Periodic solutions of the Vlasov-Poisson system with boundary conditions

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Abstract. *We study the Vlasov-Poisson system with time periodic boundary conditions. For small data we prove existence of weak periodic solutions in any space dimension. In the one dimensional case the result is stronger: we obtain existence of mild solution and uniqueness of this solution when the data are smooth. It is necessary to impose a non vanishing condition for the incoming velocities in order to control the life-time of particles in the domain.*

1 Introduction

The master system of equations of collision-less plasma physics is the Vlasov-Maxwell system. The main result in this field has been obtained in 1989 by R.J.DiPerna and P.L.Lions [13]. They prove existence of global weak solutions for the Cauchy problem with arbitrary data. The global existence of strong solution is still an open problem. The situation is much better for the Vlasov-Poisson system. This system is obtained for the first one by neglecting the magnetic field. This can be justified (at least for small time) by a non-relativistic limit [18]. It reads :

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_v f = 0, \qquad (t,x,v) \in \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v, \tag{1}$$

$$\Delta_x \varphi = \int_{\mathbb{R}^d_v} f(t,x,v)dv, \qquad (t,x) \in \mathbb{R}_t \times \mathbb{R}^d_x. \tag{2}$$

The variables (t, x, v) are respectively the time, the position and the velocity, d is the dimension of the space. The non-negative function f is the distribution of the charged particles and φ is the induced electrostatic potential. For the Cauchy problem weak global solution has been obtained by Arsenev [16]. Existence of strong solution in 2D is a result due to Degond [19] and Ukai Ohabe [20]. The same result in 3D has been proved by Pfaffelmoser [7]. A simpler and power-full method has been proposed by P.L.Lions B.Perthame [8]. However for applications like vacuum diodes, tube discharges, cold plasma, solar wind, satellite ionisation, thrusters, etc... boundary conditions have to be taken into account. For the transient regime global weak solutions of the Vlasov-Maxwell system has been proved to exist by Y.Guo [5] and independently by M.Bezart [15]. The same problem for the Vlasov-Poisson system has been investigated by Y.Guo [4] and N.Ben Abdallah [9]. Permanent regimes are particularly important. They are of two types and they are modeled by stationary solutions or time periodic solutions for boundary value problems. Results concerning stationary problems can be found in the paper of C.Greengard P.A.Raviart for the Vlasov-Poisson system in 1D, in [2] for any space dimension and in [3] for the Vlasov-Maxwell system. To our knowledge no results were available concerning time periodic solutions. One strong motivation to study such solutions is the great difficulty to compute it numerically. The analysis of the Vlasov-Maxwell system in dimension 2 or 3 in this context seems, up to now, out of reach. The situation is different in 1D because solutions of Maxwell system can be computed explicitly and the techniques introduced in this paper can be used, see [11]. We now describe precisely the boundary condition which we investigate. Let Ω be a C^1 bounded open set of $I\!R^d_x$ representing the device geometry. We denote by $\partial\Omega$ the boundary and by Σ^- the set of initial positions in phase space of incoming particles :

$$\Sigma^- \quad = \quad \{(x, v) \in \partial\Omega \times I\!R^d_v \mid v \cdot \nu(x) < 0\}, \tag{3}$$

where $\nu(x)$ is the outward normal of Ω at the point $x \in \partial\Omega$. The distribution g of incoming particles is prescribed :

$$f = g, \qquad (t, x, v) \;\; \in \;\; I\!R_t \times \Sigma^-. \tag{4}$$

We impose Dirichlet condition on the electrostatic potential φ :

$$\varphi = \varphi_0, \qquad (t, x) \;\; \in \;\; I\!R_t \times \partial\Omega. \tag{5}$$

The data are assumed to be T periodic and we look for T periodic solutions (f, φ) of the problem (VP) : (1), (2), (4) and (5). One of the key point of our proof of existence of such solutions is to control the life-time of particles in the domain Ω. It assures a dissipativeness property of the system. More precisely it allows to bound the concentration $\int f dv$. Therefore we impose a non-vanishing condition of incoming velocities which reads :

$$supp(g) \subset \{(t, x, v) \mid t \in I\!R_t, x \in \partial\Omega, v \cdot \nu(x) < 0, v_0 \leq |v| \leq v_1\}, \qquad (6)$$

for $0 < v_0 < v_1$ given. We point out that other conditions can lead to the same kind of result. Let Φ_0 be the harmonic extension of the Dirichlet data. A generalized condition could be : for any initial condition in the support of g the characteristics solving $\frac{dX}{dt} = V$, $\frac{dV}{dt} = \nabla_x \Phi_0 + F$ where F is small enough, have their life-time uniformly bounded. Our result can now be summarized as follows (see *Theorem 3* for precise assumptions). If (6) holds then for g and φ_0 small enough there exist at least one T periodic weak solution of the problem (VP). We can precise this result in dimension 1 and with supplementary smoothness assumption on the data we obtain a uniqueness result (*Theorem 1* and *Theorem 2*). Let us remark that even if the electric potential φ is "a priori" known, there is no uniqueness of the T periodic solution of the Vlasov problem (V) : (1) and (4). Indeed, the distribution function can take arbitrary (constant) values on the characteristics which remain in the domain (trapped characteristics). In order to select physical solution we introduce as in [2] and [3] the concept of minimal solution of (V) which are the solutions which vanish on the trapped characteristics. These solutions can be obtained as the limit of the (unique) solution of the modified Vlasov problem (V_α) when an absorption term $\alpha > 0$ is introduced and tends to zero :

$$\alpha f + \partial_t f + v \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_v f = 0, \qquad (t, x, v) \in I\!R_t \times \Omega \times I\!R^d_v. \qquad (7)$$

This limit absorption principle has been the starting point of the limit absorption method (LAM) which has been developed by the authors to obtain numerical periodic solutions of Partial Differential Equation, see [12]. We also stress that these results has been announced in [10].
The paper is organized as followed. In Section 2 we define weak solutions

and minimal mild solution of the Vlasov problem (V). We also proved that the weak solution of the modified Vlasov problem (V_α) is unique and coincide with the minimal mild solution. Section 3 is devoted to the 1 dimensional case. We prove existence of a mild minimal solution (f, φ) and its uniqueness in the case where the data are smooth. In Section 4 we introduce a regularized problem. The existence theorem is obtained by using Schauder's theorem for the modified problem. Then we pass to the limit in the regularization parameter to obtain our main result.

2 Definitions and bounds for the Vlasov equation.

In this section we assume that the electric field E is a T periodic function in time and we look for a solution f of the Vlasov equation:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 & (t,x,v) \in I\!R_t \times \Omega \times I\!R^d_v, \\ f(t,x,v) = g(t,x,v) & (t,x,v) \in I\!R_t \times \Sigma^-. \end{cases} \tag{8}$$

Moreover, we suppose that the given distribution function g of the in-flowing particles is T periodic in time, too. Now we briefly recall the notions of mild and weak solutions for this type of problem.

2.1 Weak solution of the Vlasov equation

We first introduce the spaces L^-, L^-_{loc} of incoming data with bounded or locally bounded fluxes :

$$L^- = \{g \quad | \quad v \cdot \nu(x) g \in L^1(I\!R_t \times \Sigma^-)\},$$

$$L^-_{loc} = \{g \quad | \quad v \cdot \nu(x) g \in L^1_{loc}(I\!R_t \times \overline{\Sigma^-})\},$$

where Σ^- is defined by (3).

Definition 1 *Let $E \in (L^\infty(I\!R_t \times \Omega))^d$ and $g \in L^1_{loc}(I\!R_t \times \Sigma^-)$ be T periodic functions in time. We say that $f \in L^1_{loc}(I\!R_t \times \Omega \times I\!R^d_v)$ is a T periodic weak solution of problem (8) iff:*

$$\int_0^T \int_\Omega \int_{\mathbb{R}^d_v} f(t,x,v)(\partial_t\theta + v\cdot\nabla_x\theta + E\cdot\nabla_v\theta)dvdxdt =$$
$$= \int_0^T \int_{\Sigma^-} v\cdot\nu(x)\cdot g(t,x,v)\cdot\theta(t,x,v)dvd\sigma dt \qquad (9)$$

for all T periodic function $\theta \in \mathcal{V}$, where:

$$\mathcal{V} = \{\eta \in W^{1,\infty}(\mathbb{R}_t \times \Omega \times \mathbb{R}^d_v) \mid \eta \textit{ is } T \textit{ periodic with respect to time },$$
$$\eta|_{\mathbb{R}_t\times\Sigma^+} = 0,\ \exists B\, bounded\, set\, of\, \mathbb{R}^d_v,\, supp(\eta) \subset \mathbb{R}_t \times \overline{\Omega} \times B\}.$$

In other words, a weak solution of problem (8) is a distribution function satisfying:

$$< f, \varphi > = \int_0^T \int_{\Sigma^-} v\cdot\nu(x)\cdot g(t,x,v)\cdot\theta(t,x,v)dvd\sigma dt \qquad (10)$$

for all T periodic function φ, where θ denote the solution of the problem:

$$\begin{cases} \partial_t\theta + v\cdot\nabla_x\theta + E\cdot\nabla_v\theta = \varphi, & (t,x,v) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^d_v, \\ \theta = 0, & (t,x,v) \in \mathbb{R}_t \times \Sigma^+ \end{cases} \qquad (11)$$

Remark 1 *In the above definition we can assume that the electric field is only in $(L^p(\mathbb{R}_t \times \Omega))^d$ by requiring more regularity on f (and g), namely f in $L^q_{loc}(\mathbb{R}_t \times \Omega)$ where q is the conjugate exponent.*

If the electric field satisfy $E \in (L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega)))^d$, we can express a solution in terms of characteristics. Let (t,x,v) belong to $\mathbb{R}_t \times \Omega \times \mathbb{R}^d_v$, we denote by $X(s;x,v,t), V(s;x,v,t)$ the solution of the system:

$$\begin{cases} \dfrac{dX}{ds} = V(s;x,v,t), & s \in [\tau_i, \tau_o] \\ X(t;x,v,t) = x, & \\ \dfrac{dV}{ds} = E(s, X(s;x,v,t)), & s \in [\tau_i, \tau_o] \\ V(t;x,v,t) = v. & \end{cases} \qquad (12)$$

where $\tau_i = \tau_i(x, v, t)$ ($\tau_o = \tau_o(x, v, t)$) is the incoming (resp. outgoing) time of the particle in the domain Ω:

$$(X(\tau_i), V(\tau_i)) \in \Sigma^- \tag{13}$$

and

$$(X(\tau_o), V(\tau_o)) \in \Sigma^+ \cup \Sigma^0. \tag{14}$$

The subsets of $\partial\Omega \times I\!R^d_v, \Sigma^+$ and Σ^0 are respectively defined by:

$$\begin{aligned} \Sigma^+ &= \{(x, v) \in \partial\Omega \times I\!R^d_v \mid v \cdot \nu(x) > 0\}, \\ \Sigma^0 &= \{(x, v) \in \partial\Omega \times I\!R^d_v \mid v \cdot \nu(x) = 0\}. \end{aligned}$$

Using the Cauchy-Lipschitz theorem, we notice that the characteristics are well defined. By integration along the characteristics curves, the solution of the problem (11) formally writes:

$$\theta(t, x, v) = -\int_t^{\tau_o} \varphi(s, X(s; x, v, t), V(s; x, v, t))ds \tag{15}$$

Now, always formally (10) implies that:

$$\begin{aligned} < f, \varphi > \quad &= \quad -\int_0^T dt \int_{\Sigma^-} d\sigma(x)\, dv\, v \cdot \nu(x) g(t, x, v) \\ & \qquad \int_t^{\tau_o(x,v,t)} \varphi(s, X(s; x, v, t), V(s; x, v, t))ds. \end{aligned} \tag{16}$$

which is equivalent to:

$$f(t, x, v) = \begin{cases} g(\tau_i, X(\tau_i; x, v, t), V(\tau_i; x, v, t)), & \text{if} \quad \tau_i > -\infty, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Definition 2 *Let $E \in (L^\infty(I\!R_t; W^{1,\infty}(\Omega)))^d$ and $g \in L^1_{loc}(I\!R_t \times \Sigma^-)$ be T periodic functions. The function $f \in L^1_{loc}(I\!R_t \times \Omega \times I\!R^d_v)$ which is the mild minimal periodic solution of problem (8) is given by (16).*

Remark 2 *There is in general no uniqueness of the weak solution because f can take arbitrarily values on the characteristics such that $\tau_i = -\infty$. But it is possible to prove that the mild solution is the ununique minimal solution of the transport equation. We refer to [3] for the concept of the minimal solution and to [6] for a proof of this assertion.*

Remark 3 *We have that $(X(s+T;x,v,t+T), V(s+T;x,v,t+T)) = (X(s;x,v,t), V(s;x,v,t))$ because of the periodicity of E. Using this equality it is easy to check that the mild solution is periodic.*

Remark 4 *If $g \in C^1(I\!R_t \times \Sigma^-)$ then the mild solution is a classical solution of (8).*

2.2 Estimation of the life-time of particles

In order to assure L^∞ estimates for the charge and current densities, we assume that the following conditions are satisfied:

$$\|E\|_{L^\infty(I\!R_t \times \Omega)} \leq \frac{1}{4} \cdot \frac{|\ v_0\ |^2}{\delta(\Omega)}, \tag{18}$$

$$E \in (L^\infty(I\!R_t; W^{1,\infty}(\Omega)))^d, \tag{19}$$

$$supp(g) \subset \{(t,x,v) \mid t \in I\!R_t, x \in \partial\Omega, v \cdot \nu(x) < 0, v_0 \leq |v| \leq v_1\}. \tag{20}$$

Here, $\delta(\Omega)$ is the diameter of Ω and the velocity v_0,v_1 are positive constants. With these assumptions, we get:

Lemma 1 *Assume that the electric field and the boundary data satisfy (18), (19) and (20). Then, the life-time in Ω of particles starting from the support of g is finite:*

$$\tau_o(x,v,t) - \tau_i(x,v,t) \leq 2 \cdot \frac{\delta(\Omega)}{v_0},\ \forall (t,x,v) \in supp(g). \tag{21}$$

Proof
Suppose that there is a particle injected in Ω at $t = \tau_i$ and which is still in the domain at $t_1 > \tau_i + 2 \cdot \dfrac{\delta(\Omega)}{v_0}$. According to (20), we have:

$$0 < v_0 \leq \mu \cdot V(\tau_i) \leq v_1,$$

where $\mu = V(\tau_i)/|V(\tau_i)|$. Integrating (12) on $[\tau_i, t] \subset [\tau_i, t_1]$, we obtain:

$$X(t) = X(\tau_i) + \int_{\tau_i}^{t} V(s)ds \tag{22}$$

$$V(t) = V(\tau_i) + \int_{\tau_i}^{t} E(s, X(s))ds \tag{23}$$

Using (23), we find for all $t \in [\tau_i, \tau_i + 2 \cdot \delta(\Omega)/v_0]$:

$$\begin{aligned} \mu \cdot V(t) &\geq \mu \cdot V(\tau_i) - \|E\|_{C^0(I\!R_t \times \Omega)} \cdot (t(t - \tau_i) \\ &\geq v_0 - \frac{1}{4} \cdot \frac{|v_0|^2}{\delta(\Omega)} \cdot 2 \cdot \frac{\delta(\Omega)}{v_0} \\ &= \frac{v_0}{2}. \end{aligned} \tag{24}$$

Hence, the particle moves in the direction μ at least with the velocity $v_0/2$ during $t \in [\tau_i, \tau_i + 2 \cdot \delta(\Omega)/v_0]$. Moreover, we can choose $\varepsilon > 0$ and $t_\varepsilon = \tau_i + 2 \cdot \delta(\Omega)/v_0 + \varepsilon < t_1$ such that:

$$\mu \cdot V(t) > 0, \forall t \in [\tau_i, t_\varepsilon]. \tag{25}$$

Using again (12), we have:

$$\begin{aligned} |X(t_\varepsilon) - X(\tau_i)| &\geq |\mu \cdot (X(t_\varepsilon) - X(\tau_i))| \\ &= \int_{\tau_i}^{t_\varepsilon} \mu \cdot V(s)ds \\ &> \int_{\tau_i}^{\tau_i + 2\cdot\delta(\Omega)/v_0} \mu \cdot V(s)ds \\ &\geq 2 \cdot \frac{\delta(\Omega)}{v_0} \cdot \frac{v_0}{2} = \delta(\Omega), \end{aligned} \tag{26}$$

which contradicts the fact that $X(t_\varepsilon) \in \Omega$.

Corollary 1 *Assuming the same hypotheses as in Lemma 1 (18), (19), (20) and let f be the mild solution of Definition 2. Then we have:*

$$supp(f) \subset \{(t, x, v) | t \in I\!R_t, x \in \Omega, \frac{v_0}{2} \leq |v| \leq v_1 + \frac{v_0}{2}\}. \tag{27}$$

Proof
The estimates (27) follow from the previous Lemma. Indeed, according to (23), we obtain:

$$V(t) = V(\tau_i) + \int_{\tau_i}^{t} E(s, X(s))ds, \tag{28}$$

and therefore:

$$\begin{aligned} |V(t)| &\geq |V(\tau_i)| - \|E\|_{C^0(\mathbb{R}_t \times \Omega)} \cdot (t - \tau_i) \\ &\geq v_0 - 2 \cdot \frac{\delta(\Omega)}{v_0} \cdot \frac{1}{4} \cdot \frac{|v_0|^2}{\delta(\Omega)} \\ &= \frac{v_0}{2}, \end{aligned} \tag{29}$$

and:

$$\begin{aligned} |V(t)| &\leq |V(\tau_i)| + \|E\|_{C^0(\mathbb{R}_t \times \Omega)} \cdot (t - \tau_i) \\ &\leq v_1 + 2 \cdot \frac{\delta(\Omega)}{v_0} \cdot \frac{1}{4} \cdot \frac{|v_0|^2}{\delta(\Omega)} \\ &= v_1 + \frac{v_0}{2}. \end{aligned} \tag{30}$$

2.3 Vlasov equation with absorption term

Lemma 2 *Let $E \in (L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega)))^d$ and $g \in L^\infty(\mathbb{R} \times \Sigma^-)$ be T periodic functions which verify (18), (20). Then a weak periodic solution in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}_v^d)$ of the modified Vlasov equation :*

$$\left\{ \begin{array}{lll} \alpha \cdot f + \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, & (t, x, v) & \in \mathbb{R}_t \times \Omega \times \mathbb{R}_v^d, \\ f = g, & (t, x, v) & \in \mathbb{R}_t \times \Sigma^-. \end{array} \right. \tag{31}$$

is unique and therefore is the mild solution.

Proof Assume that f is a solution in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}_v^d)$ with $g = 0$. We have :

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = -\alpha \cdot f \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}_v^d), \tag{32}$$

and therefore(cf. [1], [14]) we obtain:

$$\begin{aligned} -\alpha \cdot f^2 &= f(\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f) \\ &= \frac{1}{2}(\partial_t f^2 + v \cdot \nabla_x f^2 + E \cdot \nabla_v f^2). \end{aligned}$$

Integrating this relation on $]0,T[\times \Omega \times I\!R_v^d$ gives:

$$\begin{aligned} \alpha \int_0^T \int_\Omega \int_{I\!R_v^d} f^2 dv dx dt &= -\frac{1}{2}\int_0^T \int_{\partial\Omega} \int_{I\!R_v^d} v \cdot \nu(x) f^2 dv d\sigma dt \\ &\leq -\frac{1}{2}\int_0^T \int_{\Sigma^-} v \cdot \nu(x) f^2 dv d\sigma dt = 0. \end{aligned}$$

3 Mild solutions for the Vlasov-Poisson system in 1D.

In this Section we consider the 1 dimensional case and Ω is the unit interval $]0,1[$.

3.1 Continuity of the characteristics

We work under the hypotheses (18), (19),(20), which assure a finite life-time $T_{out} = 2/v_0$ and a minimal velocity $v_{min} = v_0/2$ for all particles. We prove C^0 continuity of the characteristics of Vlasov equation.

Lemma 3 *Consider $(E_n)_{n\geq 1}$ a sequence of electric fields which verify:*

$$\|E_n\|_{C^0(I\!R_t\times[0,1])} \leq \frac{|\ v_0\ |^2}{4}, \tag{33}$$

$$|\ E_n(t,x) - E_n(t,y)\ | \leq L\ |\ x-y\ |, \forall t \in I\!R_t, x,y \in [0,1], \tag{34}$$

$$\lim_{n\to\infty} E_n = E \qquad in \qquad C^0(I\!R_t \times [0,1]), \tag{35}$$

and $g \in L^\infty(I\!R_t \times \Sigma^-)$ a function such as:

$$\begin{aligned} supp(g) &\subset \{(t,0,v) \mid t \in I\!R_t,\ 0 < v_0 \leq v \leq v_1\} \\ &\cup \{(t,1,v) \mid t \in I\!R_t,\ -v_1 \leq v \leq -v_0\}. \end{aligned} \tag{36}$$

Then we have $\forall s \in (\tau_i^n, \tau_o^n) \cap (\tau_i, \tau_o)$*:*

$$
\begin{aligned}
|X_n(s) - X(s)| &\leq \left(\frac{2}{v_0}\right)^{1/2} \|E_n - E\|_{C^0(I\!R_t \times [0,1])} \exp\left(\frac{L+2}{v_0}\right), \\
|V_n(s) - V(s)| &\leq \left(\frac{2}{v_0}\right)^{1/2} \|E_n - E\|_{C^0(I\!R_t \times [0,1])} \exp\left(\frac{L+2}{v_0}\right),
\end{aligned}
$$

and also:

$$
|\tau_{o,i}^n - \tau_{o,i}| \leq \left(\frac{2}{v_0}\right)^{3/2} \cdot \|E_n - E\|_{C^0(I\!R_t \times [0,1])} \exp\left(\frac{L+2}{v_0}\right) \tag{37}
$$

Proof

We first remark that in view of (35), (34) holds also for E. Therefore the corresponding characteristics are well defined. Let $(t, x, v) \in I\!R_t \times (0,1) \times I\!R_v$. We multiply (12) by $X_n(s; x, v, t) - X(s; x, v, t)$ and $V_n(s; x, v, t) - V(s; x, v, t)$ to get $\forall s \in (\tau_i^n, \tau_o^n) \cap (\tau_i, \tau_o)$:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{ds}|X_n(s) - X(s)|^2 &+ \frac{1}{2}\frac{d}{ds}|V_n(s) - V(s)|^2 = \\
&= (X_n(s) - X(s)) \cdot (V_n(s) - V(s)) \\
&+ (E_n(s, X_n(s)) - E(s, X(s))) \cdot (V_n(s) - V(s)) \\
&\leq \frac{1}{2} \cdot |X_n(s) - X(s)|^2 + \frac{1}{2} \cdot |V_n(s) - V(s)|^2 \\
&+ (L \cdot |X_n(s) - X(s)| + \|E_n - E\|_{C^0(I\!R_t \times [0,1])}) \\
&\times |V_n(s) - V(s)| \\
&\leq \frac{L+2}{2}(|X_n(s) - X(s)|^2 + |V_n(s) - V(s)|^2) \\
&+ \frac{1}{2}\|E_n - E\|^2_{C^0(I\!R_t \times [0,1])},
\end{aligned} \tag{38}
$$

which yields:

$$
\begin{aligned}
|X_n(s) - X(s)|^2 &+ |V_n(s) - V(s)|^2 \leq \\
&\leq (L+2) \int_t^s |X_n(\tau) - X(\tau)|^2 + |V_n(\tau) - V(\tau)|^2 d\tau \\
&+ |t - s| \|E_n - E\|^2_{C^0(I\!R_t \times [0,1])}.
\end{aligned} \tag{39}
$$

By using Gronwall Lemma (and *Lemma 1*), we deduce that $\forall s \in (\tau_i^n, \tau_o^n) \cap (\tau_i, \tau_o)$:

$$|X_n(s) - X(s)|^2 + |V_n(s) - V(s)|^2 \le \frac{2}{v_0} \cdot \|E_n - E\|^2_{C^0(I\!R_t \times [0,1])} \exp\left(2\frac{L+2}{v_0}\right), \tag{40}$$

and also:

$$|X_n(s) - X(s)| \le \left(\frac{2}{v_0}\right)^{1/2} \|E_n - E\|_{C^0(I\!R_t \times [0,1])} \cdot \exp\left(\frac{L+2}{v_0}\right), \tag{41}$$

$$|V_n(s) - V(s)| \le \left(\frac{2}{v_0}\right)^{1/2} \|E_n - E\|_{C^0(I\!R_t \times [0,1])} \cdot \exp\left(\frac{L+2}{v_0}\right), \tag{42}$$

because $|t - s| \le \frac{2}{v_0}$. In order to estimate the difference of entry times, assume that $\tau_i \le \tau_i^n$ holds and write:

$$\begin{aligned} \frac{v_0}{2} \cdot |\tau_i^n - \tau_i| &\le \left| \int_{\tau_i^n}^{\tau_i} \frac{dX}{ds} ds \right| \\ &= |X(\tau_i^n) - X(\tau_i)| \\ &= |X(\tau_i^n) - X_n(\tau_i^n)| \\ &\le \left(\frac{2}{v_0}\right)^{1/2} \|E_n - E\|_{C^0(I\!R_t \times [0,1])} \cdot \exp\left(\frac{L+2}{v_0}\right), \end{aligned} \tag{43}$$

because $X(\tau_i) = X_n(\tau_i^n)$ and so:

$$|\tau_i^n - \tau_i| \le \left(\frac{2}{v_0}\right)^{3/2} \cdot \|E_n - E\|_{C^0(I\!R_t \times [0,1])} \cdot \exp\left(\frac{L+2}{v_0}\right). \tag{44}$$

If the other inequality $\tau_i > \tau_i^n$ holds, we can find the same estimates, by changing X, V, τ_i with X_n, V_n, τ_i^n respectively. The same method yields a similar estimate for the exit times :

$$|\tau_o^n - \tau_o| \le \left(\frac{2}{v_0}\right)^{3/2} \cdot \|E_n - E\|_{C^0(I\!R_t \times [0,1])} \cdot \exp\left(\frac{L+2}{v_0}\right). \tag{45}$$

Lemma 4 *With the same assumptions as in Lemma 3, if moreover* $g \in W^{1,\infty}(\mathbb{R}_t \times \Sigma^-)$ *and if we denote by* f_n,f *the solutions given by (17) which correspond to the fields* E_n,E, *we have the estimate:*

$$\begin{aligned} \|f_n - f\|_{L^\infty(\mathbb{R}_t\times(0,1)\times\mathbb{R}_v)} &\leq \|g\|_{W^{1,\infty}(\mathbb{R}_t\times\Sigma^-)} \cdot \|E_n - E\|_{C^0(\mathbb{R}_t\times[0,1])} \\ &\cdot \left(\frac{2}{v_0}\right)^{1/2}\left(1 + \frac{v_0}{2} + \frac{2}{v_0}\right) \cdot \exp\left(\frac{L+2}{v_0}\right) \end{aligned} \quad (46)$$

Proof

We assume that $\tau_i \leq \tau_i^n$ then, using *Lemma 3* and (33), the difference of the entry velocity is given by:

$$\begin{aligned} |V_n(\tau_i^n) - V(\tau_i)| &\leq |V_n(\tau_i^n) - V(\tau_i^n)| + |V(\tau_i^n) - V(\tau_i)| \\ &\leq \left(\frac{2}{v_0}\right)^{1/2} \|E_n - E\|_{C^0} \cdot \exp\left(\frac{L+2}{v_0}\right) + \left|\int_{\tau_i^n}^{\tau_i} \frac{dV}{ds} ds\right| \\ &\leq \left(\frac{2}{v_0}\right)^{1/2} \|E_n - E\|_{C^0} \cdot \exp\left(\frac{L+2}{v_0}\right) + |\tau_i^n - \tau_i| \cdot \|E\|_{C^0} \\ &\leq \left(\frac{2}{v_0}\right)^{1/2}\left(1 + \frac{v_0}{2}\right) \cdot \|E_n - E\|_{C^0} \cdot \exp\left(\frac{L+2}{v_0}\right). \end{aligned} \quad (47)$$

Assuming now that $g \in W^{1,\infty}(\mathbb{R}_t \times \Sigma^-)$. Then we easily check the statement of the lemma, using (37) and (47):

$$\begin{aligned} |f_n(t,x,v) - f(t,x,v)| &= |g(\tau_i^n, 0, V_n(\tau_i^n)) - g(\tau_i, 0, V(\tau_i))| \\ &\leq \|g\|_{W^{1,\infty}(\mathbb{R}_t\times\Sigma^-)} \\ &\times (|\tau_i^n - \tau_i| + |V_n(\tau_i^n) - V(\tau_i)|) \\ &\leq \|g\|_{W^{1,\infty}(\mathbb{R}_t\times\Sigma^-)} \|E_n - E\|_{C^0(\mathbb{R}_t\times[0,1])} \\ &\times \left(\frac{2}{v_0}\right)^{1/2}\left(1 + \frac{v_0}{2} + \frac{2}{v_0}\right) \exp\left(\frac{L+2}{v_0}\right). \end{aligned} \quad (48)$$

3.2 Existence

In this section section we establish existence result for the mild periodic solution of the 1D Vlasov-Poisson problem:

$$\begin{cases} \partial_t f + v \cdot \partial_x f + E \cdot \partial_v f = 0, & (t,x,v) \in I\!R_t \times [0,1] \times I\!R_v, \\ f = g, & (t,x,v) \in I\!R_t \times \Sigma^-, \\ E(t,x) = \partial_x \varphi_x(t,x), & (t,x) \in I\!R_t \times [0,1], \\ \partial^2_{xx}\varphi = \int_{I\!R_v} f(t,x,v)dv, & (t,x) \in I\!R_t \times [0,1], \\ \varphi(t,0) = 0, & t \in I\!R_t, \\ \varphi(t,1) = \varphi_1(t), & t \in I\!R_t,\ \varphi_1\ T - periodic. \end{cases} \tag{49}$$

We want to use the Schauder fixed point theorem. We define an application which maps a periodic electric field E to an other one E_1 where E_1 is defined as follows. Let f be the mild periodic solution of *Definition 2* corresponding to the electric field E. The electric field $E^1 = \partial_x \varphi$ is determined as the solution of the Poisson problem with the density $\rho(t,x) = \displaystyle\int_{I\!R_v} f(t,x,v)\,dv$. In order to assure the invariance of the domain, smallness assumptions of the data are required. We have:

Theorem 1 *Let $g \in L^\infty(I\!R_t \times \Sigma^-)$ be a periodic function, φ_1 a T periodic continuous function, $0 < v_0 < v_1$ such as:*

$$\begin{aligned} supp(g) \ &\subset\ \{(t,0,v) \mid t \in I\!R_t,\ 0 < v_0 \le v \le v_1\} \\ &\cup\ \{(t,1,v) \mid t \in I\!R_t,\ -v_1 \le v \le -v_0\}, \end{aligned} \tag{50}$$

$$\|\varphi_1\|_{L^\infty(I\!R_t)} + 3 \cdot v_1 \cdot \|g\|_{L^\infty(I\!R_t \times \Sigma^-)} \le \frac{|v_0|^2}{4}. \tag{51}$$

Thus, the system (49) has at least one mild periodic solution.

Proof
We denote by $\mathcal{F} : X \to X$ the map:

$$E \to f_E \to \rho_E \to E^1 = \partial_x \varphi, \tag{52}$$

where:

$$E \in X = \{e \in C^0(I\!R_t \times [0,1]);\ \|e\|_{C^0(I\!R_t \times [0,1]} \le \frac{|v_0|^2}{4},$$
$$|e(t,x) - e(t,y)| \le L \cdot |x-y|,\ \forall\, x,y \in [0,1] \tag{53}$$
$$e(t,x) = e(t+T,x),\ \forall\,(t,x) \in I\!R_t \times [0,1]\},$$

with:

$$L = 2 \cdot v_1 \cdot \|g\|_{L^\infty(I\!R_t \times \Sigma^-)}. \tag{54}$$

Here f_E is the mild periodic solution given by (16) and $E^1 = \partial_x \varphi$ is the solution of the Poisson problem:

$$\begin{cases} \partial^2_{xx}\varphi = \rho_E = \int_{I\!R_v} f(t,x,v)dv, & (t,x) \in I\!R_t \times [0,1], \\ \varphi(t,0) = 0, & t \in I\!R_t, \\ \varphi(t,1) = \varphi_1(t), & t \in I\!R_t. \end{cases} \tag{55}$$

Step 1 *The map $\mathcal{F}$ is well defined ($\mathcal{F}(X) \subset X$).*

Let $E \in X$. Using *Corolary 1* (27), we get:

$$\|\rho\|_{C^0(I\!R_t \times [0,1])} \le 2 \cdot v_1 \cdot \|g\|_{L^\infty(I\!R_t \times \Sigma^-)}. \tag{56}$$

The electric field E^1 writes:

$$E^1(t,x) = \varphi_1(t) - \int_0^1 (1-y) \cdot \rho_E(t,y)dy + \int_0^x \rho_E(t,y)dy, \tag{57}$$

and therefore:

$$\begin{aligned} \|E^1\|_{C^0(I\!R_t \times [0,1])} &\le \|\varphi_1\|_{L^\infty} + \frac{3}{2} \cdot \|\rho_E\|_{C^0(I\!R_t \times [0,1])} \\ &\le \|\varphi_1\|_{L^\infty} + 3 \cdot v_1 \cdot \|g\|_{L^\infty(I\!R_t \times \Sigma^-)} \\ &\le \frac{|v_0|^2}{4}. \end{aligned} \tag{58}$$

The electric field E^1 verifies also :

$$\begin{aligned}|E^1(t,x)-E^1(t,y)| &\le \|\partial_x E^1\|_{L^\infty(I\!R_t\times[0,1])}\cdot|x-y| \\ &= \|\rho_E\|_{L^\infty(I\!R_t\times[0,1])}\cdot|x-y| \\ &\le 2\cdot v_1\cdot\|g\|_{L^\infty(I\!R_t\times\Sigma^-)}\cdot|x-y| \\ &= L\cdot|x-y|. \end{aligned} \tag{59}$$

Moreover, because E is time periodic, f_E, ρ_E, φ and E^1 are periodic too, so $E^1=\mathcal{F}(E)\in X$.

Step 2 *The map $\mathcal{F}$ is compact for the topology of $C^0(I\!R_t\times[0,1])$.*

We prove that $\mathcal{F}(X)$ is compact. For that we derive a bound on the time derivative of $E^1=\mathcal{F}(E), E\in X$. From (57) we obtain:

$$\partial_t(E^1-\varphi_1)=-\int_0^1(1-y)\partial_t\rho_E dy+\int_0^x\partial_t\rho_E dy.$$

We use now the conservation law $\partial_t\rho_E+\partial_x j_E=0$, where $j_E=\int_v vf_E dv$. An integration by part yields:

$$\partial_t(E^1-\varphi_1)=\int_0^1 j_E(t,y)dy-j_E(t,x).$$

Therefore this relation with (53) gives that $E^1-\varphi_1$ is uniformly Lipschitz with respect to time and position. Ascoli's Theorem yields that $E^1-\varphi_1$ and then E^1 belong to a compact set of $C^0([0,T]\times[0,1])$. Because of the periodicity of E^1 we also have that it belongs to a compact set of $C^0(I\!R_t\times[0,1])$.

Step 3 *The map $\mathcal{F}:(X,C^0(I\!R_t\times[0,1]))\to(X,C^0(I\!R_t\times[0,1]))$ is continuous.*

Let $(E_n)_{n\ge1}\subset X$, $\lim_{n\to\infty}E_n=E$ in $C^0(I\!R_t\times[0,1])$. Denoting by f_n the mild solution given by (16) corresponding to E_n. Using (45), (41) et (42), we pass to the limit for $n\to\infty$ in (16) which now reads:

$$\begin{aligned}<f_n,\varphi> &= \\ &- \int_0^T\int_{v<0}\int_t^{\tau_o^n} v\cdot g(t,1,v)\cdot\varphi(s,X_n(s;1,v,t),V_n(s;1,v,t))dsdvdt \\ &+ \int_0^T\int_{v>0}\int_t^{\tau_o^n} v\cdot g(t,0,v)\cdot\varphi(s,X_n(s;0,v,t),V_n(s;0,v,t))dsdvdt.\end{aligned} \tag{60}$$

Using *Lemma 3* we can pass to the limit in this expression. So f_n is a convergent sequence in sense of distributions whose limit f is the mild solution corresponding to the field E. Moreover, from the uniform bound in $L^\infty(\mathbb{R}_t \times (0,1))$ we deduce:

$$f_n \rightharpoonup f, \qquad \text{weak } \star \text{ in } L^\infty(\mathbb{R}_t \times (0,1) \times \mathbb{R}_v). \tag{61}$$

In the same way, the densities ρ_n converge weakly, because the support of f_n are bounded with respect of velocities:

$$\rho_n = \int_{\mathbb{R}_v} f_n(t,x,v)dv \rightharpoonup \rho = \int_{\mathbb{R}_v} f(t,x,v)dv, \qquad \text{weak } \star \text{ in } L^\infty(\mathbb{R}_t \times (0,1)). \tag{62}$$

On the other hand the weak convergence of ρ_n implies the weak convergence of $\mathcal{F}(E_n)$ towards $\mathcal{F}(E)$ (for instance in $L^\infty(\mathbb{R}_t \times [0,1])$). Since $\mathcal{F}$ is compact, it implies that $\mathcal{F}(E_n) \to \mathcal{F}(E)$ in $C^0(\mathbb{R}_t \times [0,1])$. This prove the continuity of the map $\mathcal{F}$. At this point, using the Schauder fixed point theorem, we prove existence of periodic solution of 1D Vlasov-Poisson problem which concludes the proof of the *Theorem 1*.

3.3 Uniqueness

In this section we are interested in uniqueness results. We state the following:

Theorem 2 *Under the same assumptions as in Theorem 1, by requiring moreover that* $g \in W^{1,\infty}(\mathbb{R}_t \times \Sigma^-)$ *and :*

$$\|g\|_{W^{1,\infty}(\mathbb{R}_t \times \Sigma^-)} < \frac{\left(\frac{v_0}{2}\right)^{1/2} \cdot \exp\left(-\frac{L+2}{v_0}\right)}{3 \cdot v_1 \cdot \left(1 + \frac{v_0}{2} + \frac{2}{v_0}\right)}, \tag{63}$$

the system (49) has a unique mild periodic solution.

Proof
The existence result has been proved in the previous section. In order to establish uniqueness result, we show that the map $\mathcal{F}$ is a contraction. Let $E, F \in X$ two electric fields and denote by f_E, f_F the corresponding mild solutions. *Lemma 4* (46) and the fact that f_E, f_F have bounded support in velocity, allow us to write:

$$\begin{aligned}\|\rho_E-\rho_F\|_{C^0(\mathbb{R}_t\times[0,1])} &\leq 2\cdot v_1\cdot\|g\|_{W^{1,\infty}(\mathbb{R}_t\times\Sigma^-)}\cdot\|E-F\|_{C^0(\mathbb{R}_t\times[0,1])}\\ &\cdot \left(\frac{2}{v_0}\right)^{1/2}\left(1+\frac{v_0}{2}+\frac{2}{v_0}\right)\cdot\exp\left(\frac{L+2}{v_0}\right) \qquad (64)\end{aligned}$$

By formula (57), we deduce:

$$\begin{aligned}\|\mathcal{F}(E)-\mathcal{F}(F)\|_{C^0(\mathbb{R}_t\times[0,1])} &\leq \frac{3}{2}\cdot\|\rho_E-\rho_F\|_{C^0(\mathbb{R}_t\times[0,1])}\\ &\leq 3\cdot v_1\cdot\|g\|_{W^{1,\infty}(\mathbb{R}_t\times\Sigma^-)}\cdot\|E-F\|_{C^0(\mathbb{R}_t\times[0,1])}\\ &\cdot \left(\frac{2}{v_0}\right)^{1/2}\left(1+\frac{v_0}{2}+\frac{2}{v_0}\right)\cdot\exp\left(\frac{L+2}{v_0}\right). \qquad (65)\end{aligned}$$

Therefore we have:

$$\|\mathcal{F}(E)-\mathcal{F}(F)\|_{C^0(\mathbb{R}_t\times[0,1])}\leq q\cdot\|E-F\|_{C^0(\mathbb{R}_t\times[0,1])}, \qquad (66)$$

where q is given by:

$$q=3\cdot\left(\frac{2}{v_0}\right)^{1/2}\exp\left(\frac{L+2}{v_0}\right)\cdot v_1\cdot\left(1+\frac{v_0}{2}+\frac{2}{v_0}\right)\cdot\|g\|_{W^{1,\infty}(\mathbb{R}_t\times\Sigma^-)}<1. \qquad (67)$$

4 Weak solutions for the Vlasov-Poisson system in the multidimensional case.

In this section, we establish existence result for the weak periodic solution of the Vlasov-Poisson problem:

$$\begin{cases}\partial_t f+v\cdot\nabla_x f+E\cdot\nabla_v f=0, & (t,x,v)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^d_v,\\ f=g, & (t,x,v)\in\mathbb{R}_t\times\Sigma^-,\\ E(t,x)=\nabla_x\varphi, & (t,x)\in\mathbb{R}_t\times\Omega,\\ \Delta_x\varphi=\int_{\mathbb{R}^d_v}f(t,x,v)dv, & (t,x)\in\mathbb{R}_t\times\Omega,\\ \varphi=\varphi_0, & (t,x)\in\mathbb{R}_t\times\partial\Omega.\end{cases} \qquad (68)$$

Here, the boundary data g and φ_0 are $T-$ periodic functions. We look for a weak periodic solution $(f(t,x,v), \varphi(t,x,v))$. As previously, the Schauder fixed point theorem is used. We define an application which maps a periodic potential φ to an other one φ_1 where φ_1 is defined as follows. Let f be the mild periodic solution of *Definition 2* corresponding to the electric field $E = \nabla_x \varphi$. The potential φ_1 is determined as the solution of the Poisson problem with the density $\rho(t,x) = \int_{\mathbb{R}^d_v} f(t,x,v)dv$. Unfortunately this procedure cannot be used directly. Indeed the *Definiton 2* requires that the electric field is Lipschitz with respect to x and we cannot expect such a regularity in the general case. Therefore we have to regularize the potential. We also have to use an absorption term in the Vlasov equation in order to have uniqueness of the weak solution. Then the strategy of proof is as follows. We first show the existence of weak periodic solution for a regularized problem by using the Schauder fixed point theorem. Next we pass to the limit when the regularization parameter vanishes.

4.1 Fixed point for the regularized problem

Let $p > d+1$ be a positive constant and let $\mathcal{X}$ be the set of the functions φ which verify:

$$\varphi \in L^\infty(\mathbb{R}_t; W^{2,p}(\Omega)) \quad , \quad \|\varphi\|_{L^\infty(\mathbb{R}_t; W^{2,p}(\Omega))} \le C_1, \tag{69}$$

$$\partial_t \varphi \in L^\infty(\mathbb{R}_t; W^{1,p}(\Omega)) \quad , \quad \|\partial_t\varphi\|_{L^\infty(\mathbb{R}_t; W^{1,p}(\Omega))} \le C_2, \tag{70}$$

$$\|\nabla_x \varphi\|_{L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^d_v)} \le \frac{1}{4} \cdot \frac{|v_0|^2}{\delta(\Omega)}, \tag{71}$$

$$\varphi(t+T, \cdot) = \varphi(t, \cdot), \tag{72}$$

where C_1 and C_2 are fixed constants which will be chosen later on. With the definitions (69),(70), we have for any function $\varphi \in \mathcal{X}$:

$$\begin{aligned} \|\nabla_x \varphi\|^p_{W^{1,p}((0,T)\times\Omega)} &= \|\partial_t \nabla_x \varphi\|^p_{L^p((0,T)\times\Omega)} \\ &+ \sum_{i=1}^d \|\partial_{x_i} \nabla_x \varphi\|^p_{L^p((0,T)\times\Omega)} \\ &\le K(C_1, C_2) \end{aligned} \tag{73}$$

Using the compactness results embedding:

$$W^{1,p}((0,T)\times\Omega) \hookrightarrow C^0([0,T]\times\overline{\Omega}) \qquad (p > d+1,\ \delta(\Omega) < \infty),$$

we deduce that $\{\nabla_x \varphi | \varphi \in \mathcal{X}\}$ is a compact set of $C^0([0,T]\times\overline{\Omega})$. We conclude that $\mathcal{X}$ is a compact set of $C^0([0,T];C^1(\overline{\Omega}))$. We now introduce a regularization mapping:

$$\begin{aligned} &R_\alpha : C^0([0,T];C^1(\overline{\Omega})) \to C^0([0,T];C^2(\overline{\Omega})) \\ &\varphi \mapsto R_\alpha\varphi(t,x) = \int_{I\!R^d_x} \zeta_\alpha(x-y)\cdot\overline{\varphi}(t,y)dy, \end{aligned} \tag{74}$$

where $\zeta_\alpha \geq 0$ is a mollifier:

$$\zeta_\alpha(x) = \frac{1}{\alpha^d}\zeta\left(\frac{x}{\alpha}\right) \quad , \quad \zeta \in C_0^\infty(I\!R^d)$$
$$supp(\zeta) \subset B_1 \quad , \quad \int_{I\!R^d}\zeta(u)du = 1.$$

Here, $\overline{\cdot}$ is a linear extension operator from $C^1(\overline{\Omega})$ onto $C^1(I\!R^d)$ (which requires that $\partial\Omega$ is C^1). Therefore, $\overline{\varphi}$ is a T periodic extension on $I\!R_t \times I\!R^d_x$ of φ such as:

$$\|\nabla_x\overline{\varphi}\|_{L^\infty(I\!R_t\times I\!R^d_x)} \leq \|\nabla_x\varphi\|_{L^\infty(I\!R_t\times\Omega)}. \tag{75}$$

Obviously, R_α is well defined and continuous. Moreover, (71) is preserved by this application. By definition $R_\alpha\varphi$ is T periodic. Next, we consider the application:

$$\mathcal{F} : \varphi \in \mathcal{X} \mapsto \varphi^1_\alpha, \tag{76}$$

where:

$$\left\{\begin{array}{rcl} \Delta_x\varphi^1_\alpha(t) &=& \displaystyle\int_{I\!R^d_v} f_\alpha(t,x,v)dv = \rho_\alpha(t), \qquad x\in\Omega \\ \varphi^1_\alpha(t,x) &=& \varphi_0(t,x), \qquad x\in\partial\Omega. \end{array}\right. \tag{77}$$

Above f_α is the mild solution of the following modified Vlasov equation corresponding to the field $\nabla_x R_\alpha\varphi$:

$$\left\{\begin{array}{lll} \alpha\cdot f_\alpha + \partial_t f_\alpha + v\cdot\nabla_x f_\alpha + \nabla_x R_\alpha\varphi\cdot\nabla_v f_\alpha = 0, & (t,x,v) & \in I\!R_t\times\Omega\times I\!R^d_v, \\ f_\alpha = g, & (t,x,v) & \in I\!R_t\times\Sigma^-. \end{array}\right. \tag{78}$$

The term $\alpha \cdot f_\alpha$ changes the formula (16) in the following way:

$$< f_\alpha, \theta > \quad = \quad -\int_0^T dt \int_{\Sigma^-} dv d\sigma \int_t^{\tau_o^\alpha} v \cdot \nu(x) g(t,x,v)$$
$$\theta(s, X_\alpha(s;x,v,t), V_\alpha(s;x,v,t)) e^{-\alpha(s-t)} ds. \tag{79}$$

We prove now that the application $\mathcal{F}$ maps $\mathcal{X}$ into itself and is continuous on $C^0([0,T]; C^1(\overline{\Omega}))$ for convenient choices of the constants C_1 and C_2 and for small enough boundary datas.

Step 1 *Invariance of the domain.*

Let $\varphi \in \mathcal{X}$ be an electric potential and $\varphi_\alpha = R_\alpha \varphi$ its regularization. We verify (18),(19),(20), and we deduce from *Lemma 1* the existence of a finite life-time $2 \cdot \delta(\Omega)/v_0$. Using Corollary 1 we also have:

$$\begin{aligned} \|\rho_\alpha\|^p_{L^\infty(I\!R_t; L^p(\Omega))} &= \sup_{t \in I\!R_t} \int_\Omega |\rho_\alpha(t,x)|^p dx \\ &= \sup_{t \in I\!R_t} \int_\Omega \left| \int_{v_0/2 \le |v| \le v_1 + v_0/2} f_\alpha(t,x,v) dv \right|^p dx \\ &\le \|g\|^p_{L^\infty(\Sigma^-)} \, vol(\Omega) \, \omega_d^p \\ &\quad \left[\left(v_1 + \frac{v_0}{2} \right)^d - \left(\frac{v_0}{2} \right)^d \right]^p, \end{aligned}$$

where ω_d is the volume of the unit ball of $I\!R^d$. At this point, using the classical results of regularity for the Poisson equation, we get:

$$\begin{aligned} \|\varphi^1_\alpha\|_{L^\infty(I\!R_t; W^{2,p}(\Omega))} &\le C_p(\Omega) \left(\|\varphi_0\|_{L^\infty(I\!R_t; W^{2-1/p,p}(\partial\Omega))} + \|\rho_\alpha\|_{L^\infty(I\!R_t; L^p(\Omega))} \right) \\ &\le C_p(\Omega) \left(\|\varphi_0\|_{L^\infty(I\!R_t; W^{2-1/p,p}(\partial\Omega))} + vol(\Omega)^{1/p} \right. \\ &\quad \left. \|g\|_{L^\infty(I\!R_t \times \Sigma^-)} \, \omega_d \, \left[\left(v_1 + \frac{v_0}{2} \right)^d - \left(\frac{v_0}{2} \right)^d \right] \right) \\ &= C_1 \end{aligned} \tag{80}$$

Therefore we choose:

$$\begin{aligned} C_1 \quad = \quad & C_p(\Omega) \left(\|\varphi_0\|_{L^\infty(I\!R_t; W^{2-1/p,p}(\partial\Omega))} + vol(\Omega)^{1/p} \cdot \|g\|_{L^\infty(I\!R_t \times \Sigma^-)} \right. \\ & \left. \omega_d \, \left[\left(v_1 + \frac{v_0}{2} \right)^d - \left(\frac{v_0}{2} \right)^d \right] \right), \end{aligned} \tag{81}$$

in order that φ_α^1 satisfies (69). Finally, differentiating (77) with respect to t and using the equation of continuity, we obtain:

$$\begin{cases} \Delta_x \partial_t \varphi_\alpha^1 & = & \partial_t \rho_\alpha(t) = -\nabla_x \cdot j_\alpha, \qquad x \in \Omega \\ \\ \partial_t \varphi_\alpha^1(t,x) & = & \partial_t \varphi_0(t,x), \qquad x \in \partial\Omega, \end{cases} \tag{82}$$

with $j_\alpha = \int v f_\alpha(t,x,v)dv$. We observe that:

$$\begin{aligned} &\|\nabla_x \cdot j_\alpha\|_{L^\infty(I\!R_t;W^{-1,p}(\Omega))} \\ &\qquad \leq \|j_\alpha\|_{L^\infty(I\!R_t;L^p(\Omega))} \\ &\qquad \leq \|g\|_{L^\infty(I\!R_t\times\Sigma^-)}\, vol(\Omega)^{1/p}\, (v_1 + v_0/2) \\ &\qquad\quad \omega_d \left[\left(v_1 + \frac{v_0}{2}\right)^d - \left(\frac{v_0}{2}\right)^d\right] \end{aligned}$$

which yields:

$$\begin{aligned} \|\partial_t \varphi_\alpha^1\|_{L^\infty(I\!R_t;W^{1,p}(\Omega))} \leq\ & C_p(\Omega)\, (\|\partial_t \varphi_0\|_{L^\infty(I\!R_t;W^{1-1/p,p}(\partial\Omega))} + vol(\Omega)^{1/p}\, \omega_d \\ & \|g\|_{L^\infty(I\!R_t\times\Sigma^-)} \left[\left(v_1 + \frac{v_0}{2}\right)^d - \left(\frac{v_0}{2}\right)^d\right] \left(v_1 + \frac{v_0}{2}\right)). \end{aligned}$$

Therefore we choose:

$$\begin{aligned} C_2 \ = \ & C_p(\Omega)\, (\|\partial_t \varphi_0\|_{L^\infty(I\!R_t;W^{1-1/p,p}(\partial\Omega))} + vol(\Omega)^{1/p}\, \|g\|_{L^\infty(I\!R_t\times\Sigma^-)} \\ & \omega_d \left[\left(v_1 + \frac{v_0}{2}\right)^d - \left(\frac{v_0}{2}\right)^d\right] \left(v_1 + \frac{v_0}{2}\right)). \end{aligned} \tag{83}$$

We next claim that (71) holds. Indeed there is a Sobolev constant $C_s(\Omega)$ such that:

$$\begin{aligned} \|\nabla_x \varphi_\alpha^1\|_{C^0(\Omega)} &\leq C_s(\Omega) \cdot \|\nabla_x \varphi_\alpha^1\|_{W^{1,p}(\Omega)} \\ &\leq C_s(\Omega) \cdot \|\varphi_\alpha^1\|_{W^{2,p}(\Omega)} \\ &\leq C_s(\Omega) C_p(\Omega) \cdot (\|\varphi_0\|_{L^\infty(I\!R_t;W^{2-1/p,p}(\partial\Omega))} + vol(\Omega)^{1/p} \\ &\qquad \|g\|_{L^\infty(I\!R_t\times\Sigma^-)} \left[\left(v_1 + \frac{v_0}{2}\right)^d - \left(\frac{v_0}{2}\right)^d\right] \quad \omega_d) \end{aligned} \tag{84}$$

This estimates leads to the conditions on the data which allow to obtain our existence result.

Assumption
From now on we assume that data are small enough in order the following condition is satisfied:

$$\|\varphi_0\|_{L^\infty(I\!R_t;W^{2-1/p,p}(\partial\Omega))} + vol(\Omega)^{1/p}\cdot\|g\|_{L^\infty(I\!R_t\times\Sigma^-)}$$
$$\omega_d\cdot\left[\left(v_1+\frac{v_0}{2}\right)^d-\left(\frac{v_0}{2}\right)^d\right]\leq\frac{|v_0|^2}{4\cdot C_s(\Omega)\cdot C_p(\Omega)\cdot\delta(\Omega)}. \tag{85}$$

We summarized the results we have obtained above in:

Lemma 5 *Under the assumption (85) with C_1 and C_2 given by (81), (83), $\mathcal{F}$ maps $\mathcal{X}$ into itself.*

Step 2 *Continuity.*

We work with the topology of $C^0(I\!R_t;C^1(\overline{\Omega}))$ defined by the norm:

$$|\varphi| = \|\varphi\|_{L^\infty}+\|\nabla_x\varphi\|_{L^\infty}. \tag{86}$$

Let $(\varphi_n)_{n\geq1}\subset\mathcal{X}$ be a sequence $\varphi_n\to\varphi$ in $C^0(I\!R_t;C^1(\overline{\Omega}))$. By the continuity of R_α, we also have $R_\alpha\varphi_n\to R_\alpha\varphi$ in $C^0(I\!R_t;C^1(\overline{\Omega}))$. Let $f_{n,\alpha}$ be the weak solutions of the modified Vlasov equation which corresponds to the field $\nabla_x R_\alpha\varphi_n$. Obviously, we have the estimate:

$$\|f_{n,\alpha}\|_{L^\infty(I\!R_t\times\Omega\times I\!R^d_v)}\leq\|g\|_{L^\infty(I\!R_t\times\Sigma^-)}.$$

By standard compactness results, we can extract a subsequence of $(f_{n,\alpha})_{n\geq1}$ such that:

$$f_{n,\alpha}\rightharpoonup f_\alpha \text{ in } L^\infty(I\!R_t\times\Omega\times I\!R^d_v),\ \ weak\star. \tag{87}$$

Using the weak formulation, for all function $\theta\in\mathcal{V}$ we have :

$$\int_0^T\int_\Omega\int_{I\!R^d_v} f_{n,\alpha}(t,x,v)(-\alpha\cdot\theta+\partial_t\theta+v\cdot\nabla_x\theta+\nabla_x R_\alpha\varphi_n\cdot\nabla_v\theta)dvdxdt =$$
$$= \int_0^T\int_{\Sigma^-} v\cdot\nu(x)\cdot g(t,x,v)\cdot\theta(t,x,v)dvd\sigma dt$$

and we conclude that f_α is a weak solution of the modified Vlasov equation which correspond to the field $\nabla_x R_\alpha\varphi\in C^0(I\!R_t;C^1(\Omega))$:

$$\int_0^T\int_\Omega\int_{I\!R^d_v} f_\alpha(t,x,v)(-\alpha\cdot\theta+\partial_t\theta+v\cdot\nabla_x\theta+\nabla_x R_\alpha\varphi\cdot\nabla_v\theta)dvdxdt =$$
$$= \int_0^T\int_{\Sigma^-} v\cdot\nu(x)\cdot g(t,x,v)\cdot\theta(t,x,v)dvd\sigma dt$$

Now, using *Lemma 2* , we deduce that f_α is the mild solution for the modified Vlasov equation which correspond to the field $\nabla_x R_\alpha \varphi \in C^0(I\!R_t; C^1(\Omega))$:

$$\begin{aligned} < f_\alpha, \theta > \quad &= \quad -\int_0^T \int_{\Sigma^-} \int_t^{\tau_o^\alpha} v \cdot \nu(x) \cdot g(t,x,v) \\ &\cdot \quad \theta(s, X_\alpha(s;x,v,t), V_\alpha(s;x,v,t)) \cdot e^{-\alpha(s-t)} ds dv d\sigma dt. \end{aligned} \tag{88}$$

Since the limit is unique, the whole sequence converges weakly:

$$f_{n,\alpha} \rightharpoonup f_\alpha \text{ weak } \star \text{ in } L^\infty(I\!R_t \times \Omega \times I\!R_v^d).$$

Furthermore, since $f_{n,\alpha}$ have compact support in velocity, we have:

$$\rho_{n,\alpha} \rightharpoonup \rho_\alpha \text{ weak } \star \text{ in } L^\infty(I\!R_t \times \Omega),$$

which yields:

$$\rho_{n,\alpha} \rightharpoonup \rho_\alpha \text{ weak } \star \text{ in } L^r(I\!R_t \times \Omega).$$

Therefore we have:

$$\mathcal{F}(\varphi_n)(t) \rightharpoonup \mathcal{F}(\varphi)(t) \text{ weak } \star \text{ in } W^{2,r}(\Omega), \qquad a.e.t \in I\!R_t.$$

Since $(\mathcal{F}(\varphi_n))_{n\geq 1}$ is compact in $C^0(I\!R_t; C^1(\overline{\Omega}))$ (because $\mathcal{X}$ is compact), the convergence holds in $C^0(I\!R_t; C^1(\overline{\Omega}))$.
Step 3: Passing to the limit for $\alpha \to 0$.
At this point, we may apply the Schauder theorem, which yields an electric potential $\varphi_\alpha \in \mathcal{X}$ and a density f_α such that:

$$\begin{aligned} \int_0^T \int_\Omega \int_{I\!R_v^d} \quad & f_\alpha(t,x,v)(-\alpha \cdot \theta + \partial_t \theta + v \cdot \nabla_x \theta + \nabla_x R_\alpha \varphi_\alpha \cdot \nabla_v \theta) dv dx dt = \\ &= \int_0^T \int_{\Sigma^-} v \cdot \nu(x) \cdot g(t,x,v) \cdot \theta(t,x,v) dv d\sigma dt, \end{aligned} \tag{89}$$

for test function $\theta \in \mathcal{V}$ and:

$$\begin{aligned} \Delta_x \varphi_\alpha = \int_{I\!R_v^d} f_\alpha(t,x,v) dv, \quad (t,x) \quad &\in I\!R_t \times \Omega, \\ \varphi_\alpha = \varphi_0, \quad (t,x) \quad &\in I\!R_t \times \partial\Omega. \end{aligned} \tag{90}$$

In order to complete the proof, we have to pass to the limit for $\alpha \to 0$. Since $(\varphi_\alpha)_{\alpha>0} \subset \mathcal{X}$ which is compact in $C^0(I\!R_t; C^1(\overline{\Omega}))$, we may assume, extracting a subsequence if necessary, that $(\varphi_{\alpha_k})_{k\geq 1}$ converges:

$$\varphi_{\alpha_k} \to \varphi \text{ in } C^0(I\!R_t \times \overline{\Omega}), \tag{91}$$

$$\nabla_x \varphi_{\alpha_k} \to \nabla_x \varphi \text{ in } C^0(I\!R_t \times \overline{\Omega}).$$

We have the same convergences for the regularized potentials:

$$R_{\alpha_k}\varphi_{\alpha_k} \to \varphi \text{ in } C^0(I\!R_t \times \Omega),$$

$$\nabla_x R_{\alpha_k}\varphi_{\alpha_k} \to \nabla_x \varphi \text{ in } C^0(I\!R_t \times \Omega). \tag{92}$$

Indeed, because the extension operator $\bar{\cdot}$ is continuous, for all $(t,x) \in (I\!R_t \times \Omega$ we have:

$$\begin{aligned}
|R_{\alpha_k}\varphi_{\alpha_k}(t,x) - \varphi(t,x)| &= |R_{\alpha_k}\varphi_{\alpha_k}(t,x) - \overline{\varphi}(t,x)| \\
&= |\int \zeta_{\alpha_k}(x-y)[\,\overline{\varphi}_{\alpha_k}(t,y) - \overline{\varphi}(t,x)\,]dy\,| \\
&\leq |\int \zeta_{\alpha_k}(x-y)[\,\overline{\varphi}_{\alpha_k}(t,y) - \{\overline{\varphi}(t,y)\,]\,dy\,| \\
&+ |\int \zeta_{\alpha_k}(x-y)[\,\overline{\varphi}(t,y) - \overline{\varphi}(t,x)\,]\,dy\,| \\
&\leq \|\,\overline{\varphi}_{\alpha_k}(t) - \overline{\varphi}(t)\,\|_{C^0(I\!R^d)} \\
&+ \sup_{|x-y|\leq\alpha_k} |\,\overline{\varphi}(t,y) - \overline{\varphi}(t,x)\,| \longrightarrow 0.
\end{aligned} \tag{93}$$

The second convergence (92) follows in the same way.
Obviously, we have:

$$\alpha_k \int_0^T \int_\Omega \int_{I\!R^d_v} f_{\alpha_k}(t,x,v) \cdot \theta(t,x,v)dvdxdt \to 0, \tag{94}$$

and therefore we conclude that $f_{\alpha_k} \rightharpoonup f$ weak $\star$ in $L^\infty(I\!R_t \times \Omega \times I\!R^d_v)$, where f is a weak solution of the Vlasov equation which corresponds to the field $\nabla_x\varphi$ (we pass to the limit for $\alpha_k \to 0$ in (89)):

$$\int_0^T \int_\Omega \int_{I\!R^d_v} f(t,x,v)(\partial_t\theta + v\cdot\nabla_x\theta + \nabla_x\varphi\cdot\nabla_v\theta)dvdxdt =$$

$$= \int_0^T \int_{\Sigma^-} v\cdot\nu(x)\cdot g(t,x,v)\cdot\theta(t,x,v)dvd\sigma dt, \tag{95}$$

for test function $\theta \in \mathcal{V}$. We have to show that (f,φ) verifies the Poisson equation. Because $(supp f_{\alpha_k})$ are bounded (uniformly with respect to k) we have:

$$\rho_{\alpha_k} \rightharpoonup \rho = \int_{\mathbb{R}^d_v} f(t,x,v)dv, \text{ weak } \star \text{ in } L^\infty(\mathbb{R}_t \times \Omega).$$

Therefore passing to the limit in the sense of distribution in (90) gives $\Delta_x \varphi = \rho$. In view of the convergence (91) we also have $\varphi = \varphi_0$ on $\partial\Omega$. We summarize our results in the following theorem:

Theorem 3 *Let g and φ_0 be T periodic functions, Ω a bounded subset of $\mathbb{R}^d$ with $\partial\Omega \in C^1$, $p > d+1$ and $0 < v_0 < v_1$ such as:*

$$supp(g) \subset \{(t,x,v) \mid t \in \mathbb{R}_t, x \in \partial\Omega, 0 < v_0 \le -v \cdot \nu(x) \le |v|l \le v_1\},$$

$$g \in L^\infty(\mathbb{R}_t \times \Sigma^-),$$

$$\varphi_0 \in L^\infty(\mathbb{R}_t; W^{2-1/p,p}(\partial\Omega)),$$

$$\partial_t \varphi_0 \in L^\infty(\mathbb{R}_t; W^{1-1/p,p}(\partial\Omega)),$$

$$\|\varphi_0\|_{L^\infty(\mathbb{R}_t;W^{2-1/p,p}(\partial\Omega))} + K\|g\|_{L^\infty(\mathbb{R}_t\times\Sigma^-)} \le M,$$

with :

$$K = vol(\Omega)^{1/p}\, \omega_d \left[\left(v_1 + \frac{v_0}{2}\right)^d - \left(\frac{v_0}{2}\right)^d\right],\ M = \frac{|v_0|^2}{4\, C_s(\Omega)\, C_p(\Omega)\, \delta(\Omega)},$$

where ω_d is the volume of the unit ball of $\mathbb{R}^d$, $C_p(\Omega)$ is given by (80) (regularity result for the Poisson problem) and $C_s(\Omega)$ is given by (84) (Sobolev embedding). Then, the system (68) has at least one weak periodic solution.

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