# Global solutions for the one dimensional Water-Bag model

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**Abstract**

In this paper we study a special type of solution for the one dimensional Vlasov-Maxwell equations. We assume that initially the particle density is constant on its support in the phase space and we are looking for solutions with particle density having the same property at any time $t > 0$. More precisely, for each $x$ the support of the density is assumed to be an interval $[p^-, p^+]$ with end-points varying in space and time. We analyze here the case of weak and strong solutions for the effective equations verified by the end-points and the electric field (water-bag model) in the relativistic setting.

**Keywords:** Vlasov-Maxwell equations, Water-Bag model, Conservation laws

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# 1 Introduction

The Vlasov-Maxwell system governs the evolution of an ensemble of charged particles subject to electro-magnetic fields created by themselves and possibly external sources in which collisions are typically neglected. Given $f$ the density number of charged particles at time $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$, the dynamics of the particles is described by the Vlasov equation

$$\partial_t f + v(p) \cdot \nabla_x f + q(E(t,x) + v(p) \wedge B(t,x)) \cdot \nabla_p f = 0, \quad (t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \quad (1)$$

where the electro-magnetic field $(E, B)$ is defined in a self-consistent way by the Maxwell equations

$$\partial_t E - c_0^2 \mathrm{curl}_x B = -\frac{j(t,x)}{\varepsilon_0}, \quad j(t,x) = q \int_{\mathbb{R}^3} v(p) f(t,x,p)\, dp, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad (2)$$

$$\partial_t B + \mathrm{curl}_x E = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad (3)$$

$$\mathrm{div}_x E = \frac{\rho(t,x)}{\varepsilon_0}, \quad \rho(t,x) = q \int_{\mathbb{R}^3} f(t,x,p)\, dp, \quad \mathrm{div}_x B = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad (4)$$

where $q, m$ are the charge and the mass of the particles, $\varepsilon_0$ is the electric permittivity of the vacuum and $v(p)$ is the relativistic velocity associated to the momentum $p$

$$v(p) = \frac{p}{m} \left( 1 + \frac{|p|^2}{m^2 c_0^2} \right)^{-\frac{1}{2}},$$

where $c_0$ is the light speed in the vacuum. Suitable initial conditions for the particle density and the electro-magnetic field have to be prescribed verifying certain compatibility conditions. The existence of global weak solution was obtained in [9] and the existence of strong solutions has been investigated by different approaches in [11, 5, 13]. Despite of these advances in the existence theory for the Vlasov-Maxwell system, many questions concerning qualitative behavior, special solutions or regularity issues, to name a few, are completely open. Recently global existence and uniqueness results have been proved for reduced models for laser-plasma interaction [7, 4] leading to particular global solutions of the Vlasov-Maxwell system.

Neglecting the magnetic field and the relativistic corrections in the Vlasov equation leads to the Vlasov-Poisson model

$$\partial_t f + \frac{p}{m} \cdot \nabla_x f + qE(t,x) \cdot \nabla_p f = 0, \quad (t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$\mathrm{curl}_x E = 0, \quad \mathrm{div}_x E = \frac{\rho(t,x)}{\varepsilon_0}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3,$$

which is much better understood, see [1, 17, 14] for instance. The Vlasov-Poisson model can be justified as the limit of the relativistic Vlasov-Maxwell model when the characteristic speed of the particles remains small compared to the light speed [8, 3].

In this work, we elaborate on some particular type of solutions of the one-dimensional version of the Vlasov-Maxwell(Poisson) system which has received the attention in the plasma physics community [2]. Let us assume that the initial density is proportional to the characteristic function of some region of the phase space between the graphs of two functions $p_0^\pm : \mathbb{R} \to \mathbb{R}$

$$f_0(x,p) = \alpha \, \mathbf{1}_{\{p_0^-(x) < p < p_0^+(x)\}}, \quad (x,p) \in \mathbb{R}^2. \tag{5}$$

We assume that $p_0^- \le p_0^+$. We are looking for a density function of the form

$$f(t,x,p) = \alpha \, \mathbf{1}_{\{p^-(t,x) < p < p^+(t,x)\}}, \tag{6}$$

where $p^\pm : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are unknown functions to be determined such that the above density $f$ satisfies the Vlasov equation. We have the following immediate result:

**Proposition 1.1 (Smooth Water-bag Solutions)** *Let $E : [0,T[\times\mathbb{R} \to \mathbb{R}$ be a given electric field which belongs to $L^1_{\mathrm{loc}}([0,T[\times\mathbb{R})$, with $0 < T \le +\infty$. Assume that $p^\pm : [0,T[\times\mathbb{R} \to \mathbb{R}$ are smooth functions $p^\pm \in W^{1,\infty}_{\mathrm{loc}}([0,T[\times\mathbb{R})$ verifying*

$$\partial_t p^\pm + v(p^\pm)\partial_x p^\pm = qE(t,x), \quad (t,x) \in ]0,T[\times\mathbb{R},$$

*and $p^- \le p^+$. Then the density $f$ given by (6) is a weak solution (that is, a solution in distribution sense) of the Vlasov equation associated to the electric field $E$.*

Observe that the charge and current densities of the distribution in (6) are given by $\rho(t,x) = q\alpha\ (p^+(t,x) - p^-(t,x))$, $\ j(t,x) = q\alpha(\mathcal{E}(p^+(t,x)) - \mathcal{E}(p^-(t,x)))$, where the kinetic energy function is given by

$$\mathcal{E}(p) = mc_0^2 \left( \left(1 + \frac{p^2}{m^2 c_0^2}\right)^{\frac{1}{2}} - 1 \right).$$

Notice that we have $\mathcal{E}'(p) = v(p)$. Thus for the initial condition in (5) the one dimensional Vlasov-Maxwell equations reduce to the system

$$\partial_t p^\pm + \partial_x \mathcal{E}(p^\pm) = qE(t,x), \quad (t,x) \in ]0,T[\times\mathbb{R}, \tag{7}$$

$$\partial_t E = -\alpha \frac{q}{\varepsilon_0}(\mathcal{E}(p^+(t,x)) - \mathcal{E}(p^-(t,x))), \tag{8}$$

$$\partial_x E = \alpha \frac{q}{\varepsilon_0}(p^+(t,x) - p^-(t,x)), \quad (t,x) \in ]0,T[\times\mathbb{R}, \tag{9}$$

with the initial conditions

$$p^\pm(0,x) = p_0^\pm(x), \quad E(0,x) = E_0(x), \quad x \in \mathbb{R}, \tag{10}$$

satisfying

$$E_0'(x) = \alpha \frac{q}{\varepsilon_0}(p_0^+(x) - p_0^-(x)), \quad p_0^-(x) \le p_0^+(x), \quad x \in \mathbb{R}. \tag{11}$$

Let us remark that (9) is a consequence of (7), (8) and the equality in (11). The problem (7), (8), (9), (10) is called the water-bag model and has been introduced in [2]. The main goal of this paper is to establish existence and uniqueness results for the water-bag model. In Section 2 we analyze the weak solutions: we study entropy solutions of the scalar conservation laws (7) coupled to the equations (8), (9) for the electric field. Smooth solutions are constructed as well for certain class of initial conditions in Section 3.

# 2 Weak solutions

For simplicity we assume that all the physical constants $q, m, \varepsilon_0, c_0, \alpha$ are equal to the unity. We remind the reader the standard existence and uniqueness results concerning the entropy solution for scalar conservation laws. We refer to [12, 10] for details on this topic. We consider here conservation laws with right hand side terms

$$\partial_t u + \partial_x F(u) = G(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{12}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}. \tag{13}$$

**Theorem 2.1 (Entropy Solutions for Scalar Conservation Laws)** *Let us assume that $F : \mathbb{R} \to \mathbb{R}$ is a smooth function and $G$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}))$. Then for any initial condition $u_0 \in L^\infty(\mathbb{R})$ there is a unique entropy solution $u \in C(\mathbb{R}_+; L^1_{\mathrm{loc}}(\mathbb{R})) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}))$ for* (12), (13) *satisfying*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})} + \int_0^t \|G(s)\|_{L^\infty(\mathbb{R})}\, ds, \quad t \in \mathbb{R}_+. \tag{14}$$

*Moreover if $v$ is the entropy solution associated to the initial condition $v_0 \in L^\infty(\mathbb{R})$, the source term $H \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}))$ and the same smooth function $F$ then we have the inequality*

$$\begin{aligned} \int_{\mathbb{R}} |u(t,x) - v(t,x)| \mathbf{1}_{\{|x|<R\}}\, dx \quad &\le \quad \int_{\mathbb{R}} |u_0(x) - v_0(x)| \mathbf{1}_{\{|x|<R+tM(t)\}}\, dx \\ &+ \quad \int_0^t \int_{\mathbb{R}} |G(s,x) - H(s,x)| \mathbf{1}_{\{|x|<R+(t-s)M(t)\}}\, dx\, ds, \end{aligned} \tag{15}$$

*where $M(t) = \max\{M_u(t), M_v(t)\}$,*

$$M_u(t) = \sup\{|F'(\xi)| \ : \ |\xi| \le \|u_0\|_{L^\infty(\mathbb{R})} + \int_0^t \|G(s)\|_{L^\infty(\mathbb{R})}\, ds\}$$

*and*

$$M_v(t) = \sup\{|F'(\xi)| \ : \ |\xi| \le \|v_0\|_{L^\infty(\mathbb{R})} + \int_0^t \|H(s)\|_{L^\infty(\mathbb{R})}\, ds\}.$$

It is well known that for conservation laws without source term ($G = 0$) the solution operator $S(t)u_0 = u(t,\cdot)$ is order preserving on $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ that is, for any $u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $u_0 \le v_0$ a.e. we have $S(t)u_0 \le S(t)v_0$ a.e., for any $t \in \mathbb{R}_+$. This is a direct consequence of the Crandall-Tartar lemma [12, page 81]. The same result holds true for conservation laws with source terms $G \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}))$ and for initial conditions $u_0 \in L^\infty(\mathbb{R})$.

**Lemma 2.1 (Comparison Principle with Sources)** *Assume that the source $G \in L^1_{\rm loc}(\mathbb{R}_+; L^\infty(\mathbb{R}))$ and denote by $S_G(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ the solution operator given by $S_G(t)u_0 = u(t, \cdot)$ for any $u_0 \in L^\infty(\mathbb{R})$, $t \in \mathbb{R}_+$ where $u$ is the entropy solution of* (12), (13). *For any $t \in \mathbb{R}_+$ the operator $S_G(t)$ is order preserving.*

**Proof.** Since the solutions of (12) with bounded initial conditions propagate with finite speed (cf. (14), (15)), it is sufficient to prove the result for $G \in L^1_{\rm loc}(\mathbb{R}_+; L^1(\mathbb{R})) \cap L^1_{\rm loc}(\mathbb{R}_+; L^\infty(\mathbb{R}))$ and initial conditions in $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Therefore consider $u_0, v_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $u_0 \le v_0$. We claim that

$$\int_{\mathbb{R}} \{S_G(t)u_0 - S_G(t)v_0\}\, dx = \int_{\mathbb{R}} \{u_0 - v_0\}\, dx, \quad t \in \mathbb{R}_+. \tag{16}$$

Indeed, by (15) it is sufficient to prove it for compactly supported functions $u_0, v_0$ and this comes easily by interpreting $S_G(t)u_0, S_G(t)v_0$ as the limit of smooth solutions for approximating viscous problems, as the viscosity vanishes. We denote by $(\cdot)_+$ the positive part function. Combining (15), (16) yields

$$\begin{aligned} 2\int_{\mathbb{R}} (S_G(t)u_0 - S_G(t)v_0)_+\, dx &= \int_{\mathbb{R}} (S_G(t)u_0 - S_G(t)v_0)\, dx + \int_{\mathbb{R}} |S_G(t)u_0 - S_G(t)v_0|\, dx \\ &\le \int_{\mathbb{R}} (u_0 - v_0)\, dx + \int_{\mathbb{R}} |u_0 - v_0|\, dx \\ &= 2\int_{\mathbb{R}} (u_0 - v_0)_+\, dx = 0, \end{aligned}$$

implying that $S_G(t)u_0 \le S_G(t)v_0$ a.e. $x \in \mathbb{R}$, $\forall\, t \in \mathbb{R}_+$. $\square$

Consider $p_0^\pm \in L^\infty(\mathbb{R}), E_0 \in L^\infty(\mathbb{R})$ satisfying $p_0^- \le p_0^+$ and $E_0' = p_0^+ - p_0^-$. We define the application $\mathcal{F}$ on $L^1_{\rm loc}(\mathbb{R}_+; L^\infty(\mathbb{R}))$ given by $\mathcal{F}E = \tilde{E}$ where

$$\tilde{E}(t, x) = E_0(x) - \int_0^t \{\mathcal{E}(p^+(s, x)) - \mathcal{E}(p^-(s, x))\}\, ds,$$

and $p^\pm$ are the entropy solutions of

$$\partial_t p^\pm + \partial_x \mathcal{E}(p^\pm) = E(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

with the initial conditions $p_0^\pm$. It is easily seen by (14) that

$$\begin{aligned} \|\tilde{E}(t)\|_{L^\infty(\mathbb{R})} &\le \|E_0\|_{L^\infty(\mathbb{R})} + \int_0^t \{\|p^+(s)\|_{L^\infty(\mathbb{R})} + \|p^-(s)\|_{L^\infty(\mathbb{R})}\}\, ds \\ &\le \|E_0\|_{L^\infty(\mathbb{R})} + t(\|p_0^+\|_{L^\infty(\mathbb{R})} + \|p_0^-\|_{L^\infty(\mathbb{R})}) + 2t\int_0^t \|E(s)\|_{L^\infty(\mathbb{R})}\, ds. \end{aligned} \tag{17}$$

For any $t \in \mathbb{R}_+$ we denote by $e_T : [0,T] \to \mathbb{R}$ the function given by

$$e_T(t) = \left(\|E_0\|_{L^\infty(\mathbb{R})} + T(\|p_0^+\|_{L^\infty(\mathbb{R})} + \|p_0^-\|_{L^\infty(\mathbb{R})})\right) e^{2Tt}.$$

We check immediately that the set $\mathcal{D}_T = \{E \in L^1(]0,T[; L^\infty(\mathbb{R})) \ : \ \|E(t)\|_{L^\infty(\mathbb{R})} \leq e_T(t), \ \forall \ t \in [0,T]\}$ is left invariant by the application $\mathcal{F}_T$ defined by $\mathcal{F}_T E = \mathcal{F}E|_{[0,T]\times\mathbb{R}}$ for any $E \in L^1_{\text{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}))$.

A straightforward computation based on the contraction inequality (15) shows that $\mathcal{F}_T$ is continuous on $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$. We denote by $M_T$ the constant given by

$$M_T = \sup\{|\mathcal{E}'(\xi)| \ : \ |\xi| \leq \max\{\|p_0^-\|_{L^\infty(\mathbb{R})}, \|p_0^+\|_{L^\infty(\mathbb{R})}\} + \int_0^T e_T(t)\, dt\} < 1.$$

**Proposition 2.1 (Continuity of the Map)** *Assume that $p_0^\pm, E_0 \in L^\infty(\mathbb{R})$. For any $T \in \mathbb{R}_+$ we have the inequality*

$$\int_{\mathbb{R}} |\mathcal{F}_T E_1 - \mathcal{F}_T E_2|(t,x)\mathbf{1}_{\{|x|<R\}}\, dx \leq 2T \int_0^t \int_{\mathbb{R}} |E_1 - E_2|(s,x)\mathbf{1}_{\{|x|<R+(t-s)M_T\}}\, dx\, ds,$$

*for any $E_1, E_2 \in \mathcal{D}_T$, $\forall\, t \in [0,T]$, $R > 0$.*

**Proof.** Consider $E_1, E_2 \in \mathcal{D}_T$ and let us denote by $p_1^\pm$, $p_2^\pm$ the entropy solutions corresponding to the fields $E_1, E_2$ and the initial conditions $p_0^\pm$. By the definitions of $\mathcal{F}_T E_1$, $\mathcal{F}_T E_2$ and (15) we deduce easily that

$$\begin{aligned}
\int_{-R}^{R} |(\mathcal{F}_T E_1 - \mathcal{F}_T E_2)|(t,x)\, dx &\leq \int_0^t \int_{-R}^{R} \{|p_1^+ - p_2^+|(s,x) + |p_1^- - p_2^-|(s,x)\}\, ds \\
&\leq 2\int_0^t \int_0^s \int_{\mathbb{R}} |E_1 - E_2|(\tau,x)\mathbf{1}_{\{|x|<R+(s-\tau)M_T\}}\, dx d\tau ds \\
&\leq 2T \int_0^t \int_{\mathbb{R}} |E_1 - E_2|(s,x)\mathbf{1}_{\{|x|<R+(t-s)M_T\}}\, dx\, ds.
\end{aligned}$$

$\square$

**Theorem 2.2 (Global Entropy Solutions for the Water-bag model)** *Assume that $p_0^\pm, E_0 \in L^\infty(\mathbb{R})$ satisfying $E_0' = p_0^+ - p_0^-$. Then there is a global unique weak solution $(p^+, p^-, E) \in L^\infty(]0,T[\times\mathbb{R})^2 \times W^{1,\infty}(]0,T[\times\mathbb{R})$, $\forall\, T \in \mathbb{R}_+$ for the water-bag model (7), (8), (9), (10). Moreover if $p_0^- \leq p_0^+$ then $p^- \leq p^+$.*

**Proof.** It is sufficient to prove the existence of a unique solution $(p^+, p^-, E)$ on $[0,T] \times \mathbb{R}$ for any $T \in \mathbb{R}_+$. We define the sequence $(E^n)_{n\geq 0}$ given by $E^0(t,x) = E_0(x), \forall\ (t,x) \in [0,T] \times \mathbb{R}$ and $E^{n+1} = \mathcal{F}_T E^n,\ \forall\ n \in \mathbb{N}$. Observe that $(E^n)_n \subset \mathcal{D}_T$. For any $R > 0$ we consider the sequence of functions $z_R^n(t) = \int_{\mathbb{R}} |E^{n+1} - E^n|(t,x) \mathbf{1}_{\{|x|<R+(T-t)M_T\}}\ dx$, $t \in [0,T]$, $n \in \mathbb{N}$. By Proposition 2.1 it is easily seen that

$$\begin{aligned} z_R^n(t) &\leq 2T \int_0^t \int_{\mathbb{R}} |E^n - E^{n-1}|(s,x) \mathbf{1}_{\{|x|<R+(T-s)M_T\}}\ dx\ ds \\ &= 2T \int_0^t z_R^{n-1}(s)\ ds,\ \ t \in [0,T],\ \ n \geq 1, \end{aligned}$$

implying that

$$z_R^n(t) \leq \frac{(2Tt)^n}{n!} \|z_R^0\|_{L^\infty(]0,T[)},\ \ \forall n \in \mathbb{N}.$$

We deduce that $(E^n)_n$ is a Cauchy sequence in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ since

$$\int_{\mathbb{R}} |E^{n+p} - E^n|(t,x) \mathbf{1}_{\{|x|<R\}}\ dx \leq z^n(t) + z^{n+1}(t) + ... + z^{n+p-1}(t).$$

It follows that $(E^n)_n$ converges in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ towards a fixed point $E$ of $\mathcal{F}_T$. Moreover we check easily that $E \in \mathcal{D}_T$. Take now $p^+, p^-$ the unique entropy solutions of (7) corresponding to the limit field $E$ and the initial conditions $p_0^+, p_0^-$. By construction $(p^+, p^-, E)$ is a solution for the water-bag model (7), (8), (10). The equation (9) is a consequence of (7), (8) and the constraint $E_0' = p_0^+ - p_0^-$. The bounds for the derivatives of $E$ comes from the bounds of $p^\pm$ (see (14)) and (8), (9). For the inequality $p^- \leq p^+$ use Lemma 2.1. The uniqueness of the weak solution is obtained by a straightforward computation involving the Gronwall lemma. $\square$

**Remark 2.1 (Vlasov-Maxwell Solutions with Defect Measures)** *A natural question related to the previous existence result is the following: given $(p^+, p^-, E)$ a weak solution for the water-bag model, is it true that $f(t,x,p) = \mathbf{1}_{\{p^-(t,x)<p<p^+(t,x)\}}$ solves the Vlasov equation*

$$\partial_t f + v(p)\partial_x f + E(t,x)\partial_p f = 0,\ \ (t,x,p) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}\ \ ?$$

*Generally the answer to this question is negative, but we can prove that $f$ solves a Vlasov equation with a entropy defect measure. Of course we appeal here to the kinetic formulation of conservation laws [15, 16]. Indeed, observe that the function $f$ represents as $f(t,x,p) = \chi(p,p^+(t,x)) - \chi(p,p^-(t,x))$ where the function $\chi$ is given by*

$$\chi(\xi,u) = \begin{cases} +1, & 0 < \xi < u, \\ -1, & u < \xi < 0, \\ 0, & \textit{otherwise.} \end{cases}$$

*Since $p^{\pm}$ are entropy solutions we know that there are the non negative kinetic entropy defect measures $m^{\pm}$ such that*

$$\begin{cases} \partial_t \chi(p,p^{\pm}) + v(p)\partial_x \chi(p,p^{\pm}) - E(t,x)\delta_0(p-p^{\pm}) = \partial_p m^{\pm}, & (t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \chi(p,p^{\pm}(0,x)) = \chi(p,p_0^{\pm}(x)), & (x,p) \in \mathbb{R}^2, \end{cases}$$

*where the notation $\delta_0$ stands for the Dirac mass concentrated at the origin. Therefore we obtain*

$$(\partial_t + v(p)\partial_x)\{\chi(p,p^+) - \chi(p,p^-)\} - E(t,x)\{\delta_0(p-p^+) - \delta_0(p-p^-)\} = \partial_p\{m^+ - m^-\},$$

*and by taking into account that $\partial_p\{\chi(p,p^+) - \chi(p,p^-)\} = -\{\delta_0(p-p^+) - \delta_0(p-p^-)\}$ finally we can write*

$$\begin{cases} \partial_t f + v(p)\partial_x f + E(t,x)\partial_p f = \partial_p\{m^+ - m^-\}, & (t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ f(0,x,p) = \chi(p,p_0^+(x)) - \chi(p,p_0^-(x)) = \mathbf{1}_{\{p_0^-(x) < p < p_0^+(x)\}}, & (x,p) \in \mathbb{R}^2. \end{cases}$$

*Another interesting question concerns the behavior of the total energy. For example if $p^-(t,x) \le 0 \le p^+(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$ and the initial energy is finite we can prove that the total energy is not increasing on $[0,T]$. Multiplying the above Vlasov equation by $\mathcal{E}(p)$ one gets after integration*

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} \mathcal{E}(p) f \, dxdp - \int_{\mathbb{R}} E\{\mathcal{E}(p^+(t,x)) - \mathcal{E}(p^-(t,x))\} \, dx &= \int_{\mathbb{R}^2} v(p) m^-(t,x,p) \, dxdp \\ &- \int_{\mathbb{R}^2} v(p) m^+(t,x,p) \, dxdp. \end{aligned}$$

*Using* (8) *we deduce also that*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} E(t,x)^2 \; dx + \int_{\mathbb{R}} E(t,x)\{\mathcal{E}(p^+(t,x)) - \mathcal{E}(p^-(t,x))\} \; dx = 0,$$

*implying that*

$$\begin{aligned}\frac{d}{dt}\left\{\int_{\mathbb{R}^2} \mathcal{E}(p) f \; dxdp + \frac{1}{2}\int_{\mathbb{R}} E(t,x)^2 \; dx\right\} &= \int_{\mathbb{R}^2} v(p) m^-(t,x,p) \; dxdp \\ &- \int_{\mathbb{R}^2} v(p) m^+(t,x,p) \; dxdp.\end{aligned}$$

*Therefore we are done if we check that* $m^{\pm} = 0$ *on* $[0,T] \times \mathbb{R} \times \mathbb{R}^{\mp}$. *Take* $p_0 < 0$ *and let us multiply the kinetic formulation of* $\chi(p, p^+)$ *by the derivative of the convex function* $S_{p_0}(p) = (p - p_0)_-$. *After standard computations involving the usual formula* $\int_{\mathbb{R}} \chi(\xi, u) S'(\xi) \; d\xi = S(u) - S(0)$, $\forall \; S(\cdot)$, $\forall \; u \in \mathbb{R}$ *we obtain*

$$\frac{d}{dt}\int_{\mathbb{R}} S_{p_0}(p^+(t,x)) \; dx = -\int_{\mathbb{R}} m^+(t,x,p_0) \; dx.$$

*Therefore one gets for any* $t \in [0,T]$

$$\int_{\mathbb{R}} S_{p_0}(p^+(T,x)) \; dx + \int_0^T\!\!\int_{\mathbb{R}} m^+(t,x,p_0) \; dxdt = \int_{\mathbb{R}} S_{p_0}(p_0^+(x)) \; dx = 0,$$

*saying that* $m^+ = 0$ *on* $[0,T] \times \mathbb{R} \times \mathbb{R}^-$. *In a similar way we check that* $m^- = 0$ *on* $[0,T] \times \mathbb{R} \times \mathbb{R}^+$.

**Remark 2.2 (Non-relativistic setting)** *This case is a little bit more difficult since the non relativistic energy function* $\mathcal{E}(p) = \frac{p^2}{2}$ *is only locally Lipschitz. This time the analogous of the estimate* (17) *becomes quadratic*

$$\begin{aligned}\|\tilde{E}(t)\|_{L^\infty} &\leq \|E_0\|_{L^\infty} + \frac{1}{2}\int_0^t \max\{\|p^+(s)\|^2_{L^\infty}, \|p^-(s)\|^2_{L^\infty}\} \; ds \\ &\leq \|E_0\|_{L^\infty} + \int_0^t \left\{\max\{\|p_0^+\|^2_{L^\infty}, \|p_0^-\|^2_{L^\infty}\} + s\int_0^s \|E(\tau)\|^2_{L^\infty} \; d\tau\right\} \; ds \\ &\leq \|E_0\|_{L^\infty} + t\max\{\|p_0^+\|^2_{L^\infty}, \|p_0^-\|^2_{L^\infty}\} + t^2\int_0^t \|E(s)\|^2_{L^\infty} \; ds.\end{aligned}$$

*In this case for* $T > 0$ *small enough we denote by* $e_T(\cdot)$ *the unique solution of*

$$\frac{d}{dt}e_T = T^2(e_T(t))^2, \;\; 0 < t < T,$$

*with the initial condition* $e_T(0) = \|E_0\|_{L^\infty} + T \max\{\|p_0^+\|^2_{L^\infty}, \|p_0^-\|^2_{L^\infty}\}$. *It is easily seen that the set* $\mathcal{D}_T = \{E \in L^1(]0,T[;L^\infty(\mathbb{R})) \ : \ \|E(t)\|_{L^\infty} \le e_T(t), \ \forall\ t \in [0,T]\}$ *is left invariant by the application* $\mathcal{F}_T$ *and following the same arguments as in the relativistic setting we construct a local unique weak solution* $(p^+, p^-, E) \in L^\infty(]0,T[\times\mathbb{R})^2 \times W^{1,\infty}(]0,T[\times\mathbb{R})$ *for the non relativistic water-bag model. For results on the multi-water-bag model in this setting see [6].*

# 3 Strong solutions

This section is devoted to the analysis of smooth solutions for the relativistic water-bag model. We show that smooth non decreasing initial conditions launch global smooth solutions.

**Proposition 3.1 (Non-decreasing initial data for Scalar CL's)** *Assume that* $F \in W^{2,\infty}(\mathbb{R})$, $G \in L^\infty_{\text{loc}}(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}))$ *such that* $F'' \ge 0$, $\partial_x G \ge 0$. *Then for any non decreasing initial condition* $u_0 \in W^{1,\infty}(\mathbb{R})$ *the problem* (12), (13) *has a unique strong solution* $u \in W^{1,\infty}(]0,T[\times\mathbb{R})$, $\forall\, T \in \mathbb{R}_+$ *which is non decreasing with respect to* $x$.

**Proof.** We define the sequence of functions $(u^n)_{n\ge 0}$ where $u^0(t,x) = u_0(x)\ \forall\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and for any $n \in \mathbb{N}$, $u^{n+1}$ solves the problem

$$\partial_t u^{n+1} + F'(u^n(t,x))\partial_x u^{n+1} = G(t,x), \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{18}$$

$$u^{n+1}(0,x) = u_0(x), \ \ x \in \mathbb{R}. \tag{19}$$

Actually we will prove that $(u^n)_n$ are smooth and therefore the above problem is understood in the classical sense. Assume that $u^n$ belongs to $L^\infty_{\text{loc}}(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}))$, $\partial_x u^n \ge 0$ which is true for $n = 0$, and let us show that the same holds for $u^{n+1}$. We denote by $X^n(s;t,x)$ the characteristics associated to $F'(u^n)$

$$\frac{d}{ds}X^n(s;t,x) = F'(u^n(s, X^n(s;t,x))), \ \ X^n(t;t,x) = x.$$

Therefore we have

$$u^{n+1}(t,x) = u_0(X^n(0;t,x)) + \int_0^t G(s, X^n(s;t,x))\, ds, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{20}$$

We check easily that $u^{n+1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}))$ and since $x \to X^n(s;t,x)$, $u_0$ and $G(s,\cdot)$ are non decreasing we deduce that $\partial_x u^{n+1} \geq 0$. Moreover we can find bounds for the time and space derivatives uniformly with respect to $n$. For any $h > 0$ we have

$$\begin{aligned}
\partial_t\{u^{n+1}(t,x+h) - u^{n+1}(t,x)\} &+ \{F'(u^n(t,x+h)) - F'(u^n(t,x))\}\partial_x u^{n+1}(t,x+h) \\
&+ \quad F'(u^n(t,x))\partial_x\{u^{n+1}(t,x+h) - u^{n+1}(t,x)\} \\
&= \quad G(t,x+h) - G(t,x).
\end{aligned}$$

Since $\partial_x u^n \geq 0$, $\partial_x u^{n+1} \geq 0$, $F'' \geq 0$ we have

$$\{F'(u^n(t,x+h)) - F'(u^n(t,x))\}\partial_x u^{n+1}(t,x+h) \geq 0,$$

and therefore

$$\partial_t D_h u^{n+1} + F'(u^n(t,x))\partial_x D_h u^{n+1} \leq D_h G(t,x),$$

where the notation $D_h z(x)$ stands for $z(x+h) - z(x)$ for any function $z$. Integrating along the characteristics one gets

$$D_h u^{n+1}(t,x) \leq (D_h u_0)(X^n(0;t,x)) + \int_0^t D_h G(s, X^n(s;t,x))\, ds,$$

implying that

$$\frac{D_h u^{n+1}(t,x)}{h} \leq \|u_0'\|_{L^\infty(\mathbb{R})} + \int_0^t \|\partial_x G(s)\|_{L^\infty(\mathbb{R})}\, ds.$$

Since we know that $\partial_x u^{n+1} \geq 0$ finally we obtain

$$\|\partial_x u^{n+1}(t)\|_{L^\infty(\mathbb{R})} \leq \|u_0'\|_{L^\infty(\mathbb{R})} + \int_0^t \|\partial_x G(s)\|_{L^\infty(\mathbb{R})}\, ds,$$

and

$$\|\partial_t u^{n+1}(t)\|_{L^\infty(\mathbb{R})} \leq \|G(t)\|_{L^\infty(\mathbb{R})} + \|F'\|_{L^\infty(\mathbb{R})} \left( \|u_0'\|_{L^\infty(\mathbb{R})} + \int_0^t \|\partial_x G(s)\|_{L^\infty(\mathbb{R})}\, ds \right).$$

We claim that the sequence $(u^n)_n$ converges in $C([0,T]\times\mathbb{R})$, $\forall\ T\in\mathbb{R}_+$. Indeed, since $(\partial_x u^n)_n$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$, there is a constant $C_T$ depending on $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\int_0^T \|\partial_x G(s)\|_{L^\infty(\mathbb{R})}ds$ and $\|F''\|_{L^\infty}$ such that

$$|X^{n+1}(s;t,x)-X^n(s;t,x)|\le C_T\int_s^t \|u^{n+1}(\tau)-u^n(\tau)\|_{L^\infty}\,d\tau, \tag{21}$$

for any $(s,t,x)\in[0,T]^2\times\mathbb{R}$. Combining (20), (21) yields

$$\|u^{n+2}(t)-u^{n+1}(t)\|_{L^\infty(\mathbb{R})}\le \tilde{C}_T\int_0^t \|u^{n+1}(s)-u^n(s)\|_{L^\infty(\mathbb{R})}\,ds,\ \ n\in\mathbb{N},$$

implying that the sequence $(u^n)_n$ converges in $C([0,T]\times\mathbb{R})$ towards some function $u$. Since $(\partial_t u^n)_n$, $(\partial_x u^n)_n$ are bounded we deduce that $u\in W^{1,\infty}(]0,T[\times\mathbb{R})$. It remains to prove that $u$ solves (18), (19). There is a subsequence $(n_k)_k$, $\lim_{k\to+\infty} n_k=+\infty$ such that

$$\lim_{k\to+\infty}(\partial_t u^{n_k},\partial_x u^{n_k})=(\partial_t u,\partial_x u),\ \ \text{weakly}\ \star\ \text{in}\ L^\infty(]0,T[\times\mathbb{R})^2.$$

Obviously we have also the convergence $\lim_{k\to+\infty} u^{n_k-1}=u$ in $C([0,T]\times\mathbb{R})$. Multiplying (18) by a test function $\varphi\in C_c^0([0,T]\times\mathbb{R})$ one gets

$$\int_0^T\!\!\int_{\mathbb{R}}\partial_t u^{n_k}\varphi\,dxdt+\int_0^T\!\!\int_{\mathbb{R}}F'(u^{n_k-1}(t,x))\partial_x u^{n_k}\varphi\,dxdt=\int_0^T\!\!\int_{\mathbb{R}}G(t,x)\varphi(t,x)\,dxdt.$$

We can pass easily to the limit for $k\to+\infty$ and we obtain

$$\int_0^T\!\!\int_{\mathbb{R}}\partial_t u\ \varphi\,dxdt+\int_0^T\!\!\int_{\mathbb{R}}F'(u(t,x))\partial_x u\ \varphi\,dxdt=\int_0^T\!\!\int_{\mathbb{R}}G(t,x)\varphi(t,x)\,dxdt,$$

saying that $u$ is a strong solution of (18). Moreover $u$ verifies the initial condition (19) since

$$u(0,x)=\lim_{n\to+\infty}u^n(0,x)=u_0(x),\ \ x\in\mathbb{R}.$$

Since any strong solution coincides with the entropy solution, we have also the uniqueness of the strong solution. $\square$

**Theorem 3.1 (Global Smooth Solutions)** *Assume that $p_0^\pm, E_0\in W^{1,\infty}(\mathbb{R})$ satisfying $\frac{d}{dx}p_0^\pm\ge 0$, $\frac{d}{dx}E_0=p_0^+-p_0^-\ge 0$. Then there is a global unique strong solution $(p^+,p^-,E)\in W^{1,\infty}(]0,T[\times\mathbb{R})^2\times W^{2,\infty}(]0,T[\times\mathbb{R})$, $\forall\ T\in\mathbb{R}_+$ for the water-bag model.*

**Proof.** By Theorem 2.2 we know that there is a global weak solution $(p^+, p^-, E) \in L^\infty(]0,T[\times\mathbb{R})^2 \times W^{1,\infty}(]0,T[\times\mathbb{R})$, $\forall\ T \in \mathbb{R}_+$ for the water-bag model satisfying $p^- \le p^+$. By (9) we have $\partial_x E \ge 0$ and thus applying Proposition 3.1 implies that the entropy solutions $p^\pm$ belong to $W^{1,\infty}(]0,T[\times\mathbb{R})$ and are strong solutions for (7). The bounds for the second order derivatives of the electric fields follow immediately from the bounds of the first order derivatives for $p^\pm$ and (8), (9). The uniqueness of the strong solution $(p^+, p^-, E)$ for the water-bag model is a direct consequence of the uniqueness of the weak solution. $\square$

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