EXISTENCE AND UNIQUENESS OF THE MILD SOLUTION FOR THE 1D VLASOV-POISSON INITIAL-BOUNDARY VALUE PROBLEM

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Abstract. *We prove the existence and uniqueness of the mild solution for the 1D Vlasov-Poisson system with initial-boundary conditions by using iterated approximations. The same arguments yield existence and uniqueness for the free space or space periodic system. The major difficulty is the treatment of the boundary conditions. The main idea consists of splitting the velocities range by introducing critical velocities corresponding to each boundary. One of the crucial points is to estimate the critical velocity change in term of relative field. A result concerning the continuity of the mild solution upon the initial-boundary conditions is presented as well.*

Key words. Vlasov-Poisson equations, Vlasov-Maxwell equations, weak/mild formulation.

AMS subject classifications. 35Q99, 35L50.

1. Introduction.

Many studies in the physics of charged particles are modeled by kinetic equations (Vlasov, Boltzmann, etc) coupled with the electromagnetic equations (Poisson, Maxwell). A few application domains are semiconductors, particle accelerators, electron guns, etc.

Various results have been obtained for the free space systems. Weak solutions for the Vlasov-Poisson system were constructed by Arseneev [1], Horst and Hunze [16]. The existence of classical solutions has been studied in two or three dimensions by Ukai and Okabe [21], Horst [15], Batt [2], Pfaffelmoser [18]. Classical solutions for the Vlasov-Poisson equations with small initial data have been constructed by Bardos and Degond [3]. The propagation of the velocity moments for the Vlasov-Poisson system in three dimensions has been studied by Lions and Perthame in [17]. They prove also an uniqueness result under a Lipschitz continuity assumption on the initial data. Another uniqueness result has been obtained by Robert for bounded, compactly supported initial data, [20]. A uniqueness result for BV solutions was obtained by Guo, Shu and Zhou [14].

The existence of weak solutions for the Vlasov-Maxwell system in three dimensions was shown by DiPerna and Lions [9]. The relativistic Vlasov-Maxwell system was studied by Glassey and Schaeffer [10]. In one dimension, the existence and uniqueness have been obtained by Cooper and Klimas [7].

The boundary value problem have been studied as well. The existence of weak solutions for the Vlasov-Poisson initial-boundary value problem in three dimensions is a result of Abdallah [4]. The existence of weak solutions for the three dimensional Vlasov-Maxwell initial-boundary value problem has been analysed by Guo [12]. The stationary one dimensional Vlasov-Poisson system has been studied by Greengard and Raviart [11]. An asymptotic analysis of the Vlasov-Poisson system has been performed by Degond and Raviart [8] in the case of the plane diode. The stationary Vlasov-Maxwell system in three dimensions was analysed by Poupaud [19]. The regularity of the solutions for the Vlasov-Maxwell system in a half line has been studied by Guo [13]. Results for the time periodic case can be found in [6] for the Vlasov-Poisson system and in [5] for the Vlasov-Maxwell system.

In this paper we study the existence and the uniqueness of the mild solution for the Vlasov-Poisson initial-boundary value problem in one dimension :

$$\partial_t f + v \cdot \partial_x f + E(t,x) \cdot \partial_v f = 0,\ (t,x,v) \in]0,T[\times]0,1[\times\mathbb{R}_v,$$

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$$f(t=0,x,v) = f_0(x,v),\ (x,v) \in]0,1[\times \mathbb{R}_v,$$

$$f(t,x,v) = g(t,x,v),\ (t,x,v) \in]0,T[\times \Sigma^-,$$

$$E(t,x) = -\partial_x U,\ \partial_x E = -\partial_x^2 U = \rho(t,x) := \int_{\mathbb{R}_v} f(t,x,v)dv,\ \ (t,x) \in]0,T[\times]0,1[,$$

$$U(t,x=0) = U_0(t),\ \ U(t,x=1) = U_1(t),\ \ t \in]0,T[.$$

The function $f(t,x,v)$ represents the particles distribution depending on the time t, the position x and the velocity v. The electric field $E(t,x)$ derives from an electrostatic potential U verifying the Poisson equation with the charge density $\rho(t,x) := \int_{\mathbb{R}_v} f(t,x,v)dv$. Here Σ^- is the subset of boundary of the phases space $]0,1[\times\mathbb{R}_v$ corresponding to the incoming velocities :

$$\Sigma^- = \{(0,v) \mid v>0\} \cup \{(1,v) \mid v<0\} = \Sigma_0^- \cup \Sigma_1^-.$$

Similarly we define also $\Sigma^+ = \{(0,v) \mid v<0\} \cup \{(1,v) \mid v>0\} = \Sigma_0^+ \cup \Sigma_1^+$ which corresponds to the outgoing velocities and $\Sigma^0 = \{(0,0),(1,0)\}$. With the notations $g|_{]0,T[\times\Sigma_0^-} = g_0, g|_{]0,T[\times\Sigma_1^-} = g_1$ the boundary condition writes :

$$f(t,x=0,v>0) = g_0(t,v>0),\ \ f(t,x=1,v<0) = g_1(t,v<0),\ \ t\in]0,T[.$$

The existence of weak solution for the Vlasov-Poisson initial-boundary value problem has been obtained in previous works; in [4] weak solutions of finite total (kinetic and electric) energy are constructed in dimension d, $d \leq 3$ by assuming initial-boundary conditions of finite kinetic, respectively flux of kinetic energy :

$$\int_0^1\int_{\mathbb{R}_v} f_0(x,v)|v|^2 dxdv + \sup_{0\leq t\leq T}\left\{\int_{v>0} v|v|^2 g_0(t,v)dv - \int_{v<0} v|v|^2 g_1(t,v)dv\right\} < +\infty,$$

and $|v|^\lambda f_0 \in L^\infty(]0,1[\times\mathbb{R}_v), |v|^\lambda g_0 \in L^\infty(]0,T[\times\mathbb{R}_v^+), |v|^\lambda g_1 \in L^\infty(]0,T[\times\mathbb{R}_v^-)$, for some $\lambda > d+1$. The main goal of this paper is to establish the existence and uniqueness of the mild solution (or solution by characteristics) in one dimension under less restrictive hypothesis, say for initial-boundary conditions of finite charge. As usual when studying coupled equations, we search the solutions as fixed points for some nonlinear application. For the 1D Vlasov-Poisson system this application writes for example $\mathcal{F} : B_R(X_T) \to B_R(X_T)$ where :

$$\mathcal{F}E(t,x) = \int_0^x \rho_E(t,y)dy - \int_0^1 (1-y)\rho_E(t,y)dy - U_1(t) + U_0(t), (t,x) \in]0,T[\times]0,1[,$$

where $\rho_E(t,x) = \int_{\mathbb{R}_v} f_E(t,x,v)dv$ and f_E solves the linear Vlasov problem associated to the field E and $B_R(X_T)$ is the ball of radius R of some space X_T. Naturally, in order to construct solutions by characteristics, which writes :

$$\frac{d}{ds}X(s;t,x,v) = V(s;t,x,v),\ \ \frac{d}{ds}V(s;t,x,v) = E(s,X(s;t,x,v)),\ \ s_{in}(t,x,v) \leq s \leq s_{out}(t,x,v),$$

the space X_T to be considered is $L^\infty(]0,T[;W^{1,\infty}(]0,1[))$. Here s_{in}/s_{out} represent the entry/exit time of the characteristics in the domain $]0,1[$ (see the next section for exact definitions). Since by construction $\partial_x\mathcal{F}E = \rho_E$ (conforming to the Poisson equation), it is clear that $B_R(X_T)$ is preserved by $\mathcal{F}$ provided that the charge density remains uniformly bounded in $L^\infty(]0,T[\times]0,1[)$. Therefore the natural hypothesis are :

$$\int_{\mathbb{R}_v} \sup_{0<x<1} f_0(x,v)dv + \int_{v>0} \sup_{0<t<T} g_0(t,v)dv + \int_{v<0} \sup_{0<t<T} g_1(t,v)dv < +\infty,$$

and :

$$\max\{\|f_0\|_{L^\infty(]0,1[\times\mathbb{R}_v)}, \|g_0\|_{L^\infty(]0,T[\times\mathbb{R}_v^+)}, \|g_1\|_{L^\infty(]0,T[\times\mathbb{R}_v^-)}\} < +\infty.$$

We intend to show the existence of an unique fixed point for $\mathcal{F}$ by using the iterated approximations method, which requires to estimate $\mathcal{F}A-\mathcal{F}B$ in term of $A-B$ for A, B different fields of X_T. This can be done by using the mild formulation of the Vlasov problem. Indeed, by using the continuity equation $\partial_t\rho_E+\partial_x j_E = 0$, $\mathcal{F}E$ can be represented also in term of the current density. Or estimate $\int_0^t j_A(s,x)ds - \int_0^t j_B(s,x)ds$ in $L^\infty(]0,1[)$ reduces to a duality calculation by taking the product by L^1 functions φ :

$$\begin{aligned}\langle\int_0^t (j_A(s,\cdot)-j_B(s,\cdot))ds, \varphi(\cdot)\rangle =& \int_0^t\int_0^1\int_{\mathbb{R}_v}(f_A(s,x,v)-f_B(s,x,v))v\varphi(x)dsdxdv\\ =& \int_0^t\int_{v>0} vg_0(\tau,v)\int_{X_B(s^0_{out})}^{X_A(s^0_{out})}\varphi(u)dud\tau dv\\ -& \int_0^t\int_{v<0} vg_1(\tau,v)\int_{X_B(s^1_{out})}^{X_A(s^1_{out})}\varphi(u)dud\tau dv\\ +& \int_0^1\int_{\mathbb{R}_v} f_0(x,v)\int_{X_B(s^i_{out})}^{X_A(s^i_{out})}\varphi(u)dudxdv,\end{aligned}$$

where $s^0_{out} = s_{out}(\tau,0,v), s^i_{out} = s_{out}(0,x,v), s^1_{out} = s_{out}(\tau,1,v)$ represent the exit times of the characteristics (see the next sections for the exact definitions). Note that for large velocities the integrand of the left boundary term vanishes since both $X_A(s_{out}(\tau,0,v)) = X_B(s_{out}(\tau,0,v)) = 1$. This suggest the definition of some critical velocities $v^0(t;\tau,0), v^1(t;\tau,0)$ such that :

$$s_{out}(\tau,0,v)<t,\ X(s_{out}(\tau,0,v);\tau,0,v) = 0,\ 0<v<v^0(t;\tau,0),$$

$$s_{out}(\tau,0,v)=t,\ 0<X(s_{out}(\tau,0,v);\tau,0,v)<1,\ v^0(t;\tau,0)<v<v^1(t;\tau,0),$$

$$s_{out}(\tau,0,v)<t,\ X(s_{out}(\tau,0,v);\tau,0,v) = 1,\ v>v^1(t;\tau,0).$$

Similar definitions hold for the right boundary term. One of the key point of our analysis consists on estimating the relative critical velocity. For non decreasing fields with respect to x, we have :

$$|v^k_A(t;\tau,k)-v^k_B(t;\tau,k)| \le \int_\tau^t \|A(s)-B(s)\|_{L^\infty(]0,1[)}ds,\ k=0,1,$$

and finally one gets :

$$\|\mathcal{F}A(t)-\mathcal{F}B(t)\|_{L^\infty(]0,1[)} \le C\int_0^t \|A(\tau)-B(\tau)\|_{L^\infty(]0,1[)}d\tau,$$

where C depends only on the $L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ norms of A, B and the initial-boundary conditions . We prove the following existence and uniqueness result :

THEOREM *Assume that there is $n_0, h_0, h_1 : [0,+\infty[\to[0,+\infty[$ bounded non increasing functions such that $f_0(x,v) \le n_0(|v|), \forall(x,v)\in]0,1[\times\mathbb{R}_v$, $g_0(t,v)\le h_0(v), \forall(t,v)\in]0,T[\times\mathbb{R}_v^+$, $g_1(t,v)\le h_1(-v), \forall(t,v)\in]0,T[\times\mathbb{R}_v^-$ and :*

$$\int_{\mathbb{R}_v} n_0(|v|)dv + \int_{v>0} h_0(v)dv + \int_{v<0} h_1(-v)dv < +\infty,$$

$$\max\{\|n_0\|_{L^\infty(\mathbb{R}_v^+)}, \|h_0\|_{L^\infty(\mathbb{R}_v^+)}, \|h_1\|_{L^\infty(\mathbb{R}_v^+)}, \|U_1-U_0\|_{L^\infty(]0,T[)}\} < +\infty.$$

Then there is an unique mild solution for the 1D Vlasov-Poisson initial-boundary value problem .

The estimate of the relative critical velocity, which is used for the treatment of the boundary terms, relies on some comparison results for characteristics associated to non decreasing fields, presented in Section 4. This is why, when studying the Vlasov-Poisson initial-boundary value problem we consider only one species of charged particles. All the definitions concerning the weak/mild formulations for the Vlasov or Vlasov-Poisson problem are recalled in Sections 2, 3. The main result on the existence and uniqueness of the mild solution is developed in Section 5 as well as a continuity result upon the initial-boundary conditions . The same method applies when studying the free or periodic space problem. Moreover, in this cases there are no boundary terms and thus the analysis on critical velocities not need to be used. This time the existence and uniqueness result can be obtained for general electric fields (not necessarily non decreasing in space) which allows us to treat systems with two species of charged particles (plasma globally neutral). Statements and sketch of the proofs can be found in Sections 6, 7.

2. The Vlasov equation.

The equation which models the transport of charged particles is called the Vlasov equation. In one dimension, if the particles move only under the action of an electric field this equation writes :

$$\partial_t f + v \cdot \partial_x f + E(t,x) \cdot \partial_v f = 0,\ (t,x,v) \in]0,T[\times]0,1[\times\mathbb{R}_v. \tag{2.1}$$

Here $E(t,x)$ is a given electric field which derives from a potential $U(t,x)$:

$$E(t,x) = -\partial_x U,\ (t,x) \in]0,T[\times]0,1[.$$

The initial-boundary conditions for the particles distribution are given by :

$$f(t=0,x,v) = f_0(x,v),\ (x,v) \in]0,1[\times\mathbb{R}_v, \tag{2.2}$$

$$f(t,x=0,v>0) = g_0(t,v>0),\quad f(t,x=1,v<0) = g_1(t,v<0),\quad t \in]0,T[. \tag{2.3}$$

Now let us briefly recall the definitions of weak and mild solutions for the Vlasov problem (2.1), (2.2) and (2.3).

2.1. Weak solutions for the Vlasov-Poisson problem.

Definition 2.1. *Assume that* $E \in L^\infty(]0,T[\times]0,1[)$, $f_0 \in L^1_{loc}(]0,1[\times\mathbb{R}_v)$, $vg_0 \in L^1_{loc}(]0,T[\times\mathbb{R}_v^+)$, $vg_1 \in L^1_{loc}(]0,T[\times\mathbb{R}_v^-)$. *We say that* $f \in L^1_{loc}(]0,T[\times]0,1[\times\mathbb{R}_v)$ *is a weak solution for the Vlasov problem* (2.1)*,* (2.2)*,* (2.3) *iff :*

$$-\int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)\cdot(\partial_t\varphi + v\cdot\partial_x\varphi + E(t,x)\cdot\partial_v\varphi)dtdxdv = \int_0^1\int_{\mathbb{R}_v} f_0(x,v)\varphi(0,x,v)dxdv$$
$$+\int_0^T\int_{v>0} vg_0(t,v)\varphi(t,0,v)dtdv - \int_0^T\int_{v<0} vg_1(t,v)\varphi(t,1,v)dtdv,$$

for all test function $\varphi \in \mathcal{T}_w$ *where :*

$$\mathcal{T}_w = \{\varphi \in W^{1,\infty}(]0,T[\times]0,1[\times\mathbb{R}_v) \mid \varphi|_{]0,T[\times\Sigma^+} = \varphi(T,\cdot,\cdot) = 0, \exists R : supp(\varphi) \subset [0,T]\times[0,1]\times B_R\}.$$

2.2. Mild solutions for the Vlasov problem.

We need to consider also some special solutions of $(2.1), (2.2), (2.3)$ which are called mild solutions or solutions by characteristics. These solutions require more regularity on the electric field and they are particular cases of weak solutions. Assume that $E \in L^\infty(]0,T[; W^{1,\infty}(]0,1[))$ and for $(t,x,v) \in \{[0,T[\times]0,1[\times\mathbb{R}_v\} \cup \{]0,T[\times\Sigma^-\}$ let us denote by $(X(s;t,x,v), V(s;t,x,v))$ the unique solution of the ordinary differential system of equations :

$$\frac{d}{ds}X(s;t,x,v) = V(s;t,x,v), \;\; \frac{d}{ds}V(s;t,x,v) = E(s, X(s;t,x,v)), \; s_{in} \le s \le s_{out}, \tag{2.4}$$

which verify the conditions :

$$X(s=t;t,x,v) = x, \;\; V(s=t;t,x,v) = v.$$

Here $s_{in} = s_{in}(t,x,v)$ (resp. $s_{out} = s_{out}(t,x,v)$) represents the incoming (resp. outgoing) time of the characteristics in the domain $]0,1[$ defined by :

$$s_{in}(t,x,v) = \max\{0, \sup\{0 \le s \le t \; : \; X(s;t,x,v) \in \{0,1\}\}\}, \tag{2.5}$$

and :

$$s_{out}(t,x,v) = \min\{T, \inf\{T \ge s \ge t \; : \; X(s;t,x,v) \in \{0,1\}\}\}. \tag{2.6}$$

The total travel time through the domain (lifetime) writes $\tau(t,x,v) = s_{out}(t,x,v) - s_{in}(t,x,v) \le T$. Now we replace in the Definition 2.1 the function $\partial_t\varphi + v\cdot\partial_x\varphi + E(t,x)\cdot\partial_v\varphi$ by ψ, which gives after integration :

$$\varphi(t,x,v) = -\int_t^{s_{out}(t,x,v)} \psi(s, X(s;t,x,v), V(s;t,x,v))ds,$$

and we define the mild solution as follows :

Definition 2.2. *Assume that* $E \in L^\infty(]0,T[; W^{1,\infty}(]0,1[))$, $f_0 \in L^1_{loc}(]0,1[\times\mathbb{R}_v)$, $vg_0 \in L^1_{loc}(]0,T[\times\mathbb{R}^+_v)$, $vg_1 \in L^1_{loc}(]0,T[\times\mathbb{R}^-_v)$. *We say that* $f \in L^1_{loc}(]0,T[\times]0,1[\times\mathbb{R}_v)$ *is a mild solution for the Vlasov problem* $(2.1), (2.2), (2.3)$ *iff :*

$$\begin{aligned}\int_0^T\!\!\int_0^1\!\!\int_{\mathbb{R}_v} f(t,x,v)\psi(t,x,v)dtdxdv = &\int_0^1\!\!\int_{\mathbb{R}_v} f_0(x,v)\int_0^{s_{out}(0,x,v)} \psi(s, X(s;0,x,v), V(s;0,x,v))dsdxdv\\ &+\int_0^T\!\!\int_{v>0} vg_0(t,v)\int_t^{s_{out}(t,0,v)} \psi(s, X(s;t,0,v), V(s;t,0,v))dsdtdv\\ &-\int_0^T\!\!\int_{v<0} vg_1(t,v)\int_t^{s_{out}(t,1,v)} \psi(s, X(s;t,1,v), V(s;t,1,v))dsdtdv,\end{aligned}$$

for all test function $\psi \in \mathcal{T}_m$ *where :*

$$\mathcal{T}_m = \{\psi \in L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v) \mid \exists R > 0 : supp(\psi) \subset [0,T]\times[0,1]\times B_R\}.$$

In order to simplify the formulas we shall use the following notations :

$$(X(s),V(s)) = (X(s;t,x,v), V(s;t,x,v)), \;\; (X^0(s), V^0(s)) = (X(s;t,0,v), V(s;t,0,v)),$$

$$(X^1(s),V^1(s)) = (X(s;t,1,v), V(s;t,1,v)), \;\; (X^i(s), V^i(s)) = (X(s;0,x,v), V(s;0,x,v)),$$

and :

$$s_{in} = s_{in}(t,x,v), \;\; s_{out} = s_{out}(t,x,v), \;\; s^0_{out} = s_{out}(t,0,v), \;\; s^1_{out} = s_{out}(t,1,v), \;\; s^i_{out} = s_{out}(0,x,v).$$

REMARK 2.3. *It is well known that the mild solution is unique and is given by* $f(t,x,v) = g_k(s_{in}, V(s_{in}))$ *if* $s_{in}(t,x,v) > 0,\ X(s_{in}(t,x,v);t,x,v) = k, k = 0,1,\ f(t,x,v) = f_0(X(s_{in}), V(s_{in}))$ *if* $s_{in}(t,x,v) = 0$.

Note that every mild solution is also weak solution. Moreover, the existence of weak solution for the Vlasov problem with bounded initial-boundary conditions $f_0, g_0, g_1 \in L^\infty$, follows by regularization of the electric field with respect to x by convolution with $\zeta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\zeta\left(\frac{\cdot}{\varepsilon}\right),\ \zeta \in C_0^\infty,\ \mathrm{supp}(\zeta) = [-1,1], \zeta \geq 0,\ \int_{\mathbb{R}} \zeta(u)du = 1$, and by passing to the limit for $\varepsilon \searrow 0$ in the weak formulation of f^ε, the mild solution associated to $E^\varepsilon = E \star \zeta_\varepsilon$.

3. The Vlasov-Poisson system.

The self-consistent electric field solves the Poisson equation :

$$\partial_x E = -\partial_x^2 U = \rho(t,x) := \int_{\mathbb{R}_v} f(t,x,v)dv,\ (t,x) \in]0,T[\times]0,1[, \tag{3.1}$$

with the boundary conditions :

$$U(t, x = 0) = U_0(t),\ \ U(t, x = 1) = U_1(t),\ \ t \in]0,T[. \tag{3.2}$$

The system formed by $(2.1), (3.1), (2.2), (2.3), (3.2)$ is called the Vlasov-Poisson initial-boundary value problem in one dimension. Obviously the electric field writes :

$$E(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - U_1(t) + U_0(t),\ (t,x) \in]0,T[\times]0,1[, \tag{3.3}$$

and therefore we can give the following definitions :

DEFINITION 3.1. *Assume that* $f_0 \in L^1_{loc}(]0,1[\times\mathbb{R}_v), vg_0 \in L^1_{loc}(]0,T[\times\mathbb{R}_v^+), vg_1 \in L^1_{loc}(]0,T[\times\mathbb{R}_v^-)$, $U_1 - U_0 \in L^\infty(]0,T[)$. *We say that* $(f,E) \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v) \times L^\infty(]0,T[\times]0,1[)$ *(resp.* $(f,E) \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v) \times L^\infty(]0,T[;W^{1,\infty}(]0,1[))$*) is a weak (resp. mild) solution for the Vlasov-Poisson problem iff* f *is a weak (resp. mild) solution for the Vlasov problem* $(2.1), (2.2), (2.3)$ *corresponding to the electric field* (3.3) *given by the Poisson problem.*

4. Characteristics.

The main tool of our analysis is the mild formulation of the Vlasov problem. In order to estimate the charge and current densities we need more informations about the characteristics. We present here some properties of the characteristics associated to regular, non decreasing with respect to x fields.

PROPOSITION 4.1. *Assume that* $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ *is non decreasing with respect to* x*and that* $(X_1(s), V_1(s)),\ (X_2(s), V_2(s))$ *are two characteristics such that there is* $s_1 < s_2$ *verifying* $X_1(s_i) = X_2(s_i),\ i = 1,2$. *Then the characteristics coincide :* $(X_1(s), V_1(s)) = (X_2(s), V_2(s)), \forall s$.

Proof. The conclusion follows easily after multiplication of the equation $\frac{d^2}{ds^2}(X_1(s) - X_2(s)) = E(s, X_1(s)) - E(s, X_2(s))$ by $X_1(s) - X_2(s)$ and integration by parts on $[s_1, s_2]$. □

PROPOSITION 4.2. *Assume that* $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ *is non decreasing with respect to* x. *If* $v_1 < v_2$ *then we have :*

$$X(s;t,x,v_1) < X(s;t,x,v_2),\ V(s;t,x,v_1) < V(s;t,x,v_2),\ \forall s \in]t, s_{out}(t,x,v_1)] \cap]t, s_{out}(t,x,v_2)],$$

and :

$$X(s;t,x,v_1) > X(s;t,x,v_2),\ V(s;t,x,v_1) < V(s;t,x,v_2),\ \forall s \in [s_{in}(t,x,v_1), t[\cap [s_{in}(t,x,v_2)[.$$

Proof. Suppose that there is $s \in [s_{in}(t,x,v_1), s_{out}(t,x,v_1)] \cap [s_{in}(t,x,v_2), s_{out}(t,x,v_2)], s \neq t$ such that $X(s;t,x,v_1) = X(s;t,x,v_2)$. Since $X(t;t,x,v_1) = X(t;t,x,v_2) = x$, by the Proposition 4.1 it follows that the characteristics coincide, and thus $v_1 = v_2$ which is in contradiction with the hypothesis. Therefore $X(s;t,x,v_1) - X(s;t,x,v_2)$ has constant sign on the intervals $[s_{in}(t,x,v_1), t[\cap[s_{in}(t,x,v_2), t[$ and $]t, s_{out}(t,x,v_1)]\cap]t, s_{out}(t,x,v_2)]$. On the other hand we have :

$$\frac{d}{ds}(X(s;t,x,v_1) - X(s;t,x,v_2))|_{s=t} = v_1 - v_2 < 0,$$

and therefore $X(s;t,x,v_1) - X(s;t,x,v_2)$ is decreasing locally in $s = t$. We deduce that :

$$X(s;t,x,v_1) > X(s;t,x,v_2),\ s \in [s_{in}(t,x,v_1), t[\cap[s_{in}(t,x,v_2), t[,$$

and :

$$X(s;t,x,v_1) < X(s;t,x,v_2),\ s \in]t, s_{out}(t,x,v_1)]\cap]t, s_{out}(t,x,v_2)].$$

By using the characteristics equations one gets :

$$\frac{d}{ds}(V(s;t,x,v_1) - V(s;t,x,v_2)) = E(s, X(s;t,x,v_1)) - E(s, X(s;t,x,v_2)),$$

and thus $V(s;t,x,v_1) - V(s;t,x,v_2)$ is non decreasing on $[s_{in}(t,x,v_1), t[\cap[s_{in}(t,x,v_2), t[$ and non increasing on $]t, s_{out}(t,x,v_1)]\cap]t, s_{out}(t,x,v_2)]$. We deduce that :

$$V(s;t,x,v_1) - V(s;t,x,v_2) \leq v_1 - v_2 < 0,\ s \in [s_{in}(t,x,v_1), s_{out}(t,x,v_1)]\cap[s_{in}(t,x,v_2), s_{out}(t,x,v_2)].$$

□

When using the mild formulation of the Vlasov problem it is important to distinguish the characteristics with respect to the exit point. This justifies the following definitions : for $(t,x) \in \{[0,T[\times]0,1[\} \cup \{]0,T[\times\{0,1\}\}$ we denote by $\mathcal{V}^0, \mathcal{V}^1, \mathcal{V}^T$ the subsets of $\mathbb{R}_v$ given by :

$$\mathcal{V}^0(T;t,x) := \{v \in \mathbb{R}_v\ :\ s_{out}(t,x,v) < T,\ X(s_{out}(t,x,v);t,x,v) = 0\}, \tag{4.1}$$

$$\mathcal{V}^1(T;t,x) := \{v \in \mathbb{R}_v\ :\ s_{out}(t,x,v) < T,\ X(s_{out}(t,x,v);t,x,v) = 1\}, \tag{4.2}$$

$$\mathcal{V}^T(T;t,x) := \{v \in \mathbb{R}_v\ :\ s_{out}(t,x,v) = T,\ 0 < X(T;t,x,v) < 1\}. \tag{4.3}$$

Note that when E is bounded there is R large enough such that $]-\infty, -R[\subset \mathcal{V}^0(T;t,x)$ and $]R, +\infty[\subset \mathcal{V}^1(T;t,x)$ and thus $\mathcal{V}^0(T;t,x) \neq \emptyset, \mathcal{V}^1(T;t,x) \neq \emptyset$. By the definition $\mathcal{V}^0(T;t,x) \cap \mathcal{V}^1(T;t,x) = \emptyset$ and $\mathcal{V}^T(T;t,x) \cap \{\mathcal{V}^0(T;t,x) \cup \mathcal{V}^1(T;t,x)\} = \emptyset$.

PROPOSITION 4.3. *Assume that $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ is non decreasing with respect to x. Then we have :*
(1) *if $v_2 \in \mathcal{V}^0(T;t,x)$ then $v_1 \in \mathcal{V}^0(T;t,x), \forall v_1 < v_2$;*
(2) *if $v_1 \in \mathcal{V}^1(T;t,x)$ then $v_2 \in \mathcal{V}^1(T;t,x), \forall v_2 > v_1$;*
(3) *if $v_1 \in \mathcal{V}^0(T;t,x),\ v_2 \in \mathcal{V}^1(T;t,x)$, then $v_1 < v_2$.*

Proof. (1) Suppose that $s_{out}(t,x,v_1) \geq s_{out}(t,x,v_2)$. By the Proposition 4.2 we deduce that : $X(s;t,x,v_1) < X(s;t,x,v_2), \forall s \in]t, s_{out}(t,x,v_1)]\cap]t, s_{out}(t,x,v_2)] =]t, s_{out}(t,x,v_2)]$. In particular for $s = s_{out}(t,x,v_2)$ we find that : $0 \leq X(s_{out}(t,x,v_2);t,x,v_1) < X(s_{out}(t,x,v_2);t,x,v_2) = 0$, which is not possible. Finally it comes that $s_{out}(t,x,v_1) < s_{out}(t,x,v_2) < T$ and :

$$X(s_{out}(t,x,v_1);t,x,v_1) < X(s_{out}(t,x,v_1);t,x,v_2) < 1.$$

We deduce that $X(s_{out}(t,x,v_1);t,x,v_1) = 0$ or $v_1 \in \mathcal{V}^0(T;t,x)$.

(2) Similarly, if $v_1 \in \mathcal{V}^1(T;t,x,)$ and $v_1 < v_2$ we have $s_{out}(t,x,v_2) < s_{out}(t,x,v_1) < T$ (otherwise $1 = X(s_{out}(t,x,v_1);t,x,v_1) < X(s_{out}(t,x,v_1);t,x,v_2))$ and $0 < X(s_{out}(t,x,v_2);t,x,v_1) < X(s_{out}(t,x,v_2);t,x,v_2)$. We deduce that $X(s_{out}(t,x,v_2);t,x,v_2) = 1$ and $v_2 \in \mathcal{V}^1(T;t,x)$.

(3) Suppose that $v_1 \geq v_2$. Since $v_1 \in \mathcal{V}^0(T;t,x)$, by (1) it follows that $v_2 \in \mathcal{V}^0(T;t,x) \cap \mathcal{V}^1(T;t,x) = \emptyset$. Therefore we have $v_1 < v_2$. □

We introduce the critical velocities $v^0(T;t,x), v^1(T;t,x)$ given by :

$$v^0(T;t,x) := \sup \mathcal{V}^0(T;t,x), \;\; v^1(T;t,x) := \inf \mathcal{V}^1(T;t,x). \tag{4.4}$$

Obviously we have $-\infty < v^0(T;t,x) \leq v^1(T;t,x) < +\infty$.

Proposition 4.4. *Assume that $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ is non decreasing with respect to x. We have :*
(1) $]-\infty, v^0(T;t,x)[\subset \mathcal{V}^0(T;t,x) \subset]-\infty, v^0(T;t,x)]$;
(2) $]v^1(T;t,x), +\infty[\subset \mathcal{V}^1(T;t,x) \subset [v^1(T;t,x), +\infty[$;
(3) $]v^0(T;t,x), v^1(T;t,x)[\subset \mathcal{V}^T(T;t,x) \subset [v^0(T;t,x), v^1(T;t,x)]$.

Proof. From the Proposition 4.3 and the definitions of v^0, v^1 we deduce (1) and (2). By the other hand $\mathcal{V}^T(T;t,x) \subset \mathbb{R}_v - \{\mathcal{V}^0(T;t,x) \cup \mathcal{V}^1(T;t,x)\} \subset \mathbb{R}_v - \{]-\infty, v^0(T;t;x)[\cup]v^1(T;t,x), +\infty[\}$ $= [v^0(T;t,x), v^1(T;t,x)]$. Let us prove that $]v^0, v^1[\subset \mathcal{V}^T$. Consider $v^0 < v < v^1$, if $v^0 < v^1$. Suppose that $s_{out}(t,x,v) < T$ with $X(s_{out}(t,x,v);t,x,v) = 0$, or $v \in \mathcal{V}^0(T;t,x)$. By the Proposition 4.3 we deduce that $\tilde{v} \in \mathcal{V}^0(T;t,x), \forall v^0 < \tilde{v} < v$ which is in contradiction with $\tilde{v} > v^0 = \sup \mathcal{V}^0(T;t,x)$. The same arguments apply for $s_{out}(t,x,v) < T, X(s_{out}(t,x,v);t,x,v) = 1$, by taking $v < \tilde{v} < v^1$. It comes that $s_{out}(t,x,v) = T, \; \forall v^0 < v < v^1$. Suppose now that $X(T;t,x,v) = 0$. If we take $v^0 < \tilde{v} < v$ we deduce that $s_{out}(t,x,\tilde{v}) = T$ and by the Proposition 4.2 we find that $0 \leq X(T;t,x,\tilde{v}) < X(T;t,x,v) = 0$. Similarly we can show that $X(T;t,x,v) = 1$ is not possible. Finally we deduce that $X(T;t,x,v) \in]0,1[, \forall v^0 < v < v^1$, and thus $]v^0, v^1[\subset \mathcal{V}^T$. □

Let us consider two fields A, B. In order to prove the uniqueness of the mild solution for the Vlasov-Poisson problem, it will be useful to estimate the change of critical velocity $|v_A^k - v_B^k|, k = 0, 1$ with respect to the relative field $A - B$. For this we need to introduce the notion of sub/super-characteristics :

Definition 4.5. *Assume that $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ is non decreasing with respect to x. We say that $(X(s), V(s))$ is a sub-characteristic (resp. super-characteristic) iff X is twice differentiable with respect to s and :*

$$\frac{dX}{ds} = V(s), \;\; \frac{dV}{ds} \leq E(s, X(s)), \;\; s_{in} \leq s \leq s_{out},$$

(resp. :

$$\frac{dX}{ds} = V(s), \;\; \frac{dV}{ds} \geq E(s, X(s)), \;\; s_{in} \leq s \leq s_{out},)$$

with the same definitions for s_{in}, s_{out} as before.

We have the following comparison result :

Proposition 4.6. *(Forward Comparison) Assume that $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ is non decreasing with respect to x. Consider $(\underline{X}(s), \underline{V}(s)), (\overline{X}(s), \overline{V}(s))$ a sub-characteristic, resp. a super-characteristic such that : $\underline{X}(t) \leq \overline{X}(t), \;\; \underline{V}(t) \leq \overline{V}(t)$. Then we have :*

$$\underline{X}(s) \leq \overline{X}(s), \;\; \underline{V}(s) \leq \overline{V}(s), \;\; \forall s \in [t, \underline{s}_{out}] \cap [t, \overline{s}_{out}].$$

Proof. We can extend the field E to $]0,T[\times\mathbb{R}_x$ by $\tilde{E}(t,x) = E(t,0), x < 0$ and $\tilde{E}(t,x) = E(t,1), x > 1$. We have $\|\tilde{E}\|_{L^\infty(]0,T[;W^{1,\infty}(]0,1[)} \le \|E\|_{L^\infty(]0,T[;W^{1,\infty}(]0,1[))}$ and $\tilde{E}$ is non decreasing with respect to x. Consider $(x,v) \in \mathbb{R}_x \times \mathbb{R}_v$ such that $\underline{X}(t) \le x \le \overline{X}(t)$, $\underline{V}(t) \le v \le \overline{V}(t)$. Denote by $(X(s;t,x,v), V(s;t,x,v))$ the characteristic associated to the field $\tilde{E}$:

$$\frac{dX}{ds} = V(s), \;\; \frac{dV}{ds} = \tilde{E}(s, X(s)),$$

with the conditions $X(s=t;t,x,v) = x$, $V(s=t;t,x,v) = v$. We show that $\underline{X}(s) \le X(s) \le \overline{X}(s)$, $\underline{V}(s) \le V(s) \le \overline{V}(s)$, for all $s \in [t, \underline{s}_{out}] \cap [t, \overline{s}_{out}]$. For this we can use the iterated approximations method. For example, in order to prove that $\underline{X} \le X, \underline{V} \le V$ we consider as first approximation $X^0 = \underline{X}, V^0 = \underline{V}$ and we define $X^{n+1}(s) = x + \int_t^s V^n(\tau)d\tau$, $V^{n+1}(s) = v + \int_t^s \tilde{E}(\tau, X^n(\tau))d\tau$, $\forall s \in [t, \underline{s}_{out}]$, $\forall n \ge 0$. We check easily that $X^n(s) \ge \underline{X}(s), V^n(s) \ge \underline{V}(s), \forall s \in [t, \underline{s}_{out}]$ and by passing to the limit for $n \to +\infty$ we find that $X(s) \ge \underline{X}(s)$, $V(s) \ge \underline{V}(s)$, $\forall s \in [t, \underline{s}_{out}]$. In the same way, by taking as initial approximation $(X^0, V^0) = (\overline{X}, \overline{V})$ we prove that $X(s) \le \overline{X}(s), V(s) \le \overline{V}(s), \forall s \in [t, \overline{s}_{out}]$. Finally we have :

$$\underline{X}(s) \le X(s) \le \overline{X}(s), \; \underline{V}(s) \le V(s) \le \overline{V}(s), \forall s \in [t, \underline{s}_{out}] \cap [t, \overline{s}_{out}].$$

□

REMARK 4.7. *In fact, since* $0 \le \underline{X}(s), \overline{X}(s) \le 1 \;\forall\; t \le s \le \min\{\underline{s}_{out}, \overline{s}_{out}\}$ *it follows that* $0 \le X(s) \le 1 \;\forall\; t \le s \le \min\{\underline{s}_{out}, \overline{s}_{out}\}$ *and therefore* (X,V) *coincide with the characteristic associated to the field E. Moreover,* $s_{out}(t,x,v) \ge \min\{\underline{s}_{out}, \overline{s}_{out}\}$.

Now we are ready to prove a result of continuous dependence of the critical velocities with respect to the electric field. We have the following lemma :

LEMMA 4.8. *(Critical velocity change) Assume that* $A, B \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ *are non decreasing with respect to x. Then for all* $(t,x) \in [0,T[\times[0,1]$ *we have the following inequality :*

$$|v_A^k(T;t,x) - v_B^k(T;t,x)| \le \int_t^T \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds, \;\; k = 0,1. \tag{4.5}$$

Proof. Denote by $m = \|A - B\|_{L^1(]t,T[;L^\infty(]0,1[))}$. Let us prove for example that $|v_A^0 - v_B^0| \le m$. Suppose that $v_A^0 - v_B^0 > m$. Therefore there is $v > v_B^0$ such that $\tilde{v} = v + m < v_A^0$ and thus we deduce from the Proposition 4.4 that $X_B(s_{out}^B(t,x,v);t,x,v) > 0$, $X_A(s_{out}^A(t,x,\tilde{v});t,x,\tilde{v}) = 0$, $s_{out}^A(t,x,\tilde{v}) < T$. Consider the solution (X_C, V_C) of the following system of ordinary differential equations :

$$\frac{dX_C}{ds} = V_C(s), \;\; \frac{dV_C}{ds} = B(s, X_A(s)), \;\; t \le s \le s_{out}^C(t,x,v),$$

with the conditions $X_C(t) = x, V_C(t) = v$. With the notations :

$$(X_A(s), V_A(s)) = (X_A(s;t,x,\tilde{v}), V_A(s;t,x,\tilde{v})), \; t \le s \le s_{out}^A(t,x,\tilde{v}),$$

and :

$$(X_B(s), V_B(s)) = (X_B(s;t,x,v), V_B(s;t,x,v)), \; t \le s \le s_{out}^B(t,x,v),$$

we have also :

$$\frac{dX_A}{ds} = V_A(s), \;\; \frac{dV_A}{ds} = A(s, X_A(s)), \;\; t \le s \le s_{out}^A(t,x,\tilde{v}),$$

with $X_A(t) = x, V_A(t) = \tilde{v}$ and :

$$\frac{dX_B}{ds} = V_B(s), \;\; \frac{dV_B}{ds} = B(s, X_B(s)), \;\; t \le s \le s_{out}^B(t,x,v),$$

with $X_B(t) = x, V_B(t) = v$. We deduce that :

$$\frac{d}{ds}(X_A - X_C) = V_A(s) - V_C(s),\ \ \frac{d}{ds}(V_A - V_C) = (A-B)(s, X_A(s)),\ \ t \le s \le \min\{s^A_{out}, s^C_{out}\},$$

and $X_A(t) - X_C(t) = 0,\ V_A(t) - V_C(t) = \tilde{v} - v = m$. We have :

$$|V_A(s) - V_C(s) - V_A(t) + V_C(t)| \le \int_t^s \|A(\tau) - B(\tau)\|_{L^\infty(]0,1[)} d\tau \le m,\ t \le s \le \min\{s^A_{out}, s^C_{out}\},$$

and thus $V_A(s) - V_C(s) \ge V_A(t) - V_C(t) - m = 0,\ \ t \le s \le \min\{s^A_{out}, s^C_{out}\}$. Moreover, since $X_A(t) = X_C(t) = x$ it follows that $X_A(s) \ge X_C(s),\ t \le s \le \min\{s^A_{out}, s^C_{out}\}$. If we suppose that $s^A_{out} < s^C_{out}$, we deduce that $X_C(s^A_{out}; t, x, v) \le X_A(s^A_{out}; t, x, \tilde{v}) = 0$ and thus we have $s^C_{out} \le s^A_{out}$ which is in contradiction with the previous supposition. Therefore we have $s^C_{out} \le s^A_{out} < T$. In particular $X_C(s^C_{out}; t, x, v) \in \{0, 1\}$ and $X_C(s^C_{out}; t, x, v) \le X_A(s^C_{out}; t, x, \tilde{v})$. Note also that $X_A(s^C_{out}; t, x, \tilde{v}) = 1$ implies that $s^A_{out} \le s^C_{out}$ and thus it follows that $s^A_{out} = s^C_{out} < T$ which is not possible because $X_A(s^A_{out}; t, x, \tilde{v}) = 0$ and $X_A(s^C_{out}; t, x, \tilde{v}) = 1$. We obtain that $X_C(s^C_{out}; t, x, v) \le X_A(s^C_{out}; t, x, \tilde{v}) < 1$ and we deduce that $X_C(s^C_{out}; t, x, v) = 0$. On the other hand :

$$\frac{d^2}{ds^2} X_C = B(s, X_A(s)) \ge B(s, X_C(s)),\ t \le s \le s^C_{out},$$

and :

$$\frac{d^2}{ds^2} X_B = B(s, X_B(s)),\ t \le s \le s^B_{out}.$$

Note that $X_C(t) = X_B(t) = x$ and $V_C(t) = V_B(t) = v$. Thus by applying the forward comparison (see Proposition 4.6) we deduce that $X_C(s) \ge X_B(s), V_C(s) \ge V_B(s),\ t \le s \le \min\{s^B_{out}, s^C_{out}\}$. If we suppose that $s^C_{out} < s^B_{out}$, we deduce that :

$$0 = X_C(s^C_{out}; t, x, v) \ge X_B(s^C_{out}; t, x, v),$$

and thus we have $s^B_{out} \le s^C_{out}$ which is in contradiction with the previous supposition. Therefore we have $s^B_{out} \le s^C_{out} \le s^A_{out} < T$ and :

$$X_B(s) \le X_C(s) \le X_A(s),\ V_B(s) \le V_C(s) \le V_A(s),\ \ t \le s \le s^B_{out}.$$

Since $v > v^0_B$ and $s^B_{out} < T$ we have $X_B(s^B_{out}; t, x, v) = 1$. Now, by taking $s = s^B_{out}$ in the previous inequality we obtain :

$$1 = X_B(s^B_{out}; t, x, v) \le X_A(s^B_{out}; t, x, \tilde{v}),$$

which implies that $X_A(s^B_{out}; t, x, \tilde{v}) = 1$ and $s^A_{out} \le s^B_{out}$, or $s^A_{out} = s^B_{out}$. As before we obtain a contradiction because $X_A(s^A_{out}; t, x, \tilde{v}) = 0$ and $X_A(s^B_{out}; t, x, \tilde{v}) = 1$. Finally we have proved that the supposition $v^0_A - v^0_B > m$ is false and thus $v^0_A - v^0_B \le m$. By changing A with B we obtain also that $v^0_B - v^0_A \le m$, or $|v^0_A - v^0_B| \le m$. The same arguments apply for the critical velocities v^1_A, v^1_B. □

We end this section with some usual calculations concerning the continuity of the characteristics with respect to the field.

Proposition 4.9. *Assume that* $A, B \in L^\infty(]0,T[; W^{1,\infty}(]0,1[))$ *and consider* $(t, x, v) \in \{[0,T[\times]0,1[\times\mathbb{R}_v\} \cup \{]0,T[\times\Sigma^-\}$. *Then for* $s \in [s^A_{in}(t,x,v), s^A_{out}(t,x,v)] \cap [s^B_{in}(t,x,v), s^B_{out}(t,x,v)]$ *we have :*

$$|X_A(s;t,x,v) - X_B(s;t,x,v)| + |V_A(s;t,x,v) - V_B(s;t,x,v)| \le \left| \int_t^s \|A(\tau) - B(\tau)\|_{L^\infty(]0,1[)} d\tau \right| \cdot \exp\left(\left| \int_t^s (1 + \|\partial_x B(\tau)\|_{L^\infty(]0,1[)}) d\tau \right| \right).$$

5. Existence and uniqueness of the mild solution.

In this section we intend to prove the existence and the uniqueness of the mild solution for the Vlasov-Poisson initial-boundary value problem in one dimension by using the iterated approximations method. We consider the application $\mathcal{F}$ defined for regular electric field $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ as follows :

$$E \to f_E \to \rho_E = \int_{\mathbb{R}_v} f_E(t,x,v)dv \to E_1 = \mathcal{F}(E), \tag{5.1}$$

where f_E is the mild solution of the Vlasov problem associated to the field E and E_1 is the Poisson electric field corresponding to the charge density ρ_E . Before analysing the application $\mathcal{F}$ let us introduce some notations. If $u : [0,+\infty[\to [0,+\infty[$ is a bounded non increasing real function and $R > 0$ we denote by $u^R : [-R,+\infty[\to [0,+\infty[$ the function given by $u^R(t) = u(0)$ if $-R \le t \le R$ and $u^R(t) = u(t-R)$ if $t > R$. If we assume that u belongs to $L^1(\mathbb{R}^+)$ therefore :

$$\|u^R\|_{L^1(-R,+\infty)} = 2R\|u\|_{L^\infty(\mathbb{R}^+)} + \|u\|_{L^1(\mathbb{R}^+)}.$$

5.1. Estimate of $\mathcal{F}E$.

We assume that the initial-boundary conditions verify the following hypothesis denoted by (H) : there is $n_0, h_0, h_1 : [0,+\infty[\to [0,+\infty[$ bounded, non increasing functions such that :

$$f_0(x,v) \le n_0(|v|), \quad (x,v) \in]0,1[\times\mathbb{R}_v,$$

$$(H) \quad g_0(t,v) \le h_0(v), \quad (t,v) \in]0,T[\times\mathbb{R}_v^+,$$

$$g_1(t,v) \le h_1(-v), \quad (t,v) \in]0,T[\times\mathbb{R}_v^-,$$

and :

$$(H_0) \quad M_0 := \int_{\mathbb{R}_v} n_0(|v|)dv + \int_{v>0} h_0(v)dv + \int_{v<0} h_1(-v)dv < +\infty,$$

$$(H_\infty) \quad M_\infty := \max\{\|n_0\|_{L^\infty(\mathbb{R}_v^+)}, \|h_0\|_{L^\infty(\mathbb{R}_v^+)}, \|h_1\|_{L^\infty(\mathbb{R}_v^+)}\} < +\infty.$$

Under the previous hypothesis we can prove the following proposition :

Proposition 5.1. *Assume that f_0, g_0, g_1 satisfy the hypothesis (H), (H_0), (H_∞) and $U_0 - U_1 \in L^\infty(]0,T[)$. Then for every $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ we have $f_E \in L^\infty(]0,T[;L^1(]0,1[\times\mathbb{R}_v))$, $\rho_E \in L^\infty(]0,T[;L^1(]0,1[)) \cap L^\infty(]0,T[\times]0,1[), \mathcal{F}E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$. Moreover the following estimates hold :*

$$\|f_E\|_{L^\infty(]0,t[;L^1(]0,1[\times\mathbb{R}_v))} = \|\rho_E\|_{L^\infty(]0,t[;L^1(]0,1[))} \le 6 \cdot M_\infty \int_0^t \|E(\tau)\|_{L^\infty(]0,1[)} d\tau + M_0,$$

$$\|\rho_E\|_{L^\infty(]0,t[\times]0,1[)} = \|\partial_x \mathcal{F}E\|_{L^\infty(]0,t[\times]0,1[)} \le 6 \cdot M_\infty \int_0^t \|E(\tau)\|_{L^\infty(]0,1[)} d\tau + M_0,$$

$$\|\mathcal{F}E\|_{L^\infty(]0,t[;W^{1,\infty}(]0,1[))} \le 12 \cdot M_\infty \int_0^t \|E(s)\|_{L^\infty(]0,1[)} ds + 2M_0 + \|U_0 - U_1\|_{L^\infty(]0,t[)},$$

$$\lim_{R_1 \to +\infty} \int_{|v|>R_1} f_E(t,x,v)dv = 0, \ \textit{uniformly with respect to } (t,x) \in]0,T[\times]0,1[,$$

and the mild formulation of the Vlasov problem holds for test functions $\psi \in L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)$.

Proof. By the Remark 2.3 we have :

$$\rho_E(t,x)=\int_{\mathbb{R}_v} f_E(t,x,v)dv = \int_{\mathbb{R}_v} f_0(X(0;t,x,v),V(0;t,x,v))\mathbf{1}_{\{s_{in}(t,x,v)=0\}}dv$$
$$+\Sigma_{k=0}^1 \int_{\mathbb{R}_v} g_k(s_{in}(t,x,v),V(s_{in}(t,x,v);t,x,v))\mathbf{1}_{\{s_{in}(t,x,v)>0\}}\mathbf{1}_{\{X(s_{in}(t,x,v);t,x,v)=k\}}dv$$
$$=\mathcal{I}^i+\mathcal{I}^0+\mathcal{I}^1.$$

Let us estimate the first integral $\mathcal{I}^i$. For this, consider $R=\int_0^t \|E(\tau)\|_{L^\infty(]0,1[)}d\tau$ and remark that $|V(0;t,x,v)| \geq |v|-R$ which implies that $n_0(|V(0;t,x,v)|) \leq n_0^R(|v|)$. By using the hypothesis (H) we find :

$$\mathcal{I}^i \leq \int_{\mathbb{R}_v} n_0(|V(0;t,x,v)|)\mathbf{1}_{\{s_{in}(t,x,v)=0\}}dv$$
$$\leq \int_{\mathbb{R}_v} n_0^R(|v|)dv = 2R\|n_0\|_{L^\infty(\mathbb{R}_v^+)} + 2\cdot\|n_0\|_{L^1(\mathbb{R}_v^+)}.$$

In the same way, by writting $v \geq V(s_{in}(t,x,v);t,x,v)-R \geq -R$ when $X(s_{in}(t,x,v);t,x,v)=0$ and $v \leq V(s_{in}(t,x,v);t,x,v)+R \leq R$ when $X(s_{in}(t,x,v);t,x,v)=1$, one gets :

$$\mathcal{I}^0+\mathcal{I}^1 \leq \int_{v>-R} h_0^R(v)dv + \int_{v<R} h_1^R(-v)dv$$
$$\leq 2\cdot R\cdot(\|h_0\|_{L^\infty(\mathbb{R}_v^+)}+\|h_1\|_{L^\infty(\mathbb{R}_v^+)}) + \|h_0\|_{L^1(\mathbb{R}_v^+)}+\|h_1\|_{L^1(\mathbb{R}_v^+)}.$$

Finally we deduce that :

$$\rho_E(t,x) \leq 6\cdot M_\infty \int_0^t \|E(\tau)\|_{L^\infty(]0,1[)}d\tau + M_0,\ (t,x)\in]0,T[\times]0,1[,$$

and therefore :

$$|\mathcal{F}E(t,x)| = \left|\int_0^x \rho_E(t,y)dy - \int_0^1 (1-y)\rho_E(t,y)dy - U_1(t)+U_0(t)\right|$$
$$\leq \|\rho_E\|_{L^\infty(]0,t[;L^1(]0,1[))} + \|U_0-U_1\|_{L^\infty]0,t[}.$$

In order to estimate the charge outside a ball of radius R_1 just remark that, for example :

$$\mathcal{I}^i_{R_1} = \int_{|v|>R_1} f_0(X(0;t,x,v),V(0;t,x,v))\mathbf{1}_{\{s_{in}(t,x,v)=0\}}dv$$
$$\leq \int_{|v|>R_1} n_0^R(|v|)dv = \int_{|v|>R_1-R} n_0(|v|)dv,$$

for $R_1>R$. Finally one gets that :

$$\int_{|v|>R_1} f_E(t,x,v)dv \leq \int_{|v|>R_1-R} n_0(|v|)dv + \int_{v>R_1-R} h_0(v)dv + \int_{v<-R_1+R} h_1(-v)dv \to 0,$$

as $R_1\to+\infty$ uniformly with respect to $(t,x)\in]0,T[\times]0,1[$. Consider now $\psi\in L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)$ and $\psi_{R_1}=\chi_{R_1}(v)\psi(t,x,v)$ where $\chi_{R_1}(\cdot)=\chi(\cdot/R_1)$ and $\chi\in C_c^1(\mathbb{R}), \chi(u)=1, |u|\leq 1, \chi(u)=0, |u|\geq 2, 0\leq\chi(u)\leq 1, 1\leq|u|\leq 2$. Obviously $\psi_{R_1}\in\mathcal{T}_m$ and thus :

$$\int_0^T\int_0^1\int_{\mathbb{R}_v} f_E(t,x,v)\psi_{R_1}(t,x,v)dtdxdv = \int_0^1\int_{\mathbb{R}_v} f_0(x,v)\int_0^{s_{out}(0,x,v)} \psi_{R_1}(s,X(s;0,x,v),V(s;0,x,v))dsdxdv$$
$$+\int_0^T\int_{v>0} vg_0(t,v)\int_t^{s_{out}(t,0,v)} \psi_{R_1}(s,X(s;t,0,v),V(s;t,0,v))dsdtdv$$
$$-\int_0^T\int_{v<0} vg_1(t,v)\int_t^{s_{out}(t,1,v)} \psi_{R_1}(s,X(s;t,1,v),V(s;t,1,v))dsdtdv.$$

We have :

$$\left|\int_0^T\int_0^1\int_{\mathbb{R}_v} f_E\psi_{R_1}dtdxdv - \int_0^T\int_0^1\int_{\mathbb{R}_v} f_E\psi dtdxdv\right| \leq \int_0^T\int_0^1\int_{\mathbb{R}_v} f_E(1-\chi_{R_1}(v))|\psi|dtdxdv$$

$$\leq \|\psi\|_{L^\infty}\int_0^T\int_0^1\int_{|v|>R_1} f_E dtdxdv \to 0, \text{ as } R_1 \to +\infty.$$

In order to apply the dominated convergence theorem of Lebesgue remark that :

$$|f_0(x,v)\int_0^{s^i_{out}} \psi_{R_1}(s, X^i(s), V^i(s))ds| \leq f_0(x,v)\|\psi\|_{L^\infty}T \in L^1(]0,1[\times\mathbb{R}_v).$$

Note also that for $R = \|E\|_{L^1(]0,T[;L^\infty(]0,1[))}$ we have :

$$|vg_0(t,v)\int_t^{s^0_{out}}\psi_{R_1}(s, X^0(s), V^0(s))ds| \leq 2Rg_0(t,v)T\|\psi\|_{L^\infty}\mathbf{1}_{\{0<v\leq 2R\}} + vg_0(t,v)\|\psi\|_{L^\infty}\frac{1}{v-R}\mathbf{1}_{\{v>2R\}}$$

$$\leq 2RT\|\psi\|_{L^\infty}g_0(t,v)\mathbf{1}_{\{0<v\leq 2R\}} + 2\|\psi\|_{L^\infty}g_0(t,v)\mathbf{1}_{\{v>2R\}} \in L^1(]0,T[\times\mathbb{R}_v^+),$$

since $V^0(s) \geq v - R$ and $s^0_{out} - t \leq \frac{1}{v-R}$ for $v > R$. The same arguments apply for the right boundary term and finally, by passing $R_1 \to +\infty$ we deduce that the mild formulation holds for every $\psi \in L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)$. □

REMARK 5.2. *Consider* $x(t) = (M_0 + \|U_0 - U_1\|_{L^\infty(]0,T[)})\exp(6\cdot M_\infty t)$ *and :*

$$X_T = \{E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[)) \mid \|E\|_{L^\infty(]0,t[\times]0,1[)} \leq x(t), \forall\ 0 \leq t \leq T\}.$$

Then $\mathcal{F}X_T \subset X_T$ *and :*

$$\|\mathcal{F}E\|_{L^\infty(]0,T[;W^{1,\infty}(]0,1[))} \leq 2\cdot x(T) - \|U_0 - U_1\|_{L^\infty(]0,T[)}.$$

5.2. Estimate of $\mathcal{F}A - \mathcal{F}B$.

The aim of this section is to estimate the L^∞ norm of $\mathcal{F}A - \mathcal{F}B$ with respect to the L^∞ norm of $A - B$. In a first time we perform our computations by introducing also the current density $j_E(t,x) := \int_{\mathbb{R}_v} vf_E(t,x,v)dv$. This requires additional hypothesis on the initial boundary conditions. For the moment we assume also :

$$(H_1)\ \ M_1 := \int_{\mathbb{R}_v} n_0(|v|)|v|dv + \int_{v>0} h_0(v)vdv - \int_{v<0} h_1(-v)vdv < +\infty.$$

Later on we shall see that this hypothesis can be removed.

PROPOSITION 5.3. *Assume that* f_0, g_0, g_1 *satisfy* $(H), (H_1), (H_\infty)$ *and* $U_0 - U_1 \in L^\infty(]0,T[)$. *Then for every* $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ $f_E|v| \in L^\infty(]0,T[;L^1(]0,1[\times\mathbb{R}_v))$, $|j_E|(t,x) := \int_{\mathbb{R}_v} f_E(t,x,v)|v|dv \in L^\infty(]0,T[;L^1(]0,1[)) \cap L^\infty(]0,T[\times]0,1[)$, $\mathcal{F}E + U_1 - U_0 \in W^{1,\infty}(]0,T[\times]0,1[)$. *Moreover, the following estimates hold :*

$$\max\{\|\ |j_E|\ \|_{L^\infty(]0,T[;L^1(]0,1[))}, \|\ |j_E|\ \|_{L^\infty(]0,T[\times]0,1[)}\} \leq 3\cdot M_\infty\left(\int_0^t \|E(s)\|_{L^\infty(]0,1[)}ds\right)^2$$

$$+M_0\int_0^t \|E(s)\|_{L^\infty(]0,1[)}ds + M_1,$$

$$\partial_t\{\mathcal{F}E + U_1 - U_0\} = -j_E(t,x) + \int_0^1 j_E(t,y)dy,\ (t,x) \in]0,T[\times]0,1[,$$

$$\lim_{R_1\to+\infty}\int_{|v|>R_1}|v|f_E(t,x,v)dv=0,\quad \textit{uniformly with respect to } (t,x)\in]0,T[\times]0,1[,$$

and the mild formulation of the Vlasov problem holds for every function ψ *such that* $|\psi(t,x,v)|\leq C(1+|v|)$.

Proof. Exactly as before we have :

$$\begin{aligned}|j_E|(t,x)=&\int_{\mathbb{R}_v}|v|f_E(t,x,v)dv=\int_{\mathbb{R}_v}|v|f_0(X(0;t,x,v),V(0;t,x,v))\mathbf{1}_{\{s_{in}(t,x,v)=0\}}\\&+\Sigma_{k=0}^1\int_{\mathbb{R}_v}|v|g_k(s_{in}(t,x,v),V(s_{in}(t,x,v);t,x,v))\mathbf{1}_{\{s_{in}(t,x,v)>0\}}\mathbf{1}_{\{X(s_{in}(t,x,v);t,x,v)=k\}}dv\\&=\mathcal{J}^i+\mathcal{J}^0+\mathcal{J}^1.\end{aligned}$$

Consider $R=\int_0^t\|E(s)\|_{L^\infty(]0,1[)}ds$ and thus $|V(0;t,x,v)|\geq|v|-R$ which implies that :

$$\mathcal{J}^i\leq\int_{\mathbb{R}_v}|v|n_0^R(|v|)dv=R^2n_0(0)+\int_{\mathbb{R}_v}|v|n_0(|v|)dv+R\int_{\mathbb{R}_v}n_0(|v|)dv.$$

The terms $\mathcal{J}^k,k\in\{0,1\}$ can be estimated in the same manner and finally one gets :

$$|j_E|(t,x)\leq 3\cdot R^2M_\infty+RM_0+M_1,\ \ (t,x)\in]0,T[\times]0,1[.$$

By performing the same computations on $\mathbb{R}_v-B_{R_1}$ we get that $\lim_{R_1\to+\infty}\int_{|v|>R_1}|v|f_Edv=0$, uniformly with respect to $(t,x)\in]0,T[\times]0,1[$. In order to check that the mild formulation holds $\forall\psi$ such that $|\psi(t,x,v)|\leq C(1+|v|)$, consider $\psi_{R_1}=\chi_{R_1}(v)\psi\in\mathcal{T}_m$. This time we have :

$$\begin{aligned}\left|\int_0^T\!\!\int_0^1\!\!\int_{\mathbb{R}_v}f_E\psi_{R_1}dtdxdv-\int_0^T\!\!\int_0^1\!\!\int_{\mathbb{R}_v}f_E\psi dtdxdv\right|\leq&\int_0^T\!\!\int_0^1\!\!\int_{\mathbb{R}_v}f_E(1-\chi_{R_1}(v))|\psi(t,x,v)|dtdxdv\\\leq&\int_0^T\!\!\int_0^1\!\!\int_{|v|>R_1}f_E\cdot C(1+|v|)dtdxdv\to0,\ \text{as } R_1\to+\infty.\end{aligned}$$

In order to pass to the limit in the other terms of the mild formulation for the test function ψ_{R_1}, take $R=\|E\|_{L^1(]0,T[;L^\infty(]0,1[))}$ and remark that :

$$\left|f_0(x,v)\int_0^{s^i_{out}}\psi_{R_1}(s,X^i(s),V^i(s))ds\right|\leq f_0(x,v)\cdot T\cdot C(1+|v|+R)\in L^1(]0,1[\times\mathbb{R}_v),$$

$$\begin{aligned}\left|vg_k(t,v)\int_t^{s^k_{out}}\psi_{R_1}(s,X^k(s),V^k(s))ds\right|\leq&2Rg_k(t,v)\cdot T\cdot C(1+|v|+R)\mathbf{1}_{\{|v|\leq2R\}}\\&+|v|g_k(t,v)\frac{C(1+|v|+R)}{|v|-R}\mathbf{1}_{\{|v|>2R\}}\\\leq&2R\cdot T\cdot C\cdot g_k(t,v)(1+|v|+R)\mathbf{1}_{\{|v|\leq2R\}}\\&+C\left(3+\frac{1}{R}\right)|v|g_k(t,v)\mathbf{1}_{\{|v|>2R\}}\in L^1(]0,T[\times\mathbb{R}_v^{\pm}).\end{aligned}$$

By passing to the limit in the mild formulation for $R_1\to+\infty$ and using the dominated convergence theorem our conclusion follows. Let us compute now the time derivative of $\mathcal{F}E+U_1-U_0$. First of all, by using the mild formulation with the test function $\psi(t,x,v)=\partial_t\varphi+v\partial_x\varphi$, $\varphi\in C_c^1(]0,T[\times]0,1[)$ (note that $|\psi(t,x,v)|\leq C(1+|v|)$) we deduce the continuity equation $\partial_t\rho_E+\partial_xj_E=0$ in $\mathcal{D}'(]0,T[\times]0,1[)$. By direct computation, the continuity equation implies that :

$$\partial_t\{\mathcal{F}E+U_1-U_0\}=-j_E(t,x)+\int_0^1j_E(t,y)dy\in L^\infty(]0,T[\times]0,1[).$$

Obviously $\partial_x\{\mathcal{F}E+U_1-U_0\}=\rho_E\in L^\infty(]0,T[\times]0,1[)$ and thus we obtain that $\mathcal{F}E+U_1-U_0\in W^{1,\infty}(]0,T[\times]0,1[)$. □

Remark 5.4. *We have :*

$$\begin{aligned}\mathcal{F}E(t,x)+U_1(t)-U_0(t)=&-\int_0^t j_E(s,x)ds+\int_0^t\int_0^1 j_E(s,y)dsdy+\mathcal{F}E(0,x)+U_1(0)-U_0(0)\\=&-\int_0^t j_E(s,x)ds+\int_0^t\int_0^1 j_E(s,y)dsdy\\&+\int_0^x\int_{\mathbb{R}_v} f_0(y,v)dydv-\int_0^1\int_{\mathbb{R}_v}(1-y)f_0(y,v)dydv.\end{aligned}$$

By using the formula given above we can estimate $\mathcal{F}A-\mathcal{F}B$. This will be done in the following two Propositions. One of the key points is the critical velocity change result (see Lemma 4.8).

Proposition 5.5. *Assume that $A,B\in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ are non decreasing with respect to x and the hypothesis $(H),(H_1),(H_\infty)$ hold. Then for $0\le t\le T$ we have :*

$$\left\|\int_0^t j_A(s,\cdot)ds-\int_0^t j_B(s,\cdot)ds\right\|_{L^\infty(]0,1[)}\le C\cdot\int_0^t\|A(s)-B(s)\|_{L^\infty(]0,1[)}ds,$$

where C is a constant depending only on $\|A\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))},\|B\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}$, T and the initial-boundary conditions .

Proof. Consider $\varphi\in L^1(]0,1[)$ bounded and let us estimate $\int_0^1\int_0^t(j_A(s,x)-j_B(s,x))\varphi(x)dxds$. By applying the mild formulation with $\psi(t,x,v)=\varphi(x)v$ (which is possible since $|\psi(t,x,v)|\le\|\varphi\|_{L^\infty}|v|$) we have :

$$\begin{aligned}\int_0^1\int_0^t(j_A(s,x)-j_B(s,x))\varphi(x)dxds&=\int_0^t\int_0^1\int_{\mathbb{R}_v}(f_A(s,x,v)-f_B(s,x,v))v\varphi(x)dsdxdv\\&=\int_0^1\int_{\mathbb{R}_v} f_0(x,v)\left[\int_0^{s_A^i}V_A^i(\tau)\varphi(X_A^i(\tau))d\tau-\int_0^{s_B^i}V_B^i(\tau)\varphi(X_B^i(\tau))d\tau\right]dxdv\\&+\int_0^t\int_{v>0}vg_0(s,v)\left[\int_s^{s_A^0}V_A^0(\tau)\varphi(X_A^0(\tau))d\tau-\int_s^{s_B^0}V_B^0(\tau)\varphi(X_B^0(\tau))d\tau\right]dsdv\\&-\int_0^t\int_{v<0}vg_1(s,v)\left[\int_s^{s_A^1}V_A^1(\tau)\varphi(X_A^1(\tau))d\tau-\int_s^{s_B^1}V_B^1(\tau)\varphi(X_B^1(\tau))d\tau\right]dsdv\\&=\int_0^1\int_{\mathbb{R}_v} f_0(x,v)\left[\int_x^{X_A^i(s_A^i)}\varphi(u)du-\int_x^{X_B^i(s_B^i)}\varphi(u)du\right]dxdv\\&+\int_0^t\int_{v>0}vg_0(s,v)\left[\int_0^{X_A^0(s_A^0)}\varphi(u)du-\int_0^{X_B^0(s_B^0)}\varphi(u)du\right]dsdv\\&-\int_0^t\int_{v<0}vg_1(s,v)\left[\int_1^{X_A^1(s_A^1)}\varphi(u)du-\int_1^{X_B^1(s_B^1)}\varphi(u)du\right]dsdv\\&=\mathcal{I}_{AB}^i+\mathcal{I}_{AB}^0+\mathcal{I}_{AB}^1.\end{aligned}$$

We introduce the notations $\Phi_C^i=\int_x^{X_C^i(s_C^i)}\varphi(u)du,\ \Phi_C^k=\int_k^{X_C^k(s_C^k)}\varphi(u)du,\ k\in\{0,1\},\ C\in\{A,B\}$. Here s_C^i,s_C^k represent the exist times associated to the domain $]0,t[\times]0,1[\times\mathbb{R}_v$, with $k\in\{0,1\},C\in$

$\{A,B\}$. The term $\mathcal{I}^i_{AB}$ writes :

$$\begin{aligned}\mathcal{I}^i_{AB} =& \int_0^1\int_{\mathbb{R}_v} f_0(x,v)[\Phi^i_A \mathbf{1}_{\{v<v^0_A\}} - \Phi^i_B \mathbf{1}_{\{v<v^0_B\}}]dxdv \\ &+ \int_0^1\int_{\mathbb{R}_v} f_0(x,v)[\Phi^i_A \mathbf{1}_{\{v^0_A<v<v^1_A\}} - \Phi^i_B \mathbf{1}_{\{v^0_B<v<v^1_B\}}]dxdv \\ &+ \int_0^1\int_{\mathbb{R}_v} f_0(x,v)[\Phi^i_A \mathbf{1}_{\{v>v^1_A\}} - \Phi^i_B \mathbf{1}_{\{v>v^1_B\}}]dxdv \\ =& \mathcal{I}^i_0 + \mathcal{I}^i_t + \mathcal{I}^i_1,\end{aligned}$$

where $v^k_C = v^k_C(t;0,x)$ are the critical velocities corresponding to the domain $]0,t[\times]0,1[$, to the point $(0,x)$ and the field C, with $k=0,1,\ C=A,B$. The first and the third integral are easy to estimate since for $v<v^0_A$ we have $X^i_A(s^i_A)=0$ and thus $\Phi^i_A = \int_x^0 \varphi(u)du$; for $v>v^1_A$ we have $X^i_A(s^i_A)=1$ and $\Phi^i_A = \int_x^1 \varphi(u)du$. We obtain by using the critical velocity change :

$$\begin{aligned}|\mathcal{I}^i_0| \leq& \|\varphi\|_{L^1(]0,1[)} \|f_0\|_{L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)} \int_0^1 |v^0_A(t;0,x) - v^0_B(t;0,x)|dx \\ \leq& \|\varphi\|_{L^1(]0,1[)} \|f_0\|_{L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)} \int_0^t \|A(s)-B(s)\|_{L^\infty(]0,1[)}ds,\end{aligned}$$

and also :

$$|\mathcal{I}^i_1| \leq \|\varphi\|_{L^1(]0,1[)} \|f_0\|_{L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)} \int_0^t \|A(s)-B(s)\|_{L^\infty(]0,1[)}ds.$$

Let us estimate now the second integral $\mathcal{I}^i_t$. Remark that when $v^0_A<v<v^1_A$ we have $s^A_{out}(0,x,v) = t$ and thus $\Phi^i_A = \int_x^{X_A(t)}\varphi(u)du$. Similarly $\Phi^i_B = \int_x^{X_B(t)}\varphi(u)du$ when $v^0_B<v<v^1_B$. We can write :

$$\begin{aligned}|\mathcal{I}^i_t| \leq& \left|\int_0^1\int_{\mathbb{R}_v} f_0(x,v)\Phi^i_A \mathbf{1}_{\{v^0_A<v<\max\{v^0_A,v^0_B\}\}}dxdv\right| \\ &+ \left|\int_0^1\int_{\mathbb{R}_v} f_0(x,v)\Phi^i_A \mathbf{1}_{\{\min\{v^1_A,v^1_B\}<v<v^1_A\}}dxdv\right| \\ &+ \left|\int_0^1\int_{\mathbb{R}_v} f_0(x,v)\Phi^i_B \mathbf{1}_{\{v^0_B<v<\max\{v^0_A,v^0_B\}\}}dxdv\right| \\ &+ \left|\int_0^1\int_{\mathbb{R}_v} f_0(x,v)\Phi^i_B \mathbf{1}_{\{\min\{v^1_A,v^1_B\}<v<v^1_B\}}dxdv\right| \\ &+ \left|\int_0^1\int_{\mathbb{R}_v} f_0(x,v)(\Phi^i_A - \Phi^i_B) \mathbf{1}_{\{\max\{v^0_A,v^0_B\}<v<\min\{v^1_A,v^1_B\}\}}dxdv\right|.\end{aligned}$$

By using Lemma 4.8 we deduce :

$$\max\{|v^0_A - \max\{v^0_A,v^0_B\}|, |v^0_B - \max\{v^0_A,v^0_B\}|\} \leq \int_0^t \|A(s)-B(s)\|_{L^\infty(]0,1[)}ds,$$

and :

$$\max\{|v^1_A - \min\{v^1_A,v^1_B\}|, |v^1_B - \min\{v^1_A,v^1_B\}|\} \leq \int_0^t \|A(s)-B(s)\|_{L^\infty(]0,1[)}ds.$$

It comes that the first four terms can be estimated by $4\cdot\|\varphi\|_{L^1}\|f_0\|_{L^\infty}\int_0^t\|A(s)-B(s)\|_{L^\infty}ds$. When

$\max\{v_A^0, v_B^0\} < v < \min\{v_A^1, v_B^1\}$ we have :

$$|\Phi_A^i - \Phi_B^i| = \left| \int_{X_B(t)}^{X_A(t)} \varphi(u) du \right| \leq \int_0^1 |\varphi(u)| \mathbf{1}_{\{|u - X_A(t)| \leq |X_A(t) - X_B(t)|\}} du.$$

Therefore, by using the Proposition 4.9 the last term of $\mathcal{I}_t^i$ writes :

$$\begin{aligned} |\mathcal{I}_5| &= \left| \int_0^1 \int_{\mathbb{R}_v} f_0(x,v)(\Phi_A^i - \Phi_B^i) \mathbf{1}_{\{\max\{v_A^0, v_B^0\} < v < \min\{v_A^1, v_B^1\}\}} dxdv \right| \\ &\leq \int_0^1 |\varphi(u)| \int_0^1 \int_{\mathbb{R}_v} f_0(x,v) \mathbf{1}_{\{v_A^0 < v < v_A^1\}} \mathbf{1}_{\{|u - X_A(t)| \leq C \int_0^t \|A(s) - B(s)\|_{L^\infty} ds\}} dxdvdu, \end{aligned} \tag{5.2}$$

where $C = \exp\left(\int_0^t (1 + \|\partial_x B(s)\|_{L^\infty(]0,1[)}) ds\right)$. By the change of variables $y = X_A(t;0,x,v)$, $w = V_A(t;0,x,v)$ on $\{(x,v) \in]0,1[\times \mathbb{R}_v \; : \; v_A^0(t;0,x) < v < v_A^1(t;0,x)\}$ one gets :

$$\begin{aligned} |\mathcal{I}_5| &\leq \int_0^1 |\varphi(u)| \int_0^1 \int_{\mathbb{R}_w} f_0(X_A(0;t,y,w), V_A(0;t,y,w)) \mathbf{1}_{\{|u-y| \leq C \int_0^t \|A(s) - B(s)\|_{L^\infty} ds\}} dydwdu \\ &\leq \int_0^1 |\varphi(u)| \int_0^1 \int_{\mathbb{R}_w} n_0^R(|w|) \mathbf{1}_{\{|u-y| \leq C \int_0^t \|A(s) - B(s)\|_{L^\infty} ds\}} dydwdu \\ &\leq 2 \cdot C \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty} ds (2 \cdot R \cdot \|n_0\|_{L^\infty(\mathbb{R}_v^+)} + 2 \cdot \|n_0\|_{L^1(\mathbb{R}_v^+)}) \|\varphi\|_{L^1(]0,1[)}, \end{aligned}$$

where as usual $R = \int_0^t \|A(s)\|_{L^\infty(]0,1[)} ds$. Finally we proved that :

$$\begin{aligned} |\mathcal{I}_{AB}^i| &\leq \{6\|f_0\|_{L^\infty} + 4C(\int_0^t \|A(s)\|_{L^\infty} ds \|n_0\|_{L^\infty(\mathbb{R}_v^+)} + \|n_0\|_{L^1(\mathbb{R}_v^+)})\} \int_0^t \|A(s) - B(s)\|_{L^\infty} ds \cdot \|\varphi\|_{L^1} \\ &\leq C^i \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty} ds \cdot \|\varphi\|_{L^1(]0,1[)}. \end{aligned}$$

Let us analyse the term $\mathcal{I}_{AB}^0$. As before we have :

$$\begin{aligned} \mathcal{I}_{AB}^0 &= \int_0^t \int_{\mathbb{R}_v} v g_0(s,v) \left[\Phi_A^0 \mathbf{1}_{\{0 < v < v_A^0\}} - \Phi_B^0 \mathbf{1}_{\{0 < v < v_B^0\}} \right] dsdv \\ &+ \int_0^t \int_{\mathbb{R}_v} v g_0(s,v) \left[\Phi_A^0 \mathbf{1}_{\{v_A^0 < v < v_A^1\}} - \Phi_B^0 \mathbf{1}_{\{v_B^0 < v < v_B^1\}} \right] dsdv \\ &+ \int_0^t \int_{\mathbb{R}_v} v g_0(s,v) \left[\Phi_A^0 \mathbf{1}_{\{v > v_A^1\}} - \Phi_B^0 \mathbf{1}_{\{v > v_B^1\}} \right] dsdv \\ &= \mathcal{I}_0^0 + \mathcal{I}_t^0 + \mathcal{I}_1^0. \end{aligned}$$

Taking into account that for $0 < v < v_C^0(t;s,0)$ we have $X_C^0(s_{out,C}^0) = 0$ we deduce that $\Phi_C^0 = 0$ for $C = A, B$ and thus $\mathcal{I}_0^0 = 0$. By the other hand, for $v > v_C^1$ we have $X_C^0(s_{out,C}^0) = 1$ and thus $\Phi_C^0 = \int_0^1 \varphi(u) du$, for $C = A, B$. One gets :

$$|\mathcal{I}_1^0| \leq \left| \int_0^t \int_{v_A^1}^{v_B^1} v g_0(s,v) \int_0^1 \varphi(u) dsdvdu \right| \leq t \cdot \|v g_0\|_{L^\infty(]0,T[\times \mathbb{R}_v^+)} |v_A^1 - v_B^1| \cdot \|\varphi\|_{L^1(]0,1[)}.$$

By applying Lemma 4.8 we have :

$$|v_A^1(t;s,0) - v_B^1(t;s,0)| \le \int_s^t \|A(\tau) - B(\tau)\|_{L^\infty(]0,1[)} d\tau,$$

and therefore :

$$|\mathcal{I}_1^0| \le t \cdot \|vg_0\|_{L^\infty(]0,T[\times\mathbb{R}_v^+)} \cdot \|\varphi\|_{L^1(]0,1[)} \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds.$$

The term $\mathcal{I}_t^0$ writes :

$$\begin{aligned}
|\mathcal{I}_t^0| \le & \left| \int_0^t \int_{\mathbb{R}_v} vg_0(s,v) \Phi_A^0 \mathbf{1}_{\{v_A^0 < v < \max\{v_A^0, v_B^0\}\}} dsdv \right| \\
& + \left| \int_0^t \int_{\mathbb{R}_v} vg_0(s,v) \Phi_A^0 \mathbf{1}_{\{\min\{v_A^1, v_B^1\} < v < v_A^1\}} dsdv \right| \\
& + \left| \int_0^t \int_{\mathbb{R}_v} vg_0(s,v) \Phi_B^0 \mathbf{1}_{\{v_B^0 < v < \max\{v_A^0, v_B^0\}\}} dsdv \right| \\
& + \left| \int_0^t \int_{\mathbb{R}_v} vg_0(s,v) \Phi_B^0 \mathbf{1}_{\{\min\{v_A^1, v_B^1\} < v < v_B^1\}} dsdv \right| \\
& + \left| \int_0^t \int_{\mathbb{R}_v} vg_0(s,v) (\Phi_A^0 - \Phi_B^0) \mathbf{1}_{\{\max\{v_A^0, v_B^0\} < v < \min\{v_A^1, v_B^1\}\}} dsdv \right|.
\end{aligned}$$

The first four terms can be estimated as before by :

$$t \cdot \|vg_0\|_{L^\infty(]0,T[\times\mathbb{R}_v^+)} \cdot \|\varphi\|_{L^1(]0,1[)} \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds.$$

Since for $\max\{v_A^0, v_B^0\} < v < \min\{v_A^1, v_B^1\}$ we have $\Phi_A^0 - \Phi_B^0 = \int_{X_B(t)}^{X_A(t)} \varphi(u) du$ the last term writes :

$$\begin{aligned}
|\mathcal{I}_5| \le & \left| \int_0^t \int_{v>0} vg_0(s,v) \int_{X_B(t)}^{X_A(t)} \varphi(u) \mathbf{1}_{\{max\{v_A^0, v_B^0\} < v < \min\{v_A^1, v_B^1\}\}} dsdvdu \right| \\
\le & \int_0^1 |\varphi(u)| \int_0^t \int_{v>0} vg_0(s,v) \mathbf{1}_{\{|u - X_A(t;s,0,v)| < |X_A(t;s,0,v) - X_B(t;s,0,v)|\}} \mathbf{1}_{\{max\{v_A^0, v_B^0\} < v < \min\{v_A^1, v_B^1\}\}} \\
\le & \int_0^1 |\varphi(u)| \int_0^t \int_{v>0} vg_0(s,v) \mathbf{1}_{\{|u - X_A(t;s,0,v)| < C \cdot \int_0^t \|A(\tau) - B(\tau)\|_{L^\infty} d\tau\}} \mathbf{1}_{\{v_A^0 < v < v_A^1\}} dudsdv.
\end{aligned}$$

This time we perform the change of variables $(y,w) = S(s,v)$, with $y = X_A(t;s,0,v)$, $w = V_A(t;s,0,v)$ on the set $D = \{(s,v) \in]0,t[\times\mathbb{R}_v \ : \ v_A^0(t;s,0) < v < v_A^1(t;s,0)\}$. By standard computations one gets that :

$$\left| \frac{\partial(y,w)}{\partial(s,v)} \right| = |v|,$$

and thus :

$$\begin{aligned}
|\mathcal{I}_5| \le & \int_0^1 |\varphi(u)| \int_0^1 \int_{w>-R} \mathbf{1}_{\{(y,w) \in S(D)\}} g_0(s_{in}(t,y,w), V(s_{in}(t,y,w); t, y, w)) \mathbf{1}_{\{|u-y| < C \cdot \int_0^t \|A(\tau) - B(\tau)\|_{L^\infty} d\tau\}} \\
\le & \int_0^1 |\varphi(u)| \int_0^1 \int_{w>-R} h_0^R(w) \mathbf{1}_{\{|u-y| < C \cdot \int_0^t \|A(\tau) - B(\tau)\|_{L^\infty} d\tau\}} dydwdu \\
\le & 2C \int_0^t \|A(\tau) - B(\tau)\|_{L^\infty} d\tau \cdot (2R\|h_0\|_{L^\infty(\mathbb{R}_v^+)} + \|h_0\|_{L^1(\mathbb{R}_v^+)}) \cdot \|\varphi\|_{L^1(]0,1[)},
\end{aligned}$$

where $R = \int_0^t \|A(\tau)\|_{L^\infty} d\tau$. Finally one gets :

$$
\begin{aligned}
|\mathcal{I}^0_{AB}| \leq & \{5 \cdot t \cdot \|vh_0\|_{L^\infty(\mathbb{R}^+_v)} + 2\exp\left(\int_0^t (1 + \|\partial_x B(s)\|_{L^\infty(]0,1[)}) ds\right) \\
& \cdot \left(2 \cdot \|h_0\|_{L^\infty(\mathbb{R}^+_v)} \cdot \int_0^t \|A(s)\|_{L^\infty(]0,1[)} ds + \|h_0\|_{L^1(\mathbb{R}^+_v)}\right)\} \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty} ds \cdot \|\varphi\|_{L^1(]0,1[)} \\
\leq & C^0 \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds \cdot \|\varphi\|_{L^1(]0,1[)}.
\end{aligned}
$$

The same arguments apply for $\mathcal{I}^1_{AB}$ and we deduce that :

$$
\begin{aligned}
\left|\int_0^1 \left(\int_0^t j_A(s,x) ds - \int_0^t j_B(s,x) ds\right) \varphi(x) dx\right| \leq & |\mathcal{I}^i_{AB}| + |\mathcal{I}^0_{AB}| + |\mathcal{I}^1_{AB}| \\
\leq & (C^i + C^0 + C^1) \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds \cdot \|\varphi\|_{L^1},
\end{aligned}
$$

for all $\varphi \in L^1(]0,1[)$ bounded, in particular for all $\varphi \in C_0(]0,1[)$. Since $\int_0^t j_A(s,\cdot) ds - \int_0^t j_B(s,\cdot) ds$ belongs to $L^\infty(]0,1[)$ we deduce by density that the previous inequality holds for all $\varphi \in L^1(]0,1[)$ and we have the estimate :

$$
\left\|\int_0^t j_A(s,\cdot) ds - \int_0^t j_B(s,\cdot) ds\right\|_{L^\infty(]0,1[)} \leq C \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds, \quad 0 \leq t \leq T,
$$

with $C = C^i + C^0 + C^1$ a constant which depends on $\|A\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, \|B\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}$, $\|n_0\|_{L^\infty}$, $\|h_0\|_{L^\infty}$, $\|h_1\|_{L^\infty}$, $\|n_0\|_{L^1}$, $\|h_0\|_{L^1}$, $\|h_1\|_{L^1}$ but not on $\|vn_0\|_{L^1}$, $\|vh_0\|_{L^1}$, $\|vh_1\|_{L^1}$ (note also that since h_k are non increasing we have $\|vh_k\|_{L^\infty(\mathbb{R}^+_v)} \leq \|h_k\|_{L^1(\mathbb{R}^+_v)}$, $k = 0,1$). □

Proposition 5.6. *Assume that $A, B \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ are non decreasing with respect to x and that the hypothesis (H), (H_1), (H_∞) hold. Then for all $0 \leq t \leq T$ we have :*

$$
\|\mathcal{F}A(t) - \mathcal{F}B(t)\|_{L^\infty(]0,1[)} \leq 2 \cdot C \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds,
$$

with $C = C^i + C^0 + C^1$ as before.

Proof. By the Remark 5.4 we have :

$$
\begin{aligned}
|\mathcal{F}A(t,x) - \mathcal{F}B(t,x)| \leq & \left|\int_0^t j_A(s,x) ds - \int_0^t j_B(s,x) ds\right| + \int_0^1 \left|\int_0^t j_A(s,y) ds - \int_0^t j_B(s,y) ds\right| dy \\
\leq & 2 \cdot \left\|\int_0^t j_A(s,\cdot) ds - \int_0^t j_B(s,\cdot) ds\right\|_{L^\infty(]0,1[)} \\
\leq & 2 \cdot C \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds, \quad 0 \leq t \leq T.
\end{aligned}
$$

□

5.3. Existence and uniqueness of the mild solution.

Theorem 5.7. *Assume that the hypothesis (H), (H_1), (H_∞) hold and $U_1 - U_0 \in L^\infty(]0,T[)$. Then there is a unique mild solution (f, E) for the 1D Vlasov-Poisson initial-boundary value problem . Moreover we have the estimates :*

$$
\|\rho_E\|_{L^\infty(]0,T[\times]0,1[)} \leq B(\exp(TA) - 1) + C,
$$

$$\||j_E|\|_{L^\infty(]0,T[\times]0,1[)} \leq \frac{B^2}{2A}(\exp(TA)-1)^2 + \frac{BC}{A}(\exp(TA)-1) + M_1,$$

$$\|E\|_{L^\infty(]0,T[;W^{1,\infty}(]0,1[))} \leq 2B\exp(TA) + C - B,$$

where $A = 6\cdot M_\infty$, $B = M_0 + \|U_1 - U_0\|_{L^\infty(]0,T[)}$, $C = M_0$.

Proof. Consider $X_T = \{E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[)) \mid \|\partial_x E(t)\|_{L^\infty(]0,1[)} \leq B\exp(tA) + C - B, \|E(t)\|_{L^\infty(]0,1[)} \leq B\exp(tA),\ 0 \leq t \leq T\}$. By the Proposition 5.1 and the Remark 5.2 we know that $\mathcal{F}: X_T \to X_T$ is well defined and by the Proposition 5.6 there is a constant $C_1 = C_1(M_0, M_\infty, \|U_0 - U_1\|_{L^\infty(]0,T[)}, T)$ such that :

$$\|\mathcal{F}A(t) - \mathcal{F}B(t)\|_{L^\infty(]0,1[)} \leq C_1 \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds,\ \ A, B \in X_T.$$

We deduce that $\mathcal{F}$ has a unique fixed point $E \in X_T$ and therefore (f_E, E) is the unique mild solution of the 1D Vlasov-Poisson initial-boundary value problem . The estimate on $|j_E|$ follows by the Proposition 5.3. □

5.4. Existence and uniqueness of the mild solution in the general case.

In this section we study the existence and uniqueness of the mild solution when assuming only the hypothesis (H), (H_0), (H_∞). In order to do this we only need to prove that the Proposition 5.6 still holds under the above hypothesis. For $\alpha > 0$ let us consider the initial-boundary conditions given by :

$$f_0^\alpha(x,v) = \frac{f_0(x,v)}{1+\alpha|v|},\ \ (x,v) \in]0,1[\times\mathbb{R}_v,$$
$$g_0^\alpha(t,v) = \frac{g_0(t,v)}{1+\alpha v},\ \ (t,v) \in]0,T[\times\mathbb{R}_v^+,$$
$$g_1^\alpha(x,v) = \frac{g_1(t,v)}{1-\alpha v},\ \ (t,v) \in]0,T[\times\mathbb{R}_v^-.$$

It is easy to check that if (H), (H_0), (H_∞) hold, then the same hypothesis (H^α), (H_0^α), (H_∞^α), corresponding to the initial-boundary conditions $f_0^\alpha, g_0^\alpha, g_1^\alpha$, hold with the functions $n_0^\alpha(v) := \frac{n_0(v)}{1+\alpha v}$, $h_k^\alpha(v) := \frac{h_k(v)}{1+\alpha v}$, $v \in \mathbb{R}_v^+$, $k = 0, 1$ and we have $M_0^\alpha \leq M_0 < +\infty$, $M_\infty^\alpha \leq M_\infty < +\infty$. Moreover, note also that (H_1^α) is satisfied with $M_1^\alpha \leq \frac{M_0}{\alpha} < +\infty$. Since $n_0, h_0, h_1 \in L^1(\mathbb{R}_v^+)$ are non increasing we check easily that $n_0^\alpha, h_0^\alpha, h_1^\alpha$ are non increasing and :

$$\|vh_k^\alpha\|_{L^\infty(\mathbb{R}_v^+)} \leq \|vh_k\|_{L^\infty(\mathbb{R}_v^+)} \leq \|h_k\|_{L^1(\mathbb{R}_v^+)},\ k = 0, 1, \alpha > 0.$$

PROPOSITION 5.8. *Assume that* $A, B \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ *are non decreasing with respect to* x *and that* $(H), (H_0), (H_\infty)$ *hold. Then for all* $0 \leq t \leq T$ *we have :*

$$\|\mathcal{F}A(t) - \mathcal{F}B(t)\|_{L^\infty(]0,1[)} \leq C \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds,$$

where C *depends only on* $\|A\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, \|B\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, M_0, M_\infty, T$.

Proof. By the Proposition 5.6 we have :

$$\|\mathcal{F}^\alpha A(t) - \mathcal{F}^\alpha B(t)\|_{L^\infty(]0,1[)} \leq C^\alpha \cdot \int_0^t \|A(s) - B(s)\|_{L^\infty(]0,1[)} ds, \tag{5.3}$$

where $\mathcal{F}^\alpha$ corresponds to the initial-boundary conditions $f_0^\alpha, g_0^\alpha, g_1^\alpha$. Remark that $(C^\alpha)_{\alpha>0}$ is bounded since we have :

$$\begin{aligned} C^\alpha =& C(\|A\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, \|B\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, M_0^\alpha, M_\infty^\alpha, T) \\ \leq & C(\|A\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, \|B\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, M_0, M_\infty, T). \end{aligned}$$

The conclusion follows by passing to the limit in the inequality 5.3 for $\alpha \to 0$ and by using the monotone convergence theorem. □

Now we can state the existence and uniqueness result in the general case :

THEOREM 5.9. *Assume that the hypothesis* $(H), (H_0), (H_\infty)$ *hold and* $U_1 - U_0 \in L^\infty(]0,T[)$. *Then there is a unique mild solution of the 1D Vlasov-Poisson initial-boundary value problem* (f_E, E) *which verifies the estimates :*

$$\|\partial_x E\|_{L^\infty} = \|\rho_E\|_{L^\infty} \leq (M_0 + \|U_1 - U_0\|_{L^\infty}) \exp(6 \cdot T M_\infty) - \|U_1 - U_0\|_{L^\infty},$$

$$\|E\|_{L^\infty} \leq (M_0 + \|U_1 - U_0\|_{L^\infty}) \exp(6 \cdot T M_\infty).$$

5.5. Continuity upon the initial-boundary conditions .

The goal of this section is to estimate the difference between two mild solutions $(f^k, E^k), k = 1, 2$ with respect to the initial-boundary conditions . Consider two sets of initial-boundary conditions $f_0^k, g_0^k, g_1^k, U_0^k - U_1^k \in L^\infty$ verifying the hypothesis $(H^k), (H_0^k), (H_\infty^k),\ k = 1, 2$. We define the applications $\mathcal{F}^k$ as before. We have for $t \in [0, T]$:

$$\|\partial_x \mathcal{F}^k E(t)\|_{L^\infty} = \|\rho_E^k\|_{L^\infty} \leq 6 \cdot M_\infty^k \int_0^t \|E(s)\|_{L^\infty} ds + M_0^k,$$

and :

$$\|\mathcal{F}^k E(t)\|_{L^\infty} \leq 6 \cdot M_\infty^k \int_0^t \|E(s)\|_{L^\infty} ds + M_0^k + |U_0^k(t) - U_1^k(t)|.$$

First of all let us assume the hypothesis (H), (H_1) and (H_∞). We have :

PROPOSITION 5.10. *Assume that* $E \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ *is non decreasing with respect to* x *and that the hypothesis* $(H^k), (H_1^k), (H_\infty^k)$ *hold. We suppose also that the functions :*

$$(H_i)\ \ l_k(v) = \sup_{0\leq t\leq T} |g_k^1(t, (-1)^k v) - g_k^2(t, (-1)^k v)|,\ k = 0, 1$$

are non increasing with respect to $v \in \mathbb{R}_v^+$, *or :*

$$(H_{ii})\ \ \int_0^T\int_{v>0} v|g_0^1(t,v) - g_0^2(t,v)|dtdv - \int_0^T\int_{v<0} v|g_1^1(t,v) - g_1^2(t,v)|dtdv < +\infty.$$

Then for all $0 \leq t \leq T$ *we have :*

$$\begin{aligned} \|\mathcal{F}^1 E(t) - \mathcal{F}^2 E(t)\|_{L^\infty} \leq & C(\|E\|_{L^1(]0,t[;L^\infty(]0,1[))})(\|f_0^1 - f_0^2\|_{L^1(]0,1[\times\mathbb{R}_v)} + \Sigma_{k=0}^1(\|l_k\|_{L^1} + \|l_k\|_{L^\infty})) \\ & + |U_1^1(t) - U_0^1(t) - U_1^2(t) + U_0^2(t)|, \end{aligned}$$

in the case (i) or :

$$\begin{aligned} \|\mathcal{F}^1 E(t) - \mathcal{F}^2 E(t)\|_{L^\infty} \leq & 2(\|f_0^1 - f_0^2\|_{L^1(]0,1[\times\mathbb{R}_v)} + \Sigma_{k=0}^1 \|v(g_k^1 - g_k^2)\|_{L^1(]0,t[\times\mathbb{R}_v^+)}) \\ & + |U_1^1(t) - U_0^1(t) - U_1^2(t) + U_0^2(t)|, \end{aligned}$$

in the case (ii).

Proof. Consider $\varphi \in L^1(]0,1[)$ and let us calculate :

$$\begin{aligned}\int_0^1\int_0^t (j_E^1(s,x) - j_E^2(s,x))\varphi(x)dxds =& \int_0^t\int_0^1\int_{\mathbb{R}_v} (f_E^1 - f_E^2)v\varphi(x)dsdxdv \\ =& \int_0^1\int_{\mathbb{R}_v} (f_0^1(x,v) - f_0^2(x,v)) \int_0^{s_{out}^i} V^i(s)\varphi(X^i(s))dxdvds \\ &+ \Sigma_{k=0}^1 (-1)^k \int_0^t \int_{(-1)^k v>0} v(g_k^1 - g_k^2) \int_s^{s_{out}^k} V^k(\tau)\varphi(X^k(\tau))dsdvd\tau \\ =& \int_0^1\int_{\mathbb{R}_v} (f_0^1(x,v) - f_0^2(x,v)) \int_x^{X^i(s_{out}^i)} \varphi(u)dxdvdu \\ &+ \Sigma_{k=0}^1 (-1)^k \int_0^t \int_{(-1)^k v>0} v(g_k^1(s,v) - g_k^2(s,v)) \int_k^{X^k(s_{out}^k)} \varphi(u)dsdvdu \\ =& \mathcal{I}^i + \Sigma_{k=0}^1 \mathcal{I}^k.\end{aligned}$$

Obviously we have :

$$|\mathcal{I}^i| \leq \|f_0^1 - f_0^2\|_{L^1(]0,1[\times\mathbb{R}_v)} \cdot \|\varphi\|_{L^1(]0,1[)}.$$

On the other hand, with the notation $\Phi^k = \int_k^{X^k(s_{out}^k)} \varphi(u)du$ we have :

$$\begin{aligned}\mathcal{I}^0 =& \int_0^t\int_{\mathbb{R}_v} v(g_0^1(s,v) - g_0^2(s,v))dsdv\Phi^0 \mathbf{1}_{\{0<v<v_E^0\}} \\ &+ \int_0^t\int_{\mathbb{R}_v} v(g_0^1(s,v) - g_0^2(s,v))dsdv\Phi^0 \mathbf{1}_{\{v_E^0<v<v_E^1\}} \\ &+ \int_0^t\int_{\mathbb{R}_v} v(g_0^1(s,v) - g_0^2(s,v))dsdv\Phi^0 \mathbf{1}_{\{v>v_E^1\}} \\ =& \mathcal{I}_0^0 + \mathcal{I}_t^0 + \mathcal{I}_1^0,\end{aligned}$$

where $v_E^k = v_E^k(t;s,k)$ are the critical velocities corresponding to the domain $]0,t[\times]0,1[$, to the point (s,k) and the field E. Let us calculate now :

$$\begin{aligned}\int_0^1\left(\int_0^1\int_0^t (j_E^1(s,y) - j_E^2(s,y))dsdy\right)\varphi(x)dx =& \int_0^t\int_0^1\int_{\mathbb{R}_v} v(f_E^1(s,y,v) - f_E^2(s,y,v))dsdydv \cdot \int_0^1 \varphi(u)du \\ =& \int_0^1 \varphi(u)du \cdot \{\int_0^1\int_{\mathbb{R}_v} (f_0^1 - f_0^2) \int_0^{s_{out}^i} V^i(s)dxdvds \\ &+ \Sigma_{k=0}^1 (-1)^k \int_0^t \int_{(-1)^k v>0} v(g_k^1 - g_k^2) \int_s^{s_{out}^k} V^k(\tau)dsdvd\tau\} \\ =& \int_0^1 \varphi(u)du \cdot \{\int_0^1\int_{\mathbb{R}_v} (f_0^1 - f_0^2)(X^i(s_{out}^i) - x)dxdv \\ &+ \Sigma_{k=0}^1 (-1)^k \int_0^t \int_{(-1)^k v>0} v(g_k^1 - g_k^2)(X^k(s_{out}^k) - k)dsdv\} \\ =& \mathcal{J}^i + \Sigma_{k=0}^1 \mathcal{J}^k.\end{aligned}$$

Obviously we have $|\mathcal{J}^i| \leq \|f_0^1 - f_0^2\|_{L^1(]0,1[\times\mathbb{R}_v)} \cdot \|\varphi\|_{L^1(]0,1[)}$. On the other hand we have :

$$\begin{aligned}\mathcal{J}^0 =&\int_0^t\int_{\mathbb{R}_v} v(g_0^1 - g_0^2)(X^0(s_{out}^0) - 0)\mathbf{1}_{\{0<v<v_E^0\}}dsdv\int_0^1\varphi(u)du\\ &+\int_0^t\int_{\mathbb{R}_v} v(g_0^1 - g_0^2)(X^0(s_{out}^0) - 0)\mathbf{1}_{\{v_E^0<v<v_E^1\}}dsdv\int_0^1\varphi(u)du\\ &+\int_0^t\int_{\mathbb{R}_v} v(g_0^1 - g_0^2)(X^0(s_{out}^0) - 0)\mathbf{1}_{\{v>v_E^1\}}dsdv\int_0^1\varphi(u)du\\ =&\mathcal{J}_0^0 + \mathcal{J}_t^0 + \mathcal{J}_1^0.\end{aligned}$$

For $0 < v < v_E^0$ we have $X^0(s_{out}^0) = 0$ and thus $\mathcal{I}_0^0 = \mathcal{J}_0^0$. For $v > v_E^1$ we have $X^0(s_{out}^0) = 1$ and thus $\mathcal{I}_1^0 = \mathcal{J}_1^0$. In order to evaluate $\mathcal{I}_t^0$ and $\mathcal{J}_t^0$ we can perform the change of variables $(y, w) = S(s, v)$:

$$y = X^0(t; s, 0, v),\ \ w = V^0(t; s, 0, v),\ \ \left|\frac{\partial(y, w)}{\partial(s, v)}\right| = |v|,$$

on $D = \{(s, v) \in]0, t[\times\mathbb{R}_v^+ \mid v_E^0(t; s, 0) < v < v_E^1(t; s, 0)\}$. In the case (i) one gets :

$$\begin{aligned}|\mathcal{I}_t^0|\leq&\int_0^1|\varphi(u)|du\cdot\int_0^t\int_{\mathbb{R}_v} v|g_0^1(s, v) - g_0^2(s, v)|\mathbf{1}_{\{v_E^0<v<v_E^1\}}dsdv\\ =&\int_0^1|\varphi(u)|du\cdot\int_0^1\int_{w>-R}|g_0^1 - g_0^2|(s_{in}^0(t, y, w), V^0(s_{in}^0(t, y, w); t, y, w))\mathbf{1}_{S(D)}dydw\\ \leq&\int_{w>-R} l_0^R(w)dw\cdot\|\varphi\|_{L^1(]0,1[)} = (2R\|l_0\|_{L^\infty(\mathbb{R}_v^+)} + \|l_0\|_{L^1(\mathbb{R}_v^+)})\cdot\|\varphi\|_{L^1(]0,1[)},\end{aligned}$$

where $R = \int_0^t \|E(s)\|_{L^\infty(]0,1[)}ds$. In a similar manner we find that :

$$\begin{aligned}|\mathcal{J}_t^0|\leq&\int_0^t\int_{\mathbb{R}_v} v|g_0^1 - g_0^2|\mathbf{1}_{\{v_E^0<v<v_E^1\}}dsdv\cdot\|\varphi\|_{L^1(]0,1[)}\\ \leq&\int_{w>-R} h_0^R(w)dw\cdot\|\varphi\|_{L^1(]0,1[)}.\end{aligned}$$

Finally one gets that :

$$\begin{aligned}|\mathcal{I} - \mathcal{J}|=&\left|\int_0^1\left(\int_0^t(j_E^1(s, x) - j_E^2(s, x))ds - \int_0^1\int_0^t(j_E^1(s, y) - j_E^2(s, y))dyds\right)\varphi(x)dx\right|\\ =&|\mathcal{I}^i + \mathcal{I}^0 + \mathcal{I}^1 - \mathcal{J}^i - \mathcal{J}^0 - \mathcal{J}^1|\\ \leq&|\mathcal{I}^i| + |\mathcal{J}^i| + |\mathcal{I}_t^0| + |\mathcal{J}_t^0| + |\mathcal{I}_t^1| + |\mathcal{J}_t^1|\\ \leq&C(R)(\|f_0^1 - f_0^2\|_{L^1} + \Sigma_{k=0}^1(\|l_k\|_{L^\infty} + \|l_k\|_{L^1}))\cdot\|\varphi\|_{L^1(]0,1[)},\end{aligned}$$

and the conclusion follows in the case (i) by using the Remark 5.4. For the case (ii) it is sufficient to remark that :

$$\max\{|\mathcal{I}_t^k|, |\mathcal{J}_t^k|\} \leq \int_0^t\int_{(-1)^k v>0}(-1)^k v|g_k^1(s, v) - g_k^2(s, v)|dsdv\cdot\|\varphi\|_{L^1(]0,1[)},\ k = 0, 1.$$

□

REMARK 5.11. *The conclusion of Proposition 5.10 still holds if we replace the hypothesis* (H_1^k) *by* (H_0^k), $k = 0, 1$ *(proceed like in the proof of the Proposition 5.8).*

PROPOSITION 5.12. *Assume that $E^1, E^2 \in L^\infty(]0,T[;W^{1,\infty}(]0,1[))$ are non decreasing with respect to x and that $(H^k), (H_0^k), (H_\infty^k)$ hold. We suppose also that (H_i) or (H_{ii}) is verified. Then for all $0 \le t \le T$ we have :*

$$\|\mathcal{F}^1 E^1(t) - \mathcal{F}^2 E^2(t)\|_{L^\infty(]0,1[)} \le C_1 + C_2 \int_0^t \|E^1(s) - E^2(s)\|_{L^\infty(]0,1[)} ds,$$

where $C_2 = C_2(\|E^k\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, M_0^k, M_\infty^k, T)$ and :

$$C_1 = C_1(\|E^k\|_{L^1(]0,T[;L^\infty(]0,1[))}) \left(\|f_0^1 - f_0^2\|_{L^1} + \Sigma_{k=0}^1(\|l_k\|_{L^1} + \|l_k\|_{L^\infty})\right) + |U_1^1 - U_0^1 - U_1^2 + U_0^2|(t),$$

in the case (i) or :

$$C_1 = 2\left(\|f_0^1 - f_0^2\|_{L^1} + \|v(g_0^1 - g_0^2)\|_{L^1} + \|v(g_1^1 - g_1^2)\|_{L^1}\right) + |U_1^1 - U_0^1 - U_1^2 + U_0^2|(t),$$

in the case (ii).

Proof. We can write :

$$\|\mathcal{F}^1 E^1(t) - \mathcal{F}^2 E^2(t)\|_{L^\infty} \le \|\mathcal{F}^1 E^1(t) - \mathcal{F}^1 E^2(t)\|_{L^\infty} + \|\mathcal{F}^1 E^2(t) - \mathcal{F}^2 E^2(t)\|_{L^\infty}.$$

By using the Proposition 5.8 we find :

$$\|\mathcal{F}^1 E^1(t) - \mathcal{F}^1 E^2(t)\|_{L^\infty(]0,1[)} \le C_2 \int_0^t \|E^1(s) - E^2(s)\|_{L^\infty(]0,1[)} ds,$$

where C_2 depends on $\|E^k\|_{L^1(]0,T[;W^{1,\infty}(]0,1[))}, M_0^1, M_\infty^1, T$. The conclusion follows by the Proposition 5.10 and the Remark 5.11. □

THEOREM 5.13. *Assume that $f_0^k, g_0^k, g_1^k, U_1^k - U_0^k \in L^\infty(]0,T[)$, $k = 1,2$ are two sets of initial-boundary conditions verifying the hypothesis $(H^k), (H_0^k), (H_\infty^k)$ and (H_i) or (H_{ii}). Denote by (f^k, E^k), $k = 1,2$ the corresponding unique mild solutions. Then we have for all $0 \le t \le T$:*

$$\|E^1(t) - E^2(t)\|_{L^\infty(]0,1[)} \le C\{\|f_0^1 - f_0^2\|_{L^1} + \Sigma_{k=0}^1(\|l_k\|_{L^1} + \|l_k\|_{L^\infty}) + |U_1^1 - U_0^1 - U_1^2 + U_0^2|(t)\},$$

in the case (i) or :

$$\begin{aligned}\|E^1(t) - E^2(t)\|_{L^\infty(]0,1[)} \le & C\{\|f_0^1 - f_0^2\|_{L^1(]0,1[\times\mathbb{R}_v)} + \Sigma_{k=0}^1\|v(g_k^1 - g_k^2)\|_{L^1(]0,T[\times\mathbb{R}_v^+)} \\ & + |U_1^1 - U_0^1 - U_1^2 + U_0^2|(t)\},\end{aligned}$$

in the case (ii) where C is a constant depending on $M_0^k, M_\infty^k, \|U_0^k - U_1^k\|_{L^\infty}, T$.

Proof. Since (f^k, E^k) are mild solutions we have $\mathcal{F}^k E^k = E^k$, E^k are non decreasing with respect to x and we know that :

$$\|E^k\|_{L^\infty(]0,T[;W^{1,\infty}(]0,1[))} \le C(M_0^k, M_\infty^k, \|U_1^k - U_0^k\|_{L^\infty(]0,T[)}, T).$$

By the Proposition 5.12 we have for all $0 \le t \le T$:

$$\|E^1(t) - E^2(t)\|_{L^\infty(]0,1[)} = \|\mathcal{F}^1 E^1(t) - \mathcal{F}^2 E^2(t)\|_{L^\infty(]0,1[)} \le C_1 + C_2 \int_0^t \|E^1(s) - E^2(s)\|_{L^\infty(]0,1[)} ds,$$

with C_1, C_2 as before. The conclusion of the theorem follows by using the Gronwall lemma. □

6. The 1D Vlasov-Maxwell system.

This section is devoted to the study of the 1D Vlasov-Maxwell system with initial condition by adapting the method used previously. Since the proofs are quite similar we only sketch them. Moreover, as explained in the introduction, in this case we can consider different species of particles. Recall that results on the existence and uniqueness have already been obtained by Cooper and Klimas [7]. Let us introduce the equations :

$$\partial_t f^\pm + v\cdot\partial_x f^\pm \pm E\cdot\partial_v f^\pm = 0,\ \ (t,x,v)\in]0,T[\times\mathbb{R}_x\times\mathbb{R}_v, \tag{6.1}$$

$$\partial_t E = -j(t,x) := -j^+ + j^- = -\int_{\mathbb{R}_v} v(f^+(t,x,v) - f^-(t,x,v))dv,\ \ (t,x)\in]0,T[\times\mathbb{R}_x, \tag{6.2}$$

with the initial conditions :

$$f^\pm(t=0,x,v) = f_0^\pm(x,v),\ \ (x,v)\in\mathbb{R}_x\times\mathbb{R}_v, \tag{6.3}$$

$$E(t=0,x) = E_0(x) = \int \rho_0(y)dy,\ \ x\in\mathbb{R}_x, \tag{6.4}$$

where $\rho_0 = \rho_0^+ - \rho_0^- = \int_{\mathbb{R}_v}(f_0^+ - f_0^-)dv$ and $\int\rho_0(y)dy$ denotes an arbitrary primitive of ρ_0. Assume that $E\in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x)), f_0^\pm\in L^1_{loc}(\mathbb{R}_x\times\mathbb{R}_v)$. We denote by $(X^\pm(s),V^\pm(s))$ the characteristics associated to $\pm E$. As usual we say that $f^\pm\in L^1_{loc}(]0,T[\times\mathbb{R}_x\times\mathbb{R}_v)$ is a mild solution for the Vlasov problem (6.1), (6.3) iff :

$$\int_0^T\int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f^\pm(t,x,v)\psi(t,x,v)dtdxdv = \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^\pm\int_0^T\psi(s,X^\pm(s;0,x,v),V^\pm(s;0,x,v))dxdvds,$$

for all test function $\psi\in L^\infty(]0,T[\times\mathbb{R}_x\times\mathbb{R}_v)$ compactly supported in $[0,T]\times\mathbb{R}_x\times\mathbb{R}_v$. Assume now that $f_0^\pm\in L^1(\mathbb{R}_x\times\mathbb{R}_v)$. We say that $(f^\pm,E)\in L^1(]0,T[\times\mathbb{R}_x\times\mathbb{R}_v)\times L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x))$ is a mild solution of the 1D Vlasov-Maxwell problem iff $f^\pm$ is a mild solution for the Vlasov problem (6.1), (6.3) corresponding to the electric field $\pm E$ such that :

$$\int_{\mathbb{R}_x} E(t,x)\varphi(x)dx = -\int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^+(x,v)\int_x^{X^+(t;0,x,v)}\varphi(u)dudxdv + \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^-(x,v)\int_x^{X^-(t;0,x,v)}\varphi(u)dudxdv$$
$$+\int_{\mathbb{R}_x} E_0(x)\varphi(x)dx,\ \ \forall\varphi\in L^1(\mathbb{R}_x).$$

REMARK 6.1. *Note that the previous formula defines a unique function $E\in L^\infty(]0,T[\times\mathbb{R}_x)$. This definition can be derived formally from the equation (6.2) by using the mild formulation :*

$$\begin{aligned}\int_{\mathbb{R}_x} E(t,x)\varphi(x) &= -\int_0^t\int_{\mathbb{R}_x}\int_{\mathbb{R}_v} v(f^+(s,x,v) - f^-(s,x,v))\varphi(x)dsdxdv + \int_{\mathbb{R}_x} E_0(x)\varphi(x)dx\\
&= -\int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^+(x,v)\int_0^t V^+(s)\varphi(X^+(s))dsdxdv + \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^-(x,v)\int_0^t V^-(s)\varphi(X^-(s))dsdxdv\\
&+\int_{\mathbb{R}_x} E_0(x)\varphi(x)dx\\
&= -\int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^+(x,v)\int_x^{X^+(t;0,x,v)}\varphi(u)dudxdv + \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^-(x,v)\int_x^{X^-(t;0,x,v)}\varphi(u)dudxdv\\
&+\int_{\mathbb{R}_x} E_0(x)\varphi(x)dx.\end{aligned}$$

As before we define the application $\mathcal{F}$ for $E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x))$ by :

$$E \to f_E^\pm \to E_1(t) = \mathcal{F}E(t) = E_0 - \int_0^t \int_{\mathbb{R}_v} v(f_E^+ - f_E^-) ds dv,$$

where $f_E^\pm$ are the mild solutions of the Vlasov problem (6.1),(6.3) associated to the field $\pm E$, E_0 is given by (6.4) and $-\int_0^t\int_{\mathbb{R}_v} v(f_E^+ - f_E^-)dsdv$ is defined as in Remark 6.1.

6.1. Estimate of $\mathcal{F}E$.

We assume that there is $n_0^\pm : [0,+\infty[\to [0,+\infty[$ non increasing, such that :

$$(H^\pm) \;\; f_0^\pm(x,v) \le n_0^\pm(|v|), \;\; (x,v) \in \mathbb{R}_x \times \mathbb{R}_v,$$

$$(H_0^\pm) \;\; M_0^\pm := \int_{\mathbb{R}_v} n_0^\pm(|v|)dv < +\infty,$$

$$(H_\infty^\pm) \;\; M_\infty^\pm := \|n_0^\pm\|_{L^\infty(\mathbb{R}_v^+)} < +\infty,$$

$$(H_{\rho_0}) \;\; M_{\rho_0} := \sup_{x\in\mathbb{R}_x} \left| \int_0^x (\rho_0^+(y) - \rho_0^-(y))dy \right| < +\infty.$$

Proposition 6.2. *Assume that $f_0^\pm \in L^1(\mathbb{R}_x \times \mathbb{R}_v)$ satisfy $(H^\pm), (H_0^\pm), (H_\infty^\pm)$. Then for every $E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x))$ we have $f_E^\pm \in L^\infty(]0,T[;L^1(\mathbb{R}_x \times \mathbb{R}_v))$, $\rho_E^\pm \in L^\infty(]0,T[\times\mathbb{R}_x)$, $\mathcal{F}E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x))$. Moreover the following estimates hold :*

$$\|f_E^\pm\|_{L^\infty(]0,T[;L^1(\mathbb{R}_x\times\mathbb{R}_v))} = \|\rho_E^\pm\|_{L^\infty(]0,T[;L^1(\mathbb{R}_x))} = \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^\pm(x,v)dxdv,$$

$$\|\rho_E^\pm\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \le 2M_\infty^\pm \int_0^t \|E(s)\|_{L^\infty(\mathbb{R}_x)}ds + M_0^\pm,$$

$$\|\mathcal{F}E\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \le C + M_{\rho_0} + \|f_0^+\|_{L^1(\mathbb{R}_x\times\mathbb{R}_v)} + \|f_0^-\|_{L^1(\mathbb{R}_x\times\mathbb{R}_v)},$$

$$\|\partial_x\mathcal{F}E\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \le 2(M_\infty^+ + M_\infty^-) \int_0^t \|E(s)\|_{L^\infty(\mathbb{R}_x)}ds + M_0^+ + M_0^-,$$

$$\lim_{R_1\to+\infty} \int_{|v|>R_1} f_E^\pm(t,x,v)dv = 0, \text{ uniformly with respect to } (t,x) \in]0,T[\times\mathbb{R}_x,$$

and the mild formulation of the Vlasov problem holds for all $\psi \in L^\infty(]0,T[\times\mathbb{R}_x\times\mathbb{R}_v)$.

Proof. We have :

$$\begin{aligned}\rho_E^\pm(t,x) &= \int_{\mathbb{R}_v} f_E^\pm dv = \int_{\mathbb{R}_v} f_0^\pm(X^\pm(0;t,x,v), V^\pm(0;t,x,v))dv \\ &\le \int_{\mathbb{R}_v} n_0^\pm(|V^\pm(0;t,x,v)|)dv \le \int_{\mathbb{R}_v} n_0^{\pm,R}(|v|)dv \\ &= 2RM_\infty^\pm + M_0^\pm,\end{aligned}$$

where $R = \int_0^t \|E(s)\|_{L^\infty(\mathbb{R}_x)} ds$. By the definition of $\mathcal{F}E$, taking into account that $E_0(x) = C + \int_0^x \rho_0(y)dy$, we deduce that :

$$\|\mathcal{F}E(t)\|_{L^\infty(\mathbb{R}_x)} \leq C + M_{\rho_0} + \|f_0^+\|_{ofL^1(\mathbb{R}_x\times\mathbb{R}_v)} + \|f_0^-\|_{L^1(\mathbb{R}_x\times\mathbb{R}_v)}, \;\; 0 \leq t \leq T.$$

By using the definition of $\mathcal{F}E(t)$ and the mild formulation we check that $\partial_x \mathcal{F}E(t) = \rho(t)$ in $\mathcal{D}'(\mathbb{R}_x)$, $0 \leq t \leq T$ and we deduce that $\|\partial_x \mathcal{F}E\|_{L^\infty} \leq \|\rho_E^+\|_{L^\infty} + \|\rho_E^-\|_{L^\infty} \leq 2R(M_\infty^+ + M_\infty^-) + M_0^+ + M_0^-$. The last two assertions follow by standard calculations as it was done for the Vlasov-Poisson problem. □

Remark 6.3. *If we note* $X_T = \{E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x)) \mid \|E\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \leq \|E_0\|_{L^\infty(\mathbb{R}_x)} + \|f_0^+\|_{L^1} + \|f_0^-\|_{L^1}\}$, *then* $\mathcal{F}(X_T) \subset X_T$ *and :*

$$\|\partial_x \mathcal{F}E\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \leq 2(M_\infty^+ + M_\infty^-) \cdot T \cdot (\|E_0\|_{L^\infty(\mathbb{R}_x)} + \|f_0^+\|_{L^1} + \|f_0^-\|_{L^1}) + M_0^+ + M_0^-.$$

6.2. Estimate of $\mathcal{F}A - \mathcal{F}B$.

Proposition 6.4. *Assume that* $A, B \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x))$ *and* $f_0^\pm \in L^1(\mathbb{R}_x \times \mathbb{R}_v)$ *verify the hypothesis* $(H^\pm), (H_0^\pm), (H_\infty^\pm)$. *Then for all* $0 \leq t \leq T$ *we have :*

$$\|\mathcal{F}A(t) - \mathcal{F}B(t)\|_{L^\infty(\mathbb{R}_x)} \leq C \int_0^t \|A(s) - B(s)\|_{L^\infty(\mathbb{R}_x)} ds,$$

with C *a constant depending on* $\|A\|_{L^1(]0,T[;W^{1,\infty}(\mathbb{R}_x))}, \|B\|_{L^1(]0,T[;W^{1,\infty}(\mathbb{R}_x))}, M_0^\pm, M_\infty^\pm, T$.

Proof. Take $\varphi \in L^1(\mathbb{R}_x)$ and calculate :

$$\begin{aligned}\left|\int_{\mathbb{R}_x} (\mathcal{F}A(t,x) - \mathcal{F}B(t,x))\varphi(x)dx\right| =& \left| -\int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^+ \int_{X_B^+(t)}^{X_A^+(t)} \varphi(u)dudxdv + \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^- \int_{X_B^-(t)}^{X_A^-(t)} \varphi(u)dudxdv\right| \\ \leq& \Sigma_{k=\pm} \int_{\mathbb{R}_u} |\varphi(u)| \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^k(x,v) \mathbf{1}_{\{|u - X_A^k(t)| < |X_B^k(t) - X_A^k(t)|\}} dudxdv \\ \leq& \Sigma_{k=\pm} \int |\varphi(u)| \int\int f_0^k(X_A^k(0;t,y,w), V_A^k(0;t,y,w)) \mathbf{1}_{\{|u-y| \leq C\cdot R\}} \\ \leq& \|\varphi\|_{L^1(\mathbb{R}_x)} 2CR(M_0^+ + M_0^- + 2\int_0^t \|A(s)\|_{L^\infty} ds (M_\infty^+ + M_\infty^-)),\end{aligned}$$

where $C = \exp\left(\int_0^t (1 + \|\partial_x B(s)\|_{L^\infty(\mathbb{R}_x)}) ds\right)$ and $R = \int_0^t \|A(s) - B(s)\|_{L^\infty(\mathbb{R}_x)} ds$. □

We can prove by using the iterated approximations method the theorem :

Theorem 6.5. *Assume that* $f_0^\pm \in L^1(\mathbb{R}_x \times \mathbb{R}_v)$ *verify the hypothesis* $(H^\pm), (H_0^\pm), (H_\infty^\pm)$. *Then, for a fixed choice of primitive in* (6.4), *there is an unique mild solution for the 1D Vlasov-Maxwell initial value problem.*

Remark 6.6. *If in addition we assume that* $|v|^p f_0^\pm \in L^1(\mathbb{R}_x \times \mathbb{R}_v)$ *and :*

$$(H_p^\pm) \;\; M_p^\pm := \int_{\mathbb{R}_v} |v|^p n_0^\pm(|v|) dv < +\infty,$$

for some integer $p \geq 1$ *we can prove that :*

$$|v|^p f^\pm \in L^\infty(]0,T[;L^1(\mathbb{R}_x \times \mathbb{R}_v)), \;\; \int_{\mathbb{R}_v} |v|^p f^\pm(t,x,v) dv \in L^\infty(]0,T[\times\mathbb{R}_x).$$

In particular $j^\pm = \int_{\mathbb{R}_v} v f^\pm dv \in L^\infty(]0,T[\times\mathbb{R}_x)$ *and* $\partial_t E = -j$, $\lim_{R_1\to+\infty} \int_{|v|>R_1} |v|^p f^\pm dv = 0$ *uniformly with respect to* $(t,x) \in]0,T[\times\mathbb{R}_x$ *and the mild formulation of the Vlasov problem holds for all function* $|\psi(t,x,v)| \le C(1+|v|^p)$.

Proof. By multiplying the Vlasov equation by $|v|^p$ we get :

$$\frac{d}{dt}\int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f^\pm(t,x,v)|v|^p dxdv = \pm \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} E f^\pm p|v|^{p-2} v dxdv.$$

Therefore we deduce that :

$$\int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f^\pm(t,x,v)|v|^p dxdv \le \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f_0^\pm(x,v)|v|^p dxdv + p\|E\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \int_0^t \int_{\mathbb{R}_x}\int_{\mathbb{R}_v} f^\pm |v|^{p-1} dxdvds,$$

and the conclusion follows by induction on p. On the other hand :

$$\begin{aligned}\int_{\mathbb{R}_v} |v|^p f^\pm(t,x,v)dv &= \int_{\mathbb{R}_v} |v|^p f_0^\pm(X^\pm(0;t,x,v), V^\pm(0;t,x,v))dv \\ &\le \int_{\mathbb{R}_v} |v|^p n_0^{\pm,R}(|v|)dv \\ &\le C(R)(\|n_0^\pm\|_{L^\infty(\mathbb{R}_v^+)} + \||v|^p n_0^\pm(|v|)\|_{L^1(\mathbb{R}_v)}),\end{aligned}$$

with $R = \int_0^t \|E(s)\|_{L^\infty(\mathbb{R}_x)} ds$. In order to verify that $\partial_t E = -j$ in $\mathcal{D}'(]0,T[\times\mathbb{R}_x)$, take $\varphi \in C_0^1(]0,T[\times\mathbb{R}_x)$ and use the mild formulation with the test function $\psi(t,x,v) = v\varphi(t,x)$. □

7. The periodic 1D Vlasov-Poisson problem.

In this section we analyse the space periodic 1D Vlasov-Poisson problem :

$$\partial_t f^\pm + v\cdot\partial_x f^\pm \pm E\cdot\partial_v f^\pm = 0, \;\; (t,x,v) \in]0,T[\times]0,1[\times\mathbb{R}_v, \tag{7.1}$$

$$\partial_x E = \rho(t,x) := \rho^+ - \rho^- = \int_{\mathbb{R}_v} (f^+(t,x,v) - f^-(t,x,v))dv, \;\; (t,x) \in]0,T[\times]0,1[, \tag{7.2}$$

with the space periodic initial conditions :

$$f^\pm(t=0,x,v) = f_0^\pm(x,v), \;\; (x,v) \in]0,1[\times\mathbb{R}_v. \tag{7.3}$$

The electric field derives from a space periodic potential and thus $\int_0^1 E(t,x)dx = 0$. In this case the Poisson field can be written as :

$$E(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy, \;\; x \in [0,1], \;\; t \in [0,T]. \tag{7.4}$$

We introduce the mild formulation as before, by taking space periodic test functions. This time is convenient to define the application $\mathcal{F}$ for 1-periodic with respect to x fields $E \in L^\infty(]0,T[\times\mathbb{R}_x)$ by :

$$E \to f_E^\pm \to \rho_E^\pm \to \int_0^x \rho_E(t,y)dy - \int_0^1 (1-y)\rho_E(t,y)dy = \mathcal{F}E.$$

Remark 7.1. *$\mathcal{F}E$ is 1-periodic in x iff $\int_0^1 \rho_E(t,y)dy = 0$, $0 \le t \le T$ and therefore, by the conservation of the total charge, iff $\int_0^1\int_{\mathbb{R}_v} f_0^+(x,v)dxdv = \int_0^1\int_{\mathbb{R}_v} f_0^-(x,v)dxdv$.*

7.1. Estimate of $\mathcal{F}E$.

We assume that $f_0^\pm$ verify the hypothesis $(H^\pm), (H_0^\pm), (H_\infty^\pm)$. We suppose also that the neutrality condition holds :

$$(N) \quad \int_0^1\int_{\mathbb{R}_v} f_0^+(x,v)dxdv = \int_0^1\int_{\mathbb{R}_v} f_0^-(x,v)dxdv.$$

Proposition 7.2. *Assume that $f_0^\pm$ are 1-periodic in x and satisfy $(H^\pm), (H_0^\pm), (H_\infty^\pm)$ and (N). Then for every $E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x))$ 1-periodic in x we have :*

$$\|\rho_E^\pm\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \le 2M_\infty^\pm \int_0^t \|E(s)\|_{L^\infty(\mathbb{R}_x)}ds + M_0^\pm,$$

$$\|\mathcal{F}E\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \le \int_0^1\int_{\mathbb{R}_v} f_0^+(x,v)dxdv + \int_0^1\int_{\mathbb{R}_v} f_0^-(x,v)dxdv \le M_0^+ + M_0^-,$$

$$\|\partial_x\mathcal{F}E\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \le 2(M_\infty^+ + M_\infty^-)\int_0^t \|E(s)\|_{L^\infty(\mathbb{R}_x)}ds + M_0^+ + M_0^-.$$

Moreover $\lim_{R_1\to+\infty}\int_{|v|>R_1} f_E^\pm(t,x,v)dv = 0$ uniformly with respect to $(t,x) \in]0,T[\times\mathbb{R}_x$ and the mild formulation of the Vlasov problem holds for all function $\psi \in L^\infty(]0,T[\times\mathbb{R}_x\times\mathbb{R}_v)$, 1-periodic in x.

7.2. Estimate of $\mathcal{F}A - \mathcal{F}B$.

Proposition 7.3. *Assume that $A, B \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}_x))$ are 1-periodic in x and the hypothesis $(H^\pm), (H_0^\pm), (H_\infty^\pm), (N)$ hold. Then for all $0 \le t \le T$ we have :*

$$\|\mathcal{F}A(t) - \mathcal{F}B(t)\|_{L^\infty(\mathbb{R}_x)} \le C\int_0^t \|A(s)-B(s)\|_{L^\infty(\mathbb{R}_x)}ds,$$

where the constant C depends on $\|A\|_{L^1(]0,T[;W^{1,\infty}(\mathbb{R}_x))}, \|B\|_{L^1(]0,T[;W^{1,\infty}(\mathbb{R}_x))}, M_0^\pm, M_\infty^\pm, T$.

Proof. Take $\varphi \in L^1_{loc}(\mathbb{R}_x)$ and calculate :

$$\begin{aligned}
\mathcal{I}_1^\pm =& \left|\int_0^1 \varphi(x)\int_0^x (\rho_A^\pm(t,y) - \rho_B^\pm(t,y))dydx\right| \\
=& \left|\int\int (f_0^\pm(X_A^\pm(0;t,y,v), V_A^\pm(0;t,y,v)) - f_0^\pm(X_B^\pm(0;t,y,v), V_B^\pm(0;t,y,v)))\int_y^1 \varphi(x)dxdydv\right| \\
=& \left|\int\int f_0^\pm(\xi,\eta)\int_{X_A^\pm(t;0,\xi,\eta)}^{X_B^\pm(t;0,\xi,\eta)} \varphi(x)dxd\xi d\eta\right| \\
\le& \int_0^1 |\varphi(u)|\int\int f_0^\pm(\xi,\eta)\mathbf{1}_{\{|u-X_A^\pm(t)|<|X_A^\pm(t)-X_B^\pm(t)|\}}d\xi d\eta du \\
\le& \int_0^1 |\varphi(u)|\iint f_0^\pm(X_A^\pm(0;t,y,w), V_A^\pm(0;t,y,w))\mathbf{1}_{\{|u-y|<CR\}}dydw \\
\le& 2CR(2\int_0^t \|A(s)\|_{L^\infty}ds \cdot M_\infty^\pm + M_0^\pm)\cdot\|\varphi\|_{L^1(]0,1[)},
\end{aligned}$$

where $C = \exp\left(\int_0^t (1+\|\partial_x B(s)\|_{L^\infty})ds\right)$ and $R = \int_0^t \|A(s)-B(s)\|_{L^\infty}ds$. In order to estimate $\mathcal{I}_2^\pm = \left|\int_0^1 (1-y)(\rho_A^\pm(t,y) - \rho_B^\pm(t,y))dy\right|$ take $\varphi \equiv 1$ in the previous computation. □

Finally we obtain the existence and uniqueness of the space periodic mild solution :

THEOREM 7.4. *Assume that* $f_0^\pm$ *are* 1*-periodic in* x *and satisfy the hypothesis* $(H^\pm), (H_0^\pm)$, $(H_\infty^\pm)$, (N). *Then there is an unique mild solution for the space periodic 1D Vlasov-Poisson problem. Moreover we have the estimates :*

$$\|\rho^\pm\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \leq 2M_\infty^\pm \cdot T \cdot (M_0^+ + M_0^-) + M_0^\pm,$$

$$\|E\|_{L^\infty(]0,T[\times\mathbb{R}_x)} \leq M_0^+ + M_0^-.$$

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