Multi-scale analysis for linear first order PDEs. The finite Larmor radius regime

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Abstract

The subject matter of this paper concerns the asymptotic analysis of mathematical models for strongly magnetized plasmas. We concentrate on the finite Larmor radius regime with non uniform magnetic field in three dimensions. We determine the limit model and establish convergence results for any initial conditions, not necessarily well prepared. This study relies on a two-scale approach, based on the mean ergodic theorem, which allows us to separate between the fast and slow dynamics. The method adapts to many models. In particular it is possible to incorporate collision operators and to compute the effective diffusion matrices of the limit models. The average advection field and average diffusion matrix field appear as the long time limit for some parabolic problems, which allows us to obtain good approximations of the limit models, in the case of non uniform magnetic fields (when exact formulae are not available).

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1 Introduction

This work is devoted to the mathematical analysis of asymptotic regimes for the transport of charged particles under the action of strong external magnetic fields. The main application of such models concerns the energy production through the magnetic fusion, which is a very important research topic in plasma physics.

The presence density of a population of charged particles with mass m and charge q, satisfies the Vlasov equation

$$\partial_t f + v(p) \cdot \nabla_x f + q(\ E(t,x) + v(p) \wedge B(t,x)\ ) \cdot \nabla_p f = 0,\ \ (t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \quad (1)$$

Here we denote by $f = f(t,x,p)$ the particle distribution depending on time t, position x and momentum p, and by $v(p)$ the velocity function. In the relativistic case, the function to be considered is $v(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2 c_0^2}\right)^{-1/2}$, where c_0 is the light speed in the vacuum. When the typical momentum is negligible with respect to mc_0, we can use the non relativistic velocity

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$v(p) = \frac{p}{m}$. The evolution of the electro-magnetic field $(E(t,x), B(t,x))$ is described by the Maxwell equations

$$\partial_t E - c_0^2 \operatorname{rot}_x B = -\frac{j}{\varepsilon_0},\ \ \partial_t B + \operatorname{rot}_x E = 0,\ \ \operatorname{div}_x E = \frac{\rho}{\varepsilon_0},\ \ \operatorname{div}_x B = 0,\ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3,\ (2)$$

where ε_0 is the electric permittivity of the vacuum, $\rho = q \int_{\mathbb{R}^3} f \mathrm{d}p$ is the charge density, $j = q \int_{\mathbb{R}^3} v(p) f \mathrm{d}p$ is the current density. The equations (1), (2) are usually called the Vlasov-Maxwell system. Several approximations can be done, depending on the application in mind. In the magnetic confinement framework, we assume that the velocity is non relativistic, and that the magnetic field is stationary. Accordingly, the electric field derives from a potential $E = -\nabla_x \phi$, and this potential satisfies the Poisson equation

$$-\varepsilon_0 \Delta_x \phi = \rho(t,x),\ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3. \tag{3}$$

We obtain the simplified model (1), (3), which is called the Vlasov-Poisson system. For the moment we assume that the electric potential ϕ is a given function. Later on we will make some considerations on the fully non linear Vlasov-Poisson system. The external magnetic field writes

$$B^\varepsilon = \frac{B(x)}{\varepsilon} e(x),\ \ |e(x)| = 1,\ \ x \in \mathbb{R}^3,$$

where $\varepsilon > 0$ is a small parameter, characterizing strong magnetic fields. The scalar function $B(x) > 0$ is the rescaled magnitude of the magnetic field and $e(x)$ denotes its direction. The presence density of the population of charged particles, in the phase space $(x, v = p/m)$, verifies the Vlasov problem

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m}(E + v \wedge B^\varepsilon) \cdot \nabla_v f^\varepsilon = 0,\ \ (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{4}$$

$$f^\varepsilon(0,x,v) = f^{\text{in}}(x,v),\ \ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

One of the most interesting model for tokamak plasmas is the finite Larmor radius regime : the typical length in the perpendicular directions (with respect to the magnetic lines) is of the same order as the Larmor radius and the typical length in the parallel direction is much larger. This means that the dominant advection field in the Vlasov equation (4) is

$$(v - (v \cdot e)e) \cdot \nabla_x + \frac{q}{m} v \wedge B^\varepsilon \cdot \nabla_v\ .$$

In the case of a parallel magnetic field, let us say $e = e_{x_3}$, we have $B^\varepsilon = (0, 0, B(x_1, x_2)/\varepsilon)$, since by the magnetic Gauss law, $\operatorname{div}_x B^\varepsilon = 0$, and the finite Larmor radius regime is given by

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v_1 \partial_{x_1} f^\varepsilon + v_2 \partial_{x_2} f^\varepsilon) + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon + \frac{qB}{m\varepsilon}(v_2 \partial_{v_1} f^\varepsilon - v_1 \partial_{v_2} f^\varepsilon) = 0. \tag{5}$$

The dominant advection field in this case is

$$\frac{1}{\varepsilon}(v_1 \partial_{x_1} + v_2 \partial_{x_2}) + \frac{\omega_c}{\varepsilon}(v_2 \partial_{v_1} - v_1 \partial_{v_2}), \tag{6}$$

where $\omega_c = \frac{qB}{m}$ stands for the rescaled cyclotronic frequency. Multiplying (5) by ε and passing to the limit when $\varepsilon \searrow 0$, it is easily seen that the limit density $f = \lim_{\varepsilon \searrow 0} f^\varepsilon$ satisfies in distribution sense

$$(v_1 \partial_{x_1} + v_2 \partial_{x_2}) f + \omega_c(x_1, x_2)(v_2 \partial_{v_1} - v_1 \partial_{v_2}) f = 0.$$

In other words, at any time $t \in \mathbb{R}_+$, the density $f(t,\cdot,\cdot)$ depends only on the invariants of the dominant advection field. For example, if the magnetic field is uniform, $\nabla_{x_1,x_2}\omega_c = 0$, a complete family of functional independent invariants is given by the center of the Larmor circle $\left(x_1 + \frac{v_2}{\omega_c}, x_2 - \frac{v_1}{\omega_c}\right)$, the parallel position x_3, the Larmor radius $\frac{\sqrt{(v_1)^2+(v_2)^2}}{|\omega_c|}$ (or the modulus of the perpendicular velocity) and the parallel velocity v_3. Actually, with respect to our units, the center of the Larmor circle is $\left(\varepsilon x_1 + \frac{v_2}{\omega_c/\varepsilon}, \varepsilon x_2 - \frac{v_1}{\omega_c/\varepsilon}\right)$ and the radius of the Larmor circle is $\frac{\sqrt{(v_1)^2+(v_2)^2}}{\omega_c/\varepsilon}$. The limit density evolves in a reduced phase space (given by the five invariants of the dominant advection field) and the description of the finite Larmor radius regime is simpler than the original model. Indeed, the Vlasov equation (4) incorporates the cyclotronic motion around the magnetic lines, while the limit density captures only the average effect of this fast dynamics. Notice that the explicit form of the invariants may play a central role in the derivation of the limit model. Actually, the time evolution of the limit density follows by testing (5) against all functions which are constant along the flow of the dominant advection field, which are precisely the functions of the invariants of this advection field. But such global invariants may not exist, or, when their existence is guaranteed, explicit representation formula may not be available. As seen before, when the magnetic field is uniform, the invariants follow immediately. This is why most of the mathematical analysis for the finite Larmor radius regime are realized in the framework of uniform magnetic fields [3, 21, 22, 23, 24, 25, 27, 28] while, clearly, the tokamak plasmas require general toroidal (eventually axisymmetric) geometry. Other techniques used for eliminating the fast orbital time scale rely on the Lie transform perturbation method [16, 32, 33, 34, 31]. Our goal here is to study the finite Larmor radius regime with non uniform magnetic field. We are looking for strong convergence toward a density profile, by considering general initial conditions, not necessarily well-prepared. We derive the asymptotic regime, we study its well-posedness and properties, and justify rigorously the convergence results. As said before, the average advection field in the limit model is not explicit for a general magnetic shape. One of the key point is to look at the average advection field as the long time limit for some parabolic problem cf. Theorem 2.2. In such way, we are able to determine good approximations of the finite Larmor radius regime for any magnetic field.

Another interesting point when analyzing tokamak plasmas, concerns the effect of collisions [36]. It is well known that, averaging with respect to the fast cyclotronic motion, leads to diffusion not only in velocity, but also with respect to the perpendicular space directions. These facts have been clearly emphasized when studying the Fokker-Planck-Landau kernel, but in the framework of uniform magnetic fields [7, 8]. We propose to generalize the analysis of the collision effects in the setting of non uniform magnetic fields, see also [15, 19]. In particular we are interested in the description of the gyro-kinetic equilibria [9, 11]. As for transport operators, the average diffusion matrix field appears as the long time limit of some parabolic problem, which allows us to determine the average collision kernel for general magnetic shapes, cf. Theorem 7.2.

Our paper is organized as follows. The framework together with the main results are illustrated in Section 2. We introduce the main lines of our analysis based on formal computations. Section 3 is devoted to the construction of the average operators for functions and vector fields. It is shown that these average operators are orthogonal projections in Hilbert spaces and can also be obtained as long time limits for the solutions of some parabolic problems. Several uniform estimates are derived in Section 4 and two results, dealing with multi-scale analysis are presented in Section 5. The limit model, together with two convergence results are obtained in Section 6 for general first order linear PDEs. In Section 7 we discuss the finite Larmor radius regime, incorporating collisional models as well. Some standard results are detailed in Appendix A.

2 Presentation of the models and main results

We investigate asymptotic regimes of the Vlasov equation, which is a linear transport equation, since we neglect the self-consistent electro-magnetic field. We perform our analysis in the general framework of linear transport equations and we apply these results for studying the finite Larmor radius regime. Let us consider the problem

$$\partial_t u^\varepsilon + \mathrm{div}_y\{u^\varepsilon a\} + \frac{1}{\varepsilon}\mathrm{div}_y\{u^\varepsilon b\} = 0,\ \ (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m, \tag{7}$$

$$u^\varepsilon(0,y) = u^{\mathrm{in}}(y),\ \ y \in \mathbb{R}^m. \tag{8}$$

The vector fields $a = a(t,y) : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m, b = b(y) : \mathbb{R}^m \to \mathbb{R}^m$ are supposed smooth and divergence free

$$a \in L^1_{\mathrm{loc}}(\mathbb{R}_+; W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)),\ \ \mathrm{div}_y a = 0, \tag{9}$$

$$b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m),\ \ \mathrm{div}_y b = 0. \tag{10}$$

We also assume that the following growth conditions hold true

$$\forall\, T > 0\ \exists C_T > 0 \text{ such that } |a(t,y)| \le C_T(1+|y|),\ \ (t,y) \in [0,T] \times \mathbb{R}^m, \tag{11}$$

and

$$\exists C > 0 \text{ such that } |b(y)| \le C(1+|y|),\ \ y \in \mathbb{R}^m. \tag{12}$$

Under the previous hypotheses, the problem (7), (8) admits a unique weak solution for initial conditions in any Lebesgue space. For example, if $u^{\mathrm{in}} \in L^2(\mathbb{R}^m)$, the divergence conditions $\mathrm{div}_y a = 0, \mathrm{div}_y b = 0$ guarantee the conservation of the L^2 norm

$$\int_{\mathbb{R}^m} (u^\varepsilon(t,y))^2\,\mathrm{d}y = \int_{\mathbb{R}^m} (u^{\mathrm{in}}(y))^2\,\mathrm{d}y,\ \ t \in \mathbb{R}_+, \varepsilon > 0.$$

In particular we have $u^\varepsilon \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, for any $u^{\mathrm{in}} \in L^2(\mathbb{R}^m), \varepsilon > 0$. We denote by $Y(s;y)$ the characteristic flow associated to b

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = b(Y(s;y)),\ \ Y(0;y) = y,\ \ (s,y) \in \mathbb{R} \times \mathbb{R}^m.$$

The hypotheses (10), (12) on the vector field b guarantee the regularity $Y \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^m)$ and the fact that the flow $Y(s;y)$ is measure preserving. The reason for considering the above flow is that we expect that the weak limit $u = \lim_{\varepsilon \searrow 0} u^\varepsilon$ will satisfy, at any time $t \in \mathbb{R}_+$, the constraint $b(y) \cdot \nabla_y u(t,y) = 0, y \in \mathbb{R}^m$, or equivalently $u(Y(s;y)) = u(y), (s,y) \in \mathbb{R} \times \mathbb{R}^m$. A clear way to understand the interplay between the fields a and $\frac{1}{\varepsilon}b$ is to appeal to the filtering technique. Let us introduce the functions

$$v^\varepsilon(t,z) = u^\varepsilon\left(t, Y(t/\varepsilon;z)\right),\ \ (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m,\ \ \varepsilon > 0.$$

These functions satisfy the same initial condition

$$v^\varepsilon(0,z) = u^\varepsilon(0,z) = u^{\mathrm{in}}(z),\ \ z \in \mathbb{R}^m,\ \ \varepsilon > 0,$$

and by direct computations one gets, at least formally

$$\partial_t v^\varepsilon = \partial_t u^\varepsilon(t,Y(t/\varepsilon;z)) + \frac{b}{\varepsilon}(Y(t/\varepsilon;z))\cdot\nabla_y u^\varepsilon(t,Y(t/\varepsilon;z)), \nabla_z v^\varepsilon = {}^t\partial_z Y(t/\varepsilon;z)\,\nabla_y u^\varepsilon(t,Y(t/\varepsilon;z)). \tag{13}$$

Taking the derivative with respect to z of the identity $Y(-t/\varepsilon; Y(t/\varepsilon; z)) = z$ leads to the formula

$$\partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z))\, \partial_z Y(t/\varepsilon; z) = I_m, \tag{14}$$

and therefore we obtain

$$\begin{aligned} a(t, Y(t/\varepsilon; z)) \cdot \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) &= a(t, Y(t/\varepsilon; z)) \cdot {}^t\partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z))\, \nabla_z v^\varepsilon(t, z) \\ &= \partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z))\, a(t, Y(t/\varepsilon; z)) \cdot \nabla_z v^\varepsilon(t, z). \end{aligned} \tag{15}$$

Finally (7), (13), (15) yield the following transport equation for the new unknown v^ε

$$\partial_t v^\varepsilon(t, z) + \partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z))\, a(t, Y(t/\varepsilon; z)) \cdot \nabla_z v^\varepsilon(t, z) = 0, \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m, \tag{16}$$

which is to be completed by the initial condition

$$v^\varepsilon(0, z) = u^{\text{in}}(z), \quad z \in \mathbb{R}^m.$$

Notice that once we have solved for v^ε, the original unknown u^ε comes immediately by the formula

$$u^\varepsilon(t, y) = v^\varepsilon(t, Y(-t/\varepsilon; y)), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m, \quad \varepsilon > 0.$$

The advantage of considering (16) instead of (7) is that the family $(v^\varepsilon)_\varepsilon$ is stable, while clearly $(u^\varepsilon)_\varepsilon$ is not (it contains the oscillations of the fast dynamics $Y(-t/\varepsilon; y)$ induced by the dominant advection field $\frac{1}{\varepsilon}b$). We are left with the difficult task of identifying the limit advection field

$$\lim_{\varepsilon \searrow 0} \partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z))\, a(t, Y(t/\varepsilon; z)). \tag{17}$$

This computation will characterize the interaction between the slow and fast dynamics, associated to the advection fields a and $\frac{1}{\varepsilon}b$. This interaction is easily understood when these fields are in involution, that is, their Poisson bracket vanishes

$$[b, a(t)] := (b \cdot \nabla_y)a(t) - (a(t) \cdot \nabla_y)b = 0, \quad t \in \mathbb{R}_+.$$

It is well known that the above condition expresses the commutation between the flows associated to b and $a(t)$ cf. [2]

$$Z(h; Y(s; z)) = Y(s; Z(h; z)), \quad h, s \in \mathbb{R}, \quad z \in \mathbb{R}^m, \tag{18}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}h} Z(h; z) = a(t, Z(h; z)), \quad (h, z) \in \mathbb{R} \times \mathbb{R}^m.$$

Notice that the parameter t is fixed. Actually the flow Z depends also on t, since it is associated to the advection field $a(t)$. Taking the derivative of (18) with respect to h at $h = 0$, we obtain

$$a(t, Y(s; z)) = \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} Z(h; Y(s; z)) = \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} Y(s; Z(h; z)) = \partial_z Y(s; z) a(t, z), \quad (s, z) \in \mathbb{R} \times \mathbb{R}^m. \tag{19}$$

Conversely, (19) ensures the involution of the fields b and $a(t)$. Now let us come back to (16) and observe that, thanks to (19), (14) we can write

$$\partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z))\, a(t, Y(t/\varepsilon; z)) = \partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z))\, \partial_z Y(t/\varepsilon; z)\, a(t, z) = a(t, z).$$

In this case (16) reduces to

$$\partial_t v^\varepsilon + a(t, z) \cdot \nabla_z v^\varepsilon = 0, \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m.$$

Since the initial conditions $v^\varepsilon(0) = u^{\rm in}$ do not depend on ε, the family $(v^\varepsilon)_\varepsilon$ is constant with respect to $\varepsilon > 0$. More exactly $v^\varepsilon = v$ where

$$\partial_t v + a(t,z) \cdot \nabla_z v = 0, \ \ (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m,$$

$$v(0,z) = u^{\rm in}(z), \ \ z \in \mathbb{R}^m.$$

The family $(u^\varepsilon)_\varepsilon$ is given by

$$u^\varepsilon(t,y) = v(t, Y(-t/\varepsilon; y)), \ \ (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m, \ \varepsilon > 0,$$

and $u^\varepsilon(t)$ is obtained by advecting the initial condition $u^{\rm in}$ along the vector field a on the time interval $[0,t]$, followed by a second advection of duration t, along the vector field $\frac{1}{\varepsilon}b$. This is the splitting method, which applies when the advection fields are in involution.

In the general case we have to deal with the limit in (17). Clearly, it has to be handled by a two-scale approach, which distinguishes between the slow time variable t and the fast time variable $s = t/\varepsilon$. If the dependency with respect to the fast variable s would be periodic, therefore we may appeal to the concept of two-scale convergence [1, 26, 6, 17]. In many real life applications it happens that periodicity does not occur. This is exactly the case of tokamak plasmas. It is easily seen that the flow associated to the dominant advection field (6) is periodic when the magnetic field is uniform, but this fails to be true for non uniform magnetic fields. Therefore we perform our analysis without any periodicity assumption on the fast dynamics. Of course, a particular case will be that of periodic fast motions.

As usual when dealing with two scales, we freeze the slow time variable and average with respect to the fast time variable. Since we allow non periodicity, the average should be understood in the ergodic sense, that is, $\lim_{T\to+\infty} \frac{1}{T}\int_0^T \dots \, {\rm d}s$. Motivated by (17), we investigate the limit

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \partial_y Y(-s; Y(s;z)) a(t, Y(s;z)) \, {\rm d}s. \tag{20}$$

Actually, the slow time variable being fixed, we define for any $s \in \mathbb{R}$, the following transformation of the vector fields $a = a(y)$ of $\mathbb{R}^m$

$$\varphi(s)a(\cdot) = \partial_y Y(-s; Y(s;\cdot)) a(Y(s;\cdot)).$$

It is easily seen that $\varphi(0)a = a$. Moreover we prove that the family of transformations $(\varphi(s))_{s\in\mathbb{R}}$ is a group of unitary transformations on a Hilbert space. We assume that there is a matrix field $P(y)$ such that

$$^tP = P, \ \ P(y)\xi\cdot\xi > 0, \ \ \xi \in \mathbb{R}^m \setminus \{0\}, \ \ y \in \mathbb{R}^m, \ \ P^{-1}, P \in L^1_{\rm loc}(\mathbb{R}^m), \tag{21}$$

$$[b, P] := (b\cdot\nabla_y)P - \partial_y b P(y) - P(y)^t\partial_y b = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^m). \tag{22}$$

When the vector field b is uniform, we can take $P = I_m$. In the general case, a matrix field P satisfying (21), (22) can be constructed using vector fields in involution with b. Notice that we have the following characterization for (22), cf. Proposition 3.8 [10]

Proposition 2.1 *Consider $b \in W^{1,\infty}_{\rm loc}(\mathbb{R}^m)$ (not necessarily divergence free) with linear growth and $A(y) \in L^1_{\rm loc}(\mathbb{R}^m)$. Then $[b, A] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ iff*

$$A(Y(s;y)) = \partial_y Y(s;y) A(y)\, {}^t\partial_y Y(s;y), \ \ s \in \mathbb{R}, \ \ y \in \mathbb{R}^m. \tag{23}$$

For any two vectors c, d the notation $c \otimes d$ stands for the matrix whose entry (i,j) is $c_i d_j$, and for any two matrices A, B the notation $A : B$ stands for $\mathrm{tr}({}^t AB) = A_{ij}B_{ij}$ (using Einstein summation convention). Observe that any vector field $c \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ in involution with b *i.e.*, $(b \cdot \nabla_y)c - \partial_y b\, c = 0$ in $\mathcal{D}'(\mathbb{R}^m)$, provides a symmetric matrix field $P_c(y) = c(y) \otimes c(y)$ satisfying $[b, P_c] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Indeed, thanks to (19) we can write

$$\begin{aligned} P_c(Y(s;y)) &= c(Y(s;y)) \otimes c(Y(s;y)) \\ &= (\partial_y Y(s;y)c(y)) \otimes (\partial_y Y(s;y)c(y)) \\ &= \partial_y Y(s;y)(c(y) \otimes c(y))\, {}^t\partial_y Y(s;y) \\ &= \partial_y Y(s;y) P_c(y)\, {}^t\partial_y Y(s;y), \quad s \in \mathbb{R}, y \in \mathbb{R}^m, \end{aligned}$$

and therefore, by Proposition 2.1, we have $[b, P_c] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. When a family $\{c_i\}_{1 \le i \le m}$ of vector fields in involution with b is available, and $\{c_i(y)\}_{1 \le i \le m}$ form a basis of $\mathbb{R}^m$ at any point $y \in \mathbb{R}^m$, it is easily seen that the symmetric matrix field $P(y) = \sum_{i=1}^m c_i(y) \otimes c_i(y)$ is positive definite and satisfies $[b, P] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Given a matrix field P satisfying (21), (22), we consider the set

$$X_Q = \{c(y) : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable } : \int_{\mathbb{R}^m} Q(y) : c(y) \otimes c(y)\, \mathrm{d}y < +\infty\},$$

where $Q = P^{-1}$, and the scalar product (see Section 3)

$$(c, d)_Q = \int_{\mathbb{R}^m} Q(y) : c(y) \otimes d(y)\, \mathrm{d}y, \quad c, d \in X_Q.$$

We prove that the family of transformations $\varphi(s) : X_Q \to X_Q, s \in \mathbb{R}$ is a C^0- group of unitary operators on X_Q, cf. Proposition 3.1. Consequently, thanks to the von Neumann's Theorem [35] we define the average of a vector field cf. (20).

Theorem 2.1 *Assume that* (10), (12), (21), (22) *hold true. We denote by $\mathcal{L}$ the infinitesimal generator of the group $(\varphi(s))_{s \in \mathbb{R}}$. Then for any vector field $a \in X_Q$, we have the strong convergence in X_Q*

$$\lim_{T \to +\infty} \frac{1}{T} \int_r^{r+T} \partial_y Y(-s; Y(s;\cdot)) a(Y(s;\cdot))\, \mathrm{d}s = \mathrm{Proj}_{\ker \mathcal{L}} a,$$

uniformly with respect to $r \in \mathbb{R}$. If a is divergence free, then so is $\mathrm{Proj}_{\ker \mathcal{L}} a$ and for any function $\psi \in C^1(\mathbb{R}^m) \cap \ker(b \cdot \nabla_y)$ such that $\nabla_y \psi \in X_P$, that is, $\int_{\mathbb{R}^m} P \nabla_y \psi \cdot \nabla_y \psi\, \mathrm{d}y < +\infty$, we have

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T (a \cdot \nabla_y \psi)(Y(s;\cdot))\, \mathrm{d}s = \mathrm{Proj}_{\ker \mathcal{L}} a \cdot \nabla_y \psi, \quad \textit{weakly in } L^1(\mathbb{R}^m).$$

If $a \in X_Q \cap X_Q^\infty$ (see Remark 3.1 for definition of the Banach space X_Q^∞), then $\langle a \rangle := \mathrm{Proj}_{\ker \mathcal{L}} a \in X_Q \cap X_Q^\infty$ and

$$|\langle a \rangle|_{X_Q} \le |a|_{X_Q}, \quad |\langle a \rangle|_{X_Q^\infty} \le |a|_{X_Q^\infty}.$$

The average vector field constructed before will play a crucial role in the study of tokamak plasmas. The asymptotic regimes we are looking for will rely on transport equations whose effective advection fields come by averaging along a dominant flow. Therefore, we should be able to compute, or at least to approximate, the average of a vector field. Theorem 2.1 expresses the average of a vector field in terms of a ergodic mean. Another possibility is to obtain the average of the vector field a by studying the long time behavior of a parabolic problem, taking as initial condition the vector field a. The advantage is that we do not need anymore to compute ergodic means with large T, but only states for large T. The price to be paid is to solve a parabolic problem on X_Q.

Theorem 2.2 *Assume that* (10), (12), (21), (22) *hold true. We denote by* $\mathcal{L}$ *the infinitesimal generator of the group* $(\varphi(s))_{s\in\mathbb{R}}$. *For any vector field* $a \in X_Q$, *we consider the problem*

$$\partial_t c - \mathcal{L}^2 c = 0, \;\; t \in \mathbb{R}_+, \tag{24}$$

$$c(0,\cdot) = a(\cdot). \tag{25}$$

Then the solution of (24), (25) *converges weakly in* X_Q, *as* $t \to +\infty$, *toward the orthogonal projection on* $\ker \mathcal{L}$

$$\lim_{t\to+\infty} c(t) = \mathrm{Proj}_{\ker \mathcal{L}} a, \;\; \textit{weakly in } X_Q.$$

Moreover, if the range of $\mathcal{L}$ *is closed, then the previous convergence holds strongly in* X_Q *and has exponential rate.*

The construction of the average vector field, cf. Theorem 2.1 legitimates the convergence of the solutions $(v^\varepsilon)_{\varepsilon>0}$ in (16), as $\varepsilon \searrow 0$, toward the solution of the transport equation

$$\partial_t v + \langle a(t,\cdot)\rangle \cdot \nabla_z v = 0, \;\; (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m,$$

which satisfies the same initial condition as v^ε, that is u^{in}. Making some technical assumptions (see Section 4) allows us to obtain a strong convergence result for the family $(v^\varepsilon)_{\varepsilon>0}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$ and to describe the oscillations of the family $(u^\varepsilon)_{\varepsilon>0}$ in terms of the composition between the profile v and the flow Y.

Theorem 2.3 *Assume that the hypotheses* (9), (10), (11), (12), (36), (38), (39), (42) *hold true. Moreover, we suppose that* $u^{\mathrm{in}} \in H^1_R$ *and*

$$a \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; X_Q^\infty), \;\; a, \partial_t a \in L^1_{\mathrm{loc}}(\mathbb{R}_+; X_Q) \;\; (\textit{for example } a \in C^1_c(\mathbb{R}_+ \times \mathbb{R}^m)\ ).$$

We denote by $(u^\varepsilon)_{\varepsilon>0}$ *the solutions (by characteristics) of* (7), (8) *and by* $(v^\varepsilon)_{\varepsilon>0}$ *the functions*

$$v^\varepsilon(t,z) = u^\varepsilon(t, Y(t/\varepsilon; z)), \;\; (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m, \;\; \varepsilon > 0.$$

Therefore the family $(v^\varepsilon)_{\varepsilon>0}$ *converges strongly in* $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$ *to a weak solution* $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ *of the transport problem*

$$\partial_t v + \langle a(t,\cdot)\rangle \cdot \nabla_z v = 0, \;\; (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m, \tag{26}$$

$$v(0,z) = u^{\mathrm{in}}(z), \;\; z \in \mathbb{R}^m. \tag{27}$$

The function v *has the regularity* $\partial_t v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)), \nabla_z v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; X_P)$ *and it is the unique weak solution of* (26), (27) *with this regularity.*

Under additional hypotheses (see Section 6) we justify the convergence rate $\|v^\varepsilon - v\| = \mathcal{O}(\varepsilon)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as expected.

Theorem 2.4 *Suppose that all the hypotheses of Theorem 2.3 hold true. Moreover, we assume that the solution* v *of the limit model* (26), (27) *satisfies*

$$b_i \cdot \nabla_z (b_j \cdot \nabla_z v) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)), \;\; b_i \cdot \nabla_z \partial_t v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)), \;\; i,j \in \{1,...,m\},$$

and that there is a vector field $c = c(t,y)$ *verifying*

$$Rc \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}^m)), \;\; \partial_t(Rc), \;\; b_i \cdot \nabla_z(Rc) \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}^m)), \;\; i \in \{1,...,m\},$$

such that the following decomposition holds true at any time $t \in \mathbb{R}_+$

$$a(t) = \langle a(t)\rangle + \mathcal{L}c(t).$$

Then, for any $T > 0$ *there is a constant* C_T *such that*

$$\|u^\varepsilon(t,\cdot) - v(t, Y(-t/\varepsilon;\cdot))\|_{L^2(\mathbb{R}^m)} \le C_T \varepsilon, \;\; t \in [0,T], \; \varepsilon > 0.$$

In Section 7 we present some applications to gyrokinetic theory. We describe the finite Larmor radius regime, under non uniform magnetic fields. Another interesting issue concerns the treatment of the collisions. We consider the Fokker-Planck kernel, but our method adapts to other collision operators. As for vector fields, we define an average operator, acting on matrix fields, see Theorem 7.1, and we characterize the average matrix field as the long time limit of a parabolic problem cf. Theorem 7.2. We determine the effective Fokker-Planck kernel under strong magnetic fields, and we investigate its equilibria cf. Proposition 7.1. It happens that the kernel of the diffusion matrix in the average Fokker-Planck operator reduces to a mono-dimensional space. The average Fokker-Planck operator contains not only diffusion in velocity, but also in space (in the orthogonal directions with respect to the magnetic lines), as predicted by experiments and numerical simulations.

3 The average of a vector field

Let $P(y)$ be a matrix field satisfying (21), (22). We consider the inverse matrix field $Q = P^{-1}$ and we introduce the set

$$X_Q = \{c(y) : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable } : \int_{\mathbb{R}^m} |Q^{1/2}(y)c(y)|^2 \, \mathrm{d}y < +\infty\},$$

and the bilinear map

$$(\cdot,\cdot)_Q : X_Q \times X_Q \to \mathbb{R}, \;\; (c,d)_Q = \int_{\mathbb{R}^m} Q^{1/2}(y)c(y) \cdot Q^{1/2}(y)d(y) \, \mathrm{d}y, \;\; c, d \in X_Q.$$

Notice that the above map is symmetric and positive definite. Indeed, for any $c \in X_Q$ we have

$$(c,c)_Q = \int_{\mathbb{R}^m} Q(y)c(y) \cdot c(y) \, \mathrm{d}y \geq 0,$$

with equality iff $Qc \cdot c = 0$ and thus iff $c = 0$. The set X_Q endowed with the scalar product $(\cdot,\cdot)_Q$ becomes a Hilbert space, whose norm is denoted by $|c|_Q = (c,c)_Q^{1/2}, c \in X_Q$. Clearly $\{c(y) : c \in C_c^0(\mathbb{R}^m)\} \subset X_Q$. Observe that $X_Q \subset \{c(y) : c \in L^1_{\mathrm{loc}}(\mathbb{R}^m)\}$. This comes easily, observing that

$$\begin{aligned}
|c(y)| &= \sup_{\xi \neq 0} \frac{c(y) \cdot \xi}{|\xi|} \\
&= \sup_{\xi \neq 0} \frac{Q^{1/2}(y)c(y) \cdot P^{1/2}(y)\xi}{|P^{1/2}(y)\xi|} \; \frac{|P^{1/2}(y)\xi|}{|\xi|} \\
&\leq |Q^{1/2}(y)c(y)| \; \frac{(P(y)\xi \cdot \xi)^{1/2}}{|\xi|} \\
&\leq |Q^{1/2}(y)c(y)| \; |P(y)|^{1/2}.
\end{aligned}$$

Therefore, for any $R > 0$ one gets, thanks to the condition $P \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$

$$\begin{aligned}
\int_{B_R} |c(y)| \, \mathrm{d}y &\leq \int_{B_R} |Q^{1/2}(y)c(y)| \, |P(y)|^{1/2} \, \mathrm{d}y \\
&\leq \left(\int_{B_R} |Q^{1/2}(y)c(y)|^2 \, \mathrm{d}y \right)^{1/2} \left(\int_{B_R} |P(y)| \, \mathrm{d}y \right)^{1/2} \\
&\leq |c|_Q \left(\int_{B_R} |P(y)| \, \mathrm{d}y \right)^{1/2},
\end{aligned}$$

saying that $X_Q \subset \{c(y) : c \in L^1_{\rm loc}(\mathbb{R}^m)\}$. For any matrix M, the notation $|M|$ stands for the norm subordonated to the Euclidean norm of $\mathbb{R}^m$

$$|M| = \sup_{\xi \in \mathbb{R}^m \setminus \{0\}} \frac{|M\xi|}{|\xi|} \leq (M : M)^{1/2}.$$

Similarly, we consider $X_P = \{d(y) : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable} : \int_{\mathbb{R}^m} |P^{1/2}(y)d(y)|^2 \; {\rm d}y < +\infty\}$. The transformation $d \in X_P \to Pd \in X_Q$ is a unitary operator between Hilbert spaces. We introduce also the bilinear continuous map $\langle \cdot, \cdot \rangle_{P,Q} : X_P \times X_Q \to \mathbb{R}$ defined by

$$\langle d, c \rangle_{P,Q} = \int_{\mathbb{R}^m} d(y) \cdot c(y) \; {\rm d}y = \int_{\mathbb{R}^m} P^{1/2}(y)d(y) \cdot Q^{1/2}(y)c(y) \; {\rm d}y, \;\; (d, c) \in X_P \times X_Q.$$

It is easily seen that $d \in X_P \to \langle d, \cdot \rangle_{P,Q} \in (X_Q)'$ is a linear isomorphism and thus we identify $(X_Q)'$ to X_P through the duality $\langle \cdot, \cdot \rangle_{P,Q}$.

We intend to apply the von Neumann's ergodic theorem. For doing that we need a C^0-group of unitary operators on X_Q.

Proposition 3.1 *The family of linear transformations $c \to \varphi(s)c = \partial_y Y(-s; Y(s; \cdot))c(Y(s; \cdot))$, $s \in \mathbb{R}$, is a C^0-group of unitary operators on X_Q. For any $a \in X_Q, s \in \mathbb{R}$, we have ${\rm div}_y(\varphi(s)a) = ({\rm div}_y a)(Y(s; \cdot))$ in $\mathcal{D}'(\mathbb{R}^m)$. In particular, if ${\rm div}_y a = 0$ in $\mathcal{D}'(\mathbb{R}^m)$, then ${\rm div}_y(\varphi(s)a) = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ for any $s \in \mathbb{R}$.*

Proof. For any $c \in X_Q, s \in \mathbb{R}$ we have, thanks to Proposition 2.1

$$\begin{aligned}
|\varphi(s)c|_Q^2 &= \int_{\mathbb{R}^m} Q(y)\partial_y Y(-s; Y(s; y))c(Y(s; y)) \cdot \partial_y Y(-s; Y(s; y))c(Y(s; y)) \; {\rm d}y \\
&= \int_{\mathbb{R}^m} {}^t\partial_y Y(-s; Y(s; y))Q(y)\partial_y Y(-s; Y(s; y))c(Y(s; y)) \cdot c(Y(s; y)) \; {\rm d}y \\
&= \int_{\mathbb{R}^m} {}^t\partial_y Y^{-1}(s; y)P^{-1}(y)\partial_y Y^{-1}(s; y)c(Y(s; y)) \cdot c(Y(s; y)) \; {\rm d}y \\
&= \int_{\mathbb{R}^m} [\partial_y Y(s; y)P(y) \; {}^t\partial_y Y(s; y)]^{-1}c(Y(s; y)) \cdot c(Y(s; y)) \; {\rm d}y \\
&= \int_{\mathbb{R}^m} P^{-1}(Y(s; y))c(Y(s; y)) \cdot c(Y(s; y)) \; {\rm d}y \\
&= \int_{\mathbb{R}^m} Q(Y(s; y))c(Y(s; y)) \cdot c(Y(s; y)) \; {\rm d}y \\
&= \int_{\mathbb{R}^m} Q(y)c(y) \cdot c(y) \; {\rm d}y = |c|_Q^2.
\end{aligned}$$

Obviously, we have $\varphi(0)c = c, c \in X_Q$, and for any $s, t \in \mathbb{R}, c \in X_Q$, we can write

$$\begin{aligned}
\varphi(s)\varphi(t)c &= \partial_y Y(-s; Y(s; \cdot))(\varphi(t)c)(Y(s; \cdot)) \\
&= \partial_y Y(-s; Y(s; \cdot))\partial_y Y(-t; Y(t; Y(s; \cdot)))c(Y(t; Y(s; \cdot))) \\
&= \partial_y Y(-s; Y(s; \cdot))\partial_y Y(-t; Y(t + s; \cdot))c(Y(t + s; \cdot)) \\
&= \partial_y Y^{-1}(t + s; \cdot)c(Y(t + s; \cdot)) \\
&= \partial_y Y(-t - s; Y(t + s; \cdot))c(Y(t + s; \cdot)) \\
&= \varphi(s + t)c.
\end{aligned}$$

In the previous computations we have used the equalities

$$Y(-s; Y(s; y)) = y, \;\; Y(-t; Y(t + s; y)) = Y(s; y),$$

which imply after differentiation with respect to y

$$\partial_y Y(-s;Y(s;\cdot))\partial_y Y(s;\cdot)=I_m,\ \ \partial_y Y(-t;Y(t+s;\cdot))\partial_y Y(t+s;\cdot)=\partial_y Y(s;\cdot).$$

We check now the continuity of the group, *i.e.,* $\lim_{s\to 0}\varphi(s)c=c$, strongly in X_Q, for any $c\in X_Q$. For any $s\in\mathbb{R}$ we have

$$|\varphi(s)c-c|_Q^2=|\varphi(s)c|_Q^2+|c|_Q^2-2(\varphi(s)c,c)_Q=2|c|_Q^2-2(\varphi(s)c,c)_Q,$$

and thus it is enough to prove the weak convergence $\lim_{s\to 0}\varphi(s)c=c$ in X_Q. As $|\varphi(s)|=1$ for any $s\in\mathbb{R}$, we are done if we prove that for any $d\in C_c^0(\mathbb{R}^m)\subset X_Q$ we have

$$\lim_{s\to 0}(\varphi(s)c,d)_Q=(c,d)_Q.$$

It is easily seen that $\lim_{s\to 0}\varphi(-s)d=d$ strongly in X_Q, for $d\in C_c^0(\mathbb{R}^m)$ (use the dominated convergence theorem and the fact that $Q\in L^1_{\text{loc}}(\mathbb{R}^m)$) and therefore

$$\lim_{s\to 0}(\varphi(s)c,d)_Q=\lim_{s\to 0}(c,\varphi(-s)d)_Q=(c,d)_Q,\ \ d\in C_c^0(\mathbb{R}^m).$$

We analyze now how the divergence propagates along the trajectories of the group $(\varphi(s))_{s\in\mathbb{R}}$. Notice that, since $X_Q\subset L^1_{\text{loc}}(\mathbb{R}^m)$, for any $a\in X_Q$, the linear form $\theta\in C_c^1(\mathbb{R}^m)\to -\int_{\mathbb{R}^m}a\cdot\nabla_y\theta\ \mathrm{d}y$ defines an element of $\mathcal{D}'(\mathbb{R}^m)$. By the definition of the group, we can write for any test function $\theta\in C_c^1(\mathbb{R}^m)$

$$\begin{aligned}\langle \mathrm{div}_y(\varphi(s)a),\theta\rangle_{\mathcal{D}',\mathcal{D}}&=-\int_{\mathbb{R}^m}\varphi(s)a\cdot\nabla_z\theta(z)\ \mathrm{d}z\\&=-\int_{\mathbb{R}^m}\partial_y Y(-s;Y(s;z))a(Y(s;z))\cdot\nabla_z\theta(z)\ \mathrm{d}z\\&=-\int_{\mathbb{R}^m}a(Y(s;z))\cdot{}^t\partial_y Y(-s;Y(s;z))\nabla_z\theta(z)\ \mathrm{d}z\\&=-\int_{\mathbb{R}^m}a(y)\cdot{}^t\partial_y Y(-s;y)(\nabla_z\theta)(Y(-s;y))\ \mathrm{d}y\\&=-\int_{\mathbb{R}^m}a(y)\cdot\nabla_y\{\theta(Y(-s;y))\}\ \mathrm{d}y\\&=\langle \mathrm{div}_y a,\theta\circ Y(-s;\cdot)\rangle_{\mathcal{D}',\mathcal{D}}\\&=\langle(\mathrm{div}_y a)\circ Y(s;\cdot),\theta\rangle_{\mathcal{D}',\mathcal{D}}\,.\end{aligned}$$

We deduce that $\mathrm{div}_y(\varphi(s)a)=(\mathrm{div}_y a)\circ Y(s;\cdot)$. In particular, the subspace of divergence free vector fields of X_Q is left invariant by the group. □

Remark 3.1 *We introduce also the set*

$$X_Q^\infty=\{c(y):\mathbb{R}^m\to\mathbb{R}^m\ \textit{measurable}\ :\ |Q^{1/2}(\cdot)c(\cdot)|\in L^\infty(\mathbb{R}^m)\}.$$

It is a Banach space with respect to the norm $|c|_{X_Q^\infty}=\mathrm{ess\ sup}_{y\in\mathbb{R}^m}|Q^{1/2}(y)c(y)|$. *This space is left invariant by* $(\varphi(s))_{s\in\mathbb{R}}$. *Indeed, let us consider* $c\in X_Q^\infty$ *and observe that*

$$\begin{aligned}|Q^{1/2}(y)\varphi(s)c(y)|^2&=Q(y)\varphi(s)c(y)\cdot\varphi(s)c(y)\\&=Q(y)\partial_y Y(-s;Y(s;y))c(Y(s;y))\cdot\partial_y Y(-s;Y(s;y))c(Y(s;y))\\&={}^t\partial_y Y(-s;Y(s;y))Q(y)\partial_y Y(-s;Y(s;y))c(Y(s;y))\cdot c(Y(s;y))\\&=Q(Y(s;y))c(Y(s;y))\cdot c(Y(s;y))\\&=|Q^{1/2}(Y(s;y))c(Y(s;y))|^2,\ \ (s,y)\in\mathbb{R}\times\mathbb{R}^m.\end{aligned}$$

We deduce that $|\varphi(s)c|_{X_Q^\infty}=|c|_{X_Q^\infty}$, *saying that* $\varphi(s)c\in X_Q^\infty$.

We consider the infinitesimal generator of the group $(\varphi(s))_{s\in\mathbb{R}}$

$$\mathcal{L} : \mathrm{dom}\mathcal{L} \subset X_Q \to X_Q, \;\; \mathrm{dom}\mathcal{L} = \{c \in X_Q \;:\; \exists \lim_{s\to 0} \frac{\varphi(s)c - c}{s} \text{ in } X_Q\},$$

and

$$\mathcal{L}c = \lim_{s\to 0} \frac{\varphi(s)c - c}{s}, \;\; c \in \mathrm{dom}\mathcal{L}.$$

Observe that $\{c(y) : c \in C^1_c(\mathbb{R}^m)\} \subset \mathrm{dom}\mathcal{L}$ and $\mathcal{L}c = [b, c], c \in C^1_c(\mathbb{R}^m)$. Indeed, if $c \in C^1_c(\mathbb{R}^m)$, we have for any $y \in \mathbb{R}^m$

$$\begin{aligned}
\frac{\varphi(s)c(y) - c(y)}{s} &= \frac{\partial_y Y(-s; Y(s; y))c(Y(s; y)) - c(y)}{s} \\
&= \partial_y Y(-s; Y(s; y)) \frac{c(Y(s; y)) - c(y) + c(y) - \partial_y Y(s; y)c(y)}{s} \\
&= \partial_y Y(-s; Y(s; y)) \left[\frac{c(Y(s; y)) - c(y)}{s} - \frac{\partial_y Y(s; y) - I_m}{s} c(y) \right] \\
&\underset{s\to 0}{\longrightarrow} (b \cdot \nabla_y)c - \partial_y b \; c \\
&= [b, c](y).
\end{aligned}$$

We deduce thanks to the dominated convergence theorem and to the hypothesis $Q \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ that $\lim_{s\to 0} \frac{\varphi(s)c-c}{s} = [b, c]$ in X_Q, saying that $c \in \mathrm{dom}\mathcal{L}$ and $\mathcal{L}c = [b, c]$. The following proposition summarizes the main properties of the operator $\mathcal{L}$. The arguments are standard and the proof is postponed to Appendix A.

Proposition 3.2 *Consider a vector field b satisfying* (10), (12), *and a matrix field P verifying* (21), (22).

1. *The domain of $\mathcal{L}$ is dense in X_Q and $\mathcal{L}$ is closed.*
2. *The vector field $c \in X_Q$ belongs to* $\mathrm{dom}\mathcal{L}$ *iff there is a constant $K > 0$ such that*

$$|\varphi(s)c - c|_Q \le K|s|, \;\; s \in \mathbb{R}. \tag{28}$$

3. *The operator $\mathcal{L}$ is skew-adjoint.*

We come back to the limit in (20). The existence of this limit is a direct consequence of the mean ergodic theorem [35] pp.57. The arguments are standard, the reader can find the main lines in Appendix A.

Theorem 3.1 *(von Neumann's ergodic Theorem)*
Let $(G(s))_{s\in\mathbb{R}}$ be a C^0-group of unitary operators on a Hilbert space $(H, (\cdot, \cdot))$ and A be the infinitesimal generator of the C^0-group. Then, for any $x \in H$, we have

$$\lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} G(s)x \,\mathrm{d}s = \mathrm{Proj}_{\ker A} x, \;\; \textit{strongly in } H,$$

uniformly with respect to $r \in \mathbb{R}$.

Remark 3.2 *The fact that the convergence in the von Neumann's ergodic Theorem is uniform with respect to t comes easily, by observing that*

$$\begin{aligned}
\left\| \frac{1}{T} \int_t^{t+T} G(s)x \,\mathrm{d}s - \mathrm{Proj}_{\ker A} x \right\| &= \left\| \frac{G(t)}{T} \int_0^T G(s)x \,\mathrm{d}s - \mathrm{Proj}_{\ker A} x \right\| \\
&= \left\| \frac{G(t)}{T} \int_0^T G(s)x \,\mathrm{d}s - G(t)\mathrm{Proj}_{\ker A} x \right\| \\
&= \left\| \frac{1}{T} \int_0^T G(s)x \,\mathrm{d}s - \mathrm{Proj}_{\ker A} x \right\|.
\end{aligned}$$

Two direct consequences of the von Neumann's ergodic Theorem are the construction of the average operators for functions [4, 5] and vector fields.

Theorem 3.2 *Assume that* (10), (12) *hold true. Then for any function* $u \in L^2(\mathbb{R}^m)$*, we have the strong convergence in* $L^2(\mathbb{R}^m)$

$$\lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} u(Y(s;\cdot))\, \mathrm{d}s = \mathrm{Proj}_{\ker(b\cdot\nabla_y)} u,$$

uniformly with respect to $r \in \mathbb{R}$.

Proof. We consider the family of linear transformations $\tau(s)u = u(Y(s;\cdot)), s \in \mathbb{R}$. It is easily seen that $(\tau(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators on $L^2(\mathbb{R}^m)$ and that its infinitesimal generator is $b \cdot \nabla_y$, given by (see also Proposition 3.2)

$$\begin{aligned}\mathrm{dom}(b\cdot\nabla_y) &= \left\{ u \in L^2(\mathbb{R}^m) \ : \ \exists \lim_{s\to 0} \frac{\tau(s)u - u}{s} \text{ strongly in } L^2(\mathbb{R}^m) \right\} \\ &= \{ u \in L^2(\mathbb{R}^m) \ : \ \exists C > 0 \ \text{ such that } \|\tau(s)u - u\|_{L^2(\mathbb{R}^m)} \le C|s|, s \in \mathbb{R}\},\end{aligned}$$

and

$$(b\cdot\nabla_y)u = \lim_{s\to 0} \frac{\tau(s)u - u}{s}, \ \text{ strongly in } L^2(\mathbb{R}^m).$$

It is well known that $b\cdot\nabla_y$ coincides with the weak derivative along the vector field b, that is

$$\mathrm{dom}(b\cdot\nabla_y) = \left\{ u \in L^2(\mathbb{R}^m) \ : \ \exists v \in L^2(\mathbb{R}^m), \int_{\mathbb{R}^m} b\cdot\nabla_y\theta \ u \ \mathrm{d}y + \int_{\mathbb{R}^m} \theta v \ \mathrm{d}y = 0 \ , \forall \theta \in C_c^1(\mathbb{R}^m) \right\},$$

and $(b\cdot\nabla_y)u = v, u \in \mathrm{dom}(b\cdot\nabla_y)$. By the von Neumann's ergodic Theorem we have

$$\lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} u(Y(s;\cdot))\, \mathrm{d}s = \lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} \tau(s)u \ \mathrm{d}s = \mathrm{Proj}_{\ker(b\cdot\nabla_y)} u,$$

strongly in $L^2(\mathbb{R}^m)$, uniformly with respect to $r \in \mathbb{R}$. □

Remark 3.3 *For any function* $u \in L^2(\mathbb{R}^m)$*, we introduce the notation*

$$\langle u \rangle = \lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} u(Y(s;\cdot))\mathrm{d}s,$$

and we call it the average of u *along the flow of* b*. Theorem 3.2 says that the average along the flow of* b *coincides with the orthogonal projection on the subspace of* L^2 *functions, which are constant along the same flow. Since* $b\cdot\nabla_y$ *is skew-adjoint, we have the orthogonal decomposition*

$$L^2(\mathbb{R}^m) = \ker(b\cdot\nabla_y) \oplus \overline{\mathrm{Range}(b\cdot\nabla_y)},$$

saying that $\overline{\mathrm{Range}(b\cdot\nabla_y)} = \{u \in L^2(\mathbb{R}^m) : \langle u \rangle = 0\}$.

Similarly we introduce the average of a vector field. We use the same notation $\langle\cdot\rangle$ for both scalar functions and vector fields, but it should be understood in the sense of the C^0-group $(\tau(s))_{s\in\mathbb{R}}$ and $(\varphi(s))_{s\in\mathbb{R}}$ respectively.

3.1 Proof of Theorem 2.1

By the von Neumann's ergodic Theorem we have

$$\lim_{T\to+\infty}\frac{1}{T}\int_r^{r+T}\partial_y Y(-s;Y(s;\cdot))a(Y(s;\cdot))\ \mathrm{d}s = \lim_{T\to+\infty}\frac{1}{T}\int_r^{r+T}\varphi(s)a\ \mathrm{d}s = \mathrm{Proj}_{\ker\mathcal{L}}a,$$

strongly in X_Q, uniformly with respect to $r\in\mathbb{R}$. Consider now $\theta\in C_c^1(\mathbb{R}^m)$. Thanks to the hypothesis $P\in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, we deduce that $Q^{-1}\nabla\theta\in X_Q$ and therefore we have

$$\begin{aligned}\int_{\mathbb{R}^m}\langle a\rangle\cdot\nabla_z\theta\ \mathrm{d}z &= (\langle a\rangle, Q^{-1}\nabla\theta)_Q \\ &= \left(\lim_{T\to+\infty}\frac{1}{T}\int_0^T\varphi(s)a\ \mathrm{d}s, Q^{-1}\nabla\theta\right)_Q \\ &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T(\varphi(s)a, Q^{-1}\nabla\theta)_Q\ \mathrm{d}s \\ &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T(a,\varphi(-s)Q^{-1}\nabla\theta)_Q\ \mathrm{d}s.\end{aligned}\tag{29}$$

By Proposition 2.1 we know that

$$\begin{aligned}P(\cdot) &= \partial_y Y(s;Y(-s;\cdot))P(Y(-s;\cdot))\ {}^t\partial_y Y(s;Y(-s;\cdot)) \\ &= \partial_y Y(s;Y(-s;\cdot))P(Y(-s;\cdot))\ {}^t\partial_y Y^{-1}(-s;\cdot),\end{aligned}$$

and therefore we can write

$$\begin{aligned}\varphi(-s)Q^{-1}\nabla\theta =& \partial_y Y(s;Y(-s;\cdot))P(Y(-s;\cdot))(\nabla\theta)(Y(-s;\cdot)) \\ &= \partial_y Y(s;Y(-s;\cdot))P(Y(-s;\cdot))\ {}^t\partial_y Y^{-1}(-s;\cdot)\nabla(\theta\circ Y(-s;\cdot)) \\ &= P(\cdot)\nabla(\theta\circ Y(-s;\cdot)).\end{aligned}\tag{30}$$

Combining (29) and (30) yields

$$\begin{aligned}\int_{\mathbb{R}^m}\langle a\rangle\cdot\nabla_z\theta\ \mathrm{d}z &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T(a, P(\cdot)\nabla(\theta\circ Y(-s;\cdot)))_Q\ \mathrm{d}s \\ &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T\int_{\mathbb{R}^m}a(y)\cdot\nabla_y(\theta\circ Y(-s;\cdot))\ \mathrm{d}y\ \mathrm{d}s,\end{aligned}$$

saying that $\mathrm{div}_y\langle a\rangle = 0$, provided that $\mathrm{div}_y a = 0$. Actually, the above computation shows that $\mathrm{div}_y\langle a\rangle = \langle\mathrm{div}_y a\rangle$. Indeed, if we assume that $\mathrm{div}_y a\in L^2(\mathbb{R}^m)$, as before, we have

$$\begin{aligned}\int_{\mathbb{R}^m}\langle a\rangle\cdot\nabla_z\theta\ \mathrm{d}z &= -\lim_{T\to+\infty}\frac{1}{T}\int_0^T\int_{\mathbb{R}^m}\mathrm{div}_y a\ \theta(Y(-s;y))\ \mathrm{d}y\ \mathrm{d}s \\ &= -\lim_{T\to+\infty}\frac{1}{T}\int_0^T\int_{\mathbb{R}^m}(\mathrm{div}_y a)(Y(s;y))\theta(y)\ \mathrm{d}y\ \mathrm{d}s \\ &= -\int_{\mathbb{R}^m}\lim_{T\to+\infty}\left(\frac{1}{T}\int_0^T\tau(s)\mathrm{div}_y a\ \mathrm{d}s\right)\theta(y)\ \mathrm{d}y \\ &= -\int_{\mathbb{R}^m}\langle\mathrm{div}_y a\rangle\ \theta(y)\ \mathrm{d}y,\end{aligned}$$

and our statement follows.

Consider now $\psi\in C^1(\mathbb{R}^m)\cap\ker(b\cdot\nabla_y)$, such that $\nabla\psi\in X_P$ and $\theta\in L^\infty(\mathbb{R}^m)$. We claim that for any vector field $c \in X_Q$, the scalar function $\theta c \cdot \nabla \psi$ belongs to $L^1(\mathbb{R}^m)$. We are done if we prove that $c \cdot \nabla \psi \in L^1(\mathbb{R}^m)$. Indeed, we have

$$\begin{aligned}\int_{\mathbb{R}^m} |c \cdot \nabla_y \psi| \,\mathrm{d}y &= \int_{\mathbb{R}^m} |Q^{1/2}(y)c(y) \cdot P^{1/2}(y)\nabla_y \psi| \,\mathrm{d}y \\ &\leq \left(\int_{\mathbb{R}^m} |Q^{1/2}(y)c(y)|^2 \,\mathrm{d}y\right)^{1/2} \left(\int_{\mathbb{R}^m} |P^{1/2}(y)\nabla_y \psi|^2 \,\mathrm{d}y\right)^{1/2} = |c|_Q |\nabla \psi|_P < +\infty.\end{aligned}$$

As in the computations (29), (30) and by taking into account that $\psi \circ Y(-s;\cdot) = \psi$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^m} \langle a \rangle \cdot \theta \nabla_z \psi \,\mathrm{d}z &= (\langle a \rangle, Q^{-1}\theta \nabla_z \psi)_Q \\ &= \left(\lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi(s)a \,\mathrm{d}s, Q^{-1}\theta\nabla_z\psi \right)_Q \\ &= \lim_{T\to+\infty} \frac{1}{T}\int_0^T (\varphi(s)a, Q^{-1}\theta\nabla_z\psi)_Q \,\mathrm{d}s \\ &= \lim_{T\to+\infty} \frac{1}{T}\int_0^T (a, P(\cdot)\theta(Y(-s;\cdot))\nabla(\psi \circ Y(-s;\cdot)))_Q \,\mathrm{d}s \\ &= \lim_{T\to+\infty} \int_0^T (a, P(\cdot)\theta(Y(-s;\cdot))\nabla\psi)_Q \,\mathrm{d}s \\ &= \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_{\mathbb{R}^m} a(y)\cdot\nabla\psi(y)\; \theta(Y(-s;y)) \,\mathrm{d}y \,\mathrm{d}s \\ &= \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_{\mathbb{R}^m} a(Y(s;z))\cdot(\nabla\psi)(Y(s;z))\theta(z) \,\mathrm{d}z \,\mathrm{d}s \\ &= \lim_{T\to+\infty} \int_{\mathbb{R}^m} \theta(z) \left(\frac{1}{T}\int_0^T (a\cdot\nabla\psi)(Y(s;z)) \,\mathrm{d}s \right) \mathrm{d}z.\end{aligned}$$

As the previous computations hold true for any $\theta \in L^\infty(\mathbb{R}^m)$, we deduce the convergence

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T (a\cdot\nabla\psi)(Y(s;\cdot)) \,\mathrm{d}s = \langle a \rangle \cdot \nabla\psi,$$

weakly in $L^1(\mathbb{R}^m)$.

Assume that $a \in X_Q \cap X_Q^\infty$. Clearly $\langle a \rangle = \mathrm{Proj}_{\ker \mathcal{L}} a \in X_Q$ and $|\langle a \rangle|_Q \leq |a|_Q$. It remains to prove that $\langle a \rangle \in X_Q^\infty$ and $|\langle a \rangle|_{X_Q^\infty} \leq |a|_{X_Q^\infty}$. We know that $\frac{1}{T}\int_0^T \varphi(s)a \,\mathrm{d}s \to \langle a \rangle$, strongly in X_Q, as $T \to +\infty$, or equivalently, $\frac{1}{T}\int_0^T Q^{1/2}\varphi(s)a \,\mathrm{d}s \to Q^{1/2}\langle a \rangle$, strongly in $L^2(\mathbb{R}^m)$, as $T \to +\infty$. Therefore, there is a sequence $(T_k)_k$, $T_k \to +\infty$ as $k \to +\infty$, such that

$$\lim_{k\to+\infty} \frac{1}{T_k}\int_0^{T_k} Q^{1/2}(y)(\varphi(s)a)(y) \,\mathrm{d}s = Q^{1/2}(y)\langle a \rangle(y), \quad \text{a.a. } y \in \mathbb{R}^m.$$

Observing that

$$\begin{aligned}\left| \frac{1}{T_k}\int_0^{T_k} Q^{1/2}(y)(\varphi(s)a)(y) \,\mathrm{d}s \right| &\leq \frac{1}{T_k}\int_0^{T_k} |Q^{1/2}(y)(\varphi(s)a)(y)| \,\mathrm{d}s \\ &\leq \frac{1}{T_k}\int_0^{T_k} \|Q^{1/2}\varphi(s)a\|_{L^\infty} \,\mathrm{d}s \\ &= \frac{1}{T_k}\int_0^{T_k} |\varphi(s)a|_{X_Q^\infty} \,\mathrm{d}s \\ &= |a|_{X_Q^\infty}, \quad \text{a.a. } y \in \mathbb{R}^m,\end{aligned}$$

we deduce by letting $k \to +\infty$ that

$$|Q^{1/2}(y) \langle a \rangle (y)| \le |a|_{X_Q^\infty}, \text{ a.a. } y \in \mathbb{R}^m,$$

saying that $\langle a \rangle \in X_Q^\infty$ and $| \langle a \rangle |_{X_Q^\infty} \le |a|_{X_Q^\infty}$.

Remark 3.4 *Theorem 2.1 states that the average of a vector field along the flow of b coincides with its orthogonal projection on the subspace of the vector fields in involution with b. By construction, any average of a vector field is in involution with b. The property which says that the average of a vector field a is divergence free, if a is divergence free, will guarantee that our limit model is conservative, if so is the original model. The last statement in Theorem 2.1*

$$\langle a \cdot \nabla\psi \rangle = \langle a \rangle \cdot \nabla\psi, \quad \psi \in C^1(\mathbb{R}^m) \cap \ker(b \cdot \nabla_y), \quad \nabla\psi \in X_P,$$

where the average operator in the left hand side should be understood in the L^1 setting (see [4]), allows us to get some informations about the average vector field $\langle a \rangle$ anytime a global invariant ψ is available. In particular, any function which is left invariant along b and a, is also left invariant along $\langle a \rangle$.

We analyze now the parabolic problem (24), (25) and we prove that the average of the initial vector field coincides with its long time limit. In order to use variational methods, we introduce the space $Y_Q = \mathrm{dom}\mathcal{L} \subset X_Q$ endowed with the scalar product

$$((c, d))_Q = (c, d)_Q + (\mathcal{L}c, \mathcal{L}d)_Q, \quad c, d \in Y_Q.$$

It is easily seen that $(Y_Q, ((\cdot, \cdot))_Q)$ is a Hilbert space (use the fact that $\mathcal{L}$ is closed) and the inclusion $Y_Q \subset X_Q$ is continuous, with dense image. The norm associated to the scalar product $((\cdot, \cdot))_Q$ is denoted by $\|\cdot\|_Q$

$$\|c\|_Q^2 = ((c, c))_Q = (c, c)_Q + (\mathcal{L}c, \mathcal{L}c)_Q = |c|_Q^2 + |\mathcal{L}c|_Q^2, \quad c \in Y_Q.$$

We consider the bilinear form $\sigma : Y_Q \times Y_Q \to \mathbb{R}$

$$\sigma(c, d) = (\mathcal{L}c, \mathcal{L}d)_Q, \quad c, d \in Y_Q.$$

Observe that σ is coercive on Y_Q, with respect to X_Q

$$\sigma(c, c) + |c|_Q^2 = \|c\|_Q^2, \quad c \in Y_Q.$$

Thanks to Theorems 1, 2 of [18] pp. 620 we deduce that, for any $a \in X_Q$, there is a unique variational solution for (24), (25), that is $c \in C_b(\mathbb{R}_+; X_Q), \mathcal{L}c \in L^2(\mathbb{R}_+; X_Q), \partial_t c \in L^2(\mathbb{R}_+; Y_Q')$ and

$$(c(t), w)_Q|_{t=0} = (a, w)_Q, \quad \frac{\mathrm{d}}{\mathrm{d}t}(c(t), w)_Q + \sigma(c(t), w) = 0, \text{ in } \mathcal{D}'(\mathbb{R}_+), \ \forall w \in Y_Q.$$

3.2 Proof of Theorem 2.2

The variational solution of (24) satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|c(t)|_Q^2 + |\mathcal{L}c(t)|_Q^2 = 0, \quad t \in \mathbb{R}_+, \tag{31}$$

saying that $t \to |c(t)|_Q$ is decreasing on $\mathbb{R}_+$ and $t \to |\mathcal{L}c(t)|_Q^2$ is integrable on $\mathbb{R}_+$. We claim that for any $s \in \mathbb{R}$, $\varphi(s)c$ is the variational solution of (24), corresponding to the initial condition $\varphi(s)a$. Indeed, for any $\tilde{w} \in Y_Q, \eta \in C_c^1(\mathbb{R}_+)$ we know that

$$-\int_{\mathbb{R}_+} (c(t), \tilde{w})_Q \, \eta'(t) \, \mathrm{d}t - (a, \tilde{w})_Q \, \eta(0) + \int_{\mathbb{R}_+} \sigma(c(t), \tilde{w})\eta(t) \, \mathrm{d}t = 0. \tag{32}$$

For any $w \in Y_Q$, we take in the above formulation $\tilde{w} = \varphi(-s)w \in Y_Q$ and observe that

$$(c(t), \varphi(-s)w)_Q = (\varphi(s)c(t), w)_Q, \quad (a, \varphi(-s)w)_Q = (\varphi(s)a, w)_Q,$$

$$\begin{aligned}\sigma(c(t), \varphi(-s)w) &= (\mathcal{L}c(t), \mathcal{L}\varphi(-s)w)_Q = (\mathcal{L}c(t), \varphi(-s)\mathcal{L}w)_Q = (\varphi(s)\mathcal{L}c(t), \mathcal{L}w)_Q \\ &= (\mathcal{L}\varphi(s)c(t), \mathcal{L}w)_Q = \sigma(\varphi(s)c(t), w).\end{aligned}$$

Therefore (32) becomes

$$-\int_{\mathbb{R}_+} (\varphi(s)c(t), w)_Q \, \eta'(t) \, \mathrm{d}t - (\varphi(s)a, w)_Q \, \eta(0) + \int_{\mathbb{R}_+} \sigma(\varphi(s)c(t), w)\eta(t) \, \mathrm{d}t = 0,$$

or equivalently

$$(\varphi(s)c(t), w)_Q|_{t=0} = (\varphi(s)a, w)_Q, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\varphi(s)c(t), w)_Q + \sigma(\varphi(s)c(t), w) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+),$$

saying that $\varphi(s)c$ is the variational solution of (24) satisfying the initial condition $\varphi(s)a$. The identity (31) written for the solution $\varphi(s)c - c$ ensures that $t \to |\varphi(s)c(t) - c(t)|_Q$ is decreasing on $\mathbb{R}_+$, that is

$$|\varphi(s)c(t+h) - c(t+h)|_Q \le |\varphi(s)c(t) - c(t)|_Q, \quad t, h \in \mathbb{R}_+, \quad s \in \mathbb{R}.$$

Passing to the limit, when $s \to 0$, in the inequality

$$\left| \frac{\varphi(s)c(t+h) - c(t+h)}{s} \right|_Q \le \left| \frac{\varphi(s)c(t) - c(t)}{s} \right|_Q,$$

leads to $|\mathcal{L}c(t+h)|_Q \le |\mathcal{L}c(t)|_Q, t, h \in \mathbb{R}_+$, and thus the function $t \to |\mathcal{L}c(t)|_Q$ is decreasing on $\mathbb{R}_+$. But we know that $t \to |\mathcal{L}c(t)|_Q^2$ is integrable on $\mathbb{R}_+$ and we deduce that $\lim_{t\to+\infty} \mathcal{L}c(t) = 0$, strongly in X_Q.
Consider now a sequence $(t_k)_k$, such that $t_k \to +\infty$ as $k \to +\infty$ and $(c(t_k))_k$ converges weakly toward some vector field d in X_Q. For any $z \in \ker \mathcal{L}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(c(t), z)_Q = -\sigma(c(t), z) = -(\mathcal{L}c(t), \mathcal{L}z)_Q = 0,$$

and thus $(c(t_k), z)_Q = (a, z)_Q$ for any k. Passing to the limit when $k \to +\infty$ yields

$$(a - d, z)_Q = 0, \quad z \in \ker \mathcal{L}. \tag{33}$$

Take now $w \in Y_Q$ and observe that

$$(d, \mathcal{L}w)_Q = \lim_{k\to+\infty} (c(t_k), \mathcal{L}w)_Q = -\lim_{k\to+\infty} (\mathcal{L}c(t_k), w)_Q = 0,$$

thanks to the strong convergence of $\mathcal{L}c(t)$ toward 0 in X_Q, when $t \to +\infty$. We deduce that

$$d \in \mathrm{dom}\mathcal{L}^\star = \mathrm{dom}\mathcal{L}, \quad \mathcal{L}d = -\mathcal{L}^\star d = 0. \tag{34}$$

The equalities in (33), (34) exactly say that $d = \mathrm{Proj}_{\ker \mathcal{L}} a$. The boundedness of the function $t \to |c(t)|_Q, t \in \mathbb{R}_+$ and the uniqueness of the weak limit guarantee that $\lim_{t\to+\infty} c(t) = \mathrm{Proj}_{\ker \mathcal{L}} a$ weakly in X_Q. When the range of $\mathcal{L}$ is closed, the strong convergence, with exponential rate, follows thanks to the Poincaré inequality, see also Remark 3.5.

Remark 3.5 *The average of L^2 functions, see Theorem 3.2, can be obtained as the long time limit for the solution of some parabolic problem. More exactly, for any function $u \in L^2(\mathbb{R}^m)$, let us denote by v the unique variational solution of the problem*

$$\begin{cases} \partial_t v - b \cdot \nabla_y (b \cdot \nabla_y v) = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m, \\ v(0,y) = u(y), & y \in \mathbb{R}^m, \end{cases}$$

that is $v \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)), b \cdot \nabla_y v \in L^2(\mathbb{R}_+; L^2(\mathbb{R}^m))$ and for any $w \in \mathrm{dom}(b \cdot \nabla_y)$ we have

$$(v(t),w)_{L^2(\mathbb{R}^m)}|_{t=0} = (u,w)_{L^2(\mathbb{R}^m)}, \ \frac{\mathrm{d}}{\mathrm{d}t}(v(t),w)_{L^2(\mathbb{R}^m)} + (b\cdot\nabla_y v(t), b\cdot\nabla_y w)_{L^2(\mathbb{R}^m)} = 0, \ \textit{in } \mathcal{D}'(\mathbb{R}_+).$$

Therefore the solution $v(t)$ converges weakly in $L^2(\mathbb{R}^m)$, as $t \to +\infty$, toward the orthogonal projection of u on $\ker(b \cdot \nabla_y)$, and thus toward the average of u

$$\lim_{t \to +\infty} v(t) = \mathrm{Proj}_{\ker(b\cdot\nabla_y)} u = \langle u \rangle \,, \quad \textit{weakly in } L^2(\mathbb{R}^m).$$

In particular, when $\mathrm{Range}(b \cdot \nabla_y)$ is closed, which is equivalent to the Poincaré inequality cf. [14] pp.29

$$\exists\, C_P > 0 \ \textit{ such that } \left(\int_{\mathbb{R}^m} (u - \langle u \rangle)^2 \,\mathrm{d}y \right)^{1/2} \le C_P \left(\int_{\mathbb{R}^m} (b \cdot \nabla_y u)^2 \,\mathrm{d}y \right)^{1/2}, \ \ u \in \mathrm{dom}(b \cdot \nabla_y), \tag{35}$$

the above long time convergence holds strongly in $L^2(\mathbb{R}^m)$, and has exponential rate. Indeed, for any $w \in \ker(b \cdot \nabla_y)$ we have $\frac{\mathrm{d}}{\mathrm{d}t}(v(t),w)_{L^2(\mathbb{R}^m)} = 0$ saying that

$$(v(t),w)_{L^2(\mathbb{R}^m)} = (u,w)_{L^2(\mathbb{R}^m)} = (\langle u \rangle, w)_{L^2(\mathbb{R}^m)}, \ \ t \in \mathbb{R}_+,$$

and thus $\langle v(t) \rangle = \langle u \rangle$ for any $t \in \mathbb{R}_+$. Observing that

$$\partial_t (v - \langle u \rangle) - b \cdot \nabla_y (b \cdot \nabla_y (v - \langle u \rangle)) = 0,$$

we deduce by the Poincaré inequality that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} (v(t,y) - \langle u \rangle)^2 \,\mathrm{d}y + \frac{1}{C_P^2} \int_{\mathbb{R}^m} (v(t,y) - \langle u \rangle)^2 \,\mathrm{d}y \le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} (v(t,y) - \langle u \rangle)^2 \,\mathrm{d}y + \int_{\mathbb{R}^m} (\, b \cdot \nabla_y (v - \langle u \rangle) \,)^2 \,\mathrm{d}y = 0,$$

implying that

$$\| v(t) - \langle u \rangle \|_{L^2(\mathbb{R}^m)} \le e^{-t/C_P^2} \| u - \langle u \rangle \|_{L^2(\mathbb{R}^m)}, \ \ t \in \mathbb{R}_+.$$

A sufficient condition, ensuring that $\mathrm{Range}(b \cdot \nabla_y)$ is closed is that all trajectories of the vector field b are closed, uniformly in time (see Proposition 2.8 [4]) i.e.,

$$\exists\, 0 < T_\infty < +\infty \ : \ \forall\, y \in \mathbb{R}^m, \exists\, T_y \in]0, T_\infty] \ \textit{ such that } Y(T_y; y) = y.$$

In this case, the inequality (35) holds true with $C_P = T_\infty$.

4 Uniform estimates

We look for uniform estimates for the family of solutions in (7), (8). Clearly, if the initial condition u^{in} belongs to $L^2(\mathbb{R}^m)$, the family $(u^\varepsilon)_{\varepsilon>0}$ in bounded in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$

$$\int_{\mathbb{R}^m} (u^\varepsilon(t,y))^2 \, \mathrm{d}y = \int_{\mathbb{R}^m} (u^{\text{in}}(y))^2 \, \mathrm{d}y, \;\; t \in \mathbb{R}_+, \varepsilon > 0.$$

We search now uniform bounds for the space derivatives. We make the following assumption : there is a matrix field $R(y)$ such that

$$\det R(y) \neq 0, \;\; y \in \mathbb{R}^m, \;\; (b \cdot \nabla_y) R + R \partial_y b = 0 \;\; \text{in } \mathcal{D}'(\mathbb{R}^m). \tag{36}$$

The equality in the above hypothesis is equivalent to

$$R(Y(s;y)) \partial_y Y(s;y) = R(y), \;\; (s,y) \in \mathbb{R} \times \mathbb{R}^m,$$

which can be also written

$$\partial_y Y(s;y) R^{-1}(y) = R^{-1}(Y(s;y)), \;\; (s,y) \in \mathbb{R} \times \mathbb{R}^m. \tag{37}$$

This exactly says that the columns of R^{-1} are vector fields in involution with b cf. (19). The vector fields in the columns of R^{-1} are denoted $b_i, 1 \leq i \leq m$, and are supposed smooth

$$b_i \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^m), \;\; 1 \leq i \leq m. \tag{38}$$

We assume that any field b_i satisfies the growth condition

$$\forall i \in \{1, ..., m\}, \; \exists C_i > 0 \text{ such that } |b_i(y)| \leq C_i(1 + |y|), \;\; y \in \mathbb{R}^m, \tag{39}$$

which guarantees the existence of the global flows $Y_i(s;y) \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^m), i \in \{1, ..., m\}$. Clearly $R^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^m)$, since b_i, which are the columns of R^{-1}, are supposed locally bounded on $\mathbb{R}^m$. Actually $y \to R^{-1}(y)$ is continuous and invertible for any $y \in \mathbb{R}^m$. Therefore $\det R^{-1}(y)$ remains away from 0 on any compact set of $\mathbb{R}^m$ (*i.e.*, for any $M > 0, \exists C_M$ such that $|\det R^{-1}(y)| \geq C_M > 0$ if $|y| \leq M$), implying that $R = (R^{-1})^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^m)$. In particular ${}^t RR, ({}^t RR)^{-1}$ are locally bounded, and therefore locally integrable on $\mathbb{R}^m$. Notice that under the assumptions (36), (38), (39), the conditions (21) and (22) hold true with $Q = {}^t RR, P = Q^{-1} = R^{-1} \, {}^t R^{-1}$. Indeed, P is symmetric, definite positive, locally integrable on $\mathbb{R}^m$, together with $Q = P^{-1}$ and, thanks to (37), we have

$$\begin{aligned} P(Y(s;y)) &= R^{-1}(Y(s;y)) \, {}^t R^{-1}(Y(s;y)) \\ &= \partial_y Y(s;y) R^{-1}(y) \, {}^t R^{-1}(y) \, {}^t \partial_y Y(s;y) \\ &= \partial_y Y(s;y) P(y) \, {}^t \partial_y Y(s;y). \end{aligned}$$

Therefore, by Proposition 2.1 we deduce that (22) is verified, that is $[b, P] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Observe that, at any point $y \in \mathbb{R}^m$, the family $(b_i(y))_{1 \leq i \leq m}$ forms a orthonormal basis of $\mathbb{R}^m$, with respect to the scalar product induced by the symmetric matrix $Q(y) = {}^t R(y) R(y)$. Indeed, by construction we have $R(y) b_i(y) = e_i, i \in \{1, ..., m\}$ (here $(e_i)_{1 \leq i \leq m}$ stands for the canonical basis of $\mathbb{R}^m$) and

$$Q(y) b_i(y) \cdot b_j(y) = {}^t R(y) R(y) b_i(y) \cdot b_j(y) = R(y) b_i(y) \cdot R(y) b_j(y) = e_i \cdot e_j = \delta_{ij}, \; i,j \in \{1, ..., m\}.$$

The reason why we introduce the involution vector fields $(b_i)_{1 \leq i \leq m}$ is to construct a H^1 type space on $\mathbb{R}^m$. For any $i \in \{1, ..., m\}$, we consider the C^0-group of linear operators on $L^2(\mathbb{R}^m)$ given by

$$\tau_i(s) u = u \circ Y_i(s; \cdot),$$

and its infinitesimal generator, denoted $b_i \cdot \nabla_y$

$$\mathrm{dom}(b_i \cdot \nabla_y) = \{u \in L^2(\mathbb{R}^m) \ : \ \exists \lim_{s\to 0} \frac{\tau_i(s)u - u}{s} \ \text{ strongly in } L^2(\mathbb{R}^m)\},$$

$$b_i \cdot \nabla_y u = \lim_{s\to 0} \frac{\tau_i(s)u - u}{s}, \ \ u \in \mathrm{dom}(b_i \cdot \nabla_y).$$

It is well known (see [14]) that if $\mathrm{div}_y b_i \in L^\infty(\mathbb{R}^m)$, then

$$\mathrm{dom}(b_i \cdot \nabla_y) = \{u \in L^2(\mathbb{R}^m) \ : \ \exists C > 0 \text{ such that } \|\tau_i(s)u - u\|_{L^2(\mathbb{R}^m)} \le C|s| \text{ for any } s \in \mathbb{R}\}$$
$$= \{u \in L^2 \ : \ \exists v_i \in L^2 \text{ such that } \int_{\mathbb{R}^m} [u(b_i \cdot \nabla_y \theta + \mathrm{div}_y b_i \ \theta) + v_i \theta] \ \mathrm{d}y = 0, \forall \theta \in C_c^1(\mathbb{R}^m)\},$$

and $b_i \cdot \nabla_y u = v_i, u \in \mathrm{dom}(b_i \cdot \nabla_y)$. The hypothesis $\mathrm{div}_y b_i \in L^\infty(\mathbb{R}^m)$ plays a crucial role when establishing estimates like $\|\tau_i(s)u - u\|_{L^2(\mathbb{R}^m)} \le \exp(|s| \ \|\mathrm{div}_y b_i\|_\infty)\|v_i\|_{L^2(\mathbb{R}^m)}$.
We introduce the Hilbert space

$$H_R^1 = \cap_{i=1}^m \mathrm{dom}(b_i \cdot \nabla_y) = \{u \in L^2(\mathbb{R}^m) \ : \ {}^t R^{-1} \nabla_y u := {}^t(b_1 \cdot \nabla_y u, ..., b_m \cdot \nabla_y u) \in L^2(\mathbb{R}^m)^m\},$$

with the scalar product

$$(u, v)_R = \int_{\mathbb{R}^m} u(y)v(y) \ \mathrm{d}y + \sum_{i=1}^m \int_{\mathbb{R}^m} (b_i \cdot \nabla_y u)(b_i \cdot \nabla_y v) \ \mathrm{d}y, \ u, v \in H_R^1.$$

We denote by $|\cdot|_R$ the induced norm. It is easily seen that any $u \in H_R^1$ has a weak gradient, that is, there is $V = V(y) \in (L^2_{\mathrm{loc}}(\mathbb{R}^m))^m$ such that

$$\int_{\mathbb{R}^m} V(y) \cdot \xi(y) \ \mathrm{d}y + \int_{\mathbb{R}^m} u(y) \mathrm{div}_y \xi(y) \ \mathrm{d}y = 0, \tag{40}$$

for any smooth vector field $\xi \in (C_c^1(\mathbb{R}^m))^m$. The weak gradient comes by solving, at any point $y \in \mathbb{R}^m$, the linear system

$$b_1 \cdot \nabla_y u = b_1(y) \cdot V(y), ..., b_m \cdot \nabla_y u = b_m(y) \cdot V(y).$$

This system also writes

$${}^t R^{-1}(y) V(y) = {}^t(b_1 \cdot \nabla_y u, ..., b_m \cdot \nabla_y u),$$

and its unique solution is $V(y) = {}^t R \ {}^t(b_1 \cdot \nabla_y u, ..., b_m \cdot \nabla_y u), y \in \mathbb{R}^m$. The equality (40) comes immediately, for any $\xi \in (C_c^1(\mathbb{R}^m))^m$, using the dual basis $\{c_1, ..., c_m\}$ of $\{b_1, ..., b_m\}$

$$\int_{\mathbb{R}^m} V(y) \cdot \xi(y) \ \mathrm{d}y = \int_{\mathbb{R}^m} \sum_{i=1}^m (V(y) \cdot b_i(y)) \ (c_i(y) \cdot \xi(y)) \ \mathrm{d}y$$
$$= - \int_{\mathbb{R}^m} u(y) \mathrm{div}_y \left(\sum_{i=1}^m (c_i(y) \cdot \xi(y)) b_i(y) \right) \ \mathrm{d}y = - \int_{\mathbb{R}^m} u(y) \mathrm{div}_y \xi \ \mathrm{d}y.$$

The norm of H_R^1 can be written using the X_P norm of the gradient

$$\begin{aligned} |u|_R^2 &= \|u\|^2_{L^2(\mathbb{R}^m)} + \|{}^t R^{-1} \nabla_y u\|^2_{L^2(\mathbb{R}^m)} \\ &= \int_{\mathbb{R}^m} \{(u(y))^2 + (R^{-1} \ {}^t R^{-1} V \cdot V)\} \ \mathrm{d}y \\ &= \int_{\mathbb{R}^m} \{(u(y))^2 + (P(y)V(y) \cdot V(y))\} \ \mathrm{d}y \\ &= \|u\|^2_{L^2(\mathbb{R}^m)} + |V|_P^2. \end{aligned} \tag{41}$$

We decompose the vector fields $[a(t,\cdot), b_i]$ with respect to the basis $(b_j)_{1\leq j\leq m}$

$$[a(t), b_i] = \sum_{j=1}^{m} \alpha_i^j(t,y) b_j(y), \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m. \tag{42}$$

Proposition 4.1 *Assume that the vector fields $a(t,y), b(y)$ satisfy* (9),(10),(11),(12) *and that the coefficients $(\alpha_i^j(t,y))_{1\leq i,j\leq m}$ in* (42) *belong to $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}^m))$. If the initial condition u^{in} belongs to H^1_R, then the family $(u^\varepsilon(t))_{t\in\mathbb{R}_+, \varepsilon>0}$ remains in H^1_R and for any $T > 0$, there is a constant $C(T, (\alpha_i^j)_{i,j})$ such that*

$$|u^\varepsilon(t,\cdot)|_R \leq C(T, (\alpha_i^j)_{i,j}) |u^{\mathrm{in}}|_R, \quad t \in [0,T], \varepsilon > 0,$$

where $(u^\varepsilon)_\varepsilon$ are the solutions of (7), (8).

Proof. The idea is to take the derivatives along the vector fields $(b_i)_{1\leq i\leq m}$

$$\partial_t(b_i\cdot\nabla_y u^\varepsilon) + a(t,y)\cdot\nabla_y(b_i\cdot\nabla_y u^\varepsilon) + \frac{1}{\varepsilon} b\cdot\nabla_y(b_i\cdot\nabla_y u^\varepsilon) = [a, b_i]\cdot\nabla_y u^\varepsilon = \sum_{j=1}^{m} \alpha_i^j(t,y) b_j\cdot\nabla_y u^\varepsilon. \tag{43}$$

Observe that the only singular part is $\frac{1}{\varepsilon} b\cdot\nabla_y(b_i\cdot\nabla_y u^\varepsilon)$. As usual, we will get rid of it, after multiplication by $b_i\cdot\nabla_y u^\varepsilon$ and using the anti-symmetry of $b\cdot\nabla_y$. If the fields $(b_i)_{1\leq i\leq m}$ were not in involution with b, an additional singular term appears in (43), $\frac{1}{\varepsilon}[b, b_i]\cdot\nabla_y u^\varepsilon$, and there is not possible to obtain uniform estimates with respect to $\varepsilon > 0$. After standard computations one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b_i\cdot\nabla_y u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \leq \sum_{j=1}^{m} \|\alpha_i^j(t)\|_{L^\infty} \|b_j\cdot\nabla_y u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}, \quad 1\leq i\leq m,$$

and therefore

$$\begin{aligned}
\sum_{i=1}^{m} \|b_i\cdot\nabla_y u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} &\leq \sum_{i=1}^{m} \|b_i\cdot\nabla_y u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)} + \int_0^t \sum_{j=1}^{m} \|b_j\cdot\nabla_y u^\varepsilon(\tau)\|_{L^2(\mathbb{R}^m)} \sum_{i=1}^{m} \|\alpha_i^j(\tau)\|_{L^\infty} \mathrm{d}\tau \\
&\leq \sum_{i=1}^{m} \|b_i\cdot\nabla_y u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)} + \int_0^t \left(\max_{1\leq j\leq m} \sum_{i=1}^{m} \|\alpha_i^j(\tau)\|_{L^\infty}\right) \\
&\quad \times \sum_{j=1}^{m} \|b_j\cdot\nabla_y u^\varepsilon(\tau)\|_{L^2(\mathbb{R}^m)} \mathrm{d}\tau.
\end{aligned}$$

By Gronwall's lemma we deduce that

$$\begin{aligned}
\sum_{i=1}^{m} \|b_i\cdot\nabla_y u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} &\leq \sum_{i=1}^{m} \|b_i\cdot\nabla_y u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)} \exp\left(\int_0^t \max_{1\leq j\leq m} \sum_{i=1}^{m} \|\alpha_i^j(\tau)\|_{L^\infty} \mathrm{d}\tau\right) \\
&\leq \widetilde{C}(T, (\alpha_i^j)_{i,j}) \sum_{i=1}^{m} \|b_i\cdot\nabla_y u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)}, \quad t\in[0,T], \varepsilon > 0.
\end{aligned}$$

It remains to observe that, thanks to (41), we can write

$$\begin{aligned}
|u^\varepsilon(t)|_R^2 &= \|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} + |\nabla_y u^\varepsilon(t)|_P^2 \\
&\leq \|u^{\mathrm{in}}\|^2_{L^2(\mathbb{R}^m)} + m^2 \widetilde{C}^2(T, (\alpha_i^j)_{i,j}) |\nabla_y u^{\mathrm{in}}|_P^2 \\
&\leq C^2(T, (\alpha_i^j)_{i,j}) |u^{\mathrm{in}}|_R^2, \quad t\in[0,T], \varepsilon > 0, C(T, (\alpha_i^j)_{i,j}) = \max\{1, m\widetilde{C}(T, (\alpha_i^j)_{i,j})\}.
\end{aligned}$$

□

Recall that we have introduced the functions

$$v^\varepsilon(t,z) = u^\varepsilon(t, Y(t/\varepsilon; z)),\ \ (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m, \varepsilon > 0.$$

Since the flow $Y(s;y)$ is measure preserving, the L^2 norms of u^ε and v^ε coincide

$$\int_{\mathbb{R}^m} (v^\varepsilon(t,z))^2 \,\mathrm{d}z = \int_{\mathbb{R}^m} (u^\varepsilon(t,y))^2 \,\mathrm{d}y,\ \ t \in \mathbb{R}_+, \varepsilon > 0.$$

The $|\cdot|_P$ norms of the gradients $\nabla_y u^\varepsilon, \nabla_z v^\varepsilon$ coincide as well, thanks to the involution between b_i and b, $i \in \{1,...,m\}$. Indeed, we have

$$\begin{aligned} b_i(z) \cdot \nabla_z v^\varepsilon(t,z) &= b_i(z) \cdot {}^t\partial_y Y(t/\varepsilon; z)\, \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) \\ &= \partial_y Y(t/\varepsilon; z)\, b_i(z) \cdot \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) \\ &= b_i(Y(t/\varepsilon; z)) \cdot \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)), \end{aligned}$$

and therefore

$$|\nabla_z v^\varepsilon(t)|_P^2 = \sum_{i=1}^m \int_{\mathbb{R}^m} (b_i(z) \cdot \nabla_z v^\varepsilon(t,z))^2 \,\mathrm{d}z = \sum_{i=1}^m \int_{\mathbb{R}^m} (b_i(y) \cdot \nabla_y u^\varepsilon(t,y))^2 \,\mathrm{d}y = |\nabla_y u^\varepsilon(t)|_P^2.$$

Finally the family $(v^\varepsilon)_{\varepsilon>0}$ is stable in H^1_R, locally uniformly in time

$$\begin{aligned} |v^\varepsilon(t)|_R^2 = \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} + |\nabla_z v^\varepsilon(t)|_P^2 &= \|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^m)} + |\nabla_y u^\varepsilon(t)|_P^2 \\ &\le (C(T, (\alpha_i^j)_{i,j}))^2\, |u^{\mathrm{in}}|_R^2,\ \ t \in [0,T], \varepsilon > 0. \end{aligned}$$

The family $(u^\varepsilon)_{\varepsilon>0}$ presents fast time oscillations, when ε becomes small. The definition of the family $(v^\varepsilon)_{\varepsilon>0}$ was designed in order to filter out these time oscillations. In the next proposition we establish uniform estimates for the time derivatives of $(v^\varepsilon)_{\varepsilon>0}$.

Proposition 4.2 *Assume that the vector fields $a(t,y), b(y)$ satisfy (9), (10), (11), (12). For any $\varepsilon > 0$, we consider $v^\varepsilon(t,z) = u^\varepsilon(t, Y(t/\varepsilon; z)), (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m$, where $(u^\varepsilon)_\varepsilon$ are the solutions (by characteristics) of (7), (8), with $u^{\mathrm{in}} \in L^2(\mathbb{R}^m)$. Then $(v^\varepsilon)_\varepsilon$, which belong to $C(\mathbb{R}_+; L^2(\mathbb{R}^m))$, are weak solutions for*

$$\partial_t v^\varepsilon + \partial_y Y\left(-t/\varepsilon; Y\left(t/\varepsilon; z\right)\right) a(t, Y(t/\varepsilon; z)) \cdot \nabla_z v^\varepsilon(t,z) = 0,\ \ (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m, \tag{44}$$

$$v^\varepsilon(0,z) = u^{\mathrm{in}}(z),\ \ z \in \mathbb{R}^m. \tag{45}$$

If the initial condition belongs to H^1_R, then for any $T > 0$

$$\sup_{t\in[0,T],\varepsilon>0} |v^\varepsilon(t)|_R \le C(T, (\alpha_i^j)_{i,j}) |u^{\mathrm{in}}|_R.$$

Moreover, if $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; X^\infty_Q)$, then $\partial_t v^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, that is, there is a function, denoted $\partial_t v^\varepsilon$, which belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, such that for any $\theta \in L^2(\mathbb{R}^m)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} v^\varepsilon(t,z)\theta(z) \,\mathrm{d}z = \int_{\mathbb{R}^m} \partial_t v^\varepsilon(t,z)\theta(z) \,\mathrm{d}z \ \ \textit{in } \mathcal{D}'(\mathbb{R}_+),$$

and for any $T > 0$ we have

$$\sup_{\varepsilon>0} \|\partial_t v^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \le C(T, (\alpha_i^j)_{i,j}) |\nabla_y u^{\mathrm{in}}|_P \|a\|_{L^\infty([0,T];X^\infty_Q)}.$$

Proof. The fact that $(v^\varepsilon)_\varepsilon$ verify the weak formulations of (44), (45) is trivial : use the weak formulations of (7), (8) with the test functions $\theta(t, Y(-t/\varepsilon; y))$, where $\theta \in C^1_c(\mathbb{R}_+ \times \mathbb{R}^m)$. Recall that $\mathrm{div}\{\varphi(t/\varepsilon)a(t,\cdot)\} = 0$, cf. Proposition 3.1, and thus (44) is equivalent to its conservative form. It remains to establish the uniform estimates for the time derivatives of $(v^\varepsilon)_\varepsilon$. Assume now that $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; X^\infty_Q)$. Thanks to Remark 3.1 we know that

$$\partial_y Y(-t/\varepsilon; Y(t/\varepsilon;\cdot))a(t, Y(t/\varepsilon;\cdot)) = \varphi(t/\varepsilon)a(t) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; X^\infty_Q).$$

For any $\eta \in C^1_c(\mathbb{R}_+)$ and $\theta \in C^1_c(\mathbb{R}^m)$, we have by the weak formulation of (44), (45)

$$-\eta(0)\int_{\mathbb{R}^m} u^{\mathrm{in}}(z)\theta(z)\,\mathrm{d}z - \int_{\mathbb{R}_+}\eta'(t)\int_{\mathbb{R}^m} v^\varepsilon\theta(z)\,\mathrm{d}z\mathrm{d}t - \int_{\mathbb{R}_+}\eta(t)\int_{\mathbb{R}^m} v^\varepsilon\varphi(t/\varepsilon)a(t)\cdot\nabla_z\theta\,\mathrm{d}z\mathrm{d}t = 0. \tag{46}$$

Since any function in H^1_R has a weak gradient, we obtain cf. (40)

$$\begin{aligned}-\int_{\mathbb{R}^m} v^\varepsilon(t,z)(\varphi(t/\varepsilon)a(t,\cdot))(z)\cdot\nabla_z\theta(z)\,\mathrm{d}z &= -\int_{\mathbb{R}^m} v^\varepsilon(t,z)\mathrm{div}_z\{\theta(z)(\varphi(t/\varepsilon)a(t,\cdot))(z)\}\,\mathrm{d}z\\ &= \int_{\mathbb{R}^m}(\varphi(t/\varepsilon)a(t,\cdot))(z)\cdot\nabla_z v^\varepsilon\,\theta(z)\,\mathrm{d}z,\end{aligned}$$

and (46) becomes

$$\begin{aligned}-\eta(0)\int_{\mathbb{R}^m} u^{\mathrm{in}}(z)\theta(z)\,\mathrm{d}z - \int_{\mathbb{R}_+}\eta'(t)\int_{\mathbb{R}^m} v^\varepsilon(t,z)\theta(z)\,\mathrm{d}z\,\mathrm{d}t&\\ + \int_{\mathbb{R}_+}\eta(t)\int_{\mathbb{R}^m}\theta(z)\nabla_z v^\varepsilon(t,z)\cdot(\varphi(t/\varepsilon)a(t,\cdot))(z)\,\mathrm{d}z\,\mathrm{d}t = 0.&\end{aligned} \tag{47}$$

Observe that $\theta\varphi(t/\varepsilon)a(t,\cdot) \in X_Q$, for any $t \in \mathbb{R}_+$ and

$$\begin{aligned}|\theta\varphi(t/\varepsilon)a(t,\cdot)|_Q &= \left(\int_{\mathbb{R}^m}(\theta(z))^2 Q(z)(\varphi(t/\varepsilon)a(t,\cdot))(z)\cdot(\varphi(t/\varepsilon)a(t,\cdot))(z)\,\mathrm{d}z\right)^{1/2}\\ &\le \|\theta\|_{L^2(\mathbb{R}^m)}|\varphi(t/\varepsilon)a(t,\cdot)|_{X^\infty_Q}\\ &= \|\theta\|_{L^2(\mathbb{R}^m)}|a(t,\cdot)|_{X^\infty_Q}.\end{aligned}$$

Therefore, the space integral in the last term of (47) can be written as a duality pairing $\langle\cdot,\cdot\rangle_{P,Q}$ and we obtain for any $T > 0, t \in [0,T], \varepsilon > 0$

$$\begin{aligned}\left|\int_{\mathbb{R}^m}\theta(z)\nabla_z v^\varepsilon(t,z)\cdot(\varphi(t/\varepsilon)a(t,\cdot))(z)\,\mathrm{d}z\right| &= \left|\langle\nabla_z v^\varepsilon(t,\cdot), \theta\varphi(t/\varepsilon)a(t,\cdot)\rangle_{P,Q}\right|\\ &\le |\nabla_z v^\varepsilon(t,\cdot)|_P\ |\theta\varphi(t/\varepsilon)a(t,\cdot)|_Q\\ &\le C(T,(\alpha_i^j)_{i,j})|\nabla_y u^{\mathrm{in}}|_P\ |a(t,\cdot)|_{X^\infty_Q}\ \|\theta\|_{L^2(\mathbb{R}^m)}.\end{aligned}$$

It is easily seen that (47) holds true for any test function $\theta \in L^2(\mathbb{R}^m)$ and finally we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} v^\varepsilon(t,z)\theta(z)\,\mathrm{d}z = -\int_{\mathbb{R}^m}\nabla_z v^\varepsilon(t,z)\cdot(\varphi(t/\varepsilon)a(t,\cdot))(z)\theta(z)\,\mathrm{d}z \ \text{ in } \mathcal{D}'(\mathbb{R}_+).$$

The above equality says that $\partial_t v^\varepsilon = -\nabla_z v^\varepsilon\cdot\varphi(t/\varepsilon)a(t,\cdot)$, and by the previous computations we have for any $T > 0$

$$\|\partial_t v^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \le C(T,(\alpha_i^j)_{i,j})\ |\nabla_y u^{\mathrm{in}}|_P\ \|a\|_{L^\infty([0,T];X^\infty_Q)},\ \ \varepsilon > 0.$$

□

5 Multi-scale analysis

We intend to prove that the family $(v^\varepsilon)_{\varepsilon>0}$ converges, when $\varepsilon \searrow 0$. For that, we need to pass to the limit, when $\varepsilon \searrow 0$, in the weak formulation of (44), (45). The most difficult term to handle is

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v^\varepsilon(t,z)(\varphi(t/\varepsilon)a(t,\cdot))(z)\cdot\nabla_z\xi(t,z)\,\mathrm{d}z\,\mathrm{d}t,\ \ \xi\in C_c^1(\mathbb{R}_+\times\mathbb{R}^m), \tag{48}$$

since we deal with two time variables : the slow time variable t and the fast time variable $s=\frac{t}{\varepsilon}$. If the dependency with respect to the fast time variable was periodic, then classical arguments from the homogenization theory would apply : we could separate the time scales, *i.e.,* we can freeze the slow time variable, and average with respect to the fast time variable over one period. We will see that similar results occur in the general case (that is, not necessarily periodic), provided that we replace the average over one period by the ergodic mean. Up to our knowledge these results have not been reported yet and we detail them here. A very easy example is the following.

Proposition 5.1 *Consider $c\in L^\infty(\mathbb{R}_+;X_Q), d\in L^1(\mathbb{R}_+;X_P)$ such that the family of means $\left(\frac{1}{S}\int_{s_0}^{s_0+S}c(s)\mathrm{d}s\right)_{S>0}$ converges strongly in X_Q, toward some $\overline{c}\in X_Q$, uniformly with respect to $s_0\in\mathbb{R}_+$, when $S\to+\infty$. Then we have the convergence*

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}_+}\langle d(t),c(t/\varepsilon)\rangle_{P,Q}\ \mathrm{d}t=\int_{\mathbb{R}_+}\langle d(t),\overline{c}\rangle_{P,Q}\ \mathrm{d}t.$$

Proof. First of all let us clarify our hypotheses. Saying that $c\in L^\infty(\mathbb{R}_+;X_Q)$ means that for any $w\in X_Q$, the function $s\to(w,c(s))_Q=\langle Qw,c(s)\rangle_{P,Q}$ is measurable on $\mathbb{R}_+$ and $|c(\cdot)|_Q\in L^\infty(\mathbb{R}_+)$. Similarly, $d\in L^1(\mathbb{R}_+;X_P)$ means that for any $z\in X_P$, the function $s\to(d(s),z)_P=\langle d(s),Pz\rangle_{P,Q}$ is measurable on $\mathbb{R}_+$ and $|d(\cdot)|\in L^1(\mathbb{R}_+)$. The integral $\int_{s_0}^{s_0+S}c(s)\mathrm{d}s$ stands for the unique element in X_Q such that

$$\left(\int_{s_0}^{s_0+S}c(s)\mathrm{d}s,w\right)_Q=\int_{s_0}^{s_0+S}(c(s),w)_Q\ \mathrm{d}s,\ \ \forall w\in X_Q.$$

Notice that

$$\left|\frac{1}{S}\int_{s_0}^{s_0+S}c(s)\mathrm{d}s\right|_Q=\frac{1}{S}\sup_{w\neq0,w\in X_Q}\frac{\left|\int_{s_0}^{s_0+S}(c(s),w)_Q\ \mathrm{d}s\right|}{|w|_Q}\leq\frac{1}{S}\int_{s_0}^{s_0+S}|c(s)|_Q\mathrm{d}s\leq\|c\|_{L^\infty(\mathbb{R}_+;X_Q)},$$

implying also that $|\overline{c}|_Q\leq\|c\|_{L^\infty(\mathbb{R}_+;X_Q)}$. We consider first $d\in C_c(\mathbb{R}_+;X_P)$, with supp $d\subset[0,l]$. For any $\delta>0$, there is $S_\delta>0$ such that

$$\left|\frac{1}{S}\int_{s_0}^{s_0+S}c(s)\ \mathrm{d}s-\overline{c}\right|_Q<\delta,\ \text{ for any } S\geq S_\delta \text{ and } s_0\in\mathbb{R}_+.$$

Performing the change of variable $s=\frac{t}{\varepsilon}$, the above condition writes

$$\left|\frac{1}{T}\int_{t_0}^{t_0+T}c(t/\varepsilon)\ \mathrm{d}t-\overline{c}\right|_Q<\delta,\ \text{ for any } T\geq\varepsilon S_\delta=T_{\delta,\varepsilon}\ \text{ and } t_0\in\mathbb{R}_+. \tag{49}$$

Since d has compact support in $\mathbb{R}_+$, the integral of $t\to\langle d(t),c(t/\varepsilon)\rangle_{P,Q}$ over $\mathbb{R}_+$ reduces to the integral of the same function over a finite number of intervals

$$[kT_{\delta,\varepsilon},(k+1)T_{\delta,\varepsilon}[,\ \ 0\leq k\leq k_{\delta,\varepsilon}:=\left[\frac{l}{T_{\delta,\varepsilon}}\right].$$

Therefore we can write

$$
\begin{aligned}
\left| \int_{\mathbb{R}_+} \langle d(t), c(t/\varepsilon) \rangle_{P,Q} \ \mathrm{d}t - \int_{\mathbb{R}_+} \langle d(t), \overline{c} \rangle_{P,Q} \ \mathrm{d}t \right| &= \left| \int_{\mathbb{R}_+} \langle d(t), c(t/\varepsilon) - \overline{c} \rangle_{P,Q} \ \mathrm{d}t \right| \qquad (50)\\
&\leq \sum_{k=0}^{k_{\delta,\varepsilon}} \left| \int_{kT_{\delta,\varepsilon}}^{(k+1)T_{\delta,\varepsilon}} \langle d(t), c(t/\varepsilon) - \overline{c} \rangle_{P,Q} \, \mathrm{d}t \right| \\
&\leq \sum_{k=0}^{k_{\delta,\varepsilon}} \left| \int_{kT_{\delta,\varepsilon}}^{(k+1)T_{\delta,\varepsilon}} \langle d(t) - d(kT_{\delta,\varepsilon}), c(t/\varepsilon) - \overline{c} \rangle_{P,Q} \, \mathrm{d}t \right| \\
&+ \sum_{k=0}^{k_{\delta,\varepsilon}} \left| \int_{kT_{\delta,\varepsilon}}^{(k+1)T_{\delta,\varepsilon}} \langle d(kT_{\delta,\varepsilon}), c(t/\varepsilon) - \overline{c} \rangle_{P,Q} \ \mathrm{d}t \right| \\
&= \Sigma_1 + \Sigma_2.
\end{aligned}
$$

For the estimate of Σ_1 we use the uniform continuity of d. We introduce the function

$$
\omega : \mathbb{R}_+ \to \mathbb{R}_+, \ \ \omega(\alpha) = \sup_{t,t' \in \mathbb{R}_+, |t-t'| \leq \alpha} |d(t) - d(t')|_P, \ \ \alpha \in \mathbb{R}_+.
$$

The function ω is non decreasing and satisfies $\lim_{\alpha \searrow 0} \omega(\alpha) = 0$. Thus we obtain the estimate

$$
\begin{aligned}
\Sigma_1 &\leq \sum_{k=0}^{k_{\delta,\varepsilon}} \int_{kT_{\delta,\varepsilon}}^{(k+1)T_{\delta,\varepsilon}} \omega(|t - kT_{\delta,\varepsilon}|) \ |c(t/\varepsilon) - \overline{c}|_Q \ \mathrm{d}t \qquad (51)\\
&\leq \sum_{k=0}^{k_{\delta,\varepsilon}} \omega(T_{\delta,\varepsilon}) T_{\delta,\varepsilon} \ 2\|c\|_{L^\infty(\mathbb{R}_+;X_Q)} \\
&= (k_{\delta,\varepsilon} + 1) T_{\delta,\varepsilon} \omega(T_{\delta,\varepsilon}) 2\|c\|_{L^\infty(\mathbb{R}_+;X_Q)} \\
&\leq 2\|c\|_{L^\infty(\mathbb{R}_+;X_Q)} \omega(T_{\delta,\varepsilon})(l + T_{\delta,\varepsilon}).
\end{aligned}
$$

The estimate for Σ_2 comes by using (49)

$$
\begin{aligned}
\Sigma_2 &= \sum_{k=0}^{k_{\delta,\varepsilon}} \left| \int_{kT_{\delta,\varepsilon}}^{(k+1)T_{\delta,\varepsilon}} (Pd(kT_{\delta,\varepsilon}), c(t/\varepsilon) - \overline{c})_Q \ \mathrm{d}t \right| \qquad (52)\\
&= \sum_{k=0}^{k_{\delta,\varepsilon}} \left| \left(Pd(kT_{\delta,\varepsilon}), \int_{kT_{\delta,\varepsilon}}^{(k+1)T_{\delta,\varepsilon}} (c(t/\varepsilon) - \overline{c}) \ \mathrm{d}t \right)_Q \right| \\
&= \sum_{k=0}^{k_{\delta,\varepsilon}} \left| \left\langle d(kT_{\delta,\varepsilon}), \int_{kT_{\delta,\varepsilon}}^{(k+1)T_{\delta,\varepsilon}} (c(t/\varepsilon) - \overline{c}) \ \mathrm{d}t \right\rangle_{P,Q} \right| \\
&\leq \sum_{k=0}^{k_{\delta,\varepsilon}} \delta T_{\delta,\varepsilon} |d(kT_{\delta,\varepsilon})|_P \\
&\leq \delta \left[\int_{\mathbb{R}_+} |d(t)|_P \ \mathrm{d}t + \omega(T_{\delta,\varepsilon})(l + T_{\delta,\varepsilon}) \right].
\end{aligned}
$$

Putting together (50), (51), (52) yields

$$
\begin{aligned}
\left| \int_{\mathbb{R}_+} \langle d(t), c(t/\varepsilon) \rangle_{P,Q} \ \mathrm{d}t - \int_{\mathbb{R}_+} \langle d(t), \overline{c} \rangle_{P,Q} \ \mathrm{d}t \right| &\leq 2\|c\|_{L^\infty(\mathbb{R}_+;X_Q)} \omega(T_{\delta,\varepsilon})(l + T_{\delta,\varepsilon}) \\
&+ \delta \left[\|d\|_{L^1(\mathbb{R}_+;X_P)} + \omega(T_{\delta,\varepsilon})(l + T_{\delta,\varepsilon}) \right].
\end{aligned}
$$

We keep $\delta > 0$ fixed and we pass to the limit with respect to $\varepsilon > 0$. Observing that $\lim_{\varepsilon \searrow 0} T_{\delta,\varepsilon} = \lim_{\varepsilon \searrow 0} \varepsilon S_\delta = 0$, and $\lim_{\varepsilon \searrow 0} \omega(T_{\delta,\varepsilon}) = 0$, we deduce

$$\limsup_{\varepsilon \searrow 0} \left| \int_{\mathbb{R}_+} \langle d(t), c(t/\varepsilon) \rangle_{P,Q} \, \mathrm{d}t - \int_{\mathbb{R}_+} \langle d(t), \overline{c} \rangle_{P,Q} \, \mathrm{d}t \right| \leq \delta \|d\|_{L^1(\mathbb{R}_+;X_P)}, \quad \text{for any } \delta > 0.$$

Letting now $\delta \searrow 0$, we obtain

$$\limsup_{\varepsilon \searrow 0} \left| \int_{\mathbb{R}_+} \langle d(t), c(t/\varepsilon) \rangle_{P,Q} \, \mathrm{d}t - \int_{\mathbb{R}_+} \langle d(t), \overline{c} \rangle_{P,Q} \, \mathrm{d}t \right| = 0,$$

and therefore the conclusion holds, for any $d \in C_c(\mathbb{R}_+; X_P)$. The general case follows by density arguments. $\square$

Remark 5.1 *The Proposition 5.1 applies in particular for time periodic $c \in L^\infty(\mathbb{R}_+; X_Q)$, with $\overline{c}$ given by the average of c over one period.*

Remark 5.2 *The proof of Proposition 5.1 shows a little bit more. Assume that $c \in L^\infty(\mathbb{R}_+; X_Q)$ such that the family of means $\left(\frac{1}{S} \int_{s_0}^{s_0+S} c(s) \mathrm{d}s \right)_{S>0}$ converges strongly in X_Q, toward some $\overline{c} \in X_Q$, uniformly with respect to $s_0 \in \mathbb{R}_+$, when $S \to +\infty$ and that for some $T \in \mathbb{R}_+$ the function d remains into a bounded set $\mathcal{A}$ in $L^1([0,T]; X_P)$, of functions which admit as modulus of continuity in $C([0,T]; X_P)$ the same function $\omega : [0,T] \to \mathbb{R}_+$. Then*

$$\lim_{\varepsilon \searrow 0} \left\{ \int_0^T \langle d(t), c(t/\varepsilon) \rangle_{P,Q} \, \mathrm{d}t - \int_0^T \langle d(t), \overline{c} \rangle_{P,Q} \, \mathrm{d}t \right\} = 0, \quad \textit{uniformly with respect to } d \in \mathcal{A}.$$

The previous result clearly emphasizes the scale separation and the crucial role of the ergodic mean hypothesis. Nevertheless this result will not be enough for analyzing the behavior of (48) as $\varepsilon \searrow 0$, since the second term of the duality pairing, $\varphi(t/\varepsilon) a(t, \cdot)$, depends on both slow and fast time variables. A more general result is given by the following proposition. All derivatives should be understood in the weak sense.

Proposition 5.2 *Consider the measurable function $c = c(t,s) : \mathbb{R}_+ \times \mathbb{R}_+ \to X_Q$ possessing derivative with respect to the slow time variable t such that the functions $t \to \sup_{s \in \mathbb{R}_+} |c(t,s)|_Q$, $t \to \sup_{s \in \mathbb{R}_+} |\partial_t c(t,s)|_Q$ belong to $L^1(\mathbb{R}_+)$. We suppose also that there is a function $\overline{c} : \mathbb{R}_+ \to X_Q$ such that for any $t \in \mathbb{R}_+$, we have*

$$\forall s_0 \in \mathbb{R}_+, \quad \left| \frac{1}{S} \int_{s_0}^{s_0+S} c(t,s) \, \mathrm{d}s - \overline{c}(t) \right|_Q = \left| \frac{1}{S} \int_0^S c(t,s) \, \mathrm{d}s - \overline{c}(t) \right|_Q \to 0, \quad \textit{when } S \to +\infty.$$

1. *Let $d = d(t) : \mathbb{R}_+ \to X_P$ be a measurable function such that $t \to |d(t)|_P, t \to |d'(t)|_P$ belong to $L^\infty(\mathbb{R}_+)$. Then we have the convergence*

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \langle d(t), c(t, t/\varepsilon) \rangle_{P,Q} \, \mathrm{d}t = \int_{\mathbb{R}_+} \langle d(t), \overline{c}(t) \rangle_{P,Q} \, \mathrm{d}t.$$

2. *Let $d^\varepsilon = d^\varepsilon(t) : \mathbb{R}_+ \to X_P$ be measurable functions such that*

$$\sup_{\varepsilon > 0} \|d^\varepsilon\|_{L^\infty(\mathbb{R}_+;X_P)} < +\infty, \sup_{\varepsilon > 0} \|d^{\varepsilon\prime}\|_{L^\infty(\mathbb{R}_+;X_P)} < +\infty.$$

   Then we have the convergence

$$\lim_{\varepsilon \searrow 0} \left[\int_{\mathbb{R}_+} \langle d^\varepsilon(t), c(t, t/\varepsilon) \rangle_{P,Q} \, \mathrm{d}t - \int_{\mathbb{R}_+} \langle d^\varepsilon(t), \overline{c}(t) \rangle_{P,Q} \, \mathrm{d}t \right] = 0.$$

Proof.
1. Observe that for any $\varepsilon > 0$ and $z \in X_Q$, the function $t \to (c(t,t/\varepsilon), z)_Q$ is measurable, saying that $t \to c(t,t/\varepsilon)$ is measurable as application from $\mathbb{R}_+$ to X_Q. Actually it is a $L^1(\mathbb{R}_+; X_Q)$ function, since

$$\int_{\mathbb{R}_+} |c(t,t/\varepsilon)|_Q \, \mathrm{d}t \le \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \, \mathrm{d}t < +\infty.$$

As d is a $L^\infty(\mathbb{R}_+; X_P)$ function, we deduce that the function $t \in \mathbb{R}_+ \to \langle d(t), c(t,t/\varepsilon)\rangle_{P,Q} \in \mathbb{R}$ is measurable and

$$\int_{\mathbb{R}_+} |\langle d(t), c(t,t/\varepsilon)\rangle_{P,Q}| \, \mathrm{d}t \le \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)| \, \mathrm{d}t.$$

Similarly, for any $S > 0, z \in X_Q$, the function $t \to \frac{1}{S}\int_0^S (c(t,s), z)_Q \mathrm{d}s$ is measurable and therefore the function $t \to \lim_{S\to+\infty} \frac{1}{S}\int_0^S (c(t,s), z)_Q \mathrm{d}s = (\overline{c}(t), z)_Q$ is measurable, saying that $t \to \overline{c}(t)$ is measurable from $\mathbb{R}_+$ to Q. Moreover, $\overline{c}$ is a $L^1(\mathbb{R}_+; X_Q)$ function, since

$$\int_{\mathbb{R}_+} |\overline{c}(t)|_Q \, \mathrm{d}t = \int_{\mathbb{R}_+} \left| \frac{1}{S} \int_0^S c(t,s)\, \mathrm{d}s \right|_Q \mathrm{d}t \le \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \, \mathrm{d}t.$$

Since $d \in L^\infty(\mathbb{R}_+; X_P)$, we deduce that $t \to \langle d(t), \overline{c}(t)\rangle_{P,Q}$ belongs to $L^1(\mathbb{R}_+;\mathbb{R})$ and therefore the conclusion makes sense. We adapt the arguments in the proof of Proposition 5.1. We claim that the following convergence holds true

$$\lim_{S\to+\infty} \int_{\mathbb{R}_+} \left| \frac{1}{S}\int_0^S c(t,s)\, \mathrm{d}s - \overline{c}(t) \right| \mathrm{d}t = 0. \tag{53}$$

Indeed, by the ergodic mean hypothesis, we have for any $t \in \mathbb{R}_+$

$$\lim_{S\to+\infty} \left| \frac{1}{S}\int_0^S c(t,s)\mathrm{d}s - \overline{c}(t) \right|_Q = 0.$$

Observe also that for any $t \in \mathbb{R}_+$ we have

$$\left| \frac{1}{S}\int_0^S c(t,s)\, \mathrm{d}s - \overline{c}(t) \right|_Q \le \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q + |\overline{c}(t)|_Q \le 2 \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q.$$

The function $t \to \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q$ being integrable on $\mathbb{R}_+$, we deduce (53), thanks to the dominated convergence theorem. We fix now $\delta > 0$ and let S_δ be such that

$$\int_{\mathbb{R}_+} \left| \frac{1}{S}\int_0^S c(t,s)\, \mathrm{d}s - \overline{c}(t) \right|_Q \mathrm{d}t \le \delta, \quad \text{for any } S \ge S_\delta.$$

For any measurable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ and $S \ge S_\delta$, we have

$$\begin{aligned}\left| \int_{\mathbb{R}_+} \left\langle d(t), \frac{1}{S}\int_{r(t)/\varepsilon}^{r(t)/\varepsilon+S} c(t,s)\, \mathrm{d}s - \overline{c}(t) \right\rangle_{P,Q} \mathrm{d}t \right| &\le \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \left| \frac{1}{S}\int_0^S c(t,s)\, \mathrm{d}s - \overline{c}(t) \right|_Q \mathrm{d}t \\ &\le \delta \|d\|_{L^\infty(\mathbb{R}_+;X_P)}.\end{aligned}$$

After the change of variable $s = \frac{\tau}{\varepsilon}$, we obtain

$$\left| \int_{\mathbb{R}_+} \left\langle d(t), \frac{1}{T}\int_{r(t)}^{r(t)+T} c(t,\tau/\varepsilon)\, \mathrm{d}\tau - \overline{c}(t) \right\rangle_{P,Q} \mathrm{d}t \right| \le \delta \|d\|_{L^\infty(\mathbb{R}_+;X_P)}, \quad \text{for any } T \ge T_{\delta,\varepsilon} := \varepsilon S_\delta.$$

In particular we have

$$\left| \int_{\mathbb{R}_+} \left\langle d(t), \frac{1}{T_{\delta,\varepsilon}} \int_{r(t)}^{r(t)+T_{\delta,\varepsilon}} \{c(t,\tau/\varepsilon) - \overline{c}(t)\} \, \mathrm{d}\tau \right\rangle_{P,Q} \mathrm{d}t \right| \le \delta \|d\|_{L^\infty(\mathbb{R}_+;X_P)}, \tag{54}$$

for any measurable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ and any $\varepsilon > 0$. We consider the uniform grid $(t_k)_{k\in\mathbb{N}}, t_k = kT_{\delta,\varepsilon}$ and approximate the integral $\int_{\mathbb{R}_+} \langle d(t), c(t,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t$ by a Riemann serie

$$\begin{aligned}
&\left| \int_{\mathbb{R}_+} \langle d(t), c(t,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t - \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} \langle d(t_k), c(t_k,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t \right| \qquad (55)\\
&\le \sum_{k\in\mathbb{N}} \left| \int_{t_k}^{t_{k+1}} \langle d(t) - d(t_k), c(t,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t \right| + \sum_{k\in\mathbb{N}} \left| \int_{t_k}^{t_{k+1}} \langle d(t_k), c(t,t/\varepsilon) - c(t_k,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t \right| \\
&= \sum_{k\in\mathbb{N}} \left| \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \langle d'(r), c(t,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}r\mathrm{d}t \right| + \sum_{k\in\mathbb{N}} \left| \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \langle d(t_k), \partial_t c(r,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}r\mathrm{d}t \right| \\
&\le \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} |d'(r)|_P |c(t,t/\varepsilon)|_Q \, \mathrm{d}r\mathrm{d}t + \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} |d(t_k)|_P |\partial_t c(r,t/\varepsilon)|_Q \mathrm{d}r\mathrm{d}t \\
&\le \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} (t-t_k) \|d'\|_{L^\infty(\mathbb{R}_+;X_P)} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \, \mathrm{d}t \\
&+ \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{t_k}^{t} \sup_{s\in\mathbb{R}_+} |\partial_t c(r,s)|_Q \, \mathrm{d}r\mathrm{d}t \\
&\le T_{\delta,\varepsilon} \|d'\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \, \mathrm{d}t + T_{\delta,\varepsilon} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_0^{+\infty} \sup_{s\in\mathbb{R}_+} |\partial_t c(r,s)| \, \mathrm{d}r.
\end{aligned}$$

The previous computations also show that the Riemann series converges. Indeed, exactly as before we obtain for any $N \le M$

$$\begin{aligned}
&\left| \int_{t_N}^{t_M} \langle d(t), c(t,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t - \sum_{k=N}^{M-1} \int_{t_k}^{t_{k+1}} \langle d(t_k), c(t_k,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t \right| \\
&\le T_{\delta,\varepsilon} \|d'\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{t_N}^{t_M} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \, \mathrm{d}t + T_{\delta,\varepsilon} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{t_N}^{t_M} \sup_{s\in\mathbb{R}_+} |\partial_t c(t,s)|_Q \, \mathrm{d}t,
\end{aligned}$$

and therefore $\sum_{k=N}^{M-1} \int_{t_k}^{t_{k+1}} \langle d(t_k), c(t_k,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t$ becomes small as $N, M \to +\infty$, thanks to the convergence of the integrals

$$\int_{\mathbb{R}_+} \langle d(t), c(t,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t, \quad \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \, \mathrm{d}t, \quad \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |\partial_t c(t,s)|_Q \, \mathrm{d}t.$$

Next, notice that we can write

$$\begin{aligned}
\Sigma_3 &:= \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} \langle d(t_k), c(t_k,t/\varepsilon)\rangle_{P,Q} \, \mathrm{d}t = \sum_{k\in\mathbb{N}} \left\langle d(t_k), \int_{t_k}^{t_{k+1}} c(t_k,t/\varepsilon) \, \mathrm{d}t \right\rangle_{P,Q} \qquad (56)\\
&= \sum_{k\in\mathbb{N}} T_{\delta,\varepsilon} \left\langle d(t_k), \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_{k+1}} \{c(t_k,t/\varepsilon) - \overline{c}(t_k)\} \, \mathrm{d}t \right\rangle_{P,Q} + \sum_{k\in\mathbb{N}} T_{\delta,\varepsilon} \langle d(t_k), \overline{c}(t_k)\rangle_{P,Q} \\
&=: \Sigma_4 + \Sigma_5.
\end{aligned}$$

We compare Σ_4 with the integral in (54). Indeed, the difference between the integral in (54) with the function $r(t) = t_k, t \in [t_k, t_{k+1}[, k \in \mathbb{N}$ and Σ_4 is estimated by

$$\left| \int_{\mathbb{R}_+} \left\langle d(t), \frac{1}{T_{\delta,\varepsilon}} \int_{r(t)}^{r(t)+T_{\delta,\varepsilon}} \{c(t, \tau/\varepsilon) - \overline{c}(t)\} \,\mathrm{d}\tau \right\rangle_{P,Q} \mathrm{d}t \right. \tag{57}$$

$$\left. - \sum_{k\in\mathbb{N}} T_{\delta,\varepsilon} \left\langle d(t_k), \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_k+T_{\delta,\varepsilon}} \{c(t_k, \tau/\varepsilon) - \overline{c}(t_k)\} \,\mathrm{d}\tau \right\rangle_{P,Q} \right|$$

$$\leq \sum_{k\in\mathbb{N}} \left| \int_{t_k}^{t_{k+1}} \left\langle d(t) - d(t_k), \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_k+T_{\delta,\varepsilon}} \{c(t, \tau/\varepsilon) - \overline{c}(t)\} \,\mathrm{d}\tau \right\rangle_{P,Q} \mathrm{d}t \right|$$

$$+ \sum_{k\in\mathbb{N}} \left| \int_{t_k}^{t_{k+1}} \left\langle d(t_k), \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_k+T_{\delta,\varepsilon}} c(t, \tau/\varepsilon) \,\mathrm{d}\tau - \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_k+T_{\delta,\varepsilon}} c(t_k, \tau/\varepsilon) \,\mathrm{d}\tau \right\rangle_{P,Q} \mathrm{d}t \right|$$

$$+ \sum_{k\in\mathbb{N}} \left| \int_{t_k}^{t_{k+1}} \langle d(t_k), -\overline{c}(t) + \overline{c}(t_k) \rangle_{P,Q} \,\mathrm{d}t \right|$$

$$=: \Sigma_6 + \Sigma_7 + \Sigma_8.$$

The estimate for Σ_6 comes easily by using the derivative of d, that is, $d(t) - d(t_k) = \int_{t_k}^t d\,'(r)\mathrm{d}r$

$$\Sigma_6 \leq \sum_{k\in\mathbb{N}} T_{\delta,\varepsilon} \|d\,'\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{t_k}^{t_{k+1}} \left| \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_k+T_{\delta,\varepsilon}} \{c(t, \tau/\varepsilon) - \overline{c}(t)\} \,\mathrm{d}\tau \right|_Q \mathrm{d}t \tag{58}$$

$$\leq 2T_{\delta,\varepsilon} \|d\,'\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \,\mathrm{d}t.$$

Similarly, by using the derivative of c (with respect to the slow variable), we have for any $k \in \mathbb{N}$

$$\left| \left\langle d(t_k), \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_k+T_{\delta,\varepsilon}} \{c(t, \tau/\varepsilon) - c(t_k, \tau/\varepsilon)\} \,\mathrm{d}\tau \right\rangle_{P,Q} \right|$$

$$= \left| \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_k+T_{\delta,\varepsilon}} \langle d(t_k), c(t, \tau/\varepsilon) - c(t_k, \tau/\varepsilon) \rangle_{P,Q} \,\mathrm{d}\tau \right|$$

$$= \left| \frac{1}{T_{\delta,\varepsilon}} \int_{t_k}^{t_k+T_{\delta,\varepsilon}} \int_{t_k}^{t} \langle d(t_k), \partial_t c(r, \tau/\varepsilon) \rangle_{P,Q} \,\mathrm{d}r\mathrm{d}\tau \right|$$

$$\leq \left| \int_{t_k}^{t_{k+1}} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \sup_{s\in\mathbb{R}_+} |\partial_t c(r,s)|_Q \,\mathrm{d}r \right|,$$

implying that

$$\Sigma_7 \leq \sum_{k\in\mathbb{N}} T_{\delta,\varepsilon} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{t_k}^{t_{k+1}} \sup_{s\in\mathbb{R}_+} |\partial_t c(r,s)|_Q \mathrm{d}r$$

$$= T_{\delta,\varepsilon} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |\partial_t c(r,s)|_Q \mathrm{d}r. \tag{59}$$

For estimating the last sum Σ_8, we write for any $S > 0$ and any $k \in \mathbb{N}, t \in [t_k, t_{k+1}[$

$$\left| \left\langle d(t_k), \frac{1}{S} \int_0^S \{c(t,s) - c(t_k, s)\} \,\mathrm{d}s \right\rangle_{P,Q} \right| = \left| \frac{1}{S} \int_0^S \int_{t_k}^t \langle d(t_k), \partial_t c(r,s) \rangle_{P,Q} \,\mathrm{d}r\mathrm{d}s \right|$$

$$\leq \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{t_k}^{t_{k+1}} \sup_{s\in\mathbb{R}_+} |\partial_t c(r,s)|_Q \,\mathrm{d}r.$$

Passing to the limit, when $S \to +\infty$, one gets for any $k \in \mathbb{N}$ and any $t \in [t_k, t_{k+1}[$

$$| \langle d(t_k), \overline{c}(t) - \overline{c}(t_k) \rangle_{P,Q} | \le \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{t_k}^{t_{k+1}} \sup_{s\in\mathbb{R}_+} |\partial_t c(r,s)|_Q \,\mathrm{d}r,$$

leading to

$$\Sigma_8 \le T_{\delta,\varepsilon} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |\partial_t c(r,s)|_Q \,\mathrm{d}r. \tag{60}$$

Putting together (54), (57), (58), (59), (60) we deduce that

$$\Sigma_4 \le \delta \|d\|_{L^\infty(\mathbb{R}_+;X_P)} + 2T_{\delta,\varepsilon} \|d\,'\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \,\mathrm{d}t \tag{61}$$
$$+ 2T_{\delta,\varepsilon} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |\partial_t c(t,s)|_Q \,\mathrm{d}t.$$

By similar computations we estimate $\int_{\mathbb{R}_+} \langle d(t), \overline{c}(t) \rangle_{P,Q} \,\mathrm{d}t - \Sigma_5$ and we find

$$\left| \int_{\mathbb{R}_+} \langle d(t), \overline{c}(t) \rangle_{P,Q} \,\mathrm{d}t - \Sigma_5 \right| \le T_{\delta,\varepsilon} \|d\,'\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \,\mathrm{d}t \tag{62}$$
$$+ T_{\delta,\varepsilon} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |\partial_t c(t,s)|_Q \,\mathrm{d}t.$$

Our conclusion follows by combining (55), (56), (61), (62). More exactly we obtain for any $\varepsilon, \delta > 0$

$$\left| \int_{\mathbb{R}_+} \langle d(t), c(t,t/\varepsilon) \rangle_{P,Q} \,\mathrm{d}t - \int_{\mathbb{R}_+} \langle d(t), \overline{c}(t) \rangle_{P,Q} \,\mathrm{d}t \right|$$
$$\le \left| \int_{\mathbb{R}_+} \langle d(t), c(t,t/\varepsilon) \rangle_{P,Q} \,\mathrm{d}t - \Sigma_3 \right| + \left| \Sigma_4 + \Sigma_5 - \int_{\mathbb{R}_+} \langle d(t), \overline{c}(t) \rangle_{P,Q} \,\mathrm{d}t \right|$$
$$\le \left| \int_{\mathbb{R}_+} \langle d(t), c(t,t/\varepsilon) \rangle_{P,Q} \,\mathrm{d}t - \Sigma_3 \right| + |\Sigma_4| + \left| \Sigma_5 - \int_{\mathbb{R}_+} \langle d(t), \overline{c}(t) \rangle_{P,Q} \,\mathrm{d}t \right|$$
$$\le \delta \|d\|_{L^\infty(\mathbb{R}_+;X_P)} + 4T_{\delta,\varepsilon} \|d\,'\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \,\mathrm{d}t$$
$$+ 4T_{\delta,\varepsilon} \|d\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |\partial_t c(t,s)|_Q \,\mathrm{d}t.$$

Passing first to the limit when $\varepsilon \searrow 0$, we deduce that for any $\delta > 0$

$$\limsup_{\varepsilon \searrow 0} \left| \int_{\mathbb{R}_+} \langle d(t), c(t,t/\varepsilon) \rangle_{P,Q} \,\mathrm{d}t - \int_{\mathbb{R}_+} \langle d(t), \overline{c}(t) \rangle_{P,Q} \,\mathrm{d}t \right| \le \delta \|d\|_{L^\infty(\mathbb{R}_+;X_P)},$$

saying that

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \langle d(t), c(t,t/\varepsilon) \rangle_{P,Q} \,\mathrm{d}t = \int_{\mathbb{R}_+} \langle d(t), \overline{c}(t) \rangle_{P,Q} \,\mathrm{d}t.$$

2. We follow exactly the same lines as before. The key point is that, once that $\delta > 0$ is fixed, S_δ is associated only to $c(\cdot,\cdot)$ and thus $T_{\delta,\varepsilon} = \varepsilon S_\delta$ will fit to any pair $(d^\varepsilon, c), \varepsilon > 0$. We obtain the estimate

$$\left| \int_{\mathbb{R}_+} \langle d^\varepsilon(t), c(t,t/\varepsilon) \rangle_{P,Q} \,\mathrm{d}t - \int_{\mathbb{R}_+} \langle d^\varepsilon(t), \overline{c}(t) \rangle_{P,Q} \,\mathrm{d}t \right| \le \delta \|d^\varepsilon\|_{L^\infty(\mathbb{R}_+;X_P)}$$
$$+ 4T_{\delta,\varepsilon} \|d^{\varepsilon\prime}\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q \,\mathrm{d}t$$
$$+ 4T_{\delta,\varepsilon} \|d^\varepsilon\|_{L^\infty(\mathbb{R}_+;X_P)} \int_{\mathbb{R}_+} \sup_{s\in\mathbb{R}_+} |\partial_t c(t,s)|_Q \,\mathrm{d}t,$$

and we obtain our conclusion passing first to the limit with respect to ε (thanks to the uniform estimates of $d^\varepsilon, d^{\varepsilon\prime}$ and secondly with respect to δ. □

Remark 5.3 *It is possible to formulate a local version of Proposition 5.2 with respect to the slow time variable. Consider $c = c(t,s)$ such that $t \to \sup_{s\in\mathbb{R}_+} |c(t,s)|_Q$, $t \to \sup_{s\in\mathbb{R}_+} |\partial_t c(t,s)|_Q$ belong to $L^1([0,T])$. We suppose that there is a function $\overline{c} : [0,T] \to X_Q$ such that for any $t \in [0,T]$, we have*

$$\forall s_0 \in \mathbb{R}_+, \quad \left| \frac{1}{S} \int_{s_0}^{s_0+S} c(t,s)\ \mathrm{d}s - \overline{c}(t) \right|_Q = \left| \frac{1}{S} \int_0^S c(t,s)\ \mathrm{d}s - \overline{c}(t) \right|_Q \to 0, \quad \textit{when } S \to +\infty.$$

1. *Let $d = d(t) : [0,T] \to X_P$ be a measurable function such that $t \to |d(t)|_P, t \to |d'(t)|_P$ belong to $L^\infty([0,T])$. Then we have the convergence*

$$\lim_{\varepsilon \searrow 0} \int_0^T \langle d(t), c(t,t/\varepsilon)\rangle_{P,Q}\ \mathrm{d}t = \int_0^T \langle d(t), \overline{c}(t)\rangle_{P,Q}\ \mathrm{d}t.$$

2. *Let $d^\varepsilon = d^\varepsilon(t) : [0,T] \to X_P$ be measurable functions such that*

$$\sup_{\varepsilon>0} \|d^\varepsilon\|_{L^\infty([0,T];X_P)} < +\infty, \sup_{\varepsilon>0} \|d^{\varepsilon\prime}\|_{L^\infty([0,T];X_P)} < +\infty.$$

   Then we have the convergence

$$\lim_{\varepsilon \searrow 0} \left[\int_0^T \langle d^\varepsilon(t), c(t,t/\varepsilon)\rangle_{P,Q}\ \mathrm{d}t - \int_0^T \langle d^\varepsilon(t), \overline{c}(t)\rangle_{P,Q}\ \mathrm{d}t \right] = 0.$$

6 The limit model

We are ready to establish the convergence results stated in Theorem 2.3, Theorem 2.4. The proofs essentialy appeal to the properties of the average vector field (see Theorem 2.1) combined to the two-scale approach in the ergodic setting (cf. Proposition 5.2).

Proof. (of Theorem 2.3)
Consider a sequence $(\varepsilon_k)_k$ converging to 0, such that $(v^{\varepsilon_k})_k$ converges weakly $\star$ in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, toward some function $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ (recall that $\|v^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} = \|u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)}, \varepsilon > 0$). Thus we have also the weak convergence $v^{\varepsilon_k} \rightharpoonup v$ in $L^2([0,T] \times \mathbb{R}^m)$, for any $T > 0$. We pick a smooth test function ξ, for example $\xi \in C^2_c(\mathbb{R}_+ \times \mathbb{R}^m)$. The weak formulation (44), (45) written for ξ yields

$$\begin{aligned} - \int_{\mathbb{R}^m} \xi(0,z) u^{\mathrm{in}}(z)\ \mathrm{d}z - \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} \partial_t \xi(t,z) v^{\varepsilon_k}(t,z)\ \mathrm{d}z\ \mathrm{d}t \\ - \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z) \nabla_z \xi(t,z) \cdot (\varphi(t/\varepsilon_k) a(t))(z)\ \mathrm{d}z\ \mathrm{d}t = 0. \end{aligned} \tag{63}$$

Since $\partial_t \xi \in L^1(\mathbb{R}_+; L^2(\mathbb{R}^m))$, the weak $\star$ convergence of $(v^{\varepsilon_k})_k$ implies

$$\lim_{k\to+\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} \partial_t \xi(t,z) v^{\varepsilon_k}(t,z)\ \mathrm{d}z\ \mathrm{d}t = \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} \partial_t \xi(t,z) v(t,z)\ \mathrm{d}z\ \mathrm{d}t.$$

For the convergence of the last integral in (63) we appeal to Proposition 5.2, see also Remark 5.3. We take $c(t,s) = \varphi(s) a(t)$, $d_k(t) = v^{\varepsilon_k}(t,\cdot) \nabla_z \xi(t,\cdot)$ and $\overline{c}(t) = \langle a(t) \rangle, t, s \in \mathbb{R}_+$. We fix

$T > 0$ such that supp $\xi \subset [0, T[\times \mathbb{R}^m$ and thus we can write

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi(t,z)\cdot\varphi(t/\varepsilon_k)a(t)\;\mathrm{d}z\;\mathrm{d}t = \int_0^T\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi(t,z)\cdot\varphi(t/\varepsilon_k)a(t)\;\mathrm{d}z\;\mathrm{d}t$$
$$= \int_0^T \langle d_k(t), c(t,t/\varepsilon_k)\rangle_{P,Q}\;\mathrm{d}t.$$

Clearly we have

$$\sup_{s\in\mathbb{R}_+}|c(\cdot,s)|_Q = \sup_{s\in\mathbb{R}_+}|\varphi(s)a(\cdot)|_Q = |a(\cdot)|_Q \in L^1([0,T]),$$

$$\sup_{s\in\mathbb{R}_+}|\partial_t c(\cdot,s)|_Q = \sup_{s\in\mathbb{R}_+}|\varphi(s)\partial_t a(\cdot)|_Q = |\partial_t a(\cdot)|_Q \in L^1([0,T]),$$

and by Theorem 2.1 we know that for any $s_0 \in \mathbb{R}$

$$\left|\frac{1}{S}\int_{s_0}^{s_0+S} c(t,s)\;\mathrm{d}s - \bar{c}(t)\right|_Q = \left|\frac{1}{S}\int_0^S c(t,s)\;\mathrm{d}s - \bar{c}(t)\right|_Q = \left|\frac{1}{S}\int_0^S \varphi(s)a(t)\;\mathrm{d}s - \langle a(t)\rangle\right|_Q \to 0, S\to+\infty.$$

The uniform estimates for $(d_k)_k$ in $L^\infty([0,T];X_P)$ come from the uniform estimates of $(v^{\varepsilon_k})_k$ in $L^\infty([0,T];L^2(\mathbb{R}^m))$

$$|d_k(t)|_P^2 = \int_{\mathbb{R}^m}(v^{\varepsilon_k}(t,z))^2(P(z)\nabla_z\xi(t,z)\cdot\nabla_z\xi(t,z))\;\mathrm{d}z$$
$$\leq \|(P\nabla_z\xi(t)\cdot\nabla_z\xi(t))\|_{L^\infty}\|v^{\varepsilon_k}(t)\|^2_{L^2(\mathbb{R}^m)}$$
$$\leq \|(P\nabla_z\xi(t)\cdot\nabla_z\xi(t))\|_{L^\infty}\|u^{\mathrm{in}}\|^2_{L^2(\mathbb{R}^m)}.$$

But $(P\nabla_z\xi\cdot\nabla_z\xi) = \sum_{i=1}^m(b_i\cdot\nabla_z\xi)^2 \in L^\infty([0,T]\times\mathbb{R}^m)$, since ξ has compact support and $(b_i)_{1\leq i\leq m}$ are locally bounded. Therefore the sequence $(d_k)_k$ is bounded in $L^\infty([0,T];X_P)$. The uniform estimates for $(d_k')_k$ in $L^\infty([0,T];X_P)$ come from the uniform estimates of $(\partial_t v^{\varepsilon_k})_k$ in $L^\infty([0,T];L^2(\mathbb{R}^m))$, see Proposition 4.2 and use the hypothesis $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;X_Q^\infty)$. Indeed, observe that

$$\|v^{\varepsilon_k}\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} = \|u^{\mathrm{in}}\|_{L^2(\mathbb{R}^m)}, \tag{64}$$

$$\|\partial_t v^{\varepsilon_k}\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \leq C(T,(\alpha_i^j)_{i,j})|\nabla_y u^{\mathrm{in}}|_P\|a\|_{L^\infty([0,T];X_Q^\infty)}. \tag{65}$$

Take now $w \in X_Q$ and notice that

$$\|\nabla_z\xi\cdot w\|_{L^1([0,T];L^2(\mathbb{R}^m))} \leq \|(P\nabla_z\xi\cdot\nabla_z\xi)^{1/2}\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}|w|_Q,$$

and

$$\|\nabla_z\partial_t\xi\cdot w\|_{L^1([0,T];L^2(\mathbb{R}^m))} \leq \|(P\nabla_z\partial_t\xi\cdot\nabla_z\partial_t\xi)^{1/2}\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}|w|_Q.$$

Therefore the derivative, in distribution sense, of the product function $t \to (v^{\varepsilon_k}(t), \nabla_z\xi(t,\cdot)\cdot w)_{L^2(\mathbb{R}^m)} = \int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi(t,z)\cdot w(z)\;\mathrm{d}z$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi(t,z)\cdot w(z)\;\mathrm{d}z = (\partial_t v^{\varepsilon_k}(t), \nabla_z\xi(t)\cdot w)_{L^2(\mathbb{R}^m)} + (v^{\varepsilon_k}(t), \nabla_z\partial_t\xi\cdot w)_{L^2(\mathbb{R}^m)}$$
$$= \langle\partial_t v^{\varepsilon_k}(t)\nabla_z\xi(t) + v^{\varepsilon_k}(t)\nabla_z\partial_t\xi(t), w\rangle_{P,Q}.$$

We deduce that

$$d_k'(t) = \partial_t\{v^{\varepsilon_k}(t)\nabla_z\xi(t)\} = \partial_t v^{\varepsilon_k}\nabla_z\xi(t) + v^{\varepsilon_k}(t)\nabla_z\partial_t\xi,$$

and the estimates in (64), (65) yield

$$\sup_{k\in\mathbb{N}} \|d_k'\|_{L^\infty([0,T];X_P)} \leq \|(P\nabla_z\xi\cdot\nabla_z\xi)^{1/2}\|_{L^\infty([0,T]\times\mathbb{R}^m)} \sup_{k\in\mathbb{N}}\|\partial_t v^{\varepsilon_k}\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}$$
$$+ \|(P\nabla_z\partial_t\xi\cdot\nabla_z\partial_t\xi)^{1/2}\|_{L^\infty([0,T]\times\mathbb{R}^m)} \sup_{k\in\mathbb{N}}\|v^{\varepsilon_k}\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}.$$

By Remark 5.3 we obtain the convergence

$$\lim_{k\to+\infty}\left\{\int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi\cdot\varphi(t/\varepsilon_k)a(t)\ \mathrm{d}z\ \mathrm{d}t - \int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi\cdot\langle a(t)\rangle\ \mathrm{d}z\ \mathrm{d}t\right\} \tag{66}$$
$$= \lim_{k\to+\infty}\left\{\int_0^T \langle d_k(t), c(t,t/\varepsilon_k)\rangle_{P,Q}\ \mathrm{d}t - \int_0^T \langle d_k(t),\overline{c}(t)\rangle_{P,Q}\ \mathrm{d}t\right\} = 0.$$

The limit of the sequence $(\int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi\cdot\langle a(t)\rangle\ \mathrm{d}z\ \mathrm{d}t)_k$ follows easily by the weak $\star$ convergence of $(v^{\varepsilon_k})_k$ in $L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))$. Indeed, the function $\nabla_z\xi\cdot\langle a(t)\rangle$ belongs to $L^1(\mathbb{R}_+;L^2(\mathbb{R}^m))$

$$\|\nabla_z\xi\cdot\langle a(t)\rangle\|_{L^1(\mathbb{R}_+;L^2(\mathbb{R}^m))} = \|\nabla_z\xi\cdot\langle a(t)\rangle\|_{L^1([0,T];L^2(\mathbb{R}^m))}$$
$$\leq \|(P\nabla_z\xi\cdot\nabla_z\xi)^{1/2}\|_{L^\infty([0,T]\times\mathbb{R}^m)}\|\langle a\rangle\|_{L^1([0,T];X_Q)}$$
$$\leq \|(P\nabla_z\xi\cdot\nabla_z\xi)^{1/2}\|_{L^\infty([0,T]\times\mathbb{R}^m)}\|a\|_{L^1([0,T];X_Q)},$$

and thus

$$\lim_{k\to+\infty}\int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi\cdot\langle a(t)\rangle\ \mathrm{d}z\ \mathrm{d}t = \int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v(t,z)\nabla_z\xi\cdot\langle a(t)\rangle\ \mathrm{d}z\ \mathrm{d}t. \tag{67}$$

The convergences in (66), (67) allow us to pass to the limit in the last integral of (63)

$$\lim_{k\to+\infty}\int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v^{\varepsilon_k}(t,z)\nabla_z\xi\cdot\varphi(t/\varepsilon_k)a(t)\ \mathrm{d}z\ \mathrm{d}t = \int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v(t,z)\nabla_z\xi\cdot\langle a(t)\rangle\ \mathrm{d}z\ \mathrm{d}t.$$

Therefore, the weak $\star$ limit v satisfies the weak formulation, at least for $\xi\in C_c^2(\mathbb{R}_+\times\mathbb{R}^m)$

$$-\int_{\mathbb{R}^m}\xi(0,z)u^{\mathrm{in}}(z)\ \mathrm{d}z - \int_{\mathbb{R}_+}\int_{\mathbb{R}^m}\partial_t\xi\ v(t,z)\ \mathrm{d}z\ \mathrm{d}t - \int_{\mathbb{R}_+}\int_{\mathbb{R}^m}\nabla_z\xi\cdot\langle a(t)\rangle\, v(t,z)\ \mathrm{d}z\ \mathrm{d}t = 0.$$

It is easily seen that the above formulation holds true for any $\xi\in C_c^1(\mathbb{R}_+\times\mathbb{R}^m)$. By weak $\star$ convergence, we deduce that the limit function v has time and space derivatives, satisfying the estimates

$$\|\partial_t v\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \leq C(T,(\alpha_i^j)_{i,j})|\nabla u^{\mathrm{in}}|_P\|a\|_{L^\infty([0,T];X_Q^\infty)},$$

$$\|\nabla_z v\|_{L^\infty([0,T];X_P)} \leq C(T,(\alpha_i^j)_{i,j})|\nabla u^{\mathrm{in}}|_P.$$

We deduce that the function $\frac{v^2}{2}$ satisfies

$$\partial_t\frac{v^2}{2} + \langle a(t)\rangle\cdot\nabla_z\frac{v^2}{2} = 0,\ \text{ in } \mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^m).$$

In particular we have for any function $\theta\in C_c^1(\mathbb{R}^m)$

$$\int_{\mathbb{R}^m} v^2(t,z)\theta(z)\ \mathrm{d}z - \int_{\mathbb{R}^m}(u^{\mathrm{in}}(z))^2\theta(z)\ \mathrm{d}z - \int_0^t\int_{\mathbb{R}^m}\langle a(s)\rangle\cdot\nabla_z\theta\ v^2(s,z)\ \mathrm{d}z\mathrm{d}s = 0. \tag{68}$$

We claim that the L^2 norm of $v(t)$ is conserved in time. For doing that, we consider a smooth function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$, non increasing, such that $\chi(r) = 1$ if $r \in [0,1]$, $\chi(r) = 0$ if $r \geq 2$ and we pick in (68) the test function $\theta_M(z) = \chi(|z|/M), z \in \mathbb{R}^m, M \geq 1$. Thanks to the growth conditions satisfied by the vector fields b_i, we have

$$
\begin{aligned}
|\langle a(s)\rangle \cdot \nabla_z \theta_M(z)| &= |R\langle a(s)\rangle \cdot {}^tR^{-1}\nabla_z\theta_M(z)| \\
&\leq ({}^tRR\langle a(s)\rangle \cdot \langle a(s)\rangle)^{1/2} \left| \frac{1}{M}\chi'\left(\frac{|z|}{M}\right) {}^tR^{-1}\frac{z}{|z|}\right| \\
&\leq |\langle a(s)\rangle|_{X_Q^\infty} \|\chi'\|_{L^\infty} \frac{\left(\sum_{i=1}^m (b_i(z)\cdot z)^2\right)^{1/2}}{|z|M} \mathbf{1}_{M\leq |z|\leq 2M} \\
&\leq |a(s)|_{X_Q^\infty} \|\chi'\|_{L^\infty} \frac{\sum_{i=1}^m |b_i(z)\cdot z|}{|z|M} \mathbf{1}_{M\leq |z|\leq 2M} \\
&\leq |a(s)|_{X_Q^\infty} \|\chi'\|_{L^\infty} \frac{\sum_{i=1}^m C_i(1+2M)}{M} \\
&\leq 3|a(s)|_{X_Q^\infty} \|\chi'\|_{L^\infty} \sum_{i=1}^m C_i.
\end{aligned}
$$

Using the dominated convergence theorem, it is easily seen that

$$
\lim_{M\to+\infty} \int_0^t \int_{\mathbb{R}^m} \langle a(s)\rangle \cdot \nabla_z\theta_M \, v^2(s,z) \, \mathrm{d}z\mathrm{d}s = 0,
$$

$$
\lim_{M\to+\infty} \int_{\mathbb{R}^m} v^2(t,z)\theta_M(z)\,\mathrm{d}z = \int_{\mathbb{R}^m} v^2(t,z)\,\mathrm{d}z, \quad \lim_{M\to+\infty} \int_{\mathbb{R}^m} (u^{\mathrm{in}})^2(z)\theta_M(z)\,\mathrm{d}z = \int_{\mathbb{R}^m} (u^{\mathrm{in}})^2(z)\,\mathrm{d}z,
$$

implying that

$$
\int_{\mathbb{R}^m} v^2(t,z)\,\mathrm{d}z = \int_{\mathbb{R}^m} (u^{\mathrm{in}})^2(z)\,\mathrm{d}z, \quad t \in \mathbb{R}_+. \tag{69}
$$

The above conservation implies the strong convergence of $(v^{\varepsilon_k})_k$ in $L^2([0,T]\times\mathbb{R}^m)$, for any $T > 0$. Indeed, for any $T > 0$ we have

$$
\limsup_{k\to+\infty} \int_0^T \int_{\mathbb{R}^m} (v^{\varepsilon_k})^2\,\mathrm{d}z\,\mathrm{d}t = T\int_{\mathbb{R}^m} (u^{\mathrm{in}})^2(z)\,\mathrm{d}z = \int_0^T \int_{\mathbb{R}^m} (v(t,z))^2\,\mathrm{d}z\,\mathrm{d}t,
$$

and since we already know that $v^{\varepsilon_k} \rightharpoonup v$ weakly in $L^2([0,T]\times\mathbb{R}^m)$, we deduce that $v^{\varepsilon_k} \to v$ strongly in $L^2([0,T]\times\mathbb{R}^m)$. Notice that (69) also guarantees the uniqueness of the limit v, that is, any other convergent sequence $(v^{\widetilde{\varepsilon}_k})_k$ will converge toward the same limit v. This says that all the family $(v^\varepsilon)_{\varepsilon>0}$ converges to v strongly in $L^2([0,T]\times\mathbb{R}^m)$, when $\varepsilon \searrow 0$, for any $T > 0$.

Actually we can prove that the family $(v^\varepsilon)_\varepsilon$ converges to v, as $\varepsilon \searrow 0$, strongly in $L^\infty([0,T];L^2(\mathbb{R}^m))$ for any $T > 0$. Taking the difference between the equations satisfied by v^ε, v leads to

$$
\partial_t(v^\varepsilon - v) + (\varphi(t/\varepsilon)a(t))(z)\cdot\nabla_z(v^\varepsilon - v) + (\,\varphi(t/\varepsilon)a(t) - \langle a(t)\rangle\,)\cdot\nabla_z v = 0,
$$

and after multiplication by $v^\varepsilon - v$ we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} (v^\varepsilon(t,z) - v(t,z))^2\,\mathrm{d}z + \int_{\mathbb{R}^m} (\,\varphi(t/\varepsilon)a(t) - \langle a(t)\rangle\,)\cdot\nabla_z v\,(v^\varepsilon(t,z) - v(t,z))\,\mathrm{d}z = 0.
$$

Notice that for any $(t,z) \in \mathbb{R}_+\times\mathbb{R}^m$ we have the inequality

$$
|(\,\varphi(t/\varepsilon)a(t) - \langle a(t)\rangle\,)\cdot\nabla_z v| \leq |\varphi(t/\varepsilon)a(t) - \langle a(t)\rangle\,|_{X_Q^\infty}|P^{1/2}\nabla_z v| \leq 2|a(t)|_{X_Q^\infty}|P^{1/2}\nabla_z v|,
$$

which implies that for any $t \in [0,T]$

$$\frac{1}{2}\int_{\mathbb{R}^m}(v^\varepsilon(t,z)-v(t,z))^2\,\mathrm{d}z \le 2\int_0^T\int_{\mathbb{R}^m}|a(r)|_{X_Q^\infty}|P^{1/2}\nabla_z v(r,z)|\,|(v^\varepsilon - v)(r,z)|\,\mathrm{d}z\mathrm{d}r$$
$$\le 2\|a\|_{L^\infty([0,T];X_Q^\infty)}\|\nabla_z v\|_{L^2([0,T];X_P)}\|v^\varepsilon - v\|_{L^2([0,T];L^2(\mathbb{R}^m))}.$$

Therefore the family $(v^\varepsilon)_\varepsilon$ converges to v strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^m))$, as $\varepsilon \searrow 0$. □

The previous result says that for any $T>0$

$$\lim_{\varepsilon\searrow 0}\int_0^T\!\!\int_{\mathbb{R}^m}(u^\varepsilon(t,Y(t/\varepsilon;z))-v(t,z))^2\,\mathrm{d}z\,\mathrm{d}t = \lim_{\varepsilon\searrow 0}\int_0^T\!\!\int_{\mathbb{R}^m}(u^\varepsilon(t,y)-v(t,Y(-t/\varepsilon;y)))^2\,\mathrm{d}y\,\mathrm{d}t = 0,$$

but does not provide any information about the convergence rate, with respect to $\varepsilon > 0$. The next result establishes the convergence of the family $(v^\varepsilon)_\varepsilon$ at the expected rate $\mathcal{O}(\varepsilon)$, under suitable smoothness hypotheses. Basically we need second order space derivatives for the solution of the limit model (26), (27) (and thus second order derivatives for the initial condition) and the existence of a smooth vector field $c = c(t,y)$ such that the following decomposition holds true

$$a(t) = \langle a(t)\rangle + \mathcal{L}c(t),\ \ t\in\mathbb{R}_+.$$

Notice that generally we do not have $a(t)-\langle a(t)\rangle \in \mathrm{Range}\mathcal{L}$, but only

$$a(t)-\langle a(t)\rangle \in \ker\langle\cdot\rangle = \ker(\mathrm{Proj}_{\ker\mathcal{L}}) = (\ker\mathcal{L})^\perp = (\ker\mathcal{L}^\star)^\perp = \overline{\mathrm{Range}\mathcal{L}}.$$

Proof. (of Theorem 2.4)
We introduce the function $\widetilde{u}^\varepsilon(t,y) = v(t,Y(-t/\varepsilon;y))$ where v solves (26), (27). These functions satisfy transport equations very similar to those verified by u^ε. Indeed we have

$$\partial_t\widetilde{u}^\varepsilon + \frac{1}{\varepsilon}b\cdot\nabla_y\widetilde{u}^\varepsilon = \partial_t v(t,Y(-t/\varepsilon;y)) - \frac{1}{\varepsilon}b(Y(-t/\varepsilon;y))\cdot\nabla_z v(t,Y(-t/\varepsilon;y)) \tag{70}$$
$$+\frac{1}{\varepsilon}b(y)\cdot\ {}^t\partial_y Y(-t/\varepsilon;y)\,\nabla_z v(t,Y(-t/\varepsilon;y)) = (\partial_t v)(t,Y(-t/\varepsilon;y)),$$

$$\langle a(t)\rangle\cdot\nabla_y\widetilde{u}^\varepsilon = \langle a(t)\rangle\cdot\ {}^t\partial_y Y(-t/\varepsilon;y)\,\nabla_z v(t,Y(-t/\varepsilon;y)) \tag{71}$$
$$= \partial_y Y(-t/\varepsilon;y)\,\langle a(t)\rangle\cdot\nabla_z v(t,Y(-t/\varepsilon;y)).$$

But $\langle a(t)\rangle = \varphi(-t/\varepsilon)\,\langle a(t)\rangle$ and thus

$$\partial_y Y(-t/\varepsilon;y)\,\langle a(t)\rangle(y) = \partial_y Y(-t/\varepsilon;y)\,(\varphi(-t/\varepsilon)\,\langle a(t)\rangle)(y) = \langle a(t)\rangle(Y(-t/\varepsilon;y)). \tag{72}$$

Putting together (70), (71), (72) implies

$$\partial_t\widetilde{u}^\varepsilon + \langle a(t)\rangle(y)\cdot\nabla_y\widetilde{u}^\varepsilon + \frac{1}{\varepsilon}b(y)\cdot\nabla_y\widetilde{u}^\varepsilon = (\partial_t v)(t,Y(-t/\varepsilon;y)) \tag{73}$$
$$+\langle a(t)\rangle(Y(-t/\varepsilon;y))\cdot\nabla_z v(t,Y(-t/\varepsilon;y)) = 0.$$

Recall that the functions $(u^\varepsilon)_\varepsilon$ satisfy the transport equation

$$\partial_t u^\varepsilon + a(t,y)\cdot\nabla_y u^\varepsilon + \frac{1}{\varepsilon}b(y)\cdot\nabla_y u^\varepsilon = 0. \tag{74}$$

Both families verify $u^\varepsilon(0,y) = \widetilde{u}^\varepsilon(0,y) = u^{\mathrm{in}}(y), y\in\mathbb{R}^m$. We also need to introduce a corrector. We consider the function

$$u^1(t,s,y) = (c(t)\cdot\nabla_z v(t))(Y(-s;y)) - c(t,y)\cdot\nabla_y\{v(t,Y(-s;y))\}$$
$$= \tau(-s)\{c(t)\cdot\nabla_z v(t)\} - c(t)\cdot\nabla_y\{\tau(-s)v(t)\},\ \ (t,s,y)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^m.$$

A straightforward computation shows that

$$u^1(t,s,Y(s;z)) = c(t,z)\cdot\nabla_z v(t,z) - c(t,Y(s;z))\cdot {}^t\partial_y Y(-s;Y(s;z))\nabla_z v(t,z) \tag{75}$$
$$= c(t,z)\cdot\nabla_z v(t,z) - \partial_y Y(-s;Y(s;z))c(t,Y(s;z))\cdot\nabla_z v(t,z)$$
$$= c(t,z)\cdot\nabla_z v(t,z) - (\varphi(s)c(t))(z)\cdot\nabla_z v(t,z)$$
$$= (\ c(t) - \varphi(s)c(t)\ )(z)\cdot\nabla_z v(t,z).$$

Taking the derivatives with respect to s one gets

$$\partial_s u^1(t,s,Y(s;z)) + b(Y(s;z))\cdot\nabla_y u^1(t,s,Y(s;z)) = \frac{\mathrm{d}}{\mathrm{d}s}\{c(t)-\varphi(s)c(t)\}(z)\cdot\nabla_z v(t,z)$$
$$= -(\varphi(s)\mathcal{L}c)(z)\cdot\nabla_z v(t,z)$$
$$= -\partial_y Y(-s;Y(s;z))(\mathcal{L}c)(Y(s;z))\cdot\nabla_z v(t,z)$$
$$= -(\mathcal{L}c)(Y(s;z))\cdot\nabla_y\{v(t,Y(-s;\cdot))\}(Y(s;z)),$$

and therefore the corrector u^1 satisfies

$$(\mathcal{L}c)(y)\cdot\nabla_y\{v(t,Y(-s;\cdot))\} + \partial_s u^1(t,s,y) + b(y)\cdot\nabla_y u^1(t,s,y) = 0,\ \ (t,s,y)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^m.$$

In particular, for $s = t/\varepsilon$, one gets

$$(\mathcal{L}c)(y)\cdot\nabla_y\widetilde{u}^\varepsilon + (\partial_s + b\cdot\nabla_y)u^1(t,t/\varepsilon,y) = 0,\ \ (t,y)\in\mathbb{R}_+\times\mathbb{R}^m,\ \varepsilon>0,$$

which also writes

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\varepsilon u^1(t,t/\varepsilon,y)\} + (\mathcal{L}c)(y)\cdot\nabla_y\widetilde{u}^\varepsilon + \frac{1}{\varepsilon}b\cdot\nabla_y(\varepsilon u^1)(t,t/\varepsilon,y) = \varepsilon\partial_t u^1(t,t/\varepsilon,y). \tag{76}$$

Taking the difference between (74) and the sum of (73) and (76), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\{u^\varepsilon(t,y) - \widetilde{u}^\varepsilon(t,y) - \varepsilon u^1(t,t/\varepsilon,y)\} + a(t,y)\cdot\nabla_y\{u^\varepsilon(t,y) - \widetilde{u}^\varepsilon(t,y) - \varepsilon u^1(t,t/\varepsilon,y)\}$$
$$+ \frac{1}{\varepsilon}b(y)\cdot\nabla_y\{u^\varepsilon(t,y) - \widetilde{u}^\varepsilon(t,y) - \varepsilon u^1(t,t/\varepsilon,y)\} = -\varepsilon\{\partial_t u^1 + a(t,y)\cdot\nabla_y u^1\}(t,t/\varepsilon,y),$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon(t,\cdot) - \widetilde{u}^\varepsilon(t,\cdot) - \varepsilon u^1(t,t/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)} \le \varepsilon\|\partial_t u^1(t,t/\varepsilon,\cdot) + a(t,\cdot)\cdot\nabla_y u^1\|_{L^2(\mathbb{R}^m)}.$$

Notice that

$$[\ u^\varepsilon(t,\cdot) - \widetilde{u}^\varepsilon(t,\cdot) - \varepsilon u^1(t,t/\varepsilon,\cdot)\ ]_{t=0} = u^{\mathrm{in}} - u^{\mathrm{in}} = 0,$$

and therefore, after integration with respect to the slow variable t one gets

$$\|u^\varepsilon(t,\cdot) - \widetilde{u}^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^m)} \le \|u^\varepsilon(t,\cdot) - \widetilde{u}^\varepsilon(t,\cdot) - \varepsilon u^1(t,t/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)} + \varepsilon\|u^1(t,t/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)}$$
$$\le \varepsilon\int_0^t\left\{\|\partial_t u^1(\tau,\tau/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)} + \|a(\tau,\cdot)\nabla_y u^1(\tau,\tau/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)}\right\}\,\mathrm{d}\tau$$
$$+ \varepsilon\|u^1(t,t/\varepsilon,\cdot)\|_{L^2(\mathbb{R}^m)}. \tag{77}$$

We are done if the corrector u^1 verifies uniform estimates with respect to the fast time variable s. It is easily seen, using (75), that

$$\|u^1(t,s,\cdot)\|_{L^2(\mathbb{R}^m)} = \|\{c(t)-\varphi(s)c(t)\}\cdot\nabla_z v(t)\|_{L^2(\mathbb{R}^m)} \tag{78}$$
$$\le |c(t)|_{X_Q^\infty}|\nabla_z v(t)|_P + |\varphi(s)c(t)|_{X_Q^\infty}|\nabla_z v(t)|_P$$
$$\le 2|c(t)|_{X_Q^\infty}|\nabla_z v(t)|_P$$
$$= 2\|Rc(t)\|_{L^\infty(\mathbb{R}^m)}\|\,{}^tR^{-1}\nabla_z v(t)\|_{L^2(\mathbb{R}^m)},\ \ (t,s)\in\mathbb{R}_+\times\mathbb{R}.$$

Taking the derivative with respect to t of (75) we obtain

$$\partial_t u^1(t,s,Y(s;z)) = (\partial_t c(t) - \varphi(s)\partial_t c(t))(z)\cdot\nabla_z v(t,z) + (c(t)-\varphi(s)c(t))\cdot\nabla_z\partial_t v,$$

and thus

$$\begin{aligned}\|\partial_t u^1(t,s,\cdot)\|_{L^2(\mathbb{R}^m)} &\le 2|\partial_t c(t)|_{X_Q^\infty}|\nabla_z v(t)|_P + 2|c(t)|_{X_Q^\infty}|\nabla_z\partial_t v(t)|_P \\ &= 2\|R\partial_t c(t)\|_{L^\infty(\mathbb{R}^m)}\|\,{}^tR^{-1}\nabla_z v(t)\|_{L^2(\mathbb{R}^m)} \\ &+ 2\|Rc(t)\|_{L^\infty(\mathbb{R}^m)}\|\,{}^tR^{-1}\nabla_z\partial_t v(t)\|_{L^2(\mathbb{R}^m)}.\end{aligned} \tag{79}$$

We need to apply the operator ∇_z to (75), in order to estimate $a\cdot\nabla_y u^1$. We will use the matrix field R, see (37). Notice that

$$\begin{aligned}R(z)(\varphi(s)c(t))(z) &= R(z)\partial_y Y(-s;Y(s;z))c(t,Y(s;z)) = R(z)\partial_z Y^{-1}(s;z)c(t,Y(s;z)) \\ &= R(Y(s;z))c(t,Y(s;z)),\end{aligned}$$

which implies

$$\begin{aligned}(c(t)-\varphi(s)c(t))(z)\cdot\nabla_z v(t,z) &= R(z)(c(t)-\varphi(s)c(t))(z)\cdot{}^tR^{-1}\nabla_z v(t,z) \\ &= \{R(z)c(t,z) - R(Y(s;z))c(t,Y(s;z))\}\cdot{}^tR^{-1}\nabla_z v(t,z).\end{aligned}$$

We introduce the notation $\nabla^R = {}^tR^{-1}\nabla_z$, that is

$$\nabla^R v = {}^tR^{-1}\nabla_z v = {}^t(b_1\cdot\nabla_z v, ..., b_m\cdot\nabla_z v).$$

Taking the gradient with respect to z of (75) yields

$${}^t\partial_z Y(s;z)\nabla_y u^1(t,s,Y(s;z)) = \nabla_z\{\,[R(z)c(t,z) - R(Y(s;z))c(t,Y(s;z))\,]\cdot\nabla^R v(t)\},$$

and thus

$$\begin{aligned}a(t,Y(s;z))\cdot\nabla_y u^1(t,s,Y(s;z)) &= a(t,Y(s;z)) \\ &\cdot\,{}^t\partial_y Y(-s;Y(s;z))\nabla_z\{\,[R(z)c(t,z) - R(Y(s;z))c(t,Y(s;z))\,]\cdot\nabla^R v(t)\} \\ &= (\varphi(s)a(t))(z)\cdot\nabla_z\{\,[R(z)c(t,z) - R(Y(s;z))c(t,Y(s;z))\,]\cdot\nabla^R v(t)\} \\ &= R(z)(\varphi(s)a(t))(z)\cdot\nabla^R\{\,[R(z)c(t,z) - R(Y(s;z))c(t,Y(s;z))\,]\cdot\nabla^R v(t)\}.\end{aligned}$$

We deduce that for any $(t,s)\in\mathbb{R}_+\times\mathbb{R}$ we have

$$\begin{aligned}\|a(t,\cdot)\cdot\nabla_y u^1(t,s,\cdot)\|_{L^2(\mathbb{R}^m)} &\le |\varphi(s)a(t)|_{X_Q^\infty} \\ &\|\nabla^R\{\,[R(\cdot)c(t,\cdot) - R(Y(s;\cdot))c(t,Y(s;\cdot))\,]\cdot\nabla^R v(t)\}\|_{L^2(\mathbb{R}^m)} \\ &\le |a(t)|_{X_Q^\infty}\|\nabla^R\{\,[R(\cdot)c(t,\cdot) - R(Y(s;\cdot))c(t,Y(s;\cdot))\,]\cdot\nabla^R v(t)\}\|_{L^2(\mathbb{R}^m)} \\ &\le |a(t)|_{X_Q^\infty}\|\nabla^R\{R(\cdot)c(t,\cdot)\cdot\nabla^R v(t)\}\|_{L^2(\mathbb{R}^m)} \\ &+ |a(t)|_{X_Q^\infty}\|\nabla^R\{R(Y(s;\cdot))c(t,Y(s;\cdot))\cdot\nabla^R v(t)\}\|_{L^2(\mathbb{R}^m)}.\end{aligned} \tag{80}$$

Recall that we are looking for uniform estimates with respect to s. The key point is that ∇^R commutes with $\tau(s)$, *i.e.*,

$$\nabla^R\tau(s)f = \nabla^R\{f(Y(s;\cdot))\} = (\nabla^R f)(Y(s;\cdot)) = \tau(s)\nabla^R f,$$

for any smooth function f. Indeed, for any $i \in \{1, ..., m\}$, we have

$$
\begin{aligned}
(b_i \cdot \nabla_z)(\tau(s) f)(z) &= \lim_{h \to 0} \frac{f(Y(s; Y_i(h; z))) - f(Y(s; z))}{h} \\
&= \lim_{h \to 0} \frac{f(Y_i(h; Y(s; z))) - f(Y(s; z))}{h} \\
&= b_i(Y(s; z)) \cdot (\nabla_z f)(Y(s; z)) \\
&= \tau(s)(b_i \cdot \nabla_z f)(z).
\end{aligned}
$$

Based on the previous remark, we will estimate the last two terms of (80), uniformly with respect to s. For the first term (which does not depend on s), we can write

$$
\nabla^R \{Rc(t) \cdot \nabla^R v(t)\} = (\nabla^R \otimes Rc(t)) \nabla^R v(t) + (\nabla^R \otimes \nabla^R) v(t) Rc(t),
$$

where $\nabla^R \otimes Rc(t)$ is the matrix, whose entry (i, j) is $b_i \cdot \nabla_z (Rc(t))_j$ and $\nabla^R \otimes \nabla^R v(t)$ is the matrix whose entry (i, j) is $b_i \cdot \nabla_z (b_j \cdot \nabla_z v(t))$. Therefore we obtain

$$
\|\nabla^R \{Rc(t) \cdot \nabla^R v(t)\}\|_{L^2(\mathbb{R}^m)} \leq \|\nabla^R \otimes Rc(t)\|_{L^\infty} \|\nabla^R v(t)\|_{L^2(\mathbb{R}^m)} + \|\nabla^R \otimes \nabla^R v(t)\|_{L^2(\mathbb{R}^m)} \|Rc(t)\|_{L^\infty}.
$$

We claim that a similar estimate holds true for the second term

$$
\begin{aligned}
\|\nabla^R \{R(Y(s; \cdot)) c(t, Y(s; \cdot)) \cdot \nabla^R v(t)\}\|_{L^2(\mathbb{R}^m)} &\leq \|(\nabla^R \otimes Rc(t, \cdot))(Y(s; \cdot))\|_{L^\infty} \|\nabla^R v(t)\|_{L^2(\mathbb{R}^m)} \\
&\quad + \|\nabla^R \otimes \nabla^R v(t)\|_{L^2(\mathbb{R}^m)} \|(Rc(t, \cdot))(Y(s; \cdot))\|_{L^\infty} \\
&= \|\nabla^R \otimes Rc(t, \cdot)\|_{L^\infty} \|\nabla^R v(t)\|_{L^2(\mathbb{R}^m)} \\
&\quad + \|\nabla^R \otimes \nabla^R v(t)\|_{L^2(\mathbb{R}^m)} \|Rc(t, \cdot)\|_{L^\infty}.
\end{aligned}
$$

The above computations lead to the following estimate for $a \cdot \nabla_y u^1$

$$
\begin{aligned}
\int_0^T \|a(t, \cdot) \cdot \nabla_y u^1(t, t/\varepsilon, \cdot)\|_{L^2(\mathbb{R}^m)} \, \mathrm{d}s &\leq 2\|a\|_{L^\infty([0,T]; X_Q^\infty)} \{ \|\nabla^R \otimes Rc\|_{L^1([0,T]; L^\infty(\mathbb{R}^m))} \qquad (81) \\
&\times \|\nabla^R v\|_{L^\infty([0,T]; L^2(\mathbb{R}^m))} + \|Rc\|_{L^1([0,T]; L^\infty(\mathbb{R}^m))} \|\nabla^R \otimes \nabla^R v\|_{L^\infty([0,T]; L^2(\mathbb{R}^m))} \}.
\end{aligned}
$$

Using the estimates (78), (79), (81), (77) yields the convergence rate of the family $(u^\varepsilon)_\varepsilon$, for any $t \in [0, T], \varepsilon > 0$

$$
\begin{aligned}
\|u^\varepsilon(t) - \widetilde{u}^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} &\leq 2\varepsilon \{ \|Rc\|_{L^\infty([0,T]; L^\infty(\mathbb{R}^m))} + \|R\partial_t c\|_{L^1([0,T]; L^\infty(\mathbb{R}^m))} \} \|\nabla^R v\|_{L^\infty([0,T]; L^2(\mathbb{R}^m))} \\
&\quad + 2\varepsilon \|Rc\|_{L^1([0,T]; L^\infty(\mathbb{R}^m))} \|\nabla^R \partial_t v\|_{L^\infty([0,T]; L^2(\mathbb{R}^m))} \\
&\quad + 2\varepsilon \|a\|_{L^\infty([0,T]; X_Q^\infty)} \|\nabla^R \otimes Rc\|_{L^1([0,T]; L^\infty(\mathbb{R}^m))} \|\nabla^R v\|_{L^\infty([0,T]; L^2(\mathbb{R}^m))} \\
&\quad + 2\varepsilon \|a\|_{L^\infty([0,T]; X_Q^\infty)} \|Rc\|_{L^1([0,T]; L^\infty(\mathbb{R}^m))} \|\nabla^R \otimes \nabla^R v\|_{L^\infty([0,T]; L^2(\mathbb{R}^m))}.
\end{aligned}
$$

□

7 Gyrokinetic models

We mention that our method applies in many other contexts, like the behavior of living organisms (flocks of birds, school of fish, swarms of insects, myxobacteria,...). Considering populations of individuals driven by self-propelling forces and pairwise attractive and repulsive interaction [29, 30] leads to the equation

$$
\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon - \nabla_x (U \star \rho^\varepsilon(t, \cdot)) \cdot \nabla_v f^\varepsilon + \frac{1}{\varepsilon} \mathrm{div}_v \{ f^\varepsilon (\alpha - \beta |v|^2) v \} = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3,
$$

where U is the pairwise interaction potential and $\rho^\varepsilon = \int f^\varepsilon \, \mathrm{d}v$. For example the Cucker-Smale model with diffusion can be reduced to a Vicsek like model [12]. Hydrodynamic models, based on relaxation toward the mean velocity [20], can be treated as well.

We come back to magnetic confinement and we investigate the asymptotic behavior of the Vlasov equation when the magnetic field becomes very strong. For the moment, we restrict our attention to the linear transport equation (5), assuming that the electric field $E = -\nabla_x \phi$ derives from a given bounded smooth electric potential. In this case we work in a 6 dimensional phase space, $y = (x, v)$ and

$$b(x,v)\cdot\nabla_{x,v} = v_1\partial_{x_1} + v_2\partial_{x_2} + \omega_c(x_1,x_2)(v_2\partial_{v_1} - v_1\partial_{v_2}),\;\; a(x,v)\cdot\nabla_{x,v} = v_3\partial_{x_3} + \frac{q}{m}E\cdot\nabla_v.$$

Notice that $\mathrm{div}_{x,v} a = \mathrm{div}_{x,v} b = 0$. We denote by $Y = (X, V)$ the flow of the vector field b

$$\frac{\mathrm{d}X}{\mathrm{d}s} = (V_1(s;x,v), V_2(s;x,v), 0),\;\; \frac{\mathrm{d}V}{\mathrm{d}s} = \omega_c(X_1,X_2)(V_2(s;x,v), -V_1(s;x,v), 0).$$

Under the hypotheses of Theorem 2.3, Theorem 2.4, we know that

$$f^\varepsilon(t,x,v) - g(t, X(-t/\varepsilon;x,v), V(-t/\varepsilon;x,v)) = o(1) \;\text{ in }\; L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^6)),$$

respectively

$$f^\varepsilon(t,x,v) - g(t, X(-t/\varepsilon;x,v), V(-t/\varepsilon;x,v)) = \mathcal{O}(\varepsilon) \;\text{ in }\; L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^6)),$$

where g solves the problem

$$\begin{cases} \partial_t g + \langle a(t)\rangle \cdot \nabla_{x,v} g = 0, & (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^6, \\ g(0,x,v) = f^{\mathrm{in}}(x,v), & (x,v) \in \mathbb{R}^6. \end{cases} \tag{82}$$

Therefore we need to determine the average vector field $\langle a(t)\rangle$. Notice that the components of $\langle a(t)\rangle$ along x_3 and v_3 come immediately, thanks to the property

$$\langle a\rangle \cdot \nabla\psi = \langle a \cdot \nabla\psi\rangle,$$

for any smooth function ψ which belongs to $\ker(b\cdot\nabla_y)$, cf. Theorem 2.1. Obviously x_3, v_3 remain constant along the flow of b and we obtain

$$\langle a(t)\rangle_{x_3} = \langle a(t)\rangle \cdot \nabla_{x,v} x_3 = \langle a(t)\cdot\nabla_{x,v} x_3\rangle = \langle\, (a(t))_{x_3}\rangle = \langle v_3\rangle = v_3,$$

$$\langle a(t)\rangle_{v_3} = \langle a(t)\rangle \cdot \nabla_{x,v} v_3 = \langle a(t)\cdot\nabla_{x,v} v_3\rangle = \langle\, (a(t))_{v_3}\rangle = \left\langle \frac{q}{m} E_3(t)\right\rangle.$$

The formula for computing $\langle a(t)\rangle$, cf. Theorem 2.1, requires the computation of the Jacobian matrix $\partial_y Y(-s; Y(s;\cdot))$

$$\langle a(t)\rangle = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \partial_y Y(-s;Y(s;\cdot)) a(t, Y(s;\cdot))\,\mathrm{d}s, \;\text{ strongly in } X_Q. \tag{83}$$

For example, when the magnetic field is uniform, that is $\nabla_{x_1,x_2}\omega_c = 0$, it is easily seen that

$$\overline{X}(s;\overline{x},\overline{v}) = \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{\mathcal{R}(-\omega_c s)}{\omega_c}\,{}^\perp\overline{v},\;\; X_3(s;x_3) = x_3,\;\; \overline{V}(s;\overline{v}) = \mathcal{R}(-\omega_c s)\overline{v},\;\; V_3(s;v_3) = v_3,$$

where we have used the notations $\overline{x} = (x_1,x_2), \overline{v} = (v_1,v_2), {}^\perp\overline{v} = (v_2,-v_1)$ and $\mathcal{R}(\theta)$ stands for the rotation of angle $\theta \in \mathbb{R}$. The Jacobian matrix writes

$$\partial_{x,v} Y(s;x,v) = \begin{pmatrix} I_2 & O_{2\times1} & \frac{I_2 - \mathcal{R}(-\omega_c s)}{\omega_c}\mathcal{E} & O_{2\times1} \\ O_{1\times2} & 1 & O_{1\times2} & 0 \\ O_{2\times2} & O_{2\times1} & \mathcal{R}(-\omega_c s) & O_{2\times1} \\ O_{1\times2} & 0 & O_{1\times2} & 1 \end{pmatrix}.$$

where $O_{m\times n}$ stands for the null matrix with m lines and n columns, and $\mathcal{E} = \mathcal{R}(-\pi/2)$. We obtain the expressions

$$\begin{aligned}\langle a(t)\rangle_{\overline{x}} &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \frac{I_2-\mathcal{R}(\omega_c s)}{\omega_c}\frac{q}{m}\,{}^{\perp}\overline{E}(t,\overline{X}(s;\overline{x},\overline{v}),x_3)\,\mathrm{d}s\\ &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \frac{{}^{\perp}\overline{E}(t,\overline{X}(s;\overline{x},\overline{v}),x_3)}{B}\,\mathrm{d}s - \lim_{T\to+\infty}\frac{1}{T}\int_0^T \mathcal{R}(\omega_c s)\frac{{}^{\perp}\overline{E}(t,\overline{X}(s;\overline{x},\overline{v}),x_3)}{B}\,\mathrm{d}s,\end{aligned}$$

$$\langle a(t)\rangle_{x_3} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T V_3(s;v_3)\,\mathrm{d}s = v_3,$$

$$\langle a(t)\rangle_{\overline{v}} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \mathcal{R}(\omega_c s)\frac{q}{m}\overline{E}(t,\overline{X}(s;\overline{x},\overline{v}),x_3)\,\mathrm{d}s,$$

$$\langle a(t)\rangle_{v_3} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \frac{q}{m}E_3(t,\overline{X}(s;\overline{x},\overline{v}),x_3)\,\mathrm{d}s.$$

Therefore, when the magnetic field is uniform, the flow (X,V) is linear with respect to (x,v), the Jacobian matrix $\partial_y Y(-s;Y(s;\cdot))$ depends only on s and the average vector field $\langle a(t)\rangle$ comes by averaging the electric field along $\overline{X} = \overline{X}(\cdot;\overline{x},\overline{v})$. Observe that $\langle a(t)\rangle_{\overline{v}}$ and $\overline{v}$ are orthogonals

$$\begin{aligned}\frac{1}{T}\int_0^T \mathcal{R}(\omega_c s)\overline{E}(t,\overline{X}(s;\overline{x},\overline{v}),x_3)\,\mathrm{d}s\cdot\overline{v} &= \frac{1}{T}\int_0^T \overline{E}(t,\overline{X}(s;\overline{x},\overline{v}),x_3)\cdot\mathcal{R}(-\omega_c s)\overline{v}\,\mathrm{d}s\\ &= \frac{1}{T}\int_0^T \overline{E}(t,\overline{X}(s;\overline{x},\overline{v}),x_3)\cdot\frac{\mathrm{d}\overline{X}}{\mathrm{d}s}\,\mathrm{d}s\\ &= -\frac{1}{T}\int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\phi(t,\overline{X}(s;\overline{x},\overline{v}),x_3)\,\mathrm{d}s\to 0,\ \text{ as } T\to+\infty.\end{aligned}$$

Actually the same identity holds true in the general case, and comes by using the invariant $\psi = \frac{|\overline{v}|^2}{2}$

$$\langle a(t)\rangle_{\overline{v}}\cdot\overline{v} = \langle a(t)\rangle\cdot\nabla_{x,v}\frac{|\overline{v}|^2}{2} = \left\langle a(t)\cdot\nabla_{x,v}\frac{|\overline{v}|^2}{2}\right\rangle = \left\langle\frac{q}{m}\overline{E}\cdot\overline{v}\right\rangle = -\frac{q}{m}\langle b\cdot\nabla_{x,v}\phi(t)\rangle = 0.$$

The computation of the average vector field $\langle a(t)\rangle$ through the formula (83) seems out of reach since, in general, the expression of the flow $Y=(X,V)$ is not available. We can approximate the flow Y, but the computation of the Jacobian matrix $\partial_y Y(-s;Y(s;\cdot))$ still remains a difficult task. Another possibility is to appeal to Theorem 2.2 which says that for any fixed t_0, the average vector field $\langle a(t_0)\rangle$ is obtained as the long time limit of the solution c for the problem

$$\begin{cases}\partial_t c - \mathcal{L}^2 c = 0, & t\in\mathbb{R}_+,\\ c(0,\cdot) = a(t_0,\cdot),\end{cases}\tag{84}$$

where $\mathcal{L}c = [b,c] = (b\cdot\nabla_y)c - (c\cdot\nabla_y)b$. More exactly, we consider $\{t_k = k\Delta t, k\in\mathbb{N}\}$ a grid of points in $\mathbb{R}_+$. For any $k\in\mathbb{N}$, taking as initial condition $c(0,\cdot) = a(t_k,\cdot)$ and solving numerically the previous parabolic problem will lead to a numerical approximation for $\langle a(t_k,\cdot)\rangle$. Recall that, at least when the range of $\mathcal{L}$ is closed, the long time convergence is strong (in X_Q) and has exponential rate. Therefore, we expect to obtain a good numerical approximation for $\langle a(t_k,\cdot)\rangle$ after a reduced number of time steps in (84). Once that the approximation for $\langle a(t_k,\cdot)\rangle$ is available, we can solve for one time step in (82), and so on. We think that computing the average vector field $\langle a(t,\cdot)\rangle$ through the long time limit of (84) is a much robust method than computing it by appealing to the average formula (83). Same remark for averaging matrix fields, see Theorem 7.2. We will not go further in these directions here. This numerical analysis will be the topic of future works.

Our approach applies as well for non linear transport equations. For example, let us consider the finite Larmor radius regime for the Vlasov-Poisson system, that is, the electric potential is not given anymore, but satisfies the Poisson equation [23, 24, 27]. As before, we introduce the densities $(g^\varepsilon)_\varepsilon$ such that

$$f^\varepsilon(t,x,v) = g^\varepsilon(t, X(-t/\varepsilon;x,v), V(-t/\varepsilon;x,v)),\ \ (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^6,$$

and we study the behavior of the family $(g^\varepsilon)_\varepsilon$ when ε becomes small. Obviously, the asymptotic analysis requires much effort, since we deal with a non linear case : the electric potential writes as a convolution of f^ε or g^ε, and therefore the Vlasov equation, written in terms of g^ε, becomes non linear. Nevertheless, it does not contain singular terms with respect to ε, which legitimates searching for a limit profile $g = \lim_{\varepsilon\searrow 0} g^\varepsilon$. Combining some classical results of homogenization theory and multi-scale analysis [1, 26], it is possible to pass to the limit and to find out the model satisfied by the profile g, see the very recent analysis in [13].

We inquire now about collisional models. For example, we replace the Vlasov equation (5) by the Fokker-Planck equation

$$\partial_t f^\varepsilon + a\cdot\nabla_{x,v} f^\varepsilon + \frac{b}{\varepsilon}\cdot\nabla_{x,v} f^\varepsilon = C_{FP}(f^\varepsilon),\ \ C_{FP}(f) = \nu \mathrm{div}_v\{\Theta\nabla_v f + vf\}. \tag{85}$$

For simplicity we assume that the coefficients ν,Θ entering the Fokker-Planck collision operator do not depend on the small parameter ε. Nevertheless, it is possible to handle collision frequencies $\nu(\varepsilon)$ which become very large as ε goes to 0, leading to fluid models [9, 11]. Multiplying (85) by the test function $(t,y)\to\varphi(t,Y(-t/\varepsilon;y)), y=(x,v), Y=(X,V), \varphi\in C^1_c(\mathbb{R}_+\times\mathbb{R}^6)$ we obtain

$$\begin{aligned}
-\int_{\mathbb{R}^6} f^{\mathrm{in}}(y)\varphi(0,y)\,\mathrm{d}y - \int_{\mathbb{R}_+}\int_{\mathbb{R}^6} & f^\varepsilon(t,y)\{\partial_t\varphi(t) - \frac{1}{\varepsilon}b\cdot(\nabla_z\varphi)(t)\}(Y(-t/\varepsilon;y))\,\mathrm{d}y\mathrm{d}t \\
&- \int_{\mathbb{R}_+}\int_{\mathbb{R}^6} f^\varepsilon(t,y)a(t,y)\cdot {}^t\partial_y Y(-t/\varepsilon;y)\,(\nabla_z\varphi)(t,Y(-t/\varepsilon;y))\,\mathrm{d}y\mathrm{d}t \\
&- \int_{\mathbb{R}_+}\int_{\mathbb{R}^6} f^\varepsilon(t,y)\frac{b(y)}{\varepsilon}\cdot {}^t\partial_y Y(-t/\varepsilon;y)\,(\nabla_z\varphi)(t,Y(-t/\varepsilon;y))\,\mathrm{d}y\mathrm{d}t \\
&= -\nu\int_{\mathbb{R}_+}\int_{\mathbb{R}^6}(\Theta\nabla_v f^\varepsilon + vf^\varepsilon)\cdot {}^t\partial_v Y(-t/\varepsilon;y)\,(\nabla_z\varphi)(t,Y(-t/\varepsilon;y))\,\mathrm{d}y\mathrm{d}t.
\end{aligned} \tag{86}$$

We introduce the functions $(g^\varepsilon)_\varepsilon$ given by

$$f^\varepsilon(t,y) = g^\varepsilon(t,Y(-t/\varepsilon;y)),\ \ (t,y)\in\mathbb{R}_+\times\mathbb{R}^6,$$

and, as before, we use the identity

$$b(Y(-t/\varepsilon;y)) - \partial_y Y(-t/\varepsilon;y)\,b(y) = 0.$$

After performing the change of variables $z = Y(-t/\varepsilon;y)$, the weak formulation (86) reduces to

$$
\begin{aligned}
&-\int_{\mathbb{R}^6} f^{\mathrm{in}}(z)\varphi(0,z) - \int_{\mathbb{R}_+}\int_{\mathbb{R}^6} g^\varepsilon(t,z)\partial_t\varphi(t,z)\;\mathrm{d}z\mathrm{d}t \\
&\qquad\qquad - \int_{\mathbb{R}_+}\int_{\mathbb{R}^6} g^\varepsilon \partial_y Y(-t/\varepsilon; Y(t/\varepsilon;z))a(t, Y(t/\varepsilon;z))\cdot\nabla_z\varphi(t,z)\;\mathrm{d}z\mathrm{d}t \\
&\qquad\qquad = -\nu\int_{\mathbb{R}_+}\int_{\mathbb{R}^6} \Theta\partial_v Y(-t/\varepsilon;Y(t/\varepsilon;z))\;{}^t\partial_v Y(-t/\varepsilon;Y(t/\varepsilon;z))\nabla_z g^\varepsilon\cdot\nabla_z\varphi\;\mathrm{d}z\mathrm{d}t \\
&\qquad\qquad - \nu\int_{\mathbb{R}_+}\int_{\mathbb{R}^6} g^\varepsilon(t,z)\partial_v Y(-t/\varepsilon;Y(t/\varepsilon;z))V(t/\varepsilon;z)g^\varepsilon(t,z)\cdot\nabla_z\varphi\;\mathrm{d}z\mathrm{d}t.
\end{aligned}
$$

Motivated by the arguments in the proof of Theorem 2.3, we expect that $(g^\varepsilon)_\varepsilon$ converges in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^6))$ to the solution of the problem

$$
\begin{cases}
\partial_t g + \langle a(t)\rangle\cdot\nabla_z g = \nu \mathrm{div}_z\{\Theta\mathcal{D}(z)\nabla_z g + \mathcal{V}(z)g\}, & (t,z)\in\mathbb{R}_+\times\mathbb{R}^6,\\
g(0,z) = f^{\mathrm{in}}(z), & z\in\mathbb{R}^6,
\end{cases}
$$

where the matrix field $\mathcal{D}$ and vector field $\mathcal{V}$ are given by

$$
\mathcal{D}(z) = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \partial_v Y(-s;Y(s;z))\;{}^t\partial_v Y(-s;Y(s;z))\;\mathrm{d}s,
$$

$$
\mathcal{V}(z) = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \partial_v Y(-s;Y(s;z))V(s;z)\;\mathrm{d}s. \tag{87}
$$

Notice that the vector field in (87) is exactly the average of the vector field $y=(x,v)\to\;{}^t(0,v)$

$$
\begin{aligned}
\left\langle {}^t(0,v)\right\rangle(z) &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \partial_y Y(-s;Y(s;z))\;{}^t(0,V(s,z))\;\mathrm{d}s \\
&= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \partial_v Y(-s;Y(s;z))V(s,z)\;\mathrm{d}s = \mathcal{V}(z).
\end{aligned}
$$

Therefore the vector field $\mathcal{V}(z)$ can be approximated by appealing to Theorem 2.2. We concentrate now on the matrix field $\mathcal{D}$. Its construction is another consequence of the von Neumann's ergodic theorem. We introduce the Hilbert space of matrix fields

$$
H_Q = \left\{A(y)\;:\;\int_{\mathbb{R}^m} Q(y)A(y):A(y)Q(y)\;\mathrm{d}y < +\infty\right\},
$$

endowed with the scalar product

$$
(\cdot,\cdot)_Q : H_Q\times H_Q\to\mathbb{R},\;\;(A,B)_Q = \int_{\mathbb{R}^m} Q(y)A(y):B(y)Q(y)\;\mathrm{d}y.
$$

Here, $Q=Q(y)$ is a matrix field such that $P=Q^{-1}$ satisfies

$$
{}^tP = P,\;\; P(y)\xi\cdot\xi>0,\;\;\xi\in\mathbb{R}^m\setminus\{0\},\;\; y\in\mathbb{R}^m,\;\; P^{-1},P\in L^2_{\mathrm{loc}}(\mathbb{R}^m), \tag{88}
$$

and (22). The equality (23) suggests to consider the family of linear transformations $G(s): H_Q\to H_Q$

$$
G(s)A = \partial_y Y^{-1}(s;\cdot)A(Y(s;\cdot))\;{}^t\partial_y Y^{-1}(s;\cdot),\, s\in\mathbb{R},
$$

which is a C^0-group of unitary operators on H_Q (cf. Proposition 3.12 [10]). We denote by L the infinitesimal generator of the group $(G(s))_{s\in\mathbb{R}}$

$$
L:\mathrm{dom}(L)\subset H_Q\to H_Q,\;\;\mathrm{dom}L = \left\{A\in H_Q\;:\;\exists\lim_{s\to0}\frac{G(s)A-A}{s}\;\;\text{in}\;\;H_Q\right\},
$$

and $L(A) = \lim_{s\to 0} \frac{G(s)A-A}{s}$ for any $A \in \mathrm{dom}(L)$. Notice that $C^1_c(\mathbb{R}^m) \subset \mathrm{dom}(L)$ and $L(A) = b\cdot\nabla_y A - \partial_y b A - A\,{}^t\partial_y b$, $A \in C^1_c(\mathbb{R}^m)$ (use the hypothesis $Q \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ and the dominated convergence theorem). In other words $L(A)$ coincides with the bracket between b and A (see (22)) for any smooth matrix field A. The operator L, being the infinitesimal generator of a C^0-group of unitary transformations, is skew-adjoint on H_Q (see Proposition 3.13 [10] for other properties of L). As for vector fields, we can define the average of a matrix field along the C^0-group $(G(s))_{s\in\mathbb{R}}$.

Theorem 7.1 *Assume that* (10), (12), (88), (22) *hold true. Then for any matrix field* $A \in H_Q$ *we have the strong convergence in* H_Q

$$\lim_{T\to+\infty}\frac{1}{T}\int_r^{r+T}\partial_y Y(-s;Y(s;\cdot))A(Y(s;\cdot))\,{}^t\partial_y Y(-s;Y(s;\cdot))\,\mathrm{d}s = \mathrm{Proj}_{\ker L}A,$$

uniformly with respect to $r\in\mathbb{R}$. *If* $A \in H_Q$ *is a field of symmetric positive matrices, then so is* $\mathrm{Proj}_{\ker L}A$. *Moreover, if there is* $\alpha > 0$ *such that*

$$Q^{1/2}(y)A(y)Q^{1/2}(y) \geq \alpha,\ \ y\in\mathbb{R}^m,$$

therefore we have

$$Q^{1/2}(y)(\mathrm{Proj}_{\ker L}A)(y)Q^{1/2}(y) \geq \alpha,\ \ y\in\mathbb{R}^m,$$

and in particular, $(\mathrm{Proj}_{\ker L}A)(y)$ *is positive definite for any* $y\in\mathbb{R}^m$.

Proof. We detail only the last statement. By hypothesis we know that $G(s)P = P, s\in\mathbb{R}$, or equivalently $Q = {}^t\partial_y Y(s;y)Q_s\partial_y Y(s;y)$, with the notation $Q_s(\cdot) = Q(Y(s;\cdot))$. We introduce the matrix field $\mathcal{O}(s;\cdot) = Q_s^{1/2}\partial_y Y(s;\cdot)Q^{-1/2}$. Notice that

$${}^t\mathcal{O}(s;\cdot)\mathcal{O}(s;\cdot) = Q^{-1/2}\,{}^t\partial_y Y(s;\cdot)Q_s^{1/2}Q_s^{1/2}\partial_y Y(s;\cdot)Q^{-1/2} = I_m,$$

and therefore $\mathcal{O}(s;\cdot)$ is a field of orthogonal matrices. For any $\xi\in\mathbb{R}^m, \psi\in C^0_c(\mathbb{R}^m), \psi\geq 0$ we have $\psi(\cdot)P^{1/2}\xi\otimes P^{1/2}\xi \in H_Q$ and we can write

$$\begin{aligned}
(G(s)A,\psi(\cdot)P^{1/2}\xi\otimes P^{1/2}\xi)_Q &= \int_{\mathbb{R}^m} Q^{1/2}G(s)AQ^{1/2} : \psi(y)\xi\otimes\xi\,\mathrm{d}y\\
&= \int_{\mathbb{R}^m} Q^{1/2}(\partial_y Y)^{-1}(s;y)A_s\,{}^t(\partial_y Y)^{-1}(s;y)Q^{1/2} : \psi(y)\xi\otimes\xi\,\mathrm{d}y\\
&= \int_{\mathbb{R}^m}\psi(y)\,{}^t\mathcal{O}(s;y)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;y)\xi\cdot\xi\,\mathrm{d}y\\
&= \int_{\mathbb{R}^m}\psi(y)Q_s^{1/2}A_sQ_s^{1/2} : \mathcal{O}(s;y)\xi\otimes\mathcal{O}(s;y)\xi\,\mathrm{d}y\\
&\geq \alpha\int_{\mathbb{R}^m}|\mathcal{O}(s;y)\xi|^2\psi(y)\,\mathrm{d}y\\
&= \alpha|\xi|^2\int_{\mathbb{R}^m}\psi\,\mathrm{d}y.
\end{aligned}$$

Taking the average over $[0,T]$ and letting $T\to+\infty$ yield

$$\begin{aligned}
\int_{\mathbb{R}^m} Q^{1/2}\mathrm{Proj}_{\ker L}AQ^{1/2} : \xi\otimes\xi\psi(y)\,\mathrm{d}y &= (\mathrm{Proj}_{\ker L}A,\psi P^{1/2}\xi\otimes P^{1/2}\xi)_Q\\
&= \lim_{T\to+\infty}\frac{1}{T}\int_0^T (G(s)A,\psi P^{1/2}\xi\otimes P^{1/2}\xi)_Q\,\mathrm{d}s\\
&\geq \int_{\mathbb{R}^m}\alpha|\xi|^2\psi(y)\,\mathrm{d}y,
\end{aligned}$$

implying that

$$Q^{1/2}\mathrm{Proj}_{\ker L} A Q^{1/2} \geq \alpha, \;\; y \in \mathbb{R}^m.$$

□

The previous result allows us to interpret the matrix field $\mathcal{D}$ as the average of the diffusion matrix $D = \sum_{i=1}^{3} e_{v_i} \otimes e_{v_i}$ (written in variables $y = (x, v)$) of the Fokker-Planck operator

$$\langle D \rangle = \left\langle\!\!\left\langle \begin{pmatrix} O_3 & O_3 \\ O_3 & I_3 \end{pmatrix} \right\rangle\!\!\right\rangle = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \partial_y Y(-s; Y(s;z)) \begin{pmatrix} O_3 & O_3 \\ O_3 & I_3 \end{pmatrix} {}^t\partial_y Y(-s; Y(s;z))\,\mathrm{d}s$$
$$= \lim_{T\to+\infty} \frac{1}{T} \int_0^T \partial_v Y(-s; Y(s;z))\, {}^t\partial_v Y(-s; Y(s;z))\,\mathrm{d}s = \mathcal{D}(z).$$

For the numerical approximation of the average matrix field, we can solve for large time a parabolic problem. Following the arguments in the proof of Theorem 2.2, we obtain

Theorem 7.2 *Assume that* (10), (12), (88), (22) *hold true. For any matrix field* $A \in H_Q$ *we consider the problem*

$$\begin{cases} \partial_t C - L^2 C = 0, & t \in \mathbb{R}_+, \\ C(0,\cdot) = A(\cdot). \end{cases}$$

Then the solution converges weakly in H_Q*, as* $t \to +\infty$*, toward the orthogonal projection on* $\ker L$

$$\lim_{t\to+\infty} C(t) = \mathrm{Proj}_{\ker L} A, \quad \textit{weakly in } H_Q.$$

Moreover, if the range of L is closed, then the previous convergence holds strongly in H_Q*, and has exponential rate.*

Remark 7.1 *Assume that* (10), (12), (36) *hold true and take* $Q = {}^tRR$*. We prove that the range of L is closed iff the range of* $b \cdot \nabla_y$ *is closed (see Proposition 5.2 [10]).*

The above analysis leads to the following expression for the average Fokker-Planck collision kernel

$$\mathcal{C}_{FP}(g) = \nu \mathrm{div}_z \{\Theta \langle D \rangle \nabla_z g + \langle {}^t(0,v) \rangle g\}.$$

A natural question concerns the equilibria of the collision kernel $\mathcal{C}_{FP}$. Recall that the equilibria of the Fokker-Planck collision operator are given by $\rho(x)M(v)$, where M is the absolute Maxwellian, of temperature Θ

$$M(v) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|v|^2}{2\Theta}\right),$$

and ρ is a function of x. We claim that any function $\psi(y) = \rho(x_3)M(v)$ is a equilibrium of $\mathcal{C}_{FP}$. The key point is that $x_3, \frac{|\overline{v}|^2}{2}, v_3$ are left invariant by the flow $Y(s;\cdot)$, implying that

$$\psi(Y(-s;y)) = \psi(y), \;\; (s,y) \in \mathbb{R} \times \mathbb{R}^6.$$

For any test function $\chi = \chi(z) \in C_c^1(\mathbb{R}^6)$ and any $s \in \mathbb{R}$ we can write

$$\int_{\mathbb{R}^6} C_{FP}(\psi)(y)\chi(Y(-s;y))\,\mathrm{d}y = 0,$$

which implies after integration by parts

$$\int_{\mathbb{R}^6} \left\{\Theta D \nabla_y \psi + {}^t(0,v)\psi(y)\right\} \cdot {}^t\partial_y Y(-s;y)(\nabla_z \chi)(Y(-s;y))\,\mathrm{d}y = 0.$$

Taking into account that

$$\psi(Y(-s;y)) = \psi(y), \ \ {}^t\partial_y Y(-s;y)(\nabla_z\psi)(Y(-s;y)) = \nabla_y\psi(y),$$

and performing the change of coordinates $Y(-s;y) = z$, lead to

$$\int_{\mathbb{R}^6}\{\Theta\partial_y Y(-s;Y(s;z))D\,{}^t\partial_y Y(-s;Y(s;z))\nabla_z\psi + \psi\partial_y Y(-s;Y(s;z))\,{}^t(0,V(s;z))\}\cdot\nabla_z\chi\,\mathrm{d}z = 0.$$

Averaging with respect to s yields

$$\int_{\mathbb{R}^6}\{\Theta\langle D\rangle\nabla_z\psi + \psi(z)\left\langle {}^t(0,v)\right\rangle\}\cdot\nabla_z\chi\,\mathrm{d}z = 0, \ \ \chi\in C_c^1(\mathbb{R}^6),$$

and therefore

$$\mathcal{C}_{FP}(\psi) = \nu\mathrm{div}_z\{\Theta\langle D\rangle\nabla_z\psi + \psi(z)\left\langle {}^t(0,v)\right\rangle\} = 0, \ \ \psi = \rho(x_3)M(v),$$

saying that any function of the form $\rho(x_3)M(v)$ is a equilibrium for $\mathcal{C}_{FP}$.

Remark 7.2 *The Maxwellian M verifies $\Theta\langle D\rangle\nabla M + M\mathcal{V} = 0$ which also writes*

$$\mathcal{V}(z) = \langle D\rangle(z)\,{}^t(0,z_v),$$

where the notation z_v stands for the last three components of z. Indeed, the equality $|V(-s;y)|^2 = |v|^2$ implies ${}^t\partial_y Y(-s;Y(s;z))\,{}^t(0,z_v) = {}^t(0,V(s;z))$. Therefore we obtain

$$\begin{aligned}\langle D\rangle(z)\,{}^t(0,z_v) &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \partial_y Y(-s;Y(s;z))D\,{}^t\partial_y Y(-s;Y(s;z))\,{}^t(0,z_v)\,\mathrm{d}s\\ &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \partial_y Y(-s;Y(s;z))\,{}^t(0,V(s;z))\,\mathrm{d}s = \mathcal{V}(z).\end{aligned}$$

The converse statement (any equilibrium of $\mathcal{C}_{FP}$ is of the form $\rho(x_3)M(v)$) is much subtle and relies on the characterization of the kernel for the average diffusion matrix $\langle D\rangle$. By Theorem 7.1 we know that if a matrix field A is uniformly definite positive, then so is its average $\langle A\rangle$. But it may happens that a field of positive matrices generates, by average, a field of definite positive matrices. More generally we may have $\dim\ker\langle A\rangle < \dim\ker A$. This is why the average Fokker-Planck kernel will contain diffusion terms not only in velocity variables (as the original Fokker-Planck kernel), but also in space variables (orthogonal with respect to the magnetic lines). At least in the case of a periodic flow

$$\forall\, y\in\mathbb{R}^6, \ \exists\, T_y > 0 \ \text{ such that } Y(T_y;y) = y,$$

we prove that $\dim\ker\langle D\rangle = 1 < 3 = \dim\ker D$, and that any equilibrium of $\mathcal{C}_{FP}$ is of the form $\rho(x_3)M(v)$.

Proposition 7.1 *Assume that* (10), (12), (88), (22) *hold true. If the flow $Y(s;\cdot)$ is periodic, then for any $y = (x,v)\in\mathbb{R}^6$ we have* $\ker\langle D\rangle(y) = \mathbb{R}e_{x_3}$. *In particular, all the equilibria of $\mathcal{C}_{FP}$ are of the form $\rho(x_3)M(v)$.*

Proof. We have the strong convergence in H_Q

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T G(s)D\,\mathrm{d}s = \langle D\rangle,$$

and therefore

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T (G(s)D)(y)\,\mathrm{d}s = \langle D\rangle(y),\;\; y\in\mathbb{R}^6.$$

As the flow is periodic, the C^0-group $(G(s))_{s\in\mathbb{R}}$ is periodic and

$$\frac{1}{T_y}\int_0^{T_y}(G(s)D)(y)\,\mathrm{d}s = \lim_{T\to+\infty}\frac{1}{T}\int_0^T (G(s)D)(y)\,\mathrm{d}s = \langle D\rangle(y),\;\; y\in\mathbb{R}^6.$$

Since the matrices $D, G(s)D, \langle D\rangle$ are symmetric and positive, we can write

$$\xi\in\ker\langle D\rangle(y) \Leftrightarrow \langle D\rangle(y)\xi\cdot\xi = 0 \Leftrightarrow \frac{1}{T_y}\int_0^{T_y}(G(s)D)(y)\xi\cdot\xi\,\mathrm{d}s = 0$$
$$\Leftrightarrow (G(s)D)(y)\xi\cdot\xi = 0, s\in\mathbb{R} \Leftrightarrow (G(s)D)(y)\xi = 0, s\in\mathbb{R} \Leftrightarrow D\,{}^t\partial_y Y(-s;Y(s;y))\xi = 0, s\in\mathbb{R}$$
$$\Leftrightarrow ({}^t\partial_y Y(-s;Y(s;y))\xi)_v = 0, s\in\mathbb{R} \Leftrightarrow ({}^t(\partial_y Y)^{-1}(s;y)\xi)_v = 0, s\in\mathbb{R}.$$

In particular, if $\xi\in\ker\langle D\rangle(y)$, we have $\xi_v = 0$ (take $s=0$ in the previous equality) and

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\left({}^t(\partial_y Y)^{-1}(s;y)\xi\right)_v = -({}^t(\partial_y b)\xi)_v = -{}^t(\xi_{\overline{x}}, 0).$$

Conversely, any vector of $\mathbb{R}e_{x_3}$ belongs to $\ker G(s)D$, for any $s\in\mathbb{R}$ (observe that $G(s)D$ has zero entries on the third line and column) and thus any vector in $\mathbb{R}e_{x_3}$ belongs to $\ker\langle D\rangle$.

Consider now ψ such that $\mathcal{C}_{FP}(\psi)=0$. Using the relation $\Theta\langle D\rangle\nabla M + M\mathcal{V} = 0$ (see Remark 7.2), we can write

$$\Theta\langle D\rangle\nabla_z\psi + \psi\mathcal{V} = \Theta\langle D\rangle M\nabla_z\left(\frac{\psi}{M}\right),$$

and thus

$$0 = \int_{\mathbb{R}^6}\mathcal{C}_{FP}(\psi)\frac{\psi}{M}\,\mathrm{d}z = -\nu\int_{\mathbb{R}^6}\Theta\langle D\rangle : \nabla_z\left(\frac{\psi}{M}\right)\otimes\nabla_z\left(\frac{\psi}{M}\right)M\,\mathrm{d}z.$$

Since $\ker\langle D\rangle = \mathbb{R}e_{x_3}$, we deduce that $\frac{\psi(x,v)}{M(v)} = \rho(x_3), (x,v)\in\mathbb{R}^6$. □

The exact expression of the average diffusion matrix field $\langle D\rangle$ comes immediately, when the magnetic field is uniform. In this case, averaging $\partial_v Y(-s;Y(s;z))\,{}^t\partial_v Y(-s;Y(s;z))$, where

$$\partial_v Y(-s;Y(s;z)) = \begin{pmatrix} \frac{I_2-\mathcal{R}(\omega_c s)}{\omega_c}\mathcal{E} & O_{2\times1}\\ O_{1\times2} & 0\\ \mathcal{R}(\omega_c s) & O_{2\times1}\\ O_{1\times2} & 1\end{pmatrix},$$

leads to the average diffusion matrix

$$\langle D\rangle = \begin{pmatrix} \frac{2I_2}{\omega_c^2} & O_{2\times1} & -\frac{\mathcal{E}}{\omega_c} & O_{2\times1}\\ O_{1\times2} & 0 & O_{1\times2} & 0\\ \frac{\mathcal{E}}{\omega_c} & O_{2\times1} & I_2 & O_{2\times1}\\ O_{1\times2} & 0 & O_{1\times2} & 1\end{pmatrix}.$$

A C^0-groups of unitary operators

We detail here some technical arguments about C^0-groups of unitary opertors. We recall also the proof of von Neumann's ergodic theorem.

Proof. (of Proposition 3.2)
1. The operator $\mathcal{L}$ is the infinitesimal generator of a C^0-group, and therefore dom$\mathcal{L}$ is dense in X_Q and $\mathcal{L}$ is closed.
The statements 2 and 3 are general results, which hold true for the infinitesimal generator of any C^0-group of unitary operators.
2. For any $c \in \mathrm{dom}\mathcal{L}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s)c = \mathcal{L}\varphi(s)c = \varphi(s)\mathcal{L}c,$$

implying that

$$|\varphi(s)c - c|_Q = \left|\int_0^s \varphi(\tau)\mathcal{L}c \,\mathrm{d}\tau\right|_Q \le \left|\int_0^s |\varphi(\tau)\mathcal{L}c|_Q \,\mathrm{d}\tau\right| = \left|\int_0^s |\mathcal{L}c|_Q \,\mathrm{d}\tau\right| = |s||\mathcal{L}c|_Q, \ \ s \in \mathbb{R}.$$

Conversely, assume that (28) holds true. Let $(s_k)_k$ be a sequence converging to 0 such that

$$\lim_{k\to+\infty} \frac{\varphi(s_k)c - c}{s_k} = d \ \text{ weakly in } X_Q.$$

For any $w \in \mathrm{dom}\mathcal{L}$ we have

$$(\varphi(s_k)c - c, w)_Q = (c, \varphi(-s_k)w - w)_Q,$$

and thus

$$(d, w)_Q = \lim_{k\to+\infty}\left(\frac{\varphi(s_k)c - c}{s_k}, w\right)_Q = \lim_{k\to+\infty}\left(c, \frac{\varphi(-s_k)w - w}{s_k}\right)_Q = -(c, \mathcal{L}w)_Q. \tag{89}$$

Notice that (89) uniquely determines the weak limit d, since dom$\mathcal{L}$ is dense in X_Q. We estimate now the norms of $\left(\frac{\varphi(s_k)c-c}{s_k}\right)_k$, in order to convert the weak convergence to strong convergence. As before, we write for any $w \in \mathrm{dom}\mathcal{L}$

$$\begin{aligned}(\varphi(s_k)c - c, w)_Q = (c, \varphi(-s_k)w - w)_Q &= \left(c, \int_0^{-s_k} \mathcal{L}\varphi(\tau)w \,\mathrm{d}\tau\right)_Q \\ &= \int_0^{-s_k} (c, \mathcal{L}\varphi(\tau)w)_Q \,\mathrm{d}\tau = -\int_0^{-s_k} (d, \varphi(\tau)w)_Q \,\mathrm{d}\tau.\end{aligned}$$

In the last equality, we have used (89), with the element $\varphi(\tau)w \in \mathrm{dom}\mathcal{L}$, that is

$$(c, \mathcal{L}\varphi(\tau)w)_Q = -(d, \varphi(\tau)w)_Q.$$

Therefore, we obtain the estimate

$$\left(\frac{\varphi(s_k)c - c}{s_k}, w\right)_Q = -\frac{1}{s_k}\int_0^{-s_k} (d, \varphi(\tau)w)_Q \,\mathrm{d}\tau \le |d|_Q|w|_Q, \ \ w \in \mathrm{dom}\mathcal{L},$$

implying that

$$\limsup_{k\to+\infty}\left|\frac{\varphi(s_k)c - c}{s_k}\right|_Q \le |d|_Q.$$

Since d is the weak limit in X_Q of $\left(\frac{\varphi(s_k)c-c}{s_k}\right)_k$, we deduce that $\lim_{k\to+\infty}\frac{\varphi(s_k)c-c}{s_k}=d$ strongly in X_Q. Actually, all the family $\left(\frac{\varphi(s)c-c}{s}\right)_{s\in\mathbb{R}}$ converges strongly, when $s\to 0$, toward d in X_Q, thanks to the uniqueness of the limit cf. (89). Therefore $c\in \mathrm{dom}\mathcal{L}$ and $\mathcal{L}c=d$.

3. Take any two elements $c,d\in\mathrm{dom}\mathcal{L}$. Since $(\varphi(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators, we have

$$\left(\frac{\varphi(s)c-c}{s},d\right)_Q+\left(c,\frac{d-\varphi(-s)d}{s}\right)_Q=0,\ \ s\in\mathbb{R}.$$

Passing to the limit when $s\to 0$, implies

$$(\mathcal{L}c,d)_Q+(c,\mathcal{L}d)_Q=0,$$

saying that $d\in\mathrm{dom}\mathcal{L}^\star$ and $\mathcal{L}^\star d=-\mathcal{L}d$, for any $d\in\mathrm{dom}\mathcal{L}$. Therefore $\mathcal{L}\subset(-\mathcal{L}^\star)$. Consider now $d\in\mathrm{dom}\mathcal{L}^\star$, *i.e.*, $\exists K>0$ such that

$$|(\mathcal{L}c,d)_Q|\le K|c|_Q,\ \ c\in\mathrm{dom}\mathcal{L}. \tag{90}$$

In order to prove that $d\in\mathrm{dom}\mathcal{L}$, we use the characterization of the second statement. For any $c\in\mathrm{dom}\mathcal{L}$ we have

$$\begin{aligned}(\varphi(s)d-d,c)_Q&=(d,\varphi(-s)c-c)_Q\\&=\left(d,\int_0^{-s}\mathcal{L}\varphi(\tau)c\,\mathrm{d}\tau\right)_Q\\&=\int_0^{-s}(d,\mathcal{L}\varphi(\tau)c)_Q\,\mathrm{d}\tau.\end{aligned}$$

Thanks to (90), we have

$$|(d,\mathcal{L}\varphi(\tau)c)_Q|\le K|\varphi(\tau)c|_Q=K|c|_Q,$$

and therefore

$$(\varphi(s)d-d,c)_Q\le K|s||c|_Q,\ \ c\in\mathrm{dom}\mathcal{L}.$$

By the density of $\mathrm{dom}\mathcal{L}$, we deduce that $|\varphi(s)d-d|_Q\le K|s|, s\in\mathbb{R}$, saying that $d\in\mathrm{dom}\mathcal{L}$. Finally $\mathrm{dom}\mathcal{L}=\mathrm{dom}\mathcal{L}^\star$ and $\mathcal{L}^\star d=-\mathcal{L}d, d\in\mathrm{dom}\mathcal{L}=\mathrm{dom}\mathcal{L}^\star$. □

Proof. (of von Neumann's Theorem)

By Proposition 3.2 we know that A is skew-adjoint, and therefore $\overline{\mathrm{Range}A}=(\ker A^\star)^\perp=(\ker A)^\perp$. Thus we have the orthogonal decomposition $H=\ker A\oplus\overline{\mathrm{Range}A}$. We consider successively the cases $x\in\ker A$, $x\in\mathrm{Range}A$ and $x\in\overline{\mathrm{Range}A}$.

1. Assume that $x\in\ker A$. In this case we have $G(s)x=x$ for any $s\in\mathbb{R}$ and thus

$$\frac{1}{T}\int_r^{r+T}G(s)x\,\mathrm{d}s=x=\mathrm{Proj}_{\ker A}x,\ \ r\in\mathbb{R}.$$

2. Assume that $x\in\mathrm{Range}A$, that is $x=Ay, y\in\mathrm{dom}A$. In this case we write

$$\frac{1}{T}\int_r^{r+T}G(s)x\,\mathrm{d}s=\frac{1}{T}\int_r^{r+T}G(s)Ay\,\mathrm{d}s=\frac{1}{T}\int_r^{r+T}\frac{\mathrm{d}}{\mathrm{d}s}G(s)y\,\mathrm{d}s=G(r)\frac{G(T)y-y}{T}.$$

As $(G(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators, we deduce that

$$\left\|\frac{1}{T}\int_r^{r+T}G(s)x\,\mathrm{d}s\right\|\le 2\frac{\|y\|}{T},$$

saying that

$$\lim_{T\to+\infty} \frac{1}{T}\int_r^{r+T} G(s)x \, \mathrm{d}s = 0 = \mathrm{Proj}_{\ker A} x,$$

strongly in H, uniformly with respect to $r \in \mathbb{R}$.
3. Assume now that $x \in \overline{\mathrm{Range}A}$. For any $\varepsilon > 0$, there is $x_\varepsilon \in \mathrm{Range}A$ such that $\|x - x_\varepsilon\| < \varepsilon$. For any $r \in \mathbb{R}, T > 0$ we have

$$\left\|\frac{1}{T}\int_r^{r+T} G(s)x \, \mathrm{d}s\right\| \le \left\|\frac{1}{T}\int_r^{r+T} G(s)(x - x_\varepsilon) \, \mathrm{d}s\right\| + \left\|\frac{1}{T}\int_r^{r+T} G(s)x_\varepsilon \, \mathrm{d}s\right\|$$
$$\le \varepsilon + \left\|\frac{1}{T}\int_r^{r+T} G(s)x_\varepsilon \, \mathrm{d}s\right\|.$$

As $x_\varepsilon \in \mathrm{Range}A$, we deduce, thanks to the second case, that there is $T_\varepsilon > 0$ such that

$$\left\|\frac{1}{T}\int_r^{r+T} G(s)x_\varepsilon \, \mathrm{d}s\right\| < \varepsilon, \quad \text{for any } T > T_\varepsilon \text{ and any } r \in \mathbb{R}.$$

Therefore, for any $T > T_\varepsilon$ and any $r \in \mathbb{R}$, we have

$$\left\|\frac{1}{T}\int_r^{r+T} G(s)x \, \mathrm{d}s\right\| < 2\varepsilon,$$

saying that $\lim_{T\to+\infty} \frac{1}{T}\int_r^{r+T} G(s)x \, \mathrm{d}s = 0 = \mathrm{Proj}_{\ker A} x$, uniformly with respect to $r \in \mathbb{R}$. The general case, when $x \in H$, follows immediately, using the orthogonal decomposition $H = \ker A \oplus \overline{\mathrm{Range}A}$, and the first and third statements. □

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