The effective Vlasov-Poisson system for strongly magnetized plasmas

Le système de Vlasov-Poisson effectif pour les plasmas fortement magnétisés

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Résumé

Nous étudions le régime du rayon de Larmor fini pour le système de Vlasov-Poisson. Le champ magnétique est supposé uniforme. Nous restreignons l'étude de ce problème non linéaire au cas bi-dimensionnel. Nous obtenons le modèle limite en appliquant les méthodes de gyro-moyenne cf. [1,2]. Nous donnons l'expression explicite du champ d'advection effectif de l'équation de Vlasov, dans laquelle nous avons substitué le champ électrique auto-consistant, via la résolution de l'équation de Poisson moyennée à l'échelle cyclotronique. Nous mettons en évidence la structure hamiltonienne du modèle limite et présentons ses propriétés : conservations de la masse, de l'énergie cinétique, de l'énergie électrique, etc.

Abstract

We study the finite Larmor radius regime for the Vlasov-Poisson system. The magnetic field is assumed to be uniform. We investigate this non linear problem in the two dimensional setting. We derive the limit model by appealing to gyro-average methods cf. [1,2]. We indicate the explicit expression of the effective advection field, entering the Vlasov equation, after substituting the self-consistent electric field, obtained by the resolution of the averaged (with respect to the cyclotronic time scale) Poisson equation. We emphasize the Hamiltonian structure of the limit model and present its properties : conservationss of the mass, kinetic energy, electric energy, etc.

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Motivated by the magnetic confinement fusion, which is one of the main application in plasma physics, we analyse the dynamics of a population of charged particles, under the action of a strong uniform magnetic field. The goal of this note is to study the finite Larmor radius regime, that is, we assume that the particle distribution fluctuates at the Larmor radius scale along the orthogonal directions, with respect to the magnetic field [5,6,7]. To simplify, we consider the two dimensional setting, *i.e.*, $x = (x_1, x_2), v = (v_1, v_2)$, with a magnetic field orthogonal to x_1Ox_2

$$\partial_t f + v \cdot \nabla_x f + \frac{qB}{m} {}^\perp v \cdot \nabla_v f - \frac{q}{m} \nabla_x \phi \cdot \nabla_v f = 0, \ \ (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

$$-\varepsilon_0 \Delta_x \phi = q \int_{\mathbb{R}^2} f(t, x, v) \, \mathrm{d}v \ \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2.$$

Here m is the particle mass, q is the particle charge and ε_0 is the electric permittivity of the vacuum. We study the finite Larmor radius regime (FLR in short), which consists in choosing as reference length, not that of the overall system, but the average Larmor circle length $l = 2\pi \frac{mV_{\mathrm{th}}}{qB}$, where V_{th} is the thermal velocity. We pick the following dimensionless variables

$$t = Tt', \ \ x = lx', \ \ v = V_{\mathrm{th}} v'$$

and introduce the dimensionless unknowns

$$f(t, x, v) = \frac{n}{V_{\mathrm{th}}^2} f'(t', x', v'), \ \ \phi = \phi_0 \phi', \ \ \text{where} \ \ \phi_0 = \frac{mlV_{\mathrm{th}}}{qT}$$

where n is the average concentration of charged particles. The ratio between the cyclotronic period and the reference time appears naturally as a small parameter, when the magnetic field is strong. Its value represents also the ratio between the Larmor circle length and the reference length

$$\varepsilon = \frac{T_c}{T} = \frac{2\pi m}{qBT} = \frac{l}{TV_{\mathrm{th}}}.$$

Notice that the choice of the reference potential corresponds to a small ratio between the potential and kinetic energies : $q\phi_0/(mV_{\mathrm{th}}^2) = \varepsilon$. The Poisson equation becomes

$$-\varepsilon \frac{\lambda_D^2}{l^2} \Delta_{x'} \phi' = \int_{\mathbb{R}^2} f'(t', x', v') \, \mathrm{d}v', \ \ (t', x') \in \mathbb{R}_+ \times \mathbb{R}^2$$

where $\lambda_D := \left(\frac{\varepsilon_0 m V_{\mathrm{th}}^2}{nq^2} \right)^{1/2}$ is the Debye length. We assume moreover that $\sqrt{\varepsilon}\lambda_D = l$ (which also writes $\varepsilon_0 \phi_0/(nq) = l^2$). However, notice that the Debye length defined above is not very relevant in our scaling. Actually the previous Poisson equation says that the typical length of the electric phenomena will be $\sqrt{\varepsilon}\lambda_D$ here. Accordingly, in the case where the Larmor circle length coincides with the typical length of the electric phenomena, the FLR regime leads to the following ε dependent system of equations, up to a multiplicative constant ω_c of order one (for simplicity we drop the primes)

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v \cdot \nabla_x f^\varepsilon + \omega_c \, {}^\perp v \cdot \nabla_v f^\varepsilon) - \nabla_x \phi^\varepsilon \cdot \nabla_v f^\varepsilon = 0, \ \ (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \tag{1}$$

$$-\Delta_x \phi^\varepsilon = \rho^\varepsilon := \int_{\mathbb{R}^2} f^\varepsilon(t, x, v) \, \mathrm{d}v, \ \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \tag{2}$$

$$f^\varepsilon(0, x, v) = f^{\mathrm{in}}(x, v), \ \ (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{3}$$

We introduce the notations $T_c = 2\pi/\omega_c, \omega_c^\varepsilon = \omega_c/\varepsilon$, $T_c^\varepsilon = \frac{2\pi}{\omega_c^\varepsilon} = \varepsilon \frac{2\pi}{\omega_c} = \varepsilon T_c$. For any $v = (v_1, v_2) \in \mathbb{R}^2$, we denote by ${}^\perp v$ the vector ${}^\perp v = (v_2, -v_1) \in \mathbb{R}^2$. We study the stability of the family $(f^\varepsilon, \phi^\varepsilon)_{\varepsilon>0}$, when ε becomes small. The asymptotic behavior follows by filtering out the fast oscillations of the caracteristic equations for (1). It is easily seen that the changes over one cyclotronic period of the quantities $\tilde{x} = x + \frac{{}^\perp v}{\omega_c}$, $\tilde{v} = \mathcal{R}(\omega_c t/\varepsilon)v$, are negligible. We expect that the family $\tilde{f}^\varepsilon(t, \tilde{x}, \tilde{v}) = f^\varepsilon(t, \tilde{x} - \mathcal{R}(-\omega_c t/\varepsilon)\, {}^\perp\tilde{v}/\omega_c, \mathcal{R}(-\omega_c t/\varepsilon)\tilde{v})$ converges, as ε becomes small, toward some profile $\tilde{f}(t, \tilde{x}, \tilde{v})$.

Théorème 0.1 *Let $f^{\rm in} = f^{\rm in}(x, v)$ be a non negative presence density satisfying*

H1 $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^{\rm in}(x, v)\, {\rm d}v{\rm d}x < +\infty$

H2 $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|v|^2}{2} f^{\rm in}(x, v)\, {\rm d}v{\rm d}x < +\infty$

H3 *there is a bounded, non increasing function $F^{\rm in} = F^{\rm in}(r) \in L^\infty \cap L^1(\mathbb{R}_+; r{\rm d}r)$, such that $f^{\rm in}(x, v) \leq F^{\rm in}(|v|), \ (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2$.*

We consider the family $(f^\varepsilon, \phi^\varepsilon)_{\varepsilon>0}$ of weak solutions for the Vlasov-Poisson system

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v \cdot \nabla_x f^\varepsilon + \omega_c\, {}^\perp v \cdot \nabla_v f^\varepsilon) - \nabla_x \phi^\varepsilon \cdot \nabla_v f^\varepsilon = 0, \ \ (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \tag{4}$$

$$-\Delta_x \phi^\varepsilon = \rho^\varepsilon(t, x) := \int_{\mathbb{R}^2} f^\varepsilon(t, x, v)\, {\rm d}v, \ \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \tag{5}$$

$$f^\varepsilon(0, x, v) = f^{\rm in}(x, v), \ \ (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2 \tag{6}$$

and we denote by $(\tilde{f}^\varepsilon)_{\varepsilon>0}$ the densities

$$\tilde{f}^\varepsilon(t, \tilde{x}, \tilde{v}) = f^\varepsilon\left(t, \tilde{x} - \frac{\mathcal{R}\left(-\frac{\omega_c t}{\varepsilon}\right)}{\omega_c}\, {}^\perp\tilde{v}, \mathcal{R}\left(-\frac{\omega_c t}{\varepsilon}\right)\tilde{v}\right), \ \ (t, \tilde{x}, \tilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2, \ \ \varepsilon > 0.$$

Therefore there is a sequence $(\varepsilon_k)_k$ converging to 0 such that $(\tilde{f}^{\varepsilon_k})_k$ converges strongly in $L^2([0, T]; L^2(\mathbb{R}^2 \times \mathbb{R}^2))$, for any $T \in \mathbb{R}_+$, toward a solution $\tilde{f}$ of the problem

$$\partial_t \tilde{f} + \mathcal{V}[\tilde{f}(t)](\tilde{x}, \tilde{v}) \cdot \nabla_{\tilde{x}} \tilde{f} + \mathcal{A}[\tilde{f}(t)](\tilde{x}, \tilde{v}) \cdot \nabla_{\tilde{v}} \tilde{f} = 0, \ \ (t, \tilde{x}, \tilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \tag{7}$$

with the initial condition

$$\tilde{f}(0, \tilde{x}, \tilde{v}) = f^{\rm in}\left(\tilde{x} - \frac{{}^\perp\tilde{v}}{\omega_c}, \tilde{v}\right), \ \ (\tilde{x}, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2 \tag{8}$$

where the velocity and acceleration vector fields $\mathcal{V}, \mathcal{A}$ are given by

$$\mathcal{V}[\tilde{f}(t)](\tilde{x}, \tilde{v}) = -\omega_c^{-1}\, {}^\perp\nabla_{\tilde{x}} \tilde{\phi}[\tilde{f}(t)], \quad \mathcal{A}[\tilde{f}(t)](\tilde{x}, \tilde{v}) = \omega_c\, {}^\perp\nabla_{\tilde{v}} \tilde{\phi}[\tilde{f}(t)] \tag{9}$$

$$\tilde{\phi}[\tilde{f}(t)] = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left\{\ln \frac{|\tilde{v} - \tilde{w}|}{|\omega_c|}\, \mathbf{1}_{\{|\tilde{x} - \tilde{y}| \leq \frac{|\tilde{v} - \tilde{w}|}{|\omega_c|}\}} + \ln |\tilde{x} - \tilde{y}|\, \mathbf{1}_{\{|\tilde{x} - \tilde{y}| > \frac{|\tilde{v} - \tilde{w}|}{|\omega_c|}\}}\right\} \tilde{f}(t, \tilde{y}, \tilde{w})\, {\rm d}\tilde{w}{\rm d}\tilde{y}. \tag{10}$$

That asymptotic regime has already been studied before. In [5,6,7] the authors appeal to the two scale convergence method. Nevertheless, the fast time variable persists in the limit model, and the computation of the velocity and acceleration vector fields of the limit Vlasov equation requires the resolution of a Poisson equation for every couple of slow/fast time variables, and some averaging procedure. In [1] the author obtained a convergence result towards a simpler model, which is valid only for well-prepared initial data. Our result applies to general initial data, and the limit model is a rather simple equation. It is a fully explicit non linear transport equation, whose characteristic system is Hamiltonian (with respect to the appropriate variables) and which can be studied in a much simpler way. Roughly speaking, the fast time variable appearing in the previous works is averaged in a fully explicit way.

1. Trajectoires effectives

Ce travail s'inscrit dans le cadre de la modélisation des plasmas de fusion. Nous concentrons notre étude au régime du rayon de Larmor fini pour le système de Vlasov-Poisson bi-dimensionnel décrit par (1), (2), (3). La méthode développée ici consiste à exprimer le potentiel électrique à l'aide de la solution fondamentale de l'opérateur de Laplace dans $\mathbb{R}^2$, puis insérer cette expression dans les trajectoires de l'équation de Vlasov. Nous obtenons alors, à l'aide des méthodes classiques de gyro-moyenne [1,2,3] les trajectoires limites et ainsi, les expressions effectives des champs vitesse et accélération de la nouvelle équation de Vlasov, décrivant le régime asymptotique considéré. Pour plus de détails sur les preuves de ces résultats, nous renvoyons à [4].

Notons e la solution fondamentale de l'opérateur de Laplace dans $\mathbb{R}^2$

$$e(z) = -\frac{1}{2\pi} \ln |z|, \quad z \in \mathbb{R}^2 \setminus \{0\}$$

c'est-à-dire $-\Delta e = \delta_0$ dans $\mathcal{D}'(\mathbb{R}^2)$. Le potentiel électrique, solution de l'équation de Poisson (2), s'écrit donc

$$\phi^\varepsilon(t,x) = \int_{\mathbb{R}^2} e(x-y)\rho^\varepsilon(t,y)\,\mathrm{d}y = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(x-y) f^\varepsilon(t,y,w)\,\mathrm{d}w\mathrm{d}y. \tag{11}$$

Les équations caractéristiques de l'équation de transport (1) sont données par

$$\frac{\mathrm{d}X^\varepsilon}{\mathrm{d}t} = \frac{V^\varepsilon(t)}{\varepsilon}, \quad \frac{\mathrm{d}V^\varepsilon}{\mathrm{d}t} = \omega_c \frac{{}^\perp V^\varepsilon(t)}{\varepsilon} - \nabla_x \phi^\varepsilon(t, X^\varepsilon(t)), \quad (X^\varepsilon(0), V^\varepsilon(0)) = (x,v).$$

Nous cherchons des quantités qui varient peu sur une période cyclotronique. Plus exactement, à tout instant fixé $t > 0$, on introduit le changement de coordonnées

$$\tilde{x} = x + \frac{{}^\perp v}{\omega_c}, \quad \tilde{v} = \mathcal{R}\left(\frac{\omega_c t}{\varepsilon}\right) v$$

où $\mathcal{R}(\theta)$ désigne la rotation de $\mathbb{R}^2$ d'angle θ. On vérifie aisément que le déterminant jacobien vaut 1 et alors ces transformations préservent la mesure de Lebesgue de $\mathbb{R}^4$ *i.e.,* $\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = \mathrm{d}v\mathrm{d}x$. En effet, $\tilde{x}$ est le centre du cercle de Larmor d'écrit par une particule passant par x avec la vitesse v. Ce centre ne varie pas à l'échelle du mouvement rapide cyclotronique, correspondant à la fréquence cyclotronique $\frac{\omega_c}{\varepsilon}$. Plus exactement on a

$$\frac{\mathrm{d}\tilde{X}^\varepsilon}{\mathrm{d}t} = -\frac{{}^\perp \nabla_x \phi^\varepsilon}{\omega_c}(t, X^\varepsilon(t)) = -\frac{{}^\perp \nabla_x \phi^\varepsilon}{\omega_c}\left(t, \tilde{X}^\varepsilon(t) - \frac{\mathcal{R}}{\omega_c}\left(-\frac{\omega_c t}{\varepsilon}\right) {}^\perp \tilde{V}^\varepsilon(t)\right) \tag{12}$$

$$\frac{\mathrm{d}\tilde{V}^\varepsilon}{\mathrm{d}t} = -\mathcal{R}\left(\frac{\omega_c t}{\varepsilon}\right) \nabla_x \phi^\varepsilon(t, X^\varepsilon(t)) = -\mathcal{R}\left(\frac{\omega_c t}{\varepsilon}\right) \nabla_x \phi^\varepsilon\left(t, \tilde{X}^\varepsilon(t) - \frac{\mathcal{R}}{\omega_c}\left(-\frac{\omega_c t}{\varepsilon}\right) {}^\perp \tilde{V}^\varepsilon(t)\right). \tag{13}$$

On souhaite remplacer le potentiel électrique par l'éxpression de (11). On introduit les densités de présence en les coordonnées $(\tilde{x}, \tilde{v})$

$$f^\varepsilon(t,x,v) = \tilde{f}^\varepsilon(t, \tilde{x}, \tilde{v}), \quad \tilde{x} = x + \frac{{}^\perp v}{\omega_c}, \tilde{v} = \mathcal{R}\left(\frac{\omega_c t}{\varepsilon}\right) v.$$

Ainsi, (11) conduit à

$$\phi^\varepsilon\left(t, \tilde{X}^\varepsilon(t) - \frac{\mathcal{R}}{\omega_c}\left(-\frac{\omega_c t}{\varepsilon}\right) {}^\perp \tilde{V}^\varepsilon(t)\right) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e\left(\tilde{X}^\varepsilon(t) - \tilde{y} - \frac{\mathcal{R}}{\omega_c}\left(-\frac{\omega_c t}{\varepsilon}\right)^\perp (\tilde{V}^\varepsilon(t) - \tilde{w})\right) \tilde{f}^\varepsilon(t, \tilde{y}, \tilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}$$

et par conséquent, (12), (13) deviennent

$$\frac{\mathrm{d}\tilde{X}^\varepsilon}{\mathrm{d}t} = -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(\tilde{X}^\varepsilon(t) - \tilde{y} - \frac{1}{\omega_c}\mathcal{R}\left(-\frac{\omega_c t}{\varepsilon}\right){}^\perp(\tilde{V}^\varepsilon(t) - \tilde{w})\right)\tilde{f}^\varepsilon(t,\tilde{y},\tilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y} \tag{14}$$

$$\frac{\mathrm{d}\tilde{V}^\varepsilon}{\mathrm{d}t} = -\mathcal{R}\left(\frac{\omega_c t}{\varepsilon}\right)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla e\left(\tilde{X}^\varepsilon(t) - \tilde{y} - \frac{1}{\omega_c}\mathcal{R}\left(-\frac{\omega_c t}{\varepsilon}\right){}^\perp(\tilde{V}^\varepsilon(t) - \tilde{w})\right)\tilde{f}^\varepsilon(t,\tilde{y},\tilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}. \tag{15}$$

En prenant la moyenne de (14) sur la période cyclotronique $[t, t+T_c^\varepsilon]$, avec $T_c^\varepsilon = \varepsilon\frac{2\pi}{\omega_c}$, et en introduisant la variable rapide $s = (\tau - t)/\varepsilon$, $\tau \in [t, t+T_c^\varepsilon]$, nous obtenons

$$\frac{\tilde{X}^\varepsilon(t+T_c^\varepsilon) - \tilde{X}^\varepsilon(t)}{T_c^\varepsilon} = -\frac{1}{\omega_c T_c^\varepsilon}\int_t^{t+T_c^\varepsilon}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(\tilde{X}^\varepsilon(\tau) - \tilde{y} - \mathcal{R}\left(-\frac{\omega_c\tau}{\varepsilon}\right)\frac{{}^\perp(\tilde{V}^\varepsilon(\tau)-\tilde{w})}{\omega_c}\right)\tilde{f}^\varepsilon(\tau)\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\tau$$

$$= -\frac{1}{\omega_c T_c}\int_0^{T_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(\tilde{X}^\varepsilon(t+\varepsilon s) - \tilde{y} - \mathcal{R}\left(-\frac{\omega_c t}{\varepsilon} - \omega_c s\right)\frac{{}^\perp(\tilde{V}^\varepsilon(t+\varepsilon s)-\tilde{w})}{\omega_c}\right)\tilde{f}^\varepsilon(t+\varepsilon s)\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}s$$

$$\approx -\frac{{}^\perp\nabla_\xi}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{E}(\tilde{X}^\varepsilon(t)-\tilde{y}, \tilde{V}^\varepsilon(t)-\tilde{w})\tilde{f}^\varepsilon(t,\tilde{y},\tilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\theta \tag{16}$$

où la fonction $\mathcal{E}$ est définie par

$$\mathcal{E}(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} e\left(\xi - \omega_c^{-1}\mathcal{R}(\theta)^\perp\eta\right)\,\mathrm{d}\theta,\quad \xi,\eta\in\mathbb{R}^2.$$

Nous procédons de la manière identique pour obtenir, à partir de (15)

$$\frac{\tilde{V}^\varepsilon(t+T_c^\varepsilon) - \tilde{V}^\varepsilon(t)}{T_c^\varepsilon} = -\frac{1}{T_c^\varepsilon}\int_t^{t+T_c^\varepsilon}\mathcal{R}\left(\frac{\omega_c\tau}{\varepsilon}\right)\int\nabla e\left(\tilde{X}^\varepsilon(\tau) - \tilde{y} - \mathcal{R}\left(-\frac{\omega_c\tau}{\varepsilon}\right)\frac{{}^\perp(\tilde{V}^\varepsilon(\tau)-\tilde{w})}{\omega_c}\right)\tilde{f}^\varepsilon(\tau)\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\tau$$

$$= -\frac{1}{T_c}\int_0^{T_c}\mathcal{R}\left(\frac{\omega_c t}{\varepsilon} + \omega_c s\right)\int\nabla e\left(\tilde{X}^\varepsilon(t+\varepsilon s) - \tilde{y} - \mathcal{R}\left(-\frac{\omega_c t}{\varepsilon} - \omega_c s\right)\frac{{}^\perp(\tilde{V}^\varepsilon(t+\varepsilon s)-\tilde{w})}{\omega_c}\right)\tilde{f}^\varepsilon(t+\varepsilon s)\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}s$$

$$\approx \omega_c{}^\perp\nabla_\eta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{E}(\tilde{X}^\varepsilon(t)-\tilde{y}, \tilde{V}^\varepsilon(t)-\tilde{w})\tilde{f}^\varepsilon(t,\tilde{y},\tilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\theta. \tag{17}$$

En passant à la limite dans (16), (17), quand $\varepsilon \searrow 0$, nous obtenons les trajectoires après filtration du mouvement rapide cyclotronique

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}t} = -\frac{{}^\perp\nabla_\xi}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{E}(\tilde{X}(t)-\tilde{y}, \tilde{V}(t)-\tilde{w})\tilde{f}(t)\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y},\quad \frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} = \omega_c{}^\perp\nabla_\eta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{E}(\tilde{X}(t)-\tilde{y}, \tilde{V}(t)-\tilde{w})\tilde{f}(t)\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}$$

où $\tilde{f} = \lim_{\varepsilon\searrow 0}\tilde{f}^\varepsilon$ est la distribution limite. Par la suite nous déterminons une expression pour $\mathcal{E}(\xi,\eta)$. Cela résulte de la propriété de la moyenne pour les fonctions harmoniques. En effet, si $|\xi| > |\eta|/|\omega_c|$, la fonction $z \to e(z)$ est harmonique dans l'ouvert $\mathbb{R}^2 \setminus \{0\}$, contenant le disque fermé, de centre ξ et de rayon $|\eta|/|\omega_c|$, et par conséquent nous avons, grâce à la formule de la moyenne

$$\mathcal{E}(\xi,\eta) = e(\xi) = -\frac{1}{2\pi}\ln|\xi|,\quad |\xi| > \frac{|\eta|}{|\omega_c|}.$$

Plus exactement, on démontre cf. [4]

Lemme 1.1 *Pour tout $\xi,\eta\in\mathbb{R}^2$, nous avons*

$$\mathcal{E}(\xi,\eta) = e\left(\frac{\eta}{\omega_c}\right)\mathbf{1}_{\{|\xi|\le|\eta|/|\omega_c|\}} + e(\xi)\mathbf{1}_{\{|\xi|>|\eta|/|\omega_c|\}}$$

$$\nabla_\xi\mathcal{E}(\xi,\eta) = \nabla e(\xi)\,\mathbf{1}_{\{|\xi|>|\eta|/|\omega_c|\}},\quad \nabla_\eta\mathcal{E}(\xi,\eta) = \omega_c^{-1}\nabla e\left(\frac{\eta}{\omega_c}\right)\mathbf{1}_{\{|\xi|\le|\eta|/|\omega_c|\}}\ \text{au sens des distributions.}$$

2. Le modèle limite

Nous introduisons le potentiel électrique

$$
\begin{aligned}
\tilde{\phi}[\tilde{f}(t)](\tilde{x},\tilde{v}) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\tilde{x}-\tilde{y},\tilde{v}-\tilde{w})\tilde{f}(t,\tilde{y},\tilde{w})\;\mathrm{d}\tilde{w}\mathrm{d}\tilde{y} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left\{ e\left(\frac{\tilde{v}-\tilde{w}}{\omega_c}\right) \mathbf{1}_{\{|\tilde{x}-\tilde{y}|\leq|\tilde{v}-\tilde{w}|/|\omega_c|\}} + e(\tilde{x}-\tilde{y})\;\mathbf{1}_{\{|\tilde{x}-\tilde{y}|>|\tilde{v}-\tilde{w}|/|\omega_c|\}} \right\} \tilde{f}(t,\tilde{y},\tilde{w})\;\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}
\end{aligned}
$$

et les fonctions

$$
\mathcal{V}[\tilde{f}(t)](\tilde{x},\tilde{v}) = -\frac{{}^\perp\nabla_\xi}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\tilde{x}-\tilde{y},\tilde{v}-\tilde{w})\tilde{f}(t,\tilde{y},\tilde{w})\;\mathrm{d}\tilde{w}\mathrm{d}\tilde{y} = -\frac{{}^\perp\nabla_{\tilde{x}}}{\omega_c}\tilde{\phi}[\tilde{f}(t)]
$$

$$
\mathcal{A}[\tilde{f}(t)](\tilde{x},\tilde{v}) = \omega_c{}^\perp\nabla_\eta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\tilde{x}-\tilde{y},\tilde{v}-\tilde{w})\tilde{f}(t,\tilde{y},\tilde{w})\;\mathrm{d}\tilde{w}\mathrm{d}\tilde{y} = \omega_c{}^\perp\nabla_{\tilde{v}}\tilde{\phi}[\tilde{f}(t)].
$$

En dérivant sous le signe intégral, il est également possible de représenter les champs vitesse et accélération sous la forme (cf. Lemme 1.1)

$$
\begin{aligned}
\mathcal{V}[\tilde{f}(t)](\tilde{x},\tilde{v}) &= -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla_\xi\mathcal{E}(\tilde{x}-\tilde{y},\tilde{v}-\tilde{w})\tilde{f}(t,\tilde{y},\tilde{w})\;\mathrm{d}\tilde{w}\mathrm{d}\tilde{y} \\
&= -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e(\tilde{x}-\tilde{y})\;\mathbf{1}_{\{|\tilde{x}-\tilde{y}|>|\tilde{v}-\tilde{w}|/|\omega_c|\}}\tilde{f}(t,\tilde{y},\tilde{w})\;\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\mathcal{A}[\tilde{f}(t)](\tilde{x},\tilde{v}) &= \omega_c\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla_\eta\mathcal{E}(\tilde{x}-\tilde{y},\tilde{v}-\tilde{w})\tilde{f}(t,\tilde{y},\tilde{w})\;\mathrm{d}\tilde{w}\mathrm{d}\tilde{y} \\
&= \omega_c\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(\frac{\tilde{v}-\tilde{w}}{\omega_c}\right)\mathbf{1}_{\{|\tilde{x}-\tilde{y}|\leq|\tilde{v}-\tilde{w}|/|\omega_c|\}}\tilde{f}(t,\tilde{y},\tilde{w})\;\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}.
\end{aligned}
\tag{19}
$$

Les trajectoires limites sont déterminées par les champs vitesse et accélération $\mathcal{V}[\tilde{f}],\mathcal{A}[\tilde{f}]$

$$
\frac{\mathrm{d}\tilde{X}}{\mathrm{d}t} = \mathcal{V}[\tilde{f}(t)](\tilde{X}(t),\tilde{V}(t)),\quad \frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} = \mathcal{A}[\tilde{f}(t)](\tilde{X}(t),\tilde{V}(t)).
$$

Les densités de présence étant conservées le long des trajectoires, nous obtenons

$$
\tilde{f}^\varepsilon(t,\tilde{X}^\varepsilon(t),\tilde{V}^\varepsilon(t)) = f^\varepsilon(t,X^\varepsilon(t),V^\varepsilon(t)) = f(0,x,v) = f(0,\tilde{x}-\omega_c^{-1}\,{}^\perp\tilde{v},\tilde{v})
$$

et par conséquent la densité limite $\tilde{f}$ est solution de (7), (8). Notons que les équations caractéristiques limites forment un système hamiltonien, en les variables conjuguées $(\tilde{x}_2,\omega_c^{-1}\tilde{v}_1)$ et $(\omega_c\tilde{x}_1,\tilde{v}_2)$

$$
\frac{\mathrm{d}\tilde{X}_2}{\mathrm{d}t} = \frac{\partial\tilde{\phi}[\tilde{f}(t)]}{\partial(\omega_c\tilde{x}_1)}(\tilde{X}(t),\tilde{V}(t)),\quad \frac{\mathrm{d}(\omega_c^{-1}\tilde{V}_1)}{\mathrm{d}t} = \frac{\partial\tilde{\phi}[\tilde{f}(t)]}{\partial\tilde{v}_2}(\tilde{X}(t),\tilde{V}(t))
$$

$$
\frac{\mathrm{d}(\omega_c\tilde{X}_1)}{\mathrm{d}t} = -\frac{\partial\tilde{\phi}[\tilde{f}(t)]}{\partial\tilde{x}_2}(\tilde{X}(t),\tilde{V}(t)),\quad \frac{\mathrm{d}\tilde{V}_2}{\mathrm{d}t} = -\frac{\partial\tilde{\phi}[\tilde{f}(t)]}{\partial(\omega_c^{-1}\tilde{v}_1)}(\tilde{X}(t),\tilde{V}(t)).
$$

3. Quelques propriétés du modèle limite

Les champs de vitesse et accélération étant à divergence nulle

$$
\mathrm{div}_{\tilde{x}}\mathcal{V}[\tilde{f}(t)] = -\frac{1}{\omega_c}\mathrm{div}_{\tilde{x}}\,{}^\perp\nabla_{\tilde{x}}\tilde{\phi}[\tilde{f}(t)] = 0,\quad \mathrm{div}_{\tilde{v}}\mathcal{A}[\tilde{f}(t)] = \omega_c\mathrm{div}_{\tilde{v}}\,{}^\perp\nabla_{\tilde{v}}\tilde{\phi}[\tilde{f}(t)] = 0
$$

l'équation (7) s'écrit aussi sous la forme conservative

$$\partial_t \tilde{f} + \mathrm{div}_{\tilde{x}}\{\tilde{f}\mathcal{V}[\tilde{f}(t)]\} + \mathrm{div}_{\tilde{v}}\{\tilde{f}\mathcal{A}[\tilde{f}(t)]\} = 0.$$

En particulier nous obtenons la conservation de la masse. Plus généralement, nous démontrons le résultat suivant.

Proposition 3.1 *Soit* $\tilde{f} = \tilde{f}(t, \tilde{x}, \tilde{v})$ *la solution du problème* (7), (8) *et* $\psi = \psi(\tilde{x}, \tilde{v})$ *une fonction intégrable par rapport à* $\tilde{f}(0, \tilde{x}, \tilde{v})\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = f^{\mathrm{in}}(\tilde{x} - \omega_c^{-1\perp}\tilde{v}, \tilde{v})\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}$.

(i) Pour tout $t \in \mathbb{R}_+$ *nous avons*

$$\begin{aligned} 2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi(\tilde{x}, \tilde{v})\tilde{f}(t, \tilde{x}, \tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{f}(t, \tilde{y}, \tilde{w})\tilde{f}(t, \tilde{x}, \tilde{v}) \\ &\times \left[\frac{1}{\omega_c}\left(\nabla_{\tilde{y}}\psi(\tilde{y}, \tilde{w}) - \nabla_{\tilde{x}}\psi(\tilde{x}, \tilde{v})\right) \cdot {}^{\perp}\nabla e(\tilde{x} - \tilde{y})\, \mathbf{1}_{\{|\tilde{x}-\tilde{y}|>|\tilde{v}-\tilde{w}|/|\omega_c|\}} \right. \\ &\left. + \left(\nabla_{\tilde{v}}\psi(\tilde{x}, \tilde{v}) - \nabla_{\tilde{w}}\psi(\tilde{y}, \tilde{w})\right) \cdot {}^{\perp}\nabla e\left(\frac{\tilde{v} - \tilde{w}}{\omega_c}\right) \mathbf{1}_{\{|\tilde{x}-\tilde{y}|<|\tilde{v}-\tilde{w}|/|\omega_c|\}}\right] \mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}. \end{aligned} \tag{20}$$

(ii) En particulier, pour tout $t \in \mathbb{R}_+$ *nous avons*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \{1, \tilde{x}, \tilde{v}, |\tilde{x}|^2, |\tilde{v}|^2\}\tilde{f}(t, \tilde{x}, \tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \{1, \tilde{x}, \tilde{v}, |\tilde{x}|^2, |\tilde{v}|^2\} f^{\mathrm{in}}(\tilde{x} - \omega_c^{-1}\, {}^{\perp}\tilde{v}, \tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x}.$$

Preuve.

(i) Pour tout $t \in \mathbb{R}_+$, nous obtenons, grâce aux formules de représentation (18), (19)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi(\tilde{x}, \tilde{v})\tilde{f}(t, \tilde{x}, \tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi(\tilde{x}, \tilde{v})\partial_t\tilde{f}\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[\nabla_{\tilde{x}}\psi \cdot \mathcal{V}[\tilde{f}(t)] + \nabla_{\tilde{v}}\psi \cdot \mathcal{A}[\tilde{f}(t)]\right] \tilde{f}(t, \tilde{x}, \tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x} \\ &= -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla_{\tilde{x}}\psi(\tilde{x}, \tilde{v}) \cdot {}^{\perp}\nabla e(\tilde{x} - \tilde{y})\, \mathbf{1}_{\{|\tilde{x}-\tilde{y}|>|\tilde{v}-\tilde{w}|/|\omega_c|\}}\tilde{f}(t, \tilde{y}, \tilde{w})\tilde{f}(t, \tilde{x}, \tilde{v})\, \mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} \\ &+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla_{\tilde{v}}\psi(\tilde{x}, \tilde{v}) \cdot {}^{\perp}\nabla e\left(\frac{\tilde{v} - \tilde{w}}{\omega_c}\right) \mathbf{1}_{\{|\tilde{x}-\tilde{y}|<|\tilde{v}-\tilde{w}|/|\omega_c|\}}\tilde{f}(t, \tilde{y}, \tilde{w})\tilde{f}(t, \tilde{x}, \tilde{v})\, \mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}. \end{aligned}$$

La formule (20) résulte en interchangeant $(\tilde{x}, \tilde{v})$ contre $(\tilde{y}, \tilde{w})$, combiné à Fubini.

(ii) Les conservations résultent facilement, par (20) appliquée successivement aux fonctions $1, \tilde{x}, \tilde{v}, |\tilde{x}|^2, |\tilde{v}|^2$.

□

Etant donné que l'énergie cinétique est conservée, et comme on s'attend à ce que l'énergie globale soit conservée, nous devrions retrouver aussi la conservation de l'énergie électrique. Effectivement nous démontrons

Proposition 3.2 *Pour tout* $t \in \mathbb{R}_+$ *nous avons*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{\phi}[\tilde{f}(t)](\tilde{x}, \tilde{v})\tilde{f}(t, \tilde{x}, \tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = 0.$$

Preuve. L'énergie électrique s'écrit sous la forme

$$\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{\phi}[\tilde{f}(t)](\tilde{x}, \tilde{v})\tilde{f}(t, \tilde{x}, \tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\tilde{x} - \tilde{y}, \tilde{v} - \tilde{w})\tilde{f}(t, \tilde{x}, \tilde{v})\tilde{f}(t, \tilde{y}, \tilde{w})\, \mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}$$

et en utilisant la parité de $\mathcal{E}(\xi,\eta)$ en les variables ξ et η, nous obtenons facilement, par Fubini, que

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{\phi}[\tilde{f}(t)](\tilde{x},\tilde{v})\tilde{f}(t,\tilde{x},\tilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{\phi}[\tilde{f}(t)](\tilde{x},\tilde{v})\partial_t\tilde{f}\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\nabla_{\tilde{x}}\tilde{\phi}[\tilde{f}(t)]\cdot\mathcal{V}[\tilde{f}(t)]+\nabla_{\tilde{v}}\tilde{\phi}[\tilde{f}(t)]\cdot\mathcal{A}[\tilde{f}(t)]\right]\tilde{f}\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}=0.\end{aligned}$$

$\square$

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