|\nabla_y^R u^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}$$
$$+ |D|_{H_Q^\infty}\sup_{\varepsilon>0}\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))},\ \ T\in\mathbb{R}_+.$$

□

Performing similar computations, we can propagate more regularity. The goal is to obtain a uniform bound for $(\partial_t \nabla_z^R v^\varepsilon)_{\varepsilon>0}$ in $L^2_{\text{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^m))$. This can be achieved for any initial condition $u^{\text{in}}\in H_R^2$. The proof is postponed to Appendix A.

Proposition 4.4 *Assume that the hypotheses* (4), (5), (22), (23), (28), (29), (32) *hold true. Moreover we assume that for any $i,j,k\in\{1,...,m\}$*

$$\nabla_y^R\alpha_{ij}^k\in L^\infty(\mathbb{R}^m),\quad \nabla_y^R\operatorname{div}_y b_j\in L^\infty(\mathbb{R}^m),\quad \nabla_y^R\otimes\nabla_y^R\operatorname{div}_y b_j\in L^\infty(\mathbb{R}^m),$$

$$\operatorname{div}_y(R[b_i,D])\in L^\infty(\mathbb{R}^m),\ \ R[b_i,D]\,{}^tR\in L^\infty(\mathbb{R}^m),\ \ \sum_{i=1}^m b_i\cdot\nabla_y(R[b_i,D]\,{}^tR)\in L^\infty(\mathbb{R}^m)$$

$$R[b_j,[b_i,D]]\,{}^tR\in L^\infty(\mathbb{R}^m),\ \ \operatorname{div}_y(R[b_j,[b_i,D]])\in L^\infty(\mathbb{R}^m)$$
$$\nabla_y^R(RD\,{}^tR)\in L^\infty(\mathbb{R}^m),\quad \nabla_y^R\otimes\nabla_y^R(RD\,{}^tR)\in L^\infty(\mathbb{R}^m).$$

If the initial condition u^{in} belongs to H_R^2, then for any $T\in\mathbb{R}_+$ we have

$$\sup_{\varepsilon>0}\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} = \sup_{\varepsilon>0}\|\nabla_z^R\otimes\nabla_z^R v^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}<+\infty$$

$$\sup_{\varepsilon>0}\|\nabla_y^R\otimes\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} = \sup_{\varepsilon>0}\|\nabla_z^R\otimes\nabla_z^R\otimes\nabla_z^R v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))}<+\infty$$

and

$$\sup_{\varepsilon>0}\|\partial_t\nabla_z^R v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))}<+\infty.$$

Here the notation $\nabla^R\otimes\nabla^R\otimes\nabla^R w$ stands for the tensor whose entry (i,j,k) is $b_k\cdot\nabla(b_j\cdot\nabla(b_i\cdot\nabla w))$.

Similar computations allow us to estimate the solution of the limit model (15). The arguments are a little bit tedious and we refer to Appendix A for details.

Proposition 4.5 *Assume that all the hypotheses of Proposition 4.3 hold true. Then we have for any $T\in\mathbb{R}_+$*

$$\nabla_z^R v\in L^\infty([0,T];L^2(\mathbb{R}^m)),\ \nabla_z^R\otimes\nabla_z^R v\in L^2([0,T];L^2(\mathbb{R}^m)),\ \ \partial_t v\in L^2([0,T];L^2(\mathbb{R}^m)).$$

Proposition 4.6 *Assume that all the hypotheses of Proposition 4.4 hold true. Then for any $T\in\mathbb{R}_+$, we have*

$$\nabla_z^R\otimes\nabla_z^R v\in L^\infty([0,T];L^2),\quad \nabla_z^R\otimes\nabla_z^R\otimes\nabla_z^R v\in L^2([0,T];L^2),\ \ \partial_t\nabla_z^R v\in L^2([0,T];L^2).$$

Proof. Apply exactly the same arguments as in the proof of Proposition 4.4, after observing that the matrix field $\langle D\rangle$ satisfies the same hypotheses as the matrix field D (see the proof of Proposition 4.5).

□

5 Two-scale analysis

We intend to investigate the asymptotic behavior of (1), or equivalently (7). For any smooth, compactly supported function $\psi(t,z)$ we have to pass to the limit, when $\varepsilon \searrow 0$, in the formulation

$$-\int_{\mathbb{R}^m} u^{\rm in}(z)\psi(0,z)\,\mathrm{d}z - \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} v^\varepsilon(t,z)\partial_t\psi\,\mathrm{d}z\mathrm{d}t + \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} G(t/\varepsilon)D\nabla_z v^\varepsilon\cdot\nabla_z\psi\,\mathrm{d}z\mathrm{d}t = 0.$$

Clearly, the main difficulty comes from the last integral, which presents two time scales : a slow time variable t and also a fast time variable $s = t/\varepsilon$ (not necessarily periodic). We detail here a general two-scale convergence result, based on ergodic means. Let us introduce some notations. We denote by $\langle\cdot,\cdot\rangle_{P,Q} : H_P \times H_Q \to \mathbb{R}$ the bilinear continuous application defined for any $(A,B) \in H_P \times H_Q$ by

$$\langle A,B\rangle_{P,Q} = \int_{\mathbb{R}^m} A(y) : B(y)\,\mathrm{d}y = \int_{\mathbb{R}^m} P^{1/2}AP^{1/2} : Q^{1/2}BQ^{1/2}\,\mathrm{d}y \le |A|_P|B|_Q.$$

It is easily seen that $A \in H_P \to \langle A,\cdot\rangle_{P,Q} \in H_Q'$ is a linear isomorphism, $\|\langle A,\cdot\rangle_{P,Q}\|_{H_Q'} = |A|_P$. Therefore we identify H_Q' to H_P through the duality $\langle\cdot,\cdot\rangle_{P,Q}$. Notice also that $A \in H_P \to PAP \in H_Q$, $B \in H_Q \to QBQ \in H_P$ are linear isomorphisms, $|PAP|_Q = |A|_P, |QBQ|_P = |B|_Q$.

Proposition 5.1 *Let T be a positive real number. Consider $C \in L^\infty(\mathbb{R};H_Q)$, such that the family of means $\left(\frac{1}{S}\int_{s_0}^{s_0+S} C(s)\,\mathrm{d}s\right)_{S>0}$ converges strongly in H_Q toward some $\overline{C} \in H_Q$, uniformly with respect to $s_0 \in \mathbb{R}$, when $S \to +\infty$ and $\mathcal{B}_\omega \subset C([0,T];H_P)$ a bounded set in $L^1([0,T];H_P)$, of functions which admit as modulus of continuity in $C([0,T];H_P)$ the same function $\omega : [0,T] \to \mathbb{R}_+$ i.e.,*

$$|B(t) - B(t')|_P \le \omega(|t-t'|),\ \ t,t' \in [0,T],\ \ B \in \mathcal{B}_\omega$$

with ω non decreasing and $\lim_{\lambda\searrow 0}\omega(\lambda) = 0$. Then

$$\lim_{\varepsilon\searrow 0}\int_0^T \langle B(t), C(t/\varepsilon)\rangle_{P,Q}\,\mathrm{d}t = \int_0^T \left\langle B(t),\overline{C}\right\rangle_{P,Q}\,\mathrm{d}t$$

uniformly with respect to $B \in \mathcal{B}_\omega$.

Proof. For any $\delta > 0$, there is $S_\delta > 0$ such that

$$\left|\frac{1}{S}\int_{s_0}^{s_0+S} C(s)\,\mathrm{d}s - \overline{C}\right|_Q < \delta,\ \text{ for any } S \ge S_\delta \text{ and } s_0 \in \mathbb{R}.$$

Performing the change of variable $s = \frac{t}{\varepsilon}$ in the above integral, leads to

$$\left|\frac{1}{T}\int_{t_0}^{t_0+T} C(t/\varepsilon)\,\mathrm{d}t - \overline{C}\right|_Q < \delta,\ \text{ for any } T \ge \varepsilon S_\delta = T_{\delta,\varepsilon} \ \text{ and } t_0 \in \mathbb{R}. \tag{43}$$

We split the interval $[0,T[$ in a finite number of intervals of size great or equal to $T_{\delta,\varepsilon}$. For example let $k_{\delta,\varepsilon}$ be $\left[\frac{T}{T_{\delta,\varepsilon}}\right]$. If $T/T_{\delta,\varepsilon}$ is an integer, that is $T/T_{\delta,\varepsilon} = k_{\delta,\varepsilon}$, we consider the intervals

$$[kT_{\delta,\varepsilon},(k+1)T_{\delta,\varepsilon}[,\ \ 0 \le k \le k_{\delta,\varepsilon} - 1$$

and if $T/T_{\delta,\varepsilon}$ is not an integer, we take the intervals

$$[kT_{\delta,\varepsilon}, (k+1)T_{\delta,\varepsilon}[,\ \ 0 \le k \le k_{\delta,\varepsilon} - 2, \quad \text{and} \ \ [(k_{\delta,\varepsilon}-1)T_{\delta,\varepsilon}, T[.$$

Notice that in both cases we have $k_{\delta,\varepsilon}$ intervals, whose sizes are between $T_{\delta,\varepsilon}$ and $2T_{\delta,\varepsilon}$. We denote by $(t_{k,\delta,\varepsilon})_{0\le k\le k_{\delta,\varepsilon}}$, or simply $(t_k)_{0\le k\le k_{\delta,\varepsilon}}$, the end points of these intervals. The last point is allways $t_{k_{\delta,\varepsilon}} = T$. Therefore we can write for any $B \in \mathcal{B}_\omega$

$$\begin{aligned}
\left|\int_0^T \langle B(t), C(t/\varepsilon)\rangle_{P,Q}\ \mathrm{d}t - \int_0^T \left\langle B(t), \overline{C}\right\rangle_{P,Q}\ \mathrm{d}t\right| &= \left|\int_0^T \left\langle B(t), C(t/\varepsilon) - \overline{C}\right\rangle_{P,Q}\ \mathrm{d}t\right| \\
&\le \sum_{k=0}^{k_{\delta,\varepsilon}-1} \left|\int_{t_k}^{t_{k+1}} \left\langle B(t), C(t/\varepsilon) - \overline{C}\right\rangle_{P,Q} \mathrm{d}t\right| \\
&\le \sum_{k=0}^{k_{\delta,\varepsilon}-1} \left|\int_{t_k}^{t_{k+1}} \left\langle B(t) - B(t_k), C(t/\varepsilon) - \overline{C}\right\rangle_{P,Q} \mathrm{d}t\right| \\
&+ \sum_{k=0}^{k_{\delta,\varepsilon}-1} \left|\int_{t_k}^{t_{k+1}} \left\langle B(t_k), C(t/\varepsilon) - \overline{C}\right\rangle_{P,Q}\ \mathrm{d}t\right| \\
&=: \Sigma_1 + \Sigma_2. \qquad (44)
\end{aligned}$$

Since the function $t \in [0,T] \to B(t) \in H_P$ admits ω as modulus of continuity, we obtain the following estimate for Σ_1

$$\begin{aligned}
\Sigma_1 &\le \sum_{k=0}^{k_{\delta,\varepsilon}-1} \int_{t_k}^{t_{k+1}} \omega(|t-t_k|)\ |C(t/\varepsilon) - \overline{C}|_Q\ \mathrm{d}t \qquad (45) \\
&\le \sum_{k=0}^{k_{\delta,\varepsilon}-1} \omega(2T_{\delta,\varepsilon})(t_{k+1} - t_k)\ 2\|C\|_{L^\infty(\mathbb{R};H_Q)} \\
&= 2\|C\|_{L^\infty(\mathbb{R};H_Q)}\omega(2T_{\delta,\varepsilon})T.
\end{aligned}$$

The estimate for Σ_2 comes by using (43)

$$\begin{aligned}
\Sigma_2 &= \sum_{k=0}^{k_{\delta,\varepsilon}-1} \left|\int_{t_k}^{t_{k+1}} (PB(t_k)P, C(t/\varepsilon) - \overline{C})_Q\ \mathrm{d}t\right| \qquad (46) \\
&= \sum_{k=0}^{k_{\delta,\varepsilon}-1} \left|\left(PB(t_k)P, \int_{t_k}^{t_{k+1}} (C(t/\varepsilon) - \overline{C})\ \mathrm{d}t\right)_Q\right| \\
&= \sum_{k=0}^{k_{\delta,\varepsilon}-1} \left|\left\langle B(t_k), \int_{t_k}^{t_{k+1}} (C(t/\varepsilon) - \overline{C})\ \mathrm{d}t\right\rangle_{P,Q}\right| \\
&\le \sum_{k=0}^{k_{\delta,\varepsilon}-1} \delta(t_{k+1} - t_k)|B(t_k)|_P \\
&\le \delta\left[\|B\|_{L^1([0,T];H_P)} + \omega(2T_{\delta,\varepsilon})T\right].
\end{aligned}$$

Thanks to (44), (45), (46) we deduce

$$\begin{aligned}
\left|\int_0^T \langle B(t), C(t/\varepsilon)\rangle_{P,Q}\ \mathrm{d}t - \int_0^T \left\langle B(t), \overline{C}\right\rangle_{P,Q}\ \mathrm{d}t\right| &\le 2\|C\|_{L^\infty(\mathbb{R};H_Q)}\omega(2T_{\delta,\varepsilon})T \\
&+ \delta\left[\|B\|_{L^1([0,T];H_P)} + \omega(2T_{\delta,\varepsilon})T\right].
\end{aligned}$$

Let η be a positive real number and $\delta > 0$ small enough such that $\delta \|B\|_{L^1([0,T];H_P)} < \eta/2$ uniformly with respect to $B \in \mathcal{B}_\omega$ (which is possible since $\mathcal{B}_\omega$ is bounded in $L^1([0,T];H_P)$). Observing that $\lim_{\varepsilon\searrow 0} T_{\delta,\varepsilon} = \lim_{\varepsilon\searrow 0} \varepsilon S_\delta = 0$, and $\lim_{\varepsilon\searrow 0} \omega(2T_{\delta,\varepsilon}) = 0$, we deduce that there is $\varepsilon = \varepsilon(\eta)$ such that for any $0 < \varepsilon < \varepsilon(\eta)$

$$2\|C\|_{L^\infty(\mathbb{R};H_Q)}\omega(2T_{\delta,\varepsilon})T + \delta\omega(2T_{\delta,\varepsilon})T < \frac{\eta}{2}.$$

Finally we obtain

$$\left| \int_0^T \langle B(t), C(t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t - \int_0^T \left\langle B(t), \overline{C}\right\rangle_{P,Q} \, \mathrm{d}t \right| \leq \delta \|B\|_{L^1([0,T];H_P)} + \frac{\eta}{2} < \eta$$

for any $0 < \varepsilon < \varepsilon(\eta)$, uniformly with respect to $B \in \mathcal{B}_\omega$. □

Remark 5.1 *The conclusion of Proposition 5.1 holds true for any pair* $(B, C) \in L^1([0,T];H_P) \times L^\infty(\mathbb{R};H_Q)$ *such that* $\left(\frac{1}{S}\int_{s_0}^{s_0+S} C(s)\,\mathrm{d}s\right)_{S>0}$ *converges strongly in* H_Q *toward some* $\overline{C} \in H_Q$*, uniformly with respect to* $s_0 \in \mathbb{R}$*, when* $S \to +\infty$*. Indeed, observe that*

$$\left| \int_0^T \langle B(t), C(t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t - \int_0^T \left\langle B(t), \overline{C}\right\rangle_{P,Q} \, \mathrm{d}t \right| \leq 2\|B\|_{L^1([0,T];H_P)} \|C\|_{L^\infty(\mathbb{R};H_Q)}$$

and thus, by using the density of $C([0,T];H_P)$ *in* $L^1([0,T];H_P)$*, it is enough to consider* $B \in C([0,T];H_P)$*. But in this case, the uniform continuity of* B *allows us to pick a modulus of continuity* $\omega : [0,T] \to \mathbb{R}_+$

$$\omega(\lambda) = \sup_{t,t' \in [0,T], |t-t'| \leq \lambda} |B(t) - B(t')|_P, \quad \lambda \in [0,T]$$

and all the arguments in the proof of Proposition 5.1 apply.

In the sequel, we present some consequences of Proposition 5.1 which will be used when justifying the main result in Theorem 2.2.

Proposition 5.2 *Let* T *be a positive real number. Consider* $D \in H_Q \cap H_Q^\infty$ *a symmetric matrix field and* $\mathcal{W}_\omega \subset C([0,T];X_P)$ *a bounded set in* $L^2([0,T];X_P)$ *of functions which admit as modulus of continuity in* $C([0,T];X_P)$ *the same function* $\omega : [0,T] \to \mathbb{R}_+$*, i.e.,*

$$|w(t) - w(t')|_P \leq \omega(|t - t'|), \quad t, t' \in [0,T], \quad w \in \mathcal{W}_\omega$$

with ω *non decreasing and* $\lim_{\lambda\searrow 0} \omega(\lambda) = 0$*. Then for any family* $(w^\beta)_{\beta>0} \subset \mathcal{W}_\omega$ *which converges weakly in* $L^2([0,T];X_P)$ *toward* w^0 *when* $\beta \searrow 0$*, we have*

$$\lim_{(\beta,\varepsilon)\to(0,0)} \int_0^T \left\langle \theta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \, \mathrm{d}t = \int_0^T \left\langle \theta(t) \otimes w^0(t), \langle D \rangle \right\rangle_{P,Q} \, \mathrm{d}t \qquad (47)$$

for any $\theta \in L^2([0,T];X_P)$*.*

Proof. Notice that for any $\theta, w \in L^2([0,T];X_P)$ we have

$$\begin{aligned}
\int_0^T \langle \theta(t) \otimes w(t), G(t/\varepsilon)D\rangle_{P,Q} \, \mathrm{d}t &= \int_0^T\!\!\int_{\mathbb{R}^m} \theta(t,y) \otimes w(t,y) : G(t/\varepsilon)D \, \mathrm{d}y\mathrm{d}t \\
&= \int_0^T\!\!\int_{\mathbb{R}^m} (P^{1/2}(y)\theta(t,y)) \otimes (P^{1/2}(y)w(t,y)) : Q^{1/2}(y)G(t/\varepsilon)DQ^{1/2}(y) \, \mathrm{d}y\mathrm{d}t \\
&\leq \int_0^T |G(t/\varepsilon)D|_{H_Q^\infty} \int_{\mathbb{R}^m} |P^{1/2}(y)\theta(t,y)| \, |P^{1/2}(y)w(t,y)| \, \mathrm{d}y\mathrm{d}t \\
&\leq |D|_{H_Q^\infty} \left(\int_0^T\!\!\int_{\mathbb{R}^m} P(y)\theta(t,y) \cdot \theta(t,y) \, \mathrm{d}y\mathrm{d}t \right)^{1/2} \left(\int_0^T\!\!\int_{\mathbb{R}^m} P(y)w(t,y) \cdot w(t,y) \, \mathrm{d}y\mathrm{d}t \right)^{1/2} \\
&= |D|_{H_Q^\infty} \|\theta\|_{L^2([0,T];X_P)} \|w\|_{L^2([0,T];X_P)}
\end{aligned}$$

and similarly, by using $| \langle D \rangle |_{H_Q^\infty} \le |D|_{H_Q^\infty}$

$$\int_0^T \langle \theta(t) \otimes w(t), \langle D \rangle \rangle_{P,Q} \, \mathrm{d}t \le |D|_{H_Q^\infty} \|\theta\|_{L^2([0,T];X_P)} \|w\|_{L^2([0,T];X_P)}. \tag{48}$$

As the family $(w^\beta)_{\beta>0}$ is bounded in $L^2([0,T];X_P)$, it is enough to check (47) for any θ in a dense subset of $L^2([0,T];X_P)$, for example for any θ such that $P^{1/2}\theta \in C_c^0([0,T] \times \mathbb{R}^m)$. We appeal to Proposition 5.1 with $C(s) = G(s)D, \overline{C} = \langle D \rangle$ and $\mathcal{B} = \{\theta \otimes w \;:\; w \in \mathcal{W}_\omega\}$. By Proposition 3.1 we know that $(G(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators on H_Q, implying that $C \in L^\infty(\mathbb{R}; H_Q)$. By Theorem 2.1 we deduce that

$$\lim_{S\to+\infty} \frac{1}{S} \int_{s_0}^{s_0+S} C(s) \, \mathrm{d}s = \overline{C}, \text{ uniformly with respect to } s_0 \in \mathbb{R}.$$

For any $w \in \mathcal{W}_\omega$ we write

$$\begin{aligned}
\|\theta \otimes w\|_{L^1([0,T];H_P)} &= \int_0^T \left(\int_{\mathbb{R}^m} (P^{1/2}\theta) \otimes (P^{1/2}w) : (P^{1/2}\theta) \otimes (P^{1/2}w) \, \mathrm{d}y \right)^{1/2} \, \mathrm{d}t \\
&\le \int_0^T |\theta(t)|_{X_P^\infty} |w(t)|_P \, \mathrm{d}t \\
&\le \|P^{1/2}\theta\|_{L^2([0,T];L^\infty(\mathbb{R}^m))} \|w\|_{L^2([0,T];X_P)}
\end{aligned}$$

and therefore the boundedness of $\mathcal{W}_\omega$ in $L^2([0,T];X_P)$ implies the boundedness of $\mathcal{B}$ in $L^1([0,T];H_P)$ (here use $P^{1/2}\theta \in C_c^0([0,T] \times \mathbb{R}^m)$). We search now for a continuity modulus of $\mathcal{B}$. For any $w \in \mathcal{W}_\omega$, $t, t' \in [0,T]$, we have

$$\begin{aligned}
|\theta(t) \otimes w(t) - \theta(t') \otimes w(t')|_P &\le |\theta(t) - \theta(t')|_{X_P^\infty} |w(t)|_P + |\theta(t')|_{X_P^\infty} |w(t) - w(t')|_P \\
&\le \|P^{1/2}\theta(t) - P^{1/2}\theta(t')\|_{L^\infty(\mathbb{R}^m)} |w(t)|_P + \|P^{1/2}\theta(t')\|_{L^\infty(\mathbb{R}^m)} \omega(|t-t'|) \\
&\le \omega_\theta(|t-t'|) \|w\|_{C([0,T];X_P)} + \omega(|t-t'|) \|P^{1/2}\theta\|_{C^0([0,T]\times\mathbb{R}^m)}
\end{aligned}$$

where ω_θ is a continuity modulus for $P^{1/2}\theta \in C_c^0([0,T] \times \mathbb{R}^m)$. We are done if we show that $\mathcal{W}_\omega$ is also bounded in $C([0,T];X_P)$. This comes easily by noticing that for any $t, t' \in [0,T], w \in \mathcal{W}_\omega$ we have

$$|w(t)|_P^2 \le (|w(t')| + \omega(|t-t'|))^2 \le 2|w(t')|_P^2 + 2\omega^2(T).$$

Integrating with respect to $t' \in [0,T]$ one gets for any $t \in [0,T]$

$$|w(t)|_P^2 \le \frac{2}{T} \|w\|_{L^2([0,T];X_P)}^2 + 2\omega^2(T)$$

saying that $\mathcal{W}_\omega$ is bounded in $C([0,T];X_P)$. By Proposition 5.1, for any $\eta > 0$, there is $\varepsilon(\eta) > 0$ such that for any $0 < \varepsilon < \varepsilon(\eta), \beta > 0$

$$\left| \int_0^T \left\langle \theta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \, \mathrm{d}t - \int_0^T \left\langle \theta(t) \otimes w^\beta(t), \langle D \rangle \right\rangle_{P,Q} \, \mathrm{d}t \right| < \frac{\eta}{2}$$

By (48) we know that $w \to \int_0^T \langle \theta(t) \otimes w(t), \langle D \rangle \rangle_{P,Q} \, \mathrm{d}t$ is a linear continuous application on $L^2([0,T];X_P)$, and since $(w^\beta)_{\beta>0}$ converges weakly in $L^2([0,T];X_P)$, toward w^0, when $\beta \searrow 0$, there is $\beta(\eta) > 0$ such that for any $0 < \beta < \beta(\eta)$

$$\left| \int_0^T \left\langle \theta(t) \otimes w^\beta(t), \langle D \rangle \right\rangle_{P,Q} \, \mathrm{d}t - \int_0^T \langle \theta(t) \otimes w^0, \langle D \rangle \rangle_{P,Q} \, \mathrm{d}t \right| < \frac{\eta}{2}$$

Therefore, for any $\eta > 0$, there is $\beta(\eta) > 0, \varepsilon(\eta) > 0$ such that for any $0 < \beta < \beta(\eta), 0 < \varepsilon < \varepsilon(\eta)$

$$\left| \int_0^T \left\langle \theta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \, \mathrm{d}t - \int_0^T \left\langle \theta(t) \otimes w^0, \langle D \rangle \right\rangle_{P,Q} \, \mathrm{d}t \right| < \eta.$$

□

Remark 5.2 *The previous arguments show that if $D \in H_Q \cap H_Q^\infty$, then*

$$\lim_{\varepsilon \searrow 0} \int_0^T \langle w(t) \otimes w(t), G(t/\varepsilon)D \rangle_{P,Q} \, \mathrm{d}t = \int_0^T \langle w(t) \otimes w(t), \langle D \rangle \rangle_{P,Q} \, \mathrm{d}t \tag{49}$$

for any $w \in L^2([0,T]; X_P)$. Indeed, taking into account that the bilinear application

$$(\theta, w) \in L^2([0,T]; X_P) \times L^2([0,T]; X_P) \to \int_0^T \langle \theta(t) \otimes w(t), \langle D \rangle \rangle_{P,Q} \, \mathrm{d}t \in \mathbb{R}$$

is continuous, it is enough to establish (49) for w in the set $\{\theta \in L^2([0,T]; X_P) : P^{1/2}\theta \in C_c^0([0,T] \times \mathbb{R}^m)\}$, which is dense in $L^2([0,T]; X_P)$. And this is a direct consequence of Remark 5.1, since for any $\theta \in L^2([0,T]; X_P)$ such that $P^{1/2}\theta \in C_c^0([0,T] \times \mathbb{R}^m)$, we have

$$\begin{aligned} \|\theta \otimes \theta\|_{L^1([0,T];H_P)} &= \int_0^T \left(\int_{\mathbb{R}^m} |P^{1/2}\theta|^4 \, \mathrm{d}y \right)^{1/2} \mathrm{d}t \le \int_0^T \|P^{1/2}\theta(t)\|_{C^0(\mathbb{R}^m)} \|P^{1/2}\theta(t)\|_{L^2(\mathbb{R}^m)} \, \mathrm{d}t \\ &\le \|\theta\|_{L^2([0,T];X_P^\infty)} \|\theta\|_{L^2([0,T];X_P)} < +\infty. \end{aligned}$$

When the matrix field D is definite positive, the behavior of the upper limit with respect to (β, ε) for the quadratic term $\int_0^T \left\langle \theta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \, \mathrm{d}t$ characterizes the strong convergence of the family $(w^\beta)_{\beta>0}$ as shown in the following result.

Proposition 5.3 *Assume that all the hypotheses in Proposition 5.2 hold true.*

1. *If the matrix field D is positive, then we have*

$$\int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} \, \mathrm{d}t \le \liminf_{(\beta,\varepsilon) \to (0,0)} \int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \, \mathrm{d}t.$$

2. *If $(w^\beta)_{\beta>0}$ converges strongly in $L^2([0,T]; X_P)$ toward w^0 when $\beta \searrow 0$ (the existence of a modulus of continuity ω in $C([0,T]; X_P)$ for the family $(w^\beta)_{\beta>0}$ is not necessary here), then we have*

$$\int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} \, \mathrm{d}t = \lim_{(\beta,\varepsilon) \to (0,0)} \int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \, \mathrm{d}t.$$

3. *If there is $\alpha > 0$ such that $Q^{1/2}DQ^{1/2} \ge \alpha I_m$, and*

$$\limsup_{(\beta,\varepsilon) \to (0,0)} \int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \, \mathrm{d}t \le \int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} \, \mathrm{d}t$$

then the family $(w^\beta)_{\beta>0}$ converges strongly in $L^2([0,T]; X_P)$ toward w^0 when $\beta \searrow 0$.

Proof.
1. As the matrix field D is symmetric and positive, so is the matrix field $G(t/\varepsilon)D$ for any $t \in [0,T]$ and $\varepsilon > 0$, and thus

$$\int_0^T \left\langle w^0(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t = \int_0^T\!\!\int_{\mathbb{R}^m} w^0(t,y) \otimes w^\beta(t,y) : G(t/\varepsilon)D \,\mathrm{d}y\mathrm{d}t$$
$$\leq \int_0^T\!\!\int_{\mathbb{R}^m} (w^0(t,y) \otimes w^0(t,y) : G(t/\varepsilon)D)^{1/2} \,(w^\beta(t,y) \otimes w^\beta(t,y) : G(t/\varepsilon)D)^{1/2} \,\mathrm{d}y\mathrm{d}t$$
$$\leq \left(\int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D\rangle_{P,Q} \,\mathrm{d}t\right)^{1/2} \left(\int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t\right)^{1/2}.$$

Passing to the lower limit with respect to (β,ε) yields, thanks to Proposition 5.2

$$\int_0^T \langle w^0(t) \otimes w^0(t), \langle D\rangle\rangle_{P,Q} \,\mathrm{d}t \leq \liminf_{(\beta,\varepsilon)\to(0,0)} \left\{ \left(\int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D\rangle_{P,Q} \,\mathrm{d}t\right)^{1/2} \right.$$
$$\left. \times \left(\int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t\right)^{1/2} \right\}. \tag{50}$$

Thanks to Remark 5.2, we know that

$$\lim_{\varepsilon\searrow 0} \int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D\rangle_{P,Q} \,\mathrm{d}t = \int_0^T \langle w^0(t) \otimes w^0(t), \langle D\rangle\rangle_{P,Q} \,\mathrm{d}t. \tag{51}$$

Using the equality (51) in the inequality (50) leads to

$$\int_0^T \langle w^0(t) \otimes w^0(t), \langle D\rangle\rangle_{P,Q} \,\mathrm{d}t \leq \left(\int_0^T \langle w^0(t) \otimes w^0(t), \langle D\rangle\rangle_{P,Q} \,\mathrm{d}t\right)^{1/2}$$
$$\times \liminf_{(\beta,\varepsilon)\to(0,0)} \left(\int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t\right)^{1/2}$$

which is equivalent to our assertion.
2. Pick η a positive real number. By Remark 5.2, there is $\varepsilon(\eta)$ such that for any $0 < \varepsilon < \varepsilon(\eta)$

$$\left|\int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D\rangle_{P,Q} \,\mathrm{d}t - \int_0^T \langle w^0(t) \otimes w^0(t), \langle D\rangle\rangle_{P,Q} \,\mathrm{d}t\right| < \frac{\eta}{2}.$$

It is easily seen, thanks to the strong convergence of $(w^\beta)_{\beta>0}$ in $L^2([0,T];X_P)$ toward w^0, that there is $\beta(\eta) > 0$ such that for any $0 < \beta < \beta(\eta), \varepsilon > 0$

$$\left|\int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t - \int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D\rangle_{P,Q} \,\mathrm{d}t\right|$$
$$\leq |D|_{H_Q^\infty} \|w^\beta - w^0\|_{L^2([0,T];X_P)} \left(\|w^\beta\|_{L^2([0,T];X_P)} + \|w^0\|_{L^2([0,T];X_P)}\right) < \frac{\eta}{2}.$$

Therefore the second assertion holds true, that is, for any $\eta > 0$, there is $\beta(\eta) > 0, \varepsilon(\eta) > 0$ such that

$$\left|\int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t - \int_0^T \langle w^0(t) \otimes w^0(t), \langle D\rangle\rangle_{P,Q} \,\mathrm{d}t\right| < \eta$$

for any $0 < \beta < \beta(\eta), 0 < \varepsilon < \varepsilon(\eta)$.
3. We know by Proposition 3.1 that $Q^{1/2}G(t/\varepsilon)Q^{1/2} \geq \alpha I_m$, for any $t \in \mathbb{R}_+, \varepsilon > 0$ and therefore

$$\begin{aligned}\alpha\|w^\beta - w^0\|^2_{L^2([0,T];X_P)} &\leq \int_0^T \left\langle [w^\beta(t) - w^0(t)] \otimes [w^\beta(t) - w^0(t)], G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t \\ &= \int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t + \int_0^T \left\langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t \\ &- \int_0^T \left\langle w^\beta(t) \otimes w^0(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t - \int_0^T \left\langle w^0(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t.\end{aligned}$$

By Proposition 5.2 we know that

$$\begin{aligned}\lim_{(\beta,\varepsilon)\to(0,0)} \int_0^T \left\langle w^\beta(t) \otimes w^0(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t &= \lim_{(\beta,\varepsilon)\to(0,0)} \int_0^T \left\langle w^0(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t \\ &= \int_0^T \left\langle w^0(t) \otimes w^0(t), \langle D \rangle \right\rangle_{P,Q} \,\mathrm{d}t\end{aligned}$$

and by Remark 5.2 we have

$$\lim_{\varepsilon\searrow 0} \int_0^T \left\langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t = \int_0^T \left\langle w^0(t) \otimes w^0(t), \langle D \rangle \right\rangle_{P,Q} \,\mathrm{d}t.$$

Finally we obtain

$$\begin{aligned}\alpha \limsup_{\beta\searrow 0} \|w^\beta - w^0\|^2_{L^2([0,T];X_P)} &\leq \limsup_{(\beta,\varepsilon)\to(0,0)} \int_0^T \left\langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \right\rangle_{P,Q} \,\mathrm{d}t \\ &- \int_0^T \left\langle w^0(t) \otimes w^0(t), \langle D \rangle \right\rangle_{P,Q} \,\mathrm{d}t \leq 0\end{aligned}$$

saying that $(w^\beta)_{\beta>0}$ converges strongly in $L^2([0,T];X_P)$ toward w^0 when $\beta \searrow 0$. □

6 Proofs of the main theorems

We establish two convergence results. In Theorem 2.2 we prove strong convergence results for the families $(v^\varepsilon)_{\varepsilon>0}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$ and $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ in $L^2_{\mathrm{loc}}(\mathbb{R}_+; X_P)$. In Theorem 2.3 we study the order of the above convergences, by introducing a corrector, that is, we justify the dominant term in the developement (9).

Proof. (of Theorem 2.2)
As u^ε is the variational solution of (1), we have for any $\Phi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}^m)$

$$\begin{aligned}-\int_0^{+\infty}\!\!\int_{\mathbb{R}^m} u^\varepsilon(t,y)\partial_t\Phi \,\mathrm{d}y\mathrm{d}t - \int_{\mathbb{R}^m} u^{\mathrm{in}}(y)\Phi(0,y)\,\mathrm{d}y + \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} D(y)\nabla_y u^\varepsilon \cdot \nabla_y \Phi \,\mathrm{d}y\mathrm{d}t \\ - \frac{1}{\varepsilon}\int_0^{+\infty}\!\!\int_{\mathbb{R}^m} u^\varepsilon(t,y)b(y)\cdot\nabla_y\Phi\,\mathrm{d}y\mathrm{d}t = 0. \qquad (52)\end{aligned}$$

Actually the above formulation holds true for any compactly supported function in $\mathbb{R}_+ \times \mathbb{R}^m$, which belongs to $W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^m)$. Pick a test function $\psi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}^m)$ and let us introduce the function $\Phi^\varepsilon(t,y) = \psi(t, Y(-t/\varepsilon; y))$, $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^m$. Thanks to the hypotheses (4), (5), the function Φ^ε is compactly supported in $\mathbb{R}_+ \times \mathbb{R}^m$, belongs to $W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^m)$ and thus satisfies (52). We perform the change of variable $y = Y(t/\varepsilon; z)$. Taking the time and space derivatives of the equalities $\psi(t,z) = \Phi^\varepsilon(t, Y(t/\varepsilon; z))$ and $v^\varepsilon(t,z) = u^\varepsilon(t, Y(t/\varepsilon; z))$ gives

$$\partial_t \psi(t,z) = \partial_t \Phi^\varepsilon(t, Y(t/\varepsilon; z)) + \frac{1}{\varepsilon} b(Y(t/\varepsilon; z)) \cdot \nabla_y \Phi^\varepsilon(t, Y(t/\varepsilon; z))$$

$$\nabla_z \psi(t,z) = {}^t\partial Y(t/\varepsilon; z)\, \nabla_y \Phi^\varepsilon(t, Y(t/\varepsilon; z)),\ \ \nabla_z v^\varepsilon(t,z) = {}^t\partial Y(t/\varepsilon; z)\, \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z))$$

and the weak formulation (52), written with the test function $\Phi^\varepsilon(t,y)$ becomes

$$- \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} v^\varepsilon(t,z) \partial_t \psi \,\mathrm{d}z\mathrm{d}t - \int_{\mathbb{R}^m} u^{\mathrm{in}}(z) \psi(0,z)\,\mathrm{d}z$$
$$+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} \partial Y^{-1}(t/\varepsilon; z) D(Y(t/\varepsilon; z))\, {}^t\partial Y^{-1}(t/\varepsilon; z) \nabla_z v^\varepsilon \cdot \nabla_z \psi \,\mathrm{d}z\mathrm{d}t = 0.$$

Therefore v^ε is the variational solution of (7). By Propositions 4.3, 4.4 we have, for any $T \in \mathbb{R}_+$

$$\sup_{\varepsilon>0} \{ \|v^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))} + \|\nabla_z^R v^\varepsilon\|_{L^\infty([0,T]; L^2(\mathbb{R}^m))} + \|\nabla_z^R \otimes \nabla_z^R v^\varepsilon\|_{L^\infty([0,T]; L^2(\mathbb{R}^m))} \} < +\infty$$

$$\sup_{\varepsilon>0} \{ \|\partial_t v^\varepsilon\|_{L^2([0,T]; L^2(\mathbb{R}^m))} + \|\partial_t \nabla_z^R v^\varepsilon\|_{L^2([0,T]; L^2(\mathbb{R}^m))} \} < +\infty.$$

Let us consider a sequence $(\varepsilon_k)_k$ converging to 0 such that

$$\lim_{k\to+\infty} v^{\varepsilon_k} = v^0 \ \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m)) \tag{53}$$

$$\lim_{k\to+\infty} \nabla_z v^{\varepsilon_k} = \nabla_z v^0 \ \text{ weakly } \star \text{ in } L^\infty([0,T]; X_P),\ \ T \in \mathbb{R}_+. \tag{54}$$

We claim that v^0 is the variational solution of (15). For any $\eta \in C_c^1(\mathbb{R}_+)$ and $\Phi \in H_R^1$, the variational formulation of (7) yields

$$- \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z) \eta'(t) \Phi(z)\,\mathrm{d}z\mathrm{d}t - \int_{\mathbb{R}^m} u^{\mathrm{in}}(z) \eta(0) \Phi(z)\,\mathrm{d}z$$
$$+ \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} G(t/\varepsilon_k) D \nabla_z v^{\varepsilon_k} \cdot \eta(t) \nabla_z \Phi \,\mathrm{d}z\mathrm{d}t = 0.$$

As $\eta'\Phi$ belongs to $L^1(\mathbb{R}_+; L^2(\mathbb{R}^m))$, the weak $\star$ convergence in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of $(v^{\varepsilon_k})_k$ gives

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z) \eta'(t) \Phi(z)\,\mathrm{d}z\mathrm{d}t \underset{k\to+\infty}{\longrightarrow} \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} v^0(t,z) \eta'(t) \Phi(z)\,\mathrm{d}z\mathrm{d}t.$$

We use now Proposition 5.2 with $T > 0$ such that $\operatorname{supp}\eta \subset [0,T[$, and $\mathcal{W}_\omega = \{w^k = \nabla_z v^{\varepsilon_k}|_{[0,T]\times\mathbb{R}^m} \ :\ k \in \mathbb{N}\}$. Obviously, $\mathcal{W}_\omega$ is bounded in $L^2([0,T]; X_P)$ and for any $k \in \mathbb{N}$, $t, t' \in [0,T]$, we can write

$$|\nabla_z v^{\varepsilon_k}(t) - \nabla_z v^{\varepsilon_k}(t')|_P = \|\nabla_z^R v^{\varepsilon_k}(t) - \nabla_z^R v^{\varepsilon_k}(t')\|_{L^2} \leq \sqrt{|t-t'|}\, \|\partial_t \nabla_z^R v^{\varepsilon_k}\|_{L^2([0,T]; L^2(\mathbb{R}^m))}.$$

Therefore $\mathcal{W}_\omega$ is contained in $C([0,T]; X_P)$ and admits the continuity modulus

$$\omega(\lambda) = \sqrt{\lambda} \sup_{\varepsilon>0} \|\partial_t \nabla_z^R v^\varepsilon\|_{L^2([0,T]; L^2(\mathbb{R}^m))}.$$

Applying Proposition 5.2 with $\theta(t,z) = \eta(t)\nabla_z\Phi(z) \in L^2([0,T];X_P)$ we deduce that

$$\begin{aligned}\int_0^{+\infty}\!\!\int_{\mathbb{R}^m} G(t/\varepsilon_k)D\nabla_z v^{\varepsilon_k}\cdot\eta(t)\nabla_z\Phi\,\mathrm{d}z\mathrm{d}t &= \int_0^T \left\langle \eta(t)\nabla_z\Phi\otimes\nabla_z v^{\varepsilon_k}(t), G(t/\varepsilon_k)D\right\rangle_{P,Q}\,\mathrm{d}t\\ &\underset{k\to+\infty}{\longrightarrow} \int_0^T \left\langle \eta(t)\nabla_z\Phi\otimes\nabla_z v^0(t), \langle D\rangle\right\rangle_{P,Q}\,\mathrm{d}t\\ &= \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} \langle D\rangle\,\nabla_z v^0\cdot\eta(t)\nabla_z\Phi\,\mathrm{d}z\mathrm{d}t.\end{aligned}$$

Therefore, passing to the limit, when $k\to+\infty$, in the variational formulation of v^{ε_k}, implies

$$\begin{aligned}-\int_0^{+\infty}\!\!\int_{\mathbb{R}^m} v^0(t,z)\eta'(t)\Phi(z)\,\mathrm{d}z\mathrm{d}t - \int_{\mathbb{R}^m} u^{\mathrm{in}}(z)\eta(0)\Phi(z)\,\mathrm{d}z&\\ + \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} \langle D\rangle\,\nabla_z v^0\cdot\eta(t)\nabla_z\Phi\,\mathrm{d}z\mathrm{d}t &= 0\end{aligned}$$

and thus v^0 is the variational solution of (15) ($v^0 = v$). By the uniqueness of the solution for the limit model (15), we deduce that the convergences in (53), (54) hold with respect to $\varepsilon \searrow 0$

$$\lim_{\varepsilon\searrow 0} v^\varepsilon = v \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m)),\quad \lim_{\varepsilon\searrow 0}\nabla_z v^\varepsilon = \nabla_z v \text{ weakly } \star \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}_+;X_P).$$

The regularity of v follows by Propositions 4.5, 4.6, in particular $\partial_t v \in L^2_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^m))$. Actually the time derivative $\partial_t v$ belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^m))$. This comes immediately by the regularity of $\langle D\rangle$. Indeed, by the proofs of Propositions 4.5, 4.6 we know that $\mathrm{div}_z(R\langle D\rangle) \in L^\infty(\mathbb{R}^m)$, $R\langle D\rangle\,{}^tR \in L^\infty(\mathbb{R}^m)$ and we obtain

$$\begin{aligned}\partial_t v = \mathrm{div}_z(\langle D\rangle\nabla_z v) &= \mathrm{div}_z(\langle D\rangle\,{}^tR\,\nabla_z^R v) = \mathrm{div}_z(R\langle D\rangle)\cdot\nabla_z^R v + R\langle D\rangle : \partial\,\nabla_z^R v\\ &= \mathrm{div}_z(R\langle D\rangle)\cdot\nabla_z^R v + R\langle D\rangle\,{}^tR : \nabla_z^R\otimes\nabla_z^R v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^m)).\end{aligned}$$

We concentrate now on the strong convergence of $(v^\varepsilon)_{\varepsilon>0}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^m))$ and $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;X_P)$. By the energy balance associated with (7) we deduce

$$\|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} + 2\int_0^t \langle\nabla_z v^\varepsilon(\tau)\otimes\nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D\rangle_{P,Q}\,\mathrm{d}\tau = \|u^{\mathrm{in}}\|^2_{L^2(\mathbb{R}^m)}, t\in\mathbb{R}_+. \quad (55)$$

Similarly, the energy balance associated with (15) gives

$$\|v(t)\|^2_{L^2(\mathbb{R}^m)} + 2\int_0^t \langle\nabla_z v(\tau)\otimes\nabla_z v(\tau), \langle D\rangle\rangle_{P,Q}\,\mathrm{d}\tau = \|u^{\mathrm{in}}\|^2_{L^2(\mathbb{R}^m)}, t\in\mathbb{R}_+. \quad (56)$$

By the first statement in Proposition 5.3 we know that

$$\int_0^t \langle\nabla_z v(\tau)\otimes\nabla_z v(\tau), \langle D\rangle\rangle_{P,Q}\,\mathrm{d}\tau \le \liminf_{\varepsilon\searrow 0}\int_0^t \langle\nabla_z v^\varepsilon(\tau)\otimes\nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D\rangle_{P,Q}\,\mathrm{d}\tau. \quad (57)$$

Combining (55), (56), (57) one gets

$$\begin{aligned}\frac{1}{2}\limsup_{\varepsilon\searrow 0}\{\|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} - \|v(t)\|^2_{L^2(\mathbb{R}^m)}\} &= \limsup_{\varepsilon\searrow 0}\left\{\int_0^t \langle\nabla_z v(\tau)\otimes\nabla_z v(\tau), \langle D\rangle\rangle_{P,Q}\,\mathrm{d}\tau\right.\\ &\quad\left.- \int_0^t \langle\nabla_z v^\varepsilon(\tau)\otimes\nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D\rangle_{P,Q}\,\mathrm{d}\tau\right\}\\ &= \int_0^t \langle\nabla_z v(\tau)\otimes\nabla_z v(\tau), \langle D\rangle\rangle_{P,Q}\,\mathrm{d}\tau\\ &\quad - \liminf_{\varepsilon\searrow 0}\int_0^t \langle\nabla_z v^\varepsilon(\tau)\otimes\nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D\rangle_{P,Q}\,\mathrm{d}\tau \le 0\end{aligned}$$

saying that at any time $t \in \mathbb{R}_+$ we have

$$\limsup_{\varepsilon \searrow 0} \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} \le \|v(t)\|^2_{L^2(\mathbb{R}^m)}. \tag{58}$$

Applying Fatou lemma to the family of non negative functions $t \to \|u^{\rm in}\|^2_{L^2(\mathbb{R}^m)} - \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}$ we deduce that

$$\int_0^T \liminf_{\varepsilon \searrow 0} \{\|u^{\rm in}\|^2_{L^2(\mathbb{R}^m)} - \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}\} \;{\rm d}t \le \liminf_{\varepsilon \searrow 0} \int_0^T \{\|u^{\rm in}\|^2_{L^2(\mathbb{R}^m)} - \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}\} \;{\rm d}t$$

or equivalently

$$\limsup_{\varepsilon \searrow 0} \int_0^T \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} \;{\rm d}t \le \int_0^T \limsup_{\varepsilon \searrow 0} \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} \;{\rm d}t.$$

Therefore, the above inequality, together with the weak convergence of the family $(v^\varepsilon)_{\varepsilon>0}$ in $L^2([0,T]; L^2(\mathbb{R}^m))$ toward v and (58) imply

$$\limsup_{\varepsilon \searrow 0} \int_0^T \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} \;{\rm d}t \le \int_0^T \|v(t)\|^2_{L^2(\mathbb{R}^m)} \;{\rm d}t$$

saying that $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2([0,T]; L^2(\mathbb{R}^m))$ toward v for any $T \in \mathbb{R}_+$

$$\lim_{\varepsilon \searrow 0} \int_0^T \|v^\varepsilon(t) - v(t)\|^2_{L^2(\mathbb{R}^m)} \;{\rm d}t = 0.$$

There is a sequence $(\tilde{\varepsilon}_k)_k$ converging to 0 such that

$$\lim_{k \to +\infty} \|v^{\tilde{\varepsilon}_k}(t) - v(t)\|^2_{L^2(\mathbb{R}^m)} = 0, \;\; \text{for a.a. } t \in [0,T]. \tag{59}$$

As $\partial_t v \in L^2([0,T]; L^2(\mathbb{R}^m))$ and $\sup_{\varepsilon>0} \|\partial_t v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} < +\infty$, it is easily seen that (59) holds true for any $t \in [0,T]$, $T \in \mathbb{R}_+$, and thus for any $t \in \mathbb{R}_+$. Actually we have

$$\lim_{\varepsilon \searrow 0} \|v^\varepsilon(T) - v(T)\|^2_{L^2(\mathbb{R}^m)} = 0, \;\; T \in \mathbb{R}_+$$

which implies, thanks to (55), (56)

$$\begin{aligned}
\limsup_{\varepsilon \searrow 0} \int_0^T \langle \nabla_z v^\varepsilon(t) \otimes \nabla_z v^\varepsilon(t), G(t/\varepsilon) D\rangle_{P,Q} \;{\rm d}t &= \frac{1}{2}\|u^{\rm in}\|^2_{L^2(\mathbb{R}^m)} - \frac{1}{2}\lim_{\varepsilon \searrow 0} \|v^\varepsilon(T)\|^2_{L^2(\mathbb{R}^m)} \\
&= \frac{1}{2}\|u^{\rm in}\|^2_{L^2(\mathbb{R}^m)} - \frac{1}{2}\|v(T)\|^2_{L^2(\mathbb{R}^m)} \\
&= \int_0^T \langle \nabla_z v(t) \otimes \nabla_z v(t), \langle D \rangle\rangle_{P,Q} \;{\rm d}t.
\end{aligned}$$

By the third statement of Proposition 5.3 we deduce that $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2([0,T]; X_P)$ toward $\nabla_z v$, for any $T \in \mathbb{R}_+$. Finally, in order to prove the convergence of $(v^\varepsilon)_{\varepsilon>0}$ in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ toward v we take the difference between the equations (7) and (15)

$$\partial_t(v^\varepsilon - v) - {\rm div}_z\{G(t/\varepsilon) D \nabla_z v^\varepsilon - \langle D \rangle \nabla_z v\} = 0, \;\; (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m.$$

Writing the energy balance, we obtain for any $t \in \mathbb{R}_+$

$$\begin{aligned}
\frac{1}{2}\|v^\varepsilon(t) - v(t)\|^2_{L^2(\mathbb{R}^m)} &+ \int_0^t \langle [\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)] \otimes \nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon) D\rangle_{P,Q} \;{\rm d}\tau \\
&- \int_0^t \langle [\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)] \otimes \nabla_z v(\tau), \langle D \rangle\rangle_{P,Q} \;{\rm d}\tau = 0.
\end{aligned}$$

As in the proof of Proposition 5.2 we have

$$\left|\langle[\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)] \otimes \nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon) D\rangle_{P,Q}\right| \le |D|_{H_Q^\infty} |\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)|_P |\nabla_z v^\varepsilon(\tau)|_P$$

and

$$\left|\langle[\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)] \otimes \nabla_z v(\tau), \langle D\rangle\rangle_{P,Q}\right| \le |D|_{H_Q^\infty} |\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)|_P |\nabla_z v(\tau)|_P$$

and we deduce that for any $t \in [0,T]$ we have

$$\|(v^\varepsilon - v)(t)\|_{L^2}^2 \le 2|D|_{H_Q^\infty} \|\nabla_z v^\varepsilon - \nabla_z v\|_{L^2([0,T];X_P)} (\|\nabla_z v^\varepsilon\|_{L^2([0,T];X_P)} + \|\nabla_z v\|_{L^2([0,T];X_P)}).$$

The strong convergence of $(v^\varepsilon)_{\varepsilon>0}$ in $L^\infty([0,T];L^2(\mathbb{R}^m))$ toward v comes by the strong convergence of $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ in $L^2([0,T];X_P)$ toward $\nabla_z v$, when $\varepsilon \searrow 0$. □

Remark 6.1 *The strong convergence of* $(v^\varepsilon)_{\varepsilon>0}$ *in* $L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^m))$*, when* $\varepsilon \searrow 0$*, holds true for initial conditions* $u^{\rm in} \in L^2(\mathbb{R}^m)$*. Indeed, for any* $u^{\rm in} \in L^2(\mathbb{R}^m)$*,* $T \in \mathbb{R}_+, \delta > 0$*, let us consider* $u^{\rm in}_\delta \in H^2_R$ *such that* $\|u^{\rm in} - u^{\rm in}_\delta\|_{L^2(\mathbb{R}^m)} \le \delta/2$*. We denote by* v^ε_δ *(resp.* v_δ*) the variational solution of* (7) *(resp.* (15)*) with the initial condition* $u^{\rm in}_\delta$*. Thanks to the energy balance we obtain easily*

$$\begin{aligned}\|v^\varepsilon - v\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} &\le \|v^\varepsilon - v^\varepsilon_\delta\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} + \|v^\varepsilon_\delta - v_\delta\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\ &\quad + \|v_\delta - v\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\ &\le 2\|u^{\rm in} - u^{\rm in}_\delta\|_{L^2(\mathbb{R}^m)} + \|v^\varepsilon_\delta - v_\delta\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}.\end{aligned}$$

By Theorem 2.2 we know that

$$\lim_{\varepsilon\searrow 0} \|v^\varepsilon_\delta - v_\delta\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} = 0$$

and therefore

$$\limsup_{\varepsilon\searrow 0} \|v^\varepsilon - v\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \le \delta, \ \ \delta > 0$$

saying that $\lim_{\varepsilon\searrow 0} \|v^\varepsilon - v\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} = 0$*, for any* $T \in \mathbb{R}_+$*.*

Remark 6.2 *The computations in the proof of Theorem 2.2 show that for any smooth matrix field* C *and any locally integrable function* $v = v(z)$ *we have*

$$({\rm div}_z(G(s) C\, \nabla_z v))_{-s} = {\rm div}_y\{C \nabla_y (v_{-s})\} \quad \textit{in } \mathcal{D}'(\mathbb{R}^m).$$

The above considerations show that $v^\varepsilon = v + o(1)$ in $L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^m))$, when $\varepsilon \searrow 0$. As suggested by (9), we expect a convergence rate in $\mathcal{O}(\varepsilon)$. This can be achieved assuming that the limit solution v is smooth enough and that there is a smooth matrix field C such that

$$D = \langle D\rangle + L(C). \tag{60}$$

The existence of the matrix field C is essential when constructing the corrector term u^1, see (63). Notice that Proposition 3.2 guarantees that $D - \langle D\rangle \in \overline{\text{Range}\, L}$, and thus (60) holds true if the range of L is closed. Moreover, we will assume without loss of generality that $C \in (\ker L)^\perp$, which implies also that ${}^tC \in (\ker L)^\perp$. As D is symmetric, so is $\langle D\rangle$, and thus $L(C - {}^tC) = 0$. Finally $C - {}^tC \in \ker L \cap (\ker L)^\perp = \{0\}$, saying that C is symmetric.

Proof. (of Theorem 2.3)
We introduce the functions $\tilde{u}^\varepsilon(t,y) = v(t, Y(-t/\varepsilon; y)), (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m, \varepsilon > 0$. As in the proof of Theorem 2.2 we check that $\tilde{u}^\varepsilon$ is the variational solution of the problem

$$\begin{cases} \partial_t \tilde{u}^\varepsilon - \mathrm{div}_y\{G(t/\varepsilon)\langle D\rangle \nabla_y \tilde{u}^\varepsilon\} + \frac{1}{\varepsilon} b(y)\cdot \nabla_y \tilde{u}^\varepsilon = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ \tilde{u}^\varepsilon(0,y) = u^{\mathrm{in}}(y), & y \in \mathbb{R}^m. \end{cases}$$

For doing that, pick a smooth compactly supported test function $\Phi(t,y)$ and appeal to the weak formulation of v, with the test function $\psi^\varepsilon(t,z) = \Phi(t, Y(t/\varepsilon; z)), (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m$. By construction, the average matrix field $\langle D\rangle$ belongs to $\ker L$ implying that $G(t/\varepsilon)\langle D\rangle = \langle D\rangle$. Therefore the functions $(\tilde{u}^\varepsilon)_{\varepsilon>0}$ solve the problems

$$\begin{cases} \partial_t \tilde{u}^\varepsilon - \mathrm{div}_y\{\langle D\rangle \nabla_y \tilde{u}^\varepsilon\} + \frac{1}{\varepsilon} b(y)\cdot \nabla_y \tilde{u}^\varepsilon = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ \tilde{u}^\varepsilon(0,y) = u^{\mathrm{in}}(y), & y \in \mathbb{R}^m. \end{cases} \tag{61}$$

Recall that the functions $(u^\varepsilon)_{\varepsilon>0}$ satisfy

$$\begin{cases} \partial_t u^\varepsilon - \mathrm{div}_y\{D \nabla_y u^\varepsilon\} + \frac{1}{\varepsilon} b(y)\cdot \nabla_y u^\varepsilon = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ u^\varepsilon(0,y) = u^{\mathrm{in}}(y), & y \in \mathbb{R}^m. \end{cases} \tag{62}$$

Notice that both families $(\tilde{u}^\varepsilon)_{\varepsilon>0}, (u^\varepsilon)_{\varepsilon>0}$ verify the same initial condition. The key point for obtaining a convergence rate is to introduce a corrector term. We consider the function

$$\begin{aligned} u^1(t,s,y) &= -\mathrm{div}_z(C\nabla_z v(t))(Y(-s;y)) + \mathrm{div}_y\{C(y)\nabla_y v(t, Y(-s;y))\} \qquad (63) \\ &= -\tau(-s)\mathrm{div}_z(C\nabla_z v(t)) + \mathrm{div}_y\{C(y)\nabla_y[\tau(-s)v(t)]\}, \quad (t,s,y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^m \end{aligned}$$

where we use the notation $\tau(s)f = f \circ Y(s;\cdot)$ for any function f. By Remark 6.2 we have

$$u^1(t,s,Y(s;z)) = \mathrm{div}_z(G(s)C\nabla_z v(t)) - \mathrm{div}_z(C\nabla_z v(t)) \tag{64}$$

and taking the derivative with respect to s (here L is the infinitesimal generator of the group G) leads to

$$\begin{aligned} \partial_s u^1(t,s,Y(s;z)) + b(Y(s;z))\cdot\nabla_y u^1(t,s,Y(s;z)) &= \mathrm{div}_z\left\{\frac{\mathrm{d}}{\mathrm{d}s}G(s)C\,\nabla_z v(t)\right\} \\ &= \mathrm{div}_z\{G(s)L(C)\nabla_z v(t)\} \\ &= \{\mathrm{div}_y[L(C)\nabla_y \tau(-s)v(t)]\}(Y(s;z)). \end{aligned}$$

Notice that for the last equality we have used one more time Remark 6.2. Therefore the corrector u^1 verifies

$$(\partial_s + b(y)\cdot\nabla_y)u^1(t,s,y) - \mathrm{div}_y\{L(C)\nabla_y v(t, Y(-s;\cdot))\}(y) = 0, \quad (t,s,y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^m \tag{65}$$

and by definition $u^1(t,0,y) = 0,\ (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m$. The equation (65) is exactly the equality coming out at the leading order when plugging the Ansatz $u^\varepsilon(t,y) = v(t, Y(-t/\varepsilon; y)) + \varepsilon u^1(t, t/\varepsilon, y) + ...$ into (62). Indeed, the above Ansatz also writes

$$u^\varepsilon(t, Y(t/\varepsilon; z)) = v(t,z) + \varepsilon u^1(t, t/\varepsilon, Y(t/\varepsilon; z)) + ...$$

and by observing that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u^\varepsilon(t, Y(t/\varepsilon;z)) &= \partial_t u^\varepsilon(t, Y(t/\varepsilon;z)) + \frac{1}{\varepsilon} b(Y(t/\varepsilon;z))\cdot\nabla_y u^\varepsilon(t, Y(t/\varepsilon;z)) \\ &= [\mathrm{div}_y(D\nabla_y u^\varepsilon(t))](Y(t/\varepsilon;z)) = \mathrm{div}_z[G(t/\varepsilon)D\nabla_z u^\varepsilon(t, Y(t/\varepsilon;z))] \end{aligned}$$

we obtain

$$
\begin{aligned}
&\partial_t v(t,z) + \varepsilon \partial_t u^1(t,t/\varepsilon,Y(t/\varepsilon;z)) + \partial_s u^1(t,t/\varepsilon,Y(t/\varepsilon;z)) \\
&\quad + b(Y(t/\varepsilon;z))\cdot\nabla_y u^1(t,t/\varepsilon,Y(t/\varepsilon;z)) + ... = \mathrm{div}_z(G(t/\varepsilon)D\nabla_z v) \\
&\quad + \mathrm{div}_z[G(t/\varepsilon)D\nabla_z(\varepsilon u^1(t,t/\varepsilon,Y(t/\varepsilon;z)))] + ... \,.
\end{aligned} \tag{66}
$$

Taking into account that $\partial_t v = \mathrm{div}_z(\langle D\rangle \nabla_z v)$, we deduce from (66), thanks to Remark 6.2, that

$$
\begin{aligned}
(\partial_s + b(y)\cdot\nabla_y)u^1(t,s,y) &= \tau(-s)\mathrm{div}_z[G(s)D\nabla_z v(t)] - \tau(-s)\mathrm{div}_z[\langle D\rangle \nabla_z v(t)] \\
&= \mathrm{div}_y[(D-\langle D\rangle)\nabla_y \tau(-s)v(t)] \\
&= \mathrm{div}_y[L(C)\nabla_y \tau(-s)v(t)]
\end{aligned}
$$

which corresponds to (65). In particular, for $s = t/\varepsilon$, one gets

$$
(\partial_s + b(y)\cdot\nabla_y)u^1(t,t/\varepsilon,y) - \mathrm{div}_y(L(C)\nabla_y \tilde{u}^\varepsilon(t))(y) = 0,\ (t,y)\in\mathbb{R}_+\times\mathbb{R}^m, \varepsilon>0
$$

and we obtain the following equation for $\tilde{u}^1_\varepsilon := u^1(t,t/\varepsilon,y)$

$$
\partial_t(\varepsilon\tilde{u}^1_\varepsilon)(t,y) - \mathrm{div}_y(L(C)\nabla_y\tilde{u}^\varepsilon(t)) + \frac{1}{\varepsilon}b(y)\cdot\nabla_y(\varepsilon\tilde{u}^1_\varepsilon)(t,y) = \varepsilon\partial_t u^1(t,t/\varepsilon,y). \tag{67}
$$

Taking the sum between the equation in (61) and (67) yields

$$
\partial_t(\tilde{u}^\varepsilon + \varepsilon\tilde{u}^1_\varepsilon) - \mathrm{div}_y[(\langle D\rangle + L(C))\nabla_y\tilde{u}^\varepsilon] + \frac{1}{\varepsilon}b(y)\cdot\nabla_y(\tilde{u}^\varepsilon+\varepsilon\tilde{u}^1_\varepsilon) = \varepsilon\partial_t u^1(t,t/\varepsilon,y) \tag{68}
$$

which also writes, thanks to (60)

$$
\partial_t(\tilde{u}^\varepsilon + \varepsilon\tilde{u}^1_\varepsilon) - \mathrm{div}_y[D\nabla_y(\tilde{u}^\varepsilon+\varepsilon\tilde{u}^1_\varepsilon)] + \frac{1}{\varepsilon}b(y)\cdot\nabla_y(\tilde{u}^\varepsilon+\varepsilon\tilde{u}^1_\varepsilon) = \varepsilon[\partial_t u^1 - \mathrm{div}_y(D\nabla_y u^1)](t,t/\varepsilon,y).
$$

Combining (62) and (68), it is easily seen that

$$
\begin{aligned}
\partial_t(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon) - \mathrm{div}_y[D\nabla_y(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon)] + \frac{1}{\varepsilon}b(y)\cdot\nabla_y(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon)& \\
= -\varepsilon[\partial_t u^1 - \mathrm{div}_y(D\nabla_y u^1)](t,t/\varepsilon,y)&.
\end{aligned}
$$

Using the energy balance together with the hypothesis $Q^{1/2}DQ^{1/2}\geq \alpha I_m$ we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon\|^2_{L^2(\mathbb{R}^m)} + \alpha|\nabla_y(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon)|^2_P \leq \varepsilon\|u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon\|_{L^2(\mathbb{R}^m)}& \\
\times\|\partial_t u^1(t,t/\varepsilon,\cdot) - \mathrm{div}_y(D\nabla_y u^1(t,t/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)},\ \ t\in\mathbb{R}_+&.
\end{aligned}
$$

Notice that $(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon)|_{t=0} = u^{\mathrm{in}} - u^{\mathrm{in}} - 0 = 0$ and therefore, after integration with respect to $t\in[0,T]$, one gets

$$
\|u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \leq \varepsilon\int_0^T \|\partial_t u^1(t,t/\varepsilon,\cdot) - \mathrm{div}_y(D\nabla_y u^1(t,t/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)}\,\mathrm{d}t
$$

and

$$
\begin{aligned}
\alpha\int_0^T |\nabla_y(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon)|^2_P\,\mathrm{d}t &\leq \varepsilon\|u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon\tilde{u}^1_\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \\
&\quad\times\int_0^T \|\partial_t u^1(t,t/\varepsilon,\cdot) - \mathrm{div}_y(D\nabla_y u^1(t,t/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)}\,\mathrm{d}t \\
&\leq \varepsilon^2\left(\int_0^T \|\partial_t u^1(t,t/\varepsilon,\cdot) - \mathrm{div}_y(D\nabla_y u^1(t,t/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)}\,\mathrm{d}t\right)^2.
\end{aligned}
$$

We are done if the corrector $u^1(t,s,y)$ satisfies uniform estimates with respect to the fast variable s

$$u^1 \in L^\infty([0,T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m))), \;\; \partial_t u^1 \in L^1([0,T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m)))$$

$$\mathrm{div}_y(D\nabla_y u^1) \in L^1([0,T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m))), \;\; \nabla_y u^1 \in L^2([0,T]; L^\infty(\mathbb{R}_s; X_P)).$$

Let us estimate the $L^2(\mathbb{R}^m)$ norm of u^1, uniformly with respect to $(t,s) \in [0,T]\times\mathbb{R}$. Thanks to (64) we have

$$\begin{aligned}\|u^1(t,s,\cdot)\|_{L^2(\mathbb{R}^m)} &\le \|\mathrm{div}_z(G(s)C\nabla_z v(t)) - \mathrm{div}_z(C\nabla_z v(t))\|_{L^2(\mathbb{R}^m)}\\ &\le 2\sup_{s\in\mathbb{R}} \|\mathrm{div}_z(G(s)C\nabla_z v(t))\|_{L^2(\mathbb{R}^m)}.\end{aligned}$$

For any $s \in \mathbb{R}$ we can write, using the formula $\mathrm{div}_z(X\xi) = \mathrm{div}_z\, {}^tX \cdot \xi + {}^tX : \partial_z \xi$, for any smooth matrix field X and vector field ξ

$$\begin{aligned}\mathrm{div}_z(G(s)C\nabla_z v(t)) &= \mathrm{div}_z(G(s)C\,{}^tR\,\nabla_z^R v(t))\\ &= \mathrm{div}_z(RG(s)C)\cdot\, \nabla_z^R v(t) + RG(s)C\,{}^tR : \partial_z\, \nabla_z^R v(t) R^{-1}\\ &= \mathrm{div}_z(RG(s)C)\cdot\, \nabla_z^R v(t) + RG(s)C\,{}^tR :\, \nabla_z^R \otimes \nabla_z^R v(t).\end{aligned} \tag{69}$$

We claim that $\mathrm{div}_z(RG(s)C) = \tau(s)\mathrm{div}_y(RC)$. Indeed, for any smooth compactly supported vector field $\Phi = \Phi(y)$ we have, thanks to (21)

$$\begin{aligned}\int_{\mathbb{R}^m} \mathrm{div}_z(RG(s)C)\cdot\Phi(Y(s;z))\,\mathrm{d}z &= -\int_{\mathbb{R}^m} RG(s)C : \partial_z\{\Phi(Y(s;z))\}\,\mathrm{d}z\\ &= -\int_{\mathbb{R}^m} RG(s)C\,{}^tR : (\partial_y\Phi)(Y(s;z))\partial Y(s;z)\,R^{-1}\,\mathrm{d}z\\ &= -\int_{\mathbb{R}^m} (RC\,{}^tR)(Y(s;z)) : (\partial_y\Phi R^{-1})(Y(s;z))\,\mathrm{d}z\\ &= -\int_{\mathbb{R}^m} RC\,{}^tR : \partial_y\Phi R^{-1}\,\mathrm{d}y\\ &= -\int_{\mathbb{R}^m} RC : \partial_y\Phi\,\mathrm{d}y\\ &= \int_{\mathbb{R}^m} \mathrm{div}_y(RC)\cdot\Phi(y)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^m} \tau(s)[\mathrm{div}_y(RC)]\cdot\Phi(Y(s;z))\,\mathrm{d}z.\end{aligned} \tag{70}$$

Coming back to (69) we obtain

$$\mathrm{div}_z(G(s)C\nabla_z v(t)) = \tau(s)[\mathrm{div}_y(RC)]\cdot\, \nabla_z^R v(t) + \tau(s)(RC\,{}^tR) :\, \nabla_z^R \otimes \nabla_z^R v(t) \tag{71}$$

and therefore

$$\|\mathrm{div}_z(G(s)C\nabla_z v(t))\|_{L^2(\mathbb{R}^m)} \le \|\mathrm{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)}\|\, \nabla_z^R v(t)\|_{L^2(\mathbb{R}^m)} + |C|_{H_Q^\infty}\|\, \nabla_z^R \otimes \nabla_z^R v(t)\|_{L^2(\mathbb{R}^m)}$$

saying that

$$\begin{aligned}\|u^1\|_{L^\infty([0,T];L^\infty(\mathbb{R}_s;L^2(\mathbb{R}^m)))} &\le 2\|\mathrm{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)}\|\, \nabla_z^R v\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\ &\quad + 2|C|_{H_Q^\infty}\|\, \nabla_z^R \otimes \nabla_z^R v(t)\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}.\end{aligned}$$

Similarly, taking the derivative of (64) with respect to t yields

$$\|\partial_t u^1\|_{L^1([0,T];L^\infty(\mathbb{R}_s;L^2(\mathbb{R}^m)))} \le 2\|\mathrm{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)}\| \nabla^R_z \partial_t v\|_{L^1([0,T];L^2(\mathbb{R}^m))}$$
$$+ 2|C|_{H^\infty_Q}\| \nabla^R_z \otimes \nabla^R_z \partial_t v(t)\|_{L^1([0,T];L^2(\mathbb{R}^m))}.$$

It remains to estimate the space derivatives of u^1. The key point is that ∇^R commutes with $\tau(s)$, i.e.

$$\nabla^R_z(\tau(s)f) = \nabla^R_z\{f(Y(s;\cdot))\} = (\nabla^R_y f)(Y(s;\cdot)) = \tau(s)(\nabla^R_y f)$$

for any smooth function $f = f(y)$. Indeed, for any $i \in \{1, ..., m\}$ we have

$$\begin{aligned} b_i \cdot \nabla_z(\tau(s)f)(z) &= \lim_{h\to 0} \frac{f(Y(s;Y_i(h;z))) - f(Y(s;z))}{h} \\ &= \lim_{h\to 0} \frac{f(Y_i(h;Y(s;z))) - f(Y(s;z))}{h} \\ &= b_i(Y(s;z)) \cdot (\nabla_y f)(Y(s;z)) = \tau(s)(b_i \cdot \nabla_y f)(z). \end{aligned}$$

Applying the operator ∇^R in (64) and using (69), (71) lead to

$$\begin{aligned} (\nabla^R_y u^1(t,s,\cdot))(Y(s;\cdot)) &= \nabla^R_z u^1(t,s,Y(s;\cdot)) = \nabla^R_z[\mathrm{div}_z(G(s)C\nabla_z v(t)) - \mathrm{div}_z(C\nabla_z v(t))] \\ &= \nabla^R_z[\tau(s)(\mathrm{div}_y(RC)) \cdot \nabla^R_z v(t) + \tau(s)(RC\,{}^tR) : \nabla^R_z \otimes \nabla^R_z v(t)] \\ &- \nabla^R_z[\mathrm{div}_z(RC) \cdot \nabla^R_z v(t) + (RC\,{}^tR) : \nabla^R_z \otimes \nabla^R_z v(t)]. \end{aligned} \tag{72}$$

Appealing one more time to the commutation between $\tau(s)$ and ∇^R we deduce that for any $k \in \{1, ..., m\}$

$$\begin{aligned} &b_k \cdot \nabla_z[\tau(s)(\mathrm{div}_y(RC)) \cdot \nabla^R_z v(t) + \tau(s)(RC\,{}^tR) : \nabla^R_z \otimes \nabla^R_z v(t)] \\ &= \tau(s)(b_k \cdot \nabla_y \mathrm{div}_y(RC)) \cdot \nabla^R_z v(t) + \tau(s)\mathrm{div}_y(RC) \cdot (b_k \cdot \nabla_z \nabla^R_z v(t)) \\ &+ \tau(s)(b_k \cdot \nabla_y(RC\,{}^tR)) : \nabla^R_z \otimes \nabla^R_z v(t) + \tau(s)(RC\,{}^tR) : b_k \cdot \nabla_z(\nabla^R_z \otimes \nabla^R_z v(t)). \end{aligned} \tag{73}$$

Therefore there is a constant K depending on $\|\mathrm{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)} + \|RC\,{}^tR\|_{L^\infty(\mathbb{R}^m)} + \sum_{k=1}^m \|b_k \cdot \nabla_y \mathrm{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)} + \sum_{k=1}^m \|b_k \cdot \nabla_y(RC\,{}^tR)\|_{L^\infty(\mathbb{R}^m)}$ such that

$$\| \nabla^R_y u^1(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}_s;L^2(\mathbb{R}^m))} \le K\{\| \nabla^R_z v(t)\|_{L^2(\mathbb{R}^m)} + \| \nabla^R_z \otimes \nabla^R_z v(t)\|_{L^2(\mathbb{R}^m)}$$
$$+ \| \nabla^R_z \otimes \nabla^R_z \otimes \nabla^R_z v(t)\|_{L^2(\mathbb{R}^m)}\}.$$

We deduce that

$$\|\nabla_y u^1\|_{L^2([0,T];L^\infty(\mathbb{R}_s;X_P))} \le K\{\| \nabla^R_z v\|_{L^2([0,T];L^2(\mathbb{R}^m))} + \| \nabla^R_z \otimes \nabla^R_z v\|_{L^2([0,T];L^2(\mathbb{R}^m))}$$
$$+ \| \nabla^R_z \otimes \nabla^R_z \otimes \nabla^R_z v\|_{L^2([0,T];L^2(\mathbb{R}^m))}\}.$$

For the second space derivatives of u^1, we write as before

$$\mathrm{div}_y(D\nabla_y u^1) = \mathrm{div}_y(D\,{}^tR\,\nabla^R_y u^1) = \mathrm{div}_y(RD) \cdot \nabla^R_y u^1 + RD\,{}^tR : \nabla^R_y \otimes \nabla^R_y u^1.$$

Notice that $\mathrm{div}_y(RD) \cdot \nabla^R_y u^1$ belongs to $L^1([0,T];L^\infty(\mathbb{R}_s;L^2(\mathbb{R}^m)))$ since by hypotheses $\mathrm{div}_y(RD) \in L^\infty(\mathbb{R}^m)$ and we already know that $\nabla_y u^1 \in L^2([0,T];L^\infty(\mathbb{R}_s;X_P))$. As the matrix field $RD\,{}^tR$ belongs to $L^\infty(\mathbb{R}^m)$, it remains to check that $\nabla^R_y \otimes \nabla^R_y u^1$ belongs to $L^1([0,T];L^\infty(\mathbb{R}_s;L^2(\mathbb{R}^m)))$. For doing that, we apply one more time the operator ∇^R_z in

(72), or equivalently the operator $b_l \cdot \nabla_z$ in (73). Using again the commutation between $\tau(s)$ and $b_l \cdot \nabla_z$ we obtain

$$
\begin{aligned}
& b_l \cdot \nabla_z \{ b_k \cdot \nabla_z [\tau(s)(\mathrm{div}_y(RC)) \cdot \nabla_z^R v(t) + \tau(s)(RC\,{}^tR) : \nabla_z^R \otimes \nabla_z^R v(t)] \} \\
& = [b_l \cdot \nabla_y (b_k \cdot \nabla_y(\mathrm{div}_y(RC)))]_s \cdot \nabla_z^R v(t) + [b_k \cdot \nabla_y(\mathrm{div}_y(RC))]_s \cdot [b_l \cdot \nabla_z(\nabla_z^R v(t))] \\
& + [b_l \cdot \nabla_y(\mathrm{div}_y(RC))]_s \cdot (b_k \cdot \nabla_z(\nabla_z^R v(t))) + (\mathrm{div}_y(RC))_s \cdot \{ b_l \cdot \nabla_z [b_k \cdot \nabla_z(\nabla_z^R v(t))]\} \\
& + [b_l \cdot \nabla_y(b_k \cdot \nabla_y(RC\,{}^tR))]_s : \nabla_z^R \otimes \nabla_z^R v(t) + [b_k \cdot \nabla_y(RC\,{}^tR)]_s : b_l \cdot \nabla_z(\nabla_z^R \otimes \nabla_z^R v(t)) \\
& + [b_l \cdot \nabla_y(RC\,{}^tR)]_s : b_k \cdot \nabla_z(\nabla_z^R \otimes \nabla_z^R v(t)) + (RC\,{}^tR)_s : b_k \cdot \nabla_z(b_l \cdot \nabla_z(\nabla_z^R \otimes \nabla_z^R v(t)))
\end{aligned}
$$

which belongs to $L^1([0,T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m)))$, thanks to the hypotheses on the matrix field C and the solution v. $\square$

A Proofs of Propositions 4.4, 4.5

Proof. (of Proposition 4.4)
For any $i, j, k \in \{1, ..., m\}$ we introduce the notations $u_i^\varepsilon = b_i \cdot \nabla_y u^\varepsilon, v_i^\varepsilon = b_i \cdot \nabla_z v^\varepsilon, u_{ij}^\varepsilon = b_j \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon), v_{ij}^\varepsilon = b_j \cdot \nabla_z (b_i \cdot \nabla_z v^\varepsilon), u_{ijk}^\varepsilon = b_k \cdot \nabla_y (b_j \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon))$ and $v_{ijk}^\varepsilon = b_k \cdot \nabla_z (b_j \cdot \nabla_z (b_i \cdot \nabla_z v^\varepsilon))$. With these notations, the equation (31) becomes

$$
\partial_t u_i^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u_i^\varepsilon) + \frac{1}{\varepsilon} b \cdot \nabla_y u_i^\varepsilon = \mathrm{div}_y([b_i, D]\,{}^tR\,\nabla_y^R u^\varepsilon) + D\,{}^tR\,\nabla_y^R u^\varepsilon \cdot \nabla_y \mathrm{div}_y b_i.
$$

Taking now the directional derivative $b_j \cdot \nabla_y$, yields

$$
\begin{aligned}
\partial_t u_{ij}^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u_{ij}^\varepsilon) + \frac{1}{\varepsilon} b \cdot \nabla_y u_{ij}^\varepsilon &= \mathrm{div}_y([b_j, D]\,{}^tR\,\nabla_y^R u_i^\varepsilon) + RD\,{}^tR\,\nabla_y^R u_i^\varepsilon \cdot \nabla_y^R \mathrm{div}_y b_j \\
&+ b_j \cdot \nabla_y \mathrm{div}_y([b_i, D]\nabla_y u^\varepsilon) + b_j \cdot \nabla_y (D\,{}^tR\,\nabla_y^R u^\varepsilon \cdot \nabla_y \mathrm{div}_y b_i). \quad (74)
\end{aligned}
$$

Thanks to the commutation formula (34), we have

$$
\begin{aligned}
& b_j \cdot \nabla_y \mathrm{div}_y([b_i, D]\nabla_y u^\varepsilon) = [b_j \cdot \nabla_y, \mathrm{div}_y([b_i, D]\nabla_y)] u^\varepsilon + \mathrm{div}_y([b_i, D]\nabla_y(b_j \cdot \nabla_y u^\varepsilon)) \\
& = \mathrm{div}_y([b_j, [b_i, D]]\nabla_y u^\varepsilon) + [b_i, D]\nabla_y u^\varepsilon \cdot \nabla_y \mathrm{div}_y b_j + \mathrm{div}_y([b_i, D]\nabla_y(b_j \cdot \nabla_y u^\varepsilon)) \\
& = \mathrm{div}_y([b_j, [b_i, D]]\,{}^tR\,\nabla_y^R u^\varepsilon) + R[b_i, D]\,{}^tR\,\nabla_y^R u^\varepsilon \cdot \nabla_y^R \mathrm{div}_y b_j + \mathrm{div}_y([b_i, D]\,{}^tR\,\nabla_y^R u_j^\varepsilon). \quad (75)
\end{aligned}
$$

Combining (74), (75) we obtain

$$
\begin{aligned}
\partial_t u_{ij}^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u_{ij}^\varepsilon) + \frac{1}{\varepsilon} b \cdot \nabla_y u_{ij}^\varepsilon &= \mathrm{div}_y([b_j, D]\,{}^tR\,\nabla_y^R u_i^\varepsilon) + RD\,{}^tR\,\nabla_y^R u_i^\varepsilon \cdot \nabla_y^R \mathrm{div}_y b_j \\
&+ \mathrm{div}_y([b_j, [b_i, D]]\,{}^tR\,\nabla_y^R u^\varepsilon) + R[b_i, D]\,{}^tR\,\nabla_y^R u^\varepsilon \cdot \nabla_y^R \mathrm{div}_y b_j \\
&+ \mathrm{div}_y([b_i, D]\,{}^tR\,\nabla_y^R u_j^\varepsilon) + b_j \cdot \nabla_y (D\,{}^tR\,\nabla_y^R u^\varepsilon \cdot \nabla_y \mathrm{div}_y b_i).
\end{aligned}
$$

Multiplying by u_{ij}^ε and integrating on $\mathbb{R}^m$ lead to

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} (u_{ij}^\varepsilon)^2 \,\mathrm{d}y + \int_{\mathbb{R}^m} D\nabla_y u_{ij}^\varepsilon \cdot \nabla_y u_{ij}^\varepsilon \,\mathrm{d}y &= - \int_{\mathbb{R}^m} R[b_j, D]\,{}^tR\,\nabla_y^R u_i^\varepsilon \cdot \nabla_y^R u_{ij}^\varepsilon \,\mathrm{d}y \\
&+ \int_{\mathbb{R}^m} RD\,{}^tR\,\nabla_y^R u_i^\varepsilon \cdot \nabla_y^R \mathrm{div}_y b_j \, u_{ij}^\varepsilon \,\mathrm{d}y + \int_{\mathbb{R}^m} \mathrm{div}_y([b_j, [b_i, D]]\,{}^tR\,\nabla_y^R u^\varepsilon) u_{ij}^\varepsilon \,\mathrm{d}y \\
&+ \int_{\mathbb{R}^m} R[b_i, D]\,{}^tR\,\nabla_y^R u^\varepsilon \cdot \nabla_y^R(\mathrm{div}_y b_j) u_{ij}^\varepsilon \,\mathrm{d}y + \int_{\mathbb{R}^m} u_{ij}^\varepsilon \mathrm{div}_y([b_i, D]\,{}^tR\,\nabla_y^R u_j^\varepsilon) \,\mathrm{d}y \\
&+ \int_{\mathbb{R}^m} u_{ij}^\varepsilon b_j \cdot \nabla_y (D\,{}^tR\,\nabla_y^R u^\varepsilon \cdot \nabla_y(\mathrm{div}_y b_i)) \,\mathrm{d}y \\
&=: K_{ij}^1 + K_{ij}^2 + K_{ij}^3 + K_{ij}^4 + K_{ij}^5 + K_{ij}^6. \quad (76)
\end{aligned}
$$

By hypothesis (29) we have, cf. (30)

$$D\nabla_y u^\varepsilon_{ij} \cdot \nabla_y u^\varepsilon_{ij} \geq \alpha |P^{1/2}\nabla_y u^\varepsilon_{ij}|^2 = \alpha |\nabla^R_y u^\varepsilon_{ij}|^2. \tag{77}$$

Exactly as before we obtain

$$\nabla^R_y(u^\varepsilon_{ij}) = b_j \cdot \nabla_y(\nabla^R_y u^\varepsilon_i) - \mathcal{A}_j \nabla^R_y u^\varepsilon_i$$

which allows us to replace K^1_{ij} by

$$\begin{aligned} K^1_{ij} &= \int_{\mathbb{R}^m} R[b_j, D]\,{}^t R\, \nabla^R_y u^\varepsilon_i(t) \cdot \mathcal{A}_j \nabla^R_y u^\varepsilon_i(t)\, \mathrm{d}y \\ &+ \frac{1}{2}\int_{\mathbb{R}^m} b_j \cdot \nabla_y(R[b_j, D]\,{}^t R) : \nabla^R_y u^\varepsilon_i(t) \otimes \nabla^R_y u^\varepsilon_i(t)\, \mathrm{d}y \\ &+ \frac{1}{2}\int_{\mathbb{R}^m} (\mathrm{div}_y b_j) R[b_j, D]\,{}^t R : \nabla^R_y u^\varepsilon_i(t) \otimes \nabla^R_y u^\varepsilon_i(t)\, \mathrm{d}y. \end{aligned}$$

Thanks to our hypotheses, there is a constant C_3 (not depending on ε or t) such that

$$\sum_{j=1}^m \sum_{i=1}^m |K^1_{ij}| \leq C_3 \| \nabla^R_y \otimes \nabla^R_y u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

Obviously, there is a constant C_4 (not depending on ε or t) such that

$$\sum_{j=1}^m \sum_{i=1}^m |K^2_{ij}| \leq C_4 \| \nabla^R_y \otimes \nabla^R_y u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

We consider now the term K^3_{ij}, which writes

$$\begin{aligned} K^3_{ij} &= \int_{\mathbb{R}^m} u^\varepsilon_{ij}(t) \mathrm{div}_y(R[b_j, [b_i, D]]) \cdot \nabla^R_y u^\varepsilon(t)\, \mathrm{d}y \\ &+ \int_{\mathbb{R}^m} u^\varepsilon_{ij}(t) R[b_j, [b_i, D]]\,{}^t R : \nabla^R_y \otimes \nabla^R_y u^\varepsilon(t)\, \mathrm{d}y. \end{aligned}$$

It is easily seen that there is a constant C_5 (not depending on ε or t) such that

$$\begin{aligned} \sum_{j=1}^m \sum_{i=1}^m (|K^3_{ij}| + |K^4_{ij}| + |K^6_{ij}|) \leq C_5 \big(&\| \nabla^R_y \otimes \nabla^R_y u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \| \nabla^R_y u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \\ &+ \| \nabla^R_y \otimes \nabla^R_y u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}\Big), \quad t \in \mathbb{R}_+, \quad \varepsilon > 0. \end{aligned}$$

It remains to estimate the term K^5_{ij}. For any $i, j \in \{1, ..., m\}$ we have

$$u^\varepsilon_{ij}(t) = u^\varepsilon_{ji}(t) - \sum_{k=1}^m \alpha^k_{ij} u^\varepsilon_k(t)$$

and therefore K^5_{ij} writes

$$\begin{aligned} K^5_{ij} &= -\sum_{k=1}^m \int_{\mathbb{R}^m} \alpha^k_{ij} u^\varepsilon_k(t) \mathrm{div}_y([b_i, D]\,{}^t R\, \nabla^R_y u^\varepsilon_j(t))\, \mathrm{d}y \\ &+ \int_{\mathbb{R}^m} u^\varepsilon_{ji}(t) \mathrm{div}_y([b_i, D]\,{}^t R\, \nabla^R_y u^\varepsilon_j(t))\, \mathrm{d}y \\ &= \sum_{k=1}^m \int_{\mathbb{R}^m} \nabla^R_y(\alpha^k_{ij} u^\varepsilon_k(t)) \cdot R[b_i, D]\,{}^t R\, \nabla^R_y u^\varepsilon_j(t)\, \mathrm{d}y \\ &- \int_{\mathbb{R}^m} \nabla^R_y(b_i \cdot \nabla_y u^\varepsilon_j(t)) \cdot R[b_i, D]\,{}^t R\, \nabla^R_y u^\varepsilon_j(t)\, \mathrm{d}y =: K^7_{ij} + K^8_{ij}. \end{aligned}$$

Clearly, there is a constant C_6 (not depending on ε or t) such that

$$\sum_{j=1}^{m}\sum_{i=1}^{m}|K_{ij}^7| \le C_6\left(\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}\|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}\right.$$
$$\left.+\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)}\right),\ \ t\in\mathbb{R}_+,\ \ \varepsilon>0.$$

For the last term K_{ij}^8 we use (39) and we get as before

$$\begin{aligned}K_{ij}^8 &= \int_{\mathbb{R}^m}\mathcal{A}_i\nabla_y^R u_j^\varepsilon(t)\cdot R[b_i,D]\,{}^tR\,\nabla_y^R u_j^\varepsilon(t)\,\mathrm{d}y\\
&-\int_{\mathbb{R}^m} b_i\cdot\nabla_y(\nabla_y^R u_j^\varepsilon(t))\cdot R[b_i,D]\,{}^tR\,\nabla_y^R u_j^\varepsilon(t)\,\mathrm{d}y\\
&=\int_{\mathbb{R}^m}\mathcal{A}_i\nabla_y^R u_j^\varepsilon(t)\cdot R[b_i,D]\,{}^tR\,\nabla_y^R u_j^\varepsilon(t)\,\mathrm{d}y\\
&+\frac{1}{2}\int_{\mathbb{R}^m}(\mathrm{div}_y b_i)R[b_i,D]\,{}^tR:\ \nabla_y^R u_j^\varepsilon(t)\otimes\nabla_y^R u_j^\varepsilon(t)\,\mathrm{d}y\\
&+\frac{1}{2}\int_{\mathbb{R}^m} b_i\cdot\nabla_y(R[b_i,D]\,{}^tR):\ \nabla_y^R u_j^\varepsilon(t)\otimes\nabla_y^R u_j^\varepsilon(t)\,\mathrm{d}y\end{aligned}$$

implying that there is a constant C_7 (not depending on ε or t) such that

$$\sum_{j=1}^{m}\sum_{i=1}^{m}|K_{ij}^8|\le C_7\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)},\ \ t\in\mathbb{R}_+,\ \ \varepsilon>0.$$

Putting together (76), (77) and the estimates for all the terms $K_{ij}^r, i,j\in\{1,...,m\}, r\in\{1,...,8\}$ we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|^2_{L^2(\mathbb{R}^m)}+\alpha\|\nabla_y^R\otimes\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|^2_{L^2(\mathbb{R}^m)}\le C\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|_{L^2(\mathbb{R}^m)}$$
$$\times\left(\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|_{L^2(\mathbb{R}^m)}+\|\nabla_y^R u^\varepsilon\|_{L^2(\mathbb{R}^m)}\right),\ C=\sum_{r=3}^{7}C_r.$$

Applying Gronwall's lemma yields, for some constant C_T depending only on T and the coefficients α_{ij}^k, the vector fields b_i and the matrix field D

$$\begin{aligned}\|\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}&=\|\nabla_z^R\otimes\nabla_z^R v^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\
&\le C_T(\|\nabla_y^R u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)}+\|\nabla_y^R\otimes\nabla_y^R u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)}),\ \ \varepsilon>0\end{aligned}$$

and

$$\begin{aligned}\|\nabla_y^R\otimes\nabla_y^R\otimes\nabla_y^R u^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))}&=\|\nabla_z^R\otimes\nabla_z^R\otimes\nabla_z^R v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))}\\
&\le C_T(\|\nabla_y^R u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)}+\|\nabla_y^R\otimes\nabla_y^R u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)}),\varepsilon>0.\end{aligned}$$

For estimating $\partial_t\nabla_z^R v^\varepsilon$ we take the directional derivative $b_i\cdot\nabla_z$ in (42). As the vector fields b_i, b are in involution, that is $[b_i,b]=0$, the flows Y_i, Y are commuting [2, 3], and therefore the derivative along b_i commutes with the translation along the flow of b (take the derivative with respect to h, at $h=0$, of the equality $f(Y(s;Y_i(h;\cdot)))=f(Y_i(h;Y(s;\cdot)))$). We deduce that

$$\partial_t(b_i\cdot\nabla_z v^\varepsilon)(t,z)=[b_i\cdot\nabla_y\mathrm{div}_y(D\nabla_y u^\varepsilon)](Y(t/\varepsilon;z))$$

which implies

$$
\begin{aligned}
\|\partial_t \nabla_z^R v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} &= \| \nabla_y^R \mathrm{div}_y(D\nabla_y u^\varepsilon(t))\|_{L^2(\mathbb{R}^m)} \\
&= \| \nabla_y^R \mathrm{div}_y(D\,{}^tR\,\nabla_y^R u^\varepsilon(t))\|_{L^2(\mathbb{R}^m)} \\
&= \| \nabla_y^R (\mathrm{div}_y(RD)\cdot \nabla_y^R u^\varepsilon(t)) + \nabla_y^R (RD\,{}^tR : \nabla_y^R \otimes \nabla_y^R u^\varepsilon(t))\|_{L^2}.
\end{aligned}
$$

We claim that for any $i \in \{1, ..., m\}$ we have the equality

$$
\mathrm{div}_y(RD)_i = \sum_{j=1}^m b_j \cdot \nabla_y (RD\,{}^tR)_{ij} + \sum_{j=1}^m (RD\,{}^tR)_{ij} \mathrm{div}_y b_j. \tag{78}
$$

Indeed, for any $i \in \{1, ..., m\}$ we can write (here $(e_k)_{1\leq k\leq m}$ stands for the canonical basis of $\mathbb{R}^m$)

$$
\begin{aligned}
(\mathrm{div}_y RD)_i = \mathrm{div}_y(RD\,{}^tR\,{}^tR^{-1})_i &= \sum_{k=1}^m \partial_{y_k} \sum_{j=1}^m (RD\,{}^tR)_{ij} R^{-1}_{kj} \\
&= \sum_{j=1}^m \sum_{k=1}^m \partial_{y_k} [(RD\,{}^tR)_{ij}(b_j \cdot e_k)] \\
&= \sum_{j=1}^m \mathrm{div}_y [(RD\,{}^tR)_{ij} b_j].
\end{aligned}
$$

Thanks to our hypotheses and formula (78), it is easily seen that there is a constant depending only on the coefficients α_{ij}^k, the vector fields b_i and the matrix field D such that

$$
\begin{aligned}
\|\partial_t \nabla_z^R v^\varepsilon\|_{L^2(\mathbb{R}^m)} \leq C(&\| \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} + \| \nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \\
&+ \| \nabla_y^R \otimes \nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}).
\end{aligned}
$$

Thanks to the uniform estimates satisfied by $\nabla_y^R u^\varepsilon$, $\nabla_y^R \otimes \nabla_y^R u^\varepsilon$ in $L^\infty([0,T]; L^2(\mathbb{R}^m))$, and by $\nabla_y^R \otimes \nabla_y^R \otimes \nabla_y^R u^\varepsilon$ in $L^2([0,T]; L^2(\mathbb{R}^m))$, we obtain that, for any $T \in \mathbb{R}_+$, there is a constant $\tilde{C}_T$ (depending only on T and α, α_{ij}^k, b_i, D) such that

$$
\|\partial_t \nabla_z^R v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} \leq \tilde{C}_T \left(\| \nabla_y^R u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)} + \| \nabla_y^R \otimes \nabla_y^R u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)}\right), \;\; \varepsilon > 0.
$$

□

Proof. (of Proposition 4.5)
We perform exactly the same computations as in the proof of Proposition 4.3 (it does not matter that (15) has no the term $\frac{1}{\varepsilon} b \cdot \nabla_z v$). Nevertheless, we have to check that all the hypotheses on the matrix field D in Proposition 4.3 are also satisfied by the matrix field $\langle D \rangle$. By Theorem 2.1 we deduce that ${}^t\langle D\rangle = \langle D \rangle$, $\langle D \rangle \in H_Q \cap H_Q^\infty$, $|\langle D \rangle|_Q \leq |D|_Q, |\langle D \rangle|_{H_Q^\infty} \leq |D|_{H_Q^\infty}$, $Q^{1/2}\langle D\rangle Q^{1/2} \geq \alpha I_m$ and therefore the hypotheses (28), (29) corresponding to the matrix field $\langle D \rangle$ hold true. We also need to show that $R[b_i, \langle D\rangle]\,{}^tR \in L^\infty(\mathbb{R}^m)$, $\mathrm{div}_y(R\langle D\rangle) \in L^\infty(\mathbb{R}^m)$, $\sum_{i=1}^m b_i \cdot \nabla_y (R[b_i, \langle D \rangle]\,{}^tR) \in L^\infty(\mathbb{R}^m)$, provided that the same conditions are satisfied by the matrix field D. The key point is that for any $i \in \{1, ..., m\}$, the groups $(G_i(h))_{h\in\mathbb{R}}$, $(G(s))_{s\in\mathbb{R}}$ are commuting, where $G_i(h)$ is defined by

$$
G_i(h)A = \partial Y_i^{-1}(h; \cdot) A(Y_i(h;\cdot))\,{}^t\partial Y_i^{-1}(h;\cdot).
$$

It is easily seen that for any $s, h \in \mathbb{R}$

$$(G(s) \circ G_i(h))A = G(s)(G_i(h)A) = \partial Y^{-1}(s;\cdot)(G_i(h)A)(Y(s;\cdot))\,{}^t\partial Y^{-1}(s;\cdot) \tag{79}$$
$$= \partial Y^{-1}(s;\cdot)\partial Y_i^{-1}(h;Y(s;\cdot))A(Y_i(h;Y(s;\cdot)))\,{}^t\partial Y_i^{-1}(h;Y(s;\cdot))\,{}^t\partial Y^{-1}(s;\cdot)$$

and

$$(G_i(h) \circ G(s))A = G_i(h)(G(s)A) = \partial Y_i^{-1}(h;\cdot)(G(s)A)(Y_i(h;\cdot))\,{}^t\partial Y_i^{-1}(h;\cdot) \tag{80}$$
$$= \partial Y_i^{-1}(h;\cdot)\partial Y^{-1}(s;Y_i(h;\cdot))A(Y(s;Y_i(h;\cdot)))\,{}^t\partial Y^{-1}(s;Y_i(h;\cdot))\,{}^t\partial Y_i^{-1}(h;\cdot).$$

By the involution between b and b_i, we know that

$$Y_i(h;Y(s;\cdot)) = Y(s;Y_i(h;\cdot)) \tag{81}$$

and by differentiation one gets

$$\partial Y_i(h;Y(s;\cdot))\partial Y(s;\cdot) = \partial Y(s;Y_i(h;\cdot))\partial Y_i(h;\cdot)$$

which also writes

$$\partial Y^{-1}(s;\cdot)\partial Y_i^{-1}(h;Y(s;\cdot)) = \partial Y_i^{-1}(h;\cdot)\partial Y^{-1}(s;Y_i(h;\cdot)). \tag{82}$$

Combining (79), (80), (81), (82) we obtain the commutation property between the groups $(G_i(h))_{h\in\mathbb{R}}, (G(s))_{s\in\mathbb{R}}$, for any $i \in \{1, ..., m\}$. Notice that the hypothesis $R[b_i, D]\,{}^tR \in L^\infty(\mathbb{R}^m)$, or equivalently $[b_i, D] \in H_Q^\infty$, should be understood in $\mathcal{D}'(\mathbb{R}^m)$, that is, there is a matrix field, denoted $[b_i, D]$, which belongs to H_Q^∞, such that for any $A \in C_c^1(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} D : (-b_i \cdot \nabla_y A - (\mathrm{div}_y b_i)A - {}^t\partial b_i A - A\partial b_i)\, \mathrm{d}y = \int_{\mathbb{R}^m} [b_i, D] : A\, \mathrm{d}y. \tag{83}$$

We introduce the operator $L_i(D) = [b_i, D]$ and its formal adjoint

$$L_i^\star(A) = -b_i \cdot \nabla_y A - (\mathrm{div}_y b_i)A - {}^t\partial b_i A - A\partial b_i,\ A \in C_c^1(\mathbb{R}^m).$$

Using these notations, (83) becomes

$$\int_{\mathbb{R}^m} D : L_i^\star(A)\, \mathrm{d}y = \int_{\mathbb{R}^m} L_i(D) : A\, \mathrm{d}y,\quad A \in C_c^1(\mathbb{R}^m).$$

Notice that H_Q^∞ is the topological dual of the space

$$H_P^1 = \left\{ A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R}) \text{ measurable } : \int_{\mathbb{R}^m} (P(y)A(y) : A(y)P(y))^{1/2}\, \mathrm{d}y < +\infty \right\}$$

and thus $[b_i, D] \in H_Q^\infty$ iff there is a constant C_i such that $\int_{\mathbb{R}^m} D : L_i^\star(A)\, \mathrm{d}y \le C_i |A|_{H_P^1}$ for any $A \in C_c^1(\mathbb{R}^m)$. A straightforward computation shows that if $B \in H_Q^\infty$ is such that $G_i(h)B \in H_Q^\infty$ for any $h \in \mathbb{R}$ and $h \in \mathbb{R}^\star \to (G_i(h)B - B)/h$ is bounded in H_Q^∞, then $(G_i(h)B - B)/h$ converges toward $L_i(B)$ weakly $\star$ in H_Q^∞, when $h \to 0$. Indeed, for any

$A \in C_c^1(\mathbb{R}^m) \subset H_P^1$, we have

$$\int_{\mathbb{R}^m} \frac{G_i(h)B - B}{h} : A \,\mathrm{d}y = \int_{\mathbb{R}^m} \frac{\partial Y_i^{-1}(h;y) B(Y_i(h;y))\, {}^t\partial Y_i^{-1}(h;y) - B(y)}{h} : A(y) \,\mathrm{d}y$$
$$= \int_{\mathbb{R}^m} \frac{\partial Y_i(-h; Y_i(h;y)) B(Y_i(h;y))\, {}^t\partial Y_i(-h; Y_i(h;y)) - B(y)}{h} : A(y)\,\mathrm{d}y$$
$$= \frac{1}{h}\int_{\mathbb{R}^m} \big\{ \det(\partial Y_i(-h;z)) \partial Y_i(-h;z) B(z)\, {}^t\partial Y_i(-h;z) : A(Y_i(-h;z)) - B(z) : A(z) \big\} \,\mathrm{d}z$$
$$= \int_{\mathbb{R}^m} \frac{\det(\partial Y_i(-h;z)) - 1}{h} \partial Y_i(-h;z) B(z)\, {}^t\partial Y_i(-h;z) : A(Y_i(-h;z)) \,\mathrm{d}z$$
$$+ \int_{\mathbb{R}^m} B(z) : \frac{{}^t\partial Y_i(-h;z) A(Y_i(-h;z)) \partial Y_i(-h;z) - A(z)}{h} \,\mathrm{d}z$$
$$\underset{h\to 0}{\longrightarrow} - \int_{\mathbb{R}^m} \mathrm{div}_z b_i \; B(z) : A(z) \,\mathrm{d}z - \int_{\mathbb{R}^m} B(z) : (b_i \cdot \nabla_z A + {}^t\partial b_i A + A \partial b_i) \,\mathrm{d}z$$
$$= \int_{\mathbb{R}^m} B(z) : L_i^\star(A) \,\mathrm{d}z.$$

We deduce that any weak $\star$ limit point in H_Q^∞ satisfies

$$\int_{\mathbb{R}^m} \mathrm{w}\star \lim_{h_k \to 0} \frac{G_i(h_k)B - B}{h_k} : A \,\mathrm{d}y = \int_{\mathbb{R}^m} B : L_i^\star(A)\,\mathrm{d}y$$

for any $A \in C_c^1(\mathbb{R}^m)$. Therefore all the family $(G_i(h)B - B)/h$ converges weakly $\star$ in H_Q^∞, as $h \to 0$, and $L_i(B) = \lim_{h\to 0} \frac{G_i(h)B - B}{h}$, weakly $\star$ in H_Q^∞.
We claim that for any $s \in \mathbb{R}$, we have $L_i(G(s)D) = G(s)(L_i(D))$, that is

$$\int_{\mathbb{R}^m} G(s)D : L_i^\star(A)\,\mathrm{d}y = \int_{\mathbb{R}^m} G(s) L_i(D) : A\,\mathrm{d}y, \;\; A \in C_c^1(\mathbb{R}^m). \tag{84}$$

By density arguments (notice that $B_n \rightharpoonup B$ weakly $\star$ in H_Q^∞, implies $G(s)B_n \rightharpoonup G(s)B$ weakly $\star$ in H_Q^∞, for any $s \in \mathbb{R}$) it is enough to show that $L_i(G(s)B) = G(s)L_i(B)$ for any smooth, compactly supported matrix field B. Let us consider a smooth, compactly supported matrix field B. Obviously, for any $h \in \mathbb{R}^\star$ we have $G_i(h)B \in H_Q^\infty$ and $\frac{G_i(h)B - B}{h} \rightharpoonup L_i(B)$ weakly $\star$ in H_Q^∞, when $h \to 0$. We deduce that $G_i(h)G(s)B = G(s)G_i(h)B \in H_Q^\infty$ for any $h \in \mathbb{R}$ and

$$\frac{G_i(h)G(s)B - G(s)B}{h} = \frac{G(s)G_i(h)B - G(s)B}{h} = G(s)\frac{G_i(h)B - B}{h} \underset{h\to 0}{\rightharpoonup} G(s)L_i(B)$$

weakly $\star$ in H_Q^∞. By the previous remark, we obtain $L_i(G(s)B) = G(s)L_i(B)$ for any $s \in \mathbb{R}$. Now it is easily seen that $L_i(\langle D \rangle) = [b_i, \langle D\rangle] \in H_Q^\infty$, if $L_i(D) = [b_i, D] \in H_Q^\infty$. Indeed, averaging (84) with respect to s one gets

$$\int_{\mathbb{R}^m} \frac{1}{S}\int_0^S G(s)D\,\mathrm{d}s : L_i^\star(A)\,\mathrm{d}y = \int_{\mathbb{R}^m} \frac{1}{S}\int_0^S G(s)L_i(D)\,\mathrm{d}s : A\,\mathrm{d}y, \;\; A \in C_c^1(\mathbb{R}^m).$$

Taking into account that $\frac{1}{S}\int_0^S G(s)D\,\mathrm{d}s \to \langle D\rangle$ in H_Q, when $S \to +\infty$, that $PL_i^\star(A)P \in H_Q$ and noticing that

$$\left| \frac{1}{S}\int_0^S G(s)L_i(D)\,\mathrm{d}s \right|_{H_Q^\infty} \leq \frac{1}{S}\int_0^S |G(s)L_i(D)|_{H_Q^\infty}\,\mathrm{d}s = |L_i(D)|_{H_Q^\infty}$$

we deduce that any weak $\star$ limit point in H^∞_Q of $\left(\frac{1}{S}\int_0^S G(s)L_i(D)\,\mathrm{d}s\right)_S$ satisfies

$$\int_{\mathbb{R}^m} \langle D\rangle : L_i^\star(A)\,\mathrm{d}y = \int_{\mathbb{R}^m} \mathrm{w}\star \lim_{S\to+\infty} \frac{1}{S}\int_0^S G(s)L_i(D)\,\mathrm{d}s : A\,\mathrm{d}y$$

for any $A \in C^1_c(\mathbb{R}^m) \subset H^1_P$, saying that $L_i(\langle D\rangle) = \lim_{S\to+\infty} \frac{1}{S}\int_0^S G(s)L_i(D)\,\mathrm{d}s$, weakly $\star$ in H^∞_Q. In particular $|L_i(\langle D\rangle)|_{H^\infty_Q} \le |L_i(D)|_{H^\infty_Q}$.
We concentrate now on the hypothesis $\sum_{i=1}^m b_i\cdot\nabla_y(R[b_i,D]\,{}^tR) \in L^\infty(\mathbb{R}^m)$. For any $s\in\mathbb{R}$ we have

$$RG(s)[b_i,D]\,{}^tR = R\partial Y^{-1}(s;\cdot)[b_i,D](Y(s;\cdot))\,{}^t\partial Y^{-1}(s;\cdot)\,{}^tR = (R[b_i,D]\,{}^tR)(Y(s;\cdot))$$

and since $[b_i,b]=0$, we obtain

$$b_i\cdot\nabla_z(RG(s)[b_i,D]\,{}^tR) = b_i\cdot\nabla_z((R[b_i,D]\,{}^tR)\circ Y(s;\cdot)) = \big(b_i\cdot\nabla_y(R[b_i,D]\,{}^tR)\big)\circ Y(s;\cdot).$$

Multiplying by a smooth compactly supported matrix field $A\in C^1_c(\mathbb{R}^m)$ and averaging with respect to s one gets

$$\begin{aligned}-\int_{\mathbb{R}^m}\frac{\mathrm{div}_z b_i}{S}\int_0^S G(s)L_i(D)\,\mathrm{d}s : {}^tRAR\,\mathrm{d}z - \int_{\mathbb{R}^m}\frac{1}{S}\int_0^S G(s)L_i(D)\,\mathrm{d}s : {}^tR(b_i\cdot\nabla_z A)R\,\mathrm{d}z\\ = \int_{\mathbb{R}^m}\frac{1}{S}\int_0^S (b_i\cdot\nabla_y(RL_i(D)\,{}^tR))(Y(s;z))\,\mathrm{d}s : A(z)\,\mathrm{d}z.\end{aligned}$$

We use now the weak $\star$ convergence in H^∞_Q

$$\lim_{S\to+\infty}\frac{1}{S}\int_0^S G(s)L_i(D)\,\mathrm{d}s = L_i(\langle D\rangle)$$

and the facts that ${}^tRAR\in H^1_P$, $\mathrm{div}_y b_i\in L^\infty(\mathbb{R}^m)$ implying that

$$\begin{aligned}-\int_{\mathbb{R}^m}\sum_{i=1}^m \mathrm{div}_z b_i\, RL_i(\langle D\rangle)\,{}^tR : A\,\mathrm{d}z - \int_{\mathbb{R}^m}\sum_{i=1}^m RL_i(\langle D\rangle)\,{}^tR : b_i\cdot\nabla_z A\,\mathrm{d}z\\ = \int_{\mathbb{R}^m}\lim_{S\to+\infty}\frac{1}{S}\int_0^S\sum_{i=1}^m (b_i\cdot\nabla_y(RL_i(D)\,{}^tR))(Y(s;z))\,\mathrm{d}s : A(z)\,\mathrm{d}z.\end{aligned} \tag{85}$$

For passing to the limit in the last integral we use the weak $\star$ convergence in $L^\infty(\mathbb{R}^m)$, since the family $\left(\sum_{i=1}^m (b_i\cdot\nabla_y(RL_i(D)\,{}^tR))\circ Y(s;\cdot)\right)_{s\in\mathbb{R}}$ is bounded in $L^\infty(\mathbb{R}^m)$. We deduce that

$$\sum_{i=1}^m b_i\cdot\nabla_y(RL_i(\langle D\rangle)\,{}^tR) = \mathrm{w}\star\lim_{S\to+\infty}\frac{1}{S}\int_0^S\sum_{i=1}^m (b_i\cdot\nabla_y(RL_i(D)\,{}^tR))\circ Y(s;\cdot)\,\mathrm{d}s \in L^\infty(\mathbb{R}^m).$$

It remains to observe that for any $s\in\mathbb{R}$, $\mathrm{div}_z(RG(s)D) = \mathrm{div}_y(RD)\circ Y(s;\cdot)$ (see (70) for details), which implies

$$\mathrm{div}_z(R\,\langle D\rangle) = \mathrm{w}\star\lim_{S\to+\infty}\frac{1}{S}\int_0^S \mathrm{div}_y(RD)\circ Y(s;\cdot)\,\mathrm{d}s \in L^\infty(\mathbb{R}^m)$$

where, as before, the last limit should be understood in the weak $\star$ $L^\infty(\mathbb{R}^m)$ sense. $\square$

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