Periodic solutions of the 1D Vlasov-Maxwell system with boundary conditions

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Summary. *We study the 1D Vlasov-Maxwell system with time periodic boundary conditions in its classical and relativistic form. We are mainly concerned with existence of periodic weak solutions. We shall begin with the definitions of weak and mild solutions in the periodic case. The main mathematical difficulty in dealing with the Vlasov-Maxwell system consist of establihing L^∞ estimates for the charge and current densities. In order to obtain this kind of estimates, we impose non vanishing conditions for the incoming velocities, which assure a finite life-time of all particles in the computational domain $]0, L[$. The definition of the mild solution requires lipschitz regularity for the electromagnetic field. Thus, in the first time, the Vlasov equation has to be regularized. This procedure leads to the study of a sequence of approximate solutions. In the same time, an absorption term is introduced in the Vlasov equation, which guarantees the uniqueness of the mild solution of the regularized problem. In order to preserve the periodicity of the solution, a time averaging vanishing condition of the incoming current is imposed:*

$$\int_0^T dt \int_{v_x>0} \int_{v_y} v_x g_0(t, v_x, v_y) dv + \int_0^T dt \int_{v_x<0} \int_{v_y} v_x g_L(t, v_x, v_y) dv = 0, \quad (1)$$

where g_0, g_L are the incoming distributions:

$$f(t, 0, v_x, v_y) = g_0(t, v_x, v_y), \qquad t \in I\!R_t \; v_x > 0, v_y \in I\!R_v, \quad (2)$$

$$f(t, L, v_x, v_y) = g_L(t, v_x, v_y), \qquad t \in I\!R_t \; v_x < 0, v_y \in I\!R_v. \quad (3)$$

The existence proof utilizes the Schauder fixed point theorem and also the velocity average lemma of DiPerna and Lions [10]. In the last section we treat the relativistic case.

1 Introduction

The coupled nonlinear system presented by the Vlasov-Maxwell equations is a classical model in the kinetic theory of plasma. The main assumption underlying the model is that collisions are so rare that they may be neglected. In one dimension of space the Vlasov-Maxwell system (VM) writes:

$$\begin{aligned}\partial_t f + v_x \cdot \partial_x f + \frac{q}{m}(E_x + v_y \cdot B_z) \cdot \partial_{v_x} f \quad &+ \quad \frac{q}{m}(E_y - v_x \cdot B_z) \cdot \partial_{v_y} f = 0, \\ (t, x, v_x, v_y) \quad &\in \quad I\!R_t \times]0, L[\times I\!R_v^2, \end{aligned} \tag{4}$$

$$\partial_t E_x = -\frac{1}{\varepsilon} j_x := -\frac{1}{\varepsilon} \int_v v_x f(t, x, v_x, v_y)\, dv, \qquad (t, x) \in I\!R_t \times]0, L[, \tag{5}$$

$$\partial_t E_y + c^2 \partial_x B_z = -\frac{1}{\varepsilon} j_y := -\frac{1}{\varepsilon} \int_v v_y f(t, x, v_x, v_y)\, dv, \qquad (t, x) \in I\!R_t \times]0, L[, \tag{6}$$

$$\partial_t B_z + \partial_x E_y = 0, \qquad (t, x) \in I\!R_t \times]0, L[. \tag{7}$$

The variables (t, x, v_x, v_y) are respectively the time, the position and the velocity. The non-negative function f is the distribution of the charged particles of charge q and mass m, (E_x, E_y, B_z) is the electro-magnetic field, ε is the electric permitivity of the vacuum and c is the light velocity in the vacuum. A reduced description of the plasma is obtained by neglecting the magnetic field B. The associated model constitutes the Vlasov-Poisson system (VP) and it can be justified (at least for small time) by a non-relativistic limit [13]. The main result in this field has been obtained in 1989 by R.J.DiPerna and P.L.Lions [10]. They prove existence of global weak solutions for the Cauchy problem with arbitrary data. The global existence of strong solution is still an open problem. In the case of the Vlasov-Poisson system weak global solution for the Cauchy problem has been obtained by Arsenev [1]. Existence of strong solution in 2D is a result due to Degond [12] and Ukai Ohabe [2]. The same result in 3D has been proved by Pfaffelmoser [18]. For applications like vacuum diodes, tube discharges, cold plasma, solar wind, satellite ionisation, thrusters, etc... boundary conditions have to be taken into account. For the transient regime global weak solutions of the Vlasov-Maxwell system has been proved to exist by Y.Guo [16] and independently by M.Bezart [5]. The same problem for the Vlasov-Poisson system has been investigated by Y.Guo [17] and N.Ben Abdallah [4]. Permanent regimes are particularly important. They are of two types and they are modeled by stationary solutions or time periodic solutions for boundary value problems. Results concerning stationary problems can be found in the paper of C.Greengard P.A.Raviart [15] for the Vlasov-Poisson system in 1D and in the paper of F.Poupaud for the Vlasov-Maxwell system [19]. For the periodic problems, results can be found in [8]. We now describe precisely the boundary condition. Let $]0, L[$ representing the device geometry. We denote by Σ^- the set of initial positions in phase space of incoming particles :

$$\begin{aligned} \Sigma^- &= \{(0, v_x, v_y) \,|v_x > 0, v_y \in I\!R\} \\ &\cup \{(L, v_x, v_y) \,|v_x < 0, v_y \in I\!R\}. \end{aligned} \tag{8}$$

The distribution of incoming particles is prescribed :

$$f = \begin{cases} g_0, & (t, x, v_x, v_y) \in I\!R_t \times \Sigma^-, x = 0, \\ g_L, & (t, x, v_x, v_y) \in I\!R_t \times \Sigma^-, x = L. \end{cases} \tag{9}$$

We impose Silver-Müller condition on the electro-magnetic field (E_x, E_y, B_z) :

$$n_0 \wedge E + c \cdot n_0 \wedge (n_0 \wedge B) = h_0, \tag{10}$$

$$n_L \wedge E + c \cdot n_L \wedge (n_L \wedge B) = h_L, \tag{11}$$

where $n_0 = (-1, 0, 0)$ and $n_L = (1, 0, 0)$ are the outward unit normals of $]0, L[$ in $x = 0$ and $x = L$. Here, the boundary data $g_0,\ g_L,\ h_0,\ h_L$ are T periodic functions and we look for T periodic solutions (f, E_x, E_y, B_z) of the (VM) problem (4), (5), (6), (7), (9), (10) and (11). The formulas (10) and (11) model incoming waves in the device and can be written:

$$E_y(t, 0) + cB_z(t, 0) = h_0(t), \qquad t \in I\!R_t, \tag{12}$$

$$E_y(t, L) - cB_z(t, L) = h_L(t), \qquad t \in I\!R_t. \tag{13}$$

One of the key point of our proof of existence of such solutions is to control the life-time of particles in the domain $]0, L[$. Therefore we impose a non-vanishing condition of incoming velocities which reads :

$$supp(g_0) \subset \{(t, x, v_x, v_y)\ ;\ t \in I\!R_t, x = 0, 0 < v_0 \le v_x, \sqrt{v_x^2 + v_y^2} \le v_1\},$$

$$supp(g_L) \subset \{(t,x,v_x,v_y);\ t \in I\!\!R_t, x = L, 0 > -v_0 \geq v_x, \sqrt{v_x^2+v_y^2} \leq v_1\}, \tag{14}$$

for $0 < v_0 < v_1$ given. On the other hand, in order to preserve the periodicity of E_x given by (5), a time averaging vanishing condition of the incoming current is imposed:

$$\int_0^T dt \int_{v_x>0}\int_{v_y} v_x g_0(t,v_x,v_y)dv + \int_0^T dt \int_{v_x<0}\int_{v_y} v_x g_L(t,v_x,v_y)dv = 0. \tag{15}$$

Let us remark that even if the electro-magnetic field (E_x, E_y, B_z) is "a priori" known, there is no uniqueness of the T periodic solution of the Vlasov problem (V) : (4) and (9). Indeed, the distribution function can take arbitrary (constant) values on the characteristics which remain in the domain (trapped characteristics). In order to select physical solution we introduce as in [15] and [19] the concept of minimal solution of (V) which are the solutions which vanish on the trapped characteristics. These solutions can be obtained as the limit of the (unique) solution of the modified Vlasov problem (V_α) when an absorption term $\alpha > 0$ is introduced and tends to zero :

$$\begin{aligned} \alpha f + \partial_t f + v_x\cdot\partial_x f + \frac{q}{m}(E_x + v_y\cdot B_z)\cdot\partial_{v_x} f \quad &+ \quad \frac{q}{m}(E_y - v_x\cdot B_z)\cdot\partial_{v_y} f = 0, \\ (t,x,v_x,v_y) \quad &\in \quad I\!\!R_t \times]0,L[\times I\!\!R_v^2. \end{aligned} \tag{16}$$

This limit absorption principle has been developped by the author to obtain numerical periodic solutions of Partial Differential Equation, see [9]. We also stress that these results has been announced in [7].
The paper is organized as followed. In Section 2 we define weak solutions and minimal mild solution of the Vlasov problem (V). We also proved that the weak solution of the modified Vlasov problem (V_α) is unique and coincide with the minimal mild solution. In section 3 we prove existence of weak periodic solution for the classical 1D Vlasov-Maxwell system. We introduce a regularized problem. The existence theorem is obtained by using Schauder's theorem for the modified problem. Then we pass to the limit in the regularization parameter to obtain our main result. Section 4 is devoted to the relativistic 1D Vlasov-Maxwell system.

2 Definitions and bounds for the Vlasov equation.

In this section we assume that the electro-magnetic field (E_x, E_y, B_z) is T periodic in time and we look for a solution f of the Vlasov equation:

$$\begin{aligned} \partial_t f + v_x \cdot \partial_x f + \frac{q}{m}(E_x + v_y \cdot B_z) \cdot \partial_{v_x} f \quad &+ \quad \frac{q}{m}(E_y - v_x \cdot B_z) \cdot \partial_{v_y} f = 0, \\ (t, x, v_x, v_y) \quad &\in \quad \mathbb{R}_t \times]0, L[\times \mathbb{R}_v^2, \end{aligned} \tag{17}$$

$$f(t, 0, v_x, v_y) = g_0(t, v_x, v_y), \qquad t \in \mathbb{R}_t \ v_x > 0, v_y \in \mathbb{R}_v, \tag{18}$$

$$f(t, L, v_x, v_y) = g_L(t, v_x, v_y), \qquad t \in \mathbb{R}_t \ v_x < 0, v_y \in \mathbb{R}_v. \tag{19}$$

Here $q(> 0)$ and m are the charge and the mass of particles. Moreover, we suppose that the given distribution functions g_0, g_L of the in-flowing particles are T periodic in time, too. Now we briefly recall the notions of mild and weak solutions for this type of problem.

2.1 Weak and mild solution of the Vlasov equation

Definition 1 *Let $E_x, E_y, B_z \in L^\infty(\mathbb{R}_t \times]0, L[)$ and $g_0, g_L \in L^1_{loc}(\mathbb{R}_t \times \Sigma^-)$ be T periodic functions in time, where:*

$$\begin{aligned} \Sigma^- \quad &= \quad \{(t, x, v_x, v_y) \mid t \in \mathbb{R}, x = 0, v_x > 0, v_y \in \mathbb{R}\} \\ &\cup \quad \{(t, x, v_x, v_y) \mid t \in \mathbb{R}, x = L, v_x < 0, v_y \in \mathbb{R}\}. \end{aligned} \tag{20}$$

We say that $f \in L^1_{loc}(\mathbb{R}_t \times]0, L[\times \mathbb{R}_v^2)$ is a T periodic weak solution of problem (17), (18), (19) iff:

$$\begin{aligned} \int_0^T \int_0^L \int_{\mathbb{R}_v^2} (\partial_t \theta + v_x \cdot \partial_x \theta \quad &+ \quad \frac{q}{m}(E_x + v_y \cdot B_z) \cdot \partial_{v_x} \theta \\ &+ \quad \frac{q}{m}(E_y - v_x \cdot B_z) \cdot \partial_{v_y} \theta) f(t, x, v_x, v_y) dv dx dt \\ &= \quad \int_0^T \int_{v_x < 0} \int_{v_y} v_x g_L(t, v_x, v_y) \theta(t, L, v_x, v_y) dv dt \\ &- \quad \int_0^T \int_{v_x > 0} \int_{v_y} v_x g_0(t, v_x, v_y) \theta(t, 0, v_x, v_y) dv dt \end{aligned} \tag{21}$$

for all T periodic function $\theta \in \mathcal{V}$, where:

$$\mathcal{V}=\{\eta \in W^{1,\infty}(I\!\!R_t\times]0,L[\times I\!\!R^2_v) \;\; ; \;\; \eta(t,0,v_x<0,v_y)=\eta(t,L,v_x>0,v_y)=0, \\ supp(\eta)\, bounded\, set\, of\, I\!\!R_t\times[0,L]\times I\!\!R^2_v\}$$

In other words, a weak solution of problem (17), (18), (19) is a distribution function satisfying:

$$\begin{aligned} <f,\varphi> &= \int_0^T\int_{v_x<0}\int_{v_y} v_x\cdot g_L(t,v_x,v_y)\cdot\theta(t,L,v_x,v_y)dvdt \\ &- \int_0^T\int_{v_x>0}\int_{v_y} v_x\cdot g_0(t,v_x,v_y)\cdot\theta(t,0,v_x,v_y)dvdt \qquad (22) \end{aligned}$$

for all T periodic function φ, where θ denote the solution of the problem:

$$\begin{aligned} \partial_t\theta+v_x\cdot\partial_x\theta+\frac{q}{m}(E_x+v_y\cdot B_z)\cdot\partial_{v_x}\theta &+ \frac{q}{m}(E_y-v_x\cdot B_z)\cdot\partial_{v_y}\theta=\varphi, \\ (t,x,v_x,v_y) &\in I\!\!R_t\times]0,L[\times I\!\!R^2_v \qquad (23) \end{aligned}$$

$$\theta(t,0,v_x,v_y)=0, \qquad t\in I\!\!R_t\; v_x<0, v_y\in I\!\!R_v, \qquad (24)$$

$$\theta(t,L,v_x,v_y)=0, \qquad t\in I\!\!R_t\; v_x>0, v_y\in I\!\!R_v. \qquad (25)$$

Remark 1 *In the above definition we can assume that the electro-magnetic field is only in $(L^p(I\!\!R_t\times]0,L[))^3$ by requiring more regularity on f (and g_0, g_L), namely f in $L^q_{loc}(I\!\!R_t\times]0,L[\times I\!\!R^2_v)$ where q is the conjugate exponent.*

If the electro-magnetic field satisfies $(E_x,E_y,B_z)\in(L^\infty(I\!\!R_t;W^{1,\infty}(]0,L[)))^3$, we can express a solution in terms of characteristics. Let (t,x,v_x,v_y) belong to $I\!\!R_t\times]0,L[\times I\!\!R^2_v$, we denote by $X(s;x,v_x,v_y,t)$, $V_x(s;x,v_x,v_y,t)$ and $V_y(s;x,v_x,v_y,t)$ the solution of the system:

$$
\begin{cases}
\dfrac{dX}{ds} & = & V_x(s;x,v_x,v_y,t), & s\in[\tau_i,\tau_o] \\
X(t;x,v_x,v_y,t) & = & x, & \\
\dfrac{dV_x}{ds} & = & \dfrac{q}{m}(E_x(s,X(s))+V_y(s)\cdot B_z(s,X(s))), & s\in[\tau_i,\tau_o] \\
V_x(t;x,v_x,v_y,t) & = & v_x, & \\
\dfrac{dV_y}{ds} & = & \dfrac{q}{m}(E_y(s,X(s))-V_x(s)\cdot B_z(s,X(s))), & s\in[\tau_i,\tau_o] \\
V_y(t;x,v_x,v_y,t) & = & v_y. &
\end{cases}
\tag{26}
$$

where $[\tau_i(x,v_x,v_y,t),\tau_o(x,v_x,v_y,t)]$ is the life-time of the particle in the domain $]0,L[$:

$$(X(\tau_i),V_x(\tau_i),V_y(\tau_i))\in\Sigma^- \tag{27}$$

and

$$(X(\tau_o),V_x(\tau_o),V_y(\tau_o))\in\Sigma^+\cup\Sigma^0. \tag{28}$$

The subsets of $\{0,L\}\times I\!R^2_v$ Σ^+ and Σ^0 are respectively defined by:

$$
\begin{array}{rcl}
\Sigma^+ & = & \{(t,x,v_x,v_y)\mid t\in I\!R, x=0, v_x<0, v_y\in I\!R\} \\
& \cup & \{(t,x,v_x,v_y)\mid t\in I\!R, x=L, v_x>0, v_y\in I\!R\}, \\
\Sigma^0 & = & \{(t,x,v_x,v_y)\mid t\in I\!R, x=0, v_x=0, v_y\in I\!R\} \\
& \cup & \{(t,x,v_x,v_y)\mid t\in I\!R, x=L, v_x=0, v_y\in I\!R\}.
\end{array}
$$

Using the Cauchy-Lipschitz theorem, we notice that the characteristics are well defined. By integration along the characteristics curves, the solution of the problem (23), (24),(25) writes:

$$\theta(t,x,v_x,v_y)=-\int_t^{\tau_o}\varphi(s,X(s;x,v_x,v_y,t),V_x(s;x,v_x,v_y,t),V_y(s;x,v_x,v_y,t)ds$$

Now, (22) implies that:

$$<f,\varphi> \quad = \quad \int_0^T dt\int_{v_x>0}\int_{v_y}dv\int_t^{\tau_o}v_x\cdot g_0(t,v_x,v_y)$$

$$
\begin{aligned}
\cdot \quad & \varphi(s, X(s;0,v_x,v_y,t), V_x(s;0,v_x,v_y,t), V_y(s;0,v_x,v_y,t)ds \\
- \quad & \int_0^T dt \int_{v_x<0} \int_{v_y} dv \int_t^{\tau_o} v_x \cdot g_L(t,v_x,v_y) \\
\cdot \quad & \varphi(s, X(s;L,v_x,v_y,t), V_x(s;L,v_x,v_y,t), V_y(s;L,v_x,v_y,t)ds,
\end{aligned} \tag{29}
$$

which is equivalent to:

$$
f(t,x,v_x,v_y) = \begin{cases} g_0(\tau_i, V_x(\tau_i;x, \ v_x,v_y,t), V_y(\tau_i;x,v_x,v_y,t)) \\ \qquad\qquad \text{if } \tau_i > -\infty \text{ and } X(\tau_i;x,v_x,v_y,t) = 0, \\ g_L(\tau_i, V_x(\tau_i;x, \ v_x,v_y,t), V_y(\tau_i;x,v_x,v_y,t)) \\ \qquad\qquad \text{if } \tau_i > -\infty \text{ and } X(\tau_i;x,v_x,v_y,t) = L, \\ 0. \qquad\qquad \text{otherwise.} \end{cases} \tag{30}
$$

Definition 2 *Let $E_x, E_y, B_z \in L^\infty(I\!\!R_t; W^{1,\infty}(]0,L[))$ and $g_0, g_L \in L^1_{loc}(I\!\!R_t \times \Sigma^-)$ be T periodic functions. The function $f \in L^1_{loc}(I\!\!R_t \times]0,L[\times I\!\!R^2_v)$ which is the mild periodic solution of problem (17), (18), (19) is given by (29).*

Remark 2 *There is in general no uniqueness of the weak solution because f can take arbitrarily values on the characteristics such that $\tau_i = -\infty$. But it is possible to prove that the mild solution is the ununique minimal solution of the transport equation. We refer to [P, VM] for the concept of the minimal solution and to [Bod, PhD] for a proof of this assertion.*

Remark 3 *We have that $X(s+T;x,v_x,v_y,t+T) = X(s;x,v_x,v_y,t)$, $V_x(s+T;x,v_x,v_y,t+T) = V_x(s;x,v_x,v_y,t)$, $V_y(s+T;x,v_x,v_y,t+T) = V_y(s;x,v_x,v_y,t)$ and $\tau_i(x,v_x,v_y,t+T) = \tau_i(x,v_x,v_y,t) + T$ because of the periodicity of E_x, E_y, B_z. Using this equality it is easy to check that the mild solution is periodic.*

Remark 4 *If $g_0, g_L \in C^1(I\!\!R_t \times \Sigma^-)$ then the mild solution is a classical solution of (17), (18), (19).*

2.2 Estimation of the life-time of particles

In order to assure L^∞ estimates for the charge and current densities, we assume that the following conditions are satisfied:

$$\|E\|_{L^\infty} + \|B_z\|_{L^\infty} \cdot \left(v_1 + \frac{q}{m} \|E\|_{L^\infty} \frac{2L}{v_0} \right) \leq \frac{m v_0^2}{4qL} \tag{31}$$

$$(E_x, E_y, B_z) \in (L^\infty(\mathbb{R}_t; W^{1,\infty}(]0, L[)))^3, \tag{32}$$

$$\begin{aligned} supp(g_0) \subset \{(t, x, v_x, v_y) \ ; \ t \in \mathbb{R}_t, x = 0, 0 < v_0 \leq v_x, \sqrt{v_x^2 + v_y^2} \leq v_1\}, \\ supp(g_L) \subset \{(t, x, v_x, v_y); \ t \in \mathbb{R}_t, x = L, 0 > -v_0 \geq v_x, \sqrt{v_x^2 + v_y^2} \leq v_1\}. \end{aligned} \tag{33}$$

Here, $|E|_{L^\infty}$ is the L^∞ norm of $\sqrt{E_x^2 + E_y^2}$ and v_0,v_1 are constants which will be chosen in the next section. With these assumptions, we get:

Lemma 1 *Assume that the electro-magnetic field and the boundary data satisfy (31),(32) and (33). Then, the life-time in $]0, L[$ of particles starting from the support of g_0 and g_L is finite:*

$$\tau_o(x, v_x, v_y, t) - \tau_i(x, v_x, v_y, t) \leq 2 \cdot \frac{L}{v_0}, \ \forall (t, x, v_x, v_y) \in supp(g_0) \cup supp(g_L). \tag{34}$$

Proof Suppose that there is a particle starting from the support of g_0 at $t = \tau_i$ and which is still in $]0, L[$ at $\tau_i + \dfrac{2L}{v_0} < \tau_o$:

$$X\left(\tau_i + \frac{2L}{v_0}\right) \in]0, L[. \tag{35}$$

According to (33), we have:

$$0 < v_0 \leq V_x(\tau_i) \tag{36}$$

$$\sqrt{V_x^2(\tau_i) + V_y^2(\tau_i)} \leq v_1. \tag{37}$$

We multiply the velocity equations of (26) by $V_x(s)$ and $V_y(s)$ to get for $s \in [\tau_i, \tau_o]$:

$$\frac{1}{2}\cdot\frac{d}{ds}|V_x(s)|^2 = \frac{q}{m}(E_x(s,X(s))\cdot V_x(s) + V_x(s)\cdot V_y(s)\cdot B_z(s,X(s))\ ),$$

$$\frac{1}{2}\cdot\frac{d}{ds}|V_y(s)|^2 = \frac{q}{m}(E_y(s,X(s))\cdot V_y(s) - V_y(s)\cdot V_x(s)\cdot B_z(s,X(s))\ ),$$

and therefore:

$$\frac{d}{ds}\sqrt{V_x^2(s)+V_y^2(s)} \leq \frac{q}{m}\sqrt{E_x^2(s,X(s))+E_y^2(s,X(s))},$$

which yields:

$$\begin{aligned}\left|\sqrt{V_x^2(s)+V_y^2(s)} - \sqrt{v_x^2+v_y^2}\right| &\leq \frac{q}{m}\cdot\left\|\sqrt{E_x^2+E_y^2}\right\|_{L^\infty}\cdot(s-\tau_i)\\ &\leq \frac{q}{m}\cdot\left\|\sqrt{E_x^2+E_y^2}\right\|_{L^\infty}\cdot\frac{2L}{v_0}. \qquad (38)\end{aligned}$$

Integrating (26) on $[\tau_i, t] \subset [\tau_i, \tau_i + 2L/v_0]$, we obtain:

$$X(t) = X(\tau_i) + \int_{\tau_i}^{t} V_x(s)ds \qquad (39)$$

$$V_x(t) = V_x(\tau_i) + \int_{\tau_i}^{t}\frac{q}{m}(E_x(s) + V_y(s)\cdot B_z(s))ds, \qquad (40)$$

$$V_y(t) = V_y(\tau_i) + \int_{\tau_i}^{t}\frac{q}{m}(E_y(s) - V_x(s)\cdot B_z(s))ds. \qquad (41)$$

From (38) and (37) we deduce that for all $s \in [\tau_i, \tau_i + 2L/v_0]$:

$$|V_y(s)| \leq v_1 + (s-\tau_i)\frac{q}{m}\|E\|_{L^\infty}. \qquad (42)$$

Now using (36), (40), (42) and (31) we find for all $t \in [\tau_i, \tau_i + 2L/v_0]$:

$$
\begin{aligned}
V_x(t) &\geq v_x - \int_{\tau_i}^{t} \frac{q}{m} \cdot (\ |E_x(s)| + |B_z(s)| \cdot |V_y(s)|\ )\ ds \\
&\geq v_0 - \int_{\tau_i}^{t} \frac{q}{m} \cdot \Big(\|E_x\|_{L^\infty} + \|B_z\|_{L^\infty} \cdot \Big(v_1 + (s-\tau_i)\frac{q}{m}\|E\|_{L^\infty} \Big) \Big)\ ds \\
&> v_0 - \frac{2L}{v_0} \cdot \frac{q}{m} \cdot \Big(\|E\|_{L^\infty} + \|B_z\|_{L^\infty} \cdot \Big(v_1 + \frac{q}{m}\|E\|_{L^\infty}\frac{2L}{v_0} \Big) \Big) \\
&\geq v_0 - \frac{v_0}{2} = \frac{v_0}{2}.
\end{aligned} \tag{43}
$$

Now, from (97) we deduce:

$$
\begin{aligned}
X(\tau_i + 2L/v_0) &= 0 + \int_{\tau_i}^{\tau_i + 2L/v_0} V_x(s) ds \\
&> \frac{2L}{v_0} \cdot \frac{v_0}{2} = L,
\end{aligned} \tag{44}
$$

which contradicts (35). If the particle starts from the support of g_L, using the same ideas as previous we prove that $\tau_o \leq \tau_i + \dfrac{2L}{v_0}$ also holds.

Corollary 1 *Assuming the same hypotheses as in Lemma 1 and let f be the mild solution of Definition 2. Then we have:*

$$
supp(f) \subset \{(t, x, v_x, v_y) | t \in I\!R_t, x \in [0, L], \frac{v_0}{2} \leq |v_x|, \sqrt{v_x^2 + v_y^2} \leq v_1 + \frac{v_0}{2}\} \tag{45}
$$

$$
\|\rho\|_{L^\infty} \leq \frac{\pi}{2}(v_1 + v_0/2)^2 \cdot q \cdot (\|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}), \tag{46}
$$

and

$$
\max\{\|j_x\|_{L^\infty}, \|j_y\|_{L^\infty}\} \leq \frac{\pi}{2}(v_1 + v_0/2)^3 \cdot q \cdot (\|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}), \tag{47}
$$

where $\rho(t, x) = q \int_{I\!R_v^2} f(t, x, v_x, v_y)\ dv$ and $j_{x,y}(t, x) = q \int_{I\!R_v^2} v_{x,y} \cdot f(t, x, v_x, v_y)\ dv$.

Proof The estimate (45) follow from the previous Lemma. Indeed, according to (38), (33) and (31), we obtain:

$$\begin{aligned}
\sqrt{V_x^2(t)+V_y^2(t)} &\leq \sqrt{V_x^2(\tau_i)+V_y^2(\tau_i)}+\int_{\tau_i}^t \frac{q}{m}|E(s,X(s))|ds \\
&\leq v_1+(t-\tau_i)\cdot\frac{q}{m}\mid E\mid_{L^\infty} \\
&\leq v_1+\frac{2Lq}{mv_0}\mid E\mid_{L^\infty} \\
&\leq v_1+\frac{v_0}{2}. \qquad (48)
\end{aligned}$$

Using (40) and (33) we get for $t\in[\tau_i,\tau_o]$:

$$\begin{aligned}
V_x(t) &\geq V_x(\tau_i)-\int_{\tau_i}^t \frac{q}{m}(|E_x(s,X(s))|+|V_y(s)||B_z(s,X(s))|)ds \\
&\geq v_0-(\tau_o-\tau_i)\cdot\frac{q}{m}\cdot\left(\|E\|_{L^\infty}+\|B_z\|_{L^\infty}\cdot\left(v_1+\frac{q}{m}\|E\|_{L^\infty}\frac{2L}{v_0}\right)\right) \\
&\geq v_0-\frac{2L}{v_0}\cdot\frac{q}{m}\cdot\left(\|E\|_{L^\infty}+\|B_z\|_{L^\infty}\cdot\left(v_1+\frac{q}{m}\|E\|_{L^\infty}\frac{2L}{v_0}\right)\right) \\
&\geq v_0-\frac{v_0}{2}=\frac{v_0}{2}. \qquad (49)
\end{aligned}$$

If the particle starts from the support of g_L, (48) are the same and (49) change in:

$$\begin{aligned}
-V_x(t) &\geq -V_x(\tau_i)-\int_{\tau_i}^t \frac{q}{m}(|E_x(s,X(s))|+|V_y(s)||B_z(s,X(s))|)ds \\
&\geq v_0-\frac{v_0}{2}=\frac{v_0}{2}. \qquad (50)
\end{aligned}$$

Now, (46) and (47) can be easily checked. For $(t,x)\in I\!R_t\times]0,L[$ we have:

$$\begin{aligned}
\rho(t,x) &= q\int_{I\!R_{v^2}} f(t,x,v_x,v_y)\,dv \\
&= q\int_{v_x>0}\int_{I\!R_v} f(t,x,v_x,v_y)\,dv+q\int_{v_x<0}\int_{I\!R_v} f(t,x,v_x,v_y)\,dv \\
&\leq \frac{\pi}{2}(v_1+v_0/2)^2\cdot q\cdot(\|g_0\|_{L^\infty}+\|g_L\|_{L^\infty}), \qquad (51)
\end{aligned}$$

and therefore $\|\rho\|_{L^\infty}\leq\frac{\pi}{2}(v_1+v_0/2)^2\cdot q\cdot(\|g_0\|_{L^\infty}+\|g_L\|_{L^\infty})$. Obviously, (47) follows in the same way.

Remark 5 *Assuming the same hypotheses as in Lemma 1. Then the mild solution f of Definition 2 can be split in two mild solutions $f = f_0 + f_L$ given by:*

$$\begin{aligned} < f_0, \varphi > &= \int_0^T dt \int_{v_x>0} \int_{v_y} dv \int_t^{\tau_o} v_x \cdot g_0(t, v_x, v_y) \\ &\cdot \quad \varphi(s, X(s; 0, v_x, v_y, t), V_x(s; 0, v_x, v_y, t), V_y(s; 0, v_x, v_y, t)ds \end{aligned} \tag{52}$$

and:

$$\begin{aligned} < f_L, \varphi > &= -\int_0^T dt \int_{v_x<0} \int_{v_y} dv \int_t^{\tau_o} v_x \cdot g_L(t, v_x, v_y) \\ &\cdot \quad \varphi(s, X(s; L, v_x, v_y, t), V_x(s; L, v_x, v_y, t), V_y(s; L, v_x, v_y, t)ds. \end{aligned} \tag{53}$$

In the same time f_0, f_L are weak periodic solutions for the problems:

$$\begin{aligned} \partial_t f_0 + v_x \cdot \partial_x f_0 + \frac{q}{m}(E_x + v_y \cdot B_z) \cdot \partial_{v_x} f_0 &+ \frac{q}{m}(E_y - v_x \cdot B_z) \cdot \partial_{v_y} f_0 = 0, \\ (t, x, v_x, v_y) &\in I\!R_t \times]0, L[\times I\!R_v^2, \end{aligned}$$

$$f_0(t, 0, v_x, v_y) = g_0(t, v_x, v_y), \qquad t \in I\!R_t \; v_x > 0, v_y \in I\!R_v,$$

$$f_0(t, L, v_x, v_y) = 0, \qquad t \in I\!R_t \; v_x < 0, v_y \in I\!R_v,$$

and:

$$\begin{aligned} \partial_t f_L + v_x \cdot \partial_x f_L + \frac{q}{m}(E_x + v_y \cdot B_z) \cdot \partial_{v_x} f_L &+ \frac{q}{m}(E_y - v_x \cdot B_z) \cdot \partial_{v_y} f_L = 0, \\ (t, x, v_x, v_y) &\in I\!R_t \times]0, L[\times I\!R_v^2, \end{aligned}$$

$$f_L(t, 0, v_x, v_y) = 0, \qquad t \in I\!R_t \; v_x > 0, v_y \in I\!R_v,$$

$$f_L(t,L,v_x,v_y) = g_L(t,v_x,v_y), \qquad t \in I\!R_t \; v_x < 0, v_y \in I\!R_v.$$

Moreover we have :

$$supp(f_0) \subset \{(t,x,v_x,v_y) | t \in I\!R_t, x \in [0,L], \frac{v_0}{2} \leq v_x, \sqrt{v_x^2+v_y^2} \leq v_1 + \frac{v_0}{2}\}, \tag{54}$$

and :

$$supp(f_L) \subset \{(t,x,v_x,v_y) | t \in I\!R_t, x \in [0,L], -\frac{v_0}{2} \geq v_x, \sqrt{v_x^2+v_y^2} \leq v_1 + \frac{v_0}{2}\}. \tag{55}$$

3 Weak periodic solutions for the modified 1D Vlasov-Maxwell system.

Our goal is to establish existence result for the weak periodic solution of the 1D Vlasov-Maxwell problem:

$$\begin{aligned} \partial_t f + v_x \cdot \partial_x f + \frac{q}{m}(E_x + v_y \cdot B_z) \cdot \partial_{v_x} f &+ \frac{q}{m}(E_y - v_x \cdot B_z) \cdot \partial_{v_y} f = 0, \\ (t,x,v_x,v_y) &\in I\!R_t \times]0,L[\times I\!R_v^2, \end{aligned} \tag{56}$$

$$\partial_t E_x = -\frac{1}{\varepsilon} j_x := -\frac{1}{\varepsilon} \int_v v_x f(t,x,v_x,v_y)\, dv, \qquad (t,x) \in I\!R_t \times]0,L[, \tag{57}$$

$$\partial_t E_y + c^2 \partial_x B_z = -\frac{1}{\varepsilon} j_y := -\frac{1}{\varepsilon} \int_v v_y f(t,x,v_x,v_y)\, dv, \qquad (t,x) \in I\!R_t \times]0,L[, \tag{58}$$

$$\partial_t B_z + \partial_x E_y = 0, \qquad (t,x) \in I\!R_t \times]0,L[, \tag{59}$$

with the boundary conditions:

$$f(t,0,v_x,v_y) = g_0(t,v_x,v_y), \qquad t \in I\!R_t \; v_x > 0, v_y \in I\!R_v, \tag{60}$$

$$f(t,L,v_x,v_y) = g_L(t,v_x,v_y), \qquad t \in I\!R_t \; v_x < 0, v_y \in I\!R_v, \tag{61}$$

$$E_y(t,0) + cB_z(t,0) = h_0(t), \qquad t \in I\!R_t, \tag{62}$$

$$E_y(t,L) - cB_z(t,L) = h_L(t), \qquad t \in I\!R_t, \tag{63}$$

Here, the boundary data g_0, g_L, h_0, h_L are $T-$ periodic functions and c is the light velocity in the vacuum. We look for a weak periodic solution $(f(t,x,v_x,v_y),\ E_x(t,x), E_y(t,x),\ B_z(t,x))$. The Schauder fixed point theorem is used. We define an application which maps a periodic electro-magnetic field (E_x, E_y, B_z) to an other one (E_x^1, E_y^1, B_z^1) where (E_x^1, E_y^1, B_z^1) is defined as follows. Let f be the mild periodic solution of *Definition 2* corresponding to the electro-magnetic field (E_x, E_y, B_z). The new electro-magnetic field (E_x^1, E_y^1, B_z^1) is determined as the solution of the Maxwell problem with the current density $j_{x,y}(t,x) = \int_{I\!R_v^2} v_{x,y} f(t,x,v_x,v_y)dv$. Unfortunately this procedure cannot be used directly. Indeed the *Definiton 2* requires that the electro-magnetic field is Lipschitz with respect to x and we cannot expect such a regularity in the general case. Therefore we have to regularize the field. We also have to use an absorption term in the Vlasov equation in order to have uniqueness of the weak solution. Then the strategy of proof is as follows. We first show the existence of weak periodic solution for the regularized problem by using the Schauder fixed point theorem. Next we pass to the limit when the regularization parameter vanishes.

3.1 Fixed point for the regularized problem

Let $\mathcal{X}$ be the set of fields (E_x, E_y, B_z) which verify:

$$\mathcal{X} = \{(E_x, E_y, B_z) \in (L^\infty(I\!R_t \times]0,L[))^3 \; ; \; \|E\|_{L^\infty} \leq K, c \cdot \|B_z\|_{L^\infty} \leq K,$$
$$(E_x, E_y, B_z)(t) = (E_x, E_y, B_z)(t+T) \; \forall t \in I\!R_t\} \tag{64}$$

where K is a positive constant. Because of time periodicity, $\mathcal{X}$ is a compact set of $(L^2_T(I\!\!R_t\times]0,L[))^3$ with the weak topology, where:

$$L^2_T(I\!\!R_t\times]0,L[) = \{u\ ;\int_0^T\int_0^L |u(t,x)|^2 dxdt < \infty, u(t,\cdot) = u(t+T,\cdot)\ \forall t\in I\!\!R_t\} \tag{65}$$

We now introduce a regularization mapping:

$$R_\alpha : L^\infty(I\!\!R_t\times]0,L[) \to L^\infty(I\!\!R_t; C^1([0,L]),$$

$$(R_\alpha E_x, R_\alpha E_y, R_\alpha B_z)(t,x) = \int_{-\infty}^{\infty}\int_0^L \zeta_\alpha(t-s, x-y)\cdot(E_x,E_y,B_z)(s,y)dsdy, \tag{66}$$

where $\zeta_\alpha \geq 0$ is a mollifier:

$$\zeta_\alpha(t,x) = \frac{1}{\alpha^2}\zeta\left(\frac{t}{\alpha},\frac{x}{\alpha}\right) \quad , \quad \zeta\in C_0^\infty(I\!\!R^2)$$
$$supp(\zeta)\subset[-1,1]\times[-1,1] \quad , \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta(u,v)dudv = 1$$

It is easy to see that $(R_\alpha E_x, R_\alpha E_y, R_\alpha B_z)$ are also time periodic :

$$(R_\alpha E_x, R_\alpha E_y, R_\alpha B_z)(t,x) = \int_{-\infty}^{\infty}\int_0^L \zeta_\alpha(s, x-y)\cdot(E_x,E_y,B_z)(t-s,y)dsdy, \tag{67}$$

and therefore $R_\alpha(\mathcal{X})\subset\mathcal{X}$. Next, we consider the application:

$$F : (E_x,E_y,B_z)\in\mathcal{X}\mapsto(E_x^1,E_y^1,B_z^1), \tag{68}$$

where:

$$\begin{aligned} E_x^1(t,x) = \quad & - \quad \frac{1}{\varepsilon}\int_0^t\left(j_{x,\alpha}(s,x)+\alpha\int_0^x\rho_{0,\alpha}(s,y)dy-\alpha\int_x^L\rho_{L,\alpha}(s,y)dy\right)\,ds \\ & + \quad \frac{1}{\varepsilon}\int_0^x\rho(0,y)\,dy, \qquad (t,x)\in I\!\!R_t\times]0,L[\end{aligned}$$

$$
\begin{aligned}
E_y^1(t,x) &= \frac{1}{2}(h_0(t-x/c)+h_L(t-(L-x)/c)) \\
&- \frac{1}{2\varepsilon}\int_{t-x/c}^{t} j_{y,\alpha}(s,x-c(t-s))\,ds \\
&- \frac{1}{2\varepsilon}\int_{t-(L-x)/c}^{t} j_{y,\alpha}(s,x+c(t-s))\,ds, \qquad (t,x)\in I\!\!R_t\times]0,L[\\
B_z^1(t,x) &= \frac{1}{2c}(h_0(t-x/c)-h_L(t-(L-x)/c)) \\
&- \frac{1}{2c\varepsilon}\int_{t-x/c}^{t} j_{y,\alpha}(s,x-c(t-s))\,ds \\
&+ \frac{1}{2c\varepsilon}\int_{t-(L-x)/c}^{t} j_{y,\alpha}(s,x+c(t-s))\,ds, \qquad (t,x)\in I\!\!R_t\times]0,L[
\end{aligned} \tag{69}
$$

and $j_{x,\alpha}=\int_v v_x f_\alpha\ dv$, $j_{y,\alpha}=\int_v v_y f_\alpha\ dv$ and f_α is the mild periodic solution for the following modified Vlasov problem:

$$
\begin{aligned}
\alpha f_\alpha+\partial_t f_\alpha+v_x\cdot\partial_x f_\alpha &+ \frac{q}{m}(R_\alpha E_x+v_y\cdot R_\alpha B_z)\cdot\partial_{v_x} f_\alpha \\
&+ \frac{q}{m}(R_\alpha E_y-v_x\cdot R_\alpha B_z)\cdot\partial_{v_y} f_\alpha=0, \\
&\quad (t,x,v_x,v_y)\in I\!\!R_t\times]0,L[\times I\!\!R_v^2
\end{aligned} \tag{70}
$$

$$
f_\alpha(t,0,v_x,v_y)=g_0(t,v_x,v_y), \qquad t\in I\!\!R_t\ v_x>0, v_y\in I\!\!R_v, \tag{71}
$$

$$
f_\alpha(t,L,v_x,v_y)=g_L(t,v_x,v_y), \qquad t\in I\!\!R_t\ v_x<0, v_y\in I\!\!R_v. \tag{72}
$$

The term $\alpha\cdot f_\alpha$ changes the formulas (29) and (30) in the following way:

$$
< f_\alpha,\varphi > \quad = \quad \int_0^T dt\int_{v_x>0}\int_{v_y} dv\int_t^{\tau_{o,\alpha}} v_x\cdot g_0(t,v_x,v_y)e^{-\alpha(t-s)}
$$

$$\cdot\ \varphi(s, X_\alpha(s;0,v_x,v_y,t), V_{x,\alpha}(s;0,v_x,v_y,t), V_{y,\alpha}(s;0,v_x,v_y,t)ds$$

$$- \quad \int_0^T dt \int_{v_x<0}\int_{v_y} dv \int_t^{\tau_{o,\alpha}} v_x \cdot g_L(t,v_x,v_y)e^{-\alpha(t-s)}$$

$$\cdot\ \varphi(s, X_\alpha(s;L,v_x,v_y,t), V_{x,\alpha}(s;L,v_x,v_y,t), V_{y,\alpha}(s;L,v_x,v_y,t)ds, \tag{73}$$

$$f_\alpha(t,x,v_x,v_y) = \begin{cases} g_0(\tau_i^\alpha, V_{x,\alpha}(\tau_i^\alpha; x,\ v_x,v_y,t), V_{y,\alpha}(\tau_i^\alpha;x,v_x,v_y,t))e^{-\alpha(t-\tau_i^\alpha)} \\ \qquad\qquad \text{if } \tau_i^\alpha > -\infty \text{ and } X_\alpha(\tau_i^\alpha;x,v_x,v_y,t) = 0, \\ g_L(\tau_i^\alpha, V_{x,\alpha}(\tau_i^\alpha; x,\ v_x,v_y,t), V_{y,\alpha}(\tau_i^\alpha;x,v_x,v_y,t))e^{-\alpha(t-\tau_i^\alpha)} \\ \qquad\qquad \text{if } \tau_i^\alpha > -\infty \text{ and } X_\alpha(\tau_i^\alpha;x,v_x,v_y,t) = L, \\ 0. \qquad\qquad \text{otherwise.} \end{cases} \tag{74}$$

Moreover, for the modified Vlasov problem, the law for the conservation of the total mass, obtained by multiplying and integrating over all $(v_x, v_y) \in I\!R_v^2$, produces:

$$\alpha\rho_\alpha + \partial_t\rho_\alpha + \partial_x j_{x,\alpha} = 0,\ (t,x) \in I\!R_t \times]0,L[,$$

or:

$$\partial_t\rho_\alpha + \partial_x(j_{x,\alpha} + \alpha\int \rho_\alpha dx) = 0,\ (t,x) \in I\!R_t \times]0,L[. \tag{75}$$

Now, if we want to preserve the divergence equation, it is clear that we have to add the extra term $\alpha\int\rho_\alpha dx$ in the definition of E_x^1 of (69). In order to assure the time periodicity for (E_x^1, E_y^1, B_z^1), a supplementarily condition will be assumed.

Proposition 1 *We assume that the following condition holds for $t \in I\!R_t$:*

$$\int_0^T dt \int_{v_x>0}\int_{v_y} v_x g_0(t,v_x,v_y)dv + \int_0^T dt \int_{v_x<0}\int_{v_y} v_x g_L(t,v_x,v_y)dv = 0 \tag{76}$$

Then, (E_x^1, E_y^1, B_z^1) given by (69) are T periodic and verify the Maxwell equations (58), (59) and the boundary conditions (62) and (63).

Proof Using *Remark 3*, we deduce that the mild solution of the modified Vlasov problem is T periodic too. Now it is easy to check that E_y^1 and B_z^1 given by (69) are T periodic and verify the Maxwell equations (58), (59) and the boundary conditions (62), (63). In order to prove the periodicity of E_x, we use the continuity equation (75) for problem (70) whose solution is split in f_0 and f_L as in *Remark 5*. By integration on $[0, T]$ we deduce:

$$\partial_x \left(\int_0^T \left(\alpha \int_0^x \rho_0(t,y)dy + j_{x,0}(t,x) \right) dt \right) = 0, \tag{77}$$

and therefore:

$$\begin{aligned} \alpha \int_0^T dt \int_0^x \rho_0(t,y)dy + \int_0^T j_{x,0}(t,x)dt &= \int_0^T j_{x,0}(t,0)dt \\ &= \int_0^T dt \int_{v_x>0} \int_{v_y} v_x g_0(t,v_x,v_y)dv, \end{aligned} \tag{78}$$

where we have used (54). In the same way we obtain:

$$\partial_x \left(\int_0^T \left(-\alpha \int_x^L \rho_L(t,y)dy + j_{x,L}(t,x) \right) dt \right) = 0, \tag{79}$$

and:

$$\begin{aligned} -\alpha \int_0^T dt \int_x^L \rho_L(t,y)dy + \int_0^T j_{x,L}(t,x)dt &= \int_0^T j_{x,L}(t,L)dt \\ &= \int_0^T dt \int_{v_x<0} \int_{v_y} v_x g_L(t,v_x,v_y)dv. \end{aligned} \tag{80}$$

Now, using (78), (80) and (76) we deduce:

$$\int_0^T \left(j_{x,\alpha}(t,y) + \alpha \int_0^x \rho_0(t,y)dy - \alpha \int_x^L \rho_L(t,y)dy \right) dt = 0, \tag{81}$$

and so E_x^1 given by (69) is also T periodic.

Remark 6 *The electric field verifies the divergence equation:*

$$\partial_x E_x^1 = \frac{1}{\varepsilon}\rho(t,x), \qquad (t,x) \in I\!R_t \times]0, L[\tag{82}$$

and the modified Maxwell equation:

$$\partial_t E_x = -\frac{1}{\varepsilon} j_{x,\alpha} - \frac{\alpha}{\varepsilon}\int_0^x \rho_{0,\alpha}(t,y)dy \quad + \quad \frac{\alpha}{\varepsilon}\int_x^L \rho_{L,\alpha}(t,y)dy, \qquad (t,x) \in I\!R_t \times]0,L[\tag{83}$$

Proof From (57) we have:

$$\begin{aligned}
\partial_x E_x^1 &= -\frac{1}{\varepsilon}\int_0^t \left(\partial_x j_{x,\alpha}(s,x) + \alpha\rho_0(s,x) + \alpha\rho_L(s,x)\right) ds + \frac{1}{\varepsilon}\rho(0,x) \\
&= -\frac{1}{\varepsilon}\int_0^t \left(\partial_x j_{x,\alpha}(s,x) + \alpha\rho_\alpha(s,x)\right) ds + \frac{1}{\varepsilon}\rho(0,x) \\
&= \frac{1}{\varepsilon}\int_0^t \partial_t \rho_\alpha(s,x) ds + \frac{1}{\varepsilon}\rho(0,x) \\
&= \frac{1}{\varepsilon}\rho(t,x).
\end{aligned}$$

The second formula can be easily checked using (57). We prove that the application F maps $\mathcal{X}$ into itself and is continuous in $L^2(I\!R_t \times]0,L[)$ in respect with the weak topology.

Lemma 2 *We assume (33), (76), that the constant K which defines the set $\mathcal{X}$ verifies:*

$$K + \frac{K}{c}\cdot\left(v_1 + \frac{q}{m}\cdot K \cdot \frac{2L}{v_0}\right) \le \frac{mv_0^2}{4qL} \tag{84}$$

Then if $g_0,\ g_L,\ h_0,\ h_L$ satisfy

$$\frac{1}{\varepsilon}\cdot\frac{\pi}{2}\cdot q(\ \|g_0\|_{L^\infty} + \|g_L\|_{L^\infty})\cdot(v_1+v_0/2)^2(\ T(v_1+v_0/2) + \alpha LT + L) \le \frac{K}{\sqrt{2}} \tag{85}$$

$$\frac{1}{2\varepsilon}\cdot\frac{L}{c}\cdot\frac{\pi}{2}\cdot q(\|g_0\|_{L^\infty} + \|g_L\|_{L^\infty})\cdot(v_1+v_0/2)^3 + \frac{1}{2}(\|h_0\|_{L^\infty} + \|h_L\|_{L^\infty}) \le \frac{K}{\sqrt{2}} \tag{86}$$

the set $\mathcal{X}$ is invariant by the application F ($F(\mathcal{X}) \subset \mathcal{X}$).

Proof From *Corollary 2* applied to the regularized field (66) we obtain the following estimates:

$$\begin{aligned}\|E_x^1\|_{L^\infty} &\leq \frac{1}{\varepsilon}\cdot\frac{\pi}{2}\cdot q(\ \|g_0\|_{L^\infty}+\|g_L\|_{L^\infty})\cdot(v_1+v_0/2)^2\\ &\cdot \quad (\ T(v_1+v_0/2)+\alpha LT+L)\leq\frac{K}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\|E_y^1\|_{L^\infty} &\leq \frac{1}{2\varepsilon}\cdot\frac{L}{c}\cdot\frac{\pi}{2}\cdot q(\ \|g_0\|_{L^\infty}+\|g_L\|_{L^\infty})\cdot(v_1+v_0/2)^3\\ &+ \frac{1}{2}(\ \|h_0\|_{L^\infty}+\|h_L\|_{L^\infty})\leq\frac{K}{\sqrt{2}}\end{aligned}$$

Therefore, we have:

$$\|E^1\|_{L^\infty}=\left\|\sqrt{|E_x^1|^2+|E_y^1|^2}\right\|_{L^\infty}\leq K$$

$$\begin{aligned}c\cdot\|B_z^1\|_{L^\infty} &\leq \frac{1}{2\varepsilon}\cdot\frac{L}{c}\cdot\frac{\pi}{2}\cdot q(\ \|g_0\|_{L^\infty}+\|g_L\|_{L^\infty})\cdot(v_1+v_0/2)^3\\ &+ \frac{1}{2}(\ \|h_0\|_{L^\infty}+\|h_L\|_{L^\infty})\leq K\end{aligned}$$

Moreover, using *Proposition 1* we deduce that $F(E_x,E_y,B_z)$ is also T periodic, so $F(\mathcal{X})\subset\mathcal{X}$.
For the proof of the continuity we need the following *Lemma* concerning the uniqueness of weak solution for the modified Vlasov equation:

Lemma 3 *Let* $(E_x,E_y,B_z)\in(L^\infty(I\!R_t;W^{1,\infty}(]0,L[)))^3$ *and* $g_0,g_L\in L^\infty(I\!R_t\times\Sigma^-)$ *be* T *periodic functions which verify (31), (33). Then a weak periodic solution in* $L^\infty(I\!R_t\times]0,L[\times I\!R_v^2)$ *of the modified Vlasov equation (70) is unique and therefore is the mild solution given by (73).*

Proof Assume that f_α is a solution in $L^\infty(I\!R_t\times]0,L[\times I\!R_v^2)$ with $g_0=0$ and $g_L=0$. We have :

$$\begin{aligned}\partial_t f_\alpha &+ v_x\cdot\partial_x f_\alpha+\frac{q}{m}(R_\alpha E_x+v_y\cdot R_\alpha B_z)\cdot\partial_{v_x}f_\alpha\\ &+ \frac{q}{m}(R_\alpha E_y-v_x\cdot R_\alpha B_z)\cdot\partial_{v_y}f_\alpha=-\alpha f_\alpha\in L^\infty(I\!R_t\times]0,L[\times I\!R_v^2),\end{aligned}$$

and therefore(cf. [3], [11]) we obtain:

$$
\begin{aligned}
-\alpha \cdot f_\alpha^2 = f_\alpha(\partial_t f_\alpha &+ v_x \cdot \partial_x f_\alpha + \frac{q}{m}(R_\alpha E_x + v_y \cdot R_\alpha B_z) \cdot \partial_{v_x} f_\alpha \\
&+ \frac{q}{m}(R_\alpha E_y - v_x \cdot R_\alpha B_z) \cdot \partial_{v_y} f_\alpha) \\
&= \frac{1}{2}(\partial_t f_\alpha^2 + v_x \cdot \partial_x f_\alpha^2 + \frac{q}{m}(R_\alpha E_x + v_y \cdot R_\alpha B_z) \cdot \partial_{v_x} f_\alpha^2 \\
&+ \frac{q}{m}(R_\alpha E_y - v_x \cdot R_\alpha B_z) \cdot \partial_{v_y} f_\alpha^2).
\end{aligned}
$$

Integrating this relation on $]0,T[\times]0,L[\times \mathbb{R}_v^2$ gives:

$$
\begin{aligned}
\alpha \int_0^T \int_0^L \int_{\mathbb{R}_v^2} f_\alpha^2 dv dx dt = &- \frac{1}{2}\int_0^T \int_{v_x>0} \int_{v_y} v_x f_\alpha^2(t, L, v_x, v_y) dv dt \\
&+ \frac{1}{2}\int_0^T \int_{v_x<0} \int_{v_y} v_x f_\alpha^2(t, 0, v_x, v_y) dv dt \le 0.
\end{aligned}
$$

Now we can prove the continuity of the application F. We have the following proposition:

Proposition 2 *Let $g_0, g_L, h_0, h_L \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ be T periodic functions and v_0, v_1, K constants which verify (33), (84) and (76). Then the application F is continuous with respect to the weak topology of $L_T^2(\mathbb{R}_t \times]0,L[)^3$.*

Proof. Let $(E_x^n, E_y^n, B_z^n)_{n\ge 1} \subset \mathcal{X}$ such as:

$$
(E_x^n, E_y^n, B_z^n) \rightharpoonup (E_x, E_y, B_z), \quad \text{weak in}(L_T^2)^3 \tag{87}
$$

For the regularized field we have the pointwise convergence :

$$
(R_\alpha E_x^n, R_\alpha E_y^n, R_\alpha B_z^n)(t,x) \to (R_\alpha E_x, R_\alpha E_y, R_\alpha B_z)(t,x), \quad \forall (t,x) \in [0,T] \times [0,L],
$$

and therefore, by the dominate convergence theorem we obtain:

$$
(R_\alpha E_x^n, R_\alpha E_y^n, R_\alpha B_z^n) \to (R_\alpha E_x, R_\alpha E_y, R_\alpha B_z), \quad \text{strong in}(L_T^2)^3 \tag{88}
$$

Denote by f^n, f the mild solution given by (73) associated to the field $(R_\alpha E_x^n$, $R_\alpha E_y^n$, $R_\alpha B_z^n)$ and $(R_\alpha E_x, R_\alpha E_y, R_\alpha B_z)$. We recall that g_0, g_L are bounded in L^∞, and therefore, $(f^n)_{n\geq 1}$ is uniformly bounded in $L^\infty(\mathbb{R}_t \times]0, L[\times \mathbb{R}_{v^2})$. After extracting a subsequence if necessary, we have:

$$f^n \rightharpoonup \tilde{f}, \text{ weak } \star \text{ in } L^\infty. \tag{89}$$

Moreover, because f^n have uniformly bounded support in v, we deduce that:

$$\rho^n := q \int_{\mathbb{R}_{v^2}} f^n \, dv \rightharpoonup \tilde{\rho} := q \int_{\mathbb{R}_{v^2}} \tilde{f} \, dv \text{ weak } \star \text{ in } L^\infty, \tag{90}$$

and:

$$j^n_{x,y} := q \int_{\mathbb{R}_{v^2}} v_{x,y} f^n \, dv \rightharpoonup \tilde{j}_{x,y} := q \int_{\mathbb{R}_{v^2}} v_{x,y} \tilde{f} \, dv \text{ weak } \star \text{ in } L^\infty. \tag{91}$$

Now we can prove that $\tilde{f}$ is the mild solution of the modified Vlasov problem corresponding to the field $(R_\alpha E_x, R_\alpha E_y, R_\alpha B_z)$. Because f^n is the mild solution, it is also a weak solution:

$$\begin{aligned}\int_0^T\int_0^L\int_{\mathbb{R}^2_v}(-\alpha\cdot\theta + \partial_t\theta + v_x\cdot\partial_x\theta \quad &+ \quad \frac{q}{m}(R_\alpha E^n_x + v_y\cdot R_\alpha B^n_z)\cdot\partial_{v_x}\theta \\ &+ \quad \frac{q}{m}(R_\alpha E^n_y - v_x\cdot R_\alpha B^n_z)\cdot\partial_{v_y}\theta) f^n dvdxdt \\ &= \quad \int_0^T\int_{v_x<0}\int_{v_y} v_x\cdot g_L\theta(t,L,v_x,v_y)dvdt \\ &- \quad \int_0^T\int_{v_x>0}\int_{v_y} v_x\cdot g_0\theta(t,0,v_x,v_y)dvdt\end{aligned} \tag{92}$$

for all T periodic function $\theta \in \mathcal{V}$. We have:

$$\begin{aligned}\lim_{n\to\infty} \quad & \int_0^T\int_0^L\int_{\mathbb{R}^2_v} f^n\cdot(-\alpha\cdot\theta + \partial_t\theta + v_x\cdot\partial_x\theta)\, dvdxdt \\ = \quad & \int_0^T\int_0^L\int_{\mathbb{R}^2_v} \tilde{f}\cdot \quad (-\alpha\cdot\theta + \partial_t\theta + v_x\cdot\partial_x\theta)\, dvdxdt\end{aligned} \tag{93}$$

For the other terms we remark that $\int_{\mathbb{R}^2_v}\partial_{v_x}\theta\cdot f^n dv$ and $\int_{\mathbb{R}^2_v}\partial_{v_y}\theta\cdot f^n dv$ converge in L^2_T weak. Therefore using (88) we get:

$$\lim_{n\to\infty}\int_0^T\int_0^L\int_{\mathbb{R}^2_v} \quad \frac{q}{m}(R_\alpha E^n_x + v_y\cdot R_\alpha B^n_z)\partial_{v_x}\theta\cdot f^n \, dvdxdt$$

$$
\begin{aligned}
&= \lim_{n\to\infty} < \frac{q}{m} R_\alpha E_x^n, \int_{I\!R_v^2} \partial_{v_x}\theta \cdot f^n \, dv >_{L_T^2} \\
&+ \lim_{n\to\infty} < \frac{q}{m} R_\alpha B_z^n, \int_{I\!R_v^2} v_y \cdot \partial_{v_x}\theta \cdot f^n \, dv >_{L_T^2} \\
&= < \frac{q}{m} R_\alpha E_x, \int_{I\!R_v^2} \partial_{v_x}\theta \cdot \tilde{f} \, dv >_{L_T^2} \\
&+ < \frac{q}{m} R_\alpha B_z, \int_{I\!R_v^2} v_y \cdot \partial_{v_x}\theta \cdot \tilde{f} \, dv >_{L_T^2} \\
&= \int_0^T \int_0^L \int_{I\!R_v^2} \frac{q}{m} (R_\alpha E_x + v_y \cdot R_\alpha B_z) \partial_{v_x}\theta \cdot \tilde{f} \, dv dx dt \quad (94)
\end{aligned}
$$

Therefore, $\tilde{f}$ is a weak solution for the modified Vlasov problem corresponding to the field $(R_\alpha E_x, R_\alpha E_y, R_\alpha B_z)$:

$$
\begin{aligned}
\int_0^T\int_0^L\int_{I\!R_v^2} (-\alpha\cdot\theta + \partial_t\theta + v_x\cdot\partial_x\theta &+ \frac{q}{m}(R_\alpha E_x + v_y \cdot R_\alpha B_z)\cdot\partial_{v_x}\theta \\
&+ \frac{q}{m}(R_\alpha E_y - v_x \cdot R_\alpha B_z)\cdot\partial_{v_y}\theta)\tilde{f} dv dx dt \\
&= \int_0^T \int_{v_x<0}\int_{v_y} v_x \cdot g_L \theta(t, L, v_x, v_y) dv dt \\
&- \int_0^T \int_{v_x>0}\int_{v_y} v_x \cdot g_0 \theta(t, 0, v_x, v_y) dv dt
\end{aligned} \quad (95)
$$

for all T periodic function $\theta \in \mathcal{V}$. But using *Lemma 3* we deduce that $\tilde{f}$ is the mild solution corresponding to the field $(R_\alpha E_x, R_\alpha E_y, R_\alpha B_z)$ (uniqueness of the weak solution for the modified Vlasov problem), so $\tilde{f} = f$ and we have:

$$
j_{x,y}^n := q \int_{I\!R_{v^2}} v_{x,y} f^n \, dv \to j_{x,y} := q \int_{I\!R_{v^2}} v_{x,y} f \, dv \quad \text{weak } \star \text{ in } L^\infty. \quad (96)
$$

Now, it is easy to check that we can pass to the limit in (69) to obtain:

$$
\lim_{n\to\infty} F(E_x^n, E_y^n, B_z^n) = F(E_x, E_y, B_z), \quad \text{weak in } L_T^2.
$$

Proposition 3 *Let $g_0, g_L, h_0, h_L \in L^\infty(I\!R_t \times \Sigma^-)$ be T periodic functions and v_0, v_1, K constants which verify (33), (84), (85), (86) and (76). Then the modified $1D$ Vlasov-Maxwell system has at least one weak periodic solution.*

Proof. It is an immediate consequence of Schauder fixed point theorem.

3.2 Weak periodic solutions for the classical 1D Vlasov-Maxwell system.

We prove the existence of periodic weak solution for the Vlasov-Maxwell system. Obviously, this result is a direct consequence of *Proposition 3*.

Theorem 1 *Let $g_0, g_L, h_0, h_L \in L^\infty(I\!R_t \times \Sigma^-)$ be T periodic functions and v_0, v_1, K constants which verify:*

$$supp(g_0) \subset \{(t,x,v_x,v_y)\ ;\ t \in I\!R_t, x = 0, 0 < v_0 \le v_x, \sqrt{v_x^2 + v_y^2} \le v_1\},$$
$$supp(g_L) \subset \{(t,x,v_x,v_y);\ t \in I\!R_t, x = L, 0 > -v_0 \ge v_x, \sqrt{v_x^2 + v_y^2} \le v_1\},$$

$$K + \frac{K}{c} \cdot \left(v_1 + \frac{q}{m} \cdot K \cdot \frac{2L}{v_0}\right) \le \frac{mv_0^2}{4qL},$$

$$\frac{1}{\varepsilon} \cdot \frac{\pi}{2} \cdot q(\ \|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}) \cdot (v_1 + v_0/2)^2(\ T(v_1 + v_0/2) + L) < \frac{K}{\sqrt{2}},$$

$$\frac{1}{2\varepsilon} \cdot \frac{L}{c} \cdot \frac{\pi}{2} \cdot q(\|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}) \cdot (v_1 + v_0/2)^3 + \frac{1}{2}(\|h_0\|_{L^\infty} + \|h_L\|_{L^\infty}) \le \frac{K}{\sqrt{2}},$$

and:

$$\int_0^T dt \int_{v_x > 0} \int_{v_y} v_x g_0(t, v_x, v_y) dv + \int_0^T dt \int_{v_x < 0} \int_{v_y} v_x g_L(t, v_x, v_y) dv = 0.$$

Then the classical 1D Vlasov-Maxwell system has at least one weak periodic solution.

Proof. Let $(\alpha_n)_{n \ge 1}$ be a sequence of positive numbers, whose limit is 0. We observe that for α_n small we have:

$$\frac{1}{\varepsilon} \cdot \frac{\pi}{2} \cdot q(\ \|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}) \cdot (v_1 + v_0/2)^2(\ T(v_1 + v_0/2) + \alpha_n LT + L) \le \frac{K}{\sqrt{2}}$$

Therefore, by *Proposition 3*, there is $(f^n, E_x^n, E_y^n, B_z^n)$ weak periodic solutions for the α_n regularized Vlasov-Maxwell system:

$$\begin{aligned}\alpha_n \cdot f^n + \partial_t f^n + v_x \cdot \partial_x f^n \quad &+ \quad \frac{q}{m}(R_{\alpha_n} E_x^n + v_y \cdot R_{\alpha_n} B_z^n) \cdot \partial_{v_x} f^n \\ &+ \quad \frac{q}{m}(R_{\alpha_n} E_y^n - v_x \cdot R_{\alpha_n} B_z^n) \cdot \partial_{v_y} f^n = 0, \\ &\quad (t,x,v_x,v_y) \in I\!R_t \times]0, L[\times I\!R_v^2,\end{aligned}$$

$$\partial_t E_x^n = -\frac{1}{\varepsilon} j_x^n - \frac{\alpha_n}{\varepsilon}\int_0^x \rho_0^n(t,y)dy + \frac{\alpha_n}{\varepsilon}\int_x^L \rho_L^n(t,y)dy, \qquad (t,x) \in \mathbb{R}_t \times]0,L[,$$

$$\partial_t E_y^n + c^2 \partial_x B_z^n = -\frac{1}{\varepsilon} j_y^n := -\frac{1}{\varepsilon}\int_v v_y f^n(t,x,v_x,v_y)\, dv, \qquad (t,x) \in \mathbb{R}_t \times]0,L[,$$

$$\partial_t B_z^n + \partial_x E_y^n = 0, \qquad (t,x) \in \mathbb{R}_t \times]0,L[,$$

with the boundary conditions:

$$f^n(t,0,v_x,v_y) = g_0(t,v_x,v_y), \qquad t \in \mathbb{R}_t\ v_x > 0, v_y \in \mathbb{R}_v,$$

$$f^n(t,L,v_x,v_y) = g_L(t,v_x,v_y), \qquad t \in \mathbb{R}_t\ v_x < 0, v_y \in \mathbb{R}_v,$$

$$E_y^n(t,0) + cB_z^n(t,0) = h_0(t), \qquad t \in \mathbb{R}_t,$$

$$E_y^n(t,L) - cB_z^n(t,L) = h_L(t), \qquad t \in \mathbb{R}_t,$$

After extracting subsequence, we have the convergence:

$$(E_x^n, E_y^n, B_z^n) \rightharpoonup (E_x, E_y, B_z), \text{ weak in } (L_T^2)^3,$$

and:

$$f^n \rightharpoonup f, \text{ weak } \star \text{ in } L^\infty.$$

Moreover, by regularization with $\alpha_n \to 0$, the first convergences are preserved:

$$\begin{aligned} | < R_{\alpha_n} E_x^n, \eta >_{L_T^2} - < E_x, \eta >_{L_T^2} | &= | < E_x^n, R_{\alpha_n}\eta > - < E_x, \eta > | \\ &= | < E_x^n, R_{\alpha_n}\eta - \eta > + < E_x^n - E_x, \eta > | \\ &\leq | < E_x^n - E_x, \eta > | \\ &+ \|E_x^n\| \cdot \|R_{\alpha_n}\eta - \eta\| \to 0, \end{aligned}$$

and so:

$$(R_{\alpha_n}E_x^n, R_{\alpha_n}E_y^n, R_{\alpha_n}B_z^n) \rightharpoonup (E_x, E_y, B_z), \text{ weak in } (L_T^2)^3.$$

Because f^n have uniformly bounded support in v, we deduce that:

$$\rho^n := q\int_{I\!R_{v^2}} f^n\, dv \rightharpoonup \rho := q\int_{I\!R_{v^2}} f\, dv \text{ weak } \star \text{ in } L^\infty,$$

and:

$$j_{x,y}^n := q\int_{I\!R_{v^2}} v_{x,y} f^n\, dv \rightharpoonup j_{x,y} := q\int_{I\!R_{v^2}} v_{x,y} f\, dv \text{ weak } \star \text{ in } L^\infty.$$

The velocity average lemma of DiPerna and Lions [10] allows us to write:

$$\rho^n := q\int_{I\!R_{v^2}} f^n\, dv \to \rho := q\int_{I\!R_{v^2}} f\, dv \text{ in } L_T^2, \tag{97}$$

and:

$$j_{x,y}^n := q\int_{I\!R_{v^2}} v_{x,y} f^n\, dv \to j_{x,y} := q\int_{I\!R_{v^2}} v_{x,y} f\, dv \text{ in } L_T^2. \tag{98}$$

Moreover, we have:

$$\int_{I\!R_{v^2}} \psi(v_x, v_y) f^n\, dv \to \int_{I\!R_{v^2}} \psi(v_x, v_y) f\, dv \text{ in } L_T^2, \tag{99}$$

for all continuous function $\psi \in C(I\!R_{v^2})$. We prove now that f is a weak solution for the Vlasov problem corresponding to the field (E_x, E_y, B_z). By a simple density argument, it is sufficient to consider test functions with a product structure (see [10]) :

$$\theta(t, x, v_x, v_y) = \varphi(t, x) \cdot \psi(v_x, v_y).$$

We have :

$$\begin{aligned}
\lim_{n\to\infty} \int_0^T \int_0^L \int_{I\!R_v^2} & \frac{q}{m}(R_\alpha E_x^n + v_y \cdot R_\alpha B_z^n)\partial_{v_x}\theta \cdot f^n\, dvdxdt \\
&= \lim_{n\to\infty} < \frac{q}{m} R_\alpha E_x^n, \varphi(t,x) \int_{I\!R_v^2} \partial_{v_x}\psi \cdot f^n\, dv >_{L_T^2} \\
&+ \lim_{n\to\infty} < \frac{q}{m} R_\alpha B_z^n, \varphi(t,x) \int_{I\!R_v^2} v_y \cdot \partial_{v_x}\psi \cdot f^n\, dv >_{L_T^2} \\
&= < \frac{q}{m} R_\alpha E_x, \varphi(t,x) \int_{I\!R_v^2} \partial_{v_x}\psi \cdot f\, dv >_{L_T^2} \\
&+ < \frac{q}{m} R_\alpha B_z, \varphi(t,x) \int_{I\!R_v^2} v_y \cdot \partial_{v_x}\psi \cdot f\, dv >_{L_T^2} \\
&= \int_0^T \int_0^L \int_{I\!R_v^2} \frac{q}{m}(R_\alpha E_x + v_y \cdot R_\alpha B_z)\partial_{v_x}\theta \cdot f\, dvdxdt
\end{aligned} \tag{100}$$

In addition we have :

$$\lim_{n\to\infty} \alpha_n \int_0^T \int_0^L \int_{\mathbb{R}_v^2} f^n \cdot \theta \, dvdxdt = 0.$$

for all T periodic function $\theta \in \mathcal{V}$. Furthermore, passing to the limit for $n \to \infty$ in (69) and using (97) and (98), we deduce the following equalities in L_T^2:

$$\begin{aligned}
E_x(t,x) = & - \frac{1}{\varepsilon}\int_0^t j_x(s,x)\, ds \\
& + \frac{1}{\varepsilon}\int_0^x \rho(0,y)\, dy, \qquad (t,x) \in \mathbb{R}_t \times]0,L[\\
E_y(t,x) & = \frac{1}{2}(h_0(t-x/c) + h_L(t-(L-x)/c)) \\
& - \frac{1}{2\varepsilon}\int_{t-x/c}^t j_y(s, x-c(t-s))\, ds \\
& - \frac{1}{2\varepsilon}\int_{t-(L-x)/c}^t j_y(s, x+c(t-s))\, ds, \qquad (t,x) \in \mathbb{R}_t \times]0,L[\\
B_z(t,x) & = \frac{1}{2c}(h_0(t-x/c) - h_L(t-(L-x)/c)) \\
& - \frac{1}{2c\varepsilon}\int_{t-x/c}^t j_y(s, x-c(t-s))\, ds \\
& + \frac{1}{2c\varepsilon}\int_{t-(L-x)/c}^t j_y(s, x+c(t-s))\, ds, \qquad (t,x) \in \mathbb{R}_t \times]0,L[
\end{aligned}$$

and so the field (E_x, E_y, B_z) verifies the Maxwell equations.

4 Weak periodic solutions for the relativistic 1D Vlasov-Maxwell system.

Our arguments apply also to the relativistic $1D$ Vlasov-Maxwell system:

$$\begin{aligned}
\partial_t f + V_x(p)\cdot\partial_x f + q(E_x + V_y(p)\cdot B_z)\cdot\partial_{p_x} f & + q(E_y - V_x(p)\cdot B_z)\cdot\partial_{p_y} f = 0, \\
(t,x,p_x,p_y) & \in \mathbb{R}_t \times]0,L[\times \mathbb{R}_p^2, \qquad (101)
\end{aligned}$$

$$\partial_t E_x = -\frac{1}{\varepsilon} j_x := -\frac{1}{\varepsilon}\int_p V_x(p) f(t,x,p_x,p_y)\, dp, \qquad (t,x) \in I\!R_t \times]0,L[, \tag{102}$$

$$\partial_t E_y + c^2 \partial_x B_z = -\frac{1}{\varepsilon} j_y := -\frac{1}{\varepsilon}\int_p V_y(p) f(t,x,p_x,p_y)\, dp, \qquad (t,x) \in I\!R_t \times]0,L[, \tag{103}$$

$$\partial_t B_z + \partial_x E_y = 0, \qquad (t,x) \in I\!R_t \times]0,L[, \tag{104}$$

with the boundary conditions:

$$f(t,0,p_x,p_y) = g_0(t,p_x,p_y), \qquad t \in I\!R_t\ p_x > 0, p_y \in I\!R_p, \tag{105}$$

$$f(t,L,p_x,p_y) = g_L(t,p_x,p_y), \qquad t \in I\!R_t\ p_x < 0, p_y \in I\!R_p, \tag{106}$$

$$E_y(t,0) + cB_z(t,0) = h_0(t), \qquad t \in I\!R_t, \tag{107}$$

$$E_y(t,L) - cB_z(t,L) = h_L(t), \qquad t \in I\!R_t, \tag{108}$$

where g_0, g_L, h_0, h_L are T periodic functions and the velocity $V(p)$ is given by:

$$V(p) = (V_x(p), V_y(p)) = c \cdot \frac{(p_x,p_y)}{\sqrt{m^2c^2 + \|p\|^2}}, \qquad (p_x,p_y) \in I\!R_{p^2}. \tag{109}$$

We only have to modify the preceeding proofs slightly. First, we observe that the quadratic nonlinear term $(E(t,x) + V(p) \wedge B(t,x)) \cdot \nabla_p f$ may be recast as an exact derivation:

$$(E(t,x) + V(p) \wedge B(t,x)) \cdot \nabla_p f = \nabla_p \cdot \{(E(t,x) + V(p) \wedge B(t,x)) \cdot f\}.$$

Definition 3 *Let $E_x, E_y, B_z \in L^\infty(I\!R_t \times]0,L[)$ and $g_0, g_L \in L^1_{loc}(I\!R_t \times \Sigma^-)$ be T periodic functions in time, where:*

$$\begin{aligned} \Sigma^- &= \{(t,x,p_x,p_y) \mid t \in I\!R, x = 0, p_x > 0, p_y \in I\!R\} \\ &\cup \{(t,x,p_x,p_y) \mid t \in I\!R, x = L, p_x < 0, p_y \in I\!R\}. \end{aligned} \tag{110}$$

We say that $f \in L^1_{loc}(I\!R_t \times]0,L[\times I\!R^2_p)$ is a T periodic weak solution of problem (101), (105), (106) iff:

$$\begin{aligned} \int_0^T\int_0^L\int_{I\!R^2_p}(\partial_t\theta + V_x(p)\cdot\partial_x\theta &+ q(E_x + V_y(p)\cdot B_z)\cdot\partial_{p_x}\theta \\ &+ q(E_y - V_x(p)\cdot B_z)\cdot\partial_{p_y}\theta)f(t,x,p_x,p_y)dpdxdt \\ &= \int_0^T\int_{p_x<0}\int_{p_y} V_x(p)g_L(t,p_x,p_y)\theta(t,L,p_x,p_y)dpdt \\ &- \int_0^T\int_{p_x>0}\int_{p_y} V_x(p)g_0(t,p_x,p_y)\theta(t,0,p_x,p_y)dpdt \end{aligned} \tag{111}$$

for all T periodic function $\theta \in \mathcal{V}$, where:

$$\mathcal{V}=\{\eta \in W^{1,\infty}(I\!R_t \times]0,L[\times I\!R^2_p) \ ; \ \eta(t,0,p_x<0,p_y) = \eta(t,L,p_x>0,p_y) = 0, \quad supp(\eta) \text{ bounded set of } I\!R_t \times [0,L] \times I\!R^2_p\}$$

Definition 4 *Let $E_x, E_y, B_z \in L^\infty(I\!R_t; W^{1,\infty}(]0,L[))$ and $g_0, g_L \in L^1_{loc}(I\!R_t \times \Sigma^-)$ be T periodic functions. The function $f \in L^1_{loc}(I\!R_t \times]0,L[\times I\!R^2_p)$ which is the mild periodic solution of problem (101), (105), (106) is given by (112):*

$$\begin{aligned} < f,\varphi > \quad &= \int_0^T dt \int_{p_x>0}\int_{p_y} dp \int_t^{\tau_o} V_x(p)\cdot g_0(t,p_x,p_y) \\ &\cdot \quad \varphi(s, X(s;0,p_x,p_y,t), P_x(s;0,p_x,p_y,t), P_y(s;0,p_x,p_y,t)ds \\ &- \int_0^T dt \int_{p_x<0}\int_{p_y} dp \int_t^{\tau_o} V_x(p)\cdot g_L(t,p_x,p_y) \\ &\cdot \quad \varphi(s, X(s;L,p_x,p_y,t), P_x(s;L,p_x,p_y,t), P_y(s;L,p_x,p_y,t)ds, \end{aligned} \tag{112}$$

where $(X(s), P_x(s), P_y(s))$ is the solution of the system:

$$\begin{cases} \dfrac{dX}{ds} & = & V_x(P(s;x,p_x,p_y,t)), & s \in [\tau_i, \tau_o] \\ X(t;x,p_x,p_y,t) & = & x, & \\ \dfrac{dP_x}{ds} & = & q \cdot (E_x(s,X(s)) + V_y(P(s)) \cdot B_z(s,X(s))), & s \in [\tau_i, \tau_o] \\ P_x(t;x,p_x,p_y,t) & = & p_x, & \\ \dfrac{dP_y}{ds} & = & q \cdot (E_y(s,X(s)) - V_x(P(s)) \cdot B_z(s,X(s))), & s \in [\tau_i, \tau_o] \\ P_y(t;x,p_x,p_y,t) & = & p_y. & \end{cases} \tag{113}$$

In the relativistic case, the analogue of *Lemma 1* is given by:

Lemma 4 *Assume that the electro-magnetic field and the boundary data satisfy:*

$$\|E\|_{L^\infty} + c \cdot \|B_z\|_{L^\infty} \leq \frac{m \cdot (p_0/m)^2}{4qL} \cdot \left[1 + \left(\frac{p_1 + p_0/2}{mc}\right)^2\right]^{-1/2} \tag{114}$$

$$(E_x, E_y, B_z) \in (L^\infty(I\!R_t; W^{1,\infty}(]0, L[)))^3, \tag{115}$$

$$\begin{aligned} supp(g_0) &\subset \{(t,x,p_x,p_y)\ ;\ t \in I\!R_t, x = 0, 0 < p_0 \leq p_x, \sqrt{p_x^2 + p_y^2} \leq p_1\}, \\ supp(g_L) &\subset \{(t,x,p_x,p_y);\ t \in I\!R_t, x = L, 0 > -p_0 \geq p_x, \sqrt{p_x^2 + p_y^2} \leq p_1\}. \end{aligned} \tag{116}$$

Then, the life-time in $]0, L[$ *of particles starting from the support of* g_0 *and* g_L *is finite:*

$$\begin{aligned} \tau_o(x,p_x,p_y,t) - \tau_i(x,p_x,p_y,t) \quad &\leq \quad 2 \cdot \frac{L \cdot m}{p_0} \sqrt{1 + \left(\frac{p_1 + p_0/2}{mc}\right)^2}, \\ &\forall \quad (t,x,p_x,p_y) \in supp(g_0) \cup supp(g_L). \end{aligned}$$

Corollary 2 *Assuming the same hypotheses as in Lemma 4 and let f be the mild solution of Definition 4. Then we have:*

$$supp(f) \subset \{(t,x,p_x,p_y)|t \in I\!R_t, x \in [0,L], \frac{p_0}{2} \leq |p_x|, \sqrt{p_x^2+p_y^2} \leq p_1 + \frac{p_0}{2}\} \tag{117}$$

$$\|\rho\|_{L^\infty} \leq \frac{\pi}{2}(p_1+p_0/2)^2 \cdot q \cdot (\|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}), \tag{118}$$

and

$$\max\{\|j_x\|_{L^\infty}, \|j_y\|_{L^\infty}\} \leq c \cdot \frac{\pi}{2}(p_1+p_0/2)^2 \cdot q \cdot (\|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}), \tag{119}$$

where $\rho(t,x) = q \int_{I\!R_p^2} f(t,x,p_x,p_y)\, dp$ *and* $j_{x,y}(t,x) = q \int_{I\!R_p^2} V_{x,y}(p) f(t,x,p_x,p_y)\, dp$.

Like in the classical case, we first show the existence of weak periodic solution for the regularized problem by using the Schauder fixed point theorem. Next we pass to the limit when the regularization parameter vanishes. We have the following Theorem:

Theorem 2 *Let* $g_0, g_L, h_0, h_L \in L^\infty(I\!R_t \times \Sigma^-)$ *be T periodic functions and* p_0, p_1, K *constants which verify:*

$$supp(g_0) \subset \{(t,x,p_x,p_y)\ ;\ t \in I\!R_t, x=0, 0 < p_0 \leq p_x, \sqrt{p_x^2+p_y^2} \leq p_1\},$$
$$supp(g_L) \subset \{(t,x,p_x,p_y);\ t \in I\!R_t, x=L, 0 > -p_0 \geq p_x, \sqrt{p_x^2+p_y^2} \leq p_1\},$$

$$2 \cdot K \leq \frac{m \cdot (p_0/m)^2}{4qL} \cdot \left[1 + \left(\frac{p_1+p_0/2}{mc}\right)^2\right]^{-1/2},$$

$$\frac{1}{\varepsilon} \cdot \frac{\pi}{2} \cdot q(\ \|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}) \cdot (p_1+p_0/2)^2(\ cT+L) < \frac{K}{\sqrt{2}},$$

$$\frac{L}{2\varepsilon} \cdot \frac{\pi}{2} \cdot q(\|g_0\|_{L^\infty} + \|g_L\|_{L^\infty}) \cdot (p_1+p_0/2)^2 + \frac{1}{2}(\|h_0\|_{L^\infty} + \|h_L\|_{L^\infty}) \leq \frac{K}{\sqrt{2}},$$

$$\int_0^T dt \int_{p_x>0} \int_{p_y} V_x(p) g_0(t, p_x, p_y) dp + \int_0^T dt \int_{p_x<0} \int_{p_y} V_x(p) g_L(t, p_x, p_y) dp = 0.$$

Then the relativistic 1D Vlasov-Maxwell system has at least one weak periodic solution.

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