Estimation of Pickands dependence function of bivariate extremes under mixing conditions

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Abstract

In this paper, we study some asymptotic properties of CFG's estimator of Pickands dependence function of strictly stationary absolutely regular sequences of bivariate extremes. We then propose an asymptotic test of independence of the vector's margins. The finite sample properties of the estimate and the proposed test are investigated by simulation.

Classification Mathematic Subject: 62G32, 62G10.

Keywords: Absolutely regular sequence ; Bivariate extreme value distribution; Copulas; Independence test; Non-parametric estimation; Pickands dependence function; Strictly stationary process.

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1 Introduction

Modeling dependence structures of multivariate extremes is of great interest in many application fields such as for instance risk management and environmental studies (some applications can be found in [8], [34]). A well known way to model these structures is to use Pickands dependence function [26]. Let (X_1, X_2) be a bivariate vector of extremes with marginals F_1 and F_2. Thus, Pickands dependence function A is defined via the extreme-value copula's type representation:

$$C(u,v) = \mathbb{P}(F_1(X_1) \leq u, F_2(X_2) \leq v) = \exp\left\{\log(uv)A\left(\frac{\log(u)}{\log(uv)}\right)\right\},\ 0 \leq u, v \leq 1, \tag{1}$$

and totally characterizes the joint distribution $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$ of (X_1, X_2) knowing its marginal laws. It may be shown that $A : [0,1] \to [1/2, 1]$ is a convex function such that $A(0) = A(1) = 1$ and $\max(t, 1-t) \leq A(t) \leq 1$. The upper bound $A(t) = 1$ for all $t \in [0,1]$ corresponds to the independence copula $C(u,v) = uv$ for $u, v \in [0,1]$ while the lower bound $A(t) = \max\{t, 1-t\}$ corresponds to the comonotone copula $C(u,v) = \min\{u, v\}$.

The problem of estimating Pickands dependence function by nonparametric methods has been extensively studied in the literature. From the pioneer estimator of Pickands [26], several alternative estimators have been proposed and studied (see e.g. [10], [17], [25], [21], [31], [13], [5], [7], [6] in the bivariate setting and [34], [11] [24], [2], [15], [16] in the multivariate setting). One of the assumptions of the above mentioned studies is that the sequence of extremes values used for estimation is i.i.d., which excludes a possible serial correlation of the sequence. This bias is to a certain extent supported by theoretical results on maxima of strictly stationary sequences (see [22, 23] in the univariate case and [18, 19] in the multivariate case). A key result is the condition "$D(u_n)$" of [22, 23] ensuring that under some kind of mixing condition on the underlying stationary process, the maximum of the process asymptotically follows an extreme value distribution as in the i.i.d. case, and that sufficiently separated rare events are almost independent, thereby justifying the use of the block maximum approach for most stationary time series. However, in practical situations, it is well known that temporal dependence of the underlying series leads to local temporal clusterings of its extreme values, so that the temporal independence of extremes is usually an unrealistic assumption. In this paper, we propose to study the properties of a classical estimator of Pickands dependence function, the so called CFG estimator (see [6] and [34]), based on a sequence which is assumed to be strictly stationary and absolutely regular in [30]'s sense.

Formally, let $\mathcal{P}_0 = \sigma(X_t, t \leq 0)$, $\mathcal{F}_m = \sigma(X_t, t \geq m)$ and define the decreasing sequence of absolutely regular coefficients of X by

$$\beta(m) \quad = \sup_{A_i \in \mathcal{P}_0, B_j \in \mathcal{F}_m} \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)| \tag{2}$$

where the supremum is taken over all pairs of partition $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, A_J\}$ of a set Ω such that $A_i \in \mathcal{P}_0$ for each i and $B_j \in \mathcal{F}_m$ for each j. We say that X is β-mixing if it satisfies the condition:

$$\lim_{m \to +\infty} \beta(m) = 0.$$

A lot of classical models satisfy this condition, in particular the important class of linear stochastic processes are absolutely regular, provided that they are based on innovation random variables with a Lebesgue-integrable characteristic function. In order to make the presentation clearer, we place ourselves in a bivariate setting, although the extension to the multivariate case is straightforward. The paper is organized as follows. In section 2, we recall the definition of the CFG estimator and its properties in the i.i.d setting. In Section 3, we study the consistency and asymptotic normality of the CFG estimator in our dependent setting. We moreover propose a test of independence of the vector's margins. Section 4 presents a simulation study allowing to investigate the finite sample properties of the estimate and to evaluate the performance of the test. Section 5 is devoted to the proofs.

2 CFG estimator of the dependence function

Let $X = (X_t)_{t \in \mathbb{Z}}$ with $X_t = (X_{t,1}, X_{t,2})$ be a strictly stationary process such that X_t has a bivariate extreme value distribution (BEV). To fix ideas, we can think of X_t as the pair of largest values of two characteristics observed at the same time t. We denote by F the joint distribution X_t. Recall that $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$, where C is a copula function defined by (1) and the marginals F_1 and F_2 of $X_{1,t}$ and $X_{2,t}$ belong to the parametric family of generalized extreme distributions (GEV) (see [14]). Thus C and F only depends on the one-dimensional dependence function A as soon as F_1 and F_2 are known. Among the numerous estimators of A proposed in the literature, the CFG estimator proposed in [6] has been shown to perform better than its major competitors from a theoretical point of view and a prior finite sample study seems to confirm its superiority in numerous practical situations. In order to define the CFG estimator based on a size n stationary sequence $(X_{i,1}, X_{i,2})_{1 \le i \le n}$ of X, let us define as in [34] an auxiliary bivariate sequence $Z_i = (Z_{i,1}, Z_{i,2})_{1 \le i \le n}$ by

$$Z_{i,1} = \frac{\log F_2(X_{i,2})}{\log F_1(X_{i,1}) + \log F_2(X_{i,2})}, \quad Z_{i,2} = \frac{\log F_1(X_{i,1})}{\log F_1(X_{i,1}) + \log F_2(X_{i,2})}, \quad i = 1, \ldots, n. \tag{3}$$

Notice that the Z_{ij}'s belong to $[0, 1]$. Thus, when A has a first order derivative, it may be expressed as a function of the distributions $H_1(z) = \mathbb{P}(Z_{i,1} \le z)$ or $H_2(z) = \mathbb{P}(Z_{i,2} \le z)$. More precisely, one has by [6]'s Proposition 2.1

$$H_1(z) = z + z(1-z)\frac{d}{dz}\log A\left(\frac{zs}{1-s}\right), \quad H_2(z) = z + z(1-z)\frac{d}{dz}\log A(z), \tag{4}$$

so that solving the differential equations leads to two representations of A

$$\log A_1(s) = \int_0^{1-s} \frac{H_1(z) - z}{z(1-z)} dz \text{ and } \log A_2(s) = \int_0^{s} \frac{H_2(z) - z}{z(1-z)} dz. \tag{5}$$

Replacing the unknown H_1 and H_2 by their empirical counterparts leads to the estimators

$$\hat{A}_1(s) = \exp\left\{\int_0^{1-s} \frac{\hat{H}_1(z) - z}{z(1-z)} dz\right\}, \text{ and } \hat{A}_2(s) = \exp\left\{\int_0^{s} \frac{\hat{H}_2(z) - z}{z(1-z)} dz\right\}. \tag{6}$$

Therefore, one may propose for A the weighted estimator such that:

$$\log \hat{A}(s) = \lambda(s) \int_0^{1-s} \frac{\hat{H}_1(z) - z}{z(1-z)} dz + (1-\lambda(s)) \int_0^{s} \frac{\hat{H}_2(z) - z}{z(1-z)} dz, \tag{7}$$

leading to

$$\hat{A}_n(s) = \left(\hat{A}_1(s)\right)^{\lambda(s)} \left(\hat{A}_2(s)\right)^{1-\lambda(s)}, \ \hat{A}_n(1) = 1,$$

where $\lambda(s)$ is an appropriately chosen nonnegative weight function in $(0,1)$. Notice that this definition of $\hat{A}_n$ is the particular case in our bivariate setting of [34]'s definition, given in a multivariate setting. In the bivariate case, one has $Z_{i,1} = 1 - Z_{i,2}$, $H_2(z) = 1 - H_1(1-z)$ so that (5) squares with [6]'s Equation (2), replacing λ by p, A_n^0 by A_1 and A_n^1 by A_2. When λ is a bounded function on $[0,1]$, a closed form expression for $\hat{A}_n$ is given in [6]. Namely,

$$\hat{A}_n(t) = \begin{cases} (1-t)Q_n^{1-\lambda(t)} & \text{if } 0 \le t \le Z_{(1)2} \\ t^{i/n}(1-t)^{1-i/n} Q_n^{1-\lambda(t)} Q_i^{-1} & \text{if } Z_{(i)2} \le t \le Z_{(i+1)2} \ (1 \le i \le n-1) \\ tQ_n^{-\lambda(t)} & \text{si } Z_{(n)2} \le t \le 1, \end{cases} \tag{8}$$

with $Q_i = \left\{\prod_{k=1}^{i} \frac{Z_{(k)2}}{1 - Z_{(k)2}}\right\}^{1/n}$ and $Z_{(i)2}$ the i^{th} order statistic of the sample $(Z_{1,2}, \ldots, Z_{n,2})$. Notice that since $\hat{H}_1$ and $\hat{H}_2$ are discontinuous functions, $\hat{A}_n$ is not a convex function. Moreover, $\hat{A}_n(1) \neq 1$ for arbitrary functions λ. Following [34], we can put $\lambda(s) = s$ in order to achieve this property. An optimal choice for λ is given in [34]'s Remark 3.

When the margins of $(X_{i,1}, X_{i,2})_{1 \le i \le n}$ are i.i.d., [6]'s Proposition 4.1 states that when A has a bounded first derivative, $\hat{A}_n$ is a uniformly strongly consistent estimator of A. Namely,

$$\sup_{s \in [0,1]} |\hat{A}_n(s) - A(s)| \xrightarrow{a.s.} 0. \tag{9}$$

Moreover, [6]'s Proposition 3.2 gives the weak convergence of the estimate to a Gaussian process. More precisely, using the formulation of [34],

$$\sqrt{n}(\log \hat{A}_n - \log A) \xrightarrow{\mathcal{D}[0,1]} U, \tag{10}$$

with

$$U(s) = \sum_{j=1}^{2} \lambda_j(s) \int_0^{1-s_j} \frac{B_j(z)}{z(1-z)} dz,$$

where $s_1 = s$, $s_2 = 1-s$, $t_1 = t$, $t_2 = 1-t$, $\lambda_1 = 1 - \lambda_2$, $B_1(z) = B(z_1, 1)$, $B_2(z) = B(1, z_2)$ and B is a bivariate centered Gaussian process with covariance function

$$\mathbb{E}(B(z)B(z')) = Var(\mathbb{1}_{Z_1 \le z \wedge z'}), \ (z, z') \in \mathbb{R}^4 \tag{11}$$

It may be easily shown that U is a mean zero Gaussian process with covariance function :

$$\Gamma(s,t) = \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_i(s)\lambda_j(t) \int_0^{1-s_i}\int_0^{1-t_j} \frac{H_{ij}(z_1,z_2) - H_i(z_1)H_j(z_2)}{z_1 z_2(1-z_1)(1-z_2)} dz_1 dz_2, \tag{12}$$

where $H_{ij}(z_1, z_2) = \mathbb{P}(Z_{1,i} \le z_1, Z_{1,j} \le z_2)$. In particular, one has for all $s \in [0,1]$,

$$\sqrt{n}(\log \hat{A}_n(s) - \log A(s)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma(s)), \text{ with } \Gamma(s) = \Gamma(s,s). \tag{13}$$

Notice that a consistent estimator of $\Gamma(s)$ is easily obtained by replacing H_{ij}, H_i and H_j by their empirical estimators in (22). For statistical purposes, it is possible to choose the weight functions λ_1 and λ_2 so at to minimize $\Gamma(s)$.

In the following, we propose to see what is going on with these properties if the sequences $(X_{i,1}, X_{i,2})_{1 \le i \le n}$ have some kind of weak dependence.

Remark 1. *Notice that in our bivariate setting $\Gamma(t)$ can be easily expressed as a function of H_1 only as in [6]'s Proposition 3.2, using the fact that $H_2(z) = 1 - H_1(1-z)$ and $H_{12}(z_1, z_2) = H_1(z_1 \vee (1 - z_2)) - H_1(1 - z_2)$.*

Remark 2. *Extensive numerical work suggest that the CFG estimator performs better than its classical competitors. Nevertheless, it suffers from limitations. Firstly, the margins F_1 and F_2 of X_1 and X_2 are assumed to be known, so that a sample $(F_1(X_{1,1}), F_2(X_{1,2})), \ldots, (F_1(X_{n,1}), F_2(X_{n,2}))$ from A is available. In practice, however, margins are rarely known. In [13], the authors propose to estimate F_1 and F_2 by their empirical counterparts $\hat{F}_{1n}$ and $\hat{F}_{2n}$ and to base the estimation of A on the pseudo-observations $(\hat{F}_{1n}(X_{1,1}), \hat{F}_{2n}(X_{1,2})), \ldots, (\hat{F}_{1n}(X_{n,1}), \hat{F}_{2n}(X_{n,2}))$, which amounts to working with the pairs of scaled ranks. They show that their rank-based version of CFG estimators of A has the same asymptotic properties as the classical one and assess its finite sample superiority by a simulation study. Secondly, the CFG estimator is neither convex nor do it satisfies the boundary restriction $\max\{t, 1-t\} \le A(t) \le 1$, in particular the endpoint constraints $A(0) = A(1) = 1$. In [12], the authors propose a modified version of [13] estimator which fits the above contraints, without changing the asymptotic properties.*

3 The CFG estimator for absolutely regular sequences

Hereafter, we assume that $(X_t)_{t\in\mathbb{Z}}$, $X_t = (X_{t,1}, X_{t,,2})$ is an absolutely regular strictly stationary process with BEV distribution F and margins F_1 and F_2. We denote by A, the Pickands dependence function in (1) and by $\widehat{A}_n$ the CFG estimator of A defined by (7), based on a sequence $(X_1, \ldots, X_n)$ of X. In the following, we study the asymptotic properties of $\widehat{A}_n$ in this setting and propose a test of independence for the margins of X.

3.1 Asymptotic properties

Let B^* be a bivariate centered Gaussian process with covariance function

$$\mathbb{E}(B^*(z)B^*(z')) = \sum_{k\in\mathbb{Z}} Cov(\mathbb{1}_{Z_0\le z}, \mathbb{1}_{Z_k\le z'}) \tag{14}$$

and denote by $D[0,1]$ in the usual D space on $[0,1]$ with Skorokhod topology (see. [3]). Thus, we have the following

Theorem 3.1. *Let $(X_1,\ldots,X_n)$ be an absolutely regular strictly stationary sequence with β-mixing coefficients $(\beta(n))_{n>0}$. Suppose $A(s)$ has a bounded first derivative and that λ in (7) is a bounded function on $[0,1]$. Then,*

i) If $\beta(n) = O(n^{-\theta})$ for some $\theta > 1+\sqrt{2}$ thus, one has

$$\sup_{s\in[0,1]} |\hat{A}_n(s) - A(s)| \xrightarrow{P} 0, \tag{15}$$

ii) If $\beta(n) = O(n^{-\theta})$ for some $\theta \in (1,2]$ then

$$\sqrt{n}(\log \hat{A}_n(s) - \log A(s)) \xrightarrow{D} U(s) = \sum_{j=1}^{2} \lambda_j(s) \int_0^{1-s_j} \frac{B_j^*(z)}{z(1-z)} dz = \mathcal{N}(0, \Gamma^*(s)) \tag{16}$$

where $s_1 = s$, $s_2 = 1-s$, $t_1 = t$, $t_2 = 1-t$, $\lambda_1 = 1-\lambda_2$, $B_1^*(z) = B^*(z_1, 1)$, $B_2^*(z) = B^*(1, z_2)$ and $\Gamma^*(s) = \mathbb{E}(U(s)^2) = \Gamma^*(s,s)$ with

$$\Gamma^*(s,t) = \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_i(s)\lambda_j(t) \int_0^{1-s_i}\int_0^{1-t_j} \frac{\mathbb{E}(B_i^*(z_1)B_j^*(z_2))}{z_1 z_2(1-z_1)(1-z_2)} dz_1 dz_2 < \infty \tag{17}$$

Remark 3. *Notice that asymptotic confidence intervals for $\widehat{A}_n(s)$ may be easily built as soon as we get a suitable estimator $\hat{\Gamma}_n^*(s)$ for $\Gamma^*(s)$. At the confidence level $1-\alpha$ and for large enough n , one has*

$$P\left(-q_{1-\frac{\alpha}{2}} \le \sqrt{\frac{n}{\Gamma^*(s)}} \log \frac{\hat{A}_n(s)}{A(s)} \le q_{1-\frac{\alpha}{2}}\right) \simeq 1-\alpha,$$

so that

$$CI_{1-\alpha} = \left[\hat{A}_n(s) e^{-q_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\Gamma}_n^*(s)}{n}}},\ \hat{A}_n(s) e^{q_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\Gamma}_n^*(s)}{n}}}\right],$$

where $q_{1-\alpha/2}$ is the order $(1-\alpha/2)^{th}$ quantile of the normal distribution.

3.2 Testing for independence

Several tests of independence for bivariate extremes have been studied in the i.i.d. case by [1, 6] [20], [32] and [33]. Following the same scheme as [6], we can exploit Theorem 3.1 to construct a test for pairwise independence of the extreme process. More precisely, we wish to test:

$$\begin{cases} H_0 : A(t) = 1 \; \forall t \in [0,1] \\ H_1 : \exists \; t / A(t) \neq 1, \end{cases}$$

based on a sequence $(X_{i,1}, X_{i,2})_{1 \le i \le n}$ of the strictly stationary absolutely regular bivariate extreme process X. For that task, we will use the measure of association proposed in [32] and [33]. Set $m = 2(1 - A(1/2))$. One has $m = 1$ in case of total dependance and $m = 0$ in case of independence, so that the above test may be rewritten as

$$\begin{cases} H_0 : A(\frac{1}{2}) = 1 \\ H_1 : A(\frac{1}{2}) \neq 1. \end{cases}$$

Thus, let's define the test statistic

$$U_n \quad = \quad \sqrt{\frac{n}{\widehat{\Gamma}^*_n\left(\frac{1}{2}\right)}} \log \hat{A}_n\left(\frac{1}{2}\right). \tag{18}$$

Under H_0, $U_n \xrightarrow{D} N(0,1)$ so that for a nominal level α, we can base our test on the critical region

$$R_\alpha = \left\{(X_{i,1}, X_{i,2})_{1 \le i \le n}, U_n > q_{1-\alpha/2}\right\},$$

where $q_{1-\alpha/2}$ is the order $1 - \alpha/2$ quantile of a standard Gaussian distribution.

4 A simulation study

In the sequel, we run a simulation study allowing to investigate the finite sample properties of the CFG's estimator and to evaluate the performance of the test proposed in subsection 34, based on bivariate logistic distributions (see [32]) This model is known to be flexible enough to cover a wide range of dependence functions for bivariate extremes.

4.1 Models

To generate a bivariate extremes $(X_1, \dots, X_n)$, $X_i = (X_{i,1}, X_{i,2})$ sequence which is not i.i.d. we first generate an i.i.d. bivariate sequence $(Y_1, \dots, Y_n)$, $Y_i = (Y_{i,1}, Y_{i,2})$ arising from a Gumbel copula (see Gumbel 1960), and hence with the following symmetric logistic dependence function :

$$A_Y(t) = (t^{\frac{1}{r}} + (1-t)^{\frac{1}{r}})^r, \; r \in (0,1), \tag{19}$$

with marginal distributions G_1 and G_2.

Then we set

$$X_i = \begin{pmatrix} \max(Y_{i-(k-1),1}, \dots, Y_{i-1,1}, Y_{i,1}) \\ Y_{i,2} \end{pmatrix} \; 1 \le i \le n, \tag{20}$$

Thus, $(X_i)_{i=1,\dots,n}$ is a strictly stationary k-dependent bivariate sequence with marginal distributions $F_1 = G_1^k$, $F_2 = G_2$ and dependence function given by the following

Proposition 4.1. *Let* $(Y_i)_{1\leq i\leq n}$ *be an i.i.d sequence of bivariate extremes with Gumbel copula* (19). *Thus, the bivariate sequence (20) has an asymetric logistic copula with the following Pickands dependence function.*

$$A_X(t) = \left(1-\frac{1}{k}\right)t + \left(\left(\frac{t}{k}\right)^{\frac{1}{r}} + (1-t)^{\frac{1}{r}}\right)^r, \ r\in(0,1), k\geq 1$$

Remark. Since $X = (X_i)_{i=1,\ldots,n}$ is k-dependent, it follows that the mixing coefficient defined in (2) is such that $\beta(m) = 0\ \forall\, m > k$, and hence X is β-mixing.

Independence between margins is obtained when $r = 1$ while dependence increases as r goes to zero.

To illustrate the serially correlation of X, we simulate some realisations by assuming that Y has a standard Gumbel marginal distributions $G_1 = G_2$ and that X is 1-dependent and given by the equation (20).

Note that in this case the bivariate distribution of Y is given by

$$F_Y(y_1, y_2) = \exp\left(-(e^{-y_1}+e^{-y_2})\left(\left(\frac{e^{-y_1}}{e^{-y_1}+e^{-y_2}}\right)^{\frac{1}{r}} + \left(\frac{e^{-y_2}}{e^{-y_1}+e^{-y_2}}\right)^{\frac{1}{r}}\right)^r\right).$$

Figure 1 shows the sequence $(Y_i)_{1\leq i\leq n}$ for different sizes $n = 100, 1000, 5000$, and parameters dependency $r = 0.1, 0.5, 0.9$.

Figure 2 shows that the serial correlation of the sequence $(X_i)_{1\leq i\leq n}$ is significant, observe that $X_{i,1}$ and $X_{i-1,1}$ are dependent (serial correlation), $X_{i,1}$ and $X_{i,2}$ are also dependent (the components of X_i are dependent since $r\neq 1$).

In this section we will investigate, by simulation, the behaviour of the CFG's estimator. Following the remark 2 of section 2 we will consider only the empirical version of this estimator, i.e. we consider the CFG's estimator given by the equations (8) and (3), but in the last one we replace the marginal distributions F_1 and F_2 by their empirical estimators

$$\hat{F}_1(x) = \frac{1}{n+1}\sum_{t=1}^{n}\mathbb{1}_{X_{t,1}\leq x},\quad \hat{F}_2(x) = \frac{1}{n+1}\sum_{t=1}^{n}\mathbb{1}_{X_{t,2}\leq x}$$

where $\mathbb{1}_A$ is equal to 1 if A is true and equal to 0 otherwise.

An extensive simulation shows that this empirical estimator has a good properties than the original one, moreover it has also the advantage that no parametric form is assumed for the marginal distributions F_1 and F_2.

All the following properties are based on $R = 1000$ replications.

4.2 Finite sample properties of the CFG's estimator

In this subsection we investigate, by simulation, the behaviour of the mean integrated square error (MISE) of the empirical CFG's estimator which is defined as

$$MISE = \int_0^1 E(\hat{A}_n(s) - A(s))^2 ds, \tag{21}$$

Figure 1 – Scatter plots of Y for different values of n and r

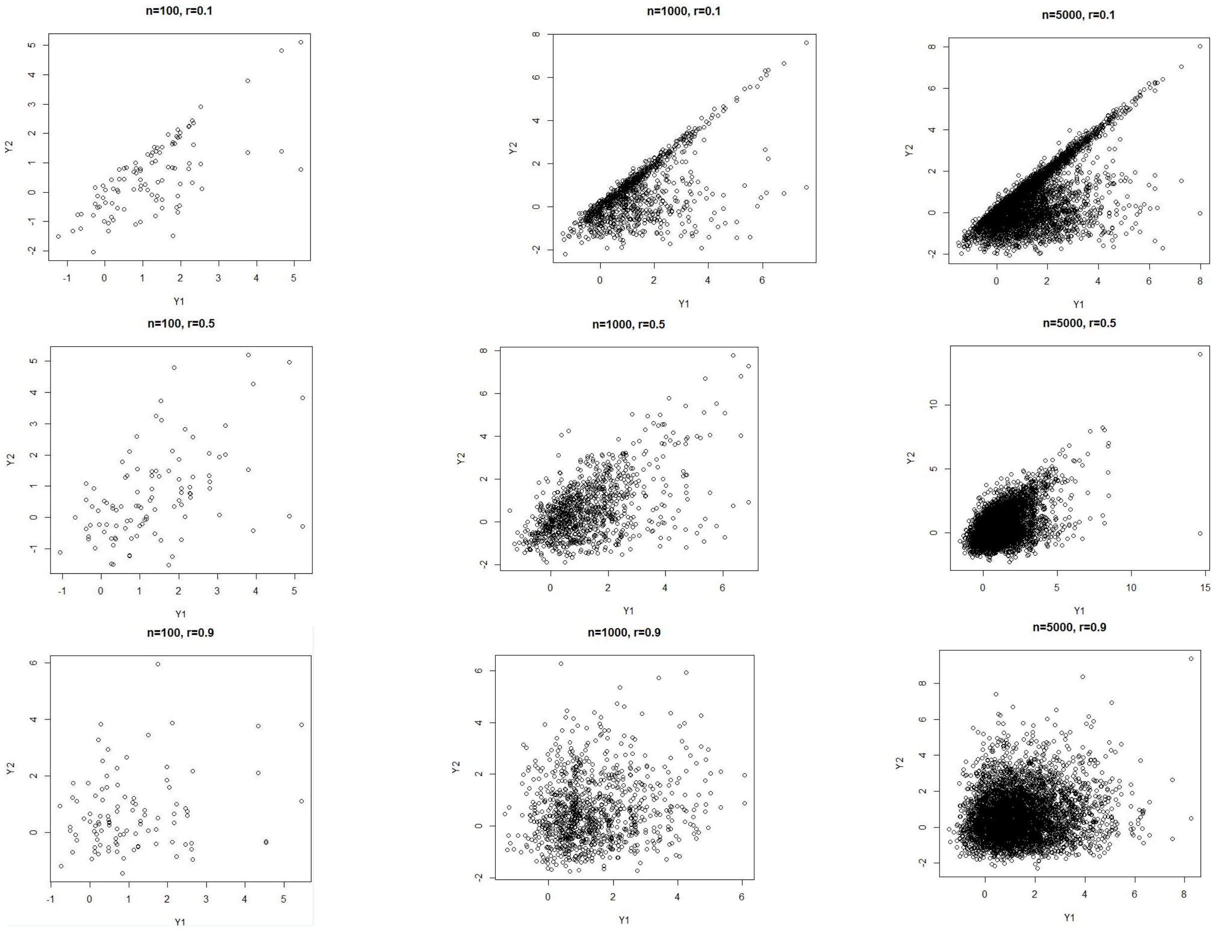
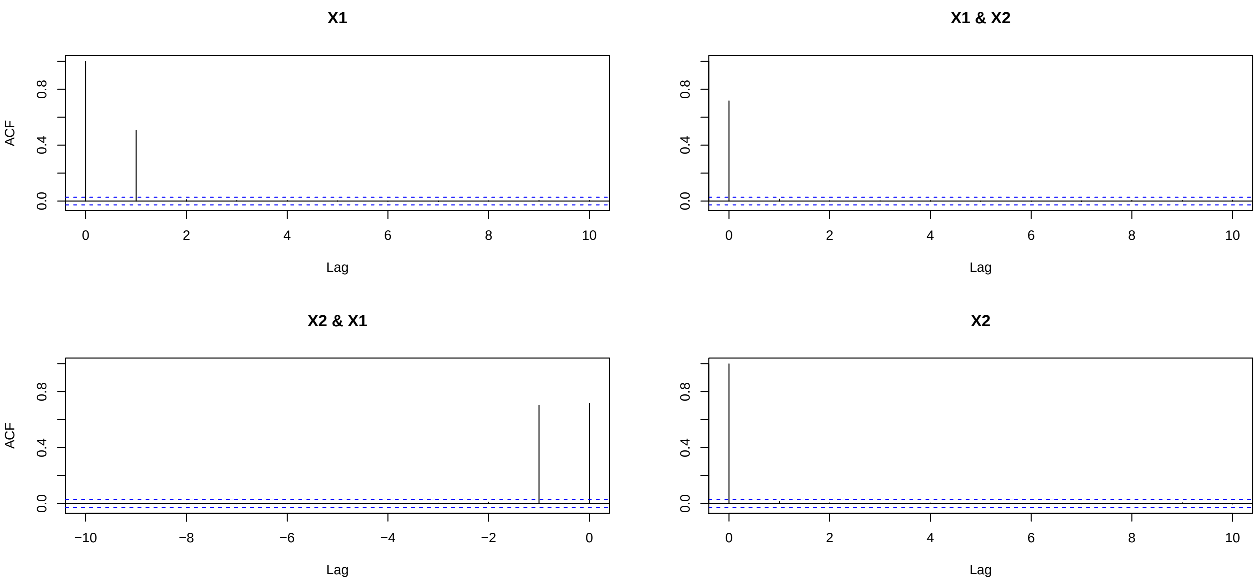


Figure 2 – Correlation of X ($n = 5000$ and $r = 0.1$)



r=	n=50	n=100	n=200	n=500
0.1	1.885537	0.9952461	0.4765154	0.1917919
0.2	6.948347	3.275925	1.698192	0.6732344
0.5	53.33643	25.70162	12.92988	5.148036
0.7	142.7788	74.63336	35.32554	12.89887
0.9	319.2063	154.8809	85.00207	32.37675
1	455.4138	273.8141	122.5095	49.02234

Table 1 – $10^5\times$ estimated MISE for the CFG's estimator with $k = 1$.

r=	n=50	n=100	n=200	n=500
0.1	106.0859	49.1597	25.32425	9.812296
0.2	111.5493	49.61342	27.13266	9.482449
0.5	162.2376	80.07837	39.35513	15.13582
0.7	275.2151	128.0954	62.669	26.31388
0.9	470.7487	229.0321	118.0711	45.28518
1	566.4828	298.0881	172.282	60.13233

Table 2 – $10^5\times$ estimated MISE for the CFG's estimator with $k = 2$.

where $\hat{A}$ is the estimator of $A(s)$.

In our simulation we estimate the MISE by

$$\widehat{MISE} = \frac{1}{RM}\sum_{i=1}^{R}\sum_{j=1}^{M}\left(\hat{A}_{n,i}(s_j) - \hat{A}(s_j)\right)^2,$$

where $\hat{A}_{n,i}$ is the empirical CFG's estimator of A in the i^{th} replication and $s_j = j/M$, and M is the size of grid on $[0,1]$ to obtain an appproximation of the integral in 21, in the following we choose $M = 1000$. We vary the within-dependence coefficient k as well as the between-dependence coefficient r, taking $k \in \{2,3,4\}$ and $r \in \{0.1, 0.5, 0.9, 1\}$ ($r = 1$ corresponds to independence between $X_{i,1}$ and $X_{i,2}$ while $k = 1$ corresponds to independence of the $X_{i,1}$'s). The weight function in (8) is taken to be equal to $\lambda(s) = 1 - s$.

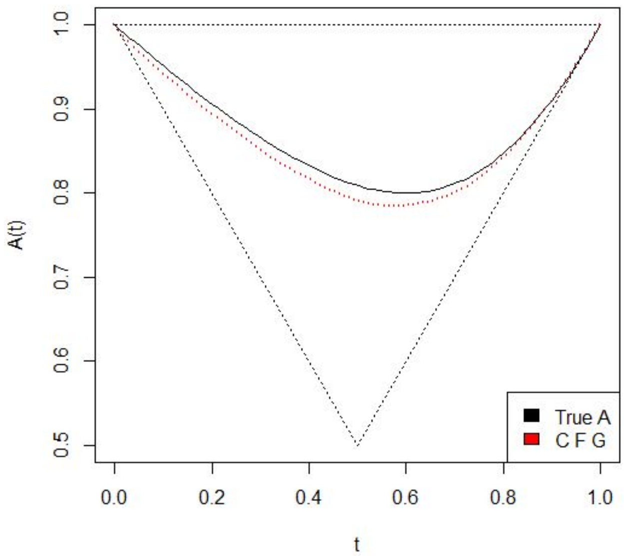
Tables 1 to 3 show that the precision of our estimate increases with the sample size and the dependence between X's components, and it decreases as the dependence within the X_{i1}'s increases.

Figure 3 shows the real dependence function and the CFG's estimator for one replication.

r=	n=50	n=100	n=200	n=500
0.1	267.7836	125.0626	55.09616	26.68735
0.2	247.1549	142.4403	67.84884	25.69461
0.5	281.8439	152.3831	76.44616	31.39142
0.7	418.3119	219.1956	99.73068	37.57211
0.9	674.0237	297.5162	153.1762	57.719
1	798.3534	386.3616	191.5574	72.67077

Table 3 – $10^5\times$ estimated MISE for the CFG's estimator with $k = 3$.

Figure 3 – The empirical CFG's estimator, n=



4.3 Size distortion and power of the proposed test for independence

To test independence by using the test statistic U_n defined by (34) we need an estimate $\widehat{\Gamma}_n^*\left(\frac{1}{2}\right)$ for the asymptotic variance $\Gamma^*\left(\frac{1}{2}\right)$. Straigtforward computations, based on multivariate smoothed periodogram see [27], lead to

$$\widehat{\Gamma}_n^*\left(\frac{1}{2}\right) = M^2 \sum_{i=1}^{M}\sum_{j=1}^{M} \frac{1}{ij(2M-i)(2M-j)} \left(F_{1,1}^{(i,j)} + F_{1,2}^{(i,j)} + F_{1,2}^{(j,i)} + F_{2,2}^{(i,j)}\right), \tag{22}$$

where

$$F_{r,s}^{(i,j)} = \sum_{|k|\le m} w\left(\frac{k}{m+1}\right)\hat{\gamma}_{r,s}^{(i,j)}(k),$$

$$\hat{\gamma}_{1,1}^{(i,j)}(k) = \frac{1}{n}\sum_{t=1}^{n-|k|}\left(\mathbb{1}_{Z_{t,1}\le t_i} - \overline{Z_{i,1}}\right)\left(\mathbb{1}_{Z_{t+k,1}\le t_j} - \overline{Z_{j,1}}\right),$$

$$\overline{Z_{j,1}} = \frac{1}{n}\sum_{t=1}^{n}\mathbb{1}_{Z_{t,1}\le t_j},$$

$$\hat{\gamma}_{1,2}^{(i,j)}(k) = \hat{\gamma}_{2,1}^{(i,j)}(k) = \frac{1}{n}\sum_{t=1}^{n-|k|}\left(\mathbb{1}_{Z_{t,1}\le t_i} - \overline{Z_{i,1}}\right)\left(\mathbb{1}_{Z_{t+k,2}\le t_j} - \overline{Z_{j,2}}\right),$$

$$\overline{Z_{j,2}} = \frac{1}{n}\sum_{t=1}^{n}\mathbb{1}_{Z_{t,2}\le t_j},$$

$$\hat{\gamma}_{2,2}^{(i,j)}(k) = \frac{1}{n}\sum_{t=1}^{n-|k|}\left(\mathbb{1}_{Z_{t,2}\le t_i} - \overline{Z_{i,2}}\right)\left(\mathbb{1}_{Z_{t+k,2}\le t_j} - \overline{Z_{j,2}}\right).$$

$t_i = \frac{i}{2M}$ and $Z_i = (Z_{i,1}, Z_{i,2})_{1\le i\le n}$ is given by (3). Following [4] we choose the Parzen window $w(x) = 1 - x^2$ and the truncation parameter is such that $1/m + m/n \to 0$ as $n \to \infty$, and M is large enough.

A simulation study shows the test statistic based on the estimator 22 has a very large size distortion for moderate sample size and hence can not be used for extreme values data.

To overcome this problem we will estimate the asymptotic variance $\Gamma^*\left(\frac{1}{2}\right)$ by using the block bootstrap which is adapted to time series with serially correlation (see appendix for more detail).

4.3.1 Size

To study the size of the test we simulate the bivariate random sequence X_i as in (20) with $r = 1, k = 1$, G_1 and G_2 are standard Gumbel distribution, hence X is a 1-dependent process but its components $X_{i,1}$ and $X_{i,2}$ are independent.

The Table 4 shows that as the sample size n increases the empirical size becomes close to the nominal level α.

Level α \ Size of the sample n	n=25	n=50	n=100	n=200
$\alpha = 1\%$	4.6	3.2	2.1	1.4
$\alpha = 5\%$	11.5	8.9	6.8	6
$\alpha = 10\%$	16.7	14.3	12.3	11.6

Table 4 – Empirical test sizes in %.

Level α \ Size of the sample n	n=25	n=50	n=100	n=200
$\alpha = 1\%$	64.9	93.1	99.7	100
$\alpha = 5\%$	83.7	97.8	100	100
$\alpha = 10\%$	90.1	99.5	100	100

Table 5 – Empirical test powers in %.

4.3.2 Power

To study the power of the test we simulate the bivariate random sequence X_i as in (20) with $r = 0.5, k = 1$, G_1 and G_2 are standard Gumbel distribution, hence X is a 1-dependent process and its components $X_{i,1}$ and $X_{i,2}$ are dependent.

The Table 5 shows that as the sample size n increases the empirical power becomes close to 100%; therefore our test is a more powerfull even for a moderate sample. Other simulation, not reported in this paper, shows that when r is close to 0, for example $r = 0.1$ then the test statistic has a very good power even for small samples.

5 Proofs

5.1 Proof of Theorem 3.1

5.1.1 Proof of i)

By (5) and (7), one has

$$\log \hat{A}(s) - \log A(s) = \lambda(s) \int_0^{1-s} \frac{\hat{H}_1(z) - H_1(z)}{z(1-z)} dz + (1 - \lambda(s)) \int_0^{s} \frac{\hat{H}_2(z) - H_2(z)}{z(1-z)} dz.$$

Following [34]'s proof of theorem 1, we write for $j = 1, 2$ and $\nu \in (0, 1/2)$

$$\frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{z(1-z)} = \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{(H_j(z)(1 - H_j(z)))^{\nu}} \left(\frac{H_j(z)(1 - H_j(z))}{z(1-z)}\right)^{\nu} (z(1-z))^{\nu - 1} \tag{23}$$

and we will show that the supremum over the integration interval of the two first terms at the right hand side of (23) are bounded in probability. Hence,

$$\sup_{s\in[0,1]}\left|\int_0^{1-s}\frac{\hat{H}_j(z)-H_j(z)}{z(1-z)}dz\right| \le \frac{C}{\sqrt{n}}\int_0^1 (z(1-z))^{\nu-1}dz = o_p(1)$$

so that since λ is a bounded function on $[0,1]$

$$\sup_{s\in[0,1]}\left|\log\hat{A}(s)-\log A(s)\right| = o_p(1),$$

and i) holds by continuity of the log function.

— To show that

$$\sup_{z\in[0,1-s_j]}\left|\frac{H_j(z)(1-H_j(z))}{z(1-z)}\right| < C,$$

let us set $D_1(z)=\frac{d}{dz}\log A\left(\frac{zs}{1-s}\right)$ and $D_2(z)=\frac{d}{dz}\log A(z)$. By (4) one has for $j=1,2$ $H_j(z)=z+z(1-z)D_j(z)$. Since $1/2\le\max(s,1-s)\le A(s)\le 1$ and A' is bounded (by K),

$$|D_1(z)| = \frac{|A'\left(\frac{zs}{1-s}\right)|}{A\left(\frac{zs}{1-s}\right)}\frac{s}{1-s}\le\frac{2Ks}{1-s}, \qquad |D_2(z)|=\frac{|A'(z)|}{A(z)}\le 2K$$

so that, for any fixed s and $1-s$,

$$\frac{H_j(z)(1-H_j(z))}{z(1-z)} = (1+(1-z)D_j(z))(1-zD_j(z))$$

is bounded too.

— Let us show that

$$\sup_{z\in[0,1-s_j]}\left|\frac{\sqrt{n}(\hat{H}_j(z)-H_j(z))}{(H_j(z)(1-H_j(z)))^{\nu}}\right| < C.$$

We use for that task [28]'s theorem 2.2 for strong mixing sequences and use the fact that absolutely regular sequences are also strong mixing so that the theorem 2.2 also applies to absolutely regular sequences. Namely,

Let $\{U_n, n\ge 1\}$ be a strong mixing stationary sequence of uniform random variables on $[0,1]$, with mixing coefficients $(\alpha_n)_{n>0}$. If there exists some $\theta\ge 1+\sqrt{2}$ and $\epsilon>0$ such that $\alpha(n)=O(n^{-\theta-\epsilon})$, then we have

$$\frac{b_n(.)}{q(.)}\xrightarrow{D[0,1]}\frac{\tilde{B}^*(.)}{q(.)} \tag{24}$$

for any weight function q satisfying $q(t)\ge C(t(1-t))^{(1-1/\theta)/2}$ for some $C>0$, where $b_n(z)=\sqrt{n}(\hat{E}_n(u)-u)$, $\hat{E}_n$ denotes the empirical cdf of the observations and $\tilde{B}^*$ is the centered Gaussian process on $[0,1]$ such that $\tilde{B}^*(0)=\tilde{B}^*(0)=1$ and

$$\mathbb{E}(\tilde{B}^*(s)\tilde{B}^*(t)) = \sum_{k\in\mathbb{Z}}\mathbb{1}_{U_0\le s}\mathbb{1}_{U_k\le t}$$

So, assume that the mixing coefficients of $(X_1, \ldots, X_n)$ satisfy $\alpha(n) = O(n^{-\theta-\epsilon})$ for some $\theta \geq 1+\sqrt{2}$ and $\epsilon > 0$. Then, the sequences $(Z_{1,j}, \ldots, Z_{n,j})$, $j = 1,2$ are mixing since they are obtained from the former sequence by a measurable transformations. Their mixing coefficients α_j satisfy $\alpha_j(n) \leq \alpha(n)$, so that they satisfy the conditions of the theorem and the same holds for the transformed uniform sequences $(H_j(Z_{1,j}) \ldots, H_j(Z_{n,j}))$, $j = 1,2$. Moreover let us set $m = 1/2 - 1/(2\theta)$ and $q(t) = (t(1-t))^{\nu}$ for some $\nu \in (0, 1/2)$ such that $\nu < m$ (note that it is still possible since $\theta > 1$). Thus, it is easily seen that $q(t) \geq C(t(1-t))^{(1-1/\theta)/2}$ with $C = \left(\frac{1}{4}\right)^m$. Hence, for $j = 1,2$ one has

$$\frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{(H_j(z)(1-H_j(z)))^{\nu}} = \frac{b_n(H_j(z))}{q(H_j(z))} \xrightarrow{D[0,1]} \frac{\tilde{B}^*(H_j(z))}{q(H_j(z))}, \tag{25}$$

so that

$$R_n = \sup_{z\in[0,1]} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{(H_j(z)(1-H_j(z)))^{\nu}} \xrightarrow{\mathcal{D}} \sup_{z\in[0,1]} \frac{\tilde{B}^*(H_j(z))}{(H_j(z)(1-H_j(z)))^{\nu}} = \sup_{u\in[0,1]} \frac{\tilde{B}^*(u)}{(u(1-u))^{\nu}}.$$

Since the sequence R_n converges in distribution, then by Prohorov theorem, it is bounded in probability:

$$\sup_{z\in[0,1]} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{(H_j(z)(1-H_j(z)))^{\nu}} = O_p(1). \tag{26}$$

5.1.2 Proof of ii)

First, the bivariate process Z is absolutely regular since it is obtained by a measurable transformation of X. Using [29]'s Theorem 1.4 and the fact that $\beta_Z(k) \leq \beta(k)$,

$$\left|Cov\left(\mathbb{1}_{Z_0\leq z}, \mathbb{1}_{Z_k\leq z'}\right)\right| \leq 2\beta(k)$$

so that (14) exists since $\sum \beta(k) < \infty$.

Now, recall that For $s = 1, A(s) = 1$ and $\sqrt{n}(\log \hat{A}(s) - \log A(s)) = 0$. For $s \neq 1$,

$$\sqrt{n}(\log \hat{A}(s) - \log A(s)) = \lambda(s) \int_0^{1-s} \frac{\sqrt{n}(\hat{H}_1(z) - H_1(z))}{z(1-z)} dz + (1-\lambda(s)) \int_0^{s} \frac{\sqrt{n}(\hat{H}_2(z) - H_2(z))}{z(1-z)} dz.$$

In order to prove the asymptotic normality, let us first show that for $j = 1,2$

$$\int_0^{1-s_j} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{z(1-z)} dz \xrightarrow{\mathcal{D}} \int_0^{1-s_j} \frac{B_j^*(z)}{z(1-z)} dz. \tag{27}$$

Set

$$\begin{aligned}\int_0^{1-s_j} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{z(1-z)} dz &= \int_0^{\frac{1}{n}} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{z(1-z)} dz + \int_{\frac{1}{n}}^{1-s_j} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{z(1-z)} dz \\ &= I_1 + I_2\end{aligned}$$

It follows from (23) and (26) that

$$I_1 = o_p(1). \tag{28}$$

We will then show that

$$\left|\int_{\frac{1}{n}}^{1-s_j} \frac{\sqrt{n}(\hat{H}_j(z)-H_j(z))}{z(1-z)}dz - \int_{\frac{1}{n}}^{1-s_j} \frac{B_j^*(z)}{z(1-z)}dz\right| = o_p(1) \tag{29}$$

so that

$$I_2 = \int_0^{1-s_j} \frac{B_j^*(z)}{z(1-z)}dz + o_p(1). \tag{30}$$

and (27) can be obtained by combining (28) and (30). For that task we will apply [9]'s theorem 3.1 to the bivariate process Z.

Let $\{Z_n, n \geq 1\}$ be an absolutely regular strictly stationary bivariate sequence with distribution function H and mixing coefficients $(\beta_n)_{n>0}$ satisfying $\beta(n) = O(n^{1-p})$ for some $p \in (2,3]$. Let us set

$$R(z,t) = \sum_{i \leq t} (\mathbb{1}_{Z_i \leq z} - H(z)),\ t \in \mathbb{R}^+,\ z \in \mathbb{R}^2.$$

Thus, there exists a centered Gaussian process $\{K(z,t),\ t \in \mathbb{R}^+,\ z \in \mathbb{R}^2\}$ with covariance function $\mathbb{E}(K(z,t)K(z',t')) = \mathbb{E}\left(B^*(z)B^*(z')\right)(t \wedge t')$ with $\mathbb{E}\left(B^*(z)B^*(z')\right)$ defined in (14), such that

$$\sup_{t \leq n} \sup_{z \in \mathbb{R}^2} |R(z,t) - K(z,t)| = O_{a.s.}(n^{1/p}(\log n)^{\eta+\epsilon+1/p})$$

for any $\epsilon > 0$ and $\eta = (5-2/(p))\mathbb{1}_{p \in (2,3)} + (14/3)\mathbb{1}_{p=3}$.

Fixing $t = n$, $K(z,t)$ turns out to be the centered Gaussian process $n^{1/2}B^*(z)$ defined in Subsection 3.1 so so that we get the Csörgö and Horváth's type result:

$$\sup_{z \in \mathbb{R}^2} \left|\sqrt{n}(\hat{H}(z) - H(z)) - B^*(z)\right| = O_{a.s.}(n^{1/p-1/2}(\log n)^{\eta+\epsilon+1/p})$$

Hence, for $j = 1, 2$,

$$\sup_{z \in \mathbb{R}} \left|\sqrt{n}(\hat{H}_j(z) - H_j(z)) - B_j^*(z)\right| = O_{a.s.}(n^{1/p-1/2}(\log n)^{\eta+\epsilon+1/p}) \tag{31}$$

so that for all $s_j > 0$,

$$\begin{aligned}
&\left|\int_{\frac{1}{n}}^{1-s_j} \frac{\sqrt{n}(\hat{H}_j(z)-H_j(t))}{z(1-z)}dz - \int_{\frac{1}{n}}^{1-s_j} \frac{B_j^*(z)}{z(1-z)}dz\right| \\
\leq\ & \sup_{z \in \mathbb{R}} \left|\sqrt{n}(\hat{H}_j(z) - H_j(z)) - B_j^*(z)\right| \int_{\frac{1}{n}}^{1-s_j} \frac{dz}{z(1-z)} \\
=\ & O(n^{\frac{1}{p}-\frac{1}{2}}(\log n)^{\eta+\epsilon+1/p}) \left(\log\left(\frac{1-s_j}{s_j}\right) + \log(n-1)\right) \\
=\ & O(n^{\frac{1}{p}-\frac{1}{2}}(\log n)^{\eta+\epsilon+1/p+1})
\end{aligned}$$

according to $\frac{1}{p} - \frac{1}{2} < 0$, then we obtain

$$\left|\int_{\frac{1}{n}}^{1-s_j} \frac{\sqrt{n}(\hat{H}_j(z)-H_j(z))}{z(1-z)}dz - \int_{\frac{1}{n}}^{1-s_j} \frac{B_j^*(z)}{z(1-z)}dz\right| = o_p(1) \tag{32}$$

as soon as $\epsilon > 0$ and $\eta = (5 - 2/p)\mathbb{1}_{p\in(2,3)} + (14/3)\mathbb{1}_{p=3}$. When $s_j = 0$, we may show that (32) still holds writing

$$I_2 = \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{z(1-z)} dz + \int_{1-\frac{1}{n}}^{1} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{z(1-z)} dz.$$

Finally,

$$I_1 + I_2 = \int_0^{1-s_j} \frac{B_j^*(z)}{z(1-z)} dz + o_p(1)$$

so that

$$\int_{\frac{1}{n}}^{1-s_j} \frac{\sqrt{n}(\hat{H}_j(z) - H_j(z))}{z(1-z)} dz \int_0^{1-s_j} \frac{B_j^*(z)}{z(1-z)} dz + o_p(1).$$

Therefore, for all $s \in [0,1]$

$$\sqrt{n}\left(\log \hat{A}_n(s) - \log A(s)\right) \xrightarrow{P} \lambda(s) \int_0^{1-s} \frac{B_1^*(z)}{z(1-z)} dz + (1-\lambda(s)) \int_0^{s} \frac{B_2^*(z)}{z(1-z)} dz$$

and then

$$\sqrt{n}\left(\log \hat{A}_n(s) - \log A(s)\right) \xrightarrow{\mathcal{D}} \lambda(s) \int_0^{1-s} \frac{B_1^*(z)}{z(1-z)} dz + (1-\lambda(s)) \int_0^{s} \frac{B_2^*(z)}{z(1-z)} dz,$$

which achieves the proof.

Finally, it remains to show that the limiting process U has the desired covariance function (17) and that it exists. This may be done by applying Fubini's theorem.

5.2 Proof of proposition 4.1

Let F and G be the joint distributions of the vectors (X_1, X_2) and (Y_1, Y_2) respectively. Moreover, denote by G_1 and G_2 the margins of (Y_1, Y_2) and C_X and A_X (resp. C_Y and A_Y) the copula and dependence function of (X_1, X_2) (resp. (Y_1, Y_2)). One has

$$\begin{aligned}
C_X(u,v) &= \mathbb{P}\left(F_1(X_{i,1}) \le u, F_2(X_{i,2}) \le v\right) \\
&= \mathbb{P}\left(G_1^k(X_{i,1}) \le u, G_2(X_{i,2}) \le v\right) \\
&= \mathbb{P}\left(X_{i,1} \le G_1^{-1}(u^{1/k}), Y_{i,2} \le G_2^{-1}(v)\right) \\
&= \mathbb{P}\left(Y_{i-j,1} \le G_1^{-1}(u^{1/k}), j = 0, \dots, k-1, Y_{i,2} \le G_2^{-1}(v) | Y_{i-j,1} \le G_1^{-1}(u^{1/k}), j = 1, \dots, k-1\right) \\
&\times \left(\mathbb{P}\left(Y_{i,1} \le G_1^{-1}(u^{1/k})\right)\right)^{k-1} \\
&= \mathbb{P}\left(G_1(Y_{i,1}) \le u^{1/k}, G_2(Y_{i,2}) \le v\right)(u^{1/k})^{k-1} \\
&= u^{(k-1)/k} C_Y(u^{1/k}, v). \qquad (33)
\end{aligned}$$

Using (33) and (1)

$$
\begin{aligned}
C_X(u,v) &= u^{(k-1)/k}\exp\left(\log u^{1/k} v A_Y\left(\frac{\log u^{1/k}}{\log u^{1/k} v}\right)\right)\\
&= \exp\left(\log u^{(k-1)/k} + \log u^{1/k} v A_Y\left(\frac{\log u^{1/k}}{\log u^{1/k} v}\right)\right)\\
&= \exp\left(\log uv\left(\frac{\log u^{(k-1)/k}}{\log uv} + \frac{\log u^{1/k} v}{\log uv} A_Y\left(\frac{\log u^{1/k}}{\log u^{1/k} v}\right)\right)\right)
\end{aligned}
$$

Let us set for all $0 \leq u, v \leq 1$, $t = \dfrac{\log u}{\log uv}$. Using (19), then

$$
\begin{aligned}
\frac{\log u^{(k-1)/k}}{\log uv} + \frac{\log u^{1/k} v}{\log uv} A_Y\left(\frac{\log u^{1/k}}{\log u^{1/k} v}\right) &= \frac{k-1}{k}t + \frac{k-(k-1)t}{k} A_Y\left(\frac{t}{k-(k-1)-t}\right)\\
&= \frac{k-1}{k}t + \frac{k-(k-1)t}{k}\left(\left(\frac{t}{k-(k-1)t}\right)^{1/r}\right.\\
&\quad \left. + \left(1 - \frac{t}{k-(k-1)t}\right)^{1/r}\right)^r\\
&= \frac{k-1}{k}t + \left(\left(\frac{t}{k}\right)^{1/r} + (1-t)^{1/r}\right)^r
\end{aligned}
$$

Finally,

$$
C_X(u,v) = \exp\left(\log uv A_X\left(\frac{\log u}{\log uv}\right)\right),
$$

with

$$
A_X(t) = \frac{k-1}{k}t + \left(\left(\frac{t}{k}\right)^{\frac{1}{r}} + (1-t)^{\frac{1}{r}}\right)^r,
$$

for all $t \in [0,1]$, which completes the proof.

6 Appendix: The bootstrap test statistic

Fix

B : the number on the bootstrap samples

T : the size of the blocs

Define $N = [n/T]$, the number of blocs, where $[x]$ is the integer part of x

Generate a bivariate extremes $(X_1, \ldots, X_n)$, $X_i = (X_{i,1}, X_{i,2})$ and compute the statistic

$$
U_n = \log \hat{A}_n\left(\frac{1}{2}\right). \tag{34}
$$

-Let IB={ 1,2,...,N }

for $b = 1, ...B$ do

*Generate a set index $I = \{i_1, i_2, ..., i_N\}$ from IB with replacement

* for $k = 1, ..., N$ for $j = 1, ..., T$, compute $X^b_{(k-1)*T+j} = X_{(i_k-1)*T+j}$

*Compute $UB_b = \log \hat{A}^b_n\left(\frac{1}{2}\right)$

where $\hat{A}_n^b$ is the empirical CFG's estimator based on the bootstrap sample $X_i^b, i = 1, .., n$.

-Compute the bootsrap statistic

$$U_n^b = U_n / (\sum_{b=1}^{B} (UB_b - \overline{UB})^2 / B)^{1/2} \tag{35}$$

where $\overline{UB} = \frac{1}{B} \sum_{b=1}^{B} UB_b$.

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