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Nathan HUGUENIN

Théories conformes des champs de Toda à bord

Toda conformal field theories with boundary

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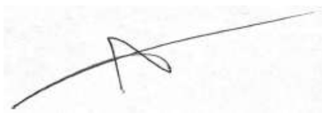

Liste de publications et participation aux conférences

Liste des publications réalisées dans le cadre du projet de thèse :

- Boundary Toda conformal field theory from the path integral [28]. *Annales Henri Poincaré.* (Avec B. Cerclé.)
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- Higher-spin symmetry in the $\mathfrak{sl}_3$ Toda conformal field theory II: Singular vectors and BPZ equations [27]. *Transactions of the AMS (à paraître).* (Avec B. Cerclé.)

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En tant qu'orateur

- Séminaire d'Analyse, Institut de Mathématiques de Marseille (février 2026)
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- Séminaire de l'équipe de Probabilités - Chair of Random Geometry, EPFL Lausanne (novembre 2024)
- Séminaire de l'équipe de Probabilités, Institut de Mathématiques de Marseille (mai 2024)
- Séminaire des doctorants, Institut de Mathématiques de Marseille (décembre 2023)

En tant que participant

- Annual meeting - Simons collaboration "Probabilistic paths to Quantum Field Theory" - New York City (avril/mai 2026)
- Workshop "Integrable Quantum Field Theory", Hausdorff research Institute for Mathematics - Bonn (juin 2025)
- Workshop "Random geometry and its connections to Quantum Field Theory", Hausdorff research Institute for Mathematics - Bonn (mai 2025)

— Winter school “Mathematical aspects of quantum mechanics", Institut de Mathématiques de Toulouse (janvier 2024)
— Spring school “3 facets of gravity", Humboldt University of Berlin (mai 2023)
— Workshop “Probability and Conformal field theory", IGESA Agay (novembre 2022)

Résumé

Dans le cadre de l'approche probabiliste des théories quantiques des champs, en dimension 2 et dans le cadre Euclidien, on s'intéresse à des modèles présentant certaines propriétés d'invariance par transformations conformes.

Les théories conformes des champs (CFT) de Toda sont des modèles dans lesquels le champ sous-jacent est à valeurs vectorielles, et généralisent la théorie de Liouville. Les modèles de Toda sont censés présenter, en plus de l'invariance conforme, un plus haut niveau de symétrie, encodée par les algèbres W. Dans cette thèse nous nous intéressons au cas où le champ est défini sur une surface de Riemann (compacte, connexe) à bord.

Un premier apport de cette thèse est la construction des modèles de Toda sur les surfaces hyperboliques avec ou sans bords (généralisant la construction de Cerclé-Rhodes-Vargas sur la sphère), grâce à des outils probabilistes. La présence d'une frontière entraîne ici des phénomènes nouveaux par rapport à la CFT de Liouville : différentes conditions au bord sont possibles pour le champ sous-jacent, en lien avec la structure du groupe d'automorphisme de l'algèbre de Lie cible. Nous explorons ces différentes constructions et identifions dans différents cas l'algèbre de symétrie des modèles proposés, au sens des algèbres de *vertex*.

Dans une seconde partie, nous spécialisons l'étude au cas le plus simple de CFT de Toda après Liouville, c'est-à-dire quand le champ est à valeurs dans l'algèbre $\mathfrak{sl}_3\mathbb{C}$. Nous étudions ce modèle sur le demi-plan complexe. Dans ce cadre, nous établissons la présence de symétries augmentées sous la forme d'identités de Ward au bord, ce qui était jusqu'alors inconnu en Physique. Ce résultat passe par la définition rigoureuse des descendants associés à l'algèbre W sur le bord, à l'aide d'une procédure de régularisation et d'outils probabilistes. Nous étudions ensuite l'existence de vecteurs singuliers dans le modèle, et découvrons la présence de vecteurs singuliers non nuls au bord jusqu'au niveau 3, ce qui est également un phénomène nouveau par rapport à la littérature existante. Ces propriétés permettent d'établir de nouvelles équations différentielles hypergéométriques de type BPZ, pour certaines fonctions de corrélations de la théorie, à savoir le corrélateur *bulk/boundary* et la fonction à trois points au bord.

Ces travaux ouvrent la voie au calcul de ces fonctions de corrélation, étape clé de l'implémentation du *bootstrap* conforme pour les théories de Toda. Ils mettent par ailleurs en évidence des phénomènes nouveaux dans l'étude des théories conformes des champs en dimension deux en présence d'une frontière.

Mots clés : théories conformes des champs, théories de Toda, surfaces de Riemann à bord, champ libre Gaussien, chaos multiplicatif Gaussien, algèbres W.

Abstract

In the framework of the probabilistic approach to quantum field theories, in dimension 2 and in the Euclidean setting, we are interested in models exhibiting certain invariance properties under conformal transformations.

Toda conformal field theories (CFTs) are models in which the underlying field takes values in a vector space and generalize Liouville theory. Toda models are expected to exhibit, in addition to conformal invariance, a higher level of symmetry encoded by W-algebras. In this thesis, we focus on the case where the field is defined on a (compact, connected) Riemann surface with boundary.

A first contribution of this thesis is the construction of Toda models on hyperbolic surfaces, with or without boundary (generalizing the construction by Cerclé-Rhodes-Vargas on the sphere), using probabilistic tools. The presence of a boundary gives rise to new phenomena compared to Liouville CFT: different boundary conditions are possible for the underlying field, depending on the structure of the automorphism group of the target Lie algebra. We explore these different constructions and identify, in various cases, the symmetry algebra of the proposed models, in the sense of vertex algebras.

In a second part, we specialize the study to the simplest Toda CFT after Liouville, that is, when the field takes values in the algebra $\mathfrak{sl}_3\mathbb{C}$. We study this model on the complex half-plane. In this context, we establish the presence of enhanced symmetries in the form of boundary Ward identities, which was previously unknown in Physics. This result relies on the rigorous definition of descendants associated with the W-algebra on the boundary, using a regularization procedure and probabilistic tools. We then study the existence of singular vectors in the model and discover the presence of nonzero singular vectors on the boundary up to level 3, which is also a new phenomenon compared to the existing literature. These properties allow us to derive new hypergeometric differential equations of BPZ type for certain correlation functions of the theory, namely the bulk/boundary correlator and the boundary three-point function.

These results pave the way for the computation of these correlation functions, a key step in the implementation of the conformal bootstrap for Toda theories. They also highlight new phenomena in the study of two-dimensional conformal field theories in the presence of a boundary.

Keywords: conformal field theories, Toda theories, Riemann surfaces with boundary, Gaussian free field, Gaussian multiplicative chaos, W-algebras.

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Table des matières

1 Introduction

Sommaire

1.1 Two-dimensional Conformal Field Theory

This thesis lies in the framework of the probabilistic formulation of two-dimensional Conformal Quantum Field Theory (CFT hereafter), and is interested with the mathematical study of enhanced symmetries within this context. In the first part of this introduction we will try to shed light on some aspects of these notions. We keep the discussion at a very informal level and certainly do not try to provide the reader with a comprehensive treatment of the Physics of CFT.

1.1.1 Conformal symmetry in two dimensions

The notion of symmetry in Physics has carried quite different meanings and experimented different formulations through the ages. Nowadays, in our context, symmetry refers to a form of invariance of a physical system with local interactions under certain transformations. These transformations are often encoded in an algebraic structure. A famous example is gauge symmetry, that expresses invariance under a local group action, and that among many other things underlies the classification of elementary particles, the so-called *standard model*.

One other natural notion of symmetry is scale invariance, that is invariance of a system under dilations $g \mapsto kg$ where g is the metric and k a constant positive coefficient. This phenomenon can happen for a physical system only at certain particular configurations (possibly depending on external parameters), where the relevant characteristic length scales at which the interactions occur are either 0 or infinite. This is the case for example for models of Statistical Physics at criticality, when a phase transition of order two or higher[1] is observed (e.g. the Ising or Potts models). In addition to global scale invariance, it is natural to expect in these cases a form of invariance under *local* dilations, i.e. when the scaling coefficient k is a function of the position. If so, one has to ensure that the metric g scales accordingly, thus allowing only local dilations that preserve angles. This is conformal invariance.

This notion of conformal invariance appears not only in critical Statistical Physics, but also e.g. in the Quantum Physics of strong interactions (Quantum Chromo-Dynamics), and in string theory, as in the latter it is tantamount to the invariance under reparametrization of the string's world-sheet.

Identifying the symmetries of a model amounts to finding constraints on the model, with the hope that these constraints allow one to solve the model (at least partially), which for us will mean basically to describe the correlation functions between observables[2], evaluated at different points of the manifold on which the model is defined. The conformal symmetry is encoded in an algebraic structure, called the conformal group. In $d \geq 3$ dimensions, this group is finite-dimensional, of dimension $\frac{1}{2}(d+1)(d+2)$. Due to this fact, in general the constraints brought about by the conformal symmetry are not enough to solve the models. In dimension two however, the conformal group is enriched with what are called local conformal transformations, that are nothing but locally analytic functions[3], making of this group an infinite-dimensional one. The conformal symmetry is then encoded in the so-called Witt algebra, a infinite-dimensional Lie algebra, whose generators are the modes of the infinitesimal conformal transformations. When considering a quantum conformal theory, i.e. when promoting the observables to self-adjoint operators on some Hilbert space, the symmetry algebra becomes the unique non-trivial central extension of the Witt algebra: the *Virasoro algebra*. Before defining this object of central importance let us explain briefly what we mean by the symmetry algebra of a model. A symmetry algebra is, at first glance, any algebraic structure encoding some conservation laws.

1. “Order two or higher" means that at least the first derivative of the free energy is continuous across the phase transition threshold.

2. By observable we mean a certain functional of the physical variables of the model. In a field theory, these are functionals of the field.

3. These are holomorphic maps from (part of) the manifold —which is a Riemann surface— onto itself, but that may not be defined everywhere.

The model, which is a given set of observables, realizes the symmetry algebra in the sense that it carries a representation of the latter: the symmetry algebra acts on the observables in a certain way. It is important to distinguish properties that are model-dependent (e.g. higher equations of motion, see below), from those that derive directly from the structure of the symmetry algebra (as the Ward identities). The Virasoro algebra, that encodes the conformal symmetry in two dimensions, is defined by its generators L_n, $n \in \mathbb{Z}$, its central element c and the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad \text{and} \quad [L_n, c] = 0 \tag{1.1}$$

for all $m, n \in \mathbb{Z}$. The operators L_{-1}, L_0, L_1 are the generators of global conformal transformations. The image of c in a representation of the Virasoro algebra is a real number called the *central charge* of the Conformal Field Theory under consideration. The Virasoro operators L_n can be seen as the modes of (the holomorphic part of) the *stress-energy tensor*[4], denoted $\mathbf{T}$:

$$\mathbf{T}(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}. \tag{1.2}$$

The commutation relation (1.1) then takes the form of the *Operator Product Expansion*

$$\mathbf{T}(z)\mathbf{T}(w) = \frac{c/2}{(z-w)^4} + \frac{2\mathbf{T}(w)}{(z-w)^2} + \frac{\partial\mathbf{T}(w)}{(z-w)} + \text{reg.} \tag{1.3}$$

which is *a priori* a relation between formal power series, but that can be given a mathematical sense, see Subsection 1.2.2.1.

Due to this rich algebraic structure, in two dimensions, physical systems enjoying conformal symmetry are highly constrained, and this particular feature makes them all the more interesting to study because they provide examples of fully solvable models.

1.1.2 The conformal bootstrap formalism

In 1984, Belavin, Polyakov and Zamolodchikov (BPZ in the sequel) published a seminal paper [12] in which they described a novel method for solving two-dimensional CFTs[5], based on purely algebraic considerations. At the heart of this method is the Operator Product Expansion hypothesis, that postulates that every product of local quantum operators can be expressed as a linear combination of basic local operators. The bootstrap assumption is the associativity of such products, thus allowing one to compute all the correlation functions of quantum operators of a model with the only knowledge of a few basic correlation functions named *structure constants*, together with some extra data called the *spectrum* of the theory, and some universal[6] special

4. This terminology comes from the fact that when $c = 0$, $\mathbf{T}(z)$ transforms as a $(2,0)$-tensor under coordinate changes.

5. In fact, the conformal bootstrap method can be generalized to other dimensions.

6. Universal means here that the conformal blocks depend only on the algebra of symmetry.

functions named *conformal blocks*. In practice, one focuses on correlation functions of a particular subset of these local operators, called *primary fields*. The latter are characterized by the conditions

$$L_n\mathcal{O} = 0 \quad \text{for all} \quad n > 0 \quad \text{and} \quad L_0\mathcal{O} = \Delta\mathcal{O} \tag{1.4}$$

where $\Delta \in \mathbb{R}$ is called the *conformal weight* of the primary field $\mathcal{O}$. All correlation functions can then be recovered by letting the negative Virasoro modes act on these primary fields.

The form of the structure constants is fixed by conformal invariance. On the sphere, the structure constants are the correlation functions of three primary fields, that we can choose to take at particular points:

$$\langle\mathcal{O}_1(0)\mathcal{O}_2(1)\mathcal{O}_3(\infty)\rangle. \tag{1.5}$$

On the disk (or equivalently the complex upper half-plane), the structure constants are the boundary three-point functions

$$\langle\mathcal{B}_1(0)B_2(1)B_3(+\infty)\rangle \tag{1.6}$$

and the bulk-boundary correlators

$$\langle\mathcal{B}(0)\mathcal{O}(\mathrm{i})\rangle. \tag{1.7}$$

The bootstrap approach has proven to be particularly efficient when the number of these primary fields is finite: these models are called minimal and were successfully solved by BPZ using the bootstrap formalism (for example, in the critical Ising model, the primary fields are the spin and the energy density). In these cases the knowledge of the Virasoro algebra and its representation theory (together with a few general CFT axioms) allow one to entirely solve the models. Other examples of solvable models were successively found by generalizing the Virasoro minimal models in the following years (see below), and since 1984 and BPZ's breakthrough, two-dimensional CFT have been a particularly active field of research in Mathematical Physics. For an in-depth treatment of the topic, we refer to the "Big Yellow Book" [44].

1.1.3 Liouville Conformal Field Theory

Liouville Conformal Field Theory, introduced by Polyakov [78], is the first example of a non-compact (i.e. with a continuum of primary fields) CFT that have been solved in Physics using the bootstrap formalism of BPZ (relying on the Virasoro algebra). Liouville theory is central in the probabilistic approach to CFT. But before diving into the definition of Liouville CFT, it is worth saying a few words about the Lagrangian formulation of Quantum Field Theory, the so-called *path integrals*.

In a classical Field Theory, the equations of motion are obtained by searching for the minimizer of an action functional, that is itself the integral of a Lagrangian density (or just Lagrangian). The Physics are then described by this minimizer. There is an important distinction between the Lorentzian and Euclidean formulations. The Lorentzian case corresponds to the physically relevant Minkowsky space-time setting, and features hyperbolic equations of motion. The Euclidean case, although less natural from a physical point of view, is often simpler to handle mathematically, which is partly due to the fact that the Euclidean signature of space-time makes the models feature elliptic equations of motion (the time being treated as a spatial variable). Both formulations are related by the Wick rotation $t \mapsto -\mathrm{i}\tau$.

In Quantum Field Theory, the Physics are described by a probability measure over the space of all possible paths (hence the name *path integral*), which is given by the exponential of a *quantized* action functional, the latter still being the integral of a Lagrangian, constructed from the classical one by introducing a *coupling constant* encoding the level of randomness. The field thus defined is random, and fluctuates around the classical solution of the equations of motion. When the coupling constant is taken to 0, one recovers the classical Physics (this is the (semi-)classical limit). The same distinction exists between Lorentzian and Euclidean QFTs as in the classical setting, although the relation between them is way less clear, as in addition to the Wick rotation it requires analytic continuation of the correlation functions (see e.g. [92, Chapter 3] and [73, 74]). In the Lorentzian setting the path integrals are oscillatory and often ill-defined even from a physical perspective, while in the Euclidean case they feature exponential decay with respect to the action, allowing one to make sense of them as statistical quantities.

The classical Liouville theory is formulated in the Euclidean setting, and is concerned with the problem of finding a metric with constant negative curvature in a given conformal class of metrics. It is of importance as well in Mathematics, where it is known as the uniformization problem, as in Physics where it appears in the realm of two-dimensional gravity coupled with matter fields. On the sphere $\mathbb{S}^2$, the Liouville equation reads

$$-2\Delta_g \Phi + R_g + 2\Lambda e^{2\Phi} = 0. \tag{1.8}$$

Here Δ_g is the Laplacian (Laplace-Beltrami operator) in the metric g, and R_g is the Ricci scalar curvature, while $\Lambda > 0$ is a constant. Solving this equation amounts to finding a metric $g' = e^{2\Phi} g$ conformally equivalent to the background metric g of constant curvature -2Λ. The weak formulation of this problem is to find the minimizer of the classical Liouville action:

$$S_L^{cla}(\phi, g) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(\left|\nabla_g \phi\right|_g^2 + R_g \phi + \Lambda e^{2\phi} \right) dv_g. \tag{1.9}$$

Note that on the sphere this action admits no minimizer because of the Gauss-Bonnet theorem. A way to overcome this issue is to assume that Φ has (conical) singularities. The quantization of the action (1.9) leads to Liouville CFT. We introduce the coupling constant $\gamma > 0$ and the background charge $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, and set

$$S_L(\phi, g) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(\left| \nabla_g \phi \right|_g^2 + Q R_g \phi + \Lambda e^{\gamma \phi} \right) d v_g. \tag{1.10}$$

The Liouville field is then the random field whose law is given on a suitable space of functionals $\mathscr{F}$ by the Euclidean path integral

$$\langle F(\Phi) \rangle_{L,g} = \frac{1}{\mathcal{Z}} \int_{\mathscr{F}} F(\phi) e^{-S_L(\phi,g)} D_g \phi \tag{1.11}$$

where $\mathcal{Z}$ is a normalizing factor called *partition function*, and where the infinitesimal element $D_g \phi$ stands for a theoretical uniform measure on the space $\mathscr{F}$ (which mathematically doesn't exist as such, of course). The Euclidean path integral formalism is reminiscent of the Gibbs measures in Statistical Mechanics: the Lagrangian density becoming the Hamiltonian of Statistical Physics, and the path integral being traduced by the sum over all possible configurations of the field.

The Quantum Conformal Liouville theory is a central model in random geometry as it describes a “typical" negatively curved random surface, but also in string theory as the CFT of the world-sheet, in two-dimensional quantum gravity, and several other topics. From the conformal bootstrap point of view, the situation is much more complicated than in the case of minimal models. Indeed, Liouville CFT counts an infinite number of primary fields, named Vertex Operators, indexed by a parameter α [7], formally defined by [8]

$$V_\alpha(z) = e^{\alpha \Phi(z)} \tag{1.12}$$

where z is a point of the surface on which the model is defined, called an *insertion*. The goal is then to compute correlation functions of Vertex Operators:

$$\langle V_{\alpha_1}(z_1) \dots V_{\alpha_N}(z_N) \rangle_{L,g} \tag{1.13}$$

where $z_1, \dots, z_N$ are distinct insertions. It is important to stress that unlike the minimal models, Liouville theory is not completely fixed by Virasoro symmetry, although the latter is still of crucial importance in the model. One needs extra input, stemming from the analysis of the path integral, to solve the theory.

The first step in the conformal bootstrap procedure, that is the computation of the structure constants of Liouville theory, the celebrated DOZZ formula, has been conducted in the Physics literature in [33, 102]. Let us briefly sketch here the main concepts related to this derivation, which will allow us to introduce some of the

7. The physically relevant domain of parameters is the spectrum of the theory, that is $Q + i\mathbb{R}_+$, although we will see that the probabilistically defined correlation functions *a priori* allow us for α real.

8. The dependence in the field Φ is usually omitted in the literature.

notions that we will be interested in later. Consider the Liouville three-point function[9]

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)\rangle_L. \tag{1.14}$$

As already mentioned the form of this correlation function is partly fixed by conformal invariance, in that the dependency in the insertions z_i can be factorized out by applying a Möbius transformation, thus writing (1.14) as

$$C(\alpha_1,\alpha_2,\alpha_3)\prod_{i<j}\left|z_i-z_j\right|^{2\Delta_{ij}}$$

with $\Delta_{12} := \Delta_{\alpha_3}-\Delta_{\alpha_1}-\Delta_{\alpha_2}$ (and likewise for Δ_{13} and Δ_{23}, $\Delta_{\alpha_i}=\frac{\alpha_i}{2}(Q-\frac{\alpha_i}{2})$ being the conformal weights), and where $C(\alpha_1,\alpha_2,\alpha_3)$ is the DOZZ structure constant. In addition to global conformal invariance, a second constraint on the correlation functions is provided by the Operator Product Expansions of the Virasoro modes with the Vertex Operators, which form depends only on the symmetry algebra:

$$L_{-n}(z)V_\alpha(w) = \frac{-\partial_w V_\alpha(w)}{(w-z)^{n-1}}+\frac{(n-1)\Delta_\alpha V_\alpha}{(w-z)^n}+\text{reg}. \tag{1.15}$$

Once inserted in a correlation function, this OPE becomes a *Ward identity*, see Subsection 1.2.2 and Chapter 3. In order to compute the three-point structure constant one first considers a particular four-point function

$$\langle V_{\alpha^*}(z)V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)\rangle_L \tag{1.16}$$

where α^* is a particular weight, called *degenerate*, whose value is predicted by the representation theory of the Virasoro algebra: $-\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$. At the level of the Virasoro algebra, a degenerate weight gives rise to what is called a *singular vector*, which means that it entails a non-trivial linear relation between some of the Virasoro operators. In our case, this singular vector becomes a *null vector*, which means that at the level of the representation of the Virasoro algebra, one has

$$\left(L_{-2}+\frac{1}{\gamma^2}L_{-(1,1)}\right)V_{-\frac{\gamma}{2}}=0 \tag{1.17}$$

where $L_{-(1,1)}$ stands for the composite operator $L_{-1}L_{-1}$. We stress that a degenerate weight (a singular vector) does not always give rise to a null vector —unless it is the case for Liouville theory on the sphere— in the sense that the right-hand side of (1.17) can be not 0: the latter relation is then called a *higher equation of motion*, see Subsection 1.2.3 and Chapter 4. At the level of the correlation function (1.16), the null vector equation (1.17), together with the OPE (1.15), result in the following: the four

9. The dependency in the metric is not relevant here, since no change of metric is considered. On the sphere one usually chooses to work in the round metric.

point function (1.16) can be written as

$$|z-z_3|^{-4\Delta_{\alpha^*}}|z_1-z_2|^{2(\Delta_{12}-\Delta_{\alpha^*})}|z_1-z_3|^{2(\Delta_{13}+\Delta_{\alpha^*})}|z_2-z_3|^{2(\Delta_{23}+\Delta_{\alpha^*})}\mathcal{T}(\eta) \tag{1.18}$$

where η is the cross-ratio of the four points z_i, and $\mathcal{T}$ is solution of the Fuschian *BPZ equation* [94]

$$\left(-\frac{1}{\gamma^2}\partial^2+\left(\frac{1}{z-1}+\frac{1}{z}\right)\partial-\frac{\Delta_{\alpha_2}}{(z-1)^2}-\frac{\Delta_{\alpha_1}}{z^2}+\frac{\Delta_{\alpha^*}-\Delta_{12}}{z(z-1)}\right)\mathcal{T}(z)=0. \tag{1.19}$$

Careful analysis of this equation, together with Operator Product Expansions, then allow one to derive periodicity equations for the DOZZ constant $C(\alpha_1,\alpha_2,\alpha_3)$, that in turn determine this quantity.

The boundary case (on the disk) was later settled in [39, 80, 57]. These major advances led to the full implementation of the BPZ method. For more details on the physical formulation of the conformal bootstrap in Liouville theory we refer for instance to [95] and to the review [70].

1.1.4 Higher-spin symmetry and Toda Conformal Field Theories

The year after the article by BPZ [12], in 1985, Zamolodchikov [103] put forward the question of what can be said if the symmetry algebra of a model strictly contains the Virasoro algebra —the algebra of conformal symmetry— as a sub-algebra. For example it was pointed out in [34] that the critical three-states Potts model of Statistical Physics does enjoy a higher level of symmetry. The algebras of symmetry encoding this *higher-spin* symmetry are called W-algebras, and depend on a complex Lie algebra $\mathfrak{g}$. For example the Virasoro algebra is the W-algebra associated with $\mathfrak{g}=\mathfrak{sl}_2\mathbb{C}$. Some W-minimal models —models with a finite number of primary fields with respect to the W-algebra— were studied subsequently to their introduction by Zamolodchikov.

Although we will be interested in this thesis in W-algebras, it is worth noting that there are still other ways to enlarge the Virasoro algebra. Celebrated examples are the (affine) Kac-Moody algebras, that encode the symmetries of Wess-Zumino-Witten models, or superconformal algebras that rule models of supersymmetric CFT and string theory, and several other extensions. Once again we refer to [44] for an overview together with an in-depth treatment of some of these topics.

Although they are not Lie algebras in general, W-algebras can in principle be described thanks to their commutation relations as in Equation (1.1). Nonetheless, the latter rapidly become intractable as the rank of the underlying Lie algebra increases. As an example we can write these commutation relations in the simplest (beyond Virasoro) $\mathfrak{g}=\mathfrak{sl}_3\mathbb{C}$ case, corresponding to Zamolodchikov's W_3 algebra. In addition to (1.1), one has (see [16])

$$[L_m, W_n] = (2m-n)W_{m+n}; \tag{1.20}$$

$$[W_m, W_n] = (m-n)\left(\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right)L_{m+n} \tag{1.21}$$

$$+\frac{16}{22+5c}(m-n)\Lambda_{m+n} + \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0} \tag{1.22}$$

where

$$\Lambda_m = \sum_{n\leq -2} L_n L_{m-n} + \sum_{n>-2} L_{m-n}L_n - \frac{3}{10}(m+3)(m+2)L_m.$$

In the same fashion as for the Virasoro algebra, the W_n operators can be seen as the modes of conserved holomorphic currents, but the framework of Vertex Operator Algebras is necessary to give them a general rigorous mathematical sense, contrary to the stress-energy tensor of conformal symmetry (that has a geometrical interpretation, see Subsection 1.2.2.1).

Toda Conformal Field Theories are natural generalizations of Liouville theory enjoying the higher level of symmetry encoded in W-algebras. Let us describe now how they are defined. Given a simple complex Lie algebra $\mathfrak{g}$, its Cartan sub-algebra is the maximal sub-algebra $\mathfrak{h}$ made of semi-simple elements[10]. The Lie algebra $\mathfrak{g}$ can be written using its *split real form* $\mathfrak{g}_\mathbb{R}$ as $\mathfrak{g} = \mathfrak{g}_\mathbb{R} + i\mathfrak{g}_\mathbb{R}$, and this descends to the Cartan sub-algebra $\mathfrak{h}$ in the sense that there exists a real vector space $\mathfrak{a}$, unique up to isomorphism, such that $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$. $\mathfrak{a}$ is the split real form of the Cartan sub-algebra of $\mathfrak{g}$. The Euclidean space obtained by equipping $\mathfrak{a}$ with a particular scalar product, that is proportional to the restriction of the Killing form to $\mathfrak{g}$, is the target space of the field in Toda CFTs. Liouville theory can be recovered in this context by choosing $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$, thus having $\mathfrak{a} \simeq \mathbb{R}$ (in general $\mathfrak{a}$ is isomorphic to $\mathbb{R}^r$ where r is the rank of $\mathfrak{g}$). The Toda action functional is given, on a Riemann surface Σ (without boundary), by

$$S(\boldsymbol{\phi}, g) = \frac{1}{4\pi}\int_\Sigma \left(\left|\nabla_g \boldsymbol{\phi}\right|_g^2 + R_g\langle Q, \boldsymbol{\phi}\rangle + \sum_{i=1}^r \Lambda_i e^{\langle \gamma e_i, \boldsymbol{\phi}\rangle}\right) dv_g \tag{1.23}$$

where e_i are a special basis of vectors called the *simple roots* of the Lie algebra $\mathfrak{g}$, and $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathfrak{a}$ inherited from the Killing form. Q is the vectorial background charge $Q = \gamma\boldsymbol{\rho} + \frac{2}{\gamma}\boldsymbol{\rho}^\vee$ with $\boldsymbol{\rho}$ the *Weyl vector*, and $\boldsymbol{\rho}^\vee$ the co-Weyl vector. For precise definitions we refer to Subsection 2.2.2. As in the Liouville case, the Toda field Φ is then described by a path integral on a space of functionals $\mathcal{F}$:

$$\langle F(\Phi)\rangle_g = \frac{1}{\mathcal{Z}}\int_\mathcal{F} F(\boldsymbol{\phi})e^{-S(\boldsymbol{\phi},g)}D_g\boldsymbol{\phi} \tag{1.24}$$

and the Vertex Operators, depending on vectorial weights, are of the form

$$V_\alpha(z) = e^{\langle\alpha, \Phi(z)\rangle}. \tag{1.25}$$

10. For $h \in \mathfrak{h}$, $\mathrm{ad}(h)$ is diagonalizable.

Toda CFTs are interesting in their own because they provide examples of concrete models where the W-symmetry can be studied, and also because they share links with many other theories, such as Wess-Zumino-Witten models, some Statistical Physics models (Ising and Potts) and through the extended AGT correspondence. The latter, discovered in [1], bridges the gap between Liouville theory and four-dimensional gauge theories. This correspondence has been extended to certain Toda theories in [100]. Moreover, the Toda path integral can also be seen as the quantization of the classical Toda action, itself deriving from what is called the Toda equations. The Toda equations are a special case of the celebrated Hitchin's equation, that describes harmonic maps from the universal cover of a surface into symmetric spaces. It is expected that this picture is recovered from Toda CFTs when taking the semi-classical limit $\gamma \to 0$ (see Chapter 5).

Despite their importance and the great amount of efforts to understand their behavior and the properties of their symmetry algebras, Toda CFTs are very far to be fully understood. The first step in the implementation of the bootstrap procedure, namely the computation of the structure constants, have been carried out on the sphere by Fateev-Litvinov [40] in the case where $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$, while in the boundary case a formula for the bulk one-point function has been proposed by Fateev-Ribault [38] and also studied in [45]. Beyond these progresses it is not clear in which form the bootstrap procedure would hold for Toda theories. Thus, and despite the groundbreaking achievements of the algebraic bootstrap method in Physics, some big open questions remain. This is the reason why it is of great importance to develop new methods in addressing these problems, and this is how has emerged the probabilistic approach to two-dimensional CFT, almost two decades ago [11].

1.1.5 The probabilistic approach

The path integral formulation of Euclidean Quantum Field Theory —although *a priori* ill-defined from a mathematical perspective— is particularly well-suited in certain cases to be given a rigorous probabilistic meaning. Before describing the probabilistic interpretation of Liouville and Toda CFTs, let us start with a well-known toy example. On the space $C_0^0([0,1])$ of continuous functions on the unit interval that vanish at both ends, define the formal Wiener measure

$$W(dx) = \frac{1}{\mathcal{Z}} e^{-S(x)} dx \tag{1.26}$$

with the action functional S given by

$$S(x) = \frac{1}{2}\int_0^1 (\partial_u x(u))^2 du. \tag{1.27}$$

11. It is worth mentioning that the notion of conformal invariance in Probability Theory was first formulated earlier by Schramm [88], and allowed a rigorous study of the links between Statistical Physics at criticality and Probability Theory, as put forward (yet non-rigorously) by Cardy.

Let B be sampled according to the measure $W(dx)$. For s and t in the unit interval, the covariance between $B(s)$ and $B(t)$ is given by

$$\frac{1}{\mathcal{Z}}\int_{C_0^0([0,1])} x(s)x(t)W(dx). \tag{1.28}$$

This is a Gaussian integral with covariance kernel given by the operator L defined by $L = -\partial_u^2$ and the boundary conditions $x(0) = x(1) = 0$, and hence formally the integral (1.28) is given by $L^{-1}(s,t)$, that is the Green's function associated with L in the sense of integral kernels. This Green's function is easy to compute, and one finds

$$L^{-1}(s,t) = s\wedge t - st \tag{1.29}$$

which is the covariance of the Brownian bridge between 0 and 1. Thus, we can give a rigorous probabilistic interpretation of the measure $W(dx)$ as a Gaussian measure on the space $H_0^1([0,1])$. As we will see (Subsection 1.2.1 and Chapter 3), one can follow the same lines to provide a rigorous interpretation of the path integrals defining the Liouville and Toda CFTs. Indeed, the kinematic part of the Liouville action gives rise to the formal path integral measure

$$\frac{1}{\mathcal{Z}}e^{-S_{FF}(\phi)}D\phi \tag{1.30}$$

where

$$S_{FF}(\phi) := \frac{1}{2}\int_{\mathbb{S}^2}\frac{1}{2\pi}\left|\nabla\phi\right|^2 dv_g.$$

By following the above example of the Brownian bridge, the measure (1.30) is naturally understood as describing a centered Gaussian field, with covariance kernel given by the inverse (in the sense of integral kernels) of the operator $-\frac{1}{2\pi}\Delta$ on the sphere, which is given by a Green's function. This Gaussian field is called the Gaussian Free Field (GFF). The problem is then to incorporate the interaction terms of the action (1.10) in this picture: the covariance of the GFF being way less regular than for the Brownian bridge, it requires particular care to interpret these terms. However, due to the exponential form of these interaction terms, this task is well achieved by another probabilistic object, the Gaussian Multiplicative Chaos (GMC).

The probabilistic formulation of Liouville CFT is due to David-Guillarmou-Kupiainen-Rhodes-Vargas [32, 31, 51], and has led to major progress in the mathematical understanding of two-dimensional Conformal Field Theory. Firstly, on the sphere, the DOZZ formula for the structure constants has been rigorously derived in [65], and the conformal boostrap procedure has been implemented in [53], while the Segal's axioms [89] were proven to be true in [54] (see Chapter 5 for more about Segal's axioms). In the boundary case, i.e. on the disk or upper half-plane, the structure constants of Liouville theory were obtained in a particular case in [82, 83] using the BPZ equations mentioned previously (see Section 1.2.3 and Chapter 4 for a probabilistic exposition of these kind of results). In the general case, the *mating-of-trees* machinery introduced in

[36] has led to the derivation of BPZ equations and to the computation of all structure constants in [5, 6]. The methods known under the name mating-of-trees rely on probabilistic couplings between the Liouville field and Schramm-Loewner Evolutions (SLE), and allow one to obtain functional equations involving probability distributions. Recently, a way to avoid the use of the mating-of-trees machinery in deriving the BPZ equations for the boundary Liouville theory has been proposed in [24]. This is the approach that we will follow in the second part of this thesis, in great part because it is closer in spirit to the methods of CFT, and also because it is unclear at this stage which form would take the SLE couplings within Toda CFTs.

Indeed, the probabilistic construction unveiled in [32] admits a generalization to Toda theories on the sphere as defined in [30]. In the case of the Lie algebra $\mathfrak{sl}_3\mathbb{C}$, the symmetries of the probabilistic model thus defined have been studied in [25] where Ward identities (see Section 1.2.2 and Chapter 3) were proven to hold, leading to a rigorous proof of the formula of Fateev-Litvinov [40] for the three-point structure constants in [21, 20], although in a particular case (when one of the insertions is semi-degenerate, see 1.2.3), the general case seeming out of reach for the time being with the only available constraints brought about by our understanding of W-symmetry.

1.1.6 A word about Vertex Operator Algebras

As it was already mentioned above, the symmetry algebras of Toda Conformal Field Theories —the W-algebras— can be defined in several ways, the naive one being to use the commutation relations, that rapidly become intractable when the rank of the underlying Lie algebra increases. The mathematical framework in which these algebraic structures are rigorously considered is the one of *Vertex Operator Algebras*, and has been introduced by Borcherds [15] and Frenkel-Lepowsky-Meurman [47]. Although the use of these concepts does not seem essential when dealing with the Virasoro algebra —that is a Lie algebra, with simple explicit commutation relations and a geometrical definition for its generators— it becomes crucial when considering W-algebras. The study of the notion of W-algebra within this framework was initiated by Feigin-Frenkel [43, 42] and subsequently by Kac-Roan-Wakimoto [63], and is nowadays a very active field of research. The representation theory of W-algebras has in particular being studied in [46, 7], see [8] for an introduction.

Although the work developed in this thesis stays mainly on the probabilistic side, and avoids (most of the time) the definition and use of the concepts of Vertex Operator Algebras, the latter is an important input to many results that we obtain, and reciprocally it is possible to derive several direct implications of our results within this framework.

1.2 Content of the thesis

This work initiates and furthers the mathematical study of Toda CFTs on Riemann surfaces with boundary. As already mentioned, less is known about these theories, although they make apparent some key notions in the study of CFT in the presence of a boundary. More precisely, and contrary to the Virasoro algebra, the W-symmetry is less understood and do not allow yet to construct Toda CFTs using algebraic constraints. The probabilistic approach at least achieves this construction as it has already been shown on the sphere [30], and as we shall see in this document for other surfaces, especially with boundary. On the sphere, the probabilistic approach has proven to be powerful in that it had led to a rigorous proof of the Fateev-Litvinov formula [40] for the three-point function [25, 21, 20]. Unfortunately we are not able yet to implement the full bootstrap procedure for Toda CFTs in the probabilistic setting. In the boundary case however, a first step towards this direction can be carried out: the derivation of the BPZ equations. As pictured before, the latter are of fundamental importance in that they supposedly allow one to compute the structure constants of the theory. This scheme is way less clear in Toda CFTs than in their Liouville counterpart, but there is good hope that these BPZ equations, together with extra input from the analysis of the probabilistic path integral, can lead to exact computations. In this thesis we preform this first step by: 1) generalizing the construction [30] to general Riemann surfaces with or without boundaries, this stage makes evident some key features of boundary CFT in general, that are interesting in their own; 2) showing the validity of the Ward identities on the boundary in the special case where the Lie algebra under consideration is $\mathfrak{sl}_3\mathbb{C}$; 3) analyzing the representation of the W_3 algebra thus constructed to identify singular vectors, that in turn give rise to higher equations of motion, the latter resulting in the BPZ equations, under additional conditions.

In Chapter 2, we generalize the construction [30] to all compact hyperbolic Riemann surfaces, with or without boundary. The hyperbolicity assumption here is just convenient but the same techniques apply to other surfaces, and especially the disk (or half-plane). To this end, we introduce a framework based on vectorial Gaussian Free Fields with different kinds of boundary conditions. In the presence of a boundary, and unlike in Liouville CFT, different boundary conditions are possible for the field of the theory, in relation with the structure of the outer automorphism group of the target Lie algebra. We propose a path integral (probabilistic) formulation in the different cases. Moreover, we show that the models thus defined enjoy the basic fundamental properties of a CFT, that are the Weyl covariance and the invariance under diffeomorphisms. We also discuss for each configuration the symmetry algebra of the model, within the framework of Vertex Operator Algebras. This part is based on the article [28], and is introduced in Section 1.2.1.

The following two Chapters specialize the study to the A_2 (or $\mathfrak{sl}_3\mathbb{C}$) Toda theory on the complex upper half-plane, with Neumann boundary conditions. In Chapter 3, we make sense of the conserved currents $\mathbf{T}$ and $\mathbf{W}$ of the W_3 symmetry algebra and of their descendants on the boundary, building on the ideas developed in [24] for boundary Liouville CFT. We then prove that these definitions effectively constitute a representation of the symmetry algebra by showing the validity of the Ward identities, both in the bulk and on the boundary. This last feature is new even compared to the Physics literature, and in fact it was unclear that such identities would hold for the boundary theory. For an introduction to this Chapter, see Section 1.2.2.

Chapter 4 is dedicated to the study of the representation of the W-algebra in the model, although we keep the exposition on the probabilistic side and do not make use of the framework of Vertex Operator Algebras (but we keep this for future work, see Chapter 5). By using the same techniques as in Chapter 3, together with inputs from the representation theory of the W_3 algebra, we prove the existence of singular vectors in the theory, that provide linear constraints between descendants. We show that in the boundary case, these singular vectors are not necessarily null vectors but rather give rise to higher equations of motion. As a byproduct, we obtain new hypergeometric differential (BPZ) equations for the three-point function and the bulk-boundary correlator of the theory, paving the way for the computation of these boundary structure constants, for which no formula is even conjectured in the Physics. In Section 1.2.3 we introduce these results in more detail. The material in these two Chapters is taken from the articles [26, 27]. All the articles [28, 26, 27] were co-authored with Baptiste Cerclé.

Finally in Chapter 5 we detail some future directions of research, at different time scales, some of them being already investigated at the time being.

1.2.1 Construction of the closed and boundary models

We sketch here the contents of Chapter 2. For precise definitions and details, we therefore refer to Chapter 2. In this section we fix a simple finite-dimensional complex Lie algebra $\mathfrak{g}$ of rank r, and denote by $\mathfrak{a}$ the real part of its Cartan sub-algebra. We equip $\mathfrak{a}$ with a scalar product $\langle\cdot,\cdot\rangle$, proportional to the restriction of the Killing form to $\mathfrak{a}$, and denote by $(e_i)_{=1,\dots,r}$ a set of simple roots. For precise definition we refer to Subsection 2.2.2.

1.2.1.1 Toda CFTs on closed compact hyperbolic Riemann surfaces

Let Σ be a compact connected Riemann surface without boundary, that we suppose of hyperbolic type ($\chi(\Sigma)<0$). We equip Σ with a smooth Riemannian metric g. Our first goal is to make sense of the path integral (1.24), and first of all, we focus on the kinetic term in the action (1.23). Namely, we want to interpret the term

$$\int F(\boldsymbol{\phi})e^{-\frac{1}{4\pi}\int_\Sigma |\nabla_g\boldsymbol{\phi}|_g^2 dv_g}D_g\boldsymbol{\phi}. \tag{1.31}$$

Following [32], this is possible thanks to the introduction of the Gaussian Free Field (GFF). Indeed, as in the toy example of the Brownian bridge given in Section 1.1.5, the integral (1.31) is a Gaussian integral that describes the law of a centered Gaussian field with covariance kernel given by $\left(-\frac{1}{2\pi}\Delta_g\right)^{-1}$, that we interpret in the sense of integral kernels by the corresponding Green's function G. The vectorial GFF $\mathbf{X}^g$ is then naturally defined by its covariance structure by setting for all $u,v\in\mathfrak{a}$ and $f_1,f_2\in H^1(\Sigma,g)$

$$\mathbb{E}[(\langle u,\mathbf{X}^g\rangle,f_1)_{H^1(\Sigma,g)}(\langle v,\mathbf{X}^g\rangle,f_2)_{H^1(\Sigma,g)}]=\langle u,v\rangle\int_{\Sigma^2}G_g(x,y)f_1(x)f_2(y)dv_g(x)dv_g(y).$$

Indeed, it is worth saying that the GFF $\mathbf{X}^g$ makes sense only as a generalized function and lives almost surely in the negative Sobolev space $H^{-1}(\Sigma \to \mathfrak{a})$ (this has to be understood component by component). We then define

$$\int F(\boldsymbol{\phi})e^{-\frac{1}{4\pi}\int_\Sigma |\nabla_g \boldsymbol{\phi}|_g^2 dv_g} D_g\boldsymbol{\phi} \propto \int_{\mathfrak{a}} \mathbb{E}[F(\Phi)]d\boldsymbol{c} \tag{1.32}$$

with the expectation taken with respect to the GFF, and where the Toda field Φ is defined by $\Phi = \mathbf{X}^g + \boldsymbol{c}$. This comes from the fact that the heuristic developed above is valid only outside the kernel of the Laplacian, so that one has to take into account the remaining *zero-mode* $\boldsymbol{c}$. The integration over $\mathfrak{a}$ is defined by

$$\int_{\mathfrak{a}} F(\boldsymbol{c})d\boldsymbol{c} = \int_{\mathbb{R}^r} F\Big(\sum_{i=1}^r \boldsymbol{v}_i c_i\Big)\prod_{i=1}^r dc_i$$

where $(\boldsymbol{v}_i)$ is any orthonormal basis of $\mathfrak{a}$. It then remains to compute the total mass of the path integral (1.31) in order to define the normalizing factor. Following the analogy with finite-dimensional Gaussian measures we interpret it as the (regularized) determinant of the Laplacian

$$\mathcal{Z}_{\mathbf{X}}(\Sigma, g) = \left(\frac{\det'\left(-\frac{1}{2\pi}\Delta_g\right)}{v_g(\Sigma)}\right)^{-\frac{r}{2}}.$$

The properties of these determinants, and especially their behavior under a conformal change of metric, have been studied by Polyakov [78, 79], and in the boundary case by Alvarez [3]. This behavior is completely explicit and is crucial in order to prove the same kind of properties for the correlation functions of Toda CFTs (see Theorem 1.2.2).

Having defined the kinetic part of the path integral, it remains to incorporate the exponential potentials (the curvature terms are not a problem because they are well-defined functions) in this definition. It is possible thanks to the introduction of Gaussian Multiplicative Chaos (GMC). Namely, the exponential term $e^{\langle \gamma e_i, \Phi\rangle} dv_g$ can be given a mathematical meaning as a random measure on Σ, through a regularization procedure (see Section 2.3.2). The theory of GMC is now well-known, and we refer e.g. to [87, 84, 13] for more details.

We finally interpret the path integral (1.24) as the limit

$$\langle F(\Phi)\rangle_g := \lim_{\varepsilon\to 0}\langle F(\Phi)\rangle_{g,\varepsilon}, \quad \text{where} \quad \langle F(\Phi)\rangle_{g,\varepsilon} := \mathcal{Z}_{\mathbf{X}}(\Sigma, g)\int_{\mathfrak{a}} \mathbb{E}\left[F(\mathbf{X}_\varepsilon^g + \mathbf{c})e^{-V_g(\mathbf{X}_\varepsilon^g+\mathbf{c})}\right]d\mathbf{c} \tag{1.33}$$

for suitable functionals F, with the regularized Toda potential given by

$$V_g(\mathbf{X}_\varepsilon^g + \mathbf{c}) = \frac{1}{4\pi}\int_\Sigma R_g\langle Q, \mathbf{X}_\varepsilon^g + \mathbf{c}\rangle dv_g + \sum_{i=1}^r \mu_{B,i} e^{\langle \gamma e_i, \mathbf{c}\rangle}\int_\Sigma \varepsilon^{\frac{\gamma_i^2}{2}} e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g\rangle} dv_g. \tag{1.34}$$

The field $\mathbf{X}_\varepsilon^g$ is a suitable regularization of the GFF. The convergence and the nontriviality of the limit are consequences of Theorem 1.2.1. We define in the same fashion correlation functions of Vertex Operators, by taking $F = V_{\alpha_1}(z_1)...V_{\alpha_N}(z_N)$, and by regularizing the GFF inside the V_α's.

1.2.1.2 Boundary Toda theories

Suppose now that Σ has a non-empty boundary $\partial\Sigma$, then the Toda action (1.23) has to be modified in order to take into account the boundary effects. Namely, one has to add boundary potentials:

$$\begin{aligned} S_T(\boldsymbol{\phi}, g) = \frac{1}{4\pi}\int_\Sigma \left(|d_g\boldsymbol{\phi}|_g^2 + R_g\langle Q, \boldsymbol{\phi}\rangle + \sum_{i=1}^r 4\pi\mu_{B,i}e^{\langle\gamma e_i, \boldsymbol{\phi}\rangle}\right)\mathrm{d}v_g \\ + \frac{1}{2\pi}\int_{\partial\Sigma}\left(k_g\langle Q, \boldsymbol{\phi}\rangle + \sum_{i=1}^r 2\pi\mu_i e^{\langle\frac{\gamma e_i}{2}, \boldsymbol{\phi}\rangle}\right)\mathrm{d}\lambda_g \end{aligned} \tag{1.35}$$

where λ_g is the line element on $\partial\Sigma$ and k_g the geodesic curvature, while the $\mu_i \in \mathbb{C}$, $i = 1, \cdots, r$ are boundary cosmological constants. In the presence of a boundary one needs to prescribe boundary conditions for the field Φ of the theory. Although the Neumann (or free) boundary conditions are the only possible choice in the case of Liouville CFT (corresponding to the $\mathfrak{sl}_2\mathbb{C}$ Toda theory), it is not generally the case for other Toda CFTs, as we will now explain. It is well known that the Riemann surface with boundary Σ can be "doubled" in order to form a closed Riemann surface Σ that is the union of Σ and its mirror copy, glued along the boundary. This gluing takes the form of an involution σ sending Σ to its double and preserving the boundary. A smooth metric g on Σ then extends to a smooth metric on the closed surface through its pullback by the involution σ. This is called the Schottky double. For example, one can start with the complex upper half-plane, and glue it along the real line with its mirror double to obtain the full complex plane. The involution σ in this case is the complex conjugate.

Inspired by the so-called *Cardy's doubling trick* [17, Section 4] in boundary Conformal Field Theory (see also [18, 19]), we may want to realize the Toda field $\Phi : \Sigma \to \mathfrak{a}$ using a field $\hat{\Phi} : \hat{\Sigma} \to \mathfrak{a}$ that lives over the closed surface $\hat{\Sigma}$. To do so we can set

$$\Phi(z) := \frac{\hat{\Phi}(z) + \left(\hat{\Phi}(\sigma(z))\right)}{\sqrt{2}} \tag{1.36}$$

where *a priori* τ is a linear involution of $\mathfrak{a}$. For example, if $\tau = \mathrm{Id}$, then Φ has Neumann boundary conditions, while if $\tau = -\mathrm{Id}$, it has Dirichlet ones. More generally Φ can be decomposed as a sum of two fields with respectively Neumann and Dirichlet boundary conditions. Now it is suggested in Physics (in the case of $\mathfrak{sl}_3\mathbb{C}$, see [38]) that not every linear involution of $\mathfrak{a}$ leads to boundary conditions that are compatible with the symmetry algebra, in the sense that the invariance property of the field $\Phi(\sigma(z)) = \tau\Phi(z)$ should lift to the conserved currents of the symmetry algebra. However, this is the case if τ is an *outer automorphism* of the underlying Lie algebra $\mathfrak{g}$. An outer automorphism can be seen as an automorphism of the root system of the Lie algebra, or equivalently as a Dynkyn diagram automorphism.

In view of the above discussion we propose in Section 2.4 the following probabilistic definition for boundary Toda CFTs. To start with fix τ and write the orthogonal decomposition $\mathfrak{a} = \mathfrak{a}_N \oplus \mathfrak{a}_D$ with $\mathfrak{a}_N$ the maximal subspace of $\mathfrak{a}$ such that $\tau_{|\mathfrak{a}_N} = \mathrm{Id}$. Then, for g a uniform metric of type 1 (see Subsection 2.2.1), we let $\mathbf{X}^{g,C} = \mathbf{X}^{g,N} + \mathbf{X}^{g,D}$ where $\mathbf{X}^{g,N} : \Sigma \to \mathfrak{a}_N$ is a GFF with Neumann boundary conditions and $\mathbf{X}^{g,D} : \Sigma \to \mathfrak{a}_D$ is a Dirichlet one. The correlation functions of the model are then defined by a limiting procedure:

$$\begin{aligned}
&\langle F(\Phi) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle_{g,\tau} := \\
&\lim_{\varepsilon \to 0} \frac{1}{\mathcal{Z}} \int_{\mathfrak{a}_N} e^{-\langle Q\chi(\Sigma), \mathbf{c}\rangle} \mathbb{E}\Big[F\left(\mathbf{X}^{g,C} + \mathbf{c}\right) \prod_{k=1}^{N} \varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{X}^{g,C}_\varepsilon(z_k) + \mathbf{c}\rangle} \prod_{l=1}^{M} \varepsilon^{\frac{|\beta_l|^2}{4}} e^{\langle \frac{1}{2}\beta_l, \mathbf{X}^{g,C}_\varepsilon(s_l) + \mathbf{c}\rangle} \\
&\times \exp\Big(-\sum_{i=1}^{r} \mu_{B,i} \int_\Sigma e^{\langle \gamma e_i, \mathbf{X}^{g,C} + \mathbf{c}\rangle} \mathrm{d}v_g - \sum_{j=1}^{d_N} \mu_j \int_{\partial\Sigma} e^{\langle \frac{\gamma}{2} f_j, \mathbf{X}^{g,C} + \mathbf{c}\rangle} \mathrm{d}\lambda_g \Big) \Big] \mathrm{d}\mathbf{c}.
\end{aligned} \tag{1.37}$$

for F continuous and bounded on a suitable functional space, where $d_N = \dim(\mathfrak{a}_N)$ and the $(f_j)_{1 \leq j \leq d_N}$ form a basis of $\mathfrak{a}_N$ (see Subsection 2.2.2.3). Here the $s_1, \cdots, s_l$ are additional insertions for Vertex Operators on the boundary $\partial\Sigma$.

We are now in position to state the main results of this chapter.

1.2.1.3 Statements of the main results

Let (Σ, g) be a smooth, connected, compact Riemannian surface with (possibly empty) boundary $\partial\Sigma$, and assume that $\chi(\Sigma) < 0$. We also fix $\mathfrak{g}$ a complex simple Lie algebra as well as τ an automorphism of the associated root system $\mathscr{R}$, with $\tau = \mathrm{Id}$ if $\partial\Sigma = \emptyset$. We further consider $z_1, \cdots, z_N$ in Σ and $s_1, \cdots, s_l$ on $\partial\Sigma$, all distinct, and $\alpha_1, \cdots, \alpha_N, \beta_1, \cdots, \beta_M$ in $\mathfrak{a}$.

Existence of the correlation functions Our first result is a necessary and sufficient condition, the *Seiberg bounds*, ensuring that the correlation functions defined by Equation (1.37), are well-defined and non-trivial (*i.e.* are finite and non-zero):

Theorem 1.2.1. *For F bounded and continuous over $H^{-1}(\Sigma \to \mathfrak{a}, g)$, the correlation function $\langle F(\Phi) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle_{g,\tau}$ exists and is non-trivial if and only if for $i = 1, \cdots, r$:*

$$\begin{aligned}
&\langle \sum_{k=1}^{N} \alpha_k + \sum_{l=1}^{M} \frac{\beta_l}{2} - Q\chi(\Sigma), \omega_i^\vee \rangle > 0; \\
&\langle \alpha_k - Q, e_i \rangle < 0 \quad \textit{for all } k = 1, ..., N \quad \textit{and} \quad \langle \beta_l - Q, e_i \rangle < 0 \quad \textit{for all } l = 1, ..., M.
\end{aligned} \tag{1.38}$$

Here the $\omega_i^\vee$ form the basis of $\mathfrak{a}$ dual to that of the simple roots e_i. These Seiberg bounds are the same as in the case of the sphere [30, Theorem 1.1].

Conformal and diffeomorphism invariance In addition to the existence of the correlation functions we would like to show that they satisfy basic assumptions related to conformal symmetry. However when $\tau \neq \mathrm{Id}$ this might no longer be the case since some of the f_j no longer have same norm as the e_i: as such general conformal invariance is broken. To overcome this issue we need to assume that the GMC terms $e^{\langle \frac{\gamma}{2} f_j, \mathbf{X}^{g,C}\rangle} d\lambda_g$ do not appear in the path integral, which means taking $\mu_j = 0$ for j such that $\left|f_j\right|^2 \neq 2$. We refer to Subsection 2.4.3.2 for a more detailed discussion. Under this additional assumption we show a Weyl anomaly under a conformal change of metric, and diffeomorphism invariance:

Theorem 1.2.2. *In the setting of Theorem 1.2.1, assume that, for $\tau \neq Id$, $\mu_j = 0$ for j such that $\left|f_j\right|^2 \neq 2$. Then for any $g' = e^{\varphi} g$ in the conformal class of g*

$$\langle F(\Phi) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{g',\tau} = \langle F(\Phi - \frac{Q}{2}\varphi) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{g,\tau} \times \tag{1.39}$$
$$\prod_{k=1}^N e^{-\Delta_{\alpha_k}\varphi(x_k)} \prod_{l=1}^M e^{-\frac{1}{2}\Delta_{\beta_l}\varphi(s_l)} \exp\left(\frac{r+6|Q|^2}{96\pi}\left(\int_\Sigma (|\nabla_g \varphi|_g^2 + 2R_g\varphi) dv_g + 4\int_{\partial\Sigma} k_g \varphi d\lambda_g\right)\right).$$

In the above the conformal weights are given by $\Delta_\alpha = \langle \frac{\alpha}{2}, Q - \frac{\alpha}{2}\rangle$. Besides, if $\psi : \Sigma \to \Sigma$ is an orientation-preserving diffeomorphism sending the boundary to itself:

$$\langle F(\Phi) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\psi^* g,\tau} = \langle F(\Phi \circ \psi) \prod_{k=1}^N V_{\alpha_k}(\psi(z_k)) \prod_{l=1}^M V_{\beta_l}(\psi(s_l))\rangle_{g,\tau}. \tag{1.40}$$

Remark 1.2.3. *The Weyl anomaly (1.39) allows one to identify the central charge of the theory which is equal to $c = r + 6|Q|^2$, in agreement with the sphere case [30].*

Implications on the symmetry algebra The construction of the field of the theory has in turn some consequences on the symmetry algebra of the latter: as mentioned above these are W-algebras, Vertex Operator Algebras (VOAs) containing the Virasoro algebra as a sub-algebra. Indeed, by construction and in view of Equation (1.36), the underlying field of boundary Toda CFT satisfies $\Phi(\sigma(z)) = \tau\Phi(z)$ (where σ is the doubling involution): this form of invariance actually lifts to the symmetry algebra of our model. Namely we prove in Subsection 2.4.4 (see this subsection for an introduction to the formalism of VOAs in this context, and Proposition 2.4.14 for a more precise statement) the following:

Proposition 1.2.4. *Assume that $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$. Then there exist currents $\mathbf{W}^{(s_i)}[\Psi]$ of spins $s_i = 2, \cdots, n$, such that for generic values of γ:*

— *the W-algebra associated to $\mathfrak{g}$ is generated using the modes of the $(\mathbf{W}^{(i)}[\Psi]])_{2\leq i\leq n}$;*
— *for $\tau \in Out(\mathscr{R})$ non-trivial we have $\mathbf{W}^{(i)}[\tau\Psi] = (-1)^i \mathbf{W}^{(i)}[\Psi]$.*

We actually have a more general statement for any $\mathfrak{g}$. The fact that boundary Toda CFTs satisfy such a property, which in view of Equation (1.36) synthetically takes the form $\overline{\mathbf{W}^{(i)}} = (-1)^i \mathbf{W}^{(i)}$, was postulated in the $\mathfrak{sl}_3\mathbb{C}$ case and in the conformal bootstrap setting in [38]. In this respect our definition generalizes this feature to general $\mathfrak{g}$ and τ and provides a concrete, path integral and probabilistic, realization of this property.

1.2.2 Ward identities

In this subsection we specialize the analysis to the Toda CFT defined on the complex upper half-plane $\mathbb{H}$, with underlying root system A_2 (or equivalently the Lie algebra $\mathfrak{sl}_3\mathbb{C}$). We also specify Neumann boundary conditions for the Toda field Φ, in the sense of Subsection 1.2.1 or Chapter 2.

Local Ward identities associated with the Virasoro algebra (or equivalently the stress-energy tensor $\mathbf{T}$) encode the local conformal symmetry enjoyed by the correlation functions. The same kind of identities for the W-algebra play the same role regarding the higher-spin symmetry. Thus, in the case of the A_2 boundary Toda theory that we consider here, in order to make explicit the symmetries enjoyed by the model, one needs to work out Ward identities associated not only with conformal invariance but also with higher-spin symmetry, both in the bulk and on the boundary. It is worth recalling at this point that unlike the Virasoro algebra, W-algebras are *not* Lie algebras and the formalism of Vertex Operator Algebras is thus necessary to make sense of this notion. As such, the study of the symmetries enjoyed by Toda CFTs is more involved than its counterpart in Liouville theory and relies on additional tools related, among other things, to the representation theory of W-algebras. Actually, it was far from clear in the Physics literature that the model we consider here has the desired symmetries (in the sense of the Ward identities). We show in Chapter 3 that it is indeed the case.

The correlation functions of Vertex Operators are defined by considering distinct bulk insertions $(z_1,\cdots,z_N)\in\mathbb{H}^N$ and distinct boundary insertions $(s_1,\cdots,s_M)\in\mathbb{R}^M$ (here $\mathbb{R}$ is viewed as the boundary of $\mathbb{H}$) with respective weights $(\alpha_1,\cdots,\alpha_N)$ and $(\beta_1,\cdots,\beta_M)$. They formally take the form

$$\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle := \frac{1}{\mathcal{Z}}\int_{\mathscr{F}}\prod_{k=1}^N e^{\langle\alpha_k,\Phi(z_k)\rangle}\prod_{l=1}^M e^{\langle\beta_l,\Phi(s_l)\rangle}.$$

Computing such correlation functions is one of the main goals in the study of Toda CFTs. In order to derive such expressions, one relies on the symmetries of the theory which are encoded here by the W_3 algebra. In the model, this W_3 algebra manifests itself via the existence of two holomorphic currents, the stress-energy tensor $\mathbf{T}$ associated to conformal invariance and the higher-spin current $\mathbf{W}$ that encodes the higher level of symmetry. At the (Vertex Operator) algebraic level, these currents admit the formal Laurent series expansion

$$\mathbf{T}(z)=\sum_{n\in\mathbb{Z}} z^{-n-2}\mathbf{L}_n \quad\text{and}\quad \mathbf{W}(z)=\sum_{n\in\mathbb{Z}} z^{-n-3}\mathbf{W}_n$$

where the modes $(\mathbf{L}_n,\mathbf{W}_m)_{n,m\in\mathbb{Z}}$ satisfy the commutation relations of the W_3 algebra. One key property of these currents is their Operator Product Expansions with Vertex

Operators, which are assumed to take the form:

$$\mathbf{T}(z_0)V_\alpha(z) = \frac{\Delta_\alpha V_\alpha(z)}{(z_0-z)^2} + \frac{\mathrm{L}_{-1}V_\alpha(z)}{z_0-z} + \text{reg.} \tag{1.41}$$

and

$$\mathbf{W}(z_0)V_\alpha(z) = \frac{w(\alpha)V_\alpha(z)}{(z_0-z)^3} + \frac{\mathrm{W}_{-1}V_\alpha(z)}{(z_0-z)^2} + \frac{\mathrm{W}_{-2}V_\alpha(z)}{z_0-z} + \text{reg.} \tag{1.42}$$

where the $\mathrm{W}_{-i}V_\alpha(z)$ and $\mathrm{L}_{-i}V_\alpha(z)$ are the *descendant fields* associated to V_α while Δ_α is the conformal weight and $w(\alpha) \in \mathbb{C}$ is called the *quantum number* associated to $\mathbf{W}$. When inserted within correlation functions these Operator Product Expansions becomes *local Ward identities*. Let us sketch here the method that we develop in Chapter 3 to prove such results.

1.2.2.1 Descendant fields and currents

The first step in the derivation of the Ward identities is to give a rigorous definition of the currents $\mathbf{T}$ and $\mathbf{W}$ and of the descendants of the primary fields $\mathbf{L}_{-n}V_\alpha(z)$ and $\mathbf{W}_{-n}V_\alpha(z)$. The currents have explicit expressions as polynomials in the derivatives of the Toda field Φ, namely

$$\mathbf{T}(z)[\Phi] = \langle Q, \partial^2\Phi(z)\rangle - \langle \partial\Phi(z), \partial\Phi(z)\rangle; \tag{1.43}$$

$$\mathbf{W}(z)[\Phi] = q^2 h_2(\partial^3\Phi(z)) - 2qB(\partial^2\Phi(z), \partial\Phi(z)) - 8C(\partial\Phi(z), \partial\Phi(z), \partial\Phi(z)). \tag{1.44}$$

Here z stands for a bulk point inside $\mathbb{H}$, in which case the derivative ∂ has to be understood as a complex (Wirtinger) derivative, or for a boundary insertion on $\mathbb{R}$, in which case the complex derivative is replaced by the usual (real) derivative. Here the h_i are particular vectors and $q = \gamma + \frac{2}{\gamma}$. All the quantities appearing in the above formulas are defined in Subsection 3.1.1.3. The expression for the stress-energy tensor $\mathbf{T}$ can be recovered from its geometrical interpretation. Indeed, formally one has

$$\langle T(z_0)V_\alpha(z)\rangle_g \propto \frac{\delta}{\delta g^{zz}(z_0)}\langle V_\alpha(z)\rangle_g \tag{1.45}$$

which means that the insertion of the stress-energy tensor inside a correlation is tantamount to an infinitesimal variation of the metric around the insertion point. This fact allows one to recover Equation (1.43), see e.g. [71] for more details. There is no such geometrical interpretation at the time being for the higher-spin current $\mathbf{W}$, although some effort has been made in this direction (see for example [48]). Instead, we find the defining expression (1.44) by means of a *Miura transform* [41]. Let us stress that the Miura transform allows one to recover the expression (1.43) too, although it seems less natural than the above geometrical interpretation.

By formally Taylor-expanding the field Φ in these expressions around singularities we also deduce similar explicit formulas for the descendant fields, and thus reduce the problem of defining all these quantities to the one of giving a meaning to correlation functions of the form

$$\langle F[\Phi] V_\beta(t) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}$$

where $F[\Phi]$ is a polynomial in the derivatives of Φ evaluated at t. Here ρ stands for the regularization of the field and δ, ε are extra regularization parameters, as pictured on Figure 1.1.

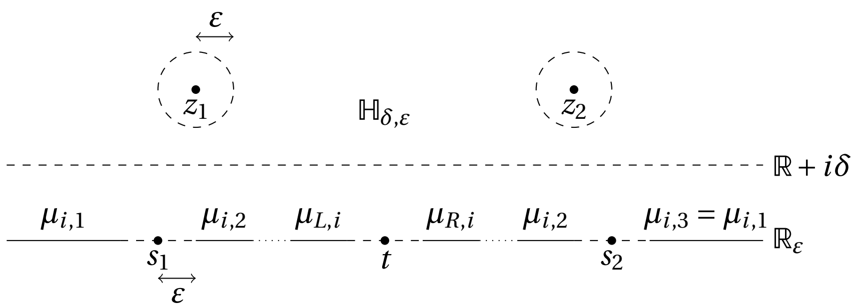

Figure 1.1 – The domains $\mathbb{H}_{\delta,\varepsilon}$ and $\mathbb{R}_\varepsilon$

This is achieved by using Gaussian integration by parts. Indeed, we can make sense of the insertion of derivatives of Φ within the correlation functions by the following lemma:

Lemma 1.2.5. *With certain assumptions on the weights* (α_k) *and* (β_l)*, for any* $t \in \mathbb{R} \setminus \{s_1, \cdots, s_M\}$*,* p *a positive integer and* $u \in \mathbb{R}^2$*:*

$$\lim_{\rho\to 0} \left\langle \frac{\langle u, \partial^p \Phi(t)\rangle}{(p-1)!} \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \right\rangle_{\rho,\varepsilon,\delta} = \sum_{k=1}^{2N+M} \frac{\langle u, \alpha_k\rangle}{2(z_k-t)^p} \langle \mathbf{V}\rangle_{\varepsilon,\delta}$$
$$- \sum_{i=1}^{2} \int_{\mathbb{R}_\varepsilon} \frac{\langle u, \gamma e_i\rangle}{2(x-t)^p} \langle V_{\gamma e_i}(x)\mathbf{V}\rangle_{\varepsilon,\delta} \mu_i(dx) + \mu_{B,i} \int_{\mathbb{H}_{\delta,\varepsilon}} \left(\frac{\langle u, \gamma e_i\rangle}{2(x-t)^p} + \frac{\langle u, \gamma e_i\rangle}{2(\bar{x}-t)^p} \right) \langle V_{\gamma e_i}(x)\mathbf{V}\rangle_{\varepsilon,\delta} d^2x.$$

Actually a more general recursive result allows one to make sense of the insertion of products of derivatives also. We see that after Gaussian integration by parts the obtained expression for the regularized correlation functions with insertion of descendant fields involves integrals over the different regularized domains of Figure 1.1. The task is now to distinguish among those different terms, the ones that converge to a well-defined limit as the regularization parameters tend to 0, and the ones that may diverge depending in the value of β. The descendant is then defined to be the limit of the converging terms.

This method is reminiscent of the definition of the weak derivative of a function, and for good reason. Indeed, we prove that the insertion of Virasoro descendants within the correlation functions can be explicitly seen as derivatives of the correlation function, that need to be understood in a weak sense on the boundary. The key point here is that, following [24], we provide explicit expressions for the weak derivatives, which in turn allows us to generalize the method to the W-descendants, yet that do not enjoy in general any interpretation as differential operators. We refer to Section 3.3 for details. We can now state the main results of this chapter.

1.2.2.2 Statements of the main results

Our first result is the proof of the validity of the local Ward identities both in the bulk and on the boundary.

Theorem 1.2.6. *Let $t \in \mathbb{R}$. For any $n \geq 2$, in the sense of weak derivatives, the conformal Ward identity holds:*

$$\begin{aligned}&\langle \mathbf{L}_{-n} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle \\ &= \left(\sum_{k=1}^{2N+M} \frac{-\partial_{z_k}}{(z_k - t)^{n-1}} + \frac{(n-1)\Delta_{\alpha_k}}{(z_k-t)^n}\right)\langle V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle.\end{aligned}$$

Moreover, for any $n \geq 3$, the higher-spin Ward identity is valid:

$$\begin{aligned}&\langle \mathbf{W}_{-n} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle = \\ &\left(\sum_{k=1}^{2N+M} \frac{-\mathcal{W}_{-2}^{(k)}}{(z_k-t)^{n-2}} + \frac{(n-2)\mathcal{W}_{-1}^{(k)}}{(z_k-t)^{n-1}} - \frac{(n-1)(n-2)w(\alpha_k)}{2(z_k-t)^n}\right)\langle V_\beta(t)\mathbf{V}\rangle\end{aligned}$$

where for $j=1,2$, $\mathcal{W}_{-j}^{(k)}\langle V_\beta(t)\mathbf{V}\rangle := \langle \mathbf{W}_{-j}V_{\alpha_k}(z_k)\prod_{l\neq k} V_{\alpha_l}(z_l)\rangle$, and $\mathcal{W}_{-j}^{(k)} = \overline{\mathcal{W}}_{-j}^{(k-N)}$ for $k \in \{N+1,\dots,2N\}$.

The descendants $\mathbf{L}_{-n}V_\beta$ and $\mathbf{W}_{-n}V_\beta$ are defined in Section 3.3.

As a corollary of the local Ward identities and the conformal covariance of the model, we also obtain global Ward identities, that provide linear relations between the descendants associated with a given correlation function:

Theorem 1.2.7. *For $0 \leq n \leq 2$ and $0 \leq m \leq 4$:*

$$\begin{aligned}&\left(\sum_{k=1}^{2N+M} z_k^n \mathcal{L}_{-1}^{(k)} + n z_k^{n-1}\Delta_{\alpha_k}\right)\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle = 0 \\ &\left(\sum_{k=1}^{2N+M} z_k^m \mathcal{W}_{-2}^{(k)} + m z_k^{m-1}\mathcal{W}_{-1}^{(k)} + \frac{m(m-1)}{2} z_k^{m-2} w(\alpha_k)\right)\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle = 0.\end{aligned}$$

Global Ward identities are a key ingredient in order to obtain differential equations for some correlation functions, as we will see in the next subsection, in that they allow one to reduce the number of variables by expressing some descendant fields as linear combinations of other ones.

1.2.3 Singular vectors and BPZ equations

The existence of Ward identities associated with the stress-energy tensor and the higher-spin current is a general feature of a CFT model that enjoys W-symmetry in addition to conformal invariance. They are concrete realizations, at the level of correlation functions, of this symmetry. There are, however, additional constraints that derive from the actual representation of the symmetry algebra that is present within the model, and thus that are model-dependent, in contrast to Ward identities. These additional constraints manifest themselves in the form of *singular vectors*, that in turn give rise to *higher equations of motion*, as we will explain now.

1.2.3.1 Singular vectors and representation theory of W-algebras

It is predicted by the representation theory of the W_3 algebra [46, 7, 43] that if the weight of the Vertex Operator V_α takes some specific values, then there are non-trivial relations between its Virasoro and W-descendants. To illustrate this suppose that $\alpha = \kappa\omega_1$ for some $\kappa \in \mathbb{R}$ (here, (ω_1, ω_2) is the dual basis of the simple roots (e_1, e_2), see Subsection 3.1.1.3). In this example, at the level of the free-field, that is, when all the cosmological constants $\mu_{B,i}$ and μ_i are identically 0 [12], we have an equality of the form $\psi_{-1,\alpha} = 0$, with

$$\psi_{-1,\alpha} := \left(\mathbf{W}_{-1} - \frac{3w(\alpha)}{2\Delta_\alpha}\mathbf{L}_{-1}\right) V_\alpha. \tag{1.46}$$

The corresponding Vertex Operator V_α is then referred to as a *semi-degenerate field*. However, this equality only holds at the level of the free-field, and is no longer valid when we consider the full boundary $\mathfrak{sl}_3\mathbb{C}$ Toda CFT (i.e. when some of the cosmological constants are not 0, which means that the action contains a non-trivial interaction term). In fact, we prove that when $V_\alpha(t)$ is a boundary semi-degenerate field, this equation may no longer be true depending on the values of the cosmological constants in a neighborhood of the insertion $t \in \mathbb{R}$. The same picture holds for higher-order descendants, when specifying further the weight α, and this gives rise to higher equations of motion. Let us describe here the method used to define and find the explicit expressions of the singular vectors.

1.2.3.2 Method for defining the singular and null vectors

Definition of the singular vectors We rely on the definition of the descendant fields performed in Chapter 3, that we generalize slightly to allow for composite descendants. For example, we will need to make sense of $\mathbf{L}_{-1}\mathbf{L}_{-1}V_\alpha$, denoted $\mathbf{L}_{-(1,1)}V_\alpha$. To incorporate these new objects in our setting we will simply write $\mathbf{L}_{-\nu}$, where ν is a Young diagram, see Subsection 4.1.1 for details. Heuristically, a singular vector is a

12. In this setting we have to make an additional assumption on the insertion weights called *neutrality condition*.

non-trivial linear combination of the form

$$\sum_{|\nu|=n} c_\nu(\alpha)\mathbf{L}_{-\nu}V_\alpha + \sum_{|\lambda|=n} d_\lambda(\alpha)\mathbf{W}_{-\lambda}V_\alpha$$

such that the associated multilinear form over $\mathbb{R}^l$

$$\sum_{|\nu|=n} c_\nu(\alpha)\mathbf{L}^\alpha_{-\nu} + \sum_{|\lambda|=n} d_\lambda(\alpha)\mathbf{W}^\alpha_{-\lambda}$$

vanishes identically. This happens for very specific values of the weight α, predicted by the representation theory of the W_3 algebra. When inserted within correlation functions such singular vectors then give rise to *higher equations of motion*, and under additional assumptions to BPZ-type differential equations. Let us explain in more details how we prove such claims.

The existence of singular vectors is predicted by the algebra and more precisely representation theory, but their explicit form is not known in general. Here we will use explicit expressions to construct singular vectors at the levels $n = 1, 2$ and 3 and that take the form

$$\psi_{n,\alpha} := \mathbf{W}_{-n}V_\alpha + \sum_{|\nu|=n} c_\nu(\alpha)\mathbf{L}_{-\nu}V_\alpha$$

where the $c_\nu(\alpha)$ are explicit, and where the weight α has to be specifically chosen. More precisely we define the singular vector within correlation functions by setting

$$\langle\psi_{n,\alpha}(t)\mathbf{V}\rangle := \langle\mathbf{W}_{-n}V_\alpha(t)\mathbf{V}\rangle + \sum_{|\nu|=n} c_\nu(\alpha)\langle\mathbf{L}_{-\nu}V_\alpha(t)\mathbf{V}\rangle,$$

where the quantities on the right-hand side are defined using the regularization procedure sketched in Subsection 1.2.2.1, and developed in Chapters 3 and 4. Heuristically, a BPZ differential equation arises when $\langle\psi_{n,\alpha}(t)\mathbf{V}\rangle = 0$, which may be the case only under certain assumptions on the cosmological constants. In the case where the right-hand side of the previous equation is not 0, we refer to as a higher equation of motion.

Explicit expression of the singular vectors For one of the singular vector considered above, we know that the associated multilinear form

$$\mathbf{W}^\alpha_{-n} + \sum_{|\nu|=n} c_\nu(\alpha)\mathbf{L}^\alpha_{-\nu}$$

vanishes identically, so it is tempting to write that $\psi_{n,\alpha} = 0$, or put differently that $\psi_{n,\alpha}$ is a *null vector*. This is however generically untrue since the descendants have been defined from distinct regularization procedures. We can however deduce from this fact that the singular vector is described using the remainder terms coming from the definition of the different descendants involved:

$$\psi_{n,\alpha} = -\lim_{\delta,\varepsilon,\rho\to 0}\left(\widetilde{\mathfrak{W}}_{-n;\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) + \sum_{|\nu|=n} c_\nu(\alpha)\widetilde{\mathfrak{L}}_{-\nu;\delta,\varepsilon,\rho}(\boldsymbol{\alpha})\right).$$

It is *a priori* far from obvious that these remainder terms admit a well-defined limit. And in fact individually these terms are divergent. There are however cancellations occurring that make the latter converge, the corresponding limit being not zero but rather given by another correlation function, that is defined from the first one (with possibly additional descendant fields) but with the weight associated to $V_\alpha(t)$ being changed. Put differently this means that the singular vector is actually not a *null vector* but is proportional to a primary or a descendant field.

We are now in position to state the main results of this chapter.

1.2.3.3 Statements of the main results

Singular vectors and higher equations of motion We first describe the singular vectors at level 1. To this end we let $t \in \mathbb{R}$ and set for $i = 1,2$, $\mu_{L,i} := \mu_i(t^-)$ and $\mu_{R,i} := \mu_i(t^+)$. Then we prove the following.

Theorem 1.2.8. *Assume that $\beta = \kappa\omega_1$ for $\kappa < q$. Then for $t \in \mathbb{R}$*

$$\psi_{-1,\beta}(t) = 2(q-\kappa)(\mu_{L,2}-\mu_{R,2})V_{\beta+\gamma e_2}(t)$$

in the sense that for any $(\beta,\boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$ and distinct insertions $(z_1,\cdots,z_N,t,s_1,\cdots,s_M)$

$$\langle\psi_{-1,\beta}(t)\mathbf{V}\rangle = 2(q-\kappa)(\mu_{L,2}-\mu_{R,2})\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle. \tag{1.47}$$

Here the left-hand side is defined by setting, in agreement with Equation (1.46)

$$\langle\psi_{-1,\beta}(t)\mathbf{V}\rangle := \langle\mathbf{W}_{-1}V_\beta(t)\mathbf{V}\rangle - \frac{3w(\alpha)}{2\Delta_\alpha}\langle\mathbf{L}_{-1}V_\beta(t)\mathbf{V}\rangle \tag{1.48}$$

where we have used the shorthand $\mathbf{V} = \prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)$.

As explained before, a semi-degenerate field is defined by making the assumption that its weight α is of the form $\alpha = \kappa\omega_1$ for some $\kappa \in \mathbb{R}$. If κ is no longer generic but takes some specific values, namely $\kappa \in \{-\gamma, -\frac{2}{\gamma}\}$, then it gives rise to a so-called *fully-degenerate field*, in which case there are two additional singular vectors in the free-field theory:

$$\begin{aligned}\psi_{-2,\chi} &:= \left(\mathbf{W}_{-2} + \frac{4}{\chi}\mathbf{L}_{-(1,1)} + \frac{4\chi}{3}\mathbf{L}_{-2}\right)V_\alpha \quad \text{and}\\ \psi_{-3,\chi} &:= \left(\mathbf{W}_{-3} + \left(\frac{\chi}{3}+\frac{2}{\chi}\right)\mathbf{L}_{-3} - \frac{4}{\chi}\mathbf{L}_{-(1,2)} - \frac{8}{\chi^3}\mathbf{L}_{-(1,1,1)}\right)V_\alpha.\end{aligned} \tag{1.49}$$

Here the notation $\mathbf{L}_{-(1,1)}V_\alpha$ refers to the second-order descendant $\mathbf{L}_{-1}\mathbf{L}_{-1}V_\alpha$, and likewise for $\mathbf{L}_{(-1,2)}$ and $\mathbf{L}_{(-1,1,1)}$. Like before in the $\mathfrak{sl}_3\mathbb{C}$ boundary Toda CFT, and in opposition to the bulk theory, these will generically not be *null vectors* and as such will give rise to higher equations of motion, which take the following form:

Theorem 1.2.9. *Assume that $\beta = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$. Then the singular vectors at level two take the form (in the sense of Theorem 1.2.8)*

$$\begin{aligned}\Psi_{-2,\frac{2}{\gamma}} &= 2\gamma(\mu_{L,2}-\mu_{R,2})\mathbf{D}_{-1,\frac{2}{\gamma}}V_{\beta+\gamma e_2} + 2\left(\frac{2}{\gamma}-\gamma\right)\left(\mu_{L,1}+\mu_{R,1}\right)V_{\beta+\gamma e_1},\\ \Psi_{-2,\gamma} &= \frac{4}{\gamma}(\mu_{R,2}-\mu_{L,2})\mathbf{D}_{-1,\gamma}V_{\beta+\gamma e_2} + 2\gamma c_\gamma(\boldsymbol{\mu})V_{\beta+2\gamma e_1} \quad \textit{for} \quad \gamma \neq 1,\end{aligned} \tag{1.50}$$

where $\mathbf{D}_{-1,\chi}$ is of the form $a\mathbf{L}_{-1}+b\mathbf{W}_{-1}$, $c_\gamma(\boldsymbol{\mu})=0$ for $\gamma>1$ and for $\gamma<1$:

$$c_\gamma(\boldsymbol{\mu}) := \left(\mu_{L,1}^2+\mu_{R,1}^2-2\mu_{L,1}\mu_{R,1}\cos\left(\pi\frac{\gamma^2}{2}\right)-\mu_{B,1}\sin\left(\pi\frac{\gamma^2}{2}\right)\right)\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma\left(1-\gamma^2\right)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}. \tag{1.51}$$

As for the singular vectors at the level three, under the additional assumption that $\mu_{L,2}-\mu_{R,2}=0$, we can find a,b for which $\widetilde{\mathbf{D}}_{-1,\chi} := a\mathbf{L}_{-1}+b\mathbf{W}_{-1}$ is such that

$$\Psi_{-3,\chi} = \widetilde{\mathbf{D}}_{-1,\chi}\Psi_{-2,\chi}. \tag{1.52}$$

We refer to Theorems 4.2.8 and 4.3.4 for more precise statements and explicit expressions for the descendant fields involved.

Differential equations for the correlation functions The derivation of higher equations of motion at level up to three is a consequence of the existence of singular vectors in the theory. Thanks to the explicit expressions of these singular vectors, we see that it is possible to choose the cosmological constants in such a way that these are zero: this leads to the existence of BPZ-type differential equations satisfied by the correlation functions as we now explain.

Let us consider a correlation function that contains a fully-degenerate field V_β with $\beta=-\chi\omega_1$, $\chi\in\{\gamma,\frac{2}{\gamma}\}$. We make the additional assumption that the cosmological constants satisfy:

$$\begin{aligned}&\mu_{L,2}=\mu_{R,2}, \quad \mu_{L,1}=-\mu_{R,1} \quad \text{for} \quad \chi=\frac{2}{\gamma}, \quad \text{while for} \quad \chi=\gamma:\\ &\mu_{L,2}=\mu_{R,2}, \quad \mu_{L,1}^2+\mu_{R,1}^2-2\mu_{L,1}\mu_{R,1}\cos\left(\frac{\pi\gamma^2}{2}\right)=\mu_{B,1}\sin\left(\frac{\pi\gamma^2}{2}\right).\end{aligned} \tag{1.53}$$

Then we see that all the singular vectors are actually null vectors:

$$\Psi_{-1,\beta}=\Psi_{-2,\beta}=\Psi_{-3,\beta}=0.$$

This allows to get three identities and express these descendants in terms of derivatives of the correlation functions. Namely we have the following:

Proposition 1.2.10. *Assume that* $\beta = -\chi\omega_1$ *where* $\chi \in \{\gamma, \frac{2}{\gamma}\}$ *and with the boundary cosmological constants satisfying the assumptions* (1.53)*. Then for* $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$*:*

$$\langle \mathbf{W}_{-1} V_\beta(t)\mathbf{V}\rangle = \frac{3w(\beta)}{2\Delta_\beta}\partial_t \langle V_\beta(t)\mathbf{V}\rangle; \tag{1.54}$$

$$\langle \mathbf{W}_{-2} V_\beta(t)\mathbf{V}\rangle = -\left(\frac{4}{\chi}\partial_t^2 + \frac{4\chi}{3}\mathscr{L}_{-2}\right)\langle V_\beta(t)\mathbf{V}\rangle; \tag{1.55}$$

$$\langle \mathbf{W}_{-3} V_\beta(t)\mathbf{V}\rangle = \left(\frac{8}{\chi^3}\partial_t^3 + \frac{4}{\chi}\partial_t\mathscr{L}_{-2} - \left(\frac{\chi}{3} + \frac{2}{\chi}\right)\mathscr{L}_{-3}\right)\langle V_\beta(t)\mathbf{V}\rangle. \tag{1.56}$$

In particular and in agreement with 1.2.6, under these assumptions we have

$$\begin{aligned}&\left(\sum_{k=1}^{2N+M}\frac{\mathcal{W}_{-2}^{(k)}}{t-z_k} + \frac{\mathcal{W}_{-1}^{(k)}}{(t-z_k)^2} + \frac{w(\alpha_k)}{(t-z_k)^3}\right)\langle V_\beta(t)\mathbf{V}\rangle\\ &= \left(\frac{8}{\chi^3}\partial_t^3 + \frac{4}{\chi}\partial_t\mathscr{L}_{-2} - \left(\frac{\chi}{3} + \frac{2}{\chi}\right)\mathscr{L}_{-3}\right)\langle V_\beta(t)\mathbf{V}\rangle.\end{aligned} \tag{1.57}$$

Based on these identities let us now consider the general form of Equation (1.57). Then we see that there are $4N+2M$ W-descendants involved in this expression (the left-hand side), the rest being differential operators acting on the correlation functions. Now the global Ward identities for the higher-spin current from Theorem 1.2.7 allow one to describe five identities between these $4N+2M$ descendants and the 2 associated to the degenerate insertion V_β. This allows one to reduce the number of W-descendants that appear in Equation (1.57) to $4N+2M-5$ plus the ones associated to V_β. Now, we already know that the descendants associated to V_β are expressible in terms of differential operators. As a consequence the total number of unknowns in Equation (1.57) can be brought down to $4N+2M-5$.

This can be done by assuming some of the other insertions to be given by semi-degenerate fields: a bulk semi-degenerate field brings about two additional constraints while a boundary semi-degenerate field yields one constraint. As a consequence we see that in order to have no W-descendants left we must be in one of the following cases:

— two generic and one semi-degenerate boundary fields correlation function;

— one generic bulk and one semi-degenerate boundary fields correlation function;

— one semi-degenerate bulk and one generic boundary fields correlation function.

Our last result consists in obtaining, in the first two cases [13], a differential equation by inserting a boundary fully-degenerate field inside the correlation function.

Theorem 1.2.11. *Take* $t \in \mathbb{R}$ *and* $\beta^{\circledast} := -\chi\omega_1$ *with* $\chi \in \{\gamma, \frac{2}{\gamma}\}$ *and assume that Equation* (1.53) *holds. Likewise for a boundary semi-degenerate field* $V_{\beta^*}(s)$ *with* $\beta^* = \kappa\omega_2$

13. Although we did not perform the computations in the last case, the method is exactly the same and one should be able to obtain a similar equation in this case too.

we assume that

$$\mu_1(s^-) = \mu_1(s^+).$$

Then we have the following BPZ-type differential equations:

— *Bulk-boundary correlator: take* $\alpha \in \mathbb{R}^2$ *such that* $(\alpha, \beta^*, \beta^{\circledast}) \in \mathscr{A}_{1,2}$. *Then:*

$$\langle V_\alpha(\mathrm{i}) V_{\beta^*}(+\infty) V_{\beta^{\circledast}}(t)\rangle = |t-\mathrm{i}|^{-\langle\beta,\alpha\rangle} \mathscr{H}(u(t))$$

where $u(t) := \frac{1}{1+t^2}$ *and* $\mathscr{H}$ *is a solution of a differential hypergeometric equation:*

$$\Big[u(A_1+u\partial)(A_2+u\partial)(A_3+u\partial) - (B_1-1+u\partial)(B_2-1+u\partial)u\partial\Big]\mathscr{H}(u) = 0. \tag{1.58}$$

Here the quantities involved are given by

$$\begin{aligned}
A_i &:= \frac{\chi}{4}\langle h_1, 2\alpha+\beta^*+\beta^{\circledast}-2Q\rangle + \frac{\chi}{2}\langle h_1-h_i, Q-\alpha\rangle,\\
B_1 &:= \frac{1}{2}, \quad B_2 := 1+\frac{\chi}{4}\langle e_2, \beta^*-2Q\rangle.
\end{aligned}$$

— *Three-point boundary correlation function: take* $\beta_1, \beta_2 \in \mathbb{R}^2$ *in such a way that* $(\beta_1, \beta_2, \beta^*, \beta^{\circledast}) \in \mathscr{A}_{0,3}$. *Then:*

$$\langle V_{\beta_1}(0) V_{\beta^*}(1) V_{\beta_2}(+\infty) V_{\beta^*}(t)\rangle = |t|^{\frac{\chi}{2}\langle\omega_1,\beta_1\rangle} |t-1|^{\frac{\chi}{2}\langle\omega_1,\beta_2\rangle} \mathscr{H}(t)$$

with $\mathscr{H}$ *is a solution of a differential hypergeometric equation:*

$$\Big[t(A_1+t\partial)(A_2+t\partial)(A_3+t\partial) - (B_1-1+t\partial)(B_2-1+t\partial)t\partial\Big]\mathscr{H}(t) = 0 \tag{1.59}$$

and where the coefficients that appear are given by

$$\begin{aligned}
A_i &:= \frac{\chi}{2}\langle h_1, \beta_1+\beta_2+\beta_3+\beta^*-2Q\rangle + \frac{\chi}{2}\langle h_i-h_1, \beta_1-Q\rangle,\\
B_i &:= 1+\frac{\chi}{2}\langle h_1-h_i, \beta_2-Q\rangle.
\end{aligned}$$

A proof of this result is found in Section 4.4.

2 Toda theories on general Riemann surfaces

Sommaire

2.1 Introduction

In this chapter we present a construction of Toda CFTs on general, compact connected hyperbolic Riemann surfaces, with or without boundaries. We also prove that these models satisfy the basic axioms of CFT, that are the conformal (Weyl) anomaly and the invariance under diffeomorphisms. In the presence of a boundary, we investigate the different boundary conditions for the field of the theory and discuss in each case the algebra of symmetry of the model.

An introduction to this Chapter as well as a presentation of the main results that it contains is found in Section 1.2.1. The content of this Chapter is taken from the article [28], co-authored with B. Cerclé.

2.2 Background and notations

This thesis deals with Toda Conformal Field Theory on Riemann surfaces with or without boundary. Our first goal is to give a precise probabilistic meaning to a certain class of these models, defined in the Physics through the path integral approach. As already mentioned in the Introduction, the path integral of Toda theories is a formal measure

$$\mu_T(D_g\boldsymbol{\phi}) = e^{-S(\boldsymbol{\phi},g)}D_g\boldsymbol{\phi} \tag{2.1}$$

on a certain space $\mathscr{F}$ of functions defined on a Riemann surface Σ (equipped with a Riemannian metric g) and with values in a particular Euclidean space $\mathfrak{a}$, with $D_g\boldsymbol{\phi}$ denoting the formal "Lebesgue" measure on $\mathscr{F}$ and with the action functional S given by

$$\begin{aligned} S(\boldsymbol{\phi},g) = &\frac{1}{4\pi}\int_\Sigma\left(\left|\nabla_g\boldsymbol{\phi}\right|_g^2 + R_g\langle Q,\boldsymbol{\phi}\rangle + \sum_{i=1}^r \mu_{B,i}e^{\langle\gamma e_i,\boldsymbol{\phi}\rangle}\right)dv_g \\ &+\frac{1}{2\pi}\int_{\partial\Sigma}\left(k_g\langle Q,\boldsymbol{\phi}\rangle + \sum_{j=1}^{d_N} e^{\langle\frac{\gamma f_j}{2},\boldsymbol{\phi}\rangle}\right)\mu(d\lambda_g). \end{aligned} \tag{2.2}$$

Equivalently, giving a sense to the measure (2.1) amounts to defining the quantities

$$\langle F(\Phi)\rangle_{\Sigma,g} := \frac{1}{\mathcal{Z}}\int_{\mathscr{F}} F(\boldsymbol{\phi})\mu_T(D_g\boldsymbol{\phi}) \tag{2.3}$$

for suitable functionals F on $\mathscr{F}$, where $\mathcal{Z}$ stands for a normalizing factor.

A certain amount of apparently disconnected mathematical background is needed in order to give a rigorous and suitable sense to the action S and to the measure μ_T.

The goal of this section is to give a gentle introduction to these notions, and to fix the notations that will be used throughout the document, while keeping the construction of the integral (2.3) as the guiding thread.

There are objects of two different nature showing up in the action (2.2): some depend on the geometry of the underlying surface (the "base space" of our field), while some are related to a Lie algebra structure (the "target space"). Let us start with the basic geometrical aspects.

2.2.1 Compact Riemann surfaces, Laplacians and Green's functions

2.2.1.1 Compact Riemann surfaces

A Riemann surface can be defined as a connected two-dimensional real manifold endowed with a *conformal structure*, i.e. an equivalence class of Riemannian metrics. Let Σ denote such a surface, with a possibly non-empty boundary $\partial\Sigma$. We denote respectively by $\boldsymbol{g}(\Sigma)$ and $\boldsymbol{k}(\Sigma)$ its genus and the number of connected components of its boundary. The Euler characteristic of the surface Σ is a classifying topological invariant that is defined by

$$\chi(\Sigma) = 2(1-\boldsymbol{g}(\Sigma)) - \boldsymbol{k}(\Sigma) \leq 2.$$

Compact Riemann surfaces are classified in three categories depending on their Euler characteristic, they can be either:

— Elliptic, that is such that $\chi(\Sigma) > 0$. The only elliptic Riemann surface is the sphere, up to biholomorphism.

— Parabolic, i.e. with $\chi(\Sigma) = 0$. The only parabolic surfaces are the cylinder and the tori.

— Hyperbolic. These are all the other Riemann surfaces, i.e. those with negative Euler characteristic.

Suppose that the compact Riemann surface Σ is equipped with a smooth Riemannian metric g [1]. There are two curvature terms appearing in the Toda action (2.2). One involves the *Ricci scalar curvature* R_g, that equals twice the Gauss' curvature, and can be defined in this 2-dimensional setting by $R_g = -\Delta_g \ln\sqrt{\det g}$. The other one depends on the *geodesic curvature* along the boundary k_g. Both are related by the celebrated Gauss-Bonnet theorem:

$$\int_\Sigma R_g dv_g + 2\int_{\partial\Sigma} k_g d\lambda_g = 4\pi\chi(\Sigma). \tag{2.4}$$

Here and after we denote by dv_g the volume form on Σ and by $d\lambda_g$ the induced measure on the boundary $\partial\Sigma$. In particular this result shows that the curvature of a

1. In what follows, unless explicitly stated, the surfaces under consideration will always be connected and compact, as well as the metrics will always be smooth, even if we forget to mention it.

metric is constrained by the topology of the surface, and we will make this precise now.

2.2.1.2 Uniform metrics

We say that two metrics g and g' are *conformally equivalent* if there exists a smooth function φ on Σ such that $g' = e^{\varphi} g$, and we denote by $[g]$ the set of all the metrics that are conformally equivalent to g. The Ricci scalar and the geodesic curvature behave explicitly under a conformal change of metric, namely

$$R_{g'} = e^{-\varphi}(R_g - \Delta_g \varphi) \quad ; \quad k_{g'} = e^{-\varphi/2}(k_g + \frac{1}{2}\partial_{\vec{n}_g}\varphi). \tag{2.5}$$

Here Δ_g is the Laplacian (Laplace-Beltrami operator) in the metric g, and $\partial_{\vec{n}_g}$ is the outer normal derivative (Neumann operator), i.e. $\partial_{\vec{n}_g} = (\nabla_g, \vec{n})_g$ where ∇_g is the gradient and $\vec{n}$ the outgoing normal vector. For a compact Riemann surface without boundary, equipped with a smooth metric g, the uniformization theorem entails that there exists a smooth metric in the conformal class $[g]$ associated to which the Ricci scalar curvature is constant. This metric is called *uniform*. This constant can be chosen to be equal to -2 for hyperbolic surfaces (the uniform metric is thus hyperbolic itself), to 0 for parabolic surfaces (such metrics are called *flat*) and to 2 for elliptic ones. A similar result holds for surfaces with boundary. Namely, on a surface with boundary equipped with a smooth metric g, one can always find:

- A metric $g_1 \in [g]$ such that R_{g_1} is constant and $k_{g_1} = 0$ on $\partial\Sigma$. The boundary of Σ is then a geodesic. Such a metric is called *uniform metric of type 1*, and is unique up to scaling and isometry (as in the closed case).
- A metric $g_2 \in [g]$ which is flat and such that k_{g_2} is constant, called *uniform metric of type 2*, also unique up to scaling and isometry.

This result is proven in the article [72], to which we refer for additional details. Again, the Gauss-Bonnet theorem forces a uniform type 1 metric on a hyperbolic surface to be itself hyperbolic, i.e. with constant negative curvature, etc. We now describe the doubling of a smooth Riemann surface with boundary.

2.2.1.3 Doubling Riemann surfaces with boundary

Suppose that Σ is a hyperbolic surface with a non-empty boundary and that it is equipped with a uniform type 1 metric. By the Gauss-Bonnet theorem, the metric g on Σ is hyperbolic — that is, its scalar curvature R_g is a negative constant. We now construct a closed surface $\hat{\Sigma}$ by gluing Σ to a copy of itself along the boundary. More precisely, define

$$\hat{\Sigma} = \Sigma^\circ \cup \partial\Sigma \cup \sigma(\Sigma^\circ),$$

where the union is disjoint, and where σ is an involution (i.e., $\sigma \circ \sigma = \mathrm{Id}$) that fixes the boundary ($\sigma|_{\partial\Sigma} = \mathrm{Id}$) and that sends Σ onto its mirror copy. Intuitively, this amounts to reflecting Σ across its boundary. Now we have the following.

Lemma 2.2.1. *The metric g on Σ extends to a smooth metric $\hat{g}$ on the closed surface $\hat{\Sigma}$.*

Proof. We define the metric on $\sigma(\Sigma)$ as the pullback $\sigma^* g$, so that both Σ and $\sigma(\Sigma)$ are equipped with isometric hyperbolic metrics. Since the boundaries of Σ and $\sigma(\Sigma)$ correspond under σ and have equal length (being geodesics), the two metrics match along the boundary. Thus, we obtain a metric $\hat{g}$ on all of $\hat{\Sigma}$, defined to agree with g on Σ and with $\sigma^* g$ on $\sigma(\Sigma)$. Further, the resulting metric $\hat{g}$ is smooth on $\hat{\Sigma}$ except possibly at the boundary $\partial\Sigma$, which now lies in the interior of $\hat{\Sigma}$. Now, since g is hyperbolic, each point $x \in \partial\Sigma$ has a neighborhood isometric to a geodesic ball $B_{\hat{g}}(x,\varepsilon) \subset \hat{\Sigma}$ that is isometric to a ball $B_{g_{\mathbb{H}}}(i,\varepsilon)$ in the upper half-plane model $\mathbb{H}$ of the hyperbolic plane. Under this isometry, the portion of the ball in Σ maps to the right of the imaginary axis, and the portion in $\sigma(\Sigma)$ maps to the left. This identification provides a local complex coordinate z near x, with respect to which the metric takes the standard hyperbolic form:

$$\hat{g} = g_{\mathbb{H}} = \frac{|dz|^2}{\mathrm{Im}(z)^2}.$$

This shows that $\hat{g}$ is smooth near $\partial\Sigma$ as well, and hence smooth on all of $\hat{\Sigma}$. □

This being established, we will often refer to the extended metric $\hat{g}$ in the same way as the original metric g, if there is no ambiguity.

2.2.1.4 Green's functions

In this chapter we will be interested in three different PDE problems on a compact Riemann surface equipped with a smooth Riemannian metric g. If the boundary of the surface is empty then we will be interested in what we call the closed Poisson problem, while if the boundary is non-empty we have to specify boundary conditions, that will be either of Neumann or Dirichlet type. In this section we define the Green's functions associated with these problems and state some useful properties. Recall that Δ_g denotes the (negative) Laplacian in the metric g.

Suppose first that $\partial\Sigma = \emptyset$. The Green's function of the closed Poisson problem on (Σ, g) is defined as the solution (see e.g. [51] for more details) G_g of the weak formulated problem[2]

$$\begin{cases} -\Delta_g G_g(\cdot, y) = 2\pi(\delta_y - \frac{1}{v_g(\Sigma)}) & \text{in } \Sigma \\ m_g(G_g(\cdot, y)) = 0 & \text{for all } y \in \Sigma \end{cases} \tag{2.6}$$

where δ is the Dirac delta function and $m_g(f)$ is the average of f over Σ, with respect to the metric g: $m_g(f) := v_g(\Sigma)^{-1} \int_\Sigma f \, dv_g$. As it is defined G_g is *a priori* a Schwartz distribution on $\Sigma \times \Sigma$, but we can see it as the integral kernel of a smoothing operator from $L^2_0(\Sigma)$, the space of square-integrable functions with vanishing mean on Σ, to $H^2(\Sigma)$ the second-order L^2-Sobolev space on Σ, through the natural L^2-pairing. In

2. The factor 2π in the equation is cosmetic and added in order to lighten the notations in the sequel.

addition, for any function f smooth over $\bar{\Sigma} := \Sigma \cup \partial\Sigma$,

$$-\int_\Sigma G_g(x,\cdot)\Delta_g f \, dv_g = 2\pi(f(x) - m_g(f)). \tag{2.7}$$

Further, G_g is symmetric. As we will see below (Lemma 2.2.3), G_g is actually a smooth function away from the diagonal, with logarithmic singularity on the diagonal. Finally, if g' is in the conformal class of the metric g then the function $G_{g'}$ defined for $x \neq y \in \Sigma$ by[3]

$$G_{g'}(x,y) := G_g(x,y) - m_{g'}(G_g(x,\cdot)) - m_{g'}(G_g(\cdot,y)) + m_{g'}(G_g) \quad \text{where} \tag{2.8}$$

$$m_{g'}(G_g) := \frac{1}{v_{g'}(\Sigma)^2}\iint_{\Sigma\times\Sigma} G_g(x,y)\,dv_{g'}(x)\,dv_{g'}(y)$$

solves the problem (2.6) in the metric g'.

We now turn to the case where $\partial\Sigma \neq \emptyset$. The Neumann and Dirichlet Green's functions are defined respectively as the solutions of

$$\begin{cases} -\Delta_g G_g^N(\cdot,y) = 2\pi(\delta_y - \frac{1}{v_g(\Sigma)}) & \text{in } \Sigma \\ \partial_{\vec{n}_g} G_g^N(x,y) = 0 & \text{for } x \in \partial\Sigma \text{ and for all } y \in \Sigma \\ m_g(G_g^N(\cdot,y)) = 0 & \text{for all } y \in \Sigma \end{cases} \tag{2.9}$$

where recall that $\partial_{\vec{n}_g}$ is defined with respect to the outward normal vector $\vec{n}_g$, and

$$\begin{cases} -\Delta_g G_g^D(\cdot,y) = 2\pi\delta_y & \text{in } \Sigma \\ G_g^D(x,y) = 0 & \text{for } x \in \partial\Sigma \text{ and for all } y \in \Sigma. \end{cases} \tag{2.10}$$

In the same way as in the closed case we see the Neumann and Dirichlet Green's functions as integral kernels of operators from an L^2 space to a second order Sobolev space on Σ. As such, for any f smooth over $\bar{\Sigma} = \Sigma \cup \partial\Sigma$, the Neumann Green's function satisfies:

$$\int_{\partial\Sigma} G_g^N(x,\cdot)\partial_{\vec{n}_g} f \, d\lambda_g - \int_\Sigma G_g^N(x,\cdot)\Delta_g f \, dv_g = 2\pi(f(x) - m_g(f)) \tag{2.11}$$

while the Dirichlet Green's function satisfies (2.7). Both also enjoy the same change of metric formula (2.8) as the closed Green's function G_g.

We now provide a relation between the closed Green's function and the boundary ones, using the doubling of surface described in the previous subsection.

Lemma 2.2.2. *Let Σ be a compact surface of hyperbolic type with boundary $\partial\Sigma \neq \emptyset$, equipped with a uniform type 1 metric g. Let $\hat{\Sigma}$ be the doubled surface equipped with the smooth hyperbolic metric $\hat{g}$ and σ be the canonical involution. Then for $x, y \in \overline{\Sigma}$ we have*

$$G_g^N(x,y) = G_{\hat{g}}(x,y) + G_{\hat{g}}(x,\sigma(y)) \tag{2.12}$$

3. In fact this defines $G_{g'}$ in the sense of integral kernels on the full $\Sigma \times \Sigma$.

and

$$G_g^D(x,y) = G_{\hat{g}}(x,y) - G_{\hat{g}}(x,\sigma(y)). \tag{2.13}$$

Proof. We first have to show that the right-hand side of (2.12) is a solution to the problem (2.9). We compute the Laplacian in the variable x:

$$-\frac{1}{2\pi}\Delta_g\big(G_{\hat{g}}(x,y)+G_{\hat{g}}(x,\sigma(y))\big) = \delta_y(x) - \frac{1}{v_{\hat{g}}(\hat{\Sigma})} + \delta_{\sigma(y)}(x) - \frac{1}{v_{\hat{g}}(\hat{\Sigma})} = \delta_y(x) - \frac{1}{v_g(\Sigma)}.$$

Then we investigate the outer normal derivative along $\partial\Sigma$. Since it holds that $G_{\hat{g}}(x,y) = G_{\hat{g}}(\sigma(x),\sigma(y))$, then for $x\in\partial\Sigma$ and $y\in\Sigma$

$$\partial_{\vec{n}_g}\big(G_{\hat{g}}(x,y)+G_{\hat{g}}(x,\sigma(y))\big) = \partial_{\vec{n}_g}G_{\hat{g}}(x,y) - \partial_{\vec{n}_g}G_{\hat{g}}(x,y) = 0 \quad \text{and likewise}$$
$$\int_\Sigma \big(G_{\hat{g}}(x,y)+G_{\hat{g}}(x,\sigma(y))\big)dv_g(y) = \int_{\hat{\Sigma}} G_{\hat{g}}(x,y)dv_{\hat{g}}(y) = 0.$$

Combining these properties with the symmetry of the Green's functions implies the result. The proof of (2.13) is immediate once one recalls that $\sigma = \mathrm{Id}$ on the boundary of Σ. □

Eventually we recall here some estimates from [51] on the Green's function G, which show that it has a logarithmic divergence on the diagonal:

Lemma 2.2.3. *On a compact surface Σ without boundary equipped with a hyperbolic metric g_0, the Green's function G_{g_0} can be written when x and y are closed to each other as*

$$G_{g_0}(x,y) = \log\frac{1}{d_{g_0}(x,y)} + f_{g_0}(x,y) \tag{2.14}$$

with f_{g_0} a smooth function on Σ^2. Moreover, for any x_0 in Σ there are coordinates $z\in B(0,1)\subset\mathbb{C}$ such that $g_0 = \frac{4|dz|^2}{(1-|z|^2)^2}$ and

$$G_{g_0}(z,z') = \log\frac{1}{|z-z'|} + F(z,z') \tag{2.15}$$

near x_0, with F smooth. Finally, if g is another metric in the conformal class of the hyperbolic metric g_0, then the estimate (2.14) holds near the diagonal with f_g continuous.

This logarithmic divergence allows to make sense of the Gaussian Multiplicative Chaos measures associated to the Gaussian Field with covariance kernel G_g in subsequent sections.

2.2.1.5 Regularized determinants

When dealing with Gaussian measures on finite-dimensional spaces, one has to compute determinants of the inverse of covariance matrices. In the infinite-dimensional setting, we will see that the natural candidate to generalize this picture is the Gaussian Free Field, whose covariance structure is given by the Green's function, so that one is naturally led to consider determinants of Laplacians. In order to define the determinant of the Laplacian on a compact and connected Riemann surface Σ one needs a regularization procedure. To this aim the classical method is to introduce the spectral ζ-function

$$\zeta(s) = \sum_{\lambda>0} \lambda^{-s}$$

where the λ's are the eigenvalues of the (positive) Laplacian[4]. On a compact surface, the spectrum of the positive Laplacian is discrete and forms an increasing sequence $0 \leq \lambda_0 < \lambda_1 \leq ... \to \infty$, with $\lambda_0 = 0$ for the closed and the Neumann Laplacian, and $\lambda_0 > 0$ for the Dirichlet Laplacian. The asymptotics of the eigenvalues are given by the Weyl's law (see e.g. [61]), which allows one to prove that the spectral ζ-function converges for $\mathrm{Re}(s)$ positive enough, and admits a meromorphic continuation to $\mathrm{Re}(s) > -1$ that is regular at 0 (see [72]). We then define the *regularized determinant* as

$$\det{}'(-\Delta_g) = e^{-\zeta'(0)}$$

and simply write det for the Dirichlet Laplacian since the full spectrum is involved in the determinant. The Polyakov-Alvarez formula gives the behavior of the determinant under a conformal change of metric, and will be very useful to compute the conformal anomaly of Toda correlation functions. The formula was originally derived in the closed case by Polyakov [78, 79] and refined in the boundary case by Alvarez [3] (see also [69]). We state it as a lemma, and refer to e.g. [72] for a concise exposition of this kind of results.

Lemma 2.2.4. *Let $g' = e^{\varphi} g$ be a smooth Riemannian metric in the conformal class of g. Then we have the following relation between determinants of the Laplacians associated with g and g':*

$$\log \frac{\det\left(-\Delta_{g'}\right)}{\det\left(-\Delta_g\right)} = -\frac{1}{48\pi}\left(\int_\Sigma (|\nabla_g \varphi|_g^2 + 2R_g\varphi)\, dv_g + 4\int_{\partial\Sigma} k_g \varphi\, d\lambda_g\right) \pm \frac{1}{8\pi}\int_{\partial\Sigma} \partial_{\vec{n}_g}\varphi\, d\lambda_g \tag{2.16}$$

where $\det\left(-\Delta_g\right) := \det{}'(-\Delta_g)/v_g(\Sigma)$ *for the closed Laplacian and the Neumann Laplacian. The sign of the corrective term is positive in case of Neumann boundary conditions and negative in case of Dirichlet boundary conditions.*[5]

We will also consider the regularized determinant $\det{}'\left(-\frac{1}{2\pi}\Delta_g\right)$ to define the partition function of the Gaussian Free Field. According to the previous discussion we have

$$\det{}'\left(-\frac{1}{2\pi}\Delta_g\right) = (2\pi)^{-\zeta_0} \det{}'\left(-\Delta_g\right) \tag{2.17}$$

4. So for us, $-\Delta_g$ on closed surfaces, and $-\Delta_g$ with Neumann or Dirichlet conditions on surfaces with boundary, which we will denote later $-\Delta_g^N$ and $-\Delta_g^D$.

5. Our convention for the conformal factor differs from the standard convention by a factor 2, i.e. $\varphi = 2\omega$ where ω stands for the usual conformal factor, which explains that the formula is slightly different.

where ζ_0 is independent of the metric g and given by $\zeta_0 = \frac{\chi(\Sigma)}{6} - 1$ for a closed surface and in the Neumann case while in the Dirichlet case the determinants are not truncated and $\zeta_0 = \frac{\chi(\Sigma)}{6}$. Note that in the closed case $\zeta_0 = \zeta(0)$. The above identity can be recovered using Lemma 2.2.4 by taking $\varphi = \ln 2\pi$ together with the Gauss-Bonnet formula (2.4). So far we have introduced all the geometric objects that we need in order to make sense of the action (2.2) and the measure (2.1). Let us now turn to the Lie algebra aspects.

2.2.2 Finite-dimensional complex simple Lie algebras

The definition of Toda CFTs relies on a Lie algebra structure. We gather in this section some classical facts about finite-dimensional complex simple Lie algebras, and introduce the notations that will be used throughout the document. Additional details can be found *e.g.* in the textbook [60].

2.2.2.1 Basic facts

In virtue of the Cartan-Killing classification, every finite-dimensional complex simple Lie algebra is isomorphic to either one of the following family of classical Lie algebras: A_n for $n \geq 1$ (corresponding to $\mathfrak{sl}_{n+1}\mathbb{C}$), B_n for $n \geq 2$ (corresponding to $\mathfrak{so}_{2n+1}\mathbb{C}$), C_n for $n \geq 3$ (corresponding to $\mathfrak{sp}_{2n}\mathbb{C}$) and D_n for $n \geq 4$ (corresponding to $\mathfrak{so}_{2n}\mathbb{C}$), or one of the five exceptional simple Lie algebras: E_6, E_7, E_8, F_4 and G_2. For any finite-dimensional complex simple Lie algebra $\mathfrak{g}$ there is a natural Euclidean space $\mathfrak{a}$ that is such that the Cartan subalgebra of $\mathfrak{g}$ decomposes as $\mathfrak{a} + i\mathfrak{a}$ (this space $\mathfrak{a}$ is the real Cartan subalgebra of the split real form of $\mathfrak{g}$). To this Euclidean space is attached a scalar product $\langle \cdot, \cdot \rangle$ which is proportional to the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{a}$. The Euclidean space $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ is unique up to isomorphism and can be realized as $\mathbb{R}^r$ equipped with its standard scalar product, where r is the rank of $\mathfrak{g}$.

The so-called simple roots $(e_i)_{i=1,...,r}$ form a particular basis of $\mathfrak{a}$; this basis is such that $\langle e_i^\vee, e_j \rangle = A_{i,j}$ where A is the Cartan matrix of $\mathfrak{g}$ (see also the next subsection) and $e_i^\vee := 2\frac{e_i}{\langle e_i, e_i \rangle}$ is the coroot of e_i. In the sequel we will assume that the scalar product is renormalized in such a way that the longest simple root has squared length 2, which is a standard assumption. The dual basis of $(e_i^\vee)_{1 \leq i \leq r}$ is given by $(\omega_i)_{1 \leq i \leq r}$ where the

$$\omega_i := \sum_{j=1}^{r} (A^{-1})_{j,i} e_j \quad \text{for} \quad i = 1, \cdots, r$$

are called the fundamental weights. They verify $\langle e_i^\vee, \omega_j \rangle = \delta_{ij}$. Another important object is the Weyl vector, defined by $\boldsymbol{\rho} = \sum_{i=1}^{r} \omega_i$, which is such that $\langle \boldsymbol{\rho}, e_i^\vee \rangle = 1$ for all i. We also define the Weyl vector associated to the coroots by $\boldsymbol{\rho}^\vee = \sum_{i=1}^{r} \omega_i^\vee$ where $\omega_i^\vee$ is defined so that $\langle \omega_i^\vee, e_j \rangle = \delta_{ij}$ for $i = 1, ... r$.

2.2.2.2 The outer automorphism group

The notion of outer automorphism [60, Section 12.2] naturally arises when discussing the admissible boundary conditions of the theory, as discussed in the introduction. It is convenient to define these transformation to introduce the notion of *root system*.

Definition 2.2.5. Given a Euclidean space $(V,(\cdot,\cdot))$, a *root system* associated to V is defined as a finite subset $\mathscr{R}$ of $V\setminus\{0\}$ such that

$$\begin{aligned}
&\text{(i) the elements of } \mathscr{R} \text{ span } V,\\
&\text{(ii) } \mathscr{R}\cap\mathbb{R}\alpha=\{\alpha,-\alpha\} \quad \text{for all } \alpha\in\mathscr{R},\\
&\text{(iii) } s_\alpha\mathscr{R}=\mathscr{R} \quad \text{for all } \alpha\in\mathscr{R}.
\end{aligned} \tag{2.18}$$

The simple roots $(e_i)_{1\leq i\leq r}$, then form a basis of V such that any $\alpha\in\mathscr{R}$ can be written as a linear combination

$$\alpha=\sum_{i=1}^{r}\lambda_i e_i \quad \text{where the } \lambda_i \text{ have same sign.}$$

A root for which the λ_i are all non-negative is called a *positive root*: their set is denoted $\mathscr{R}^+$.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. We denote $\mathscr{R}$ a root system associated to $(\mathfrak{a},\langle\cdot,\cdot\rangle)$ (such a root system characterizes the Lie algebra $\mathfrak{g}$ up to isomorphism). The automorphism group of $\mathscr{R}$ is a semi-direct product $\mathrm{Aut}(\mathscr{R})=\mathrm{Out}(\mathscr{R})\ltimes W(\mathfrak{g})$, where elements of $\mathrm{Out}(\mathscr{R})$ are called outer automorphisms and correspond to Dynkin diagram automorphisms[6], while $W(\mathfrak{g})$ is the Weyl group of $\mathfrak{g}$, i.e. automorphisms of $\mathscr{R}$ induced by inner automorphisms of $\mathfrak{g}$ (generated by elements of the form $\mathrm{Ad}(g)$ where g lies in a Lie group with Lie algebra $\mathfrak{g}$ and Ad is the adjoint map). Standard results [60, Section 12.2] then show that for a root system associated with a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ the outer automorphism group is either: trivial, this is the case for A_1, B_n, C_n, E_7, E_8, F_4 and G_2, or isomorphic to the cyclic group of order two in all the other cases, except for D_4 where the outer automorphism group is isomorphic to the group of permutations on three elements (thus of order 6). In all cases an element τ in $\mathrm{O}(\mathscr{R})$ is an isometry of $\mathfrak{a}$, i.e. $\langle u,v\rangle=\langle\tau(u),\tau(v)\rangle$ for any $u,v\in\mathfrak{a}$.

As an example, for the Lie algebra A_n ($\mathfrak{sl}_{n+1}\mathbb{C}$) the non-trivial automorphism τ is given by $e_i\mapsto e_{n+1-i}$, so that for A_2 it swaps the two simple roots: if $u=u_1e_1+u_2e_2\in\mathfrak{a}$, then $\tau(u)=u_2e_1+u_1e_2$. In general the outer automorphisms are particular permutations of the simple roots.

6. For simple complex Lie algebras, the Dynkyn diagrams are those of the same types A,B,C,D,E,F,G as in the Cartan-Killing classification.

2.2.2.3 Folding of root systems

Given τ in $\mathrm{Out}(\mathscr{R})$ let us denote by $\mathfrak{a}_N \subseteq \mathfrak{a}$ the corresponding invariant subspace and by d_N its dimension. There is a natural root system in $\mathfrak{a}_N$ defined by considering, with the $(\mathscr{O}_j)_{1\leq j\leq d_N}$ being the orbits of τ (viewed as acting on $\{1,\cdots,r\}$):

$$f_j := \frac{1}{|\mathscr{O}_j|}\sum_{i\in\mathscr{O}_j} e_i \quad \text{for } 1\leq j\leq d_N \tag{2.19}$$

Indeed one can check that $\langle f_i^\vee, f_j\rangle = A^\tau_{i,j}$ where A^τ is the Cartan matrix of the complex simple Lie algebra $\mathfrak{g}^\tau$ which is the *folding* of $\mathfrak{g}$. As a consequence $(f_j)_{1\leq j\leq d_N}$ can be viewed as simple roots for the *folded* root system $\mathscr{R}^\tau$ of $\mathfrak{g}^\tau$, with the length of the longest root being now given by some κ. The table below describes this folding in all non-trivial cases.

Root system $\mathscr{R}$	Order of $\tau\in\mathrm{Out}(\mathscr{R})$	Invariant sub-root system	Dimension of $\mathfrak{a}_N$	κ^2
A_{2n}	2	B_n	n	1
A_{2n-1}	2	C_n	n	2
$D_n, n\geq 4$	2	B_{n-1}	$n-1$	2
D_4	3	G_2	2	2
E_6	2	F_4	4	2

For outer automorphisms of order two the f_j's are of the form $f_j = e_i + \tau(e_i)$ or $f_j = \frac{e_i+\tau(e_i)}{2}$, so that $\mathfrak{a}_N = \mathrm{Span}(v+\tau(v), v\in\mathfrak{a})$. Moreover such automorphisms are self-adjoint with respect to the scalar product on $\mathfrak{a}$, which allows to decompose the space $\mathfrak{a}$ as the orthogonal sum $\mathfrak{a} = \mathfrak{a}_N \bigoplus \mathfrak{a}_D$ where $\mathfrak{a}_D = \mathrm{Span}(v-\tau(v), v\in\mathfrak{a})$. For the two 3-cycles of $\mathrm{Out}(D_4)$, $\mathfrak{a} = \mathfrak{a}_N\bigoplus\mathfrak{a}_D$ with $\mathfrak{a}_N = \mathrm{Span}\left(v+\tau(v)+\tau^2(v), v\in\mathfrak{a}\right)$ and $\mathfrak{a}_D = \mathrm{Span}(v-\tau(v), v-\tau^2(v), v\in\mathfrak{a})$.

It is important to note that in all non-trivial cases the Weyl vector $\boldsymbol{\rho}$ belongs to the invariant subspace $\mathfrak{a}_N$: this is due to the fact that for simply-laced Lie algebras it is characterized by $\langle\boldsymbol{\rho}, e_i\rangle = 1$ for all $1\leq i\leq r$, a condition which is preserved by the action of an outer automorphism. Hence the background charge $Q = \gamma\boldsymbol{\rho} + \frac{2}{\gamma}\boldsymbol{\rho}^\vee$ always belongs to $\mathfrak{a}_N$.

Hereafter for notational simplicity the outer automorphism under consideration will often be denoted by a star $\cdot^*$. Likewise we will denote by p_N (resp. p_D) the orthogonal projection on $\mathfrak{a}_N$ (resp. $\mathfrak{a}_D$). We also set for τ non-trivial

$$Q_\tau := \gamma\rho_\tau + \frac{2}{\gamma}\rho_\tau^\vee \tag{2.20}$$

where ρ_τ and $\rho_\tau^\vee$ are respectively the Weyl and co-Weyl vector of the folded Lie algebra $\mathfrak{g}_\tau$, defined uniquely by the property that $\rho_\tau\in\mathfrak{a}_N$ with, for all $1\leq j\leq d_N$,

$$\langle\rho_\tau, f_j\rangle = \frac{\|f_j\|^2}{2} \quad\text{and}\quad \langle\rho_\tau^\vee, f_j\rangle = 1.$$

Eventually we introduce the notation $I_\tau := \left\{ j \in \{1, \cdots, d_N\}, \quad \|f_j\|^2 = 2 \right\}$.

2.2.2.4 Semi-simple Lie algebras

Any finite-dimensional complex *semi-simple* Lie algebra $\mathfrak{g}$ decomposes as a direct sum $\mathfrak{g} = \bigoplus_{k=1}^{p} \mathfrak{g}_k$ where the $\mathfrak{g}_k$'s are finite-dimensional complex *simple* Lie algebras. As pointed out in [30, Remark 1.2], if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a direct sum of semi-simple Lie algebras then the Toda correlation functions factorize as $\langle \mathbf{V} \rangle^{\mathfrak{g}} = \langle \mathbf{V} \rangle^{\mathfrak{g}_1} \langle \mathbf{V} \rangle^{\mathfrak{g}_2}$ (and the Weyl anomaly thus shows that the central charges add up, see Remark 1.2.3). This allows one to reduce the study of Toda theories for semi-simple Lie algebras to those with respect to simple Lie algebras. We thus work with $\mathfrak{g}$ simple in the sequel.

2.3 Toda theories on closed Riemann surfaces

In this section, we propose a mathematical construction of Toda CFTs based on a compact hyperbolic Riemann surface without boundary, and associated to any complex simple Lie algebra $\mathfrak{g}$. For this purpose we rely on a probabilistic framework based on Gaussian Free Fields (GFFs hereafter) and Gaussian Multiplicative Chaos (GMC in the sequel). Throughout this section, Σ will denote a connected compact Riemann surface without boundary, assumed to be of hyperbolic type in the sense that $\chi(\Sigma) < 0$. We equip Σ with a smooth Riemannian metric g.

2.3.1 Gaussian Free Field

The action functional entering the path integral definition of Toda CFTs (1.24) is made of several terms. Our first goal is to interpret the quadratic component, that is, to make sense of

$$\int F(\boldsymbol{\phi}) e^{-\frac{1}{4\pi}\int_\Sigma |\nabla_g \boldsymbol{\phi}|_g^2 dv_g} D_g \boldsymbol{\phi} \tag{2.21}$$

where in the above the gradient ∇_g acts component by component on functions from Σ to $\mathfrak{a}$. This is achieved by the introduction of the vectorial Gaussian Free Field (GFF hereafter). Indeed, the formal integral (2.21) can be written as

$$\int F(\boldsymbol{\phi}) e^{-\frac{1}{2}\int_\Sigma \langle \boldsymbol{\phi}, -\frac{1}{2\pi}\Delta_g \boldsymbol{\phi} \rangle_g dv_g} D_g \boldsymbol{\phi} \tag{2.22}$$

where, again, the Laplacian acts independently on the r components of $\boldsymbol{\phi}$. Equation (2.22) describes (formally) the law of a Gaussian field with values in $\mathfrak{a}$, mean 0 and positive definite covariance kernel $(-\frac{1}{2\pi}\Delta_g)^{-1}$. There is one subtlety here in that this observation is valid only on the orthogonal of the kernel of the Laplacian, so that the above Gaussian field would be defined only on this subspace. We will come back to this issue later. The inverse of the one-dimensional Laplacian (outside of its kernel) is given in the sense of integral kernels by the Green's function, and since the

r-dimensional Laplacian in (2.22) acts independently on the r components of the field, it is natural to define our underlying field as follows.

Definition 2.3.1. We set $\mathbf{X}^g$ to be the centered Gaussian field on Σ with covariance structure

$$\mathbb{E}[(\langle u, \mathbf{X}^g\rangle, f_1)_{H^1(\Sigma,g)}(\langle v, \mathbf{X}^g\rangle, f_2)_{H^1(\Sigma,g)}] = \langle u, v\rangle \int_{\Sigma^2} G_g(x,y) f_1(x) f_2(y) dv_g(x) dv_g(y)$$

for all $u, v \in \mathfrak{a}$ and $f_1, f_2 \in H^1(\Sigma, g)$, where $(\langle u, \mathbf{X}^g\rangle, f)_{H^1(\Sigma,g)}$ is the pairing between H^1 and its dual H^{-1}.

Indeed, the field $\mathbf{X}^g$ is a random distribution, and lives almost surely in the negative index Sobolev space $H^{-1}(\Sigma \to \mathfrak{a}, g)$, which means that each component is in H^{-1}. As already mentioned, this Gaussian field is only defined up to constants, and we need to take this fact into account in order to define the path integral (2.21). We overcome this issue by tensorizing the law of the GFF by the Lebesgue measure on $\mathbb{R}^r$. That is, we interpret (2.22) as

$$\int F(\boldsymbol{\phi}) e^{-\frac{1}{4\pi}\int_\Sigma |\nabla_g \boldsymbol{\phi}|_g^2 dv_g} D_g\boldsymbol{\phi} \propto \int_{\mathfrak{a}} \mathbb{E}[F(\mathbf{X}^g + \boldsymbol{c})] d\boldsymbol{c} \tag{2.23}$$

with the expectation taken with respect to the GFF and the integration over $\mathfrak{a}$ defined by

$$\int_{\mathfrak{a}} F(\boldsymbol{c}) d\boldsymbol{c} = \int_{\mathbb{R}^r} F\Big(\sum_{i=1}^r \boldsymbol{v}_i c_i\Big) \prod_{i=1}^r dc_i$$

where $(\boldsymbol{v}_i)$ is any orthonormal basis of $\mathfrak{a}$. The theory of GFF is well-known and we refer for example to [13, 81] for a review. For the sake of simplicity we will use the notation

$$\mathbb{E}[\langle u, \mathbf{X}^g(x)\rangle \langle v, \mathbf{X}^g(y)\rangle] = \langle u, v\rangle G_g(x,y) \tag{2.24}$$

hereafter. Note that $\mathbf{X}^g$ can be represented as the sum $\mathbf{X}^g = \sum_{i=1}^r X_i^g \boldsymbol{v}_i$ where the $(X_i^g)_{1\le i\le r}$ are independent, real-valued GFFs over Σ, and $(\boldsymbol{v}_i)_{1\le i\le r}$ is any orthonormal basis of $\mathfrak{a}$.

For the picture to be complete, it misses a multiplicative normalization term called the partition function: this corresponds to the total mass of the path integral (2.21). Following the analogy with the finite-dimensional case mentioned in Subsection 2.2.1.5, we define it using the regularized determinant of the Laplacian.

Definition 2.3.2. The partition function of the GFF $\mathbf{X}^g$ is defined by

$$\mathcal{Z}_{\mathbf{X}}(\Sigma, g) = \left(\frac{\det'\left(-\frac{1}{2\pi}\Delta_g\right)}{v_g(\Sigma)}\right)^{-\frac{r}{2}}.$$

where Δ_g is the (negative) Laplacian acting on functions from Σ to $\mathbb{R}$.

An important property of the GFF is its behavior under conformal changes of metric. This has a direct implication for the measure defined by Equation (2.23).

Lemma 2.3.3. *Let g' be conformally equivalent to the metric g. Then for all positive continuous functional F over $H^{-1}(\Sigma\to\mathfrak{a},g)$,*

$$\mathbf{X}^{g'}\overset{(law)}{=}\mathbf{X}^{g}-m_{g'}(\mathbf{X}^{g}),\quad\textit{so that}\quad\int_{\mathfrak{a}}\mathbb{E}[F(\mathbf{X}^{g'}+\mathbf{c})]d\mathbf{c}=\int_{\mathfrak{a}}\mathbb{E}[F(\mathbf{X}^{g}+\mathbf{c})]d\mathbf{c}.\tag{2.25}$$

Proof. Let $\widetilde{\mathbf{X}}^g:=\mathbf{X}^g-m_{g'}(\mathbf{X}^g)$. For any $u,v\in\mathfrak{a}$ and $x,y\in\Sigma$ we have

$$\begin{aligned}\mathbb{E}[\langle u,\widetilde{\mathbf{X}}^g(x)\rangle\langle v,\widetilde{\mathbf{X}}^g(y)\rangle]=\langle u,v\rangle\Big(&G_g(x,y)-m_{g'}(G_g(x,\cdot))-m_{g'}(G_g(\cdot,y))\\&+v_{g'}(\Sigma)^{-2}\iint_{\Sigma^2}G_g(x,y)dv_{g'}(x)dv_{g'}(y)\Big)\end{aligned}$$

which is equal to $\langle u,v\rangle G_{g'}(x,y)$ thanks to Equation (2.8). Since both fields are centered Gaussian fields this is enough to conclude that $\widetilde{\mathbf{X}}^g$ has the same law as $\mathbf{X}^{g'}$. Now to prove the lemma we make a change of variables in the zero-mode:

$$\int_{\mathfrak{a}}\mathbb{E}[F(\mathbf{X}^{g'}+\mathbf{c})]d\mathbf{c}=\int_{\mathfrak{a}}\mathbb{E}[F(\mathbf{X}^{g}-m_{g'}(\mathbf{X}^g)+\mathbf{c})]d\mathbf{c}=\int_{\mathfrak{a}}\mathbb{E}[F(\mathbf{X}^{g}+\mathbf{c})]d\mathbf{c}.$$

□

Further, as a straightforward consequence of Lemma 2.2.4 one obtains the following conformal anomaly for the partition function of the GFF.

Proposition 2.3.4. *Let $g'=e^{\varphi}g$ be a metric in the conformal class of g. Then we have*

$$\mathcal{Z}_{\mathbf{X}}(\Sigma,g')=\mathcal{Z}_{\mathbf{X}}(\Sigma,g)\exp\left(\frac{r}{96\pi}\int_\Sigma(|\nabla_g\varphi|_g^2+2R_g\varphi)dv_g\right).\tag{2.26}$$

2.3.2 Gaussian Multiplicative Chaos

Having given a meaning to the quadratic term that appears in the action (1.23), we now would like to give a rigorous interpretation of the exponential potential that appears there, *i.e.* define the exponentials $e^{\langle u,\mathbf{X}^g\rangle}dv_g$ for $u\in\mathfrak{a}$. However, since $\mathbf{X}^g$ is far from being a continuous well-defined function we need to introduce a regularization procedure. This method is at the heart of the theory of Gaussian Multiplicative Chaos — for details about the theory of GMC, see for instance [87, 84, 13].

Definition 2.3.5. Given g a smooth Riemannian metric on Σ and $x\in\Sigma$, let $C_g(x,\varepsilon):=\{y\in\Sigma,d_g(x,y)=\varepsilon\}$ be the geodesic circle centered at x of (small) radius $\varepsilon>0$. We define the regularized field $\mathbf{X}^g_\varepsilon$ at x as the average of the GFF $\mathbf{X}^g$ over $C_g(x,\varepsilon)$.

Synthetically, $\mathbf{X}^g_\varepsilon(x)=\mu_{g,\varepsilon}[\mathbf{X}_g](x)$ where $\mu_{g,\varepsilon}[\cdot](x)$ is the uniform probability measure on $C_g(x,\varepsilon)$, that is $\int\cdot d\mu_{g,\varepsilon}(x)=\frac{1}{\lambda_g(C_g(x,\varepsilon))}\int_{C_g(x,\varepsilon)}\cdot d\lambda_g$. It can be shown [37] that the random variable $\mathbf{X}^g_\varepsilon(x)$ has an almost surely continuous version that we will consider from now on.

Lemma 2.3.6. *The field $\mathbf{X}_\varepsilon^g$ has the following covariance structure:*

$$\mathbb{E}[\langle u,\mathbf{X}_\varepsilon^g(x)\rangle\langle v,\mathbf{X}_\varepsilon^g(y)\rangle] = \langle u, v\rangle \iint G_g d\mu_{g,\varepsilon}(x) d\mu_{g,\varepsilon}(y) \tag{2.27}$$

for any $u, v \in \mathfrak{a}$. Moreover, if $g = e^\varphi g_0$ with g_0 hyperbolic then as $\varepsilon \to 0$ and uniformly in x

$$\mathbb{E}[\langle u,\mu_{g,\varepsilon}[\mathbf{X}^{g_0}](x)\rangle^2] = |u|^2\left(-\log\varepsilon + W_{g_0}(x) + \frac{1}{2}\varphi(x)\right) + o(1) \tag{2.28}$$

with W_{g_0} a smooth function on Σ independent from φ.

Proof. The first item follows from the definition of $\mathbf{X}_\varepsilon^g$. In the case where $\varphi = 0$ the second item is a straightforward higher-rank generalization of Lemma 3.2 in [51]. The case of non-zero φ corresponds to [51, Equation (3.10)]. □

From Lemma 2.3.6 we deduce that one needs to rescale the exponential of the regularized field by a power of ε in order to obtain a finite limit, giving rise to a GMC measure.

Definition 2.3.7. Let g be a smooth Riemannian metric on Σ and $\mathbf{X}_\varepsilon^g$ be the regularized field introduced above. We define a Radon measure over Σ by setting for $|u|^2 < 4$

$$e^{\langle u,\mathbf{X}^g(x)\rangle} dv_g(x) := \lim_{\varepsilon\to 0} \varepsilon^{\frac{|u|^2}{2}} e^{\langle u,X_\varepsilon^g(x)\rangle} dv_g(x). \tag{2.29}$$

Thanks to the theory of GMC, as soon as $|u|^2 < 4$ the above convergence stands in probability in the space of Radon measures equipped with the weak topology. Moreover, the limiting measure is finite and non-trivial. Note that as written in the action (1.23), we will only need to consider some particular vectors u in the direction of the simple roots $(e_i)_{1\le i\le r}$, that is $u = \gamma e_i$ for $i = 1, \cdots, r$. Then $|u|^2 = \gamma^2\langle e_i, e_i\rangle =: \gamma_i^2$. Since the longest roots are such that $\langle e_i, e_i\rangle = 2$ this explains the constraint that γ must belong to $(0, \sqrt{2})$.

The following Lemma is the companion result of Lemma 2.3.3 for the GMC:

Lemma 2.3.8. *Let $g = e^\varphi g_0$ be a metric in the conformal class of the hyperbolic metric g_0, and u in $\mathfrak{a}$ with $\left|u^2\right| < 4$. Then the following equality holds in distribution:*

$$e^{\langle u,\mathbf{X}^g\rangle} dv_g \overset{(law)}{=} e^{(1+\frac{|u|^2}{4})\varphi} e^{\langle u,\mathbf{X}^{g_0} - m_g(\mathbf{X}^{g_0})\rangle} dv_{g_0}.$$

Hence for any F (resp. f_i, G) positive continuous over $H^{-1}(\Sigma\to\mathfrak{a}, g)$ (resp. Σ, $\mathbb{R}^r$):

$$\begin{aligned}&\int_{\mathfrak{a}} \mathbb{E}\left[F\left(\mathbf{X}^g + \boldsymbol{c}\right) G\left(\left(\int_\Sigma f_i e^{\langle\gamma e_i,\mathbf{X}^g+\boldsymbol{c}\rangle} dv_g\right)_{1\le i\le r}\right]\right] d\boldsymbol{c}\\ &= \int_{\mathfrak{a}} \mathbb{E}\left[F\left(\mathbf{X}^{g_0} + \boldsymbol{c}\right) G\left(\left(\int_\Sigma f_i e^{\frac{\langle Q,\gamma e_i\rangle}{2}\varphi} e^{\langle\gamma e_i,\mathbf{X}^{g_0}+\boldsymbol{c}\rangle} dv_{g_0}\right)_{1\le i\le r}\right]\right] d\boldsymbol{c}.\end{aligned} \tag{2.30}$$

Proof. Thanks to Lemma 2.3.6 together with Lemma 2.3.3 we deduce that

$$e^{\langle u,\mathbf{X}^g_\varepsilon(x)\rangle}dv_g \overset{\text{(law)}}{=} \lim_{\varepsilon\to 0}\varepsilon^{\frac{|u|^2}{2}}e^{\langle u,\mu_{g,\varepsilon}[\mathbf{X}^{g_0}(x)-m_g(\mathbf{X}^{g_0})](x)\rangle}e^{\varphi(x)}dv_{g_0}(x)$$
$$= e^{-\langle u,m_g(\mathbf{X}^{g_0})\rangle}e^{\varphi(x)}e^{\frac{|u|^2}{2}(W_{g_0}(x)+\frac{1}{2}\varphi(x))}\lim_{\varepsilon\to 0}e^{\langle u,\mu_{g,\varepsilon}[\mathbf{X}^{g_0}](x)\rangle-\frac{1}{2}\mathbb{E}[\langle u,\mu_{g,\varepsilon}[\mathbf{X}^{g_0}](x)\rangle^2]}dv_{g_0}(x).$$

We conclude since the limiting GMC measure $\lim_{\varepsilon\to 0}e^{\langle u,\mu_{g,\varepsilon}[\mathbf{X}^{g_0}](x)\rangle-\frac{1}{2}\mathbb{E}[\langle u,\mu_{g,\varepsilon}[\mathbf{X}^{g_0}](x)\rangle^2]}dv_{g_0}(x)$ is independent from the regularization scheme (see e.g. [87]), so that we can replace $\mu_{g,\varepsilon}\left[\mathbf{X}^{g_0}\right]$ by $\mu_{g_0,\varepsilon}\left[\mathbf{X}^{g_0}\right]$. The case where $u=\gamma e_i$ is handled by using that $\langle Q,e_i\rangle = \gamma\frac{|e_i|^2}{2}+\frac{2}{\gamma}$ and yields the second item by the change of variable $\boldsymbol{c}\to\boldsymbol{c}+m_g(\mathbf{X}^{g_0})$. □

2.3.3 Probabilistic interpretation of the path integral

We can now give a probabilistic interpretation of the path integral of Toda CFTs (1.24). Recall that the background charge is $Q=\gamma\boldsymbol{\rho}+\frac{2}{\gamma}\boldsymbol{\rho}^\vee$ and that the cosmological constants $\mu_{B,i}$ are taken positive for $1\leq i\leq r$.

2.3.3.1 Definition of the Toda field

We first make sense of the partition function and of the Toda field. This is done as follows:

Definition 2.3.9. Let g be a smooth Riemannian metric on Σ and F any bounded continuous functional over $H^{-1}(\Sigma\to\mathfrak{a},g)$. We set

$$\langle F\rangle_g := \lim_{\varepsilon\to 0}\langle F\rangle_{g,\varepsilon},\quad \text{where}\quad \langle F\rangle_{g,\varepsilon} := \mathcal{Z}_{\mathbf{X}}(\Sigma,g)\int_{\mathfrak{a}}\mathbb{E}\left[F(\mathbf{X}^g_\varepsilon+\mathbf{c})e^{-V_g(\mathbf{X}^g_\varepsilon+\mathbf{c})}\right]d\mathbf{c} \tag{2.31}$$

with the regularized Toda potential given by

$$V_g(\mathbf{X}^g_\varepsilon+\mathbf{c}) = \frac{1}{4\pi}\int_\Sigma R_g\langle Q,\mathbf{X}^g_\varepsilon+\mathbf{c}\rangle dv_g+\sum_{i=1}^r\mu_{B,i}e^{\langle\gamma e_i,\mathbf{c}\rangle}\int_\Sigma\varepsilon^{\frac{\gamma_i^2}{2}}e^{\langle\gamma e_i,\mathbf{X}^g_\varepsilon\rangle}dv_g. \tag{2.32}$$

We refer to $\Phi=\mathbf{X}^g+\mathbf{c}$ as the Toda field.

The following statement ensures that the Toda path integral gives rise to a well-defined positive measure on $H^{-1}(\Sigma\to\mathfrak{a},g)$:

Proposition 2.3.10. *Assume that $\chi(\Sigma)<0$ and take g any smooth Riemannian metric on Σ. Let F be any continuous, bounded functional over $H^{-1}(\Sigma\to\mathfrak{a},g)$. Then the limit* (2.31) *exists and belongs to* $(0,\infty)$. *Moreover the limiting partition function satisfies a Weyl anomaly of the form* (1.39) *and is invariant under orientation-preserving diffeomorphisms* (1.40).

See Theorems 1.2.1 and 1.2.2 for a more precise statement. We defer the proof of this statement to Appendix 2.5. As an immediate consequence of this result we have the following:

Corollary 2.3.11. *The assignment of $F \mapsto \langle F \rangle_g$ for F a bounded continuous functional over $H^{-1}(\Sigma \to \mathfrak{a}, g)$ defines a unique Borel measure over $H^{-1}(\Sigma \to \mathfrak{a}, g)$.*

Remark 2.3.12. *This naturally defines a probability measure on $H^{-1}(\Sigma \to \mathfrak{a}, g)$ equipped with its Borel σ-algebra by considering the mapping $F \mapsto \frac{\langle F \rangle_g}{\langle 1 \rangle_g}$.*

Remark 2.3.13. *In the case where $\chi(\Sigma) \geq 0$ the partition function is infinite because of the behavior of the integral in the zero-mode $\mathbf{c}$ when $\langle \mathbf{c}, e_i \rangle \to -\infty$. We can still define the measure by $F \mapsto \langle F \rangle_g$ for suitable functionals, although it will have infinite total mass.*

2.3.3.2 Correlation functions of Vertex Operators

Key observables in Toda theory are the Vertex Operators, formal exponentials of the random distribution Φ under the law described by the path integral (1.24): $\langle e^{\langle \alpha, \Phi(z) \rangle} \rangle_g$ for some $\alpha \in \mathfrak{a}$ and $z \in \Sigma$. Averaging products of Vertex Operators with respect to the law defined by the path integral amounts to defining their correlation functions: we study here the existence and some properties of such objects.

Definition 2.3.14. Consider a family of $N \in \mathbb{N}$ pairs $(x_k, \alpha_k)_{k=1,\dots,N}$ with $x_k \in \Sigma$ all distinct and $\alpha_k \in \mathfrak{a}$. The regularized Vertex Operator at x_k with weight α_k is defined by

$$V_{\alpha_k,\varepsilon}(x_k) := \varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{X}_\varepsilon^g(x_k) + \mathbf{c} \rangle}. \tag{2.33}$$

The correlation function of Vertex Operators is then, if it makes sense, the limit

$$\begin{aligned} &\langle \prod_{k=1}^N V_{\alpha_k}(x_k) \rangle_g := \lim_{\varepsilon \to 0} \langle \prod_{k=1}^N V_{\alpha_k,\varepsilon}(x_k) \rangle_{g,\varepsilon}, \quad \text{where} \\ &\langle \prod_{k=1}^N V_{\alpha_k,\varepsilon}(x_k) \rangle_{g,\varepsilon} = \mathcal{Z}_{\mathbf{X}}(\Sigma, g) \int_{\mathfrak{a}} \mathbb{E}\left[\langle \prod_{k=1}^N V_{\alpha_k,\varepsilon}(x_k) \rangle_{g,\varepsilon} e^{-V_g(\mathbf{X}_\varepsilon^g + \mathbf{c})} \right] d\mathbf{c}. \end{aligned} \tag{2.34}$$

We prove in Appendix 2.5 that it is indeed the case under the Seiberg bounds:

Theorem 2.3.15. *For g a smooth Riemannian metric on Σ, the limit (2.34) exists and belongs to $(0, \infty)$ if and only if the Seiberg bounds (1.38) hold. The limiting correlation function verifies the Weyl anomaly (1.39) and is invariant under orientation-preserving diffeormorphisms (1.40).*

Remark 2.3.16. *If the Seiberg bounds hold the map $F \mapsto \langle F \prod_{k=1}^N V_{\alpha_k}(x_k) \rangle_g / \langle \prod_{k=1}^N V_{\alpha_k}(x_k) \rangle_g$ defines a probability measure on $H^{-1}(\Sigma \to \mathfrak{a}, g)$ equipped with its Borel σ-algebra.*

2.4 Boundary Toda theories

Having defined Toda CFTs when the underlying surface is a closed hyperbolic Riemann surface, we now turn to the case where the compact surface Σ has a non-empty boundary. As explained in the introduction a major difference is that the presence of a boundary implies that one needs to specify boundary conditions for the field $\boldsymbol{\phi}$ in the Toda action, thus giving rise to several possible constructions for the theory with a boundary. These boundary conditions correspond to automorphisms of the underlying root system $\mathscr{R}$.

In this section we assume that Σ has a non-empty boundary, is of hyperbolic type and is equipped with a smooth Riemannian metric g. We also fix an element τ of $\mathrm{Out}(\mathscr{R})$: as explained in Subsections 2.2.2.2 and 2.2.2.3 this allows one to write $\mathfrak{a}$ as the orthogonal sum $\mathfrak{a} = \mathfrak{a}_N \bigoplus \mathfrak{a}_D$ with the corresponding orthogonal projections denoted p_N and p_D. We will also frequently use the notation u^* for $\tau(u)$ where u is in $\mathfrak{a}$.

2.4.1 The Gaussian Free Fields considered

2.4.1.1 Neumann and Dirichlet Gaussian Free Fields

In the same way as in Section 2.3.1, it is possible to define a Gaussian field with covariance structure given by either the Neumann or Dirichlet Green's function introduced respectively in Equations (2.9) and (2.10).

Definition 2.4.1. The (vectorial) Neumann GFF $\mathbf{X}^{g,N}$ is the centered Gaussian field with covariance structure given for all $u, v \in \mathfrak{a}$ and all $x \neq y \in \overline{\Sigma}$ by

$$\mathbb{E}[\langle u, \mathbf{X}^{g,N}(x)\rangle\langle v, \mathbf{X}^{g,N}(y)\rangle] = \langle u, v\rangle G_g^N(x,y) \tag{2.35}$$

where we used the same abuse of notation as when defining the closed GFF. Likewise the Dirichlet GFF $\mathbf{X}^{g,D}$ is centered with covariance for all $u, v \in \mathfrak{a}$ and all $x \neq y \in \overline{\Sigma}$:

$$\mathbb{E}[\langle u, \mathbf{X}^{g,D}(x)\rangle\langle v, \mathbf{X}^{g,D}(y)\rangle] = \langle u, v\rangle G_g^D(x,y).$$

Due to the presence of the zero-mode, to make sense of the path integral the law of the Neumann GFF has to be tensorized with the Lebesgue measure just as in the closed case. We will take this into account when defining the partition function.

Like before we can see these fields as vectorial GFFs, e.g. $\mathbf{X}^{g,N} = \sum_{i=1}^r X_i^{g,N} \boldsymbol{v}_i$ where $(\boldsymbol{v}_i)_{1\leq i\leq r}$ is an orthonormal basis of $\mathfrak{a}$ and with the $X_i^{g,N}$ independent 1-dimensional Neumann GFFs on Σ (and likewise for $\mathbf{X}^{g,D}$). This allows to define their partition function:

Definition 2.4.2. Let g be a smooth Riemannian metric over Σ, Δ_g^N be the one-dimensional Neumann Laplacian on Σ and Δ_g^D be the one-dimensional Dirichlet Laplacian on Σ. We set:

$$\mathscr{Z}_N(\Sigma, g) := \left(\frac{\det'\left(-\frac{1}{2\pi}\Delta_g^N\right)}{v_g(\Sigma)}\right)^{-\frac{r}{2}} \exp\left(-\frac{r}{8\pi}\int_{\partial\Sigma} k_g d\lambda_g\right). \tag{2.36}$$

Likewise the partition function of the Dirichlet field is given by a regularized determinant:

$$\mathcal{Z}_D(\Sigma, g) := \det\left(-\frac{1}{2\pi}\Delta_g^D\right)^{-\frac{r}{2}} \exp\left(\frac{r}{8\pi}\int_{\partial\Sigma} k_g d\lambda_g\right). \tag{2.37}$$

The terms $\exp\left(\pm\frac{r}{8\pi}\int_{\partial\Sigma} k_g d\lambda_g\right)$ are needed for the conformal anomaly (see Proposition 2.4.5).

2.4.1.2 Gaussian Free Field with Cardy boundary conditions

Lemma 2.2.2 suggests an alternative way of defining the Neumann and Dirichlet GFFs. Indeed, let g_0 be a uniform metric of type 1 on Σ: as shown in Lemma 2.2.1 one can double the surface Σ to obtain a closed surface $\hat{\Sigma}$ equipped with a smooth metric $\hat{g}_0$. Now let $\mathbf{X}^{\hat{g}_0}$ be the vectorial GFF on $\hat{\Sigma}$. Then the fields

$$\mathbf{X}^{g_0,N} := \frac{\mathbf{X}^{\hat{g}_0} + \mathbf{X}^{\hat{g}_0}\circ\sigma}{\sqrt{2}} \quad \text{and} \quad \mathbf{X}^{g_0,D} := \frac{\mathbf{X}^{\hat{g}_0} - \mathbf{X}^{\hat{g}_0}\circ\sigma}{\sqrt{2}} \tag{2.38}$$

(where σ is the canonical involution) have respectively the law of the Neumann and Dirichlet GFFs on Σ (in virtue of Lemma 2.2.2). Based on this observation we wish to define a new field $\mathbf{X}^{g_0,C}$ over Σ based on the doubling trick discussed in the introduction. Namely $\mathbf{X}^{g_0,C}$ would be such that $\mathbf{X}^{\hat{g}_0,C}$ and $(\mathbf{X}^{\hat{g}_0,C})^*\circ\sigma$ have the same law, where $\cdot^*$ is an order two outer automorphism of the underlying root system and $\mathbf{X}^{\hat{g}_0,C}$ is the extension of $\mathbf{X}^{g_0,C}$ to $\hat{\Sigma}$ defined using σ. To this end in the metric g_0 we define a new field on $\bar{\Sigma}$ via

$$\mathbf{X}^{g_0,C} = \frac{\mathbf{X}^{\hat{g}_0} + (\mathbf{X}^{\hat{g}_0})^*\circ\sigma}{\sqrt{2}}. \tag{2.39}$$

Some key remarks have to be made at this stage:

1. a straightforward consequence of Equation (2.39) is that $\mathbf{X}^{\hat{g}_0,C} = (\mathbf{X}^{\hat{g}_0,C})^*\circ\sigma$;
2. the field $\mathbf{X}^{g_0,C}$ has the following covariance structure for $x, y \in \bar{\Sigma}$ and $u, v \in \mathfrak{a}$:

$$\mathbb{E}[\langle u, \mathbf{X}^{g_0,C}(x)\rangle\langle v, \mathbf{X}^{g_0,C}(y)\rangle] = \langle u, v\rangle G_{\hat{g}_0}(x,y) + \langle u, v^*\rangle G_{\hat{g}_0}(x,\sigma(y)); \tag{2.40}$$

3. let us decompose $\mathbf{X}^{g_0,C}$ in $\mathfrak{a} = \mathfrak{a}_N \bigoplus \mathfrak{a}_D$ (recall Subsection 2.2.2.2) by writing

$$\mathbf{X}^{g_0,C} = \mathbf{X}^{g_0,N} + \mathbf{X}^{g_0,D}, \quad \mathbf{X}^{g_0,N} := \frac{\mathbf{X}^{g_0,C} + (\mathbf{X}^{g_0,C})^*}{2} \quad \text{and} \quad \mathbf{X}^{g_0,D} := \frac{\mathbf{X}^{g_0,C} - (\mathbf{X}^{g_0,C})^*}{2}. \tag{2.41}$$

   Then $\mathbf{X}^{g_0,N}$ (resp. $\mathbf{X}^{g_0,D}$) has the law of a Neumann (resp. Dirichlet) GFF from Σ to $\mathfrak{a}_N$ (resp. $\mathfrak{a}_D$). The two fields are independent. Besides, we can rewrite (2.40) as

$$\mathbb{E}[\langle u, \mathbf{X}^{g_0,C}(x)\rangle\langle v, \mathbf{X}^{g_0,C}(y)\rangle] = \langle u, \frac{v+v^*}{2}\rangle G^N_{\hat{g}_0}(x,y) + \langle u, \frac{v-v^*}{2}\rangle G^D_{\hat{g}_0}(x,\sigma(y)). \tag{2.42}$$

This construction is limited to the uniform metric g_0 and to outer automorphisms of order two. However, these observations suggest the following definition for the Cardy

GFF:

Definition 2.4.3. Given a smooth Riemannian metric g on Σ and an automorphism τ of $\mathscr{R}$, the (vectorial) Cardy GFF $\mathbf{X}^{g,C}$ is the centered Gaussian field with covariance structure given for all $u, v \in \mathfrak{a}$ and all $x \neq y \in \overline{\Sigma}$ by

$$\mathbb{E}[\langle u, \mathbf{X}^{g,C}(x)\rangle\langle v, \mathbf{X}^{g,C}(y)\rangle] = \langle p_N u, p_N v\rangle G_g^N(x,y) + \langle p_D u, p_D v\rangle G_g^D(x,y). \tag{2.43}$$

For such a GFF $\mathbf{X}^{g,C}$ we have a decomposition similar to Equation (2.41), which allows to define its partition function: set $d_N := \dim(\mathfrak{a}_N)$ and $d_D := \dim(\mathfrak{a}_D)$ so that in particular $d_N + d_D = r$ — recall that the values of these quantities are explicit and given in Subsection 2.2.2.2.

Definition 2.4.4. Let g be a smooth Riemannian metric on Σ. We set

$$\mathcal{Z}_C(\Sigma, g) := \left(\frac{\det'\left(-\frac{1}{2\pi}\Delta_g^N\right)}{v_g(\Sigma)}\right)^{-\frac{d_N}{2}} \det\left(-\frac{1}{2\pi}\Delta_g^D\right)^{-\frac{d_D}{2}} \exp\left(\frac{d_D - d_N}{8\pi}\int_{\partial\Sigma} k_g d\lambda_g\right). \tag{2.44}$$

This formula agrees with (2.36) when the outer automorphism τ is the identity (for which $d_N = r$ and $d_D = 0$). Via Lemma 2.2.4 it satisfies the following conformal anomaly formula:

Proposition 2.4.5. *Let $g' = e^{\varphi} g$ be a smooth Riemannian metric in the conformal class of g. Then we have*

$$\mathcal{Z}_C(\Sigma, g') = \mathcal{Z}_C(\Sigma, g)\exp\left(\frac{r}{96\pi}\left(\int_\Sigma (|\nabla_g\varphi|_g^2 + 2R_g\varphi)dv_g + 4\int_{\partial\Sigma} k_g\varphi d\lambda_g\right)\right). \tag{2.45}$$

Proof. Thanks to Lemma 2.2.4 we have

$$\log\mathcal{Z}_C(\Sigma, g') - \log\mathcal{Z}_C(\Sigma, g) = \frac{r}{96\pi}\left(\int_\Sigma (|\nabla_g\varphi|_g^2 + 2R_g\varphi)dv_g + 4\int_{\partial\Sigma} k_g\varphi d\lambda_g\right) + R$$
$$\text{with}\quad R = \frac{d_N - d_D}{16\pi}\left(\int_{\partial\Sigma}\partial_{\vec{n}_g}\varphi d\lambda_g + 2\int_{\partial\Sigma} k_g d\lambda_g - 2\int_{\partial\Sigma} k_{g'} d\lambda_{g'}\right)$$

which is seen to vanish thanks to (2.5). □

2.4.2 Gaussian Multiplicative Chaos

In order to make sense of the exponential of the GFF we need to regularize the field $\mathbf{X}^{g,C}$. To this end for $x \in \overline{\Sigma}$ and $\varepsilon > 0$ small let $C_g(x,\varepsilon) := \left\{y \in \overline{\Sigma}, d_g(x,y) = \varepsilon\right\}$ be the geodesic (semi-)circle of radius $\varepsilon > 0$ centered at x, and $\mu_{g,\varepsilon}[\cdot](x)$ be the uniform measure probability on $C_g(x,\varepsilon)$.

Definition 2.4.6. The regularized GFF $\mathbf{X}_\varepsilon^{g,C}$ is the average of $\mathbf{X}^{g,C}$ over geodesic circles:

$$\mathbf{X}_\varepsilon^{g,C}(x) := \mu_{g,\varepsilon}[\mathbf{X}^{g,C}](x). \tag{2.46}$$

The averaged GFF $\mathbf{X}_\varepsilon^{g,C}$ satisfies:

Lemma 2.4.7. *The field $\mathbf{X}_\varepsilon^{g,C}$ has covariance kernel given by*

$$\mathbb{E}[\langle u,\mathbf{X}_\varepsilon^{g,C}(x)\rangle\langle v,\mathbf{X}_\varepsilon^{g,C}(y)\rangle] = \iint (\langle p_N u, p_N v\rangle G_g^N + \langle p_D u, p_D v\rangle G_g^D) d\mu_{g,\varepsilon}(x) d\mu_{g,\varepsilon}(y). \tag{2.47}$$

Besides, if $g = e^\varphi g_0$ is a smooth Riemannian metric in the conformal class of the uniform type 1 metric g_0, then as $\varepsilon \to 0$ and uniformly in $x \notin \partial\Sigma$:

$$\mathbb{E}[\langle u,\mu_{g,\varepsilon}[\mathbf{X}^{g_0,C}](x)\rangle^2] = \begin{cases} |u|^2\left(-\log\varepsilon + W_{g_0}(x) + \frac{1}{2}\varphi(x)\right) + \langle u, p_N u - p_D u\rangle G_{g_0}(x,\sigma(x)) + o(1) \\ 2\left|p_N u\right|^2\left(-\log\varepsilon + W_{g_0}(x) + \frac{1}{2}\varphi(x)\right) + o(1) \end{cases} \tag{2.48}$$

with W_{g_0} a smooth function on $\bar{\Sigma}$, $x \in \Sigma$ on the first line and $x \in \partial\Sigma$ on the second one.

Proof. This again follows from the computations conducted in [51, Lemma 3.2 and Equation (3.10)] by writing $G_{g_0}^{N/D}$ in terms of $G_{\hat{g}_0}$ as in Lemma 2.2.2. See also the proof of Lemma 2.3.6. □

Definition 2.4.8. The GMC measures associated to $\mathbf{X}^{g,C}$ are then defined by the limits:

$$e^{\langle u,\mathbf{X}^{g,C}(x)\rangle} dv_g(x) := \lim_{\varepsilon\to 0} \varepsilon^{\frac{|u|^2}{2}} e^{\langle u,\mathbf{X}_\varepsilon^{g,C}(x)\rangle} dv_g(x) \tag{2.49}$$

for x in the bulk, while for $x \in \partial\Sigma$ they are given by

$$e^{\langle u,\mathbf{X}^{g,C}(x)\rangle} d\lambda_g(x) := \lim_{\varepsilon\to 0} \varepsilon^{|p_N u|^2} e^{\langle u,\mathbf{X}_\varepsilon^{g,C}(x)\rangle} d\lambda_g(x). \tag{2.50}$$

Like before all the convergences stand in probability in the space of Radon measures equipped with the weak topology, provided $|u|^2 < 4$ for the bulk measure and $\left|p_N u\right|^2 < 1$ for the boundary one. We stress that in the case where τ is the identity, the above statement amounts to defining the vectorial Neumann GMC on Σ.

Finally we provide the analogue of Lemma 2.3.8 for the behavior of the GMC measures under a conformal change of metric. Recall that the f_j are a basis of $\mathfrak{a}_N$ of the form $f_j = p_N(e_i)$.

Lemma 2.4.9. *Let $g = e^\varphi g_0$ be a metric in the conformal class of the uniform type 1 metric g_0, and u in $\mathfrak{a}$. Then the following equalities hold in distribution:*

$$\begin{aligned} &e^{\langle u,\mathbf{X}^{g,C}\rangle} dv_g \overset{(law)}{=} e^{\left(1+\frac{|u|^2}{4}\right)\varphi} e^{\langle u,\mathbf{X}^{g_0,C} - m_g(\mathbf{X}^{g_0,C})\rangle} dv_{g_0} \\ \text{and}\quad &e^{\langle u,\mathbf{X}^{g,C}\rangle} d\lambda_g \overset{(law)}{=} e^{\left(1+|p_N u|^2\right)\frac{\varphi}{2}} e^{\langle u,\mathbf{X}^{g_0,C} - m_g(\mathbf{X}^{g_0,C})\rangle} d\lambda_{g_0}. \end{aligned}$$

Proof. It is exactly the same proof as in the closed case, using Lemma 2.4.7. Namely for x in the bulk

$$e^{\langle u,\mathbf{X}^{g,C}(x)\rangle}dv_g(x)\overset{\text{(law)}}{=}\lim_{\varepsilon\to0}\varepsilon^{\frac{|u|^2}{2}}e^{\langle u,\mathbf{X}_\varepsilon^{g_0,C}(x)-m_g(\mathbf{X}^{g_0,C})\rangle}e^{\varphi(x)}dv_{g_0}(x)$$
$$=e^{\left(1+\frac{|u|^2}{4}\right)\varphi(x)}e^{-\langle u,m_g(\mathbf{X}^{g_0,C})\rangle}F_{\hat g_0}(x)\lim_{\varepsilon\to0}e^{\langle u,\mu_{g,\varepsilon}(\mathbf{X}^{g_0,C})(x)\rangle-\frac12\mathbb{E}[\langle u,\mu_{g,\varepsilon}(\mathbf{X}^{g_0,C})(x)\rangle^2]}dv_{g_0}(x)$$

where $F_{\hat g_0}(x)=e^{\frac12\left(|u|^2W_{\hat g_0}(x)+\langle u,p_Nu-p_Du\rangle G_{\hat g_0}(x,\sigma(x))\right)}$ is independent of φ. For $x\in\partial\Sigma$

$$e^{\langle u,\mathbf{X}^{g,C}(x)\rangle}d\lambda_g(x)\overset{\text{(law)}}{=}\lim_{\varepsilon\to0}\varepsilon^{|p_Nu|^2}e^{\langle u,\mu_{g,\varepsilon}[\mathbf{X}^{g_0,C}](x)-m_g(\mathbf{X}^{g_0,C})\rangle}e^{\frac{\varphi(x)}{2}}d\lambda_{g_0}(x)$$
$$=e^{\left(1+|p_Nu|^2\right)\frac{\varphi(x)}{2}}e^{-\langle u,m_g(\mathbf{X}^{g_0,C})\rangle}e^{|p_Nu|^2W_{\hat g_0}(x)}\times\lim_{\varepsilon\to0}e^{\langle u,\mu_{g,\varepsilon}(\mathbf{X}^{g_0,C})(x)\rangle-\frac12\mathbb{E}[\langle u,\mu_{g,\varepsilon}(\mathbf{X}^{g_0,C})(x)\rangle^2]}d\lambda_{g_0}(x).$$

We end the proof as in Lemma 2.3.8, using the fact that the limiting GMC measures are independent of the regularization scheme. □

2.4.3 Different boundary models

2.4.3.1 The Toda action and invariant subspaces

Thanks to Equation (2.41), for $\tau\in\mathrm{Out}(\mathscr{R})$ we can decompose $\mathbf{X}^{g,C}$ as a sum of a Neumann GFF $\mathbf{X}^{g,N}$ and a Dirichlet GFF $\mathbf{X}^{g,D}$. Like in the closed case the Neumann GFF is defined up to an additive constant (fixed by the requirement that $m_g(\mathbf{X}^{g,N})=0$), but this is not the case for the Dirichlet GFF. As a consequence the space where one should take the constant mode $\boldsymbol{c}$ is not $\mathfrak{a}$ but rather $\mathfrak{a}_N$. Likewise functionals of the field should take into account its specific form. In particular for exponentials of the Toda field $e^{\langle\beta,\Phi(s)\rangle}$, when $s\in\partial\Sigma$ we can assume that $\beta\in\mathfrak{a}_N$ since $\Phi(s)\in\mathfrak{a}_N$ which is orthogonal to $\mathfrak{a}_D$.

Definition 2.4.10. Given cosmological constants $\mu_{B,i}>0$ and $\mu_j\geq0$ for $1\leq i\leq r$ and $1\leq j\leq d_N$, and g a smooth Riemannian metric on Σ, the path integral is defined by setting, for F a bounded continuous functional on $H^{-1}(\Sigma\to\mathfrak{a},g)$,

$$\langle F\rangle_{g,\tau}:=\lim_{\varepsilon\to0}\langle F\rangle_{g,\tau,\varepsilon},\quad\langle F\rangle_{g,\tau,\varepsilon}:=\mathscr{Z}_C(\Sigma,g)\int_{\mathfrak{a}_N}\mathbb{E}\left[F(\mathbf{X}_\varepsilon^{g,C}+\mathbf{c})e^{-V_g(\mathbf{X}_\varepsilon^{g,C}+\mathbf{c})}\right]d\mathbf{c}.\quad(2.51)$$

Here the regularized potential is given by

$$\begin{aligned}V_g(\mathbf{X}_\varepsilon^{g,C}+\mathbf{c})&=\frac{1}{4\pi}\int_\Sigma R_g\langle Q,\mathbf{X}_\varepsilon^{g,C}+\mathbf{c}\rangle dv_g+\frac{1}{2\pi}\int_{\partial\Sigma}k_g\langle Q,\mathbf{X}_\varepsilon^{g,C}+\mathbf{c}\rangle d\lambda_g\\&+\sum_{i=1}^r\mu_{B,i}e^{\langle\gamma e_i,\mathbf{c}\rangle}\int_\Sigma\varepsilon^{\frac{\gamma_i^2}{2}}e^{\langle\gamma e_i,\mathbf{X}_\varepsilon^{g,C}\rangle}dv_g+\sum_{j=1}^{d_N}e^{\langle\frac{\gamma}{2}f_j,\mathbf{c}\rangle}\int_{\partial\Sigma}\varepsilon^{\frac{\gamma^2|f_j|^2}{4}}e^{\langle\frac{\gamma}{2}f_j,\mathbf{X}_\varepsilon^{g,C}\rangle}\mu_j(d\lambda_g).\end{aligned}\quad(2.52)$$

where $\mu_j(d\lambda_g):=\mu_jd\lambda_g$ (the reason for this notation will become clear later). We then have the following analogue of Proposition 2.3.10, proved in Appendix 2.5.

Proposition 2.4.11. *In the setting of Defintion 2.4.10, assume that $\chi(\Sigma) < 0$. Then the limit (2.51) exists and belongs to $(0,\infty)$.*

2.4.3.2 Around the Weyl anomaly

Having defined the path integral, we now would like to deduce from Lemma 2.4.9 the analogue of the change of metric formula (2.30) for the path integral in the boundary case, which in turn allows to prove a Weyl anomaly for the correlation functions. However, there is a major obstruction that comes from the fact that, when $\tau \neq \mathrm{Id}$, the folded root system $(f_j)_{j=1,\dots,d_N}$ is *not* simply-laced, so that the conformal anomaly of the boundary GMC measures associated with short roots is expressed in terms of a different background charge Q_τ (2.20), thus breaking the conformal symmetry of the model. We thus need to distinguish between the cases where $\tau = \mathrm{Id}$ or not.

Neumann boundary conditions. When the outer automorphism under consideration is the identity the construction leads to Neumann (or free) boundary conditions. In this case the model is conformally covariant (Theorem 1.2.2), and its algebra of symmetry is given by the W-algebra constructed from $\mathfrak{g}$, both in the bulk and on the boundary, as shown in the A_2 ($\mathfrak{sl}_3\mathbb{C}$) case in the next chapter (taken from [26]).

Cardy boundary conditions. In view of Lemma 2.4.9 we know that the conformal anomaly for the boundary GMC measure is given by $\exp\left(\frac{\langle Q_\tau,\gamma f_j\rangle}{4}\varphi\right)$ where Q_τ is defined in Equation (2.20). It suggests that one should consider a model for which the background charge on the boundary is set to be Q_τ instead of Q. However, this spoils the construction in the bulk by breaking conformal covariance there! A way to overcome this issue is to assume that $\mu_j = 0$ for all j such that $\langle Q, f_j\rangle \neq \langle Q_\tau, f_j\rangle$, which is the case when $j \in \{1,\cdots,d_N\}\setminus I_\tau$.

The model then enjoys conformal Weyl anomaly and diffeomorphism invariance:

Proposition 2.4.12. *In the setting of Proposition 2.4.12, assume that $\mu_j \neq 0$ only if $j \in I_\tau$. Then the model satisfies the Weyl anomaly formula (1.39) and diffeomorphism invariance (1.40).*

2.4.3.3 Correlation functions

On the surface with boundary Σ we consider not only N insertion points in the bulk together with their associated weights $(x_k,\alpha_k)_{k=1,\dots,N}$ but also $M \in \mathbb{N}$ insertions on the boundary $(s_l,\beta_l)_{l=1,\dots,M}$ with $s_l \in \partial\Sigma$ and $\beta_l \in \mathfrak{a}_N$.

In the boundary case one can slightly generalize the model by introducing different cosmological constants between the boundary insertion points. Namely, suppose that the boundary of Σ is composed of $\mathbf{k}$ connected components $\mathscr{C}_n$, $n = 1,\dots,\mathbf{k}$, and that M_n insertions lie on the part $\mathscr{C}_n$ of $\partial\Sigma$ (thus $M_1+\dots+M_{\mathbf{k}} = M$). Let us introduce a family of non-negative[7] cosmological constants $(\mu_{j,l_n}^{(n)})$ with $j = 1,\cdots,d_N$, $n = 1,\dots,\mathbf{k}$ and $l_n = 1,\dots,M_n$. Then in the action (2.52) we set $\mu_j(d\lambda_g) = \sum_{n=1}^{\mathbf{k}}\sum_{l_n=1}^{M_n}\mu_{j,l_n}^{(n)}\mathbb{1}_{(s_{l_n}^{(n)},s_{l_n+1}^{(n)})}d\lambda_g$,

7. In fact, one can even choose the boundary cosmological constants in $\mathbb{C}$ with $\mathrm{Re}(\mu_{j,l_n}^{(n)}) \geq 0$.

$(s_{l_n}^{(n)}, s_{l_n+1}^{(n)})$ being the part of $\partial\Sigma$ between the two insertion points $s_{l_n}^{(n)}$ and $s_{l_n+1}^{(n)}$ and with the convention that $s_{M_n+1}^{(n)} = s_1^{(n)}$.

Definition 2.4.13. The bulk Vertex Operators are defined similarly as in Equation (2.33) by

$$V_{\alpha_k,\varepsilon}(x_k) = \varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{X}_\varepsilon^{g,C}(x_k)+\mathbf{c}\rangle} \quad \text{and} \quad V_{\beta_l,\varepsilon}(s_l) = \varepsilon^{\frac{|\beta_l|^2}{4}} e^{\langle \frac{\beta_l}{2}, \mathbf{X}_\varepsilon^{g,C}(s_l)+\mathbf{c}\rangle} \tag{2.53}$$

on the boundary. We then define the correlation function in the boundary case by the limit

$$\begin{aligned}\langle \prod_{k=1}^N V_{\alpha_k}(x_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{g,\tau} &:= \lim_{\varepsilon\to 0} \langle \prod_{k=1}^N V_{\alpha_k,\varepsilon}(x_k) \prod_{l=1}^M V_{\beta_l,\varepsilon}(s_l)\rangle_{g,\tau,\varepsilon} \\ &:= \lim_{\varepsilon\to 0} \mathcal{Z}_C(\Sigma, g) \int_{\mathfrak{a}_N} \mathbb{E}\left[\prod_{k=1}^N V_{\alpha_k,\varepsilon}(x_k) \prod_{l=1}^M V_{\beta_l,\varepsilon}(s_l) e^{-V_g(\mathbf{X}_\varepsilon^{g,C}+\mathbf{c})}\right] d\mathbf{c}.\end{aligned} \tag{2.54}$$

Theorems 1.2.1 and 1.2.2 (proved in Appendix 2.5) state that the limit is well-defined and non-trivial if and only if the Seiberg bounds hold true, and that the limiting correlation functions do enjoy diffeomorphism invariance, and satisfy the Weyl anomaly formula, under the assumptions of Proposition 2.4.12.

2.4.4 Algebra of symmetry

We briefly discuss here what is the algebra of symmetry of the models constructed, especially when τ is non-trivial. To this end let us assume that Σ is the upper-half plane $\mathbb{H}$ equipped with a metric conformally equivalent to $g = \frac{\|dz\|^2}{(1+\|z\|^2)^2}$. This discussion strongly relies on the framework developed in [22], to which we refer for more details, and definitions of the objects involved and of the notations used.

2.4.4.1 From the Heisenberg vertex algebras to W-algebras

Following [22, Section 3], inside $\mathbb{H}$ the symmetry algebra associated with the free-field $\mathbf{X}^g$ is, independently of the boundary conditions chosen, described by the Heisenberg vertex algebra of rank r. It is constructed out of a Fock space $\mathbf{V}_+$ of rank r, generated by acting on a *vacuum state* $|0\rangle$. More precisely, one can define [22, Subsection 3.1] a family of unbounded operators $(\mathrm{A}_n)_{n\in\mathbb{Z}}$, acting[8] on a dense subset of a Hilbert space $\mathcal{H}_0 = L^2(\Omega_\mathbb{T}, \mathbb{P}_\mathbb{T})$ [22, Subsection 2.2], and satisfying the commutation relations of a (rank r) Heisenberg algebra, i.e. for any $u, v \in \mathfrak{a}$ and $n, m \in \mathbb{Z}$

$$[\langle u, \mathrm{A}_n\rangle, \langle v, \mathrm{A}_n\rangle] = \langle u, v\rangle \frac{n}{2}\delta_{n,-m}. \tag{2.55}$$

8. In [22] the $(\mathrm{A}_n)_{n\in\mathbb{Z}}$ rather act on a subset of the full Hilbert space $\mathcal{H}$, but this doesn't change anything in our context and makes the exposition lighter.

The vacuum state $|0\rangle$ is then an element of $\mathscr{H}_0$ such that $\mathrm{A}_n|0\rangle = 0$ for all $n \geq 0$, and $\mathrm{A}_n|0\rangle \neq 0$ for $n < 0$, while the Fock space $\mathbf{V}_+$ is the subspace of $\mathscr{H}_0$ defined by

$$\mathbf{V}_+ := \bigoplus_{n\geq 0} \mathbf{V}_+^{(n)}, \quad \text{where} \quad \mathbf{V}_+^{(n)} := \bigoplus_{\substack{m_1,\cdots m_r \geq 1 \\ m_1+\cdots+m_r=n}} \left\{\langle u_1, \mathrm{A}_{-m_1}\rangle \cdots \langle u_n, \mathrm{A}_{-m_r}\rangle|0\rangle\right\}. \tag{2.56}$$

Here the span is algebraic, meaning that we consider finite linear combinations.

The latter allows to define for $z \in \mathbb{C}\setminus\{0\}$ the formal power series in $\mathrm{End}(\boldsymbol{V}_{+,\boldsymbol{c}})[[z, z^{-1}, \log\|z\|]]$:

$$\Psi(z) := \boldsymbol{c} + \mathrm{i}\mathrm{A}(z) + \mathrm{i}\bar{\mathrm{A}}(\bar{z}), \quad \mathrm{A}(z) := \mathrm{A}_0 \log z - \sum_{n\in\mathbb{Z}^*} \frac{1}{n}\mathrm{A}_n z^{-n}, \tag{2.57}$$

see [22, Equation (3.14)] (the latter is denoted Φ there) for more details. Its holomorphic derivative $\partial\Psi(z)$ is then the generating series of the generators of the Heisenberg algebra:

$$\partial\Psi(z) = \mathrm{i}\sum_{n\in\mathbb{Z}} \mathrm{A}_n z^{-n-1}. \tag{2.58}$$

These can be viewed as unbounded and densely defined operators. We can further define so-called bosonic Vertex Operators [22, Equation (3.31)] by setting, for $\alpha \in \mathfrak{a}$ and $z \in \mathbb{C}\setminus\{0\}$, $\mathcal{V}_\alpha^+(z) :=: e^{\langle\alpha, \boldsymbol{c}+\mathrm{i}\mathrm{A}(z)\rangle}$: where $:\cdot:$ denotes the normal ordering.

Compared to the free-field, the addition of the potential $e^{\langle\gamma e_i, \mathbf{X}\rangle}$ to the action, which gives rise to the Toda theories considered in this document, breaks the Heisenberg symmetry [22, Lemma 5.8] but preserves the W-one (as stated in [22, Theorem 5.5] under a *neutrality condition*). This is due to the definition of the W-algebra associated to $\mathfrak{g}$ as intersection of kernels of screening operators $Q_i^+ := \oint \mathcal{V}_{\gamma e_i}^+(z)\mathrm{d}z$ for $1 \leq i \leq r$, where the contour surrounds the origin. Namely, the restriction of the vertex algebra structure from $\mathbf{V}_+$ to ${}^{\mathfrak{g}}\mathscr{M}_+ := \bigcap_{1\leq i\leq r} \mathrm{Ker}_{\mathbf{V}_+}\left(Q_i^+\right)$ defines the W-algebra associated with $\mathfrak{g}$ [43, Definition 4.3.6].

It can be shown [43, Theorem 4.6.9] (see also [22, Theorem 4.3])that for γ in a dense open subset of $(0, \sqrt{2})$, this W-algebra is generated using the modes of currents $\left(\mathbf{W}^{(s_i)}[\Psi](z)\right)_{1\leq i\leq r}$ of spins s_i, where $s_i - 1$ ranges over the exponents of $\mathfrak{g}$. These currents may be thought of as polynomials in the holomorphic derivatives of Ψ [22, Theorem 1.1]. For instance $\mathbf{W}^{(2)}$ is the stress-energy tensor $\mathbf{W}^{(2)}[\Psi] = \langle Q, \partial^2\Psi\rangle - :\langle\partial\Psi, \partial\Psi\rangle:$, which thus takes the form

$$\mathbf{W}^{(2)}[\Psi](z) = \sum_{n\in\mathbb{Z}} \mathrm{L}_n z^{-n-2}, \quad \text{with} \quad \mathrm{L}_n := -\mathrm{i}(n+1)\langle Q, \mathrm{A}_n\rangle + \sum_{m\in\mathbb{Z}} :\langle \mathrm{A}_m, \mathrm{A}_{n-m}\rangle:. \tag{2.59}$$

More generally these currents are formal power series in $\mathrm{End}(\boldsymbol{V}_{+,\boldsymbol{c}})[[z, z^{-1}]]$:

$$\begin{gathered}\mathbf{W}^{(s_i)}[\Psi](z) = \sum_{n\in\mathbb{Z}} \mathbf{W}_n^{(s_i)} z^{-n-s_i}, \quad \text{which are such that} \\ {}^{\mathfrak{g}}\mathscr{M}_+ = \bigoplus_{n_{k,i}\geq s_i} \mathrm{span}\left\{\mathbf{W}_{-n_{1,r}}^{(s_r)} \cdots \mathbf{W}_{n_{l_r,r}}^{(s_r)} \cdots \mathbf{W}_{-n_{1,1}}^{(s_1)} \cdots \mathbf{W}_{n_{l_1,1}}^{(s_1)}|0\rangle\right\}.\end{gathered} \tag{2.60}$$

In agreement with [43, Theorem 4.6.9], they can be chosen to obey the commutation relations $[\mathbf{W}^{(2)}_n, \mathbf{W}^{(s_i)}_m] = (n(s_i-1)-m)\,\mathbf{W}^{(s_i)}_m$ for $1 \leq i \leq r$ and $n, m \in \mathbb{Z}$. These however do not uniquely define them and there is a certain freedom in the choice of these currents. In the next subsection we explain how this freedom can be exploited to define currents satisfying this property and that in addition satisfy a covariance property under the action of $\mathrm{Out}(\mathscr{R})$.

2.4.4.2 Covariance of the W-algebra currents under $\tau \in \mathrm{Out}(\mathscr{R})$

Compared to the bulk case, in the presence of a boundary the choice of an element τ in $\mathrm{Out}(\mathscr{R})$ has concrete implications on the properties of the symmetry algebra. Namely, when τ is of order two, in view of Equation (2.39), one can still make sense of $\mathbf{X}^{g,C}(\bar{z})$ for $z \in \mathbb{H}$, which is nothing but $\tau(\mathbf{X}^{g,C}) = (\mathbf{X}^{g,C})^*$. As such, with a slight abuse of notation, we have $\mathbf{W}^{(s_i)}[\Psi](\bar{z}) = \mathbf{W}^{(s_i)}[\Psi^*](z)$. Now, and up to redefining the currents $\mathbf{W}^{(s_i)}$, we can show that the effect of changing Ψ to Ψ^* has an explicit effect on these currents as follows:

Proposition 2.4.14. *Assume that $\mathfrak{g} \neq \mathfrak{so}_{4n}$ for some $n \geq 2$. There exist currents $\mathbf{W}^{(s_i)}[\Psi]$ of spins s_i, with $s_i - 1$ ranging over the exponents of $\mathfrak{g}$, such that for generic values of γ:*

— *${}^{\mathfrak{g}}\mathscr{M}_+$ is freely generated by acting on $|0\rangle$ with the modes $(\mathbf{W}^{(s_i)}_{-n_i})_{\substack{1\leq i\leq r\\ n_i \geq s_i}}$;*

— *$[\mathbf{W}^{(2)}_n, \mathbf{W}^{(s_i)}_m] = (n(s_i-1)-m)\,\mathbf{W}^{(s_i)}_m$ for all $1 \leq i \leq r$ and $n, m \in \mathbb{Z}$;*

— *there exists $\sigma : \{1, \cdots, r\} \to \{-1, 1\}$ such that for $\tau \in \mathrm{Out}(\mathscr{R})$ of order 2 we have $\mathbf{W}^{(s_i)}[\tau\Psi] = \sigma(i)\mathbf{W}^{(s_i)}[\Psi]$.*

In the case of $\mathfrak{g} = \mathfrak{sl}_n$ we have $\mathbf{W}^{(s_i)}[\tau\Psi] = (-1)^{s_i}\mathbf{W}^{(s_i)}[\Psi]$ for $\tau \in \mathrm{Out}(\mathscr{R})$ non-trivial.

The two first items of the above result are part of [43, Theorem 4.6.9]. The main novelty of our statement is the fact that it is possible to redefine the currents in such a way that they become covariant with respect to the action of $\mathrm{Out}(\mathscr{R})$ (which is relevant when this group is not trivial). To the best of our knowledge and apart form the $\mathfrak{g} = \mathfrak{sl}_3$ case this property is new and highlights new features of W-algebras in relation with the notion of folding.

Proof. Let τ have order 2. Since τ gives rise to a permutation of the simple roots $(e_i)_{1\leq i\leq r}$, the intersection of the kernels of the screening operators remains unchanged. Moreover, the stress-energy tensor $\mathbf{W}^{(2)}[\Psi] = \langle Q, \partial^2\Psi\rangle - \langle\partial\Psi, \partial\Psi\rangle$ is also invariant under $\Psi \to \Psi^*$. As a consequence, we see that the commutation relations $[\mathbf{W}^{(2)}_n, \mathbf{W}^{(s_i)}_m] = (n(s_i-1)-m)\,\mathbf{W}^{(s_i)}_m$ are also preserved. Thus changing the currents from $\mathbf{W}^{(s_i)}[\Psi]$ to $\mathbf{W}^{(s_i)}[\Psi^*]$ preserves the W-algebra and the property of being a Virasoro primary field, whatever basis of currents is chosen. We denote such a basis by $\left(\boldsymbol{w}^{(s_i)}\right)_{1\leq i\leq r}$.

Let us now construct the currents by showing inductively that for all $1 \leq i \leq r$, there exist currents $\mathbf{W}^{(s_j)}$, $1 \leq j \leq i$, such that ${}^{\mathfrak{g}}\mathscr{M}_+{}^{(\leq s_i)} := \bigoplus_{n\leq s_i} {}^{\mathfrak{g}}\mathscr{M}_+{}^{(n)}$ is freely generated by acting on $|0\rangle$ with the modes of these currents, and with $\mathbf{W}^{(s_j)}[\Psi^*] = \pm\mathbf{W}^{(s_j)}[\Psi]$. Here the grading is the one from the proof of [22, Theorem 4.3]. The case of $i = 1$ corresponds to the stress-energy tensor (of spin $s = 2$), so we can focus on the inductive step. Let us assume that we have been able to define currents of spins s_j, $1 \leq j \leq i-1$, satisfying these assumptions. Then, following the proof of [43, Theorem 4.6.9], the subspace ${}^{\mathfrak{g}}\mathcal{M}_+{}^{(s_i)}$ of ${}^{\mathfrak{g}}\mathcal{M}_+$ at level s_i is given, since we have assumed that $\mathfrak{g}\neq\mathfrak{so}_{4n}$, by

$$ {}^{\mathfrak{g}}\mathcal{M}_+{}^{(s_i)}=\mathbb{C}\boldsymbol{w}^{(s_i)}_{-s_i}|0\rangle\bigoplus\mathrm{span}\left\{\mathbf{W}^{(s_1)}_{-\lambda_1}\cdots\mathbf{W}^{(s_{i-1})}_{-\lambda_{i-1}}|0\rangle\right\}. $$

Since τ preserves the space ${}^{\mathfrak{g}}\mathcal{M}_+$, we can thus expand $\boldsymbol{w}^{(s_i)}_{-s_i}[\Psi^*]|0\rangle=\lambda_1\boldsymbol{w}^{(s_i)}_{-s_i}[\Psi]|0\rangle+\mathbf{D}^{<s_i}$ where $\mathbf{D}^{<s_i}$ is generated using the modes of the currents $\mathbf{W}^{(s_j)}[\Psi]$ for $j<s_i$. By induction, we can further decompose $\mathbf{D}^{<s_i}=\mathbf{D}^{<s_i}_N+\mathbf{D}^{<s_i}_D$ where $\tau\mathbf{D}^{<s_i}_N=\mathbf{D}^{<s_i}_N$ while $\tau\mathbf{D}^{<s_i}_D=-\mathbf{D}^{<s_i}_D$. Now since τ is an involution, applying τ once again we obtain

$$ (1+\lambda_1)\left(\boldsymbol{w}^{(s)}_{-s}[\Psi^*]-\boldsymbol{w}^{(s)}_{-s}[\Psi]\right)|0\rangle=-2\mathbf{D}^{<s_i}_D. $$

Here we need to distinguish according to the value of λ_1. Let us first assume that $\lambda_1\neq-1$. We can then define $\mathbf{W}^{(s)}$ to be the vertex operator (see [22, Equation (3.19)]) associated to $\boldsymbol{w}^{(s)}_{-s}[\Psi]|0\rangle+\frac{1}{1+\lambda_1}\mathbf{D}^{<s_i}_D$. Then by construction we have $\mathbf{W}^{(s_i)}[\Psi^*]=\mathbf{W}^{(s_i)}[\Psi]$. Moreover since $\boldsymbol{w}^{(s)}_{-s}[\Psi]|0\rangle$ and $\frac{1}{1+\lambda_1}\mathbf{D}^{<s_i}_N$ are linearly independent, the induction hypothesis is satisfied. If we rather assume that $\lambda_1=-1$, then $\left(\boldsymbol{w}^{(s)}_{-s}[\Psi^*]+\boldsymbol{w}^{(s)}_{-s}[\Psi]\right)|0\rangle=\mathbf{D}^{<s_i}_D$, so that we rather define $\mathbf{W}^{(s)}$ to be the vertex operator associated to $\boldsymbol{w}^{(s)}_{-s}[\Psi]|0\rangle-\frac{1}{2}\mathbf{D}^{<s_i}_D$. We then obtain $\mathbf{W}^{(s)}[\Psi^*]=-\mathbf{W}^{(s)}[\Psi]$ and the induction hypothesis is satisfied: we can always find currents that verify the two first points of the proposition, and such that $\mathbf{W}^{(s_i)}[\Psi^*]=\pm\mathbf{W}^{(s_i)}[\Psi]$.

It remains to see that this sign, in the $\mathfrak{sl}_n$ case, is determined by the spin s_i. To this end, we first choose the basis of currents defined using a Miura transform [41], that we denote $\left(\boldsymbol{w}^{(s_i)}\right)_{1\leq i\leq r}$: we refer to [22, Subsection B.2] for more details, and from which we borrow the notations. These currents admit an expansion of the form $\boldsymbol{w}^{(s_i)}=\sum_{j=0}^{s_i-1}q^jP_j(\Psi)$, where $q=\gamma+\frac{2}{\gamma}$ and $P_j(\Psi)$ is a (s_i-j)-multilinear form in the derivatives of Ψ. For instance

$$ P_0(\Psi)=\sum_{\substack{\mathcal{P}\subseteq\{1,\cdots,n\}\\ \|\mathcal{P}\|=s_i}}\prod_{j\in\mathcal{P}}\langle h_j,\partial\Psi\rangle. \tag{2.61} $$

For $\tau\neq\mathrm{Id}$, we have $\tau e_k=e_{n-k}$ so $\tau h_i=-h_{n+1-i}$ where $h_i=\omega_1-\sum_{k=1}^{i-1}e_k$ (with the notations from [22, Subsection B.2]). This implies that $P_0(\Psi^*)=(-1)^{s_i}P_0(\Psi)$. Now by contradiction assume that s_i is odd and that $\mathbf{W}^{(s_i)}[\Psi^*]=+\mathbf{W}^{(s_i)}[\Psi]$. Then the equalities $P_0[\Psi^*]=-P_0[\Psi]$ and $\tau\mathbf{D}^{<s_i}_D=-\mathbf{D}^{<s_i}_D$ imply $P_0[\Psi]+\mathbf{D}^{<s_i}_{D,0}[\Psi]=0$, where $\left(\mathbf{D}^{<s_i}_D[\Psi]\right)_{-s_i}|0\rangle=\mathbf{D}^{<s_i}_D$ and $\mathbf{D}^{<s_i}_{D,0}[\Psi]$ is the s_i-multilinear form in the expansion of $\mathbf{D}^{<s_i}_D[\Psi]$. But by construction, in the $\mathfrak{sl}_n$ case, $\mathbf{D}^{<s_i}_{D,0}[\Psi]$ is of the form

$$ \mathbf{D}^{<s_i}_{D,0}[\Psi]=\sum_{\boldsymbol{p}}c_{\boldsymbol{p}}\prod_{j=1}^{i-1}\left(\sum_{\substack{\mathcal{P}\subseteq\{1,\cdots,n\}\\ \|\mathcal{P}\|=s_j}}\prod_{l\in\mathcal{P}}\langle h_l,\partial\Psi\rangle\right)^{p_j} $$

where the sum ranges over $\boldsymbol{p} = (p_1, \cdots, p_{j-1}) \in \mathbb{N}^r$ such that $\sum_{j=1}^{j-1} p_j s_j = s_i$, and $c_{\boldsymbol{p}}$ are constants. The latter being linearly independent (for instance by viewing them as polynomials in the variables $X_i = \langle h_i, \partial\Psi\rangle\rangle$ from P_0, we cannot have $P_0[\Psi] + \mathbf{D}_{D,0}^{<s_i}[\Psi] = 0$. Hence we arrive to a contradiction: odd spin implies $\lambda = -1$. The same argument shows that if the spin is even we must have $\lambda = 1$, and this concludes the proof. We believe that a similar argument holds in the $\mathfrak{so}_{2n}$ case using the Miura transform [68].

□

2.5 Proofs of the main statements

We gather here the proofs of the statements defining the probabilistic path integral and the correlation functions. The case of a closed Riemann surface being treated in the exact same fashion as the open one we will focus on the latter. Likewise the partition function being a special case of the correlation functions our goal is to prove Theorems 1.2.1 and 1.2.2. Hereafter we denote $\boldsymbol{x} = (x_1, \cdots, x_N, s_1, \cdots, s_M)$ and $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N, \frac{1}{2}\beta_1, \cdots, \frac{1}{2}\beta_M)$.

A key ingredient is the following formulation of the Girsanov (or Cameron-Martin) theorem:

Theorem 2.5.1. *Let (Σ, g) be a Riemannian surface, $(X(x))_{x\in\Sigma} = (X_1(x), ..., X_n(x))_{x\in\Sigma}$ be a family of smooth centered Gaussian fields on Σ and Y be a random variable in the L^2-closure of $Span(X(x))_{x\in\Sigma}$. For any F bounded over the space of continuous functions,*

$$\mathbb{E}\left[e^{Y-\frac{1}{2}\mathbb{E}[Y^2]}F(X(x))_{x\in\Sigma}\right] = \mathbb{E}\left[F(X(x) + \mathbb{E}[YX(x)])_{x\in\Sigma}\right].$$

2.5.1 Existence of correlation functions (Theorem 1.2.1)

We start by establishing the Seiberg bounds (1.38). In view of the Weyl anomaly formula that we will prove next we can always assume that $g = g_0$ is a uniform metric of type 1. Without loss of generality and since, for $z \in \mathbb{C}$, $|e^z| = e^{\mathrm{Re}(z)}$ we also assume the cosmological constants to be real-valued.

Recall the decomposition (2.41) of the field $\mathbf{X}^{g,C}$ in terms of $\mathbf{X}^{N,g}$ and $\mathbf{X}^{D,g}$ two independent vectorial Neumann (resp. Dirichlet) GFFs. Then by Girsanov's theorem 2.5.1 for any F bounded continuous over $H^{-1}(\Sigma \to \mathfrak{a}, g)$ we have

$$\mathbb{E}\left[\prod_{k=1}^{N} V_{\alpha_k,\varepsilon}(x_k)\prod_{l=1}^{M} V_{\beta_l,\varepsilon}(s_l)F\left(\mathbf{X}_\varepsilon^{g,C} + \mathbf{c}\right)\right] = C_{\varepsilon,\boldsymbol{x},\boldsymbol{\alpha}}\mathbb{E}\left[F\left(\mathbf{X}_\varepsilon^{g,C} + \mathbf{c} + H_{\varepsilon,\boldsymbol{x},\boldsymbol{\alpha}}\right)\right] \quad \text{with}$$

$$C_{\varepsilon,\boldsymbol{x},\boldsymbol{\alpha}} = \prod_{k=1}^{N} e^{\frac{|\alpha_k|^2}{2}W_{g,\varepsilon}(x_k)}\prod_{l=1}^{M} e^{\frac{|\beta_l|^2}{4}W_{g,\varepsilon}(x_k)}\prod_{1\leq k<l\leq N+M} e^{\langle\alpha_k,\alpha_l\rangle G_{g,\varepsilon}(x_k,x_l)} \quad \text{and}$$

$$H_{\varepsilon,\boldsymbol{x},\boldsymbol{\alpha}} = \sum_{k=1}^{N+M}\alpha_k G_{g,\varepsilon,\varepsilon}(\cdot, x_k).$$

Here $G_{g,\varepsilon}(x,y) = \mu_{g,\varepsilon}[G_g(\cdot,y)](x)$, $G_{g,\varepsilon,\varepsilon}(x,y) = \mu_{g,\varepsilon}[z \mapsto \mu_{g,\varepsilon}[G_g(\cdot,z)](x)](y)$, while $W_{g,\varepsilon}(x_k) = W_{g_0}(x_k)+o(1)$ as $\varepsilon \to 0$ (see Equation (2.14)). Hence $C_{\varepsilon,\boldsymbol{x},\boldsymbol{\alpha}} = C_{0,\boldsymbol{x},\boldsymbol{\alpha}}(1+o(1))$. As a consequence the regularized correlation functions are

$$A_\varepsilon = C_{0,\boldsymbol{x},\boldsymbol{\alpha}} \int_{\mathfrak{a}_N} e^{\langle \sum \alpha_k + \frac{1}{2}\sum \beta_l - Q\chi(\Sigma),\mathbf{c}\rangle} \mathbb{E}\left[e^{-\sum_{i=1}^r \mu_{B,i} e^{\langle \gamma e_i,\mathbf{c}\rangle} Z_{i,\varepsilon} - \sum_{j=1}^{d_N} e^{\langle \frac{\gamma}{2} f_j,\mathbf{c}\rangle} Z^\partial_{j,\varepsilon}} \right] d\mathbf{c}(1+o(1)),$$

$$Z_{i,\varepsilon} = \int_\Sigma \varepsilon^{\frac{\gamma_i^2}{4}} e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^{C,g}(x) + H_{\varepsilon,\boldsymbol{x},\boldsymbol{\alpha}}(x)\rangle} dv_g(x), \quad Z^\partial_{j,\varepsilon} = \int_{\partial\Sigma} \varepsilon^{\frac{|\gamma f_j|^2}{2}} e^{\langle \frac{\gamma}{2} f_j, X_\varepsilon^{N,g}(s) + H_{\varepsilon,\boldsymbol{x},\boldsymbol{\alpha}}(s)\rangle} \mu_j(d\lambda_g)(s).$$

To see that the Seiberg bounds are necessary conditions, we bound for any positive $M>0$

$$A_\varepsilon \geq C_{\varepsilon,\boldsymbol{x},\boldsymbol{\alpha}} \int_{\mathfrak{a}_N} e^{\langle \sum \alpha_k + \frac{1}{2}\sum \beta_l - Q\chi(\Sigma),\mathbf{c}\rangle} e^{-\sum_{i=1}^r \left(\mu_{B,i} e^{\langle \gamma e_i,\mathbf{c}\rangle} + e^{\langle \frac{\gamma}{2} f_j,\mathbf{c}\rangle}\right) M}$$
$$\times \mathbb{P}(\forall i,j \ Z_{i,\varepsilon} \leq M \text{ and } Z^\partial_{j,\varepsilon} \leq M) d\mathbf{c}(1+o(1)).$$

The latter probability being non-zero we see that A_ε is infinite as soon as there is a $i \in \{1,\cdots,r\}$ such that $\langle \sum \alpha_k + \frac{1}{2}\sum \beta_l - Q\chi(\Sigma), \omega_i^\vee\rangle \leq 0$: the first condition of the Seiberg bounds must be satisfied. Besides, if $\langle \alpha_k - Q, e_i\rangle \geq 0$ for some $1 \leq k \leq N$ and $1 \leq i \leq r$ we have that for ε small enough and for any $s<0$, $\int_{B(z_k,\varepsilon)} |x-z_k|^{-\langle \alpha_k, \gamma e_i\rangle} e^{\langle \gamma e_i, \mathbf{X}^{g,C}\rangle} dv_g = +\infty$ almost surely (and likewise for boundary insertions) ; see for instance the paragraph between Equations (3.9) and (3.10) in [59]. This entails that the second condition in the Seiberg bounds has to be verified too.

We now assume the conditions (1.38) to hold and show that they are sufficient for the regularized partition function to converge and to be non-trivial. We actually prove a stronger statement that may be used to extend the Seiberg bounds. Let us introduce the shortcuts $\mu^N_{i,\varepsilon}(x)$ and $\mu^D_{i,\varepsilon}(x)$ for the regularized Neumann and Dirichlet GMC densities, respectively.

Lemma 2.5.2. *Let $1 \leq i \leq r$ and assume that $\langle \alpha - Q, e_i\rangle < 0$. Then for $x \in \Sigma$ and $\rho > 0$ such that $B_g(x,2\rho) \cap \partial\Sigma = \emptyset$,*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\left(\int_{B_g(x,\rho)} e^{\langle \gamma e_i, \alpha\rangle G_{g,\varepsilon}(x,y)} \mu^N_{i,\varepsilon}(y) \mu^D_{i,\varepsilon}(y) dv_g(y) \right)^p \right] < +\infty$$

if and only if $p \in (-\infty, \frac{4}{\gamma_i^2} \wedge \frac{1}{\gamma}\langle Q-\alpha, e_i^\vee\rangle)$.

Proof. By Kahane's convexity inequality this result follows from [32][Lemma A.1]; see the proof of the next lemma. □

Lemma 2.5.3. *Let $1 \leq i \leq r$ and suppose that $\langle \beta - Q, e_i\rangle < 0$, and $s \in \partial\Sigma$. Then for all $1 > \delta > 0$ and as soon as $p \in (-\infty, \frac{2}{\gamma_i^2} \wedge \frac{1}{\gamma}\langle Q-\beta, e_i^\vee\rangle)$*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\left(\int_{B_g(s,1-\delta)\cap\Sigma} e^{\frac{1}{2}\langle \gamma e_i, \beta\rangle G_{g,\varepsilon}(s,x)} \mu^N_{i,\varepsilon}(x) \mu^D_{i,\varepsilon}(x) dv_g(x) \right)^p \right] < +\infty.$$

Proof. Suppose first that $p \geq 0$. We can get rid of the Dirichlet part of the Cardy GMC measure by using Kahane's convexity inequality (see [58, Corollary 25]). Indeed, a simple computation shows that

$$\begin{aligned}\mathbb{E}[\langle e_i, X_\varepsilon^{g,C}(x)\rangle\langle e_i, X_\varepsilon^{g,C}(y)\rangle] &= |e_i|^2 G^N_{g,\varepsilon,\varepsilon}(x,y) - 2\langle e_i, p_D e_i\rangle G_{g,\varepsilon,\varepsilon}(x,\sigma(y)) \\ &= |e_i|^2 G^D_{g,\varepsilon,\varepsilon}(x,y) + 2\langle e_i, p_N e_i\rangle G_{g,\varepsilon,\varepsilon}(x,\sigma(y))\end{aligned} \tag{2.62}$$

with $\langle e_i, p_D e_i\rangle \geq 0$ and $\langle e_i, p_N e_i\rangle \geq 0$, so that we can compare

$$\begin{aligned}&\mathbb{E}[\langle e_i, X_\varepsilon^{g,C}(x)\rangle\langle e_i, X_\varepsilon^{g,C}(y)\rangle] \leq \mathbb{E}[\langle e_i, X_\varepsilon^{g,N}(x)\rangle\langle e_i, X_\varepsilon^{g,N}(y)\rangle] + C \quad \text{and}\\ &\mathbb{E}[\langle e_i, X_\varepsilon^{g,C}(x)\rangle\langle e_i, X_\varepsilon^{g,C}(y)\rangle] \geq \mathbb{E}[\langle e_i, X_\varepsilon^{g,D}(x)\rangle\langle e_i, X_\varepsilon^{g,D}(y)\rangle] - C\end{aligned}$$

for $x, y \in B_g(s, 1-\delta) \cup \Sigma$. Since the function $t \mapsto t^p$ is convex for $t \geq 0$ when $p \geq 1$ and concave for $0 < p < 1$, Kahane's inequality implies that the expectation in the statement of the lemma can be upper-bounded by (a constant times) the same integral with only Neumann ($p \geq 1$) or Dirichlet ($0 \leq p < 1$) type density. Moreover, thanks to Lemma 2.2.3, we can work on the Poincaré disk $(B(0, 1-\delta), g_P)$. Thus, the result follows from [59, Lemmas 6.3 and 6.9]. For $p < 0$ the situation is simpler since the integral in the statement of the lemma can be lower-bounded by the same integral on $B_g(s, 1-\delta) \cap \{x \in \Sigma : d_g(x, \partial\Sigma) > \eta\}$ with $\eta > 0$ small, so the situation is the same as in the bulk. □

2.5.2 Weyl anomaly and diffeomorphism invariance (Theorem 1.2.2)

First, notice that it is enough to show the result for $g = g_0$ the uniform type 1 metric. Indeed, suppose that $g' = e^{\varphi} g$ are two metrics as in Theorem 1.2.2, there exist two smooth functions ψ and ξ such that $g' = e^{\psi} g_0$ and $g_0 = e^{\xi} g$, with $\varphi = \psi + \xi$. Then we have[9] using (2.5):

$$\begin{aligned}&\int_\Sigma (|\nabla_{g_0}\psi|^2_{g_0} + 2R_{g_0}\psi)\, dv_{g_0} + 4\int_{\partial\Sigma} k_{g_0}\psi\, d\lambda_{g_0} + \int_\Sigma (|\nabla_g \xi|^2_g + 2R_g\xi)\, dv_g + 4\int_{\partial\Sigma} k_g \xi\, d\lambda_g \\ &= \int_\Sigma (|\nabla_g\varphi|^2_g + 2R_g\varphi - 2\langle\nabla_g\psi, \nabla_g\xi\rangle_g - 2\psi\Delta_g\xi)\, dv_g + \int_{\partial\Sigma}(4k_g\varphi + 2\psi\partial_{\vec{n}_g}\xi)\, d\lambda_g \\ &= \int_\Sigma (|\nabla_g\varphi|^2_g + 2R_g\varphi)\, dv_g + \int_{\partial\Sigma} 4k_g\varphi\, d\lambda_g\end{aligned}$$

by integration by parts. This shows that we can restrict ourselves to the case of $g = e^{\varphi} g_0$ with g_0 the uniform type 1 metric. We choose to present the proof in the most general case where the field under consideration is the Cardy GFF $\mathbf{X}^{g,C}$. Remember that in this case the μ_j's associated with the short roots are set to 0. One naturally recovers the Neumann case by setting $d_N = r$. We omit the ε-regularization in the following since the convergence of the correlation functions for a general smooth metric g is

9. This property is known as the *cocycle property* of the Liouville action.

justified by the computations to come. We first use Lemmas 2.3.3 and 2.4.9 and make a change of variable $\mathbf{c} \to \mathbf{c} + m_g(\mathbf{X}^{g_0,C})$ (which is possible since the Dirichlet part of $m_g(\mathbf{X}^{g_0,C})$ is 0, so that $m_g(\mathbf{X}^{g_0,C})$ lives in $\mathfrak{a}_N$) and apply the Gauss-Bonnet theorem to restrict ourselves to

$$\int_{\mathfrak{a}_N} e^{-\langle Q,\mathbf{c}\rangle\chi(\Sigma)}\mathbb{E}\left[F(\mathbf{X}^{g_0,C}+\mathbf{c})\exp\left(-\frac{1}{4\pi}\int_\Sigma R_g\langle Q,\mathbf{X}^{g_0,C}\rangle dv_g - \frac{1}{2\pi}\int_{\partial\Sigma} k_g\langle Q,\mathbf{X}^{g_0,C}\rangle d\lambda_g \right.\right.$$
$$\left.\left. -\sum_{i=1}^r \mu_{B,i}e^{\langle\gamma e_i,\mathbf{c}\rangle}\int_\Sigma e^{(1+\frac{\gamma_i^2}{4})\varphi}e^{\langle\gamma e_i,\mathbf{X}^{g_0,C}\rangle}dv_{g_0} - \sum_{j=1}^{d_N}\mu_j e^{\langle\frac{\gamma f_j}{2},\mathbf{c}\rangle}\int_{\partial\Sigma} e^{(1+\frac{\gamma^2|f_j|^2}{4})\frac{\varphi}{2}}e^{\langle\frac{\gamma f_j}{2},\mathbf{X}^{g_0,C}\rangle}d\lambda_{g_0}\right)\right]d\mathbf{c}. \tag{2.63}$$

Now we would like to apply the Girsanov transform to the term e^Y, where (using (2.5))

$$\begin{aligned} Y &:= -\frac{1}{4\pi}\int_\Sigma R_g\langle Q,\mathbf{X}^{g_0}\rangle dv_g - \frac{1}{2\pi}\int_{\partial\Sigma}k_g\langle Q,\mathbf{X}^{g_0,C}\rangle d\lambda_g \\ &= \frac{1}{4\pi}\left(\int_\Sigma \Delta_{g_0}\varphi\langle Q,\mathbf{X}^{g_0,C}\rangle dv_{g_0} - \int_{\partial\Sigma}\partial_{\vec{n}_{g_0}}\varphi\langle Q,\mathbf{X}^{g_0,C}\rangle d\lambda_{g_0}\right). \end{aligned}$$

Hence we shall compute the variance of Y. For this we use the fact that $Q \in \mathfrak{a}_N$, together with (2.11) and integration by parts to infer that

$$\begin{aligned} \mathbb{E}[Y^2] &= \frac{|Q|^2}{16\pi^2}\left(\int_\Sigma \Delta_{g_0}\varphi(x)\left(\int_\Sigma \Delta_{g_0}\varphi(y)G^N_{g_0}(x,y)dv_{g_0}(y) - \int_{\partial\Sigma}\partial_{\vec{n}_{g_0}}\varphi(y)G^N_{g_0}(x,y)d\lambda_{g_0}(y)\right)dv_{g_0}(x)\right. \\ &\left.+\int_{\partial\Sigma}\partial_{\vec{n}_{g_0}}\varphi(y)\left(\int_{\partial\Sigma}\partial_{\vec{n}_{g_0}}\varphi(x)G^N_{g_0}(x,y)d\lambda_{g_0}(x) - \int_\Sigma \Delta_{g_0}\varphi(x)G^N_{g_0}(x,y)dv_{g_0}(x)\right)d\lambda_{g_0}(y)\right) \\ &= \frac{|Q|^2}{8\pi}\left(m_{g_0}\varphi\left(\int_\Sigma\Delta_{g_0}\varphi dv_{g_0} - \int_{\partial\Sigma}\partial_{\vec{n}_{g_0}}\varphi d\lambda_{g_0}\right) + \int_\Sigma \varphi\Delta_{g_0}\varphi dv_{g_0} - \int_{\partial\Sigma}\varphi\partial_{\vec{n}_{g_0}}\varphi d\lambda_{g_0}\right) \\ &= \frac{|Q|^2}{8\pi}\int_\Sigma |\nabla_{g_0}\varphi|^2_{g_0}dv_{g_0}. \end{aligned}$$

In the same way, we obtain

$$\begin{aligned} \mathbb{E}[Y\langle u,\mathbf{X}^{g_0,C}\rangle] &= \frac{\langle Q,p_N u\rangle}{4\pi}\left(\int_\Sigma \Delta_{g_0}\varphi(x)G^N_{g_0}(x,\cdot)dv_{g_0}(x) - \int_{\partial\Sigma}\partial_{\vec{n}_{g_0}}\varphi(x)G^N_{g_0}(x,\cdot)d\lambda_{g_0}(x)\right) \\ &= -\frac{\langle Q,p_N u\rangle}{4\pi}2\pi(\varphi - m_{g_0}(\varphi)) = -\frac{\langle Q,u\rangle}{2}(\varphi - m_{g_0}(\varphi)) \end{aligned}$$

for any $u \in \mathfrak{a}$. Notice that the above Girsanov shift coincides with the conformal anomaly of the GMC measures in (2.63), since the short roots are not taken into account in the boundary potential. Hence applying Girsanov theorem allows us to rewrite (2.63) as

$$\int_{\mathfrak{a}_N} e^{\frac{|Q|^2}{16\pi}\int_\Sigma |\nabla_{g_0}\varphi|_{g_0}^2 dv_{g_0} - \langle Q,\mathbf{c}\rangle\chi(\Sigma)} \mathbb{E}\left[F(\mathbf{c}+\mathbf{X}^{g_0,C} - \frac{Q}{2}(\varphi - m_{g_0}(\varphi)))\right.$$
$$\left.\times \exp\left(-\sum_{i=1}^r \mu_{B,i} e^{\langle\gamma e_i,\mathbf{c}+\frac{Q}{2}m_{g_0}(\varphi)\rangle}\int_\Sigma e^{\langle\gamma e_i,\mathbf{X}^{g_0,C}\rangle} dv_{g_0} - \sum_{i=1}^{d_N}\mu_i e^{\langle\frac{\gamma f_j}{2},\mathbf{c}+\frac{Q}{2}m_{g_0}(\varphi)\rangle}\int_{\partial\Sigma} e^{\langle\frac{\gamma f_j}{2},\mathbf{X}^{g_0,C}\rangle} d\lambda_{g_0}\right)\right] d\mathbf{c}.$$

Now we make the change of variables in the zero-mode $\mathbf{c}\to\mathbf{c}-\frac{Q}{2}m_{g_0}(\varphi)$ (which is possible since $Q\in\mathfrak{a}_N$) to obtain that (2.63) equals

$$e^{\frac{|Q|^2}{16\pi}\int_\Sigma |\nabla_{g_0}\varphi|_{g_0}^2 dv_{g_0} + \frac{|Q|^2}{2}\chi(\Sigma)m_{g_0}(\varphi)} \mathcal{Z}_C(\Sigma,g_0)^{-1}\langle F(\cdot - \frac{Q}{2}\varphi)\rangle_{g_0}.$$

We conclude with the Gauss-Bonnet theorem (Equation (2.4)) and Proposition 2.4.5. Finally, the fact that the partition function is diffeomorphism invariant directly follows from the fact that all the objects involved are so. Indeed, if $\psi:\Sigma\to\Sigma$ is an orientation-preserving diffeomorphism preserving $\partial\Sigma$, then for any smooth Riemannian metric g we have $R_{\psi^*g}=R_g\circ\psi$ and $k_{\psi^*g}=k_g\circ\psi$, while $\mathbf{X}^{\psi^*g,C}\overset{\text{(law)}}{=}\mathbf{X}^{g,C}\circ\psi$ since we have the following diffeomorphism invariance satisfied by the Green's functions:

$$G^N_{\psi^*g}(x,y)=G^N_g(\psi(x),\psi(y))\quad G^D_{\psi^*g}(x,y)=G^D_g(\psi(x),\psi(y)).$$

3 Ward identities in the A_2 Toda theory on the half-plane

Sommaire

3.1 Introduction

In this chapter and the next one, we specialize the study to the Toda CFT based on the root system A_2, or equivalently the Lie algebra $\mathfrak{sl}_3\mathbb{C}$, defined on the complex upper half-plane $\mathbb{H}$. We also specify Neumann boundary conditions for the underlying Toda field. The present chapter is dedicated to proving that the so-called Ward identities are valid for this model, which constitutes a first manifestation of the higher-spin symmetry enjoyed by the latter. For this we rely on the probabilistic construction described in the previous chapter (that is easily adapted to the case of $\mathbb{H}$), and exhibit an explicit representation of the W-algebra W_3 (or WA_2) in the model, using a regularization procedure allowing us to define the W-descendants in a weak sense on the boundary $\mathbb{R}$. The content of this chapter is taken from [26], co-authored with B. Cerclé. An introduction to the method and the statements of the main results of this chapter is found in Subsection 1.2.2.

Let us first specify the background notions described in Section 2.2, Subsections 2.3.1 and 2.3.2, to the present context.

3.1.1 The general framework

3.1.1.1 Gaussian Free Fields

In the setting of the A_2 Toda theory on $\mathbb{H}$, with Neumann boundary conditions, the probabilistic definition of the Toda field as performed in the first chapter (taken from the article [28]) is based on the consideration of a vectorial Gaussian Free Field $\mathbf{X}:\overline{\mathbb{H}}\to\mathbb{R}^2$. This GFF is a centered Gaussian random distribution, that we define through its covariance kernel, given for any $u,v\in\mathbb{R}^2$ and x,y in $\overline{\mathbb{H}}$ by[1]

$$\mathbb{E}\left[\langle u,\mathbf{X}(x)\rangle\langle v,\mathbf{X}(y)\rangle\right]=\langle u,v\rangle G(x,y)$$

with the Neumann Green's function G[2] given by

$$G(x,y):=\ln\frac{1}{\left|x-y\right|\left|x-\bar{y}\right|}+2\ln|x|_++2\ln\left|y\right|_+$$

and where we have used the notation $|x|_+:=\max(|x|,1)$. The Green's function G can be alternatively written as

$$G(x,y)=G_{\hat{\mathbb{C}}}(x,y)+G_{\hat{\mathbb{C}}}(x,\bar{y}),\quad\text{with}\quad G_{\hat{\mathbb{C}}}(x,y)=\ln\frac{1}{\left|x-y\right|}+\ln|x|_++\ln\left|y\right|_+.$$

1. We use the same abuse of notation as explained in Subsection 2.3.1.
2. Since we will only deal with this particular function, we omit the subscript N here.

Here $G_{\hat{\mathbb{C}}}$ is the Green's function on the Riemann sphere. As we have already seen the object thus defined is not a function but rather an element of the Sobolev space with negative index $H^{-1}(\mathbb{H} \to \mathbb{R}^2)$. We can however construct a regular function out of it by using a regularization procedure, and we choose to consider a smooth mollifier η and define for any $\rho > 0$[3]:

$$\mathbf{X}_\rho(x) := \int_{\mathbb{H}} \mathbf{X}_\rho(y)\eta_\rho(x-y)dyd\bar{y} \tag{3.1}$$

where we have set $\eta_\rho(\cdot) := \frac{1}{\rho^2}\eta(\frac{\cdot}{\rho})$. This is a standard procedure, see e.g. [30].

3.1.1.2 Gaussian Multiplicative Chaos

The Gaussian Multiplicative Chaos measures are defined on either $\mathbb{H}$ or its boundary $\mathbb{R}$, as described in the previous chapter, by:

$$M_{\gamma e_i}(d^2x) := \lim_{\rho\to 0} \rho^{\gamma^2} e^{\langle \gamma e_i, \mathbf{X}_\rho(x)\rangle} dxd\bar{x}; \quad M^{\partial}_{\gamma e_i}(dx) := \lim_{\rho\to 0} \rho^{\frac{\gamma^2}{2}} e^{\langle \frac{\gamma}{2} e_i, \mathbf{X}_\rho(x)\rangle} dx$$

where these limits hold in probability, in the sense of weak convergence of measures [13, 85]. In the above the $(e_i)_{i=1,2}$ are a choice of simple roots of $\mathfrak{sl}_3\mathbb{C}$ (see just after), and as such are elements of $\mathbb{R}^2$ such that $|e_i|^2 = 2$, thus we need to make the assumption that $\gamma < \sqrt{2}$ in order for these random measures to be well-defined.

3.1.1.3 On the Lie algebra $\mathfrak{sl}_3\mathbb{C}$

For details about simple complex Lie algebras, we refer to Section 2.2.2 (and references therein). We denote by (e_1, e_2) the (in fact, a choice of) simple roots of the Lie algebra $\mathfrak{sl}_3\mathbb{C}$. They satisfy the property that

$$\left(\langle e_i, e_j\rangle\right)_{i,j} := A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

with A the Cartan matrix of $\mathfrak{sl}_3$. We will also consider the dual basis of (e_1, e_2) which we denote by (ω_1, ω_2). Explicit computations show that we have

$$\omega_1 = \frac{2e_1 + e_2}{3} \quad \text{and} \quad \omega_2 = \frac{e_1 + 2e_2}{3}.$$

In the case of $\mathfrak{sl}_3\mathbb{C}$, the Weyl vector is given by[4]

$$\boldsymbol{\rho} = \omega_1 + \omega_2 = e_1 + e_2, \quad \text{so that} \quad \langle \boldsymbol{\rho}, e_i\rangle = 1 \quad \text{for} \quad i = 1, 2.$$

3. The reason for changing the notation for the regularization parameter compared to the previous chapter will be clear later, when we will introduce extra regularization procedures.

4. For simply-laced Lie algebras, $\boldsymbol{\rho} = \boldsymbol{\rho}^\vee$.

In this setting the background charge Q can be expressed in terms of the Weyl vector and the coupling constant $\gamma \in (0, \sqrt{2})$ as

$$Q = \left(\gamma + \frac{2}{\gamma}\right)\boldsymbol{\rho}.$$

We will also consider the fundamental weights in the first fundamental representation π_1 of $\mathfrak{sl}_3\mathbb{C}$ with the highest weight ω_1:

$$h_1 := \frac{2e_1 + e_2}{3}, \quad h_2 := \frac{-e_1 + e_2}{3}, \quad h_3 := -\frac{e_1 + 2e_2}{3}. \tag{3.2}$$

Finally we introduce the following notations that will enter the definitions of the higher-spin current:

$$\begin{aligned}
&B(u, v) := (h_2 - h_1)(u)h_1(v) + (h_3 - h_2)(u)h_3(v);\\
&C(u, v, w) := h_1(u)h_2(v)h_3(w) + h_1(v)h_2(w)h_3(u) + h_1(w)h_2(u)h_3(v).
\end{aligned}$$

where $h_i(u)$ is the pairing between $\mathfrak{a}$ and its dual $\mathfrak{a}^*$ —both isomorphic to $\mathbb{R}^2$— given by $h_i(u) = \langle h_i, u\rangle$. We also introduce here for future use the following shortcut:

$$C^\sigma(u, v, w) := C(u, v, w) + C(u, w, v).$$

3.2 Description of the method and some technical estimates

3.2.1 Definition of the (regularized) correlation functions and descendant fields

In this section we sketch the construction of the Toda CFT on the upper half-plane $\mathbb{H}$ as detailed in the previous chapter. We also introduce the regularized descendant fields that we wish to define in order to prove the Ward identities.

3.2.1.1 Toda correlation functions on $\mathbb{H}$

We first define the *Toda field* on $\overline{\mathbb{H}}$ to be given by[5]

$$\Phi := \mathbf{X} - 2Q\ln|\cdot|_+ + \boldsymbol{c},$$

where $\mathbf{X}$ is the GFF defined previously, and $\boldsymbol{c}$ is distributed according to the Lebesgue measure on $\mathbb{R}^2$. We denote by Φ_ρ the regularized Toda field in the sense of Equation (3.1). The Vertex Operators are observables of the Toda field that are of fundamental importance. They depend on an insertion point $z \in \mathbb{H}$ or $s \in \mathbb{R}$ as well as a weight α

5. This definition amounts to considering the Toda CFT on $\mathbb{H}$ with the metric $g(z) = d^2z/|z|_+^4$, as it is often done.

or β in $\mathbb{R}^2$, and are formally defined by considering the functionals

$$V_\alpha(z)[\Phi] := e^{\langle \alpha, \Phi(z)\rangle} \quad \text{for } z\in\mathbb{H}, \text{while for } t\in\mathbb{R} \quad V_\beta(s)[\Phi] := e^{\langle \frac{\beta}{2}, \Phi(s)\rangle}.$$

Based on these observables we can form the correlation functions of Vertex Operators, which depend on insertion points $(z_1,...,z_N,s_1,...,s_M)\in\mathbb{H}^N\times\mathbb{R}^M$, to which are associated weights
$(\alpha_1,...,\alpha_N,\beta_1,...,\beta_M)\in(\mathbb{R}^2)^{N+M}$. The corresponding correlation function is then formally given by

$$\langle \prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle$$

where the average is made with respect to the law of the Toda field. We now recall how to provide a rigorous meaning to the latter.

For the sake of simplicity we write $\boldsymbol{z}=(z_k)_{k=1,...,2N+M}$ for $(z_1,...,z_N,\bar z_1,...,\bar z_N,s_1,...,s_M)$ and $\boldsymbol{\alpha}=(\alpha_k)_{k=1,...,2N+M}$ for $(\alpha_1,...,\alpha_N,\alpha_1,...,\alpha_N,\beta_1,...,\beta_M)$. In order to define the correlations we also introduce the bulk cosmological constants $\mu_{B,i}>0$ for $i=1,2$, as well as the boundary cosmological constants $\mu_{i,l}$, which we choose in $\mathbb{C}$ but with the additional assumption that $\mathrm{Re}(\mu_{i,l})\geq 0$, and set μ_i to be the piecewise constant measure on $\mathbb{R}$ given by

$$\mu_i(dx)=\sum_{l=1}^M \mu_{i,l+1}\mathbb{1}_{(s_l,s_{l+1})}(x)dx$$

where by convention $s_0=-\infty$, $s_{M+1}=+\infty$ and $\mu_{M+1}=\mu_1$.

We can then make sense of the correlation functions using a regularization procedure. Namely, for a suitable functional F, we can first consider the following quantity at the regularized level[6]:

$$\begin{aligned}\langle F[\Phi]\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} := \int_{\mathbb{R}^2} e^{-\langle Q,\boldsymbol{c}\rangle}\mathbb{E}\left[F[\Phi_\rho]\prod_{k=1}^N V_{\alpha_k,\rho}(z_k)\prod_{l=1}^M V_{\beta_l,\rho}(s_l)\right.\\ \left.\times\exp\left(-\sum_{i=1}^2\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\rho^{\gamma^2}e^{\langle\gamma e_i,\Phi_\rho(x)\rangle}dxd\bar x+\int_{\mathbb{R}_\varepsilon}\rho^{\frac{\gamma^2}{2}}e^{\langle\frac{\gamma}{2}e_i,\Phi_\rho(x)\rangle}\mu_i(dx)\right)\right]d\boldsymbol{c}\end{aligned} \tag{3.3}$$

where

$$V_{\alpha_k,\rho}(z_k):=\rho^{\frac{|\alpha_k|^2}{2}}e^{\langle\alpha_k,\Phi_\rho(z_k)\rangle}\quad\text{and}\quad V_{\beta_l,\rho}(s_l):=\rho^{\frac{|\beta_l|^2}{4}}e^{\langle\frac{\beta_l}{2},\Phi_\rho(s_l)\rangle}.$$

Here the domains of integration are defined by

$$\mathbb{H}_{\delta,\varepsilon}=(\mathbb{H}+\mathrm{i}\delta)\setminus\bigcup_{k=1}^N B(z_k,\varepsilon)\quad\text{and}\quad\mathbb{R}_\varepsilon=\mathbb{R}\setminus\bigcup_{l=1}^M(s_l-\varepsilon,s_l+\varepsilon),$$

6. We choose not to renormalize the correlation functions using the partition function of the GFF as we did previously, since these constant quantities are not relevant for the present exposition.

with ε and δ such that all the $B(z_k,\varepsilon)$ lie in $\mathbb{H}_{\delta,\varepsilon}$ and are disjoint, and all the intervals $(s_l-\varepsilon,s_l+\varepsilon)$ are disjoint. (see Figure 3.1).

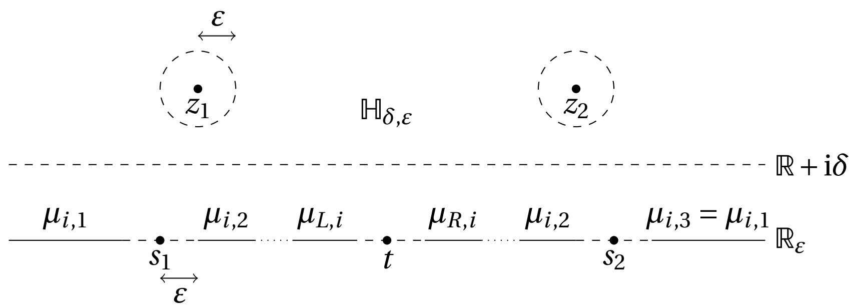

Figure 3.1 – The domains $\mathbb{H}_{\delta,\varepsilon}$ and $\mathbb{R}_\varepsilon$

Note that applying a Girsanov transform (Theorem 2.5.1) allows one to write the regularized correlation function in the following form:

$$\langle F[\Phi]\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}=C(\boldsymbol{z},\boldsymbol{\alpha})\int_{\mathbb{R}^2}e^{\langle \boldsymbol{s},\boldsymbol{c}\rangle}\mathbb{E}\left[F[\Phi_\rho+H]\right.\tag{3.4}$$
$$\left.\times\exp\left(-\sum_{i=1}^2\mu_{B,i}e^{\langle\gamma e_i,\boldsymbol{c}\rangle}\int_{\mathbb{H}_{\delta,\varepsilon}}Z_{i,\rho}\rho^{\gamma^2}e^{\langle\gamma e_i,\mathbf{X}_\rho(x)\rangle}dxd\bar x+e^{\langle\frac{\gamma}{2}e_i,\boldsymbol{c}\rangle}\int_{\mathbb{R}_\varepsilon}Z^\partial_{i,\rho}\rho^{\frac{\gamma^2}{2}}e^{\langle\frac{\gamma}{2}e_i,\mathbf{X}_\rho(x)\rangle}\mu_i(dx)\right)\right]d\boldsymbol{c}(1+o(1))$$

where $\boldsymbol{s}:=\sum\alpha_k+\frac12\sum\beta_l-Q$, and where we have set:

$$C(\boldsymbol{z},\boldsymbol{\alpha})=\prod_{k<l}|z_k-z_l|^{-\langle\alpha_k,\alpha_l\rangle}\prod_{k=1}^M|z_k-\bar z_k|^{\frac{|\alpha_k|^2}{2}},\quad H=\sum_{k=1}^{2N+M}\alpha_kG_{\hat{\mathbb{C}}}(\cdot,z_k),$$

$$Z_{i,\rho}=\prod_{k=1}^{2N+M}\left(\frac{|x|_+}{|z_k-x|}\right)^{\langle\gamma e_i,\alpha_k\rangle}\quad\text{and}\quad Z^\partial_{i,\rho}=\prod_{k=1}^{2N+M}\left(\frac{|x|_+}{|z_k-x|}\right)^{\langle\frac{\gamma}{2}e_i,\alpha_k\rangle}.$$

If we choose $F=1$, then in the same way as in Theorem 1.2.1 we can show that under some conditions on the weights, the regularized correlation functions converge as ρ,ε and then δ tend to 0, and are non-trivial in the limit. The difference here is that the boundary cosmological constants are no longer real but are rather chosen in $\mathbb{C}$, but this issue is addressed in [24]. Eventually, one shows that the limiting correlation functions exist and are non-trivial if and only if the following conditions hold for all $i=1,2$:

$$\langle\boldsymbol{s},\omega_i\rangle>0\,;\ \langle\alpha_k-Q,e_i\rangle<0\quad\text{for all }k=1,\dots,2N+M;$$

which corresponds to the Seiberg bounds (Theorem 1.2.1), supplemented by

$$\mu_{B,i}>0\text{ and }\mathrm{Re}\ \mu_{i,l}\geq 0\text{ for all }l=1,\dots,M.$$

We denote by $\mathscr{A}_{N,M}$ the set of such weights satisfying the Seiberg bounds. We define similarly the limiting correlation function for general F by the same procedure as soon as it makes sense, and denote

$$\langle F(\Phi)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle := \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\lim_{\rho\to 0}\langle F(\Phi)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}.$$

The correlation functions obey the following KPZ identity, which is key to get rid of the metric dependent terms, when performing Gaussian integration by parts, as it will be seen below.

Lemma 3.2.1 (KPZ identity)**.** *For (α_k,β_l) in $\mathscr{A}_{N,M}$, it holds that*

$$\begin{aligned}\boldsymbol{s}\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} = \sum_{i=1}^2\Bigg(&\mu_{B,i}\gamma e_i\int_{\mathbb{H}_{\delta,\varepsilon}}\langle V_{\gamma e_i}(x)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}dxd\bar{x}\\ &+\frac{\gamma e_i}{2}\int_{\mathbb{R}_\varepsilon}\langle V_{\gamma e_i}(x)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}\mu_i(dx)\Bigg).\end{aligned}$$

Proof. In the definition of the correlation function (3.3), make the change of variable $\boldsymbol{c}\to\boldsymbol{c}+\frac{\boldsymbol{\rho}}{\gamma}\log\mu_{B,1}$ to obtain

$$\begin{aligned}\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} &= \mu_{B,1}^{-\frac{\langle \boldsymbol{s},\boldsymbol{\rho}\rangle}{\gamma}}\int_{\mathbb{R}^2}e^{-\langle \boldsymbol{s},\boldsymbol{c}\rangle}\mathbb{E}\Bigg[\prod_{k=1}^N\widetilde{V}_{\alpha_k,\rho}(z_k)\prod_{l=1}^M\widetilde{V}_{\beta_l,\rho}(s_l)\\ &\times\exp\Bigg(-\sum_{i=1}^2\frac{\mu_{B,i}}{\mu_{B,1}}\int_{\mathbb{H}_{\delta,\varepsilon}}\rho^{\gamma^2}e^{\langle\gamma e_i,\Phi_\rho(x)\rangle}dxd\bar{x}+\mu_{B,1}^{-\frac{1}{2}}\int_{\mathbb{R}_\varepsilon}\rho^{\frac{\gamma^2}{2}}e^{\langle\frac{\gamma}{2}e_i,\Phi_\rho(x)\rangle}\mu_i(dx)\Bigg)\Bigg]d\boldsymbol{c}\end{aligned}$$

where the notation $\widetilde{V}$ refers to the Vertex Operator associated to the field $\Phi_\rho-\boldsymbol{c}$ (that is, without the zero-mode). Now, differentiating in $\mu_{B,1}$ and performing the opposite shift in the zero-mode yields

$$\begin{aligned}\frac{\partial}{\partial\mu_{B,1}}\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} &= -\frac{\langle \boldsymbol{s},\boldsymbol{\rho}\rangle}{\gamma\mu_{B,1}}\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}\\ &+\frac{\mu_{B,2}}{\mu_{B,1}}\int_{\mathbb{H}_{\delta,\varepsilon}}\langle V_{\gamma e_i}(x)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}dxd\bar{x}\\ &+\sum_{i=1}^2\frac{1}{2\mu_{B,1}}\int_{\mathbb{R}_\varepsilon}\langle V_{\gamma e_i}(x)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}\mu_i(dx).\end{aligned}$$

On the other hand, by differentiating directly the correlation function in $\mu_{B,1}$ one obtains

$$\frac{\partial}{\partial\mu_{B,1}}\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} = -\langle V_{\gamma e_1}(x)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}$$

hence the result by identification. □

It is also worth mentioning here that the correlation functions obey the following conformal covariance property that will be used in Subsection 3.3.4, and that is a consequence of the Weyl anomaly and of diffeomorphism invariance (Theorem 1.2.2):

Proposition 3.2.2. *Let $\psi \in PSL(2,\mathbb{R})$ be a Möbius transform of the upper-half plane $\mathbb{H}$. Then for $\boldsymbol{\alpha}$ in $\mathscr{A}_{N,M}$ and a suitable functional F:*

$$\begin{aligned}&\langle F[\Phi\circ\psi + Q\ln\left|\psi'\right|]\prod_{k=1}^N V_{\alpha_k}(\psi(z_k))\prod_{l=1}^M V_{\beta_l}(\psi(s_l))\rangle\\&=\prod_{k=1}^N\left|\psi'(z_k)\right|^{-2\Delta_{\alpha_k}}\prod_{l=1}^M\left|\psi'(s_l)\right|^{-\Delta_{\beta_l}}\langle F[\Phi]\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle.\end{aligned}\tag{3.5}$$

Proof. Any holomorphic change of coordinate $z\to\psi(z)$ acts on the metric as a Weyl rescaling $|dz|^2\to\left|\psi'(z)\right|^2|dz|^2$. Now combining invariance by diffeomorphism and the Weyl anomaly formula we have for a Möbius transform ψ

$$\begin{aligned}&\langle F[\Phi\circ\psi + Q\ln\left|\psi'\right|]\prod_{k=1}^N V_{\alpha_k}(\psi(z_k))\prod_{l=1}^M V_{\beta_l}(\psi(s_l))\rangle = e^{\frac{1+3|Q|^2}{48\pi}\int_{\mathbb{H}}\left|\nabla\ln|\psi'(z)|^2\right|^2 dz d\bar{z}}\\&\times\prod_{k=1}^N\left|\psi'(z_k)\right|^{-2\Delta_{\alpha_k}}\prod_{l=1}^M\left|\psi'(s_l)\right|^{-\Delta_{\beta_l}}\langle F[\Phi]\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle.\end{aligned}\tag{3.6}$$

and the proof is complete since the integral term in the Weyl anomaly vanishes for holomorphic functions. □

Finally, we introduce the shorthand

$$\mathbf{V}:=\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l).$$

in order to lighten the presentation.

3.2.1.2 Descendant fields and currents

Let us denote by Φ the Toda field. The way we define the regularized descendant fields is by means of expressions of the form

$$\langle F[\Phi]V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}$$

where $F[\Phi]$ is a polynomial in the derivatives of Φ evaluated at t. The expression of this polynomial is key as it encodes the symmetries of the model, and is derived from the currents $\mathbf{T}$ and $\mathbf{W}$ whose modes generate the corresponding representation of the W_3 algebra. Before providing an explicit expression for the descendants recall that the currents are defined by

$$\mathbf{T}(z)[\Phi]=\langle Q,\partial^2\Phi(z)\rangle-\langle\partial\Phi(z),\partial\Phi(z)\rangle$$

for the stress-energy tensor, while for the higher-spin current

$$\mathbf{W}(z)[\Phi] = q^2 h_2(\partial^3\Phi(z)) - 2qB(\partial^2\Phi(z),\partial\Phi(z)) - 8C(\partial\Phi(z),\partial\Phi(z),\partial\Phi(z)).$$

Here z stands for a bulk point inside $\mathbb{H}$ or for a boundary insertion on $\mathbb{R}$, in which case the complex derivative is replaced by the usual (real) derivative.

In the framework of Vertex Operators Algebras, the descendants of the Vertex Operators V_α can be found using the Operator Product Expansion between the above currents and V_α, in the sense that we have a formal expansion as $z \to w$:

$$\mathbf{W}(z)V_\alpha(w) = \sum_{n\geq 0}(z-w)^{n-3}\mathbf{W}_{-n}V_\alpha(w)$$

and likewise for $\mathbf{T}$. The translation between the two languages has been worked out *e.g.* in [22]; informally speaking in order to obtain a probabilistic interpretation of the above expansion we first shift $\Phi(z)$ to $\Phi_\alpha = \Phi + \alpha \ln \frac{1}{|\cdot - w|}$ and then formally Taylor expand the field $\Phi(z)$ as $\sum_{n\geq 0}\frac{(z-w)^n}{n!}\partial^n\Phi(w)$. By doing so we arrive at the following expressions for the descendant fields associated to the stress-energy tensor:

$$\begin{aligned}&\mathbf{L}_{-n}V_\alpha[\Phi] := \mathbf{L}^\alpha_{-n}[\Phi]V_\alpha[\Phi], \quad \text{with} \quad \mathbf{L}^\alpha_{-1}[\Phi] := \langle\alpha,\partial\Phi\rangle \qquad \text{while for } n\geq 2\\ &\mathbf{L}^\alpha_{-n}[\Phi] = \left\langle (n-1)Q+\alpha, \frac{\partial^n\Phi}{(n-1)!}\right\rangle - \sum_{i=0}^{n-2}\left\langle \frac{\partial^{i+1}\Phi}{i!}, \frac{\partial^{n-i-1}\Phi}{(n-2-i)!}\right\rangle\end{aligned} \tag{3.7}$$

The expression of the descendants associated to the higher-spin current is slightly more involved:

$$\begin{aligned}&\mathbf{W}_{-n}V_\alpha[\Phi] := \mathbf{W}^\alpha_{-n}[\Phi]V_\alpha[\Phi], \quad \text{with} \quad \mathbf{W}^\alpha_{-1}[\Phi] := -qB(\alpha,\partial\Phi) - 2C(\alpha,\alpha,\partial\Phi),\\ &\mathbf{W}^\alpha_{-2}[\Phi] := q\left(B(\partial^2\Phi,\alpha) - B(\alpha,\partial^2\Phi)\right) - 2C(\alpha,\alpha,\partial^2\Phi) + 2C^\sigma(\alpha,\partial\Phi,\partial\Phi)\\ &\mathbf{W}^\alpha_{-n}[\Phi] := (n-1)(n-2)q^2\langle h_2,\Phi^{(n)}\rangle + q\left((n-1)B(\Phi^{(n)},\alpha) - B(\alpha,\Phi^{(n)})\right)\\ &-2C(\alpha,\alpha,\Phi^{(n)}) + \sum_{i=0}^{n-2}\left(4C(\Phi^{(i+1)},\Phi^{(n-i-1)},\alpha) - 2iqB(\Phi^{(i+1)},\Phi^{(n-1-i)})\right)\\ &-\frac{8}{3}\sum_{i=0}^{n-3}\sum_{j=0}^{i}C(\Phi^{(j+1)},\Phi^{(i-j+1)},\Phi^{(n-i-2)})\end{aligned} \tag{3.8}$$

where we have set $\Phi^{(n)}(z) := \frac{\partial^n\Phi(z)}{(n-1)!}$ and recall that $C^\sigma(u,v,w) = C(u,v,w) + C(u,w,v)$.

3.2.1.3 Regularized descendant fields

Products of derivatives of the Toda field do not make sense as such since the GFF only exists in the distributional sense. However using its regularization Φ_ρ as done above we can actually make sense of its derivatives, while for the product we will rely on Wick products instead. To be more specific, we set for positive ρ

$$\begin{aligned} &\mathbf{L}_{-n}V_\alpha(z)[\Phi_\rho] := \quad :\mathbf{L}^\alpha_{-n}[\Phi]:\ V_{\alpha,\rho}[\Phi](z) \quad \text{where}\\ &:\mathbf{L}^\alpha_{-n}[\Phi]: \quad := \left\langle (n-1)Q+\alpha, \frac{\partial^n\Phi}{(n-1)!}\right\rangle - \sum_{i=0}^{n-2} :\left\langle \frac{\partial^{i+1}\Phi}{i!}, \frac{\partial^{n-i-1}\Phi}{(n-2-i)!}\right\rangle : \end{aligned} \tag{3.9}$$

with $:\cdot:$ denoting the Wick product of the Gaussian random variables involved (see [62, Chapter III] for more details). For instance for centered Gaussian variables we have

$$:\xi_1\xi_2 := \xi_1\xi_2 - \mathbb{E}[\xi_1\xi_2], \quad \text{and} \quad :\xi_1\xi_2\xi_3 := \xi_1\xi_2\xi_3 - \xi_1\mathbb{E}[\xi_2\xi_3] - \xi_2\mathbb{E}[\xi_3\xi_1] - \xi_3\mathbb{E}[\xi_1\xi_2].$$

The regularized descendant fields are then defined with correlation functions by considering

$$\langle \mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}.$$

Of course the same procedure remains valid when we consider the **W** descendant fields instead of the Virasoro ones.

3.2.2 Method for defining the descendant fields

Our goal is to define the descendant fields associated with the (boundary) vertex operators V_β, which we will denote $\mathbf{L}_{-n}V_\beta$ or $\mathbf{W}_{-m}V_\beta$. To do so we will define expressions of the form

$$\langle \mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle \quad \text{and} \quad \langle \mathbf{W}_{-m}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle$$

where the weights are chosen in such a way that the correlation function $\langle V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle$ is well-defined, that is $(\beta,\boldsymbol{\alpha})\in\mathscr{A}_{N,M+1}$.

3.2.2.1 Gaussian integration by parts

In order to make sense of such quantities we will first define them at the regularized level, that is for fixed $\delta,\varepsilon,\rho>0$ we look at the well-defined quantities

$$\langle \mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} \quad \text{and} \quad \langle \mathbf{W}_{-m}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}.$$

For this purpose we will make use of the following Gaussian integration by parts formula to provide an explicit expression for such quantities:

Lemma 3.2.3. *Assume that $\boldsymbol{\alpha}$ belongs to $\mathscr{A}_{N,M}$. Then for any $t\in\mathbb{R}\setminus\{s_1,\cdots,s_M\}$, p a* *positive integer and $u \in \mathbb{R}^2$:*

$$\lim_{\rho\to 0}\Big\langle \frac{\langle u,\partial^p\Phi(t)\rangle}{(p-1)!}\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\Big\rangle_{\rho,\varepsilon,\delta} = \sum_{k=1}^{2N+M}\frac{\langle u,\alpha_k\rangle}{2(z_k-t)^p}\langle \mathbf{V}\rangle_{\varepsilon,\delta}$$
$$-\sum_{i=1}^2\int_{\mathbb{R}_\varepsilon}\frac{\langle u,\gamma e_i\rangle}{2(x-t)^p}\langle V_{\gamma e_i}(x)\mathbf{V}\rangle_{\varepsilon,\delta}\mu_i(dx)+\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\frac{\langle u,\gamma e_i\rangle}{2(x-t)^p}+\frac{\langle u,\gamma e_i\rangle}{2(\bar{x}-t)^p}\right)\langle V_{\gamma e_i}(x)\mathbf{V}\rangle_{\varepsilon,\delta}dxd\bar{x}.$$

Proof. This statement can be adapted from [24, Lemma 2.6], the only difference being that weights now belong to $\mathbb{R}^2$ rather than $\mathbb{R}$, but all the arguments employed there readily applies in this setting by using the KPZ identity from Lemma 3.2.1 (in the bulk case this is done in [25, Lemma 3.3]). □

Lemma 3.2.3 can be generalized in a straightforward way to the following equality for $p_1,\cdots,p_m$ positive integers, using Wick products as considered above:

$$\begin{aligned}
&\lim_{\rho\to 0}\frac{1}{(p_1-1)!}\Big\langle :\prod_{l=1}^m\langle u_l,\partial^{p_l}\Phi(t)\rangle:\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\Big\rangle_{\rho,\varepsilon,\delta}\\
&=\sum_{k=1}^{2N+M}\frac{\langle u_1,\alpha_k\rangle}{2(z_k-t)^{p_1}}\langle:\prod_{l=2}^m\langle u_l,\partial^{p_l}\rangle\Phi(t):\mathbf{V}\rangle_{\varepsilon,\delta}\\
&-\sum_{i=1}^2\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\frac{\langle u_1,\gamma e_i\rangle}{2(x-t)^{p_1}}+\frac{\langle u_1,\gamma e_i\rangle}{2(\bar{x}-t)^{p_1}}\right)\langle:\prod_{l=2}^m\langle u_l,\partial^{p_l}\Phi(t)\rangle:V_{\gamma e_i}(x)\mathbf{V}\rangle_{\varepsilon,\delta}dxd\bar{x}\\
&-\sum_{i=1}^2\int_{\mathbb{R}_\varepsilon}\frac{\langle u_1,\gamma e_i\rangle}{2(x-t)^{p_1}}\langle:\prod_{l=2}^m\langle u_l,\partial^{p_l}\Phi(t)\rangle:V_{\gamma e_i}(x)\mathbf{V}\rangle_{\varepsilon,\delta}\mu_i(dx).
\end{aligned}\tag{3.10}$$

3.2.2.2 Identification of the singular terms

As explained above the expression of the regularized correlation functions with descendant fields involve integrals over the domains $\mathbb{H}_{\delta,\varepsilon}$ and $\mathbb{R}_\varepsilon$, where the integrand is given by regularized correlation functions against some explicit rational functions. Our goal is to show that such expressions can be written as a sum of two terms, the first one having a well-defined limit that depends analytically in the weights $\boldsymbol{\alpha}$ (which will be called (P)-class below), while the second one may diverge depending on the value of β: this second term will be a “remainder term". Our goal is to identify this remainder term and define the descendant field as the limit of the (P)-class term in the two-terms expansion of the regularized correlation functions.

In order to identify this remainder term we will split the integrals coming from Gaussian integration by parts between regular and singular parts, that is away and around the insertion t, and introduce in this prospect the following notations. Define the integration over the full regularized domains as

$$I_{tot}[F^i(t;x)\Psi_i(x)]:=\sum_{i=1,2}\int_{\mathbb{R}_\varepsilon}F^i(t;s)\Psi_i(s)\mu_i(ds)+\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(F^i(t;x)+F^i(t;\bar{x})\right)\Psi_i(x)dxd\bar{x},$$

or in operator-like notation

$$I_{tot}[F^i(t;x)\Psi_i(x)] = \sum_{i=1,2} I^i_{tot}(F^i(t;\cdot),\Psi_i)$$

where I^i_{tot} is the bilinear operator

$$I^i_{tot} = \int_{\mathbb{R}_\varepsilon} \cdot\otimes\cdot\,\mu_i(ds) + \mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}} \left(\cdot + \cdot\circ\mathrm{conj}\right)\otimes\cdot\, dxd\bar{x}.$$

Now, for $r>0$ independent of ρ,ε,δ small enough, define the subdomains $\mathbb{R}^1_\varepsilon := \mathbb{R}_\varepsilon\cap B(t,r)$, $\mathbb{H}^1_{\delta,\varepsilon} := \mathbb{H}_{\delta,\varepsilon}\cap B(t,r)$, $\mathbb{R}^c_\varepsilon := \mathbb{R}_\varepsilon\setminus\mathbb{R}^1_\varepsilon$ and $\mathbb{H}^c_{\delta,\varepsilon} := \mathbb{H}_{\delta,\varepsilon}\setminus\mathbb{H}^1_{\delta,\varepsilon}$. We define in the same way integration over these subdomains, and denote them by I_{sing} for the integrals over $\mathbb{R}^1_\varepsilon\times\mathbb{H}^1_{\delta,\varepsilon}$, and I_{reg} for the ones over $\mathbb{R}^c_\varepsilon\times\mathbb{H}^c_{\delta,\varepsilon}$. In the above $(F^i(t;\cdot))_{i=1,2}$ is a pair of functions coming from the Gaussian integration by parts formula from Lemma 3.2.3 and $\Psi_i(x) = \langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\rho,\varepsilon,\delta}$. The iteration of this formula naturally leads to the generalization of the above for functions of several variables $(F^{i_1,\cdots,i_p}(t;x_1,\cdots,x_p))_{i_1,\dots,i_p\in\{1,2\}}$:

$$I^{p_1}_{tot}\times I^{p_2}_{sing}\times I^{p_3}_{reg}[F^{i_1,\dots,i_p}(t;x_1,\dots,x_p)\Psi_{i_1,\dots,i_p}(x_1,\dots,x_p)] :=$$
$$\sum_{i_1,\dots,i_p\in\{1,2\}}\left(\prod_{k=1}^{p_1} I^{i_k,(k)}_{tot}\right)\left(\prod_{k=p_1+1}^{p_1+p_2} I^{i_k,(k)}_{sing}\right)\left(\prod_{k=p_1+p_2+1}^{p} I^{i_k,(k)}_{reg}\right)(F^{i_1,\dots,i_p}(t;\cdot),\Psi_{i_1,\dots,i_p})$$

where $p_1+p_2+p_3=p$, $I^{i_k,(k)}_{tot}$ (and similarly for $I^{i_k,(k)}_{sing}$, $I^{i_k,(k)}_{reg}$) is the bilinear operator

$$I^{i_k,(k)}_{tot} = \int_{\mathbb{R}_\varepsilon}\cdot\otimes\cdot\,\mu_{i_k}(ds_k) + \mu_{B,i_k}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\cdot+\cdot\circ\mathrm{conj}_k\right)\otimes\cdot\, dxd\bar{x}_k$$

with conj_k denoting complex conjugation of the k'th variable, and

$$\Psi_{i_1,\cdots,i_p}(x_1,\cdots,x_p) := \langle V_{\gamma e_{i_1}}(x_1)\cdots V_{\gamma e_{i_p}}(x_p)V_\beta(t)\mathbf{V}\rangle_{\rho,\varepsilon,\delta}. \tag{3.11}$$

We stress that in the above generalization, in the definitions of the domains of integration for I_{sing} and I_{reg}, the different radii r that separate the near-t part from the rest are independent. Eventually, we denote for $i=1,2$,

$$\hat{i} := 3-i. \tag{3.12}$$

3.2.2.3 Symmetrization identities and Stoke's formula

In order to make explicit the singular behaviour of such integrals and discard "fake singularities" we will rely on "symmetrization identites", that is equalities for noncoinciding points

$$\sum_{i=1}^{n-1} \frac{1}{(x-t)^i (y-t)^{n-i}} = \frac{1}{x-y}\left(\frac{1}{(y-t)^{n-1}} - \frac{1}{(x-t)^{n-1}}\right),$$

that we use to simplify the expression of the regularized correlation functions with descendants. This will allow us to rewrite such quantities in term of integrals of *total derivatives*. More precisely we will see that the remainder terms coming from Gaussian integration by parts can be written as a linear combination of expressions of the form

$$\mathfrak{R} := I_{sing}^{p_1} \times I_{reg}^{p_2} \times I_{tot}^{p_3}\left[\partial_{x_1}\cdots\partial_{x_l}\left(F(t,\boldsymbol{x})\langle V_{\gamma e_{i_1}}(x_1)\cdots V_{\gamma e_{i_p}}(x_p)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right)\right]. \tag{3.13}$$

These can be transformed using Stoke's formula: namely we can write that

$$\begin{aligned}
&I_{sing}\left[\partial_{x_1}\left(F(t,x)\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right)\right] = \\
&\sum_{i=1}^{2} \mu_{L,i_1}\left[F(t,x)\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right]_{t-r}^{t-\varepsilon} + \mu_{R,i_1}\left[F(t,x)\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right]_{t+\varepsilon}^{t+r} \\
&+\mu_{B,i}\int_{(t-r,t+r)}\left(F(t,x+\mathrm{i}\delta)-F(t,x-\mathrm{i}\delta)\right)\langle V_{\gamma e_i}(x+\mathrm{i}\delta)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\frac{\mathrm{i}dx}{2} \\
&+\mu_{B,i}\oint_{\mathbb{H}_{\delta,\varepsilon}\cap\partial B(t,r)}\langle V_{\gamma e_i}(\xi)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\left(F(t,\bar\xi)\frac{d\bar\xi}{2} - F(t,\xi)\frac{\mathrm{i}d\xi}{2}\right).
\end{aligned}$$

More generally applying Stoke's formula to Equation (3.13) will yield several terms of this form and involving composite integrals. Such expressions will contain the remainder terms but also some convergent quantities that also depend on r. In the above example, the remainder terms will be given by

$$\begin{aligned}
&\sum_{i=1}^{2} \mu_{L,i_1}\left(F(t,t-\varepsilon)\langle V_{\gamma e_i}(t-\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right) - \mu_{R,i_1}\left(F(t,t+\varepsilon)\langle V_{\gamma e_i}(t+\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right) \\
&+\mu_{B,i}\int_{(t-r,t+r)}\left(F(t,x+\mathrm{i}\delta)-F(t,x-\mathrm{i}\delta)\right)\langle V_{\gamma e_i}(x+\mathrm{i}\delta)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\frac{\mathrm{i}dx}{2}.
\end{aligned}$$

One sees that for $\langle\beta,e_i\rangle$ negative enough this converges to 0 as first ε and then δ go to 0. Conversely if $F=1$ and $\langle\beta,\gamma e_i\rangle$ is positive then this term is seen to be divergent.

To discard the other terms appearing in the expansion, for $\mathfrak{R}$ given by Equation (3.13) we define $\widetilde{\mathfrak{R}}$ to be the sum of the terms that involve at least one such remainder term. Put differently $\widetilde{\mathfrak{R}}$ is defined from $\mathfrak{R}$ by removing the terms involving **only** correlation functions $\langle\prod_{p=1}^n V_{\gamma e_{i_p}}(x_p)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}$ where all the x_p are at fixed distance r_p from t. This remainder term $\widetilde{\mathfrak{R}}$ will be of crucial importance in the definition of the descendant field.

3.2.2.4 Taking the limit

We then investigate the limit of these remainder terms as first $\rho \to 0$ and then ε and δ go to 0. As we will see (and for generic value of β) they will either vanish in the limit or diverge; in the latter case we need to take this into account when defining the correlation function $\langle \mathbf{L}_{-n} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle$. Namely if we denote by $\widetilde{\mathfrak{L}}_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ such remainder terms the above correlation function will be defined as the limit

$$\langle \mathbf{L}_{-n} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle := \\ \lim_{\delta,\varepsilon,\rho\to 0} \langle \mathbf{L}_{-n} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} - \widetilde{\mathfrak{L}}_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

For instance we will see that

$$\widetilde{\mathfrak{L}}_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) = \sum_{i=1}^2 \left(\mu_{L,i}\langle V_{\gamma e_i}(t-\varepsilon) V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} - \mu_{R,i}\langle V_{\gamma e_i}(t+\varepsilon) V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right)$$

so that the $\mathbf{L}_{-1} V_\beta$ descendant is defined within correlation functions by setting

$$\langle \mathbf{L}_{-1} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle := \lim_{\delta,\varepsilon,\rho\to 0} \\ \langle \mathbf{L}_{-1} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} - \sum_{i=1}^2 \Big(\mu_{L,i}\langle V_{\gamma e_i}(t-\varepsilon) V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} - \mu_{R,i}\langle V_{\gamma e_i}(t+\varepsilon) V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\Big).$$

The same applies for other Virasoro descendants, but the expressions for the remainder terms get increasingly involved. And likewise, we will proceed in the same way to make sense the W-descendants, in which case their definitions rely on the very same method but involve some additional combinatorial identities due to the higher-spin symmetry.

3.2.2.5 Some notations

In order to lighten the computations to come we introduce several shortcuts. First of all recall the definition of the symbols I_{tot}, I_{sing} and I_{reg} from Equation (3.2.2.2), as well as the shortcut $\mathbf{V} = \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)$ and the notation for the correlation functions

$$\Psi_{i_1,\cdots,i_p}(x_1,\cdots,x_p) := \langle V_{\gamma e_{i_1}(x_1)} \cdots V_{\gamma e_{i_p}(x_p)} V_\beta(t)\mathbf{V}\rangle_{\rho,\varepsilon,\delta}. \tag{3.14}$$

3.2.3 Some technical estimates

In order to implement the previous method we will need to ensure that the quantities are well-defined and for this it is necessary to understand several of their analytic properties. We gather such properties in this subsection. Since the proofs of these statements are very similar to the case of boundary Liouville CFT (for the fusion estimates [11] while for analycity we refer to [6]) and of bulk Toda CFT (fusion estimates can be found in [25, Lemma 3.2] and analycity in [20, Section 2.2]), and proceed in the exact same way, we do not include them in the present document.

3.2.3.1 Bounds at infinity

As explained above, when implementing our method we will need to consider integrals over the upper-half plane or the real line that contain correlation functions. These correlation functions feature additional Vertex Operators of the form $V_{\gamma e_i}$ for some $i = 1, 2$. In order to ensure that such integrals are indeed well-defined we first need to understand what happens when these insertions diverge. To this end we provide the following statement which provides bounds on the correlation functions when their arguments go to ∞:

Lemma 3.2.4. *Assume that $\boldsymbol{\alpha} \in \mathscr{A}_{N,M}$ and consider for $i = 1, 2$ additional insertions $\boldsymbol{x}_i := \left(x_i^{(1)}, \cdots, x_i^{(N_i)}\right) \in \mathbb{H}^{N_i}$ and $\boldsymbol{y}_i := \left(y_i^{(1)}, \cdots, y_i^{(M_i)}\right) \in \mathbb{R}^{M_i}$. Then for any $h > 0$, if for $i = 1, 2$ the $\boldsymbol{x}_i$, $\boldsymbol{y}_i$ and $\boldsymbol{z}$ stay in the domain $U_h := \left\{\boldsymbol{w}, h < \min_{n \neq m} \left|w^{(n)} - w^{(m)}\right|\right\}$, then there exists $C = C_h$ such that, uniformly in $\delta, \varepsilon, \rho$,*

$$\langle \prod_{i=1}^{2} \prod_{n=1}^{N_i} V_{\gamma e_i}\left(x_i^{(n)}\right) \prod_{m=1}^{M_i} V_{\gamma e_i}\left(y_i^{(m)}\right) \mathbf{V} \rangle_{\delta,\varepsilon,\rho} \leq C_h \prod_{i=1}^{2} \prod_{n=1}^{N_i} \left(1 + \left|x_i^{(n)}\right|\right)^{-4} \prod_{m=1}^{M_i} \left(1 + \left|y_i^{(m)}\right|\right)^{-2}.$$

Thanks to this estimate we see that the correlation functions are integrable at ∞. This will allow us in the sequel to infer that boundary terms that may come from the successive integration by parts do indeed vanish at infinity. As such we won't consider such terms in the definition of the descendant fields in the next two sections.

3.2.3.2 Fusion estimates

In order to ensure finiteness of the correlation functions and of their derivatives we will also need to ensure that the integrals that we encounter are not singular near the insertion points. For this purpose we rely on *fusion asymptotics* of the correlation functions, that is their rate of divergence when several insertion points collide:

Lemma 3.2.5. *Assume that $\boldsymbol{\alpha} \in \mathscr{A}_{N,M}$ and that all pairs of points in $\boldsymbol{z}$ are separated by some distance $h > 0$ except for one pair (z_1, z_2). Further assume that $\langle \alpha_1 + \alpha_2 - Q, e_1 \rangle < 0$. Then as $z_1 \to z_2$, for any positive η there exists a positive constant K such that, uniformly on $\delta, \varepsilon, \rho$:*

1. *if $z_1, z_2 \in \mathbb{H}$ then*

$$\langle \mathbf{V} \rangle_{\delta,\varepsilon,\rho} \leq K |z_1 - z_2|^{-\langle \alpha_1, \alpha_2 \rangle + \left(\frac{1}{2}\langle \alpha_1 + \alpha_2 - Q, e_2 \rangle^2 - \eta\right) \mathbb{1}_{\langle \alpha_1 + \alpha_2 - Q, e_2 \rangle > 0}}; \qquad (3.15)$$

2. *if $z_1, z_2 \in \mathbb{R}$ then*

$$\langle \mathbf{V} \rangle_{\delta,\varepsilon,\rho} \leq K|z_1 - z_2|^{-\frac{\langle \alpha_1,\alpha_2\rangle}{2}+\left(\frac{1}{4}(\langle \alpha_1+\alpha_2-Q,e_2\rangle)^2-\eta\right)\mathbb{1}_{\langle \alpha_1+\alpha_2-Q,e_2\rangle>0}}; \tag{3.16}$$

3. *if $z_1 \in \mathbb{H}$ while $z_2 \in \mathbb{R}$ with in addition $\langle \alpha_1 - \frac{Q}{2}, e_1\rangle < 0$ then*

$$\begin{aligned}\langle \mathbf{V} \rangle_{\delta,\varepsilon,\rho} \leq K \times |z_1 - \bar{z}_1|^{-\frac{|\alpha_1|^2}{2}+\left(\left(\langle \alpha_1-\frac{Q}{2},e_2\rangle\right)^2-\eta\right)\mathbb{1}_{\langle \alpha_1-\frac{Q}{2},e_2\rangle>0}} \\ |z_1 - z_2|^{-\langle \alpha_1,\alpha_2\rangle+\left(\left(\langle \alpha_1+\frac{\alpha_2}{2}-\frac{Q}{2},e_2\rangle\right)^2-\eta\right)\mathbb{1}_{\langle \alpha_1+\frac{\alpha_2}{2}-\frac{Q}{2},e_2\rangle>0}}.\end{aligned} \tag{3.17}$$

As a consequence for β such that $\langle \beta, e_i\rangle < 0$ we have that $|x-t|^{\frac{\langle \beta,\gamma e_i\rangle}{2}} F_i(x)$ is continuous at $x = t$, where F_i is of the form

$$F_i(x) = I^p_{sing} \times I^q_{reg}\left[r_{i,\boldsymbol{i}}(x,\boldsymbol{x})\Psi_{i,\boldsymbol{i}}(x,\boldsymbol{x})\right]$$

with $r_{i,\boldsymbol{i}} = r_{i,i_1,\dots,i_{(p+q)}}$ regular at $x = t$ and such that $r_{i,\boldsymbol{i}}(x,\boldsymbol{x})\Psi_{i,\boldsymbol{i}}(\boldsymbol{x})$ is integrable on $I^p_{sing} \times I^q_{reg}$, while $\Psi_{i,\boldsymbol{i}}(x,\boldsymbol{x}) = \Psi_{i,i_1,\dots,i_{p+q}}(x,x_1,\dots,x_{p+q})$.

Proof. The first part of the claim (fusion estimates) follows from the counterparts statements in the boundary Liouville case [11, Section 5] (see also [24, Lemma 2.4]) and bulk Toda theory [25, Lemma 3.2] (this adaptation is made possible since the two GFFs $\langle \mathbf{X}, e_i\rangle$ that appear in the definitions of the GMC measures are negatively correlated since $\langle e_1, e_2\rangle < 0$). As for the second part (continuity at $x = t$) we use the probabilistic representation to infer that $|x-t|^{\frac{\langle \beta,\gamma e_i\rangle}{2}} r_{i,\boldsymbol{i}}(x,\boldsymbol{x})\Psi_{i,\boldsymbol{i}}(x,\boldsymbol{x})$ is continuous at $x = t$. To deduce continuity of $|x-t|^{\frac{\langle \beta,\gamma e_i\rangle}{2}} F_i$ from this fact we then rely on the integrability properties of the correlation functions in the form of Lemma 3.2.4. □

3.2.3.3 A class of remainder terms

In the definition of the descendant fields as described above, we need to take a limit of a regularized quantity to which we have substracted a remainder term. In order for this procedure to define a meaningful object, we introduce a class of remainder terms that would satisfy the properties we expect from a definition of the descendant fields:

Definition 3.2.6. We will say that a regularized quantity $F_{\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ has the (P) property or is (P)-class when it satisfies the following assumptions:

1. for any $\boldsymbol{\alpha} \in \mathscr{A}_{N,M}$, the limit $F(\boldsymbol{\alpha}) := \lim_{\delta,\varepsilon,\rho\to 0} F_{\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ exists and is finite as ρ, ε and then $\delta \to 0$;
2. the map $\boldsymbol{\alpha} \mapsto F(\boldsymbol{\alpha})$ is analytic in a complex neighborhood of $\mathscr{A}_{N,M}$.

Let us now provide some examples of such quantities. And to start with we stress that the correlation functions themselves satisfy this property:

Lemma 3.2.7. *The regularized correlation functions* $\langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}$ *are* (P)*-class.*

Likewise some singular integrals (that will later on enter the definition of the descendants) satisfy this property:

Lemma 3.2.8. *For any* i *and* j *in* $\{1,2\}$ *the following integrals are* (P)*-class:*

$$\begin{aligned}
&\int_{\frac{1}{2}\mathbb{D}\times(\mathbb{D}\setminus\frac{1}{2}\mathbb{D})} \frac{1}{y-x}\langle V_{\gamma e_i}(x+i)V_{\gamma e_j}(y+i)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}d^2xd^2y,\\
&\int_{-1}^{0}\int_{0}^{1}\frac{1}{y-x}\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}dxdy \quad \textit{and}\\
&\int_{\mathbb{D}\cap\mathbb{H}}\int_{1}^{2}\frac{1}{y-x}\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}d^2xdy.
\end{aligned} \tag{3.18}$$

In particular integrals of the form

$$I_{sing}\times I_{reg}\left[\frac{1}{x-y}\Psi_{i,j}(x,y)\right]$$

are (P)*-class.*

3.3 Definition of the descendant fields and Ward identities

This section is dedicated to providing a definition of the descendant fields in the case where only one current is involved, that is to say we will provide a meaning to the quantities $\mathbf{L}_{-n}V_\beta$ and $\mathbf{W}_{-m}V_\beta$ for any positive integers n, m and for suitable β (that is $\beta\in Q+\mathscr{C}_-$). We will define such descendants for boundary Vertex Operators, the bulk case being treated via the same arguments without the technicalities that appear in the boundary case.

To define such descendants we will make sense of them within correlation functions, that is we will make sense of the quantities

$$\langle \mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle \quad \text{and} \quad \langle \mathbf{W}_{-n}V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle$$

where $(\beta,\boldsymbol{\alpha})\in\mathscr{A}_{N,M+1}$ and with $t\in\partial\mathbb{H}=\mathbb{R}$.

3.3.1 Descendant fields at the first order

To start with we will define the descendant associated to Virasoro and higher-spin symmetry at the level one. These are the easiest ones to deal with but their definition still relies on the type of argument that we will use in the more general case, in the next two subsections.

3.3.1.1 Definition of the descendant fields

In contrast with the closed case, defining the descendant fields in the presence of a boundary requires some extra care. We first define them at the regularized level and then take an appropriate limit, in which we need to subtract some additional divergent terms compared to the closed case. For this purpose let us introduce the following quantities.

Lemma 3.3.1. *For any i in $\{1,2\}$ set*

$$\mathfrak{L}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) := I_{sing}\left[\partial_x \langle V_{\gamma e_i}(x) V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}\right].$$

Then the following quantity is (P)-class:

$$\langle \mathbf{L}_{-1} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} - \sum_{i=1}^2 \mathfrak{L}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

In the same fashion, let us define for $i = 1,2$

$$\mathfrak{W}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) := -h_2(e_i)\left(q - 2\omega_{\hat{i}}(\beta)\right)\mathfrak{L}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

Then the following quantity is (P)-class:

$$\langle \mathbf{W}_{-1} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} - \sum_{i=1}^2 \mathfrak{W}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

Proof. Fix δ,ε to be positive and set D to be either $\mathbf{L}$ or $\mathbf{W}$. Using Gaussian integration by parts we have

$$\begin{aligned}\langle \mathrm{D}_{-1} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} = &\sum_{k=1}^{2N+M} \frac{\mathrm{D}^\beta_{-1}(\alpha_k)}{2(z_k - t)}\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} - \sum_{i=1}^2 \int_{\mathbb{R}_\varepsilon} \frac{\mathrm{D}^\beta_{-1}(\gamma e_i)}{2(s-t)}\langle V_{\gamma e_i}(s) V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\mu_i(ds)\\ &- \sum_{i=1}^2 \mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\frac{\mathrm{D}^\beta_{-1}(\gamma e_i)}{2(x-t)} + \frac{\mathrm{D}^\beta_{-1}(\gamma e_i)}{2(\bar{x}-t)}\right)\langle V_{\gamma e_i}(x) V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} d^2x.\end{aligned}$$

As $\varepsilon,\delta \to 0$ the first term in this expansion is (P)-class in agreement with Lemma 3.2.7. As for the integrals, we see that thanks to Lemma 3.2.5 the integrals over the subdomains $\mathbb{R}^c_\varepsilon$ and $\mathbb{H}^c_{\delta,\varepsilon}$ are (P)-class so that

$$\langle \mathrm{D}_{-1} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} = (P)\text{-class terms} - I_{sing}\left[\frac{\mathrm{D}^\beta_{-1}(\gamma e_i)}{2(x-t)}\langle V_{\gamma e_i}(x) V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right].$$

To be more explicit the (P)-class terms in the above are explicitly given by

$$\sum_{k=1}^{2N+M}\frac{\mathrm{D}^{\beta}_{-1}(\alpha_k)}{2(z_k-t)}\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}-I_{reg}\left[\frac{\mathrm{D}^{\beta}_{-1}(\gamma e_i)}{2(x-t)}\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right].$$

To treat the remaining terms we first note that we have the following explicit equalities (which can be checked based on explicit computations):

$$\mathbf{L}^{\beta}_{-1}(\gamma e_i)=\langle\beta,\gamma e_i\rangle,\quad \mathbf{W}^{\beta}_{-1}(\gamma e_i)=-h_2(e_i)\left(q-2\omega_{\hat{i}}(\beta)\right)\langle\beta,\gamma e_i\rangle.$$

In particular in both cases the ratio $\frac{\mathrm{D}^{\beta}_{-1}(\gamma e_i)}{\langle\beta,\gamma e_i\rangle}$ is well-defined for any β in $\mathbb{C}^2$. This allows to rewrite the remaining integrals as

$$\frac{\mathrm{D}^{\beta}_{-1}(\gamma e_i)}{\langle\beta,\gamma e_i\rangle}I_{sing}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(x-t)}\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right].$$

We recognize there derivatives of correlation functions in that

$$\partial_x\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}=\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}+\frac{\langle\gamma e_i,\gamma e_i\rangle}{2(\bar{x}-x)}+\sum_{k=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_k\rangle}{2(z_k-x)}\right)\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}$$
$$-I_{tot}\left[\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-x)}\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right]$$

and likewise for $\partial_{\bar{x}}$ and ∂_s, where the integral is over y. This shows that

$$I_{sing}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(x-t)}\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right]=I_{sing}\left[\left(\partial_x+\sum_{k=1}^{N}\frac{\langle\alpha_k,\gamma e_i\rangle}{2(x-z_k)}\right)\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right]$$
$$+\sum_{i,j=1}^{2}\mu_{B,i}\mu_{B,j}\int_{\mathbb{H}^1_{\delta,\varepsilon}\times\mathbb{H}_{\delta,\varepsilon}}\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-\bar{y})}+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(\bar{x}-y)}+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(\bar{x}-\bar{y})}\right)\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)\mathbf{V}\rangle_{\delta,\varepsilon}d^2xd^2y$$
$$+\sum_{i,j=1}^{2}\mu_{B,i}\int_{\mathbb{H}^1_{\delta,\varepsilon}\times\mathbb{R}_\varepsilon}\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(\bar{x}-y)}\right)\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)\mathbf{V}\rangle_{\delta,\varepsilon}d^2x\mu_j(dy)$$
$$+\sum_{i,j=1}^{2}\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}\times\mathbb{R}^1_\varepsilon}\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-x)}+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-\bar{x})}\right)\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)\mathbf{V}\rangle_{\delta,\varepsilon}d^2x\mu_j(dy)$$
$$+\sum_{i,j=1}^{2}\int_{\mathbb{R}^1_\varepsilon\times\mathbb{R}_\varepsilon}\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-\bar{y})}\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)\mathbf{V}\rangle_{\delta,\varepsilon}\mu_i(dx)\mu_j(dy).$$

Now by symmetry in the variables x and y the diagonal part of the integral over $\mathbb{H}_{\delta,\varepsilon}\times\mathbb{H}_{\delta,\varepsilon}$ (that is the subintegral over $\mathbb{H}^1_{\delta,\varepsilon}\times\mathbb{H}^1_{\delta,\varepsilon}$) vanishes so that only the integral over $\mathbb{H}^1_{\delta,\varepsilon}\times\mathbb{H}^c_{\delta,\varepsilon}$ remains. Thanks to Lemma 3.2.8 we know that this integral is (P)-class. The same argument applies to the remaining integrals, whose sum is actually non-zero only over $\mathbb{H}^1_{\delta,\varepsilon}\times\mathbb{R}^c_\varepsilon$, $\mathbb{H}^c_{\delta,\varepsilon}\times\mathbb{R}^1_\varepsilon$ and $\mathbb{R}^c_\varepsilon\times\mathbb{R}^c_\varepsilon$. Recollecting term we obtain that

$$\langle \mathrm{D}_{-1}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} = (P)\text{-class terms} + I_{sing}\left[\frac{\mathrm{D}^\beta_{-1}(\gamma e_i)}{\langle\beta,\gamma e_i\rangle}\partial_x\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right].$$

This concludes for the proof of Lemma 3.3.1. □

Based on this observation we now would like to define the descendant fields $\mathbf{L}_{-1}V_\beta$ and $\mathbf{W}_{-1}V_\beta$. However in order to remove the dependence with respect to the radius r and as explained in the previous section we actually need to remove one additional term. In this perspective let us recall that we have introduced there the notations $\widetilde{\mathfrak{L}}_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ and $\widetilde{\mathfrak{W}}_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ to discard such terms. Here we see that more explicitly the remainder term is given by (up to a $o(1)$ term)

$$\begin{aligned}
&\widetilde{\mathfrak{L}}_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) = \sum_{i=1}^2 \widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) \quad\text{and}\quad \widetilde{\mathfrak{W}}_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) = \sum_{i=1}^2 \widetilde{\mathfrak{W}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}),\ \text{where}\\
&\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) = \left(\mu_{L,i}\langle V_{\gamma e_i}(t-\varepsilon)\mathbf{V}\rangle_{\delta,\varepsilon} - \mu_{R,i}\langle V_{\gamma e_i}(t+\varepsilon)\mathbf{V}\rangle_{\delta,\varepsilon}\right) \quad\text{while}\\
&\widetilde{\mathfrak{W}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) := -h_2(e_i)\left(q-2\omega_{\hat{i}}(\beta)\right)\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).
\end{aligned}$$

Note that $\mathfrak{L}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ and $\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ only differ by (P)-class terms in that:

$$\begin{aligned}
&\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) = \mathfrak{L}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) - \left(\mu_{R,i}\langle V_{\gamma e_i}(t+r)\mathbf{V}\rangle_{\delta,\varepsilon} - \mu_{L,i}\langle V_{\gamma e_i}(t-r)\mathbf{V}\rangle_{\delta,\varepsilon}\right)\\
&-\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}\cap\partial B(t,r)}\langle V_{\gamma e_i}(\xi)\mathbf{V}\rangle_{\delta,\varepsilon}\frac{id\bar{\xi}-id\xi}{2}.
\end{aligned}$$

Definition 3.3.2. Take $t\in\mathbb{R}$ and $(\beta,\boldsymbol{\alpha})\in\mathscr{A}_{N,M+1}$. We define the descendant field $\mathbf{L}_{-1}V_\beta$ within half-plane correlation functions by the limit

$$\begin{aligned}
&\langle \mathbf{L}_{-1}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle :=\\
&\lim_{\delta,\varepsilon,\rho\to0}\langle \mathbf{L}_{-1}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} - \sum_{i=1}^2\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).
\end{aligned}$$

Under the same assumptions, the $\mathbf{W}_{-1}V_\beta$ descendent field is defined by setting

$$\begin{aligned}
&\langle \mathbf{W}_{-1}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle :=\\
&\lim_{\delta,\varepsilon,\rho\to0}\langle \mathbf{W}_{-1}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} - \sum_{i=1}^2\widetilde{\mathfrak{W}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).
\end{aligned}$$

Thanks to Lemma 3.3.1 we see that these limits are well-defined and analytic in a complex neighbourhood of $\mathscr{A}_{N,M+1}$.

Remark 3.3.3. *Going along the proof of Lemma 3.3.1 we see that $\mathbf{L}_{-1}V_\beta$ is actually such that*

$$\begin{aligned}
&\langle \mathbf{L}_{-1}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} = \sum_{k=1}^{N}\frac{\langle\beta,\alpha_k\rangle}{2(z_k-t)}\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} - I_{reg}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(x-t)}\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right]\\
&+I_{sing}\left[\left(\partial_x+\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_i\rangle}{2(x-z_k)}\right)\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right]\\
&-I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right].
\end{aligned} \tag{3.19}$$

3.3.1.2 Connection with derivatives of the correlation functions

The insertion of Virasoro descendants within correlation functions can be explicitly translated into derivatives of this correlation function. The following statement shows this connection for the $\mathbf{L}_{-1}$ descendant:

Proposition 3.3.4. *Take $t\in\mathbb{R}$ and $(\beta,\boldsymbol{\alpha})\in\mathscr{A}_{N,M+1}$. Then in the sense of weak derivatives*

$$\langle \mathbf{L}_{-1}V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle=\partial_t\langle V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle.$$

Proof. Note that for positive ε,δ,ρ, $\langle V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}$ is differentiable with respect to t with derivative given by $\langle \mathbf{L}_{-1}V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}$. In particular if we take f to be any smooth, compactly supported function over $\mathbb{R}$ then

$$\begin{aligned}
&\int_{\mathbb{R}} f(t)\langle \mathbf{L}_{-1}V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}dt\\
&=\int_{\mathbb{R}}\left(-\partial_t f(t)\right)\langle V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}dt.
\end{aligned}$$

Let us first assume that $\langle\beta,e_i\rangle<0$ for $i=1,2$: in that case $\lim_{\delta,\varepsilon,\rho\to0}\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})=0$. To see why we rely on the explicit expression provided by Lemma 3.3.1:

$$\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon}(\boldsymbol{\alpha})=\mu_{L,i}\langle V_{\gamma e_i}(t-\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}-\mu_{R,i}\langle V_{\gamma e_i}(t+\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}.$$

Under the assumption that $\langle\beta,e_i\rangle<0$ for all i, as $\delta,\varepsilon\to0$ we have using the fusion asymptotics (Lemma 3.2.5) that $\langle V_{\gamma e_i}(t\pm\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\to0$, so that the above expression vanishes in the limit. This allows to conclude that if $\langle\beta,e_i\rangle<0$ for $i=1,2$ then

$$\int_{\mathbb{R}} f(t)\langle \mathbf{L}_{-1}V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle dt=\int_{\mathbb{R}}\left(-\partial_t f(t)\right)\langle V_\beta(t)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\prod_{l=1}^{M}V_{\beta_l}(s_l)\rangle dt.$$

To extend this equality to the whole range of values of $(\beta,\boldsymbol{\alpha})\in\mathscr{A}_{N,M+1}$ we rely on the fact that both the left and right-hand sides in this equality depend analytically on β. □

The insertion of a $\mathbf{L}_{-1}$ descendant thus acts as a differential operator on the correlation functions. A major difference between Virasoro and W currents is that this is not the case, for generic values of β, for the $\mathbf{W}_{-1}$ descendant. As we will see in the next chapter (taken from [27], see also [25, Proposition 5.1]), in order to be able to express the insertion of a $\mathbf{W}_{-1}$ as a differential operator we need to specialize to special values of the weight β, in which case we end up with singular vectors.

3.3.2 Virasoro descendants and conformal Ward identities

Having defined the descendants at the first level we now turn to the definition of the Virasoro descendants at level higher than one. As we will see the proof is slightly more involved compared to that previously disclosed. As a consequence we may start using the shortcut $\Psi_{i_1,\cdots,i_p}(x_1,\cdots,x_p)$ defined in Equation (3.11).

3.3.2.1 Virasoro descendants

We first introduce the Virasoro descendants $\mathbf{L}_{-n}V_\beta$. For this purpose and like before we start by introducing remainder terms as follows.

Lemma 3.3.5. *For any positive integer n and i in $1,2$ set*

$$\mathfrak{L}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) := I_{sing}\left[\partial_x\left(\frac{1}{(x-t)^{n-1}}\langle V_{\gamma e_i}(x)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}\right)\right].$$

Then the following quantity is (P)-class:

$$\langle \mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}-\sum_{i=1}^2\mathfrak{L}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

Proof. The arguments are similar to the ones developed in the previous section. The starting point are the basic identities

$$\mathbf{L}^\beta_{-n}\left(\alpha\ln\frac{1}{|\cdot-t|}\right)=\frac{(n-1)\Delta_\alpha+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(\cdot-t)^2}\quad\text{and}\quad\mathbf{L}^\beta_{-n}\left(\gamma e_i\ln\frac{1}{|\cdot-t|}\right)=\frac{n-1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(\cdot-t)^2}.$$

Thanks to them we can write that

$$\begin{aligned}
&\langle \mathbf{L}_{-n} V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon} = \\
&\left[\sum_{k=1}^{2N+M} \frac{(n-1)\Delta_{\alpha_k}}{2(z_k-t)^n} - \sum_{k=1}^{2N+M}\left(\frac{\langle\beta,\alpha_k\rangle}{2(z_k-t)^n} + \sum_{p=1}^{n-1}\sum_{l\neq k}\frac{\langle\alpha_k,\alpha_l\rangle}{4(z_k-t)^p(z_l-t)^{n-p}}\right)\right]\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} \\
&- I_{tot}\left[\left(\frac{n-1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(x-t)^n} - \sum_{p=1}^{n-1}\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_i\rangle}{2(x-t)^p(z_k-t)^{n-p}}\right)\Psi_i(x)\right] \\
&+ \sum_{i=1}^2 \mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\sum_{p=1}^{n-1}\frac{\langle\gamma e_i,\gamma e_i\rangle}{2(x-t)^p(\bar{x}-t)^{n-p}}\Psi_i(x)d^2x - I_{tot}^2\left[\sum_{p=1}^{n-1}\frac{\langle\gamma e_i,\gamma e_j\rangle}{4(x-t)^p(y-t)^{n-p}}\Psi_{i,j}(x,y)\right].
\end{aligned}$$

Let us start with the one-fold integral. Since the integrals over $\mathbb{H}^c_{\delta,\varepsilon}$ and $\mathbb{R}^c_\varepsilon$ are (P)-class we can focus on the singular part of these integrals. For this we rely on the fact that

$$\sum_{p=1}^{n-1}\frac{1}{(x-t)^p(y-t)^{n-p}} = \frac{1}{x-y}\left(\frac{1}{(y-t)^{n-1}} - \frac{1}{(x-t)^{n-1}}\right), \tag{3.20}$$

so that the singular one-fold integrals can be rewritten as

$$\begin{aligned}
&-\sum_{k=1}^{2N+M}\frac{1}{(z_k-t)^{n-1}} I_{sing}\left[\frac{\langle\alpha_k,\gamma e_i\rangle}{2(z_k-x)}\Psi_i(x)\right] \\
&- I_{sing}\left[\left(\frac{n-1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(x-t)^n} + \sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_i\rangle}{2(x-t)^{n-1}(x-z_k)}\right)\Psi_i(x)\right] \\
&- \sum_{i=1}^2\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\frac{\langle\gamma e_i,\gamma e_i\rangle}{2(x-t)^{n-1)}(x-\bar{x})} + \frac{\langle\gamma e_i,\gamma e_i\rangle}{2(\bar{x}-t)^{n-1}(\bar{x}-x)}\right)\Psi_i(x)d^2x.
\end{aligned}$$

The first term is (P)-class; the second one can be dealt with like before by writing it as a total derivative in that

$$\begin{aligned}
&\partial_x\left(\frac{1}{(t-x)^{n-1}}\Psi_i(x)\right) \\
&= \left(\frac{n-1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(t-x)^n} + \sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_i\rangle}{2(t-x)^{n-1}(z_k-x)} + \frac{\langle\gamma e_i,\gamma e_i\rangle}{2(t-x)^{n-1)}(\bar{x}-x)}\right)\Psi_i(x) \\
&- I_{tot}\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)^{n-1}(y-x)}\Psi_i(x)\right).
\end{aligned} \tag{3.21}$$

Therefore the only remaining terms to be treated are given by

$$-I_{sing}\times I_{tot}\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-t)^{n-1}(y-x)}\Psi_i(x)\right)$$
$$+I_{tot}^2\left(\sum_{i=1}^{n-1}\frac{\langle\gamma e_i,\gamma e_j\rangle}{4(x-t)^i(y-t)^{n-i}}\Psi_{i,j}(x,y)\right).$$

We can rewrite the second integral using the identity (3.20) together with symmetry in the x,y variable. By doing so we see that the integral over $I_{sing}\times I_{tot}$ vanishes, so that only the following integral remains:

$$I_{reg}\times I_{tot}\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-t)^{n-1}(y-x)}\Psi_i(x)\right).$$

Over $I_{reg}\times I_{reg}$ we use again the identity (3.20) to write the integrand as

$$\sum_{p=1}^{n-1}\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-t)^{n-p}(y-t)^p}\Psi_{i,j}(x,y)$$

so that this term is (P)-class. As for the integral over $I_{reg}\times I_{sing}$ it is seen to be (P)-class thanks to the fusion asymptotics of Lemma 3.2.8. □

Definition 3.3.6. We then define the $L_{-n}V_\beta$ descendant within correlation functions via the limits

$$\langle\mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle:=$$
$$\lim_{\delta,\varepsilon,\rho\to0}\langle\mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}-\sum_{i=1}^2\widetilde{\mathfrak{L}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

Like before we can provide an explicit expression for the remainder terms defined above:

$$\widetilde{\mathfrak{L}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})=\left(\mu_{L,i}\frac{\langle V_{\gamma e_i}(t-\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}}{(-\varepsilon)^{n-1}}-\mu_{R,i}\frac{\langle V_{\gamma e_i}(t+\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}}{(\varepsilon)^{n-1}}\right)$$
$$-\mu_{B,i}\int_{(t-r,t+r)}\langle V_{\gamma e_i}(x+\mathrm{i}\delta)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\operatorname{Im}\left(\frac{1}{(x+\mathrm{i}\delta-t)^{n-1}}\right)dx.$$

And like before these remainder terms and the one provided in Lemma 3.3.5 differ by quantities that are (P)-class. As such the limit defining $\langle\mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle$ is well-defined and analytic in a complex neighbourhood of $\mathscr{A}_{N,M+1}$. Likewise the $\mathbf{L}_{-n}V_\beta$ descendant when inserted within correlation functions is seen to be given at the regularized level by

$$
\begin{aligned}
&\langle \mathbf{L}_{-n} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} = A_{\delta,\varepsilon} + B_{\delta,\varepsilon}, \quad \text{where} \\
&A_{\delta,\varepsilon} := \left(\sum_{k=1}^{2N+M} \frac{\langle (n-1)Q+\beta,\alpha_k\rangle}{2(z_k-t)^n} - \sum_{k,l=1}^{2N+M}\sum_{p=1}^{n-1} \frac{\langle \alpha_k,\alpha_l\rangle}{4(z_k-t)^p(z_l-t)^{n-p}}\right)\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} \\
&- I_{reg}\left[\left(\frac{n-1+\frac{\langle \beta,\gamma e_i\rangle}{2}}{(x-t)^n} - \sum_{k=1}^{2N+M}\sum_{p=1}^{n-1}\frac{\langle \gamma e_i,\alpha_k\rangle}{2(x-t)^p(z_k-t)^{n-p}}\right)\Psi_i(x)\right] \\
&- I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle \gamma e_i,\alpha_k\rangle}{2(z_k-t)^{n-1}(z_k-x)}\Psi_i(x)\right] \\
&+ I_{reg}\times I_{reg}\left[-\sum_{p=1}^{n-1}\frac{\langle \gamma e_i,\gamma e_j\rangle}{4(x-t)^p(y-t)^{n-p}}\Psi_{i,j}(x,y)\right] \\
&+ I_{sing}\times I_{reg}\left[\frac{\langle \gamma e_i,\gamma e_j\rangle}{2(y-t)^{n-1}(y-x)}\Psi_{i,j}(x,y)\right] \\
&\text{while}\quad B_{\delta,\varepsilon} := I_{sing}\left[\partial_x\left(\frac{1}{(x-t)^{n-1}}\Psi_i(x)\right)\right].
\end{aligned}
\tag{3.22}
$$

3.3.2.2 Conformal Ward identities

We have seen previously that the $\mathbf{L}_{-1}$ descendant allowed to define the (weak) derivatives of the correlation functions. As we now show the other descendants $\mathbf{L}_{-n}$, $n \geq 2$ play an analogous role based on the conformal Ward identities.

Theorem 3.3.7. *Take $t \in \mathbb{R}$ and $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$. Then for any $n \geq 2$, in the sense of weak derivatives*

$$
\begin{aligned}
&\langle \mathbf{L}_{-n} V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle \\
&= \left(\sum_{k=1}^{2N+M}\frac{-\partial_{z_k}}{(z_k-t)^{n-1}} + \frac{(n-1)\Delta_{\alpha_k}}{(z_k-t)^n}\right)\langle V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle.
\end{aligned}
$$

Proof. As explained in Proposition 3.3.4 the weak derivatives are defined using the $\mathbf{L}_{-1}$ descendant fields, themselves defined using a regularization procedure based on Lemma 3.3.1. As such our goal is to show that

$$
\begin{aligned}
&\lim_{\delta,\varepsilon,\rho\to 0}\left(\mathscr{L}_{-n} - \sum_{k=1}^{2N+M}\frac{-\partial_{z_k}}{(z_k-t)^{n-1}} + \frac{(n-1)\Delta_{\alpha_k}}{(z_k-t)^n}\right)\langle V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} \\
&\qquad - \sum_{i=1}^2 \mathfrak{R}^i(\delta,\varepsilon,\rho;\boldsymbol{\alpha}) = 0, \quad \text{with}
\end{aligned}
\tag{3.23}
$$

$$\mathscr{L}_{-n}\langle V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} := \langle \mathbf{L}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} \quad \text{and}$$

$$\mathfrak{R}^i(\delta,\varepsilon,\rho;\boldsymbol{\alpha}) = \widetilde{\mathfrak{L}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) + \sum_{l=1}^M \frac{\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l)}{(s_l-t)^{n-1}}.$$

In the above $\widetilde{\mathfrak{L}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l)$ is the remainder term from Lemma 3.3.1 but associated to the boundary descendant field $\mathbf{L}_{-1}V_{\beta_l}(s_l)$.

To see why this is true we combine the exact expressions provided in the proofs of Lemmas 3.3.1 and 3.3.5 to get that the first line in Equation (3.23) is given by

$$\begin{aligned}
&-I_{tot}\left[\left(\frac{n-1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(x-t)^n}+\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_i\rangle}{2(x-t)^{n-1}(z_k-x)}\right)\Psi_i(x)\right]\\
&-\sum_{i=1}^2\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\frac{\langle\gamma e_i,\gamma e_i\rangle}{2(x-t)^{n-1)}(x-\bar{x})}+\frac{\langle\gamma e_i,\gamma e_i\rangle}{2(\bar{x}-t)^{n-1}(\bar{x}-x)}\right)\Psi_i(x)d^2x\\
&+I_{tot}^2\left(\sum_{i=1}^{n-1}\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-t)^{n-1}(x-y)}\Psi_{i,j}(x,y)\right).
\end{aligned}$$

Along the same lines as before the latter is a total derivative. We get that the first line in Equation (3.23) is equal to

$$I_{tot}\left[\partial_x\left(\frac{1}{(x-t)^{n-1}}\Psi_i(x)\right)\right].$$

After integration by parts (that is application of Stoke's formula over the domains $\mathbb{H}_{\delta,\varepsilon}$ and $\mathbb{R}_\varepsilon$) we recover the desired remainder terms provided that for any $1\leq l\leq M$,

$$\lim_{\delta,\varepsilon,\rho\to 0}\left(\frac{1}{(s_l+\varepsilon-t)^{n-1}}-\frac{1}{(s_l-t)^{n-1}}\right)\Psi_i(s_l+\varepsilon)=0,$$

and likewise for any $1\leq k\leq N$:

$$\lim_{\delta,\varepsilon,\rho\to 0}\oint_{\partial B(z_k,\varepsilon)}\Psi_i(\xi)d\xi=0,$$

This follows from the fusion asymptotics of Lemma 3.2.5. □

3.3.3 W-descendants and higher-spin Ward identities

Having shown the validity of conformal Ward identities, we now turn to the higher-spin symmetry enjoyed by Toda CFTs and which are encoded using the current $\mathbf{W}$. To this end we first define the W-descendants at the order two and then proceed to the general case of $\mathbf{W}_{-n}$ with $n\geq 3$. We will show that the insertion of a $\mathbf{W}_{-n}$ descendant, $n\geq 3$, within correlation functions has an effect similar to that of the Virasoro descendant $\mathbf{L}_{-n}$ but that involves this time the $\mathbf{W}$ descendants corresponding to the other insertions.

3.3.3.1 Definition of the W-descendants

In order to define the W-descendants we proceed in the same fashion that we did for the Virasoro descendants. And to start with we define the descendant at the level two.

Lemma 3.3.8. *Let us set for $i = 1,2$:*

$$\mathfrak{W}^i_{-2,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) := I_{sing}\left[\partial_x\left(\left(\frac{2h_2(e_i)\omega_{\hat{i}}(\beta)}{x-t}+\sum_{k=1}^{2N+M}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{x-z_k}\right)\Psi_i(x)\right)\right]$$
$$-I_{sing}\times I_{tot}\left[\delta_{i\neq j}\partial_x\left(\frac{2\gamma h_2(e_i)}{x-y}\Psi_{i,j}(x,y)\right)\right]$$

where recall the notation $\hat{i} = 3-i$. Then the following quantity is (P)-class:

$$\langle \mathbf{W}_{-2}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}-\sum_{i=1}^2\mathfrak{W}^i_{-2,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

We define the $\mathbf{W}_{-2}$ descendant by setting

$$\langle \mathbf{W}_{-2}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle :=$$
$$\lim_{\delta,\varepsilon,\rho\to 0}\langle \mathbf{W}_{-2}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}-\sum_{i=1}^2\widetilde{\mathfrak{W}}^i_{-2,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

Proof. The proof relies on the very same arguments as the one of Lemma 3.3.5, the main difference lying in the combinatorial identities underlying the definition of the W-descendants. To be more specific we rely on the fact that (shown by explicit computations) for $i = 1,2$:

$$\begin{aligned}
&\mathbf{W}^\beta_{-2}\left(\gamma e_i\ln\frac{1}{|\cdot-t|}\right)=2h_2(e_i)w_{\hat{i}}(\beta)\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(\cdot-t)^2};\\
&C^\sigma(\beta,\gamma e_i,\alpha_k)=-h_2(e_i)\left(\omega_{\hat{i}}(\alpha_k)\langle\beta,\gamma e_i\rangle+\omega_{\hat{i}}(\beta)\langle\alpha_k,\gamma e_i\rangle\right);\\
&C^\sigma(\beta,\gamma e_i,\gamma e_j)=-\gamma h_2(e_i)\langle\beta,\gamma e_i\rangle\delta_{i\neq j}-\langle\gamma e_i,\gamma e_j\rangle h_2(e_i)\omega_{\hat{i}}(\beta).
\end{aligned}$$

As a consequence we can write that the integrals appearing at the regularized level are

$$-I_{sing}\left[\left(\frac{2h_2(e_i)\omega_{\hat{i}}(\beta)\left(1+\langle\beta,\gamma e_i\rangle/2\right)}{(t-x)^2}-\sum_{k=1}^{2N+M}\frac{h_2(e_i)\left(\omega_{\hat{i}}(\alpha_k)\langle\beta,\gamma e_i\rangle+\omega_{\hat{i}}(\beta)\langle\alpha_k,\gamma e_i\rangle\right)}{(t-x)(t-z_k)}\right)\Psi_i(x)\right]$$
$$+I^2_{tot}\left[\delta_{i=j}\frac{h_2(e_i)w_{\hat{i}}(\beta)\times-\langle\gamma e_i,\gamma e_i\rangle}{2(t-x)(t-y)}\Psi_{i,j}(x,y)\right]+I^2_{tot}\left[\delta_{i\neq j}\frac{-2\gamma^2h_2(\beta)}{(t-x)(t-y)}\Psi_{i,j}(x,y)\right].$$

Proceeding along the very same lines as in Lemma 3.3.5 using symmetrization and integration by parts, we can rewrite the latter (up to convergent terms) as

$$\begin{aligned}
&- I_{sing}\left[\partial_x\left(\left(\frac{2h_2(e_i)\omega_{\hat{i}}(\beta)}{t-x} - \sum_{k=1}^{2N+M}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{x-z_k}\right)\Psi_i(x)\right)\right]\\
&+ I_{sing}\times I_{tot}\left[\frac{h_2(e_i)\omega_j(\beta)\langle\gamma e_i,\gamma e_j\rangle}{(t-x)(x-y)}\Psi_{i,j}(x,y)\right]\\
&+ I_{tot}^2\left[\delta_{i=j}\frac{h_2(e_i)w_{\hat{i}}(\beta)\times\langle\gamma e_i,\gamma e_i\rangle}{(t-x)(y-x)}\Psi_{i,j}(x,y)\right] + I_{tot}^2\left[\delta_{i\neq j}\frac{-2\gamma^2 h_2(\beta)}{(t-x)(t-y)}\Psi_{i,j}(x,y)\right].
\end{aligned}$$

The only remaining two-fold integrals are thus given by

$$I_{sing}\times I_{tot}\left[\left(\delta_{i\neq j}\frac{-\gamma^2 h_2(e_i)\omega_j(\beta)}{(t-x)(x-y)} + \frac{-2\gamma^2 h_2(\beta)}{(t-x)(t-y)}\right)\Psi_{i,j}(x,y)\right].$$

Since for $i\neq j$, $h_2(e_i)\omega_j(\beta)+2h_2(\beta) = h_2(e_i)\langle e_i,\beta\rangle$ the latter is actually equal (again up to convergent terms) to, as expected,

$$I_{sing}\times I_{tot}\left[\delta_{i\neq j}\partial_x\left(\frac{2\gamma h_2(e_i)}{y-x}\Psi_{i,j}(x,y)\right)\right]$$

□

We now turn to the W-descendants at level higher than two.

Lemma 3.3.9. *For any positive integer n and i in $1,2$, set*

$$\begin{aligned}
\mathfrak{W}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) &:= I_{sing}\left[\partial_x\left(\left(\frac{h_2(e_i)((n-2)q+2\omega_{\hat{i}}(\beta))}{(x-t)^{n-1}} + \sum_{k=1}^{2N+M}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{(x-t)^{n-2}(x-z_k)}\right)\Psi_i(x)\right)\right] \qquad (3.24)\\
&+ I_{sing}\times I_{tot}\left[\partial_x\left(\frac{2\gamma h_2(e_i)\delta_{i\neq j}}{(x-t)^{n-2}(y-x)}\Psi_{i,j}(x,y)\right)\right].
\end{aligned}$$

Then the following quantity is (P)-class:

$$\langle \mathbf{W}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho} - \sum_{i=1}^2\mathfrak{W}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).$$

Proof. The result follows from the proof of Theorem 3.3.11 in the next subsection. □

Definition 3.3.10. Take $t\in\mathbb{R}$ and $(\beta,\boldsymbol{\alpha})\in\mathscr{A}_{N,M+1}$. For $n\geq 3$ we define the descendant field $\mathbf{W}_{-n}V_\beta$ within half-plane correlation functions by the limit

$$\begin{aligned}&\langle \mathbf{W}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle :=\\ &\lim_{\delta,\varepsilon,\rho\to 0}\langle \mathbf{W}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}-\sum_{i=1}^2\widetilde{\mathfrak{W}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).\end{aligned}$$

Like before, these limits are well-defined and analytic in a complex neighborhood of $\mathscr{A}_{N,M+1}$. The explicit expression for the remainder is

$$\begin{aligned}&\widetilde{\mathfrak{W}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})\\ &=h_2(e_i)((n-2)q+2\omega_{\hat{i}}(\beta))\left(\mu_{i,L}\frac{\Psi_i(t-\varepsilon)}{(-\varepsilon)^{n-1}}-\mu_{i,R}\frac{\Psi_i(t-\varepsilon)}{\varepsilon^{n-1}}\right)\\ &+\sum_k 2h_2(e_i)\omega_{\hat{i}}(\alpha_k)\left(\mu_{i,L}\frac{\Psi_i(t-\varepsilon)}{(-\varepsilon)^{n-2}(t-\varepsilon-z_k)}-\mu_{i,R}\frac{\Psi_i(t+\varepsilon)}{\varepsilon^{n-2}(t+\varepsilon-z_k)}\right) \qquad (3.25)\\ &-\mu_{B,i}\int_{(t-r,t+r)}\mathrm{Im}\left(\frac{h_2(e_i)((n-2)q+2\omega_{\hat{i}}(\beta))}{(x+\mathrm{i}\delta-t)^{n-1}}+\sum_k\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{(x+\mathrm{i}\delta-t)^{n-2}(x+\mathrm{i}\delta-z_k)}\right)\Psi_i(x+\mathrm{i}\delta)dx\\ &+I_{tot}\left[2\gamma h_2(e_i)\delta_{i\neq j}\left(\mu_{i,L}\frac{\Psi_{i,j}(t-\varepsilon,y)}{(s_l-\varepsilon-t)^{n-2}(y-s_l+\varepsilon)}-\mu_{i,R}\frac{\Psi_{i,j}(t+\varepsilon,y)}{(s_l+\varepsilon-t)^{n-2}(y-s_l-\varepsilon)}\right)\right.\\ &\left.+\mu_{B,i}\int_{(t-r,t+r)}\left(\frac{2\gamma h_2(e_i)\delta_{i\neq j}}{(x+\mathrm{i}\delta-t)^{n-2}(y-x-\mathrm{i}\delta)}-\frac{2\gamma h_2(e_i)\delta_{i\neq j}}{(x-\mathrm{i}\delta-t)^{n-2}(y-x+\mathrm{i}\delta)}\right)\Psi_{i,j}(x+\mathrm{i}\delta,y)dx\right]\end{aligned}$$

3.3.3.2 Ward identities for the higher-spin current

In the same fashion as the Virasoro descendants, the insertion of W-descendants at order higher than three within correlation functions gives rise to Ward identities but this time associated to higher-spin symmetry. They take the following form:

Theorem 3.3.11. *Take $t\in\mathbb{R}$ and $(\beta,\boldsymbol{\alpha})\in\mathscr{A}_{N,M+1}$. Then for any $n\geq 3$:*

$$\begin{aligned}&\langle \mathbf{W}_{-n}V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle = \qquad (3.26)\\ &\left(\sum_{k=1}^{2N+M}\frac{-\mathscr{W}^{(k)}_{-2}}{(z_k-t)^{n-2}}+\frac{(n-2)\mathscr{W}^{(k)}_{-1}}{(z_k-t)^{n-1}}-\frac{(n-1)(n-2)w(\alpha_k)}{2(z_k-t)^n}\right)\langle V_\beta(t)\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle\end{aligned}$$

where for $j=1,2$, $\mathscr{W}^{(k)}_{-j}\langle V_\beta(t)\mathbf{V}\rangle := \langle \mathbf{W}_{-j}V_{\alpha_k}(z_k)\prod_{l\neq k}V_{\alpha_l}(z_l)\rangle$, and $\mathscr{W}^{(k)}_{-j}=\overline{\mathscr{W}}^{(k-N)}_{-j}$ for $k\in\{N+1,\ldots,2N\}$.

Proof. The free-field theory. In order to make the exposition as clear as possible we start by considering the free-field theory, that is when $\mu_i=\mu_i^\partial=0$. In this setting we have to assume that $\sum_k\alpha_k+\frac{1}{2}\sum_l\beta_l-Q=0$ in order for the Gaussian integration by parts formula (Lemma 3.2.3) to be valid[7]. To start with from the explicit definitions of the W-descendants and thanks to Lemma 3.2.3 we see that the right-hand side of (3.26) is given by

$$\begin{aligned}
&\left(\sum_{k=1}^{2N+M} -\frac{\mathscr{W}_{-2}^{(k)}}{(z_k-t)^{n-2}} + \frac{(n-2)\mathscr{W}_{-1}^{(k)}}{(z_k-t)^{n-1}} - \frac{(n-1)(n-2)w(\alpha_k)}{2(z_k-t)^n}\right)\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}^{FF} \qquad (3.27)\\
&= -\left[\sum_{k=1}^{2N+M}\left(\frac{(n-1)(n-2)w(\alpha_k)-(n-2)(qB(\alpha_k,\beta)+2C(\alpha_k,\alpha_k,\beta))}{2(z_k-t)^n}\right.\right.\\
&\left.+\frac{q(B(\beta,\alpha_k)-B(\alpha_k,\beta))+2C(\beta-\alpha_k,\alpha_k,\beta)}{2(z_k-t)^n}\right)\\
&+\sum_{l\neq k}\left(\frac{-(n-2)(qB(\alpha_k,\alpha_l)+2C(\alpha_k,\alpha_k,\alpha_l))+2C(\alpha_k,\beta,\alpha_l)+2C(\alpha_k,\alpha_l,\beta)}{2(z_k-t)^{n-1}(z_k-z_l)}\right.\\
&\left.+\frac{q(B(\alpha_l,\alpha_k)-B(\alpha_k,\alpha_l))+2C(\alpha_l-\alpha_k,\alpha_k,\alpha_l)}{2(z_k-t)^{n-2}(z_k-z_l)^2}\right)\\
&\left.+\sum_{m\neq l,k}\frac{C(\alpha_k,\alpha_l,\alpha_m)}{(z_k-t)^{n-2}(z_k-z_l)(z_k-z_m)}\right]\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}^{FF}
\end{aligned}$$

where the notation $\langle\cdot\rangle^{FF}$ refers to correlation functions with respect to the free field.

On the other hand, based on the defining formula (3.8) and using again Lemma 3.2.3 we see that the left-hand side of (3.26) is equal to :

$$\begin{aligned}
&\langle \mathbf{W}_{-n}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}^{FF} \qquad (3.28)\\
&= -\left[\sum_{k=1}^{2N+M}\left(\frac{qB(\beta,\alpha_k)+2C(\beta,\beta,\alpha_k)-(n-1)(qB(\alpha_k,\beta)+2C(\alpha_k,\alpha_k,\beta))}{2(z_k-t)^n}\right.\right.\\
&\left.+\frac{(n-1)(n-2)(-q^2h_2(\alpha_k)+\frac{q}{2}B(\alpha_k,\alpha_k)+\frac{1}{3}C(\alpha_k,\alpha_k,\alpha_k))}{2(z_k-t)^n}\right)\\
&+\sum_{l\neq k}\left(\sum_{p=1}^{n-1}\frac{q(p-1)B(\alpha_k,\alpha_l)-2C(\alpha_k,\alpha_l,\beta)+2(p-1)C(\alpha_k,\alpha_k,\alpha_l)}{2(z_k-t)^p(z_l-t)^{n-p}}\right.\\
&\left.\left.+\sum_{m\neq l,k}\sum_{p_1=1}^{n-2}\sum_{p_2=1}^{p_1}\frac{C(\alpha_k,\alpha_l,\alpha_m)}{3(z_k-t)^{p_2}(z_l-t)^{p_1-p_2+1}(z_m-t)^{n-p_1-1}}\right)\right]\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}^{FF}.
\end{aligned}$$

Elementary calculations involving the explicit expressions for B and C show that the $(z_k-t)^{-n}$ terms in (3.28) and (3.27) are actually equal. In addition to the symmetrization identity from Equation (3.20) we use the following additional symmetrization identities to transform the terms involving several insertions in Equation (3.28):

$$\sum_{p=1}^{n-1}\frac{p-1}{(t-x)^p(t-y)^{n-p}} = \frac{1}{(y-x)^2}\left(\frac{1}{(t-y)^{n-2}}-\frac{1}{(t-x)^{n-2}}\right)-\frac{n-2}{(y-x)(t-x)^{n-1}}; \qquad (3.29)$$

7. This is known in Physics as a *neutrality condition*

$$\sum_{p_1=1}^{n-2}\sum_{p_2=1}^{p_1}\frac{1}{(t-x)^{p_2}(t-y)^{p_1-p_2+1}(t-z)^{n-p_1-1}}=\frac{1}{(z-y)(z-x)(t-z)^{n-2}} \\ +\frac{1}{(y-x)(y-z)(t-y)^{n-2}}+\frac{1}{(x-y)(x-z)(t-x)^{n-2}}. \tag{3.30}$$

Now we can use Equations (3.20) and (3.29) to symmetrize the terms involving k,l insertions in the left-hand side of the Ward identity (3.28). Doing so explicit computations show that this yields precisely the corresponding term in the right-hand side of Equation (3.27). For the last line of (3.28) we use the identity (3.30) together with symmetry between the z_k, z_l and z_m variables: we then see that this exactly matches the last line of Equation (3.27). Finally, we deduce that the Ward identity for the free-field theory holds true:

$$\langle \mathbf{W}_{-n}V_\beta(t)\mathbf{V}\rangle^{FF} \\ =\left(\sum_{k=1}^{2N+M}\frac{-\mathcal{W}_{-2}^{(k)}}{(z_k-t)^{n-2}}+\frac{(n-2)\mathcal{W}_{-1}^{(k)}}{(z_k-t)^{n-1}}-\frac{(n-1)(n-2)w(\alpha_k)}{2(z_k-t)^n}\right)\langle V_\beta(t)\mathbf{V}\rangle^{FF}$$

after taking the limit as $\varepsilon,\delta\to 0$.

The full theory. The reasoning is the same in the presence of the GMC potential except that we have to work at the regularized level and keep track of the remainder terms that may occur. At such our goal is to show that

$$\lim_{\delta,\varepsilon,\rho\to 0}\left(\mathcal{W}_{-n}-\left(\sum_{k=1}^{2N+M}\frac{-\mathcal{W}_{-2}^{(k)}}{(z_k-t)^{n-2}}+\frac{(n-2)\mathcal{W}_{-1}^{(k)}}{(z_k-t)^{n-1}}-\frac{(n-1)(n-2)w(\alpha_k)}{2(z_k-t)^n}\right)\right)\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} \\ -\sum_{i=1}^{2}\mathfrak{R}^i(\delta,\varepsilon,\rho;\boldsymbol{\alpha})=0,\quad\text{with} \\ \mathfrak{R}^i(\delta,\varepsilon,\rho;\boldsymbol{\alpha}):=\widetilde{\mathfrak{W}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})-\sum_{l=1}^{M}\frac{-\widetilde{\mathfrak{W}}^i_{-2,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l)}{(s_l-t)^{n-2}}+\frac{(n-2)\widetilde{\mathfrak{W}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l)}{(s_l-t)^{n-1}}. \tag{3.31}$$

In order to achieve this, we use that the correlation function $\langle \mathbf{W}_{-n}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}$ is the sum of four terms $I_0+I_1+I_2+I_3$, corresponding respectively to the free-field (i.e. not integrated) part, and 1, 2 and 3-fold integrals. Likewise the terms in the right-hand side of Equation (3.26) take the form $\widetilde{I}_0+\widetilde{I}_1+\widetilde{I}_2$. The equality $I_0=\widetilde{I}_0$ has already been checked since it amounts to treating the free-field case. Therefore our goal is to show that the difference $I_1+I_2+I_3-(\widetilde{I}_1+\widetilde{I}_2+\widetilde{I}_3)$ is given by the corresponding remainder up to vanishing terms.

One-fold integrals. And to start with let us first compute the part in the right-hand side of the Ward identity (3.26) that involves only one-fold integrals. Using again the definitions of the descendants we find by Gaussian integration by parts

$$\widetilde{I}_1 = I_{tot}\left[\sum_{k=1}^{2N+M}\left(\frac{-(n-2)(qB(\alpha_k,\gamma e_i)+2C(\alpha_k,\alpha_k,\gamma e_i))+2C(\alpha_k,\gamma e_i,\beta)+2C(\alpha_k,\beta,\gamma e_i)}{2(z_k-t)^{n-1}(z_k-x)}\right.\right.$$
$$+\frac{q(B(\gamma e_i,\alpha_k)-B(\alpha_k,\gamma e_i))+2C(\gamma e_i-\alpha_k,\alpha_k,\gamma e_i)}{2(z_k-t)^{n-2}(z_k-x)^2}$$
$$\left.\left.+\sum_{l\neq k}\frac{C(\alpha_k,\gamma e_i,\alpha_l)+C(\alpha_k,\alpha_l,\gamma e_i)}{(z_k-t)^{n-2}(z_k-z_l)(z_k-x)}\right)\Psi_i(x)\right]$$
$$+\sum_{i=1}^{2}\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\sum_{k=1}^{2N+M}\frac{2C(\alpha_k,\gamma e_i,\gamma e_i)}{(z_k-t)^{n-2}(x-z_k)(\bar{x}-z_k)}\Psi_i(x)dxd\bar{x}.$$

As for the left-hand side of Equation (3.26), we find by the same method:

$$I_1 = I_{tot}\left[\left(\frac{qB(\beta,\gamma e_i)+2C(\beta,\beta,\gamma e_i)-(n-1)(qB(\gamma e_i,\beta)+2C(\beta,\gamma e_i,\gamma e_i))}{2(x-t)^n}\right.\right.$$
$$+\frac{(n-1)(n-2)(-q^2h_2(\gamma e_i)+\frac{q}{2}B(\gamma e_i,\gamma e_i)+\frac{1}{3}C(\gamma e_i,\gamma e_i,\gamma e_i))}{2(x-t)^n}$$
$$+\sum_{k=1}^{2N+M}\left(\sum_{p=1}^{n-1}\frac{(p-1)(qB(\gamma e_i,\alpha_k)+2C(\gamma e_i,\gamma e_i,\alpha_k))-2C(\gamma e_i,\alpha_k,\beta)}{2(x-t)^p(z_k-t)^{n-p}}\right.$$
$$+\frac{(p-1)(qB(\alpha_k,\gamma e_i)+2C(\alpha_k,\alpha_k,\gamma e_i))-2C(\alpha_k,\gamma e_i,\beta)}{(z_k-t)^p(x-t)^{n-p}}$$
$$+\frac{1}{3}\sum_{l\neq k}\sum_{p_1=1}^{n-2}\sum_{p_2=1}^{p_1}\left(\frac{C(\alpha_k,\gamma e_i,\alpha_l)}{(z_k-t)^{p_2}(x-t)^{p_1-p_2+1}(z_l-t)^{n-p_1-1}}+\frac{C(\gamma e_i,\alpha_k,\alpha_l)}{(x-t)^{p_2}(z_k-t)^{p_1-p_2+1}(z_l-t)^{n-p_1-1}}\right.$$
$$\left.\left.\left.\left.+\frac{C(\alpha_k,\alpha_l,\gamma e_i)}{(z_k-t)^{p_2}(z_l-t)^{p_1-p_2+1}(x-t)^{n-p_1-1}}\right)\right)\right)\Psi_i(x)\right]$$
$$+\sum_{i=1}^{2}\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\sum_{p=1}^{n-1}\frac{(p-1)(qB(\gamma e_i,\gamma e_i)-2C(\gamma e_i,\gamma e_i,\gamma e_i))+2C(\gamma e_i,\gamma e_i,\beta)}{2(x-t)^p(\bar{x}-t)^{n-p}}\right.$$
$$+\frac{1}{3}\sum_{k=1}^{2N+M}\sum_{p_1=1}^{n-2}\sum_{p_2=1}^{p_1}\left(\frac{C(\alpha_k,\gamma e_i,\gamma e_i)}{(z_k-t)^{p_2}(x-t)^{p_1-p_2+1}(\bar{x}-t)^{n-p_1-1}}+\frac{C(\alpha_k,\gamma e_i,\gamma e_i)}{(x-t)^{p_2}(z_k-t)^{p_1-p_2+1}(\bar{x}-t)^{n-p_1-1}}\right.$$
$$\left.\left.+\frac{C(\alpha_k,\gamma e_i,\gamma e_i)}{(x-t)^{p_2}(\bar{x}-t)^{p_1-p_2+1}(z_k-t)^{n-p_1-1}}\right)\right)\Psi_i(x)dxd\bar{x}.$$

First of all algebraic computations show that the $(t-x)^{-n}$ coefficient in the above equals

$$h_2(e_i)((n-2)q+2\omega_{\hat{i}}(\beta))(n-1+\frac{\langle\beta,\gamma e_i\rangle}{2}) =: J^{\beta}_{-n}(e_i).$$

Besides, we can symmetrize the I_1 term like before to obtain

$$
\begin{aligned}
I_1 = I_{tot} \Bigg[\Bigg(& \frac{J^{\beta}_{-n}(e_i)}{(x-t)^n} + \sum_{k=1}^{2N+M} \Bigg(\frac{q(B(\gamma e_i,\alpha_k) - B(\alpha_k,\gamma e_i)) + 2C(\gamma e_i - \alpha_k,\alpha_k,\gamma e_i)}{2(z_k-t)^{n-2}(z_k-x)^2} \\
&+ \frac{q(B(\alpha_k,\gamma e_i) - B(\gamma e_i,\alpha_k)) + 2C(\alpha_k - \gamma e_i,\alpha_k,\gamma e_i)}{2(x-t)^{n-2}(z_k-x)^2} \\
&- \frac{-(n-2)(qB(\gamma e_i,\alpha_k) + 2C(\gamma e_i,\gamma e_i,\alpha_k)) + 2C(\alpha_k,\gamma e_i,\beta) + 2C(\alpha_k,\beta,\gamma e_i)}{2(x-t)^{n-1}(z_k-x)} \\
&+ \frac{-(n-2)(qB(\alpha_k,\gamma e_i) + 2C(\alpha_k,\alpha_k,\gamma e_i)) + 2C(\alpha_k,\gamma e_i,\beta) + 2C(\alpha_k,\beta,\gamma e_i)}{2(z_k-t)^{n-1}(z_k-x)} \\
&+ \sum_{l\neq k} \frac{C(\alpha_k,\gamma e_i,\alpha_l) + C(\alpha_k,\alpha_l,\gamma e_i)}{(z_k-t)^{n-2}(z_k-z_l)(z_k-x)} + \frac{C(\gamma e_i,\alpha_k,\alpha_l)}{(x-t)^{n-2}(z_k-x)(z_l-x)} \Bigg) \Psi_i(x) \Bigg] \\
&+ \sum_{i=1}^{2} \mu_{B,i} \int_{\mathbb{H}_{\delta,\varepsilon}} \Bigg(\frac{(n-2)(qB(\gamma e_i,\gamma e_i) + 2C(\gamma e_i,\gamma e_i,\gamma e_i)) - 4C(\gamma e_i,\gamma e_i,\beta)}{2(x-t)^{n-1}(x-\bar{x})} \\
&+ \sum_{k=1}^{2N+M} \frac{2C(\alpha_k,\gamma e_i,\gamma e_i)}{(x-t)^{n-2}(z_k-x)(x-\bar{x})} + \frac{2C(\alpha_k,\gamma e_i,\gamma e_i)}{(z_k-t)^{n-2}(z_k-x)(z_k-\bar{x})} \Bigg) \Psi_i(x)\, dx d\bar{x}.
\end{aligned}
$$

Therefore we can write that the difference $I_1 - \widetilde{I}_1$ is given by

$$
\begin{aligned}
I_{tot} \Bigg[\Bigg(& \frac{J^{\beta}_{-n}(e_i)}{(x-t)^n} + \sum_{k=1}^{2N+M} \Bigg(\frac{q(B(\alpha_k,\gamma e_i) - B(\gamma e_i,\alpha_k)) + 2C(\alpha_k - \gamma e_i,\alpha_k,\gamma e_i)}{2(x-t)^{n-2}(z_k-x)^2} \\
&+ \frac{(n-2)(qB(\gamma e_i,\alpha_k) + 2C(\gamma e_i,\gamma e_i,\alpha_k)) - 2C(\alpha_k,\gamma e_i,\beta) - 2C(\alpha_k,\beta,\gamma e_i)}{2(x-t)^{n-1}(z_k-x)} \\
&+ \sum_{l\neq k} \frac{C(\gamma e_i,\alpha_k,\alpha_l)}{(x-t)^{n-2}(z_k-x)(z_l-x)} \Bigg) \Psi_i(x) \Bigg] \\
&+ \sum_{i=1}^{2} \mu_{B,i} \int_{\mathbb{H}_{\delta,\varepsilon}} \Bigg(\frac{(n-2)(qB(\gamma e_i,\gamma e_i) + 2C(\gamma e_i,\gamma e_i,\gamma e_i)) - 4C(\gamma e_i,\gamma e_i,\beta)}{2(x-t)^{n-1}(x-\bar{x})} \\
&+ \sum_{k=1}^{2N+M} \frac{2C(\alpha_k,\gamma e_i,\gamma e_i)}{(x-t)^{n-2}(z_k-x)(x-\bar{x})} \Bigg) \Psi_i(x)\, dx d\bar{x}.
\end{aligned}
$$

Now we can use the following identities (proved using the explicit expressions for B and C):

$$
\begin{aligned}
& qB(\gamma e_i,\alpha_k) + 2C(\gamma e_i,\gamma e_i,\alpha_k) = qh_2(e_i)\langle\gamma e_i,\alpha_k\rangle + 4h_2(e_i)\omega_{\hat{i}}(\alpha_k); \qquad (3.32) \\
& C(\alpha_k,\gamma e_i,\beta) + C(\alpha_k,\beta,\gamma e_i) = -h_2(e_i)\left(\omega_{\hat{i}}(\beta)\langle\gamma e_i,\alpha_k\rangle + \omega_{\hat{i}}(\alpha_k)\langle\gamma e_i,\beta\rangle\right); \\
& q(B(\alpha_k,\gamma e_i) - B(\gamma e_i,\alpha_k)) + 2C(\alpha_k - \gamma e_i,\alpha_k,\gamma e_i) = -4h_2(e_i)\omega_{\hat{i}}(\alpha_k)\left(1 + \frac{\langle\gamma e_i,\alpha_k\rangle}{2}\right)
\end{aligned}
$$

to rewrite the above under the form

$$I_{tot}\left[\left(\frac{h_2(e_i)((n-2)q+2\omega_{\hat{i}}(\beta))(n-1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{(x-t)^n}+\sum_{k=1}^{2N+M}\left(\frac{-4h_2(e_i)\omega_{\hat{i}}(\alpha_k)\left(1+\frac{\langle\gamma e_i,\alpha_k\rangle}{2}\right)}{2(x-t)^{n-2}(z_k-x)^2}\right.\right.\right.$$
$$+\frac{(n-2)(qh_2(e_i)\langle\gamma e_i,\alpha_k\rangle+4h_2(e_i)\omega_{\hat{i}}(\alpha_k))+2h_2(e_i)\left(\omega_{\hat{i}}(\beta)\langle\gamma e_i,\alpha_k\rangle+\omega_{\hat{i}}(\alpha_k)\langle\gamma e_i,\beta\rangle\right)}{2(x-t)^{n-1}(z_k-x)}$$
$$\left.\left.+\sum_{l\neq k}\frac{-4h_2(e_i)\omega_{\hat{i}}(\alpha_k)\langle\gamma e_i,\alpha_l\rangle}{(x-t)^{n-2}(z_k-x)(z_l-x)}\right)\Psi_i(x)\right]$$
$$+\sum_{i=1}^{2}\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\frac{(n-2)(qB(\gamma e_i,\gamma e_i)+2C(\gamma e_i,\gamma e_i,\gamma e_i))-4C(\gamma e_i,\gamma e_i,\beta)}{2(x-t)^{n-1}(x-\bar{x})}\right.$$
$$\left.+\sum_{k=1}^{2N+M}\frac{2C(\alpha_k,\gamma e_i,\gamma e_i)}{(x-t)^{n-2}(z_k-x)(x-\bar{x})}\right)\Psi_i(x)dxd\bar{x}.$$

Like before we recognize derivatives of the correlation functions. More precisely we end up with

$$I_1-\widetilde{I}_1=-I_{tot}\left[\partial_x\left(\left(\frac{-h_2(e_i)((n-2)q+2\omega_{\hat{i}}(\beta))}{(x-t)^{n-1}}+\sum_{k=1}^{2N+M}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{(x-t)^{n-2}(z_k-x)}\right)\Psi_i(x)\right)\right] \tag{3.33}$$
$$+I_{tot}\times I_{tot}\left[\left(\frac{-h_2(e_i)((n-2)q+2\omega_{\hat{i}}(\beta))\langle\gamma e_i,\gamma e_j\rangle}{(x-t)^{n-1}(x-y)}+\sum_{k=1}^{2N+M}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)\langle\gamma e_i,\gamma e_j\rangle}{(x-t)^{n-2}(z_k-x)(x-y)}\right)\Psi_i(x)\right]$$
$$+\sum_{i=1}^{2}\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}}\left(\frac{(n-2)(qB(\gamma e_i,\gamma e_i)+2C(\gamma e_i,\gamma e_i,\gamma e_i))-4C(\gamma e_i,\gamma e_i,\beta)}{2(x-t)^{n-1}(x-\bar{x})}\right.$$
$$\left.+\sum_{k=1}^{2N+M}\frac{2C(\alpha_k,\gamma e_i,\gamma e_i)}{(x-t)^{n-2}(z_k-x)(x-\bar{x})}\right)\Psi_i(x)dxd\bar{x}.$$

The first integral corresponds exactly to the one in Lemma 3.3.9. The terms involving $x,\bar{x}$ are treated exactly the same way using the identities (3.32) with either β or α_k replaced by γe_i. The second line of the above will show up when considering the two-fold integrals, which is our next task.

Two-fold integrals. The two-fold integrals in the l.h.s. of the Ward identity (3.26) are given by (we voluntarily omit the terms involving combinations of the variables x,y

with $\bar{x}$ or $\bar{y}$, since they can be dealt with in the exact same way)

$$I_2 = -I_{tot}^2 \left[\left(\sum_{p=1}^{n-1} \frac{q(p-1)B(\gamma e_i, \gamma e_j) - 2C(\gamma e_i, \gamma e_j, \beta) + 2(p-1)C(\gamma e_i, \gamma e_i, \gamma e_j)}{2(x-t)^p (y-t)^{n-p}} \right.\right.$$
$$+ \frac{1}{3} \sum_{k=1}^{2N+M} \sum_{p_1=1}^{n-2} \sum_{p_2=1}^{p_1} \left(\frac{C(\alpha_k, \gamma e_i, \gamma e_j)}{(z_k-t)^{p_2}(x-t)^{p_1-p_2+1}(y-t)^{n-p_1-1}} + \frac{C(\gamma e_i, \alpha_k, \gamma e_j)}{(x-t)^{p_2}(z_k-t)^{p_1-p_2+1}(y-t)^{n-p_1-1}} \right.$$
$$\left.\left.\left. + \frac{C(\gamma e_i, \gamma e_j, \alpha_k)}{(x-t)^{p_2}(y-t)^{p_1-p_2+1}(z_k-t)^{n-p_1-1}} \right) \right) \Psi_{i,j}(x,y) \right].$$

Once we have symmetrized all the terms we are left with

$$-I_{tot}^2 \left[\left(\left(\frac{1}{(y-x)^2} \left(\frac{1}{(y-t)^{n-2}} - \frac{1}{(x-t)^{n-2}} \right) - \frac{n-2}{(x-y)(x-t)^{n-1}} \right) (\frac{q}{2} B(\gamma e_i, \gamma e_j) + C(\gamma e_i, \gamma e_i, \gamma e_j)) \right.\right.$$
$$- \frac{1}{y-x} \left(\frac{1}{(y-t)^{n-1}} - \frac{1}{(x-t)^{n-1}} \right) C(\gamma e_i, \gamma e_j, \beta) + \sum_{k=1}^{2N+M} \left(\frac{C(\alpha_k, \gamma e_i, \gamma e_j)}{(z_k-x)(z_k-y)(z_k-t)^{n-2}} \right.$$
$$\left.\left.\left. + \frac{C(\alpha_k, \gamma e_i, \gamma e_j) + C(\alpha_k, \gamma e_j, \gamma e_i)}{(x-y)(x-z_k)(x-t)^{n-2}} \right) \right) \Psi_{i,j}(x,y) \right].$$

Using symmetry between the integration variables allows one to reduce the latter to

$$-I_{tot}^2 \left[\left(- \frac{(n-2)(\frac{q}{2} B(\gamma e_i, \gamma e_j) + C(\gamma e_i, \gamma e_i, \gamma e_j)) - C(\gamma e_i, \gamma e_j, \beta) - C(\gamma e_j, \gamma e_i, \beta)}{(x-y)(x-t)^{n-1}} \right.\right.$$
$$- \frac{\frac{q}{2}(B(\gamma e_i, \gamma e_j) - B(\gamma e_j, \gamma e_i)) + C(\gamma e_i, \gamma e_i, \gamma e_j) - C(\gamma e_j, \gamma e_j, \gamma e_i)}{(y-x)^2 (x-t)^{n-2}}$$
$$\left.\left. + \sum_{k=1}^{2N+M} \frac{C(\alpha_k, \gamma e_i, \gamma e_j) + C(\alpha_k, \gamma e_j, \gamma e_i)}{(x-y)(x-z_k)(x-t)^{n-2}} \right) \right) \Psi_{i,j}(x,y) \right]$$
$$-I_{tot}^2 \left[\sum_{k=1}^{2N+M} \frac{C(\alpha_k, \gamma e_i, \gamma e_j)}{(z_k-t)^{n-2}(x-z_k)(y-z_k)} \Psi_{i,j}(x,y) \right].$$

The last line corresponds exactly to the two-fold integral $\widetilde{I}_2$ in the r.h.s. of the Ward identity. For the other terms we use the identities (derived from Equation (3.32))

$$(n-2)(\frac{q}{2} B(\gamma e_i, \gamma e_j) + C(\gamma e_i, \gamma e_i, \gamma e_j)) - C(\gamma e_i, \gamma e_j, \beta) - C(\gamma e_j, \gamma e_i, \beta)$$
$$= 2\gamma h_2(e_i) \delta_{i\neq j} \left((n-2) + \frac{\langle \gamma e_i, \beta \rangle}{2} \right) + h_2(e_i)((n-2)q + 2\omega_{\hat{i}}(\beta)) \frac{\langle \gamma e_i, \gamma e_j \rangle}{2};$$

$$\frac{q}{2}(B(\gamma e_i, \gamma e_j) - B(\gamma e_j, \gamma e_i)) + C(\gamma e_i, \gamma e_i, \gamma e_j) - C(\gamma e_j, \gamma e_j, \gamma e_i) = 2\gamma h_2(e_i) \left(1 + \frac{\langle \gamma e_i, \gamma e_j \rangle}{2} \right) \delta_{i\neq j};$$

$$C(\alpha_k, \gamma e_i, \gamma e_j) + C(\alpha_k, \gamma e_j, \gamma e_i) = -2\gamma h_2(e_i) \frac{\langle \gamma e_i, \alpha_k \rangle}{2} \delta_{i\neq j} - \langle \gamma e_i, \gamma e_j \rangle h_2(e_i) \omega_{\hat{i}}(\alpha_k)$$

to get that the difference $I_2 - \widetilde{I}_2$ is given by

$$-I_{tot}^2\left[-2\gamma h_2(e_i)\delta_{i\neq j}\left(\frac{n-2+\frac{\langle\gamma e_i,\beta\rangle}{2}}{(x-y)(x-t)^{n-1}}+\frac{1+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2}}{(x-y)^2(x-t)^{n-2}}\right.\right.$$
$$\left.+\sum_{k=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_k\rangle}{2(x-y)(x-z_k)(x-t)^{n-2}}\right)\Psi_{i,j}(x,y)$$
$$\left.-\left(\frac{h_2(e_i)((n-2)q+2\omega_{\hat{i}}(\beta))\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)(x-t)^{n-1}}+\sum_{k=1}^{2N+M}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)\langle\gamma e_i,\gamma e_j\rangle}{(x-y)(x-z_k)(x-t)^{n-2}}\right)\Psi_{i,j}(x,y)\right].$$

The second line corresponds to the two-fold integrals that we obtained from the treatment of one-fold integrals in Equation (3.33). As for the first line we rely on the fact that

$$\partial_x\left(\frac{1}{(x-t)^{n-2}(y-x)}\Psi_{i,j}(x,y)\right)=$$
$$\left(\frac{1+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2}}{(x-t)^{n-2}(y-x)^2}+\frac{\frac{\langle\gamma e_i,\gamma e_j\rangle}{2}}{(x-t)^{n-2}(y-x)(\bar{y}-x)}+\frac{\frac{\langle\gamma e_i,\gamma e_i\rangle}{2}}{(x-t)^{n-2}(\bar{x}-x)(y-x)}+\frac{n-2+\frac{\langle\gamma e_i,\beta\rangle}{2}}{(x-t)^{n-1}(x-y)}\right.$$
$$\left.+\sum_{k=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_k\rangle}{2(x-t)^{n-2}(y-x)(z_k-x)}\right)\Psi_{i,j}(x,y)$$
$$-I_{tot}\left[\frac{\langle\gamma e_i,\gamma e_f\rangle}{2(x-t)^{n-2}(y-x)(z-x)}\Psi_{i,j,f}(x,y,z)\right].$$

The integral term that shows up is canceled by the three-fold integrals in the left-hand side of (3.26). Namely these are given by

$$I_3=-I_{tot}^3\left[\sum_{p_1=1}^{n-2}\sum_{p_2=1}^{p_1}\frac{C(\gamma e_i,\gamma e_j,\gamma e_f)}{3(x-t)^{p_2}(y-t)^{p_1-p_2+1}(z-t)^{n-p_1-1}}\Psi_{i,j,f}(x,y,z)\right].$$

Symmetrizing the above and using symmetry between the variables yield

$$I_3=-I_{tot}^3\left[\delta_{i\neq j}\frac{C(\gamma e_i,\gamma e_j,\gamma e_j)}{(x-t)^{n-2}(x-y)(x-z)}\Psi_{i,j,j}(x,y,z)\right].$$

Again, algebraic computations show that

$$C(\gamma e_i,\gamma e_j,\gamma e_j)=-\gamma h_2(e_i)\langle\gamma e_i,\gamma e_j\rangle$$

for $i \neq j$, which allows us to conclude that

$$\langle \mathbf{W}_{-n} V_\beta(t) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon} - \tag{3.34}$$
$$\left(\sum_{k=1}^{2N+M} \frac{-\mathscr{W}_{-2}^{(k)}}{(z_k-t)^{n-2}} + \frac{(n-2)\mathscr{W}_{-1}^{(k)}}{(z_k-t)^{n-1}} - \frac{(n-1)(n-2)w(\alpha_k)}{2(z_k-t)^n}\right) \langle V_\beta(t) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon}$$
$$= I_{tot}\left[\partial_x J^i(x)\right] + I_{tot} \times I_{tot}\left[\partial_x J^{i,j}(x,y)\right].$$

where we have introduced the shorthands

$$J^i(x) = \left(\frac{h_2(e_i)((n-2)q + 2\omega_{\hat{i}}(\beta))}{(x-t)^{n-1}} + \sum_{k=1}^{2N+M} \frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{(x-t)^{n-2}(x-z_k)}\right)\Psi_i(x)$$

and

$$J^{i,j}(x,y) = \frac{2\gamma h_2(e_i)\delta_{i\neq j}}{(x-t)^{n-2}(y-x)}\Psi_{i,j}(x,y).$$

Concluding the proof. In order to wrap up the proof of Theorem 3.3.11, it only remains to show that

$$\lim_{\delta,\varepsilon\to 0} I_{tot}\left[\partial_x J^i(x)\right] + I_{tot} \times I_{tot}\left[\partial_x J^{i,j}(x,y)\right] - \sum_{i=1}^{2} \mathfrak{R}^i(\delta,\varepsilon,\rho;\boldsymbol{\alpha}) = 0.$$

For this purpose we integrate by parts the right-hand side of Equation (3.34), that is, we apply Stokes formula separately over the domains $\mathbb{H}^1_{\delta,\varepsilon} \cup \mathbb{R}^1_\varepsilon$ and $\mathbb{H}^c_{\delta,\varepsilon} \cup \mathbb{R}^c_\varepsilon$. Let us first compute the integrals on $\mathbb{R}^c_\varepsilon$. As already explained, we do not treat the terms at ∞ since they are seen to vanish thanks to Lemma 3.2.4. Around an s_l insertion, we have

$$\int_{\mathbb{R}^c_\varepsilon \cap (s_l - r_l, s_l + r_l)} \partial_x J^i(x)\mu_i(dx) + I_{tot}\left[\int_{\mathbb{R}^c_\varepsilon \cap (s_l - r_l, s_l + r_l)} \partial_x J^{i,j}(x,y)\mu_i(dx)\right]$$
$$= h_2(e_i)((n-2)q + 2\omega_{\hat{i}}(\beta))\left(\mu_i(s_l^-)\frac{\Psi_i(s_l-\varepsilon)}{(s_l-\varepsilon-t)^{n-1}} - \mu_i(s_l^+)\frac{\Psi_i(s_l+\varepsilon)}{(s_l+\varepsilon-t)^{n-1}}\right)$$
$$+ \sum_{k\neq l} 2h_2(e_i)\omega_{\hat{i}}(\alpha_k)\left(\mu_i(s_l^-)\frac{\Psi_i(s_l-\varepsilon)}{(s_l-\varepsilon-t)^{n-2}(s_l-\varepsilon-z_k)} - \mu_i(s_l^+)\frac{\Psi_i(s_l+\varepsilon)}{(s_l+\varepsilon-t)^{n-2}(s_l+\varepsilon-z_k)}\right)$$
$$+ 2h_2(e_i)\omega_{\hat{i}}(\beta_l)\left(\mu_i(s_l^-)\frac{\Psi_i(s_l-\varepsilon)}{(s_l-\varepsilon-t)^{n-2}(-\varepsilon)} - \mu_i(s_l^+)\frac{\Psi_i(s_l+\varepsilon)}{(s_l+\varepsilon-t)^{n-2}\varepsilon}\right) \tag{3.35}$$
$$+ I_{tot}\left[2\gamma h_2(e_i)\delta_{i\neq j}\left(\mu_i(s_l^-)\frac{\Psi_i(s_l-\varepsilon)}{(s_l-\varepsilon-t)^{n-2}(y-s_l+\varepsilon)} - \mu_i(s_l^+)\frac{\Psi_i(s_l+\varepsilon)}{(s_l+\varepsilon-t)^{n-2}(y-s_l-\varepsilon)}\right)\right]$$
$$-\text{same terms with } r_l \text{ instead of } \varepsilon.$$

The terms in the last line of the above expression cancel out when summing over l. Now, we recall that

$$\widetilde{\mathfrak{W}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l) = -h_2(e_i)\left(q - 2\omega_{\hat{i}}(\beta_l)\right)\left(\mu_i(s_l^-)\Psi_i(s_l-\varepsilon) - \mu_i(s_l^+)\Psi_i(s_l+\varepsilon)\right) \quad \text{and}$$

$$\begin{aligned}
&\widetilde{\mathfrak{W}}^i_{-2,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l)\\
&=\left(-\sum_{k\neq l}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{z_k-s_l}-\frac{2h_2(e_i)\omega_{\hat{i}}(\beta)}{t-s_l}\right)\left(\mu_i(s_l^-)\Psi_i(s_l-\varepsilon)-\mu_i(s_l^+)\Psi_i(s_l+\varepsilon)\right)\\
&-\frac{2h_2(e_i)\omega_{\hat{i}}(\beta_l)}{\varepsilon}\left(\mu_i(s_l^-)\Psi_i(s_l-\varepsilon)+\mu_i(s_l^+)\Psi_i(s_l+\varepsilon)\right)\\
&-\mu_{B,i}\int_{(s_l-r_l,s_l+r_l)}\mathrm{Im}\left(\frac{2\gamma h_2(e_i)\omega_{\hat{i}}(\beta_l)}{x+\mathrm{i}\delta-s_l}\right)\Psi_i(x+\mathrm{i}\delta)dx\\
&+I_{tot}\left[2\gamma h_2(e_i)\delta_{i\neq j}\left(\mu_i(s_l^-)\frac{\Psi_{i,j}(s_l-\varepsilon,y)}{y-s_l+\varepsilon}-\mu_i(s_l^+)\frac{\Psi_{i,j}(s_l+\varepsilon,y)}{y-s_l-\varepsilon}\right.\right.\\
&\left.\left.+\mu_{B,i}\int_{(s_l-r_l,s_l+r_l)}\left(\frac{1}{y-x-\mathrm{i}\delta}-\frac{1}{y-x+\mathrm{i}\delta}\right)\Psi_{i,j}(x+\mathrm{i}\delta,y)\frac{\mathrm{i}dx}{2}\right)\right].
\end{aligned}$$

Using these explicit expressions one readily sees that, up to a $o(\varepsilon)$ term, Equation (3.35) corresponds to the part that is on the real line (that is that does not contain a term of the form $x+\mathrm{i}\delta$) of the expected remainder

$$-\sum_{l=1}^{M}\frac{-\widetilde{\mathfrak{W}}^i_{-2,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l)}{(s_l-t)^{n-2}}+\frac{(n-2)\widetilde{\mathfrak{W}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l)}{(s_l-t)^{n-1}}.$$

This is true except for the term

$$\frac{2(n-2)h_2(e_i)\omega_{\hat{i}}(\beta_l)}{(s_l-t)^{n-1}}\left(\mu_i(s_l^-)\Psi_i(s_l-\varepsilon)-\mu_i(s_l^+)\Psi_i(s_l+\varepsilon)\right)$$

coming from $-(n-2)\frac{\widetilde{\mathfrak{W}}^i_{-1,\delta,\varepsilon,\rho}(\boldsymbol{\alpha};\beta_l)}{(s_l-t)^{n-1}}$. Nonetheless, one can recover this term by performing an expansion of the fourth line in Equation (3.35). Indeed, if one writes

$$\frac{1}{(s_l+\varepsilon-t)^{n-2}\varepsilon}=\frac{1}{(s_l-t)^{n-2}\varepsilon}-\frac{n-2}{(s_l-t)^{n-1}}+o(\varepsilon)$$

and

$$\frac{1}{(s_l-\varepsilon-t)^{n-2}\varepsilon}=\frac{1}{(s_l-t)^{n-2}\varepsilon}+\frac{n-2}{(s_l-t)^{n-1}}+o(\varepsilon).$$

Then replacing the fourth line in Equation (3.35) using these identities yields the missing term.

Let us now turn to the integrals over $\mathbb{H}^c_{\delta,\varepsilon}$. When integrating by parts the right-hand side of (3.34) over $\mathbb{H}^c_{\delta,\varepsilon}$ one obtains two kind of terms, see Figure 3.2 (again, we omit the terms at infinity since they are easily shown to vanish thanks to Lemma 3.2.4). The first ones are the contour integrals around the z_k's (in blue on Figure 3.2), that are of the form

$$\oint_{\partial B(z_k,\varepsilon)}\Psi_i(\xi)(\widetilde{J}^i(\xi)\frac{id\bar{\xi}}{2}-\widetilde{J}^i(\bar{\xi})\frac{id\xi}{2})+I_{tot}\left[\oint_{\partial B(z_k,\varepsilon)}\Psi_{i,j}(\xi,y)(\widetilde{J}^{i,j}(\xi,y)\frac{id\bar{\xi}}{2}-\widetilde{J}^{i,j}(\bar{\xi},y)\frac{id\xi}{2})\right].$$

These terms actually tend to 0 in the $\delta,\varepsilon\to 0$ thanks to the fusion asymptotics. The other term is the integral over $(\mathbb{R}+\mathrm{i}\delta\setminus(t-r+\mathrm{i}\delta,t+r+\mathrm{i}\delta))\cup(\mathbb{H}^c_{\delta,\varepsilon}\cap\partial B(t,r))$ (see Figure 3.2). To deal with it, one has to show that the parts of the integral on the segments $(s_l+r_l+\mathrm{i}\delta,s_{l+1}-r_{l+1}+\mathrm{i}\delta)$ vanishes in the limit, since the other parts are the singular terms

$$-\int_{(s_l-r_l,s_l+r_l)}\mathrm{Im}(\widetilde{J}^i(x+\mathrm{i}\delta))\Psi_i(x+\mathrm{i}\delta)dx$$
$$+I_{tot}\left[\int_{(s_l-r_l,s_l+r_l)}(\widetilde{J}^{i,j}(x+\mathrm{i}\delta,y)-\widetilde{J}^{i,j}(x-\mathrm{i}\delta,y))\Psi_{i,j}(x+\mathrm{i}\delta,y)\frac{\mathrm{i}dx}{2}\right]$$

stemming from the definition of the descendant $\mathbf{W}^{\beta_l}_{-2}$ (shown in red on Figure 3.2). This is true like before thanks to fusion asymptotics. The term in green on Figure 3.2 has already been treated above as it was part of the remainder terms.

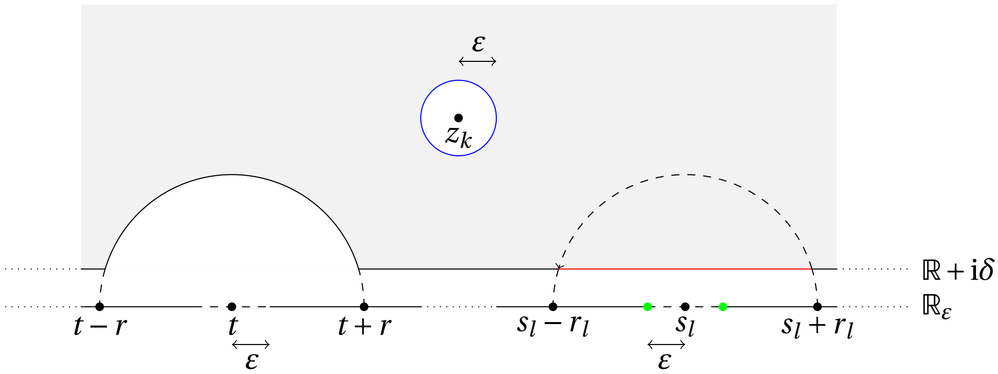

Figure 3.2 – The contours of integration in I_{reg}

At this point we have proven Lemma 3.3.9. It remains to show that $\widetilde{\mathfrak{W}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ and $\mathfrak{W}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ differ by (P)-class terms in order to remove the dependence in r and justify the Definition 3.3.10. For this purpose we integrate by parts the integrals over $\mathbb{H}^1_{\delta,\varepsilon}\cup\mathbb{R}^1_\varepsilon$ (the associated domain of integration is depicted on Figure 3.3 with the remainder terms shown in red).

We obtain

$$\begin{aligned}
&I_{sing}\left[\partial_x J^i(x)\right]+I_{sing}\times I_{tot}\left[\partial_x J^{i,j}(x,y)\right]-\widetilde{\mathfrak{W}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})\\
&=\mu_{R,i}J^i(t+r)-\mu_{L,i}J^i(t-r)+\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}\cap\partial B(t,r)}\Psi_i(\xi)(\widetilde{J}^i(\xi)\frac{\mathrm{i}d\bar{\xi}}{2}-\widetilde{J}^i(\bar{\xi})\frac{\mathrm{i}d\xi}{2})\\
&+I_{tot}\Big[\mu_{R,i}J^{i,j}(t+r,y)-\mu_{L,i}J^{i,j}(t-r,y)\\
&+\mu_{B,i}\int_{\mathbb{H}_{\delta,\varepsilon}\cap\partial B(t,r)}\Psi_{i,j}(\xi,y)(\widetilde{J}^{i,j}(\xi,y)\frac{\mathrm{i}d\bar{\xi}}{2}-\widetilde{J}^{i,j}(\bar{\xi},y)\frac{\mathrm{i}d\xi}{2})\Big].
\end{aligned}\tag{3.36}$$

Here $\widetilde{\mathfrak{W}}^i_{-n,\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ is defined in Equation (3.25), and with $\widetilde{J}$ denoting the function J

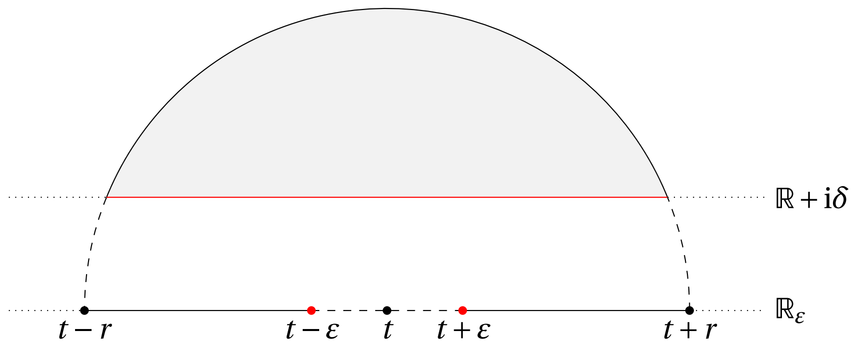

Figure 3.3 – Integration around t

without the correlation function inside. We conclude thanks to the fusion asymptotics that all the terms in the right-hand side of the equality above are (P)-class, thus justifying Definition 3.3.10 and completing the proof of Theorem 3.3.11. □

3.3.4 Global Ward identities

In the previous subsections we have provided rigorous definitions for the descendant fields, and proved that thanks to them it was possible to infer local Ward identities. As we will now see this will allow us to show that *global* Ward identities are valid for our probabilistic model, putting strong constraints on the correlation functions. Such global Ward identities are obtained by inserting *holomorphic tensors* within correlation functions, which can be seen as the $\mathbf{L}_{-2}V_\beta$ and $\mathbf{W}_{-3}V_\beta$ descendants where $\beta = 0$, and that correspond to the conserved currents of the W_3-algebra.

To be more specific, let us assume that $\beta = 0$ and denote respectively $\mathbf{T} := \mathbf{L}_{-2}V_0$ and $\mathbf{W}(t) := \mathbf{W}_{-3}V_0$. Then we see that

$$\begin{aligned}
&\mathbf{T}[\Phi] = \langle Q, \partial^2\Phi\rangle - \langle\partial\Phi,\partial\Phi\rangle \quad \text{and}\\
&\mathbf{W}[\Phi] = q^2 h_2(\partial^3\Phi) - 2qB(\partial^2\Phi,\partial,\Phi) - 8h_1(\partial\Phi)h_2(\partial\Phi)h_3(\partial\Phi).
\end{aligned}$$

They correspond respectively to the *stress-energy tensor* $\mathbf{T}$ and the *higher-spin current* $\mathbf{W}$, already encountered in Subsection 3.2.1.2. Recall that in the language of Vertex Operator Algebra these currents can be thought of as formal power series whose modes generate the W_3 algebra.

3.3.4.1 Local Ward identities for the currents

The local Ward identities are obtained by inserting these currents within correlation functions. Then we see that, in agreement with Theorems 3.3.7 and 3.3.11 we have the following:

Theorem 3.3.12. *Assume that the weights satisfy $\boldsymbol{\alpha} \in \mathscr{A}_{N,M}$ and take $t \in \mathbb{R}$. Then in the sense of weak derivatives*

$$\langle \mathbf{T}(t) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle = \left(\sum_{k=1}^{2N+M} \frac{\partial_{z_k}}{t - z_k} + \frac{\Delta_{\alpha_k}}{(t - z_k)^2} \right) \langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle.$$

As for the higher-spin current we have:

$$\begin{aligned} &\langle \mathbf{W}(t) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle \\ &= \left(\sum_{k=1}^{2N+M} \frac{\mathbf{W}_{-2}^{(k)}}{t - z_k} + \frac{\mathbf{W}_{-1}^{(k)}}{(t - z_k)^2} + \frac{w(\alpha_k)}{(t - z_k)^3} \right) \langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle \end{aligned} \tag{3.37}$$

In the above the quantities that appear are defined by taking $\beta = 0$ in the definition of the $\mathbf{L}_{-2} V_\beta$ and $\mathbf{W}_{-3} V_\beta$ descendants.

3.3.4.2 Global Ward identities

Based on the local Ward identities we will be able to derive global Ward identities. These are related to the symmetries enjoyed by the model in that they describe some conserved quantities, which translate as strong constraints put on the correlation functions. In order to show that such identities hold true we will rely on the conformal covariance of the correlation functions (Equation (3.5)) together with the fact that the stress-energy tensor and the higher-spin current behave like covariant tensors. More specifically we first show that:

Proposition 3.3.13. *Let $\psi \in PSL(2,\mathbb{R})$ be a Möbius transform of the upper-half plane $\mathbb{H}$ and $\boldsymbol{\alpha} \in \mathscr{A}_{N,M}$. Then:*

$$\begin{aligned} &\langle \mathbf{T}(t) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle = \\ &\psi'(t)^2 \prod_{k=1}^{N} \left|\psi'(z_k)\right|^{2\Delta_{\alpha_k}} \prod_{l=1}^{M} \left|\psi'(s_l)\right|^{\Delta_{\beta_l}} \langle \mathbf{T}\big(\psi(t)\big) \prod_{k=1}^{N} V_{\alpha_k}(\psi(z_k)) \prod_{l=1}^{M} V_{\beta_l}(\psi(s_l)) \rangle. \end{aligned}$$

for the stress-energy tensor, while for the higher-spin current:

$$\begin{aligned} &\langle \mathbf{W}(t) \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle = \\ &\psi'(t)^3 \prod_{k=1}^{N} \left|\psi'(z_k)\right|^{2\Delta_{\alpha_k}} \prod_{l=1}^{M} \left|\psi'(s_l)\right|^{\Delta_{\beta_l}} \langle \mathbf{W}\big(\psi(t)\big) \prod_{k=1}^{N} V_{\alpha_k}(\psi(z_k)) \prod_{l=1}^{M} V_{\beta_l}(\psi(s_l)) \rangle. \end{aligned}$$

Proof. To start with we note that for such maps, in agreement with [25, Proposition

4.1], the currents satisfy

$$\mathbf{T}\left[\Phi\circ\psi+Q\ln\left|\psi'\right|\right]=(\psi')^2\mathbf{T}[\Phi]\circ\psi \quad \text{and} \quad \mathbf{W}\left[\Phi\circ\psi+Q\ln\left|\psi'\right|\right]=(\psi')^3\mathbf{W}[\Phi]\circ\psi.$$

As a consequence at the regularized level we have that

$$\begin{aligned}&\psi'(t)^3\langle\mathbf{W}\left(\psi(t)\right)\prod_{k=1}^N V_{\alpha_k}(\psi(z_k))\prod_{l=1}^M V_{\beta_l}(\psi(s_l))\rangle_{\delta,\varepsilon}\\&=\langle\mathbf{W}\left[\Phi\circ\psi+Q\ln\left|\psi'\right|\right](t)\prod_{k=1}^N V_{\alpha_k}(\psi(z_k))\prod_{l=1}^M V_{\beta_l}(\psi(s_l))\rangle_{\delta,\varepsilon}.\end{aligned}$$

Let us now focus on the remainder terms that show up in the definition of $\langle\mathbf{W}\left(\psi(t)\right)\mathbf{V}(\psi(\boldsymbol{z}))\rangle$ (the case of the stress-energy tensor is treated in the exact same way): they are defined from

$$\begin{aligned}&I_{sing}\left[\partial_x\left(\left(\frac{qh_2(e_i)}{(x-\psi(t))^2}+\sum_{k=1}^{2N+M}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{(x-\psi(t))(x-\psi(z_k))}\right)\psi_i(x)\right)\right]\\&+I_{sing}\times I_{tot}\left[\partial_x\left(\frac{2\gamma h_2(e_i)\delta_{i\neq j}}{(x-\psi(t))(y-x)}\psi_{i,j}(x,y)\right)\right].\end{aligned}$$

We can make the changes of variables $x\leftrightarrow\psi(x)$ and $y\leftrightarrow\psi(y)$ in the above integrals. We can then combine Equation (3.5) together with the fact that $\Delta_{\gamma e_i}=1$ and the following property of the Mobiüs transform:

$$(\psi(x)-\psi(t))(\psi(x)-\psi(y))=\psi'(x)\psi'(t)^{\frac12}\psi'(y)^{\frac12}(x-t)(x-y)$$

to get that the remainder term is actually given by $\psi'(t)^{-3}\prod_{k=1}^N\left|\psi'(z_k)\right|^{-2\Delta_{\alpha_k}}\prod_{l=1}^M\left|\psi'(s_l)\right|^{-\Delta_{\beta_l}}$ times the remainder term that would come from the definition of $\langle\mathbf{W}(t)\mathbf{V}\rangle$. This allows to take the $\delta,\varepsilon\to 0$ limit and conclude for the proof. □

Based on the above property we can readily deduce that global Ward identities hold true:

Theorem 3.3.14. *Assume that $\boldsymbol{\alpha}\in\mathcal{A}_{N,M}$. Then the following identities hold true for $0\leq n\leq 2$ and $0\leq m\leq 4$:*

$$\begin{aligned}&\left(\sum_{k=1}^{2N+M}z_k^n\mathcal{L}_{-1}^{(k)}+nz_k^{n-1}\Delta_{\alpha_k}\right)\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle=0\\&\left(\sum_{k=1}^{2N+M}z_k^m\mathcal{W}_{-2}^{(k)}+mz_k^{m-1}\mathcal{W}_{-1}^{(k)}+\frac{m(m-1)}{2}z_k^{m-2}w(\alpha_k)\right)\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\prod_{l=1}^M V_{\beta_l}(s_l)\rangle=0.\end{aligned}$$

Proof. This is an immediate consequence of Proposition 3.3.13 combined with Theorem 3.3.12. Namely by taking $\psi(z) = \frac{-1}{z}$ we see that as $t \to \infty$:

$$\left(\sum_{k=1}^{2N+M} \frac{\mathbf{W}_{-2}^{(k)}}{t-z_k} + \frac{\mathbf{W}_{-1}^{(k)}}{(t-z_k)^2} + \frac{w(\alpha_k)}{(t-z_k)^3}\right) \langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle$$
$$= t^{-6} \prod_{k=1}^{2N+M} |z_k|^{-2\Delta_{\alpha_k}} \langle \mathbf{W}\left(\frac{1}{t}\right) \prod_{k=1}^{2N+M} V_{\alpha_k}\left(\frac{-1}{z_k}\right) \rangle.$$

From this we deduce that the left-hand side must scale like Ct^{-6} for some constant C. But we can expand it in negative powers of t: the coefficients that appear in this expansion are precisely given by the ones that appear in the statement of the global Ward identities, namely

$$\left(\sum_{k=1}^{2N+M} z_k^m \mathcal{W}_{-2}^{(k)} + m z_k^{m-1} \mathcal{W}_{-1}^{(k)} + \frac{m(m-1)}{2} z_k^{m-2} w(\alpha_k)\right) \langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \prod_{l=1}^{M} V_{\beta_l}(s_l) \rangle.$$

□

4 Singular vectors and higher equations of motion

Sommaire

4.1 Introduction

In the previous chapter we have defined descendant fields of the Vertex Operator V_β of the form $\mathbf{L}_{-n}V_\beta$ or $\mathbf{W}_{-n}V_\beta$ for $n \geq 1$. We proved that thanks to these definitions we were able to express the symmetries of the model as actual constraints on the correlation functions in the form of Ward identities.

In this chapter we pursue the study of the symmetries enjoyed by the A_2 Toda CFT on the upper half-plane. Based on the probabilistic framework introduced in the previous chapter, we prove that the representation theory of the W_3 algebra entails the existence of singular vectors in the probabilistic model, that in turn give rise to higher equations of motions. As a corollary we obtain new hypergeometric (BPZ) differential equations verified by some basic correlation functions of the theory. The content of this chapter is taken from [27], co-authored with B. Cerclé. An introduction to the method and results is found in Subsection 1.2.3.

4.1.1 Descendant fields

We recall here the method of construction of the descendant fields carried out in the previous chapter, in a slightly more general way that will allow us to make sens of the singular vectors.

The definition of the descendant fields is based on the consideration of regularized expressions of the form

$$\langle F[\Phi] V_\beta(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)\rangle_{\delta,\varepsilon,\rho}$$

where $F[\Phi]$ is a polynomial in the derivatives of the Toda field Φ evaluated at t, whose exact expression is determined by algebraic computations based on the probabilistic representation of the currents $\mathbf{T}$ and $\mathbf{W}$ (3.2.1.2). Conducting these manipulations leads to the consideration of functionals of the Toda field of the form

$$\mathbf{L}_{-\lambda} V_\alpha[\Phi] = \sum_{|\nu|=|\lambda|} \mathbf{L}^\alpha_{-\nu;\lambda}\left(\partial^{\nu_1}\Phi, \cdots, \partial^{\nu_l}\Phi\right) V_\alpha[\Phi]$$

where the $\mathbf{L}^\alpha_{-\nu;\lambda}$ are multilinear forms, while λ is a Young diagram, that is $\lambda = (\lambda_1, \cdots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l$, and where the sum ranges over Young diagrams ν with $|\nu| := \nu_1 + \cdots + \nu_l = |\lambda|$. We will only need here to consider descendants of the form $\mathbf{L}_{-n} V_\beta$ for $n \geq 1$ as well as $\mathbf{L}_{-(1,n)} V_\beta$ for $n = 1, 2$ and $\mathbf{L}_{-(1,1,1)}$, that, as explained in the Introduction, correspond to the composite descendants $\mathbf{L}_{-1}\mathbf{L}_{-n}$ and $\mathbf{L}_{-1}\mathbf{L}_{-1}\mathbf{L}_{-1}$ respectively. In this case these multilinear forms are given by

$$\begin{aligned}
&\mathbf{L}^\alpha_{-n}[\Phi] = \left\langle (n-1)Q + \alpha, \frac{\partial^n \Phi}{(n-1)!}\right\rangle - \sum_{i=0}^{n-2} \left\langle \frac{\partial^{i+1}\Phi}{i!}, \frac{\partial^{n-i-1}\Phi}{(n-2-i)!}\right\rangle \\
&\mathbf{L}^\alpha_{-(1,1)}[\Phi] := \langle \alpha, \partial^2\Phi\rangle + (\langle \alpha, \partial\Phi\rangle)^2, \\
&\mathbf{L}^\alpha_{-(1,2)}[\Phi] := \langle Q+\alpha, \partial^3\Phi\rangle + \langle Q+\alpha, \partial^2\Phi\rangle\langle \alpha, \partial\Phi\rangle - 2\langle \partial^2\Phi, \partial\Phi\rangle - \langle \partial\Phi, \partial\Phi\rangle\langle \alpha, \partial\Phi\rangle, \\
&\mathbf{L}^\alpha_{-(1,1,1)}[\Phi] := \langle \alpha, \partial^3\Phi\rangle + 3\langle \alpha, \partial^2\Phi\rangle\langle \alpha, \partial\Phi\rangle + (\langle \alpha, \partial\Phi\rangle)^3.
\end{aligned} \tag{4.1}$$

A similar procedure remains valid for the descendant fields associated to the higher-spin current, and the defining expression for them then involves quantities related to the Lie algebra structure (see e.g. Equation (3.8)), but we will not need their exact expression in the sequel.

Recall that to make sense of the insertion of such quantities within correlation functions we first need to define them at the regularized level. This is done by setting for $\delta, \varepsilon, \rho > 0$ fixed:

$$\langle \mathbf{L}_{-\lambda} V_\beta(t) \mathbf{V}\rangle_{\delta,\varepsilon,\rho} := \sum_{|\nu|=|\lambda|} \langle : \mathbf{L}^\alpha_{-\nu;\lambda}\left(\partial^{\nu_1}\Phi, \cdots, \partial^{\nu_l}\Phi\right) V_\beta(t) : \mathbf{V}\rangle. \tag{4.2}$$

where the notation $: F[\Phi] V_\beta(t) :$ refers to $F[\Phi + \beta \ln|\cdot - t|] V_\beta(t)$, while recall the shorthand $\mathbf{V} = \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{l=1}^M V_{\beta_l}(s_l)$. This can be expressed explicitly based on the

Gaussian integration by parts formula 3.2.3.

4.2 Singular vectors and higher equations of motion: levels one and two

We will start by describing singular vectors at the levels one and two, in which case we will conduct the computations in more details. In order to define the singular vector at level two we will also need to introduce beforehand the descendant field $\mathbf{L}_{(-1,1)}V_\beta$.

4.2.1 Singular vector at the level one

Singular vectors arise when the weight β takes specific values. For instance singular vectors at the level one appear when considering *semi-degenerate fields*, that is Vertex Operators of the form V_β where $\beta = \kappa\omega_i$ for $\kappa \in \mathbb{R}$ and $i \in \{1,2\}$. Based on this semi-degenerate field we can formally define the singular vector at level one by setting

$$\psi_{-1,\beta} := \left(\mathbf{W}_{-1} - \frac{3w(\beta)}{2\Delta_\beta}\mathbf{L}_{-1}\right)V_\beta.$$

To be more specific we define it as follows.

Definition 4.2.1. Take $t \in \mathbb{R}$ and $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$. We define the singular vector $\Psi_{-1,\beta}$ within half-plane correlation functions by setting

$$\langle \psi_{-1,\beta}(t)\mathbf{V}\rangle := \langle \mathbf{W}_{-1}V_\beta(t)\mathbf{V}\rangle - \frac{3w(\beta)}{2\Delta_\beta}\langle \mathbf{L}_{-1}V_\beta(t)\mathbf{V}\rangle.$$

This singular vector is actually not a *null* vector, that is unlike in the free-field theory or bulk Toda CFT where the above correlation functions identically vanish (see [25, Proposition 5.1]), in boundary Toda CFT this singular vector is not zero but rather given by a primary field. Namely we have the following statement:

Theorem 4.2.2. *Assume that $\beta = \kappa\omega_1$ for $\kappa < q$. Then*

$$\psi_{-1,\beta} = 2(q-\kappa)(\mu_{L,2} - \mu_{R,2})V_{\beta+\gamma e_2}$$

in the sense that for any $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$

$$\langle \psi_{-1,\beta}(t)\mathbf{V}\rangle = 2(q-\kappa)(\mu_{L,2} - \mu_{R,2})\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle.$$

Proof. To start with note that $\frac{3w(\beta)}{2\Delta_\beta} = \big(q - 2\omega_2(\beta)\big) = (q - \frac{2\kappa}{3})$ for such a $\beta = \kappa\omega_1$. Besides explicit computations show that for such a β, we have the equality $\frac{3w(\beta)}{2\Delta_\beta}\mathbf{L}_{-1}^\beta =$

$\mathbf{W}^{\beta}_{-1}$ between linear maps over $\mathbb{C}^2$ (see *e.g.* [25, Subsection 5.1]), so that for any positive δ,ε,ρ

$$\frac{3w(\beta)}{2\Delta_\beta}\langle \mathbf{L}_{-1}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}=\langle \mathbf{W}_{-1}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}.$$

As a consequence and by definition of the descendants from Definition 3.3.2

$$\langle\psi_{-1,\beta}(t)\mathbf{V}\rangle=\lim_{\delta,\varepsilon,\rho\to 0}\frac{3\omega(\beta)}{2\Delta_\beta}\widetilde{\mathfrak{L}}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})-\widetilde{\mathfrak{W}}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha}).$$

Now, as explained in the last chapter and reminded in Subsection 1.2.3.2, from the explicit expressions of the remainders in Definition 3.3.2 we infer that

$$\widetilde{\mathfrak{L}}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})=\sum_{i=1}^{2}\widetilde{\mathfrak{L}}^{i}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})\quad\text{and}\quad\widetilde{\mathfrak{W}}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})=\sum_{i=1}^{2}\widetilde{\mathfrak{W}}^{i}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha}),\text{ where}$$
$$\widetilde{\mathfrak{L}}^{i}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})=\left(\mu_{L,i}\langle V_{\gamma e_i}(t-\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}-\mu_{R,i}\langle V_{\gamma e_i}(t+\varepsilon)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}\right)\quad\text{while}$$
$$\widetilde{\mathfrak{W}}^{i}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha}):=-h_2(e_i)\left(q-2\omega_{\hat{i}}(\beta)\right)\widetilde{\mathfrak{L}}^{i}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha}).$$

Since we have $\frac{3w(\beta)}{2\Delta_\beta}=q-\frac{2\kappa}{3}$ for $\beta=\kappa\omega_1$ we see that the remainder terms are such that:

$$\widetilde{\mathfrak{W}}^{1}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})=\frac{3w(\beta)}{2\Delta_\beta}\widetilde{\mathfrak{L}}^{1}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})\quad\text{while}$$
$$\widetilde{\mathfrak{W}}^{2}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})=-\left(q-2\omega_1(\beta)\right)\widetilde{\mathfrak{L}}^{2}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha}).$$

where the exponents indicate whether we pick $i=1$ or $i=2$. Now since $\langle\beta,\gamma e_2\rangle=0$ and thanks to the probabilistic representation of the correlation functions the above remainder term is seen to be given by

$$\widetilde{\mathfrak{L}}^{2}_{-1}(\delta,\varepsilon,\rho;\boldsymbol{\alpha})=(\mu_{L,2}-\mu_{R,2})\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle+o(1).$$

As a consequence we can conclude that

$$\begin{aligned}\langle\psi_{-1,\beta}(t)\mathbf{V}\rangle&=\left(\frac{3\omega(\beta)}{2\Delta_\beta}+q-\frac{4\kappa}{3}\right)(\mu_{L,2}-\mu_{R,2})\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\\&=2\left(q-\kappa\right)(\mu_{L,2}-\mu_{R,2})\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle.\end{aligned}$$

The result follows from the definition of the descendants 3.3.2. □

4.2.2 Descendants and singular vectors at the level two

In this subsection we derive the explicit expression of the singular vector at the level two $\psi_{-2,\chi}$ introduced in Equation (1.49):

$$\psi_{-2,\chi} := \left(\mathbf{W}_{-2} + \frac{4}{\chi}\mathbf{L}_{-(1,1)} + \frac{4\chi}{3}\mathbf{L}_{-2}\right) V_\alpha.$$

To begin with we first need to define the descendant field $\mathbf{L}_{-(1,1)}V_\beta$ that appears in the expression of this singular vector. Based on this definition together with the ones already provided for $\mathbf{L}_{-2}V_\beta$ and $\mathbf{W}_{-2}V_\beta$ we will see that when the weight β is given by $\beta = -\chi\omega_1$, $\chi \in \{\gamma, \frac{2}{\gamma}\}$, the associated fully-degenerate field V_β allows to define one additional singular vector at the level two.

4.2.2.1 A first take on the descendant at the second level

As a first step we define here the descendant field $\mathbf{L}_{(-1,1)}V_\beta$, where recall that the latter is defined by considering expressions of the form

$$\mathbf{L}_{(-1,1)}V_\beta := \left(\mathbf{L}^\beta_{(-1,1)}(\partial^2\Phi) + \mathbf{L}^\beta_{(-1,1)}(\partial\Phi,\partial\Phi)\right)V_\beta,$$
$$\mathbf{L}^\beta_{(-1,1)}(u) := \langle\beta,u\rangle \quad \text{and} \quad \mathbf{L}^\beta_{(-1,1)}(u,v) := \langle\beta,u\rangle\langle\beta,v\rangle.$$

The proof is rather heavy and relies on the same method as in Section 3.3: as a consequence we will only provide a few details on the computations conducted and refer to Chapter 2 (or [26]) for more explanations on the method developed here.

Lemma 4.2.3. *Define the remainder term*

$$\begin{aligned}\mathfrak{L}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) :=& I_{sing}\left[\partial_x\langle V_{\gamma e_i}(x)\text{Ł}_{-1}^{2\beta+\gamma e_i}\overline{V_\beta(t)\mathbf{V}}\rangle_{\delta,\varepsilon}\right] - I_{sing}\left[\left(\partial_x + \partial_{\bar x}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\Psi_i(x)\right)\right]\\ &+ I_{sing}\times I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle}{2(t-y)}\Psi_{i,j}(x,y)\right)\right] \quad \textit{with}\end{aligned}$$

$$\begin{aligned}\langle V_{\gamma e_i}(x)\text{Ł}_{-1}^{2\beta+\gamma e_i}\overline{V_\beta(t)\mathbf{V}}\rangle_{\delta,\varepsilon} :=& \sum_{k=1}^{2N+M}\frac{\langle 2\beta+\gamma e_i,\alpha_k\rangle}{2(z_k-t)}\Psi_i(x) - I_{reg}\left[\frac{\langle 2\beta+\gamma e_i,\gamma e_j\rangle}{2(y-t)}\Psi_{i,j}(x,y)\right]\\ &+ I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_j\rangle}{2(y-z_k)}\Psi_{i,j}(x,y)\right] + I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_j,\gamma e_f\rangle}{2(y-z)}\Psi_{i,j,f}(x,y,z)\right].\end{aligned}$$

Then the following quantity is (P)*-class:* $\langle\text{Ł}_{-(1,1)}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} - \mathfrak{L}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$.

Remark 4.2.4. *The notation* $\langle V_{\gamma e_i}(x)\text{Ł}_{-1}^{2\beta+\gamma e_i}\overline{V_\beta(t)\mathbf{V}}\rangle_{\delta,\varepsilon}$ *is inspired by what would be the Wick's contraction between the descendant* $\text{Ł}_{-1}^{2\beta+\gamma e_i}V_\beta(t)$ *and* $\mathbf{V}$ *since, in agreement* *with* (3.22)

$$\langle \text{Ł}_{-1}^{\beta} V_\beta(t)\mathbf{V}\rangle = \sum_{k=1}^{2N+M} \frac{\langle \beta, \alpha_k\rangle}{2(z_k - t)} \langle V_\beta(t)\mathbf{V}\rangle - I_{reg}\left[\frac{\langle \beta, \gamma e_j\rangle}{2(y-t)}\Psi_j(y)\right]$$
$$+ I_{sing}\left[\sum_{k=1}^{2N+M} \frac{\langle \alpha_k, \gamma e_j\rangle}{2(y-z_k)}\Psi_j(y)\right] + I_{sing}\times I_{reg}\left[\frac{\langle \gamma e_j, \gamma e_f\rangle}{2(y-z)}\Psi_{j,f}(y,z)\right].$$

Proof. To start with we use the defining expression (4.2) together with Lemma 3.2.3 to get

$$\langle \text{Ł}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} = \sum_{k=1}^{2N+M} \frac{\langle \beta, \alpha_k\rangle}{2(t-z-k)^2)} + \sum_{k,l=1}^{2N+M} \frac{\langle \beta, \alpha_k\rangle\langle \beta, \alpha_l\rangle}{4(t-z-k)(t-z_l))}$$
$$- I_{tot}\left[\left(\frac{\langle \beta, \gamma e_i\rangle(2+\langle \beta, \gamma e_i\rangle)}{4(t-x)^2} + 2\frac{\langle \beta, \gamma e_i\rangle\langle \beta, \gamma e_i\rangle}{4(t-x)(t-\bar{x})}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}} + 2\sum_{k=1}^{2N+M}\frac{\langle \beta, \alpha_k\rangle\langle \beta, \gamma e_i\rangle}{4(t-z_k)(t-x)}\right)\Psi_i(x)\right]$$
$$+ I_{tot}^2\left[\frac{\langle \beta, \gamma e_i\rangle\langle \beta, \gamma e_j\rangle}{4(t-x)(t-y)}\Psi_{i,j}(x,y)\right].$$

After some algebraic manipulations on this expression we see that the only possibly singular terms in the expansion of $\langle \text{Ł}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}$ are given by

$$\mathfrak{I} := - I_{sing}\left[\left(\frac{\langle \beta, \gamma e_i\rangle(2+\langle \beta, \gamma e_i\rangle)}{4(t-x)^2} + 2\frac{\langle \beta, \gamma e_i\rangle\langle \beta, \gamma e_i\rangle}{4(t-x)(t-\bar{x})}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}} + 2\sum_{k=1}^{2N+M}\frac{\langle \beta, \alpha_k\rangle\langle \beta, \gamma e_i\rangle}{4(t-z_k)(t-x)}\right)\Psi_i(x)\right]$$
$$+ I_{sing}\times I_{reg}\left[2\frac{\langle \beta, \gamma e_i\rangle\langle \beta, \gamma e_j\rangle}{4(t-x)(t-y)}\Psi_{i,j}(x,y)\right] + I_{sing}^2\left[\frac{\langle \beta, \gamma e_i\rangle\langle \beta, \gamma e_j\rangle}{4(t-x)(t-y)}\Psi_{i,j}(x,y)\right].$$

The one-fold integral can be rewritten using

$$\left(\partial_x + \partial_{\bar{x}}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\left[\left(\frac{\langle \beta, \gamma e_i\rangle}{2(t-x)} + \sum_{k=1}^{2N+M}\frac{\langle 2\beta+\gamma e_i, \alpha_k\rangle}{2(t-z_k)}\right)\Psi_i(x)\right] = (P)\text{-class terms}$$
$$+\left(\frac{\langle \beta, \gamma e_i\rangle(2+\langle \beta, \gamma e_i\rangle)}{4(t-x)^2} + \frac{\langle \beta, \gamma e_i\rangle\langle \beta, \gamma e_i\rangle}{2(t-x)(t-\bar{x})}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\Psi_i(x)$$
$$+\sum_{k=1}^{2N+M}\left(\frac{\langle \beta, \alpha_k\rangle\langle \beta, \gamma e_i\rangle}{2(t-z_k)(t-x)} + \frac{\langle \beta, \alpha_k\rangle\langle \beta, \gamma e_i\rangle}{2(t-z_k)(t-\bar{x})}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\Psi_i(x)$$
$$- I_{tot}\left[\left(\frac{\langle \beta, \gamma e_i\rangle}{2(t-x)} + \sum_{k=1}^{2N+M}\frac{\langle 2\beta+\gamma e_i, \alpha_k\rangle}{2(t-z_k)}\right)\left(\frac{\langle \gamma e, \gamma e_j\rangle}{2(y-x)} + \frac{\langle \gamma e, \gamma e_j\rangle}{2(y-\bar{x})}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\Psi_{i,j}(x,y)\right].$$

Since the integral $I_{sing}\times I_{reg}\left[\frac{1}{x-y}\Psi_{i,j}(x,y)\right]$ is (P)-class in virtue of Lemma 3.2.8 this entails

$$
\begin{aligned}
\mathfrak{I} = \quad & (P)\text{-class terms} \\
& - I_{sing}\left[\left(\partial_x + \partial_{\bar x}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\Psi_i(x)\right)\right] - I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\frac{\langle 2\beta+\gamma e_i,\alpha_k\rangle}{2(t-z_k)}\Psi_i(x)\right)\right] \\
& + I_{sing}\times I_{reg}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)} + \frac{\langle\beta,\gamma e_i\rangle\langle\gamma e,\gamma e_j\rangle}{4(t-x)(x-y)}\right)\Psi_{i,j}(x,y)\right] \\
& + I_{sing}\times I_{sing}\left[\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\left(\frac{\langle\beta,\gamma e_j\rangle}{2(t-y)} + \frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}\right) + \sum_{k=1}^{2N+M}\frac{\langle 2\beta+\gamma e_i,\alpha_k\rangle\langle\gamma e,\gamma e_j\rangle}{4(t-z_k)(y-x)}\right)\Psi_{i,j}(x,y)\right].
\end{aligned}
$$

Let us now turn to the two-folds integrals: over $I_{sing}\times I_{reg}$ singular terms are given by

$$
\begin{aligned}
& I_{sing}\times I_{reg}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)} + \frac{\langle\beta,\gamma e_i\rangle\langle\gamma e,\gamma e_j\rangle}{4(t-x)(x-y)}\right)\Psi_{i,j}(x,y)\right] \\
= \quad & (P)\text{-class terms} \quad + I_{sing}\times I_{reg}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\langle 2\beta+\gamma e_i,\gamma e_j\rangle}{4(t-x)(t-y)}\right)\Psi_{i,j}(x,y)\right].
\end{aligned}
$$

To take care of the singularity at $x=t$ we use the same reasoning as in the proof of Lemma 3.3.1: this shows that

$$
\begin{aligned}
& I_{sing}\times I_{reg}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)} + \frac{\langle\beta,\gamma e_i\rangle\langle\gamma e,\gamma e_j\rangle}{4(t-x)(x-y)}\right)\Psi_{i,j}(x,y)\right] \\
= \quad & (P)\text{-class terms} \quad + I_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle 2\beta+\gamma e_i,\gamma e_j\rangle}{2(t-y)}\right)\Psi_{i,j}(x,y)\right].
\end{aligned}
$$

This corresponds exactly to the integral over I_{reg} that appears in the expansion of $\langle V_{\gamma e_i}(x)\mathbf{L}_{-1}^{2\beta+\gamma e_i}\overline{V_\beta(t)}\mathbf{V}\rangle_{\delta,\varepsilon}$.

As for the integral over $I_{sing}\times I_{sing}$ the corresponding term is

$$
\begin{aligned}
& I_{sing}\times I_{sing}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\left(\frac{\langle\beta,\gamma e_j\rangle}{2(t-y)} + \frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}\right)\Psi_{i,j}(x,y)\right] \\
& + I_{sing}\times I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle 2\beta+\gamma e_i,\alpha_k\rangle}{2(t-z_k)}\frac{\langle\gamma e,\gamma e_j\rangle}{2(y-x)}\Psi_{i,j}(x,y)\right].
\end{aligned}
$$

The integral on the second line is actually (P)-class: to see why for $i=j$ we use the symmetry in x,y to see that the term vanishes while for $i\neq j$ the singularity at $x=y$ is integrable thanks to Lemma 3.2.5 (the exponent is positive). The first integral can be rewritten as

$$
\begin{aligned}
&I_{sing} \times I_{sing}\left[\left(\partial_y + \sum_{k=1}^{2N+M} \frac{\langle \gamma e_j, \alpha_k\rangle}{2(y-z_k)}\right) \frac{\langle \beta, \gamma e_i\rangle}{2(t-x)} \Psi_{i,j}(x,y)\right] \\
&- I_{sing} \times I_{sing} \times I_{tot}\left[\frac{\langle \beta, \gamma e_i\rangle \langle \gamma e_j, \gamma e_f\rangle}{4(t-x)(y-z)} \Psi_{i,j,f}(x,y,z)\right].
\end{aligned}
$$

The derivative ∂_y appears in the remainder term. Therefore it only remains to treat

$$
I_{sing} \times I_{sing}\left[\sum_{k=1}^{2N+M} \frac{\langle \gamma e_j, \alpha_k\rangle \langle \beta, \gamma e_i\rangle}{4(y-z_k)(t-x)} \Psi_{i,j}(x,y)\right] - I_{sing} \times I_{sing} \times I_{tot}\left[\frac{\langle \beta, \gamma e_i\rangle \langle \gamma e_j, \gamma e_l\rangle}{4(t-x)(y-z)} \Psi_{i,j,f}(x,y,z)\right].
$$

In the same fashion as in the proof of Theorem 4.2.2 the first integral will give rise to the term corresponding to the integral over I_{sing} that appears in the expansion of $\langle V_{\gamma e_i}(x) Ł_{-1}^{2\beta+\gamma e_i} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}$. The second integral vanishes over I_{sing}^3 by symmetry in y, z, while the integral over $I_{sing}^2 \times I_{reg}$ yields the term $I_{sing} \times I_{reg}$ in $\langle V_{\gamma e_i}(x) Ł_{-1}^{2\beta+\gamma e_i} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon}$.

Recollecting term we see that as desired $\mathfrak{I} =$ (P)-class terms $+\mathfrak{L}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$. □

4.2.2.2 Definition of the descendants at the level two

Based on the previous statement we can now define the descendant $\mathbf{L}_{-(1,1)} V_\beta$. This is done by substracting the remainder term $\widetilde{\mathfrak{L}}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ to the regularized quantity, where the latter is obtained from $\mathfrak{L}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ by application of Stokes' formula:

$$
\begin{aligned}
&\widetilde{\mathfrak{L}}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) = -\frac{\langle \beta, \gamma e_i\rangle}{2\varepsilon}\left(\mu_{L,i}\Psi_i(t-\varepsilon) + \mu_{R,i}\Psi_i(t+\varepsilon)\right) \\
&+ \mu_{L,i}\langle V_{\gamma e_i}(t-\varepsilon)\mathbf{L}_{-1}^{2\beta+\gamma e_i} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} - \mu_{R,i}\langle V_{\gamma e_i}(t+\varepsilon)\mathbf{L}_{-1}^{2\beta+\gamma e_i} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} \\
&+ I_{sing}\left[\frac{\langle \beta, \gamma e_j\rangle}{2(t-y)}\left(\mu_{L,i}\Psi_{i,j}(t-\varepsilon,y) - \mu_{R,i}\Psi_{i,j}(t+\varepsilon,y)\right)\right].
\end{aligned}
\tag{4.3}
$$

Definition 4.2.5. For any $t \in \mathbb{R}$ and $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$, the descendant field $\mathbf{L}_{-(1,1)} V_\beta$ is defined by the following limit, well-defined and analytic in a complex neighborhood of $\mathscr{A}_{N,M+1}$:

$$
\langle \mathbf{L}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle := \lim_{\delta,\varepsilon,\rho\to 0} \langle \mathbf{L}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} - \widetilde{\mathfrak{L}}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}).
$$

Proof. We need to show that the difference $\mathfrak{L}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) - \widetilde{\mathfrak{L}}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ is (P)-class. In this expression the only term that may not be (P)-class is given by

$$
\oint_{\mathbb{H}_{\delta,\varepsilon}\cap\partial B(t,r)} I_{sing}\left[\frac{\langle \beta, \gamma e_j\rangle}{2(t-y)}\Psi_{i,j}(\xi,y)\right]\frac{\imath d\bar\xi - \imath d\xi}{2} - \oint_{\mathbb{H}_{\delta,\varepsilon}\cap\partial B(t,r)} I_{sing}\left[\partial_y \Psi_{i,j}(\xi,y)\right]\frac{\imath d\bar\xi - \imath d\xi}{2}.
$$

For this we develop the derivative in y to cancel the singularity at $y = t$: this yields only (P)-class terms. $\square$

Proposition 4.2.6. *For $t \in \mathbb{R}$ and $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M}$, in the sense of weak derivatives:*

$$\langle \mathbf{L}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle = \partial_t^2 \langle V_\beta(t)\mathbf{V}\rangle.$$

Proof. The proof follows from the very same argument as that of Proposition 3.3.4. For positive $\delta, \varepsilon, \rho$ and any smooth and compactly supported test function f:

$$\int_{\mathbb{R}} f(t)\langle \mathbf{L}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}\, dt = \int_{\mathbb{R}} \partial_t^2 f(t)\langle V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}\, dt.$$

Now it is readily seen from the fusion asymptotics of Lemma 3.2.5 (see also the reasoning developed in [24, Subsection 4.2]) that for $\langle \beta, e_i\rangle$ negative enough the remainder term $\widetilde{\mathfrak{L}}_{-(1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ that appears in the definition of the descendant field vanishes in the $\rho, \varepsilon, \delta \to 0$ limit. As a consequence for $\langle \beta, e_i\rangle$ negative enough we have the equality

$$\int_{\mathbb{R}} f(t)\langle \mathbf{L}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle\, dt = \int_{\mathbb{R}} \partial_t^2 f(t)\langle V_\beta(t)\mathbf{V}\rangle\, dt.$$

This equality can be extended by analycity of the correlation functions in a complex neighborhood of $\mathscr{A}_{N,M}$, thus completing the proof. $\square$

4.2.2.3 Degenerate fields at the level two

We have defined in the previous subsection semi-degenerate fields, which are Vertex Operators V_β where the weight is chosen to be of the form $\beta = \kappa\omega_i$ for $\kappa \in \mathbb{R}$ and $i \in \{1,2\}$. We have seen that such a semi-degenerate field allows to define a singular vector at level one $\Psi_{-1,\beta}$. We now explain how a further specialization of the weight β leads to the introduction of singular vectors at a higher level.

To this end let us now assume that $\beta = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$. The singular vector at the level two is formally defined by

$$\psi_{-2,\chi} := \left(\mathbf{W}_{-2} + \frac{4}{\chi}\mathbf{L}_{-(1,1)} + \frac{4\chi}{3}\mathbf{L}_{-2}\right) V_\beta$$

and makes sense when inserted within correlation functions as follows.

Definition 4.2.7. Assume that $\beta = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$ and take $t \in \mathbb{R}$ as well as $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$. The singular vector $\psi_{-2,\chi}$ is defined within half-plane correlation functions by

$$\langle \psi_{-2,\chi}(t)\mathbf{V}\rangle := \langle \mathbf{W}_{-2} V_\beta(t)\mathbf{V}\rangle + \frac{4}{\chi}\langle \mathbf{L}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle + \frac{4\chi}{3}\langle \mathbf{L}_{-2} V_\beta(t)\mathbf{V}\rangle.$$

In the same fashion as for the first level, the singular vector $\psi_{-2,\beta}$ is generically not a null vector but rather gives rise to *equations of motion*:

Theorem 4.2.8. *Assume that $\beta = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$, and take $a(\chi)$, $b(\chi)$ so that*

$$3\beta + \gamma e_2 = a(\chi)\mathbf{L}_{-1}^{\beta+\gamma e_2} + b(\chi)\mathbf{W}_{-1}^{\beta+\gamma e_2}. \tag{4.4}$$

If we set $\mathbf{D}_{-1,\chi} = a(\chi)\mathbf{L}_{-1} + b(\chi)\mathbf{W}_{-1}$, then

$$\Psi_{-2,\frac{2}{\gamma}} = 2\gamma(\mu_{R,2} - \mu_{L,2})\mathbf{D}_{-1,\frac{2}{\gamma}} V_{\beta+\gamma e_2} + 2\left(\frac{2}{\gamma} - \gamma\right)(\mu_{L,1} + \mu_{R,1}) V_{\beta+\gamma e_1}$$

in the sense that as soon as $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$

$$\begin{aligned}\langle \Psi_{-2,\frac{2}{\gamma}}(t)\mathbf{V}\rangle =& 2\gamma(\mu_{R,2} - \mu_{L,2})\Big(a\langle \mathbf{L}_{-1} V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle + b\langle \mathbf{W}_{-1} V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\Big) \\ &+ 2\left(\frac{2}{\gamma} - \gamma\right)(\mu_{L,1} + \mu_{R,1})\langle V_{\beta+\gamma e_1}(t)\mathbf{V}\rangle.\end{aligned} \tag{4.5}$$

When $\chi = \gamma$ we have $\Psi_{-2,\gamma} = \frac{4}{\gamma}(\mu_{R,2} - \mu_{L,2})\mathbf{D}_{-1,\gamma} V_{\beta+\gamma e_2}$ for $\gamma > 1$ while for $\gamma < 1$:

$$\Psi_{-2,\gamma} = \frac{4}{\gamma}(\mu_{R,2} - \mu_{L,2})\mathbf{D}_{-1,\gamma} V_{\beta+\gamma e_2} + 2\gamma c_\gamma(\boldsymbol{\mu}) V_{\beta+2\gamma e_1} \tag{4.6}$$

meaning the same as for $\Psi_{-2,\frac{2}{\gamma}}$. Here the constant $c_\gamma(\boldsymbol{\mu})$ is given by

$$\left(\mu_{L,1}^2 + \mu_{R,1}^2 - 2\mu_{L,1}\mu_{R,1}\cos\left(\pi\frac{\gamma^2}{2}\right) - \mu_{B,1}\sin\left(\pi\frac{\gamma^2}{2}\right)\right)\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma\left(1-\gamma^2\right)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}.$$

Remark 4.2.9. *We can rewrite the above as $\Psi_{-2,\chi} = -\frac{1}{1+\chi^2}\mathbf{D}_{-1,\chi}\Psi_{-1,\chi} + \Psi_{-2,\chi}^0$ where we have a descendant at level 1 of the singular vector $\Psi_{-1,\chi}$ and a new primary field $\Psi_{-2,\chi}^0$.*

Proof. Like before we know from [25, Proposition 5.3] (more generally see [25, Subsection 5.2]) that we have an equality at the level of multilinear maps over $\mathbb{C}$

$$\mathbf{W}_{-2}^\beta + \frac{4}{\chi}\mathbf{L}_{-(1,1)}^\beta + \frac{4\chi}{3}\mathbf{L}_{-2}^\beta = 0.$$

This entails the equality for the regularized correlation functions

$$\langle \mathbf{W}_{-2} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} + \frac{4}{\chi}\langle \mathbf{L}_{-(1,1)} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} + \frac{4\chi}{3}\langle \mathbf{L}_{-2} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} = 0$$

so that we only need to check that

$$\lim_{\delta,\varepsilon,\rho\to 0} \widetilde{\mathfrak{W}}_{-2}(\delta,\varepsilon,\rho;\boldsymbol{\alpha}) + \frac{4}{\chi}\widetilde{\mathfrak{L}}_{-(1,1)}(\delta,\varepsilon,\rho;\boldsymbol{\alpha}) + \frac{4\chi}{3}\widetilde{\mathfrak{L}}_{-2}(\delta,\varepsilon,\rho;\boldsymbol{\alpha}) = -\langle \Psi_{-2,\chi}(t)\mathbf{V}\rangle.$$

To start with according to Equations (3.3.2.1), (3.24) and Lemma 4.2.3, the left-hand side admits the representation, up to (P)-class terms that we discard to simplify the discussion,

$$\begin{aligned}\mathfrak{I} :=& I_{sing}\left[\partial_x\left(F(x)\Psi_i(x)\right)\right] + I_{sing}\left[\partial_x\left(\frac{2h_2(e_i)\omega_{\hat{i}}(\beta) + \frac{2\langle\beta,\gamma e_i\rangle}{\chi} + \frac{4\chi}{3}}{x-t}\Psi_i(x)\right)\right]\\ &+\frac{4}{\chi}\sum_{i=1}^{2}\mu_{B,i}\int_{\mathbb{H}^1_{\delta,\varepsilon}}\partial_{\bar{x}}\left(\frac{\langle\beta,\gamma e_i\rangle}{2(x-t)}\Psi_i(x)\right)+\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle}{2(\bar{x}-t)}\Psi_i(x)\right)dxd\bar{x}\\ &+I_{sing}\times I_{sing}\left[\partial_x\left(\left(\delta_{i\neq j}\frac{2\gamma h_2(e_i)}{y-x}+\frac{2\langle\beta,\gamma e_j\rangle}{\chi(t-y)}\right)\Psi_{i,j}(x,y)\right)\right]\end{aligned} \tag{4.7}$$

where $F(x)$ is regular at $x=t$ and explicitly given by

$$\begin{aligned}F(x)\Psi_i(x) =& \frac{4}{\chi}\langle V_{\gamma e_i}(x)\overline{\mathbf{L}_{-1}^{2\beta+\gamma e_i} V_\beta(t)\mathbf{V}}\rangle_{\delta,\varepsilon}\\ &+\sum_{k=1}^{2N+M}\frac{2h_2(e_i)\omega_{\hat{i}}(\alpha_k)}{x-z_k}\Psi_i(x) - I_{reg}\left[\delta_{i\neq j}\frac{2\gamma h_2(e_i)}{x-y}\Psi_{i,j}(x,y)\right].\end{aligned} \tag{4.8}$$

Let us first treat the integrals where F does not appear.

For the remaining one-fold integral I_{sing} we rely on explicit computations based on the expression of $\beta = -\chi\omega_1$ to get that

$$2h_2(e_i)\omega_{\hat{i}}(\beta) + \frac{2\langle\beta,\gamma e_i\rangle}{\chi} + \frac{4\chi}{3} = \delta_{i,1}2(\chi-\gamma).$$

In particular if $\chi=\gamma$ this term vanishes and therefore does not contribute to the limit. For $\chi=\frac{2}{\gamma}$ we rely on Lemma 4.3.6 to see that the corresponding remainder term is given by

$$-2\left(\frac{2}{\gamma}-\gamma\right)\left(\mu_{R,1}+\mu_{L,1}\right)\langle V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle.$$

Next we consider the additional one-fold integral that ranges over $\mathbb{H}_{\delta,\varepsilon}$. Then we see that for $i=2$ the numerator vanishes, while for $i=1$ we are in the setting of Lemma 4.3.7 below where similar quantities are considered, and shown to scale like

$$2\gamma\mu_{B,1}\sin\left(\pi\frac{\gamma^2}{2}\right)\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma\left(1-\gamma^2\right)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}$$

for $\chi=\gamma$ and $\gamma<1$, while these are $o(1)$ if $\chi=\frac{2}{\gamma}$ or $\chi=\gamma$ and $\gamma>1$.

Let us now turn to the two-folds integrals. By explicit computations these are equal to

$$I_{sing}\times I_{sing}\left[\partial_x\left(\left(\delta_{i,j=1}\frac{2\gamma}{y-t}-\delta_{\substack{i=1\\j=2}}\frac{2\gamma}{y-x}+\delta_{\substack{i=2\\j=1}}\left(\frac{2\gamma}{y-t}+\frac{2\gamma}{y-x}\right)\right)\Psi_{i,j}(x,y)\right)\right].$$

The first term is considered in Lemma 4.3.7, thanks to which for $\chi=\gamma$ and $\gamma<1$ we have

$$\lim_{\delta,\varepsilon,\rho\to 0} I_{sing}\times I_{sing}\left[\partial_x\left(\frac{\delta_{i,j=1}}{y-t}\Psi_{i,j}(x,y)\right)\right]$$
$$=-\left(\mu_{L,1}^2+\mu_{R,1}^2-2\mu_{L,1}\mu_{R,1}\cos\left(\pi\frac{\gamma^2}{2}\right)\right)\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma\left(1-\gamma^2\right)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}$$

while this limit is zero for $\chi=\frac{2}{\gamma}$ and $\chi=\gamma$ with $\gamma>1$. When we consider $(i,j)=(1,2)$ we use Lemma 3.2.5 to infer that the integral $I_{sing}\left[\frac{1}{y-t}\Psi_{1,2}(t,y)\right]$ is absolutely convergent. As a consequence the remainder term coming from the $I_{sing}\times I_{sing}$ integral with $(i,j)=(1,2)$ this term converges to 0 in the limit (since $\langle\beta,\gamma e_1\rangle<0$). This is however not the case when $(i,j)=(2,1)$: like in the proof of Theorem 4.2.2 we see that this term converges towards

$$(\mu_{L,2}-\mu_{R,2})I_{sing}\left[\frac{4\gamma}{y-t}\langle V_{\gamma e_1}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right].$$

Recollecting terms we see that

$$\mathfrak{I}=(P)\text{-class terms }+I_{sing}\left[\partial_x\left(F(x)\Psi_i(x)\right)\right]+(\mu_{L,2}-\mu_{R,2})I_{sing}\left[\frac{4\gamma}{y-t}\langle V_{\gamma e_1}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]$$
$$-2\left(\frac{2}{\gamma}-\gamma\right)\left(\mu_{R,1}+\mu_{L,1}\right)\langle V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\mathbb{1}_{\chi=\frac{2}{\gamma}}-2\gamma c_\gamma(\boldsymbol{\mu})\langle V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\mathbb{1}_{\chi=\gamma}.$$

Let us now consider the term $I_{sing}\left[\partial_x\left(F(x)\Psi_i(x)\right)\right]$, where F has been described in Equation (4.8). Using the very same reasoning as the one conducted in the proof of Theorem 4.2.2, we see that since $\langle\beta,\gamma e_1\rangle<0$ while $\langle\beta,\gamma e_2\rangle=0$ this term converges towards $(\mu_{L,2}-\mu_{R,2})H$,

$$\begin{aligned}H :=& \frac{4}{\chi}\left(\sum_{k=1}^{2N+M}\frac{\langle 2\beta+\gamma e_2,\alpha_k\rangle}{2(z_k-t)}\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle - I_{reg}\left[\frac{\langle 2\beta+\gamma e_2,\gamma e_j\rangle}{2(y-t)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\right)\\ &+\frac{4}{\chi}I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_j\rangle}{2(y-z_k)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\\ &+\frac{4}{\chi}I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_j,\gamma e_f\rangle}{2(z-y)}\langle V_{\gamma e_j}(y)V_{\gamma e_f}(z)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\\ &+\sum_{k=1}^{2N+M}\frac{2\omega_1(\alpha_k)}{t-z_k}\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle - I_{reg}\left[\frac{2\gamma}{t-y}\langle V_{\gamma e_1}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right].\end{aligned}$$

Now we have $2\beta+\gamma e_2-\chi\omega_1=3\beta+\gamma e_2$. As a consequence the latter can be simplified to

$$\begin{aligned}H =& \frac{4}{\chi}\left(\sum_{k=1}^{2N+M}\frac{\langle 3\beta+\gamma e_2,\alpha_k\rangle}{2(z_k-t)}\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle - I_{reg}\left[\frac{\langle 3\beta+\gamma e_2,\gamma e_j\rangle}{2(y-t)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\right)\\ &+\frac{4}{\chi}I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_j\rangle}{2(y-z_k)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\\ &+\frac{4}{\chi}I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_j,\gamma e_f\rangle}{2(z-y)}\langle V_{\gamma e_j}(y)V_{\gamma e_f}(z)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right].\end{aligned}$$

Gathering all these terms we end up with the equality

$$\begin{aligned}\mathfrak{I} =& (P)\text{-class terms } +\frac{4}{\chi}(\mu_{L,2}-\mu_{R,2})\mathfrak{I}\\ &-2\left(\frac{2}{\gamma}-\gamma\right)\left(\mu_{R,1}+\mu_{L,1}\right)\langle V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\mathbb{1}_{\chi=\frac{2}{\gamma}}-2\gamma c_\gamma(\boldsymbol{\mu})\langle V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\mathbb{1}_{\chi=\gamma}.\end{aligned}$$

where we have set

$$\begin{aligned}\mathfrak{I} :=& \sum_{k=1}^{2N+M}\frac{\langle 3\beta+\gamma e_2,\alpha_k\rangle}{2(z_k-t)}\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle - I_{reg}\left[\frac{\langle 3\beta+\gamma e_2,\gamma e_j\rangle}{2(y-t)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\\ &+I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_j\rangle}{2(y-z_k)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle+\frac{\gamma\chi}{y-t}\langle V_{\gamma e_1}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\\ &+I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_j,\gamma e_f\rangle}{2(z-y)}\langle V_{\gamma e_j}(y)V_{\gamma e_f}(z)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right].\end{aligned}$$

Now since we have that $\langle\beta+\gamma e_2,\gamma e_1\rangle<0$ we can use integration by parts (Lemma 3.2.3) to rewrite that up to (P)-class terms we have

$$
\begin{aligned}
&I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_j\rangle}{2(y-z_k)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle+\frac{\gamma\chi}{y-t}\langle V_{\gamma e_1}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\\
&+I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_j,\gamma e_f\rangle}{2(z-y)}\langle V_{\gamma e_j}(y)V_{\gamma e_f}(z)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right]\\
&=I_{sing}\left[\frac{\langle\beta+\gamma e_2,\gamma e_i\rangle-2\gamma\chi\delta_{j=1}}{2(t-y)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right].
\end{aligned}
$$

Recollecting terms we arrive to

$$
\mathfrak{J}=\sum_{k=1}^{2N+M}\frac{\langle 3\beta+\gamma e_2,\alpha_k\rangle}{2(z_k-t)}\langle V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle-I_{tot}\left[\frac{\langle 3\beta+\gamma e_2,\gamma e_j\rangle}{2(y-t)}\langle V_{\gamma e_j}(y)V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\right].
$$

This shows that this term gives rise to a linear combination of the descendants $\mathbf{L}_{-1}$ and $\mathbf{W}_{-1}$ of $V_{\beta+\gamma e_2}$. More precisely we get

$$
\mathfrak{J}=a\langle\mathbf{L}_{-1}V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle+b\langle\mathbf{W}_{-1}V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle
$$

where a and b are chosen in such a way that

$$
3\beta+\gamma e_2=a\mathbf{L}_{-1}^{\beta+\gamma e_2}+b\mathbf{W}_{-1}^{\gamma+\beta e_2}.
$$

This allows to conclude that for a and b as above,

$$
\begin{aligned}
\langle\Psi_{-2,\frac{2}{\gamma}}V_\beta(t)\mathbf{V}\rangle=&(\mu_{R,2}-\mu_{L,2})\Big(a\langle\mathbf{L}_{-1}V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle+b\langle\mathbf{W}_{-1}V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\Big)\\
&+2(\frac{2}{\gamma}-\gamma)(\mu_{L,1}+\mu_{R,1})\langle V_{\beta+\gamma e_1}(t)\mathbf{V}\rangle
\end{aligned}
$$

when $\chi=\frac{2}{\gamma}$, and likewise for $\chi=\gamma$

$$
\begin{aligned}
\langle\Psi_{-2,\gamma}V_\beta(t)\mathbf{V}\rangle=&(\mu_{R,2}-\mu_{L,2})\Big(a\langle\mathbf{L}_{-1}V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle+b\langle\mathbf{W}_{-1}V_{\beta+\gamma e_2}(t)\mathbf{V}\rangle\Big)\\
&+2\gamma c_\gamma(\boldsymbol{\mu})\langle V_{\beta+2\gamma e_1}(t)\mathbf{V}\rangle.
\end{aligned}
$$

This wraps up the proof of Theorem 4.2.8. □

4.3 Singular vectors at the level three

In the previous section we have defined the descendants at the level two of a primary field V_β, and shown that under specific assumptions on the value of the weight β they give rise to singular vectors. We now turn to the singular vectors at the third level and show that when the primary field is fully degenerate, one can define singular vectors out of them. These in turn give rise to higher equations of motion, which under additional assumptions allow to obtain BPZ-type differential equations for a family of correlation functions.

4.3.1 Descendants at the third order

As a first step in the derivation of the singular vectors at the level three, one first needs to make sense of the descendants at the third order. Namely we now define the $\mathbf{L}_{-(1,2)}$ and $\mathbf{L}_{-(1,1,1)}$ descendants.

4.3.1.1 The $\mathbf{L}_{-(1,2)}$ descendant

Following the same method as before in order to define the descendant at the third order $\mathbf{L}_{-(1,2)}$ we first need to understand the remainder term that arises in the computation of the regularized correlation functions containing these descendant fields inserted. To this end we introduce the notations for $i=1,2$:

$$G_i(x) := \sum_{k=1}^{N} \frac{\langle \beta, \alpha_k \rangle}{2(z_k - t)} \Psi_i(x) - I_{reg}\left[\frac{\langle \beta, \gamma e_j\rangle}{2(y-t)}\Psi_{i,j}(x,y)\right] \quad \text{as well as}$$

$$\begin{aligned}
&H_i(x) = \langle V_{\gamma e_i}(x)\mathbf{L}_{-2}^{\beta+\gamma e_i} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon} := \\
&\left(\sum_{k=1}^{2N+M} \frac{\langle Q+\beta+\gamma e_i, \alpha_k\rangle}{2(z_k-t)^2} - \sum_{k,l=1}^{2N+M} \frac{\langle \alpha_k, \alpha_l\rangle}{4(z_k-t)(z_l-t)}\right)\Psi_i(x) \\
&- I_{reg}\left[\left(\frac{1+\frac{\langle \beta+\gamma e_i, \gamma e_j\rangle}{2}}{(y-t)^2} - \sum_{k=1}^{2N+M}\frac{\langle \gamma e_i, \alpha_k\rangle}{2(y-t)(z_k-t)}\right)\Psi_{i,j}(x,y)\right] \\
&- I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle \gamma e_i, \alpha_k\rangle}{2(z_k-t)(z_k-y)}\Psi_{i,j}(x,y)\right] \\
&+ I_{reg}^2\left[-\frac{\langle \gamma e_j, \gamma e_f\rangle}{4(y-t)(z-t)}\Psi_{i,j,f}(x,y,z)\right] + I_{sing}\times I_{reg}\left[\frac{\langle \gamma e_j, \gamma e_f\rangle}{2(z-t)(z-y)}\Psi_{i,j,f}(x,y,z)\right].
\end{aligned}$$

The notation for H_i is motivated, like in the statement of Lemma 4.2.3, by Equation (3.22): it would correspond to the Wick's contraction between the descendant $\mathbf{L}_{-2}^{\beta+\gamma e_i} V_\beta$ and $\mathbf{V}$.

Lemma 4.3.1. *Let us set*

$$\begin{aligned}
\mathfrak{L}_{-(1,2);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) &:= I_{sing}\left[\partial_x\left(H_i(x) + \frac{1}{x-t}G_i(x)\right)\right] \\
&+ I_{sing}\left[\partial_x\left(\frac{1+\frac{\langle \beta, \gamma e_i\rangle}{2}}{(t-x)^2}\Psi_i(x)\right)\right] - I_{sing}^2\left[\partial_x\left(\frac{\langle \beta, \gamma e_j\rangle}{2(t-x)(t-y)}\Psi_{i,j}(x,y)\right)\right].
\end{aligned} \tag{4.9}$$

Then the difference $\langle \mathbf{L}_{-(1,2)} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} - \mathfrak{L}_{-(1,2);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ is (P)-class. Moreover define the $\mathbf{L}_{-(1,2)} V_\beta$ descendant within correlation function by setting for $t \in \mathbb{R}$ and $(\boldsymbol{\alpha}, \beta) \in$

$\mathscr{A}_{N,M+1}$:
$$\langle \mathbf{L}_{-(1,2)} V_\beta(t)\mathbf{V}\rangle := \lim_{\delta,\varepsilon,\rho\to 0} \langle \mathbf{L}_{-(1,2)} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} - \widetilde{\mathfrak{L}}_{-(1,2);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}). \tag{4.10}$$

Then this descendant is such that, in the weak sense of derivatives:

$$\langle \mathbf{L}_{-(1,2)} V_\beta(t)\mathbf{V}\rangle = \partial_t \left(\sum_{k=1}^{2N+M} \frac{\partial_{z_k}}{t-z_k} + \frac{\Delta_{\alpha_k}}{(t-z_k)^2} \right) \langle V_\beta(t)\mathbf{V}\rangle. \tag{4.11}$$

Proof. Algebraic manipulations following the exact same techniques as the ones developed in order to prove the results in the previous section show that the singular terms in the expression of $\langle \mathbf{L}_{-(1,2)} V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}$ are given by $I_1+I_2+I_3$ where we have set

$$\begin{aligned}
I_1 :=& I_{sing}\left[\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(t-x)^2} \left(\frac{2+\frac{\langle\beta,\gamma e_i\rangle}{2}}{t-x} + \frac{\gamma^2}{\bar{x}-x}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}} + \sum_{k=1}^{2N+M} \frac{\langle\alpha_k,\gamma e_i\rangle}{2(z_k-t)} \right) \Psi_i(x) \right] \\
&+ I_{sing}\left[\sum_{k=1}^{2N+M} \frac{\langle\beta,\alpha_k\rangle}{2(t-x)(t-z_k)} \left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{t-x} + \frac{\gamma^2}{\bar{x}-x}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}} + \sum_{l=1}^{2N+M} \frac{\langle\alpha_l,\gamma e_i\rangle}{2(z_l-t)} \right) \Psi_i(x) \right] \\
&+ I_{sing}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)} \left(\sum_{k=1}^{2N+M} \frac{\langle Q+\beta,\alpha_k\rangle}{2(t-z_k)^2} - \sum_{k,l=1}^{2N+M} \frac{\langle\alpha_k,\alpha_l\rangle}{4(t-z_k)(t-z_l)} \right) \Psi_i(x) \right] \\
&+ I_{sing}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)(t-\bar{x})}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}} \left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{t-x} + \frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{t-\bar{x}} + \sum_{k=1}^{2N+M} \frac{\langle\alpha_k,\gamma e_i\rangle}{(z_k-t)} \right) \Psi_i(x) \right] \\
&- I_{sing}\left[\sum_{k=1}^{2N+M} \frac{\langle\alpha_k,\gamma e_i\rangle}{2(t-x)(t-z_k)^2} \Psi_i(x) \right],
\end{aligned}$$

$$\begin{aligned}
I_2 :=& -I_{tot}^2\left[\frac{\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)} \left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(t-x)} + \frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-x)} + \frac{\gamma^2\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}}{\bar{x}-x} + \sum_{k=1}^{2N+M} \frac{\langle\alpha_k,\gamma e_i\rangle}{2(z_k-t)} \right) \Psi_{i,j}(x,y) \right] \\
&- I_{tot}^2\left[\left(\frac{(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{(t-x)^2} + \sum_{k=1}^{2N+M} \frac{\langle\beta,\alpha_k\rangle}{2(t-x)(t-z_k)} \right) \frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-x)} \Psi_{i,j}(x,y) \right] \\
&+ I_{tot}^2\left[\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(t-\bar{x})(t-y)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}} \Psi_{i,j}(x,y) \right] \quad \text{and}
\end{aligned}$$

$$I_3 := I_{tot}^3\left[-\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\gamma e_f\rangle}{8(t-x)(t-y)(t-z)} \Psi_{i,j,f}(x,y,z) \right].$$

We can further simplify I_1 by using the identity

$$\frac{(1+\frac{\langle\beta,\gamma e_i\rangle}{2})\langle\alpha_k,\gamma e_i\rangle}{(t-x)^2(z_k-t)} = \frac{(1+\frac{\langle\beta,\gamma e_i\rangle}{2})\langle\alpha_k,\gamma e_i\rangle}{(t-x)^2(z_k-x)} + \frac{(1+\frac{\langle\beta,\gamma e_i\rangle}{2})\langle\alpha_k,\gamma e_i\rangle}{(t-x)(z_k-t)^2} - \frac{(1+\frac{\langle\beta,\gamma e_i\rangle}{2})\langle\alpha_k,\gamma e_i\rangle}{(z_k-x)(z_k-t)^2}.$$

Likewise we can write that I_2 is given by the sum of (P)-terms and

$$
\begin{aligned}
&- I_{sing}\times I_{tot}\left[\frac{\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)}\left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(t-x)}+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-x)}+\frac{\gamma^2\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}}{\bar x-x}+\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_i\rangle}{2(z_k-t)}\right)\Psi_{i,j}(x,y)\right]\\
&- I_{sing}\times I_{reg}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\left(\frac{1+\frac{\langle\beta+\gamma e_i,\gamma e_j\rangle}{2}}{(t-y)^2}-\frac{\gamma^2\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}}{(t-y)(t-\bar y)}+\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_j\rangle}{2(t-y)(z_k-t)}\right)\Psi_{i,j}(x,y)\right]\\
&- I_{sing}\times I_{tot}\left[\left(\frac{(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{(t-x)^2}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\alpha_k\rangle}{2(t-x)(t-z_k)}\right)\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-x)}\Psi_{i,j}(x,y)\right]\\
&- I_{sing}\times I_{tot}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)(t-\bar x)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-x)}+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-\bar x)}\right)\Psi_{i,j}(x,y)\right]
\end{aligned}
$$

and that I_3 simplifies to

$$
\begin{aligned}
I_3 &= (P)\text{-class terms }+I_{sing}\times I_{tot}^2\left[\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\gamma e_f\rangle}{4(t-x)(t-y)(z-x)}\Psi_{i,j,f}(x,y,z)\right]\\
&+ I_{sing}^2\times I_{reg}\left[\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\gamma e_f\rangle}{4(t-x)(t-z)(y-z)}\Psi_{i,j,f}(x,y,z)\right]\\
&- I_{sing}\times I_{reg}^2\left[\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\gamma e_f\rangle}{8(t-x)(t-y)(t-z)}\Psi_{i,j,f}(x,y,z)\right].
\end{aligned}
$$

As a consequence we are led to the equality

$$
\begin{aligned}
&\langle\mathbf{L}_{-(1,2)}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} = (P)\text{-class terms }+I_{sing}\left[\partial_x\left(H_i(x)+\frac{1}{x-t}G_i(x)\right)\right]\\
&+ I_{sing}\left[(\partial_x+\partial_{\bar x})\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)(t-\bar x)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\Psi_i(x)\right)\right]\\
&+ I_{sing}\left[\partial_x\left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(t-x)^2}\Psi_i(x)\right)\right]-I_{sing}\times I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)}\Psi_{i,j}(x,y)\right)\right],
\end{aligned}
$$

allowing to recover the desired expression for the remainder term:

$$
\begin{aligned}
\mathfrak{L}_{-(1,2);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) &= I_{sing}\left[\partial_x\left(H_i(x)+\frac{1}{x-t}G_i(x)\right)\right]\\
&+ I_{sing}\left[\partial_x\left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{(t-x)^2}\Psi_i(x)\right)\right]-I_{sing}^2\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)}\Psi_{i,j}(x,y)\right)\right].
\end{aligned}
$$

As for the second part of the statement, we see from its expression that for $\langle\beta,e_i\rangle$ negative enough the remainder term $\widetilde{\mathfrak{L}}_{-(1,2);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ defined from Equation (4.9) goes to zero. Moreover in that case the equality is seen to hold in the sense of weak derivatives, see *e.g.* [25, Equation (5.7)]. We can conclude like in the proof of Proposition 4.2.6. □

4.3.1.2 The $\mathbf{L}_{-(1,1,1)}$ descendant

We now turn to the $\mathbf{L}_{-(1,1,1)}$ descendant, for which the analog statement is the following:

Lemma 4.3.2. *One can find a quantity $\mathfrak{L}_{-(1,1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ for which $\langle \mathbf{L}_{-(1,1,1)}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} - \mathfrak{L}_{-(1,1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})$ is (P)-class. The $\mathbf{L}_{-(1,1,1)}V_\beta$ descendant defined for $t\in\mathbb{R}$ and $(\boldsymbol{\alpha},\beta)\in\mathscr{A}_{N,M+1}$ via*

$$\langle \mathbf{L}_{-(1,1,1)}V_\beta(t)\mathbf{V}\rangle := \lim_{\delta,\varepsilon,\rho\to 0}\langle \mathbf{L}_{-(1,1,1)}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho} - \widetilde{\mathfrak{L}}_{-(1,1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) \tag{4.12}$$

satisfies the property that, in the weak sense,

$$\langle \mathbf{L}_{-(1,1,1)}V_\beta(t)\mathbf{V}\rangle = \partial_t^3\langle V_\beta(t)\mathbf{V}\rangle. \tag{4.13}$$

Along the proof of Lemma 4.3.2, we will provide an explicit expression for this remainder term. Namely we will show that it is given by

$$\begin{aligned}
&\mathfrak{L}_{-(1,1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) = I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\theta_i(x)+\vartheta_i(x)\right)\right]\\
&+I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-\bar x)}\Theta_i(x)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}-I_{sing}\left[\frac{\langle\beta,\gamma e_j\rangle}{2(t-y)}\Theta_{i,j}(x,y)\right]\right)\right]\\
&+A_{\delta,\varepsilon,\rho}+B_{\delta,\varepsilon,\rho}+C_{\delta,\varepsilon,\rho},
\end{aligned}\tag{4.14}$$

$$A_{\delta,\varepsilon,\rho} := I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)}{2(t-x)^2}\Psi_i(x)\right)\right] \tag{4.15}$$

$$\begin{aligned}
B_{\delta,\varepsilon,\rho} &:= -I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\left(1+\frac{\langle\beta,\gamma e_j\rangle}{2}\right)}{2(t-y)^2}+\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)}\right)\Psi_{i,j}(x,y)\right)\right]\\
&+I_{sing}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\left(1+\frac{\langle\beta,\gamma e_j\rangle}{2}\right)}{2(t-\bar x)}+\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-\bar x)}\right)\Psi_i(x)\right)\right]
\end{aligned}\tag{4.16}$$

$$\begin{aligned}
C_{\delta,\varepsilon,\rho} &:= I_{sing}^3\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\langle\beta,\gamma e_f\rangle}{4(t-y)(t-z)}\right)\Psi_{i,j,f}(x,y,z)\right)\right]\\
&-I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-\bar x)(t-y)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle^2}{4(t-y)(t-\bar y)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{y\in\mathbb{H}_{\delta,\varepsilon}}\right]
\end{aligned}\tag{4.17}$$

where ϑ_i is explicit but rather tedious to write (we provide its expression in Equation (4.23) below), while θ_i and $\Theta_{i,j}$ are given by

$$\begin{aligned}\theta_i(x) &:= \sum_{k=1}^{2N+M}\left(\frac{\langle 3\beta+\gamma e_i,\alpha_k\rangle}{2(t-z_k)}+\frac{\langle\gamma e_i,\alpha_k\rangle}{2(\bar{x}-z_k)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\Psi_i(x)\\ &-I_{reg}\left[\left(\frac{\langle 3\beta+\gamma e_i,\gamma e_j\rangle}{2(t-y)}+\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(\bar{x}-y)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\Psi_{i,j}(x,y)\right]\\ &-I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\gamma e_j,\alpha_k\rangle}{2(y-z_k)}\Psi_{i,j}(x,y)\right]+I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_j,\gamma e_f\rangle}{2(y-z)}\Psi_{i,j,f}(x,y,z)\right],\end{aligned}\tag{4.18}$$

$$\begin{aligned}\Theta_{i,j}(x,y) &:= \sum_{k=1}^{2N+M}\left(\frac{\langle 3\beta+\gamma e_j,\alpha_k\rangle}{2(t-z_k)}+\frac{\langle\gamma e_i,\alpha_k\rangle}{2(x-z_k)}\right)\Psi_{i,j}(x,y)\\ &-I_{reg}\left[\left(\frac{\langle 3\beta+\gamma e_j,\gamma e_f\rangle}{2(t-z)}+\frac{\langle\gamma e_i,\gamma e_f\rangle}{2(x-z)}\right)\Psi_{i,j,f}(x,y,z)\right]\\ &-I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\gamma e_f,\alpha_k\rangle}{2(z-z_k)}\Psi_{i,j,f}(x,y,z)\right]+I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_f,\gamma e_m\rangle}{2(z-w)}\Psi_{i,j,f,m}(x,y,z,w)\right].\end{aligned}\tag{4.19}$$

Finally $\Theta_i(x)$ is defined via the same expression as $\Theta_{i,j}(x,y)$ but with $\Psi_{i,j}(x,y)$ being replaced by $\Psi_i(x)$.

Proof. Like for the previous lemma we only sketch the main steps and leave the details of the computations to the (very motivated!) reader. We can write the potentially singular terms in the expression of $\langle \mathbf{L}_{-(1,1,1)}V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}$ as $I_1+I_2+I_3$, with

$$\begin{aligned}I_1 :=& I_{sing}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)\left(2+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)}{2(t-x)^3}+\sum_{k=1}^{2N+M}\frac{3\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)\langle\beta,\alpha_k\rangle}{4(t-x)^2(t-z_k)}\right)\Psi_i(x)\right]\\ &+I_{sing}\left[\frac{3\langle\beta,\gamma e_i\rangle}{2(t-x)}\left(\sum_{k=1}^{2N+M}\frac{\langle\beta,\alpha_k\rangle}{2(t-z_k)^2}+\sum_{k,l=1}^{2N+M}\frac{\langle\beta,\alpha_k\rangle\langle\beta,\alpha_l\rangle}{4(t-z_k)(t-z_l)}\right)\Psi_i(x)\right]\\ &+I_{sing}\left[\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\frac{3\langle\beta,\gamma e_i\rangle^2}{4(t-x)(t-\bar{x})}\left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{t-x}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\alpha_k\rangle}{2(t-z_k)}\right)\Psi_i(x)\right],\end{aligned}$$

$$\begin{aligned}I_2 :=& -I_{tot}^2\left[\left(\frac{3\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)\langle\beta,\gamma e_j\rangle}{4(t-x)^2(t-y)}+\frac{3\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle\langle\beta,\alpha_k\rangle}{8(t-x)(t-y)(t-z_k)}\right)\Psi_{i,j}(x,y)\right]\\ &-I_{tot}^2\left[\frac{3\langle\beta,\gamma e_i\rangle^2\langle\beta,\gamma e_j\rangle}{4(t-x)(t-\bar{x})(t-y)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\Psi_{i,j}(x,y)\right],\quad\text{and}\end{aligned}$$

$$I_3 := I_{tot}^3\left[\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle\langle\beta,\gamma e_f\rangle}{8(t-x)(t-y)(t-z)}\Psi_{i,j,f}(x,y,z)\right].$$

Our goal is, like before, to rewrite the latter to make appear derivatives of the correlation functions. And to do so we rely on the same techniques as above.

For the sake of simplicity let us first assume that $\mu_{B,1} = \mu_{B,2} = 0$ so that we can discard terms with $\bar{x}$. Then using symmetrization I_1 is given by $I_1 = (P)$-class terms $+$ I_1^1 where

$$I_1^1 := I_{sing}\left[\frac{\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)}{2(t-x)^2}\left(\frac{2+\frac{\langle\beta,\gamma e_i\rangle}{2}}{t-x}+\frac{\gamma^2\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}}{\bar{x}-x}+\sum_{k=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_k\rangle}{2(z_k-x)}\right)\Psi_i(x)\right]$$
$$+ I_{sing}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{t-x}+\frac{\gamma^2\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}}{\bar{x}-x}+\sum_{l=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_l\rangle}{2(z_l-x)}\right)\theta_i^1(x)\right]+I_{sing}\left[\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\vartheta_i^1(x)\right]$$

with $\quad \theta_i^1(x) := \sum_{k=1}^{2N+M}\frac{\langle 3\beta+\gamma e_i,\alpha_k\rangle}{2(t-z_k)}\Psi_i(x)\quad$ and

$$\vartheta_i^1(x) := \left(\sum_{k=1}^{2N+M}\frac{\langle 3\beta+(1+\frac{\langle\beta,\gamma e_i\rangle}{2})\gamma e_i,\alpha_k\rangle}{2(t-z_k)^2}+\sum_{k,l=1}^{2N+M}\frac{3\langle\beta,\alpha_k\rangle\langle\beta,\alpha_l\rangle+\langle 3\beta+\gamma e_i,\alpha_k\rangle\langle\gamma e_i,\alpha_l\rangle}{4(t-z_k)(t-z_l)}\right)\Psi_i(x).$$

Based on the same reasoning as in the previous sections we thus end up with

$$I_1 = (P)\text{-class terms} + R_1^1$$
$$+ I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)}{2(t-x)^2}\Psi_i(x)+\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\theta_i^1(x)+\vartheta_i^1(x)\right)\right]\quad \text{where we have set}$$

$$R_1^1 := -I_{sing}\times I_{tot}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)}{2(t-x)^2}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\alpha_k\rangle}{4(t-x)(t-z_k)}\right)\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}\Psi_{i,j}(x,y)\right].$$

We have thus treated the one-fold integral: let us now turn to the two-fold ones. And to start with, we first rewrite I_2 as

$$I_2 = (P)\text{-class terms}$$
$$- I_{sing}^2\left[\left(\frac{3\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)\langle\beta,\gamma e_j\rangle}{4(t-x)^2(t-y)}+\frac{3\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle\langle\beta,\alpha_k\rangle}{8(t-x)(t-y)(t-z_k)}\right)\Psi_{i,j}(x,y)\right]$$
$$- I_{sing}\times I_{reg}\left[\left(\frac{3\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)\langle\beta,\gamma e_j\rangle}{4(t-x)^2(t-y)}\right)\Psi_{i,j}(x,y)\right]$$
$$- I_{sing}\times I_{reg}\left[\left(\frac{3\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle\left(1+\frac{\langle\beta,\gamma e_j\rangle}{2}\right)}{4(t-x)(t-y)^2}+\sum_{k=1}^{2N+M}\frac{3\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle\langle\beta,\alpha_k\rangle}{8(t-x)(t-y)(t-z_k)}\right)\Psi_{i,j}(x,y)\right]$$

Then using symmetrization and expressing the above in terms of derivatives like before some tedious computations show that

$$I_2 + R_1^1 = (P)\text{-class terms } + I_2^1 + I_2^2 + R_2^1 + R_2^2, \quad \text{where:}$$

$$\begin{aligned}
I_2^1 :=& -I_{sing}^2 \left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle(1+\frac{\langle\beta,\gamma e_j\rangle}{2})}{2(t-y)^2} + \frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)}\right)\Psi_{i,j}(x,y)\right)\right]\\
&- I_{sing}^2\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-x)(y-z_k)} + \frac{\langle\beta,\gamma e_j\rangle\langle 3\beta+\gamma e_j,\alpha_k\rangle}{4(t-y)(t-z_k)} + \frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(x-z_k)}\right)\Psi_{i,j}(x,y)\right)\right]\\
&- I_{sing}^2\left[\partial_x\left(\vartheta_{i,j}^1(x,y)\right)\right] \quad \text{with}
\end{aligned}$$

$$\begin{aligned}
\vartheta_{i,j}^1(x,y) :=& \sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_j,\alpha_k\rangle}{4(y-z_k)^2} + \frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-z_k)(y-z_k)}\right)\Psi_{i,j}(x,y)\\
&+ \sum_{k,l=1}^{2N+M}\left(\frac{\langle\gamma e_j,\alpha_k\rangle\langle\gamma e_j,\alpha_l\rangle}{4(y-z_k)(y-z_l)} + \frac{\langle 3\beta+2\gamma e_i,\alpha_k\rangle\langle\gamma e_j,\alpha_l\rangle}{4(t-z_k)(y-z_l)}\right)\Psi_{i,j}(x,y) \quad \text{and}
\end{aligned}$$

$$I_2^2 := -I_{sing}\times I_{reg}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\gamma e_j\rangle}{4(t-x)(t-y)} + \vartheta_{i,j}^2(x,y)\right)\Psi_{i,j}(x,y)\right)\right] \quad \text{with}$$

$$\begin{aligned}
\vartheta_{i,j}^2(x,y) :=& \left(\frac{\langle 3\beta+\gamma e_i,\gamma e_j\rangle\left(1+\frac{\langle\beta+\gamma e_i,\gamma e_j\rangle}{2}\right)}{2(t-y)^2}\right.\\
&\left.+\sum_{k=1}^{2N+M}\frac{\langle 3\beta+\gamma e_i,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle + \langle 3\beta+\gamma e_i,\alpha_k\rangle\langle\gamma e_i,\gamma e_j\rangle + 6\langle\beta,\gamma e_j\rangle\langle\beta,\alpha_k\rangle}{4(t-y)(t-z_k)}\right)\Psi_{i,j}(x,y).
\end{aligned}$$

The extra terms R_2^1 and R_2^2 are equal to

$$\begin{aligned}
R_2^1 :=& I_{sing}^2\times I_{tot}\left[\left(\frac{\langle\beta,\gamma e_j\rangle(1+\frac{\langle\beta,\gamma e_j\rangle}{2})}{2(t-y)^2} + \frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)}\right)\frac{\langle\gamma e_i,\gamma e_f\rangle}{2(x-z)}\Psi_{i,j,f}(x,y,z)\right]\\
&+ I_{sing}^2\times I_{tot}\left[\sum_{k=1}^{2N+M}\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-x)(y-z_k)}\right)\frac{\langle\gamma e_i,\gamma e_f\rangle}{2(x-z)}\Psi_{i,j,f}(x,y,z)\right]\\
&+ I_{sing}^2\times I_{tot}\left[\sum_{k=1}^{2N+M}\left(\frac{\langle\beta,\gamma e_j\rangle\langle 3\beta+\gamma e_j,\alpha_k\rangle}{4(t-y)(t-z_k)} + \frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(x-z_k)}\right)\frac{\langle\gamma e_i,\gamma e_f\rangle}{2(x-z)}\Psi_{i,j,f}(x,y,z)\right]
\end{aligned}$$

$$R_2^2 := I_{sing} \times I_{reg} \times I_{tot} \left[\left(\frac{\langle \beta, \gamma e_i \rangle \langle 3\beta + \gamma e_i, \gamma e_j \rangle}{4(t-x)(t-y)} \right) \frac{\langle \gamma e_i, \gamma e_f \rangle}{2(x-z)} \Psi_{i,j,f}(x,y,z) \right].$$

An important point to highlight is that in the (P)-class terms we have included expressions of the form $-I_{sing}^2 \left[\frac{1}{x-y} f_{i,j}(x,y) \Psi_{i,j}(x,y) \right]$ where $f_{i,j}$ is regular at $x = y$, $x = t$ and $y = t$. At first sight it may not be clear why the singularity $\frac{1}{x-y}$ would be integrable. However we can separate between the case where $i \neq j$, for which $\Psi_{i,j}(x,y) \simeq \left| x-y \right|^{\gamma^2}$ as $\left| x-y \right| \to 0$, and when $i = j$ in which case by symmetry in x, y the latter can be reduced to a regular quantity. This allows to treat the two-fold integrals and discard such terms.

Now to transform the three-fold integrals we first split I_3 as $I_3 = (P)$-class terms $+ I_3^1$ with

$$\begin{aligned} I_3^1 := & I_{sing}^3 \left[\frac{\langle \beta, \gamma e_i \rangle \langle \beta, \gamma e_j \rangle \langle \beta, \gamma e_f \rangle}{8(t-x)(t-y)(t-z)} \Psi_{i,j,f}(x,y,z) \right] \\ & + I_{sing}^2 \times I_{reg} \left[\frac{3\langle \beta, \gamma e_i \rangle \langle \beta, \gamma e_j \rangle \langle \beta, \gamma e_f \rangle}{8(t-x)(t-y)(t-z)} \Psi_{i,j,f}(x,y,z) \right] \\ & + I_{sing} \times I_{reg}^2 \left[\frac{3\langle \beta, \gamma e_i \rangle \langle \beta, \gamma e_j \rangle \langle \beta, \gamma e_f \rangle}{8(t-x)(t-y)(t-z)} \Psi_{i,j,f}(x,y,z) \right] \end{aligned}$$

Likewise we see that $R_2^1 + R_2^2 = I_{s,s,s} + I_{s,s,r} + I_{s,r,r}$ where

$$\begin{aligned} I_{s,s,s} := & I_{sing}^3 \left[\left(\frac{\langle \beta, \gamma e_j \rangle (1 + \frac{\langle \beta, \gamma e_j \rangle}{2})}{2(t-y)^2} + \frac{\langle \beta, \gamma e_i \rangle \langle \beta, \gamma e_j \rangle}{2(t-x)(t-y)} \right) \frac{\langle \gamma e_i, \gamma e_f \rangle}{2(x-z)} \Psi_{i,j,f}(x,y,z) \right] \\ & + I_{sing}^3 \left[\sum_{k=1}^{2N+M} \left(\frac{\langle \beta, \gamma e_i \rangle \langle \gamma e_j, \alpha_k \rangle}{4(t-x)(y-z_k)} \right) \frac{\langle \gamma e_i, \gamma e_f \rangle}{2(x-z)} \Psi_{i,j,f}(x,y,z) \right] \\ & + I_{sing}^3 \left[\sum_{k=1}^{2N+M} \left(\frac{\langle \beta, \gamma e_j \rangle \langle 3\beta + \gamma e_j, \alpha_k \rangle}{4(t-y)(t-z_k)} + \frac{\langle \beta, \gamma e_j \rangle \langle \gamma e_i, \alpha_k \rangle}{4(t-y)(x-z_k)} \right) \frac{\langle \gamma e_i, \gamma e_f \rangle}{2(x-z)} \Psi_{i,j,f}(x,y,z) \right] \end{aligned}$$

$$\begin{aligned} I_{s,s,r} := & I_{sing}^2 \times I_{reg} \left[\left(\frac{\langle \beta, \gamma e_j \rangle (1 + \frac{\langle \beta, \gamma e_j \rangle}{2})}{2(t-y)^2} + \frac{\langle \beta, \gamma e_i \rangle \langle \beta, \gamma e_j \rangle}{2(t-x)(t-y)} \right) \frac{\langle \gamma e_i, \gamma e_f \rangle}{2(x-z)} \Psi_{i,j,f}(x,y,z) \right] \\ & + I_{sing}^2 \times I_{reg} \left[\left(\frac{\langle \beta, \gamma e_i \rangle \langle 3\beta + \gamma e_i, \gamma e_f \rangle}{4(t-x)(t-z)} \right) \frac{\langle \gamma e_i, \gamma e_j \rangle}{2(x-y)} \Psi_{i,j,f}(x,y,z) \right] \\ & + I_{sing}^2 \times I_{reg} \left[\sum_{k=1}^{2N+M} \left(\frac{\langle \beta, \gamma e_j \rangle \langle 3\beta + \gamma e_j, \alpha_k \rangle}{4(t-y)(t-z_k)} + \frac{\langle \beta, \gamma e_j \rangle \langle \gamma e_i, \alpha_k \rangle}{4(t-y)(x-z_k)} \right) \frac{\langle \gamma e_i, \gamma e_f \rangle}{2(x-z)} \Psi_{i,j,f}(x,y,z) \right] \\ & + I_{sing}^2 \times I_{reg} \left[\sum_{k=1}^{2N+M} \left(\frac{\langle \beta, \gamma e_i \rangle \langle \gamma e_j, \alpha_k \rangle}{4(t-x)(y-z_k)} \right) \frac{\langle \gamma e_i, \gamma e_f \rangle}{2(x-z)} \Psi_{i,j,f}(x,y,z) \right]. \end{aligned}$$

$$I_{s,r,r} := I_{sing} \times I_{reg}^2 \left[\left(\frac{\langle \beta, \gamma e_i \rangle \langle 3\beta + \gamma e_i, \gamma e_j \rangle}{4(t-x)(t-y)} \right) \frac{\langle \gamma e_i, \gamma e_f \rangle}{2(x-z)} \Psi_{i,j,f}(x,y,z) \right].$$

Gathering these terms together with that coming from I_3 we arrive to the equality

$$I_3 + R_2^1 + R_2^2 = (P)\text{-class terms } + I_3^1 + I_3^2 + I_3^3 + R_3^1 + R_3^2, \quad \text{with}$$

$$I_3^1 = I_{sing}^3 \left[\partial_x \left(\left(\frac{\langle \beta, \gamma e_j \rangle \langle \beta, \gamma e_f \rangle}{4(t-y)(t-z)} + \sum_{k=1}^{2N+M} \frac{\langle \beta, \gamma e_j \rangle \langle \gamma e_f, \alpha_k \rangle}{4(t-y)(z-z_k)} + \sum_{k,l=1}^{2N+M} \frac{\langle \gamma e_j, \alpha_k \rangle \langle \gamma e_f, \alpha_l \rangle}{4(y-z_k)(z-z_l)} \right) \Psi_{i,j,f}(x,y,z) \right) \right];$$

$$\begin{aligned} I_3^2 = I_{sing}^2 \times I_{reg} &\left[\partial_x \left(\left(\frac{\langle \beta, \gamma e_i \rangle \langle \gamma e_j, \gamma e_f \rangle}{4(t-x)(y-z)} + \vartheta_{i,j,f}(x,y,z) \right) \Psi_{i,j,f}(x,y,z) \right) \right] \\ + I_{sing}^2 \times I_{reg} &\left[\partial_x \left(\left(\frac{\langle \beta, \gamma e_j \rangle \langle 3\beta + \gamma e_j, \gamma e_f \rangle}{4(t-y)(t-z)} + \frac{\langle \beta, \gamma e_j \rangle \langle \gamma e_i, \gamma e_f \rangle}{4(t-y)(x-z)} \right) \Psi_{i,j,f}(x,y,z) \right) \right]; \end{aligned}$$

$$I_3^3 = I_{sing} \times I_{reg}^2 \left[\partial_x \left(\frac{\langle 3\beta + \gamma e_i, \gamma e_j \rangle \langle \gamma e_j, \gamma e_f \rangle + 3 \langle \beta, \gamma e_j \rangle \langle \beta, \gamma e_f \rangle}{4(t-y)(t-z)} \Psi_{i,j,f}(x,y,z) \right) \right].$$

In the above $\vartheta_{i,j,f}(x,y,z)$ and the additional terms are given by

$$\begin{aligned} \vartheta_{i,j,f}(x,y,z) :=& \left(\frac{\langle 3\beta + \gamma e_i, \gamma e_f \rangle + \langle \beta, \gamma e_j \rangle}{2(t-z)} + \frac{1 + \langle \gamma e_j, \gamma e_f \rangle}{2(y-z)} \right) \frac{\langle \gamma e_j, \gamma e_f \rangle}{2(y-z)} \\ &+ \sum_{k=1}^{2N+M} \left(\frac{\langle 3\beta + \gamma e_i, \gamma e_f \rangle}{2(t-z)} + \frac{\langle \gamma e_j, \gamma e_f \rangle}{2(x-z)} \right) \frac{\langle \gamma e_j, \alpha_k \rangle}{2(y-z_k)} + \frac{\langle \gamma e_j, \gamma e_f \rangle \langle \gamma e_i, \alpha_k \rangle}{4(y-z)(x-z_k)} \\ &+ \sum_{k=1}^{2N+M} \left(\frac{\langle 3\beta + \gamma e_i, \alpha_k \rangle}{2(t-z_k)} + \frac{\langle \gamma e_j, \alpha_k \rangle}{2(x-z_k)} \right) \frac{\langle \gamma e_j, \gamma e_f \rangle}{2(y-z)} + \frac{\langle \gamma e_j, \alpha_k \rangle \langle \gamma e_i, \gamma e_f \rangle}{4(y-z_k)(x-z)}; \end{aligned} \tag{4.20}$$

$$R_3^1 = -I_{sing}^3 \times I_{tot} \left[\left(\frac{\langle \beta, \gamma e_j \rangle \langle \beta, \gamma e_f \rangle}{4(t-y)(t-z)} + \sum_{k=1}^{2N+M} \frac{\langle \beta, \gamma e_j \rangle \langle \gamma e_f, \alpha_k \rangle}{4(t-y)(z-z_k)} \right) \frac{\langle \gamma e_i, \gamma e_m \rangle}{2(x-w)} \Psi_{i,j,f,m}(x,y,z,w) \right];$$

$$\begin{aligned} R_3^2 = &-I_{sing}^2 \times I_{reg} \times I_{tot} \left[\frac{\langle \beta, \gamma e_i \rangle \langle \gamma e_j, \gamma e_f \rangle \langle \gamma e_i, \gamma e_m \rangle}{8(t-x)(y-z)(x-w)} \Psi_{i,j,f,m}(x,y,z,w) \right] \\ &-I_{sing}^2 \times I_{reg} \times I_{tot} \left[\frac{\langle \beta, \gamma e_j \rangle \langle 3\beta + \gamma e_j, \gamma e_f \rangle \langle \gamma e_i, \gamma e_m \rangle}{8(t-y)(t-z)(x-w)} \Psi_{i,j,f,m}(x,y,z,w) \right] \\ &-I_{sing}^2 \times I_{reg} \times I_{tot} \left[\frac{\langle \beta, \gamma e_j \rangle \langle \gamma e_i, \gamma e_f \rangle \langle \gamma e_i, \gamma e_m \rangle}{8(t-y)(x-z)(x-w)} \Psi_{i,j,f,m}(x,y,z,w) \right]. \end{aligned}$$

We now have some new four-fold integrals to treat. They give rise to the remainder terms:

$$
\begin{aligned}
&-I_{sing}^3\times I_{reg}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_f,\gamma e_m\rangle}{4(t-y)(z-w)}\Psi_{i,j,f,m}(x,y,z,w)+\vartheta_{i,j,f,m}\right)\right)\right]\\
&-I_{sing}^2\times I_{reg}^2\left[\partial_x\left(\left(\frac{\langle\gamma e_i,\gamma e_f\rangle\langle\gamma e_j,\gamma e_m\rangle}{4(x-z)(y-w)}+\frac{\langle 3\beta+\gamma e_i+\gamma e_j,\gamma e_f\rangle\langle\gamma e_j,\gamma e_m\rangle}{4(t-z)(y-w)}\right)\Psi_{i,j,f,m}(x,y,z,w)\right)\right]\\
&+I_{sing}^3\times I_{reg}^2\left[\partial_x\left(\frac{\langle\gamma e_j,\gamma e_n\rangle\langle\gamma e_f,\gamma e_m\rangle}{4(y-w)(z-\varpi)}\Psi_{i,j,f,m,n}(x,y,z,w,\varpi)\right)\right],
\end{aligned}
$$

$$
\vartheta_{i,j,f,m}(x,y,z,w):=\sum_{k=1}^{2N+M}\frac{\langle\gamma e_j,\alpha_k\rangle\langle\gamma e_f,\gamma e_m\rangle}{2(y-z_k)(z-w)}\Psi_{i,j,f,m}(x,y,z,w). \tag{4.21}
$$

Recollecting all the terms that we have obtained above we recover the expression provided in Equation (4.14) up to the $\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}$ term in that we get

$$
\begin{aligned}
\mathfrak{L}_{-(1,1,1);\delta,\varepsilon,\rho}^{\mu_B=0}(\boldsymbol{\alpha}) &= I_{sing}\left[\partial_x\left(\vartheta_i^{\mu_B=0}(x)+\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\theta_i(x)\right)\right]\\
&-I_{sing}^2\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle}{2(t-y)}\Theta_{i,j}(x,y)\right)\right]\\
&+I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)}{2(t-x)^2}\Psi_i(x)\right)\right]\\
&-I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\left(1+\frac{\langle\beta,\gamma e_j\rangle}{2}\right)}{2(t-y)^2}+\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)}\right)\Psi_{i,j}(x,y)\right)\right]\\
&+I_{sing}^3\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\langle\beta,\gamma e_f\rangle}{4(t-y)(t-z)}\right)\Psi_{i,j,f}(x,y,z)\right)\right],\quad\text{with}
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
&\vartheta_i^{\mu_B=0}(x):=\vartheta_i^1(x)-I_{sing}\left[\vartheta_{i,j}^1(x,y)\right]-I_{reg}\left[\vartheta_{i,j}^2(x,y)\right]+I_{sing}\times I_{reg}\left[\vartheta_{i,j,f}(x,y,z)\right]\\
&+I_{sing}^2\left[\sum_{k,l=1}^{2N+M}\frac{\langle\gamma e_j,\alpha_k\rangle\langle\gamma e_f,\alpha_l\rangle}{4(y-z_k)(z-z_l)}\Psi_{i,j,f}(x,y,z)\right]\\
&+I_{reg}^2\left[\frac{\langle 3\beta+\gamma e_i,\gamma e_j\rangle\langle\gamma e_j,\gamma e_f\rangle+3\langle\beta,\gamma e_j\rangle\langle\beta,\gamma e_f\rangle}{4(t-y)(t-z)}\Psi_{i,j,f}(x,y,z)\right]\\
&-I_{sing}^2\times I_{reg}\left[\vartheta_{i,j,f,m}(x,y,z,w)\right]\\
&-I_{sing}\times I_{reg}^2\left[\left(\frac{\langle\gamma e_i,\gamma e_f\rangle\langle\gamma e_j,\gamma e_m\rangle}{4(x-z)(y-w)}+\frac{\langle 3\beta+\gamma e_i+\gamma e_j,\gamma e_f\rangle\langle\gamma e_j,\gamma e_m\rangle}{4(t-z)(y-w)}\right)\Psi_{i,j,f,m}(x,y,z,w)\right]\\
&+I_{sing}^2\times I_{reg}^2\left[\frac{\langle\gamma e_j,\gamma e_n\rangle\langle\gamma e_f,\gamma e_m\rangle}{4(y-w)(z-\varpi)}\Psi_{i,j,f,m,n}(x,y,z,w,\varpi)\right].
\end{aligned}
\tag{4.23}
$$

Now, let us no longer assume that $\mu_{B,1} = \mu_{B,2} = 0$. Then we get the extra term in I_1:

$$I_{sing}\left[\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\frac{3\langle\beta,\gamma e_i\rangle^2}{4(t-x)(t-\bar{x})}\left(\frac{1+\frac{\langle\beta,\gamma e_i\rangle}{2}}{t-x}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\alpha_k\rangle}{2(t-z_k)}\right)\Psi_i(x)\right],$$

and its counterpart for I_2:

$$\begin{aligned}&-I_{sing}^2\left[\frac{3\langle\beta,\gamma e_i\rangle^2\langle\beta,\gamma e_j\rangle}{4(t-x)(t-\bar{x})(t-y)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\Psi_{i,j}(x,y)\right]\\&-I_{sing}\times I_{reg}\left[\frac{3\langle\beta,\gamma e_i\rangle^2\langle\beta,\gamma e_j\rangle}{4(t-x)(t-\bar{x})(t-y)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\Psi_{i,j}(x,y)\right]\\&-I_{sing}\times I_{reg}\left[\frac{3\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle^2}{4(t-x)(t-y)(t-\bar{y})}\mathbb{1}_{y\in\mathbb{H}_{\delta,\varepsilon}}\Psi_{i,j}(x,y)\right].\end{aligned}$$

Moreover the algebraic manipulations that allowed to derive the expression of the remainder term in the case $\mu_{B,1} = \mu_{B,2} = 0$ remain valid without this assumption, the main difference lying in the fact that derivatives give rise in addition to a $\frac{\gamma^2}{x-\bar{x}}$ term. For one-fold integrals this corresponds to the additional term

$$+I_{sing}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\left(1+\frac{\langle\beta,\gamma e_i\rangle}{2}\right)}{2(t-x)^2}\Psi_i(x)+\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}\theta_i^1(x)\right)\frac{\gamma^2\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}}{x-\bar{x}}\right].$$

For two-fold integrals this yields the extra term (we have used the symmetry in $x, \bar{x}$ to cancel some quantities)

$$\begin{aligned}&-I_{sing}^2\left[\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-x)(t-y)}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-x)(y-z_k)}\right)\frac{\gamma^2\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}}{x-\bar{x}}\Psi_{i,j}(x,y)\right]\\&-I_{sing}\times I_{reg}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\gamma e_j\rangle}{4(t-x)(t-y)}\right)\frac{\gamma^2\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}}{x-\bar{x}}\Psi_{i,j}(x,y)\right].\end{aligned}$$

As for the three-fold integrals, we get the additional quantities

$$\begin{aligned}&I_{sing}^2\times I_{reg}\left[\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\gamma e_f\rangle\gamma^2}{4(t-x)(y-z)(x-\bar{x})}\Psi_{i,j,f,m}(x,y,z,w)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\+&I_{sing}^2\times I_{reg}\left[\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\gamma e_f\rangle\gamma^2}{4(t-y)(x-z)(x-\bar{x})}\Psi_{i,j,f,m}(x,y,z,w)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right].\end{aligned}$$

The four-fold and five-fold integrals only give rise to (P)-class terms.

We now treat successively these integrals. In the same fashion as the integral over

I_{sing}^2, the one-fold integral can be rewritten as (up to (P)-class terms)

$$
\begin{aligned}
&I_{sing}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{2(t-\bar{x})^2}+\frac{\langle\beta,\gamma e_i\rangle^2}{2(t-x)(t-\bar{x})}\right)\Psi_i(x)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\right]\\
&+I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-x)(\bar{x}-z_k)}+\frac{\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\alpha_k\rangle}{4(t-\bar{x})(t-z_k)}+\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-\bar{x})(x-z_k)}\right)\Psi_i(x)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right)\right]\\
&-I_{sing}\times I_{tot}\left[\left(\frac{\langle\beta,\gamma e_i\rangle(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{2(t-\bar{x})^2}+\frac{\langle\beta,\gamma e_i\rangle^2}{2(t-x)(t-\bar{x})}\right)\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}\Psi_{i,j}(x,y)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}\times I_{tot}\left[\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-x)(\bar{x}-z_k)}\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}\Psi_{i,j}(x,y)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}\times I_{tot}\left[\sum_{k=1}^{2N+M}\left(\frac{\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\alpha_k\rangle}{4(t-\bar{x})(t-z_k)}+\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-\bar{x})(x-z_k)}\right)\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}\Psi_{i,j}(x,y)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right].
\end{aligned}
$$

Likewise the remaining two-fold integrals containing the $\bar{x}$ terms are the sum of (P)-class terms and

$$
\begin{aligned}
&-I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-\bar{x})(t-y)}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-\bar{x})(y-z_k)}+\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(\bar{x}-z_k)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}^2\left[\partial_y\left(\left(\frac{\langle\beta,\gamma e_i\rangle^2}{2(t-x)(t-\bar{x})}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-x)(\bar{x}-z_k)}+\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-\bar{x})(x-z_k)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\gamma e_j\rangle}{4(t-x)(\bar{x}-y)}\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}\times I_{reg}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\gamma e_j\rangle}{4(t-\bar{x})(t-y)}+\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\gamma e_j\rangle}{4(t-\bar{x})(x-y)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]
\end{aligned}
$$

and the extra three-fold integrals given by

$$
\begin{aligned}
&I_{sing}^2\times I_{tot}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-\bar{x})(t-y)}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-\bar{x})(y-z_k)}+\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(\bar{x}-z_k)}\right)\frac{\langle\gamma e_i,\gamma e_f\rangle}{2(x-z)}\Psi_{i,j,f}(x,y,z)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&I_{sing}^2\times I_{tot}\left[\left(\frac{\langle\beta,\gamma e_i\rangle^2}{2(t-x)(t-\bar{x})}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-x)(\bar{x}-z_k)}+\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-\bar{x})(x-z_k)}\right)\frac{\langle\gamma e_j,\gamma e_f\rangle}{2(y-z)}\Psi_{i,j,f}(x,y,z)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&I_{sing}\times I_{reg}\times I_{tot}\left[\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\gamma e_j\rangle}{4(t-x)(\bar{x}-y)}\frac{\langle\gamma e_i,\gamma e_f\rangle}{2(x-z)}\Psi_{i,j,f}(x,y,z)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&I_{sing}\times I_{reg}\times I_{tot}\left[\left(\frac{\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\gamma e_j\rangle}{4(t-\bar{x})(t-y)}+\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\gamma e_j\rangle}{4(t-\bar{x})(x-y)}\right)\frac{\langle\gamma e_i,\gamma e_f\rangle}{2(x-z)}\Psi_{i,j,f}(x,y,z))\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right].
\end{aligned}
$$

By combining these three-fold integrals with the ones obtained before we get that the three-fold integrals can be rewritten as (P)-class terms plus

$$
\begin{aligned}
&I_{sing}^2 \times I_{reg}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\gamma e_f\rangle}{4(t-\bar{x})(y-z)}+\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\gamma e_f\rangle}{4(t-y)(\bar{x}-w)}\right)\Psi_{i,j,f}(x,y,z)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+I_{sing}^2 \times I_{reg}\left[\partial_y\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\gamma e_f\rangle}{4(t-x)(\bar{x}-z)}\Psi_{i,j,f}(x,y,z)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}^2 \times I_{reg}^2\left[\partial_y\left(\frac{\langle\gamma e_i,\gamma e_f\rangle\langle\gamma e_i,\gamma e_m\rangle}{4(x-z)(\bar{x}-w)}\Psi_{i,j,f,m}(x,y,z,w)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right].
\end{aligned}
$$

where we have used symmetry in $x,\bar{x}$ to remove singular terms. Recollecting terms and discarding (P)-class terms we get

$$
\begin{aligned}
&\mathfrak{L}_{-(1,1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) = \mathfrak{L}^{\mu_B=0}_{-(1,1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha}) + I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{2(t-\bar{x})^2}\Psi_i(x)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-x)(\bar{x}-z_k)}+\frac{\langle\beta,\gamma e_i\rangle}{2(t-\bar{x})}\Theta_i(x)\right)\Psi_i(x)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-\bar{x})(t-y)}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(\bar{x}-z_k)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle^2}{2(t-y)(t-\bar{y})}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-y)(\bar{y}-z_k)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{y\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\gamma e_j\rangle}{4(t-x)(\bar{x}-y)}\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+I_{sing}^2\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\gamma e_f\rangle}{4(t-y)(\bar{x}-w)}\Psi_{i,j,f}(x,y,z)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+I_{sing}^2\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_j,\gamma e_f\rangle}{4(t-y)(\bar{y}-z)}\Psi_{i,j,f}(x,y,z)\right)\mathbb{1}_{y\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-I_{sing}^2\times I_{reg}^2\left[\partial_x\left(\frac{\langle\gamma e_j,\gamma e_f\rangle\langle\gamma e_j,\gamma e_m\rangle}{4(y-z)(\bar{y}-w)}\Psi_{i,j,f,m}(x,y,z,w)\right)\mathbb{1}_{y\in\mathbb{H}_{\delta,\varepsilon}}\right].
\end{aligned}
\tag{4.24}
$$

Finally, the last part of the claim (expressing the $\mathbf{L}_{-(1,1,1)}$ descendant in terms of derivatives) follows from the very same arguments as before. □

4.3.2 Singular vector at the level three

We have seen in the previous section that fully-degenerate fields, corresponding to Vertex Operators V_β with $\beta=-\chi\omega_1$ for $\chi\in\{\gamma,\frac{2}{\gamma}\}$, give rise to singular vectors at the second level $\Psi_{-2,\beta}$. Actually from such fields we can further define singular vectors at the level three, leading to higher equations of motion.

4.3.2.1 Higher equations of motion at the level three

To do so we define the singular vector $\Psi_{-3,\beta}$ to be given by

$$\psi_{-3,\chi} := \left(\mathbf{W}_{-3} + \left(\frac{\chi}{3} + \frac{2}{\chi}\right)\mathbf{L}_{-3} - \frac{4}{\chi}\mathbf{L}_{-(1,2)} - \frac{8}{\chi^3}\mathbf{L}_{-(1,1,1)}\right)V_\beta \tag{4.25}$$

where $\beta = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$. More precisely it is defined as follows:

Definition 4.3.3. Assume that $\beta = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$ and take $t \in \mathbb{R}$ as well as $(\beta, \boldsymbol{\alpha}) \in \mathscr{A}_{N,M+1}$. The singular vector $\psi_{-3,\chi}$ is defined within half-plane correlation functions via

$$\begin{aligned}&\langle \psi_{-3,\chi}(t)\mathbf{V}\rangle := \\ &\langle \mathbf{W}_{-3}V_\beta(t)\mathbf{V}\rangle + \left(\frac{\chi}{3} + \frac{2}{\chi}\right)\langle \mathbf{L}_{-3}V_\beta(t)\mathbf{V}\rangle - \frac{4}{\chi}\langle \mathbf{L}_{-(1,2)}V_\beta(t)\mathbf{V}\rangle - \frac{8}{\chi^3}\langle \mathbf{L}_{-(1,1,1)}V_\beta(t)\mathbf{V}\rangle.\end{aligned} \tag{4.26}$$

Like before, the singular vector $\Psi_{-3,\chi}$ is actually not null in general but is seen to be equal to a sum of descendants of other primary fields. However here we will make the simplifying assumption that this singular vector gives rise to a null vector, which translates on an assumption on the cosmological constants:

Theorem 4.3.4. *Assume that $\beta = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$, and assume that $\mu_{L,2} - \mu_{R,2} = 0$. Then the singular vector at the third level $\psi_{-3,\chi}$ is equal to*

$$\psi_{-3,\chi} = -\frac{2}{\chi^2}\mathbf{D}_{-1,\chi}\psi_{-2,\chi} \tag{4.27}$$

where $\mathbf{D}_{-1,\chi}$ is of the form $\mathbf{D}_{-1,\chi} := a(\chi)\mathbf{L}_{-1} + b(\chi)\mathbf{W}_{-1}$ with a, b such that

$$a(\chi)\mathbf{L}_{-1}^{-\chi\omega_1+\gamma e_1} + b(\chi)\mathbf{W}_{-1}^{-\chi\omega_1+\gamma e_1} = \begin{cases} \gamma e_1 - \frac{2}{\gamma}\rho & \text{for} \quad \chi = \frac{2}{\gamma} \\ \gamma e_1 - \gamma e_2 & \text{for} \quad \chi = \gamma. \end{cases} \tag{4.28}$$

Like before these equalities are to be understood in the sense of Theorems 4.2.2 and 4.2.8.

Remark 4.3.5. *Though we have not conducted the computations when we no longer assume that $\mu_{L,2} - \mu_{R,2} = 0$, the arguments that we have developed along the proof of Theorem 4.3.4 suggest that in general we can write the singular vector at level three as a sum*

$$\psi_{-3,\chi} = (\mu_{L,2} - \mu_{R,2})\mathbf{D}_{-2}V_{\beta+\gamma e_2} + \begin{cases} \gamma^2\left(\gamma - \frac{2}{\gamma}\right)(\mu_{L,1} + \mu_{R,1})\mathbf{D}_{-1}V_{\beta+\gamma e_1} & \text{if} \quad \chi = \frac{2}{\gamma} \\ -\frac{4}{\gamma}c_\gamma(\boldsymbol{\mu})\mathbf{D}_{-1}V_{\beta+2\gamma e_1} & \text{if} \quad \chi = \gamma \end{cases}$$

where $\mathbf{D}_{-2}V_{\beta+\gamma e_2}$ *is a descendant at the second order of* $V_{\beta+\gamma e_2}$ *for which*

$$\mathbf{D}_{-2}^{\beta+\gamma e_2}(u) = \frac{8}{\chi^3}\langle 3\beta+\gamma e_2, u\rangle + \frac{4}{\chi}\langle Q+\beta+\gamma e_2, u\rangle$$
$$\mathbf{D}_{-2}^{\beta+\gamma e_2}(u,v) = \frac{8}{\chi^3}\left(3\langle\beta,u\rangle\langle\beta,v\rangle + \frac{3}{2}(\langle\beta,u\rangle\langle\gamma e_2,v\rangle+\langle\beta,v\rangle\langle\gamma e_2,u\rangle)+\langle\gamma e_2,u\rangle\langle\gamma e_2,v\rangle\right) - \frac{4}{\chi}\langle u,v\rangle.$$

This can be rewritten as $\psi_{-3,\chi} = \widetilde{\mathbf{D}}_{-2,\chi}\psi_{-1,\chi} - \frac{2}{\chi^2}\mathbf{D}_{-1,\chi}\psi_{-2,\chi}$ *which emphasizes that the singular vector at the level three does not give rise to a new primary field but is rather a linear combination of descendants of the singular vectors at level one and two.*

Proof. Since at the regularized level we have the equality

$$\langle\left(\mathbf{W}_{-3}+\left(\frac{\chi}{3}+\frac{2}{\chi}\right)\mathbf{L}_{-3}-\frac{4}{\chi}\mathbf{L}_{-(1,2)}-\frac{8}{\chi^3}\mathbf{L}_{-(1,1,1)}\right)V_\beta(t)\mathbf{V}\rangle_{\delta,\varepsilon,\rho}=0$$

we only need to focus on the remainder terms that occur in the limit. And more specifically we need to show that if $\mu_{L,2}-\mu_{R,2}=0$ then

$$\begin{aligned}\lim_{\delta,\varepsilon,\rho\to 0}\widetilde{\mathbf{W}}_{-3;\delta,\varepsilon,\rho}(\boldsymbol{\alpha})+&\left(\frac{\chi}{3}+\frac{2}{\chi}\right)\widetilde{\mathfrak{L}}_{-3;\delta,\varepsilon,\rho}(\boldsymbol{\alpha})\\ &-\frac{4}{\chi}\widetilde{\mathfrak{L}}_{-(1,2);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})-\frac{8}{\chi^3}\widetilde{\mathfrak{L}}_{-(1,1,1);\delta,\varepsilon,\rho}(\boldsymbol{\alpha})=-\langle\psi_{-3,\chi}\mathbf{V}\rangle.\end{aligned}\tag{4.29}$$

For this purpose we will need the explicit expressions of the remainder terms from Equations (3.3.2.1) and (3.24), as well as from Lemmas 4.3.1 and 4.3.2. Based on these expressions we can write the left-hand side in Equation (4.29) as the sum $I_1+I_2+I_3+I_4$, where in I_k we have gathered all the k-fold integrals that appear in these statements.

Before entering into the computations, let us stress that since we have made the assumption that $\mu_{L,2}-\mu_{R,2}=0$ and because $\langle\beta,\gamma e_i\rangle<0$ for $i=1$ remainder terms of the form

$$I^a_{sing}\times I^b_{reg}\left[\partial_{x_1}\left(F_{i_1,\cdots,i_{a+b}}(x_1,\cdots,x_{a+b})\Psi_{\boldsymbol{i}}(x_1,\cdots,x_{a+b})\right)\right]$$

where $F_{i_1,\cdots,i_{a+b}}$ is regular at $x_i=t$ and $x_i=x_j$ for all $i,j\le a$ vanish in the limit along the same lines as in the proofs of Theorems 4.2.2 and 4.2.8. As a consequence we will not take such terms into account later on. The proof will be divided in two parts: first of all we will assume that $\mu_{B,1}=\mu_{B,2}=0$ and then proceed to the general case. To compute the limit of these remainder terms we will rely on Lemmas 4.3.6 and 4.3.7 below, where for the sake of simplicity we will use the notations

$$\begin{aligned}c^1_\gamma(\boldsymbol{\mu}) &:= -\mu_{B,1}\sin\left(\pi\frac{\gamma^2}{2}\right)\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma\left(1-\gamma^2\right)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}\mathbb{1}_{\gamma<1}\quad\text{and}\\ c^2_\gamma(\boldsymbol{\mu}) &:= \left(\mu_{L,1}^2-2\mu_{L,1}\mu_{R,1}\cos\left(\pi\frac{\gamma^2}{2}\right)+\mu_{R,1}^2\right)\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma\left(1-\gamma^2\right)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}\mathbb{1}_{\gamma<1}\end{aligned}\tag{4.30}$$

so that $c_\gamma(\boldsymbol{\mu})\mathbb{1}_{\gamma<1} = c^1_\gamma(\boldsymbol{\mu}) + c^2_\gamma(\boldsymbol{\mu})$.

The case $\mu_{B,1} = \mu_{B,2} = 0$

Thus to start with let us assume that $\mu_{B,1} = \mu_{B,2} = 0$, so that we can discard the integrals containing $\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}$ terms. Then we have, up to (P)-class terms,

$$I_1 = I_{sing}\left[\partial_x\left(\frac{h_2(e_i)(q+2\omega_{\hat{i}}(\beta))+\left(\frac{\chi}{3}+\frac{2}{\chi}\right)-\frac{4}{\chi}(1+\frac{\langle\beta,\gamma e_i\rangle}{2})-\frac{4}{\chi^3}\langle\beta,\gamma e_i\rangle(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{(x-t)^2}\Psi_i(x)\right)\right]$$
$$+I_{sing}\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{-2h_2(e_i)\omega_{\hat{i}}(\alpha_k)-\frac{2}{\chi}\langle\beta,\alpha_k\rangle-\frac{2}{\chi^3}\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\alpha_k\rangle}{(t-x)(t-z_k)}\right)\Psi_i(x)\right)\right].$$

Likewise we can write that $I_2 = A+B$ with

$$A := I^2_{sing}\left[\partial_x\left(\left(\frac{2\gamma h_2(e_i)\delta_{i\neq j}}{(t-x)(x-y)}+\frac{\frac{2}{\chi}\langle\beta,\gamma e_j\rangle+\frac{4}{\chi^3}\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{(t-x)(t-y)}+\frac{4}{\chi^3}\frac{\langle\beta,\gamma e_j\rangle(1+\frac{\langle\beta,\gamma e_j\rangle}{2})}{(t-y)^2}\right)\Psi_{i,j}(x,y)\right)\right] \tag{4.31}$$

$$\begin{aligned}B :=& \frac{8}{\chi^3}I^2_{sing}\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-x)(y-z_k)}\right)\Psi_{i,j}(x,y)\right)\right]\\
&+\frac{8}{\chi^3}I^2_{sing}\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle 3\beta+\gamma e_j,\alpha_k\rangle}{4(t-y)(t-z_k)}+\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(x-z_k)}\right)\Psi_{i,j}(x,y)\right)\right]\\
&+I_{sing}\times I_{reg}\left[\partial_x\left(\left(\frac{2\gamma h_2(e_i)\delta_{i\neq j}}{(t-x)(x-y)}+\frac{2}{\chi}\frac{\langle\beta,\gamma e_j\rangle}{(t-x)(t-y)}+\frac{8}{\chi^3}\frac{\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\gamma e_j\rangle}{4(t-x)(t-y)}\right)\Psi_{i,j}(x,y)\right)\right].\end{aligned} \tag{4.32}$$

Finally the three- and four-fold integrals are given by

$$\begin{aligned}I_3 =& -\frac{8}{\chi^3}I^3_{sing}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\langle\beta,\gamma e_f\rangle}{4(t-y)(t-z)}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_f,\alpha_k\rangle}{4(t-y)(z-z_k)}\right)\Psi_{i,j,f}(x,y,z)\right)\right]\\
&-\frac{8}{\chi^3}I^2_{sing}\times I_{reg}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\langle 3\beta+\gamma e_j,\gamma e_f\rangle}{4(t-y)(t-z)}+\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\gamma e_f\rangle}{4(t-y)(x-z)}\right)\Psi_{i,j,f}(x,y,z)\right)\right]\\
&-\frac{8}{\chi^3}I^2_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\gamma e_f\rangle}{4(t-x)(y-z)}\Psi_{i,j,f}(x,y,z)\right)\right]\quad\text{and}\end{aligned} \tag{4.33}$$

$$I_4 = \frac{8}{\chi^3}I^3_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_f,\gamma e_m\rangle}{4(t-y)(z-w)}\Psi_{i,j,f,m}(x,y,z,w)\right)\right]. \tag{4.34}$$

We will treat successively these four types of remainder terms to see that they give rise to the right-hand side in Equation (4.29).

One-fold integrals

For this let us start by looking at the one-fold integrals, and first of all to the ones that do not contain any $\bar{x}$ term. To treat such integrals we begin with terms that contain a singularity $\frac{1}{(t-x)^2}$. The corresponding quantity is then given by

$$I_{sing}\left[\partial_x\left(\left(\frac{h_2(e_i)(q+2\omega_{\hat{i}}(\beta))+\left(\frac{\chi}{3}+\frac{2}{\chi}\right)-\frac{4}{\chi}(1+\frac{\langle\beta,\gamma e_i\rangle}{2})-\frac{4}{\chi^3}\langle\beta,\gamma e_i\rangle(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{(x-t)^2}\right)\Psi_i(x)\right)\right].$$

Using the value of $\beta=-\chi\omega_1$, the numerator is found to be equal to $\delta_{i,1}(2\gamma-\frac{4}{\chi})(1-\frac{\gamma}{\chi})$ which vanishes for $\chi\in\{\gamma,\frac{2}{\gamma}\}$. The subsequent term in one-fold integrals, corresponding to singularities of the form $\frac{1}{t-x}$, is given (up to negligible terms) by

$$I_{sing}\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{-2h_2(e_i)\omega_{\hat{i}}(\alpha_k)-\frac{2}{\chi}\langle\beta,\alpha_k\rangle-\frac{2}{\chi^3}\langle\beta,\gamma e_i\rangle\langle 3\beta+\gamma e_i,\alpha_k\rangle}{(t-x)(t-z_k)}\right)\Psi_i(x)\right)\right].$$

To start with for $i=2$ this coefficient is seen to be zero for both $\chi=\gamma$ and $\chi=\frac{2}{\gamma}$. Likewise for $\chi=\gamma$ and $i=1$ simplifications occur and this coefficient is actually seen to vanish. If however we assume that $\chi=\frac{2}{\gamma}$ and $i=1$ then the coefficient is equal to

$$2\omega_2(\alpha_k)+2\omega_1(\alpha_k)+\frac{\gamma^3}{2}\langle-\frac{6}{\gamma}\omega_1+\gamma e_1,\alpha_k\rangle=\frac{\gamma^2}{2}(\gamma-\frac{2}{\gamma})\langle\gamma e_1-\frac{2}{\gamma}\rho,\alpha_k\rangle,$$

where we have used that $\rho=3\omega_1-e_1$. As a consequence and in agreement with Lemma 4.3.6 the corresponding remainder term converges towards

$$r_1:=-\gamma^2\left(\gamma-\frac{2}{\gamma}\right)(\mu_{L,1}+\mu_{R,1})\sum_{k=1}^{2N+M}\frac{\langle\gamma e_1-\frac{2}{\gamma}\rho,\alpha_k\rangle}{2(z_k-t)}\langle V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle. \tag{4.35}$$

To summarize we have that $I_1=(P)$-class terms $+\,r_1\mathbb{1}_{\chi=\frac{2}{\gamma}}$.

Two- and three-fold integrals: the largest singularities

Let us now turn to the two- and three-fold integrals, in which case the remainder terms are given by Equations (4.31), (4.32) and (4.33). And to start with like before we first consider the largest singularities, that is we gather terms with integrands of the form $\frac{1}{(t-u)(t-v)}$ where both u and v are in I_{sing}: this corresponds to A defined in Equation (4.31) and the very first term in Equation (4.33).

These terms are a priori divergent in the $\delta,\varepsilon,\rho\to0$ limit. However simplifications allow us to show that they are actually asymptotically given by

$$\begin{aligned}A-\frac{8}{\chi^3}I^3_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle\langle\beta,\gamma e_f\rangle}{4(t-y)(t-z)}\Psi_{i,j,f}(x,y,z)\right)\right]=(P)\text{-class terms}+r_2\mathbb{1}_{\chi=\frac{2}{\gamma}}\\-\frac{4}{\chi}I^2_{sing}\Big[\partial_x\Big(\sum_{k=1}^{2N+M}\Big(\frac{\langle\gamma e_i,\alpha_k\rangle}{2(t-x)(z_k-y)}\Psi_{1,i}(x,y)-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\Psi_{i,1}(x,y)\Big)\Big)\Big]\\+\frac{4}{\chi}I^2_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)-\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)\right)\right]\end{aligned}\tag{4.36}$$

with r_2 described in Lemma 4.3.8 below, and where this limit is justified.

The remaining two-fold integrals

As a consequence we are only left with integrals that have a singularity of order 1 at $t=x$ or $t=y$. Let us thus consider the remaining two-fold integrals from Equations (4.32) and (4.36):

$$\begin{aligned}&-\frac{4}{\chi}I^2_{sing}\Big[\partial_x\Big(\sum_{k=1}^{2N+M}\Big(\frac{\langle\gamma e_i,\alpha_k\rangle}{2(t-x)(z_k-y)}\Psi_{1,i}(x,y)-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\Psi_{i,1}(x,y)\Big)\Big)\Big]\\&-\frac{8}{\chi^3}I^2_{sing}\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-x)(z_k-y)}\right)\Psi_{i,j}(x,y)\right)\right]\\&-\frac{8}{\chi^3}I^2_{sing}\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle3\beta+\gamma e_j,\alpha_k\rangle}{4(t-y)(z_k-y)}+\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(z_k-x)}\right)\Psi_{i,j}(x,y)\right)\right]\\&+I_{sing}\times I_{reg}\left[\partial_x\left(\left(\frac{2\gamma h_2(e_i)\delta_{i\neq j}}{(t-x)(x-y)}+\frac{2}{\chi}\frac{\langle\beta,\gamma e_j\rangle}{(t-x)(t-y)}+\frac{8}{\chi^3}\frac{\langle\beta,\gamma e_i\rangle\langle3\beta+\gamma e_i,\gamma e_j\rangle}{4(t-x)(t-y)}\right)\Psi_{i,j}(x,y)\right)\right].\end{aligned}$$

Let us first treat the three integrals ranging over I^2_{sing}, and to begin with let us assume that $\chi=\frac{2}{\gamma}$. Then in agreement with Lemmas 4.3.6 and 4.3.7 the integral over I^2_{sing} will contribute to the limit by

$$r_3:=-\gamma^2\left(\gamma-\frac{2}{\gamma}\right)(\mu_{L,1}+\mu_{R,1})I_{sing}\Big[\sum_{k=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_k\rangle}{2(y-z_k)}\langle V_{\gamma e_i}(y)V_{\beta+\gamma e_1}(t)\mathbf{V}\rangle\Big].\tag{4.37}$$

If we take $\chi=\gamma$ then we see that such integrals simplify to

$$\frac{4}{\gamma}I^2_{sing}\Big[\partial_x\Big(\Big(\sum_{k=1}^{2N+M}\frac{\langle3\beta+2\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}+\frac{\langle\gamma e_i,\alpha_k\rangle}{2(t-y)(z_k-x)}\Big)\Psi_{i,1}(x,y)\Big)\Big].$$

For $i=1$, we can use the equality $3\beta+2\gamma e_1+\gamma e_2=0$ to infer that the corresponding term vanishes in the limit. If we now assume that $i=1$ then thanks to Lemma 4.3.7 below we have that the remainder term converges toward (recall that $c^2_\gamma(\boldsymbol{\mu})=0$ for

$\gamma > 1$)

$$s_1 := \frac{4}{\gamma} c_\gamma^2(\boldsymbol{\mu}) \sum_{k=1}^{2N+M} \frac{\langle 3\beta + 3\gamma e_1, \alpha_k \rangle}{2(z_k - t)} \langle V_{\beta+2\gamma e_1}(t)\mathbf{V}\rangle. \tag{4.38}$$

For the integrals over $I_{sing} \times I_{reg}$ we proceed as before: in the case where $(i,j) = (2,1)$ simplifications occur and we see that the corresponding $I_{sing} \times I_{reg}$ integral is given, up to a term that vanishes in the limit, by

$$I_{sing} \times I_{reg} \left[\partial_x \left(\frac{2\gamma}{(t-y)(x-y)} \Psi_{2,1}(x,y) \right) \right]$$

which is seen to converge to zero in the limit thanks to the assumption that $\mu_{L,2} - \mu_{R,2} = 0$. For $i = j = 2$ we see that the integrand is identically zero, so we are left with the cases where $i = 1$. If we take $j = 2$ then we get

$$I_{sing} \times I_{reg} \left[\partial_x \left(\frac{2\gamma}{(t-y)(y-x)} \Psi_{1,2}(x,y) \right) \right]$$

which converges to 0 since $\langle \beta, \gamma e_1 \rangle < 0$. When $j = 1$ and $\chi = \gamma$ the integrand is actually zero. Finally when we take $j = 1$ and $\chi = \frac{2}{\gamma}$ the corresponding term is given by

$$I_{sing} \times I_{reg} \left[\partial_x \left(\left(\frac{-2\gamma\delta_{j,2}}{(t-x)(x-y)} - \frac{2\gamma\delta_{j,1}}{(t-x)(t-y)} - \frac{\gamma^3 \langle 3\beta + \gamma e_1, \gamma e_j \rangle}{2(t-x)(t-y)} \right) \Psi_{1,j}(x,y) \right) \right]$$

which in the limit (via Lemma 4.3.6) will yield a term

$$r_4 := \gamma^2 (\frac{2}{\gamma} - \gamma)(\mu_{L,1} + \mu_{R,1}) I_{reg} \left[\left(\frac{\langle \gamma e_1 - \frac{2}{\gamma}\rho, \gamma e_j \rangle}{2(t-y)} \right) \Psi_{1,j}(t,y) \right] \tag{4.39}$$

To wrap up for the two-fold integrals, we see that we arrive to the following conclusion:

$$\begin{aligned} &A + B - \frac{8}{\chi^3} I_{sing}^3 \left[\partial_x \left(\frac{\langle \beta, \gamma e_j \rangle \langle \beta, \gamma e_f \rangle}{4(t-y)(t-z)} \Psi_{i,j,f}(x,y,z) \right) \right] \\ &= (P)\text{-class terms } + (r_2 + r_3 + r_4) \mathbb{1}_{\chi = \frac{2}{\gamma}} + s_1 \mathbb{1}_{\chi = \gamma}. \end{aligned} \tag{4.40}$$

The three- and four-fold integrals

Therefore it only remains to treat the remaining three- and four-fold integrals, coming either from Equations (4.33) and (4.34), and given by:

$$\begin{aligned}
&-\frac{8}{\chi^3}I_{sing}^3\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_f,\alpha_k\rangle}{4(t-y)(z-z_k)}\right)\Psi_{i,j,f}(x,y,z)\right)\right]\\
&-\frac{8}{\chi^3}I_{sing}^2\times I_{reg}\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle\langle 3\beta+\gamma e_j,\gamma e_f\rangle}{4(t-y)(t-z)}+\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\gamma e_f\rangle}{4(t-y)(x-z)}\right)\Psi_{i,j,f}(x,y,z)\right)\right]\\
&-\frac{8}{\chi^3}I_{sing}^2\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_j,\gamma e_f\rangle}{4(t-x)(y-z)}\Psi_{i,j,f}(x,y,z)\right)\right]\\
&+\frac{4}{\chi}I_{sing}^2\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)-\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)\right)\right]\\
&+\frac{8}{\chi^3}I_{sing}^3\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_f,\gamma e_m\rangle}{4(t-y)(z-w)}\Psi_{i,j,f,m}(x,y,z,w)\right)\right].
\end{aligned}$$

The integral over I_{sing}^3 is treated using Lemma 4.3.7 below and yields

$$r_2:=\frac{4}{\gamma}c_\gamma^2(\boldsymbol{\mu})I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\gamma e_f,\alpha_k\rangle}{2(z-z_k)}\langle V_{\beta+2\gamma e_1}(t)V_{\gamma e_f}(z)\mathbf{V}\rangle\right].\tag{4.41}$$

As for the other integrals, by explicit computations they simplify to

$$\begin{aligned}
&\frac{2}{\chi^2}(\gamma-\chi)I_{sing}^2\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_j,\gamma e_f\rangle}{(t-x)(y-z)}\Psi_{1,j,f}(x,y,z)\right)\right]\\
&+\frac{2\gamma}{\chi^2}I_{sing}^2\times I_{reg}\left[\partial_x\left(\left(\frac{\langle 3\beta+\gamma e_1,\gamma e_f\rangle}{(t-y)(t-z)}+\frac{\langle\gamma e_i,\gamma e_f\rangle}{(t-y)(x-z)}+\frac{\chi}{\gamma}\frac{\langle\gamma e_1,\gamma e_f\rangle}{(t-y)(y-z)}\right)\Psi_{i,1,f}(x,y,z)\right)\right]\\
&-\frac{2\gamma}{\chi^2}I_{sing}^3\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_f,\gamma e_m\rangle}{(t-y)(z-w)}\Psi_{i,1,f,m}(x,y,z,w)\right)\right].
\end{aligned}$$

According to Lemma 4.3.6 the latter will give a remainder term for $\chi=\frac{2}{\gamma}$:

$$r_5:=\gamma^2(\gamma-\frac{2}{\gamma})(\mu_{L,1}+\mu_{R,1})I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_j,\gamma e_f\rangle}{2(y-z)}\Psi_{1,j,f}(t,y,z)\right],\tag{4.42}$$

while for $\chi=\gamma$ the corresponding remainder term is

$$\begin{aligned}
s_3:=&\frac{4}{\gamma}c_\gamma^2(\boldsymbol{\mu})I_{reg}\left[\frac{\langle 3\beta+3\gamma e_1,\gamma e_f\rangle}{2(t-z)}\langle V_{\beta+2\gamma e_1}(t)V_{\gamma e_f}(z)\mathbf{V}\rangle\right]\\
&-\frac{4}{\gamma}c_\gamma^2(\boldsymbol{\mu})I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_f,\gamma e_m\rangle}{2(z-w)}\Psi_{1,1,f,m}(t,t,z,w)\right].
\end{aligned}\tag{4.43}$$

Recollecting terms

Finally we have treated all the remainder terms that appear in the expression of the singular vectors. After this process the only remaining terms are

$$(r_1+r_2+r_3+r_4+r_5)\mathbb{1}_{\chi=\frac{2}{\gamma}}+(s_1+s_2+s_3)\mathbb{1}_{\substack{\chi=\gamma\\ \gamma<1}}.$$

To start with, in the case where $\chi=\gamma$ we obtain a remainder term given by $\frac{4}{\gamma}c_\gamma^2(\boldsymbol{\mu})\times$

$$\begin{aligned}&\sum_{k=1}^{2N+M}\frac{\langle 3\beta+3\gamma e_1,\alpha_k\rangle}{2(z_k-t)}\langle V_{2\gamma e_1-\gamma\omega_1}(t)\mathbf{V}\rangle-I_{reg}\left[\frac{\langle 3\beta+3\gamma e_1,\gamma e_f\rangle}{2(x-t)}\langle V_{\gamma e_i}(x)V_{\beta+2\gamma e_1}(t)\mathbf{V}\rangle\right]\\&+I_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_i\rangle}{2(x-z_k)}\langle V_{\gamma e_i}(x)V_{\beta+2\gamma e_1}(t)\mathbf{V}\rangle\right]\\&-I_{sing}\times I_{reg}\left[\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(x-y)}\langle V_{\gamma e_i}(x)V_{\gamma e_j}(y)V_{\beta+2\gamma e_1}(t)\mathbf{V}\rangle\right].\end{aligned}$$

The above is nothing but the descendant $(a\mathbf{L}_{-1}+b\mathbf{W}_{-1})V_{\beta+2\gamma e_1}$ where a,b satisfy

$$a\mathbf{L}_{-1}^{\beta+2\gamma e_1}+b\mathbf{W}_{-1}^{\beta+2\gamma e_1}=3(\gamma e_1-\gamma\omega_1)=-3\gamma h_2.$$

Therefore we infer that when $\mu_{B,1}=\mu_{B,2}=0$:

$$\psi_{-3,\gamma}=-\frac{4}{\gamma}c_\gamma^2(\boldsymbol{\mu})\mathbf{D}_{-1,\gamma}V_{\beta+2\gamma e_1}. \tag{4.44}$$

When $\chi=\frac{2}{\gamma}$ the remaining terms coming from the r_k are given by $\gamma^2(\frac{2}{\gamma}-\gamma)(\mu_{L,1}+\mu_{R,1})\times$

$$\begin{aligned}&\sum_{k=1}^{2N+M}\frac{\langle\gamma e_1-\frac{2}{\gamma}\rho,\alpha_k\rangle}{2(z_k-t)}\langle V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle-I_{reg}\left[\frac{\langle\gamma e_1-\frac{2}{\gamma}\rho,\gamma e_j\rangle}{2(y-t)}\langle V_{\gamma e_j}(y)V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\right]\\&+I_{sing}\Big[\frac{1}{y-t}\Psi_{1,2}(t,y)\Big]+I_{sing}\Big[\sum_{k=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_k\rangle}{2(y-z_k)}\langle V_{\gamma e_j}(y)V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\Big]\\&-I_{sing}\times I_{reg}\Big[\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(y-z)}\langle V_{\gamma e_i}(y)V_{\gamma e_j}(z)V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\Big].\end{aligned}$$

Like in the proof of Theorem 4.2.8, since $\langle\gamma e_1-\frac{2}{\gamma}\omega_1,\gamma e_i\rangle<0$ for $i=1,2$ we can rewrite the latter using integration by parts, and get

$$\begin{aligned}&\sum_{k=1}^{2N+M}\frac{\langle\gamma e_1-\frac{2}{\gamma}\rho,\alpha_k\rangle}{2(z_k-t)}\langle V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle-I_{reg}\left[\frac{\langle\gamma e_1-\frac{2}{\gamma}\rho,\gamma e_j\rangle}{2(y-t)}\langle V_{\gamma e_j}(y)V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\right]\\&-I_{sing}\Big[\frac{\langle\gamma e_1-\frac{2}{\gamma}\omega_1,\gamma e_i\rangle-2\delta_{j=2}}{2(y-t)}\langle V_{\gamma e_j}(y)V_{\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle\Big].\end{aligned}$$

We recognize here the definition of the descendant $(a\mathbf{L}_{-1}+b\mathbf{W}_{-1})V_{\gamma e_1-\frac{2}{\gamma}\omega_1}$ where a and b are chosen in such a way that

$$a\mathbf{L}_{-1}^{\gamma e_1-\frac{2}{\gamma}\omega_1}+b\mathbf{W}_{-1}^{\gamma e_1-\frac{2}{\gamma}\omega_1}=\gamma e_1-\frac{2}{\gamma}\rho.$$

As a consequence we arrive to the equality

$$\psi_{-3,\frac{2}{\gamma}}=\gamma^2\left(\gamma-\frac{2}{\gamma}\right)\left(\mu_{L,1}+\mu_{R,1}\right)\mathbf{D}_{-1,\frac{2}{\gamma}}V_{\beta+\gamma e_1}. \tag{4.45}$$

We thus conclude in this case via the expression of the singular vector from Theorem 4.2.8.

The general case

Let us now no longer make the assumption that $\mu_{B,1}=\mu_{B,2}=0$. Then the remainder terms feature extra terms, given by

$$\begin{aligned}
\mathfrak{I}:=&-\frac{8}{\chi^3}I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{2(t-\bar x)^2}\Psi_i(x)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-\frac{8}{\chi^3}I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-x)(\bar x-z_k)}+\frac{\langle\beta,\gamma e_i\rangle}{2(t-\bar x)}\Theta_i(x)\right)\Psi_i(x)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+\frac{8}{\chi^3}I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-\bar x)(t-y)}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(\bar x-z_k)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+\frac{8}{\chi^3}I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_j\rangle^2}{2(t-y)(t-\bar y)}+\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-y)(\bar y-z_k)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{y\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+\frac{8}{\chi^3}I_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle\langle\gamma e_i,\gamma e_j\rangle}{4(t-x)(\bar x-y)}\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right].
\end{aligned}\tag{4.46}$$

In the above we have removed terms that vanish in the limit. Then we can rewrite the latter via the same reasoning as in Lemma 4.3.8: up to (P)-class terms

$$\begin{aligned}
&-\frac{8}{\chi^3}I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle(1+\frac{\langle\beta,\gamma e_i\rangle}{2})}{2(t-\bar x)^2}\Psi_i(x)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+\frac{8}{\chi^3}I_{sing}^2\left[\partial_x\left(\left(\frac{\langle\beta,\gamma e_i\rangle\langle\beta,\gamma e_j\rangle}{2(t-\bar x)(t-y)}\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}+\frac{\langle\beta,\gamma e_j\rangle^2}{2(t-y)(t-\bar y)}\mathbb{1}_{y\in\mathbb{H}_{\delta,\varepsilon}}\right)\Psi_{i,j}(x,y)\right)\right]\\
&=\frac{4}{\chi}I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-x)(z_k-\bar x)}\Psi_1(x)-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-\bar x)(z_k-\bar x)}\Psi_1(x)\right)\right)\right]\\
&-\frac{4}{\chi}I_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_1,\gamma e_i\rangle}{2(t-x)(y-\bar x)}\Psi_{1,i}(x,y)-\frac{\langle\gamma e_1,\gamma e_i\rangle}{2(t-\bar x)(y-\bar x)}\Psi_{1,i}(x,y)\right)\right].
\end{aligned}$$

As a consequence we obtain

$$
\begin{aligned}
\mathfrak{I} = &\frac{8\gamma}{\chi^2} I_{sing}\left[\partial_x\left(\frac{1}{2(t-\bar{x})}\left(\sum_{k=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_k\rangle}{4(t-\bar{x})(\bar{x}-z_k)}+\Theta_i(x)\right)\Psi_i(x)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+\frac{8}{\chi^3} I_{sing}^2\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_i,\alpha_k\rangle}{4(t-y)(\bar{x}-z_k)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+\frac{8}{\chi^3} I_{sing}^2\left[\partial_x\left(\left(\sum_{k=1}^{2N+M}\frac{\langle\beta,\gamma e_j\rangle\langle\gamma e_j,\alpha_k\rangle}{4(t-y)(\bar{y}-z_k)}\right)\Psi_{i,j}(x,y)\right)\mathbb{1}_{y\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&-\frac{4}{\chi^2}(\gamma-\chi) I_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_1,\gamma e_i\rangle}{2(t-x)(\bar{x}-y)}\Psi_{1,i}(x,y)\right)\mathbb{1}_{x\in\mathbb{H}_{\delta,\varepsilon}}\right]\\
&+\frac{4}{\chi} I_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_1,\gamma e_i\rangle}{2(t-\bar{x})(y-\bar{x})}\Psi_{1,i}(x,y)\right)\right].
\end{aligned}
$$

The limit is computed in Lemma 4.3.7: for $\chi=\frac{2}{\gamma}$ it vanishes while for $\chi=\gamma$ we get

$$
\begin{aligned}
s_4 := &\frac{4}{\gamma}c_\gamma^1(\boldsymbol{\mu})\left(\sum_{k=1}^{2N+M}\frac{\langle\gamma e_i,\alpha_k\rangle}{2(z_k-t)}\langle V_{\beta+2\gamma e_1}(t)\mathbf{V}\rangle-\Theta_i(t)\right)\\
&-\frac{4}{\gamma}c_\gamma^1(\boldsymbol{\mu}) I_{reg}\left[\frac{\langle\gamma e_1,\gamma e_i\rangle}{2(y-t)}\langle V_{\beta+2\gamma e_1}(t)V_{\gamma e_i}(y)\mathbf{V}\rangle\right].
\end{aligned}
$$

By definition of Θ_i from Equation (4.19) we see that this extra term is given by

$$
s_4=\frac{4}{\gamma}c_\gamma^1(\boldsymbol{\mu})\langle\mathbf{D}_{-1,\gamma}V_{\beta+2\gamma e_1}(t)\mathbf{V}\rangle. \tag{4.47}
$$

By combining with the expression obtained above for the case with $\mu_B=0$ we arrive to the desired conclusion when this assumption is no longer made:

$$
\psi_{-3,\gamma}=-\frac{4}{\gamma}c_\gamma(\boldsymbol{\mu})\mathbf{D}_{-1,\gamma}V_{\beta+2\gamma e_1}. \tag{4.48}
$$

This allows us to conclude for the proof of Theorem 4.3.4. □

4.3.3 Auxiliary computations

We gather here some auxiliary computations that are used in the previous subsections.

Lemma 4.3.6. *Assume that $\beta=-\frac{2}{\gamma}\omega_1$ and that F is of the form*

$$
F_i(x)=I_{sing}^p\times I_{reg}^q\left[r_{i,\boldsymbol{i}}(x,\boldsymbol{x})\Psi_{i,\boldsymbol{i}}(x,\boldsymbol{x})\right]
$$

where $r_{i,\boldsymbol{i}}$ and $\Psi_{i,\boldsymbol{i}}(x,\boldsymbol{x})$ are as in Lemma 3.2.5. Then we have

$$I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-x)}F_i(x)\right)\right] = (P)\text{-class terms} - (\mu_{L,1}+\mu_{R,1})\widetilde{F}_1(t) \tag{4.49}$$

with $\widetilde{F}$ given by $\widetilde{F}_i(x) := |t-x|^{\frac{\langle\beta,\gamma e_i\rangle}{2}}F_i(x)$, continuous at $x=t$ (via Lemma 3.2.5).

Proof. Using Stokes' formula, the only terms that may not be (P)-class in the left-hand side of (4.49) are given by

$$-\frac{1}{\varepsilon}(\mu_{L,1}F_1(t-\varepsilon)+\mu_{R,1}F_1(t+\varepsilon)) + \mu_{B,1}\int_{(t-r,t+r)}\mathrm{Im}\left(\frac{1}{t-x-\imath\delta}\right)F_1(x+\imath\delta)dx.$$

Since $\langle\gamma e_1 - \frac{2}{\gamma}\omega_1 - Q, e_1\rangle < 0$ we know thanks to Lemma 3.2.5 that $\Psi_{1,\boldsymbol{i}}(t\pm\varepsilon,\boldsymbol{x})$ scales like ε when $\varepsilon\to 0$, thus we readily see that the first term of the above converges to

$$-(\mu_{L,1}+\mu_{R,1})\widetilde{F}_1(t).$$

The second term can be rewritten as

$$\mu_{B,1}\delta^{1-\gamma^2}2^{-\gamma^2}\int_{(t-r,t+r)}|t-x-\imath\delta|^{-2}(2\delta)^{\gamma^2}F_1(x+\imath\delta)dx$$

so we have to separate the cases depending on the sign of $\gamma^2-\frac{2}{3}$. If $\gamma^2<\frac{2}{3}$ then from the fusion asymptotics (Lemma 3.2.5) we have that

$$|t-x-\imath\delta|^{-2}(2\delta)^{\gamma^2}\Psi_{1,\boldsymbol{i}}(x+\imath\delta,\boldsymbol{x}) \to \langle V_{\gamma e_{i_1}}(x_1)\ldots V_{\gamma e_{i_{(p+q)}}}(x_{p+q})V_{2\gamma e_1-\frac{2}{\gamma}\omega_1}(t)\mathbf{V}\rangle$$

uniformly in $\boldsymbol{x}$ so that we can write the above as

$$\delta^{1-\gamma^2}\left(2^{-\gamma^2}\mu_{B,1}\int_{(t-r,t+r)}(t-x)F_1(t)dx + o(1)\right)$$

which tends to 0 as $\delta\to 0$. If $\gamma^2\geq\frac{2}{3}$ then again thanks to the fusion asymptotics there exists a positive constant C such that for $\eta>0$ small enough,

$$\delta\int_{(t-r,t+r)}\frac{F_1(x+\imath\delta)}{(t-x)^2+\delta^2}dx \leq C\delta^{1-\gamma^2+\left(\frac{3}{2}\gamma-\frac{1}{\gamma}\right)^2-\eta}$$

with the exponent being positive, thus finishing the proof. ◻

Lemma 4.3.7. *Assume that G is of the form*

$$G_{i,j}(x,y) = I^p_{sing}\times I^q_{reg}\left[r_{i,j,\boldsymbol{i}}(x,y,\boldsymbol{x})\Psi_{i,j,\boldsymbol{i}}(x,y,\boldsymbol{x})\right]$$

where $r_{i,j,\boldsymbol{i}} = r_{i,j,i_1,\ldots,i_{(p+q)}}$ is regular at $x=y=t$, and such that $r_{i,j,\boldsymbol{i}}(x,y,\boldsymbol{x})\Psi_{i,\boldsymbol{i}}(\boldsymbol{x})$ is integrable on $I^p_{sing}\times I^q_{reg}$, with $\Psi_{i,j,\boldsymbol{i}}(x,y,\boldsymbol{x}) = \Psi_{i,j,i_1,\ldots,i_{p+q}}(x,y,x_1,\ldots,x_{p+q})$. and that

F is as in Lemma 4.3.6. Then for $\beta = -\gamma\omega_1$:

$$I^2_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle\delta_{i=j}}{2(t-y)}G_{i,j}(x,y)\right)\right] = (P)\text{-class terms} - \frac{\gamma^2}{2}c^2_\gamma(\boldsymbol{\mu})\widetilde{G}_{1,1}(t,t) \quad \text{and}$$
$$I_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_i\rangle}{2(t-\bar{x})}F_i(x)\right)\right] = (P)\text{-class terms} + \frac{\gamma^2}{2}c^1_\gamma(\boldsymbol{\mu})\widetilde{F}_1(t) \tag{4.50}$$

where $\widetilde{G}$ is defined by

$$\widetilde{G}_{i,j}(x,y) = |t-x|^{\frac{\langle\beta,\gamma e_i\rangle}{2}}\,|t-y|^{\frac{\langle\beta,\gamma e_j\rangle}{2}}\,|y-x|^{\frac{\langle\gamma e_i,\gamma e_j\rangle}{2}}\,G_{i,j}(x,y).$$

For $\beta = -\frac{2}{\gamma}\omega_1$ these two limits are given by (P)-class terms.

Proof. We proceed as in Lemma 4.3.6 and apply Stokes' formula to the first integral to obtain

$$\begin{aligned} I^2_{sing}\left[\partial_x\left(\frac{1}{t-y}G_{1,1}(x,y)\right)\right] &= (P)\text{-class terms} \\ &+\int_{(t-r,t-\varepsilon)\cup(t+\varepsilon,t+r)}\frac{1}{t-y}\left(\mu_{L,1}G_{1,1}(t-\varepsilon,y)-\mu_{R,1}G_{1,1}(t+\varepsilon,y)\right)\mu_1(dy). \end{aligned} \tag{4.51}$$

Developing the integral yields four terms, proportional to $\mu^2_{L,1}$, $\mu^2_{R,1}$ and $\mu_{L,1}\mu_{R,1}$. We first focus on the one with coefficient $\mu^2_{L,1}$ which writes

$$\mu^2_{L,1}\int_\varepsilon^r |y|^{-1}G_{1,1}(t-\varepsilon,t-y)dy.$$

Thanks to Lemma 3.2.5, in the case where $\beta = -\gamma\omega_1$ and $\gamma < 1$ we know that $\widetilde{G}_{1,1}(x,y)$ is continuous at $x = y = t$ since $\langle 2\gamma e_1 - \gamma\omega_1 - Q, e_1\rangle < 0$. Thus we can write the latter as

$$\mu^2_{L,1}\left(\widetilde{G}_{1,1}(t,t)\int_\varepsilon^r |y|^{-1+\frac{\gamma^2}{2}}\varepsilon^{\frac{\gamma^2}{2}}|y-\varepsilon|^{-\gamma^2}dy + o(1)\right)$$

and make the change of variable $y \leftrightarrow \varepsilon y$ to obtain

$$\mu^2_{L,1}\left(\widetilde{G}_{1,1}(t,t)\int_1^{r/\varepsilon}|y|^{-1+\frac{\gamma^2}{2}}|y-1|^{-\gamma^2}dy + o(1)\right).$$

The integral is evaluated in [24, Lemma A.2] and is equal to $\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma(1-\gamma^2)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}$. For the terms proportional to $\mu_{L,1}\mu_{R,1}$ we proceed exactly the same way to obtain

$$\mu_{L,1}\mu_{R,1}\left(\widetilde{G}_{1,1}(t,t)\int_1^{r/\varepsilon}|y|^{-1+\frac{\gamma^2}{2}}|y+1|^{-\gamma^2}dy + o(1)\right).$$

We know that this integral is equal to $\cos\left(\pi\frac{\gamma^2}{2}\right)\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma(1-\gamma^2)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}$, thus recollecting the terms we conclude for the proof of the first point when $\beta=-\gamma\omega_1$ and $\gamma<1$. When $\gamma>1$ we now have $\langle 2\gamma e_1-\gamma\omega_1-Q,e_1\rangle>0$, and therefore there is an extra term in the fusion estimates. This yields an extra scaling factor of $\langle\gamma e_1-\frac{\gamma}{2}\omega_1-\frac{Q}{2},e_1\rangle^2=(\gamma-\frac{1}{\gamma})^2$, and thus an overall scaling factor of $\varepsilon^{-2+\gamma^2+\frac{1}{\gamma^2}}$. This exponent being positive we can conclude for the case $\gamma>1$. For $\beta=-\frac{2}{\gamma}\omega_1$ the very same reasoning shows that these integrals scale like $\varepsilon^{2-\gamma^2}$ and therefore vanish in the limit, concluding the proof.

Let us now turn to the integral with $\bar{x}$ instead of y. In that case we can proceed as in Lemma 4.3.6 to see that the corresponding integral, after application of Stokes' formula, is

$$-\mu_{B,1}\delta^{1-\gamma^2}2^{-\gamma^2}\int_{(t-r,t+r)}|t-x-\imath\delta|^{-2+\gamma^2}\widetilde{F}_1(x+\imath\delta)dx.$$

We make the change of variables $x\leftrightarrow t+\delta x$ to see that the latter is given by

$$-\mu_{B,1}2^{-\gamma^2}\int_{\mathbb{R}}(1+x^2)^{-1+\frac{\gamma^2}{2}}dx\widetilde{F}_1(t)+o(1).$$

The integral is evaluated in [24, Lemma A.1] and is given by $2^{\gamma^2}\sin\left(\pi\frac{\gamma^2}{2}\right)\frac{\Gamma\left(\frac{\gamma^2}{2}\right)\Gamma(1-\gamma^2)}{\Gamma\left(1-\frac{\gamma^2}{2}\right)}$, thus concluding for the proof of the lemma. □

We eventually gather here some statements we used along the proof of Theorem 4.3.4.

Lemma 4.3.8. *Recall the definition of A from Equation (4.31). In the $\delta,\varepsilon,\rho\to 0$ limit:*

$$\begin{aligned}
&A-\frac{8}{\chi^3}I^3_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle\langle\beta,\gamma e_f\rangle}{4(t-y)(t-z)}\Psi_{i,j,f}(x,y,z)\right)\right] = (P)\text{-class terms}+r_2\mathbb{1}_{\chi=\frac{2}{\gamma}}\\
&-\frac{4}{\chi}I_{sing}\times I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_i,\alpha_k\rangle}{2(t-x)(z_k-y)}\Psi_{1,i}(x,y)-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\Psi_{i,1}(x,y)\right)\right)\right]\\
&+\frac{4}{\chi}I^2_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)-\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)\right)\right]
\end{aligned}\tag{4.52}$$

where we have set

$$r_2:=\gamma^2(\frac{2}{\gamma}-\gamma)(\mu_{L,1}+\mu_{R,1})I_{sing}\left[\frac{\Psi_{1,2}(t,y)}{t-y}\right].\tag{4.53}$$

Proof. Let us write $\mathfrak{J}:=A-\frac{8}{\chi^3}I^3_{sing}\left[\partial_x\left(\frac{\langle\beta,\gamma e_j\rangle\langle\beta,\gamma e_f\rangle}{4(t-y)(t-z)}\Psi_{i,j,f}(x,y,z)\right)\right]$. To start with we use the specific value of $\beta = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$ to simplify the expression of $\mathfrak{J}$:

$$\begin{aligned}\mathfrak{J} = I_{sing}^2\Big[\partial_x\Big(\frac{2\gamma}{(t-y)(x-y)}\Psi_{2,1}(x,y) - \frac{2\gamma}{(t-x)(x-y)}\Psi_{1,2}(x,y)\Big)\Big]\\ -\frac{4}{\chi}I_{sing}^2\left[\partial_x\left(\frac{1+\frac{\langle\beta,\gamma e_1\rangle}{2}}{(t-y)^2}\Psi_{i,1}(x,y) + \frac{\frac{\gamma\chi}{2}-\gamma^2}{(t-x)(t-y)}\Psi_{1,1}(x,y)\right)\right]\\ -\frac{4}{\chi}I_{sing}^3\left[\partial_x\left(\frac{\gamma^2}{2(t-y)(t-z)}\Psi_{i,1,1}(x,y,z)\right)\right].\end{aligned} \tag{4.54}$$

Now we claim that the following identity holds true:

$$\begin{aligned}&I_{sing}^2\left[\partial_x\left(\frac{1+\frac{\langle\beta,\gamma e_1\rangle}{2}}{(t-y)^2}\Psi_{i,1}(x,y) + \frac{\frac{\gamma\chi}{2}-\gamma^2}{(t-x)(t-y)}\Psi_{1,1}(x,y)\right)\right]\\ &-I_{sing}^2\Big[\partial_x\Big(\frac{\gamma^2}{2(t-y)(x-y)}\Psi_{2,1}(x,y) - \frac{\gamma^2}{2(t-x)(x-y)}\Psi_{1,2}(x,y)\Big)\Big]\\ &-I_{sing}^2\times I_{tot}\left[\partial_x\left(\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z) - \frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)\right)\right]\\ &=I_{sing}^2\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-x)(z_k-y)} - \frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\right)\Psi_{1,1}(x,y)\right)\right]\\ &+I_{sing}^2\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_2,\alpha_k\rangle}{2(t-x)(z_k-y)}\Psi_{1,2}(x,y) - \frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\Psi_{2,1}(x,y)\right)\right)\right].\end{aligned} \tag{4.55}$$

To see why this is the case we note that by symmetry in x, y we have

$$0 = I_{sing}\times I_{sing}\left[\partial_x\partial_y\left(\frac{1}{(t-y)}\Psi_{i,1}(x,y)\right)\right] - I_{sing}\times I_{sing}\left[\partial_x\partial_y\left(\frac{1}{(t-x)}\Psi_{1,i}(x,y)\right)\right].$$

We then develop the derivative in y: this is done by means of the equality

$$\begin{aligned}&\partial_y\left(\frac{1}{t-y}\Psi_{i,1}(x,y)\right) = \left(\frac{1+\frac{\langle\beta,\gamma e_1\rangle}{2}}{(t-y)^2} + \frac{\langle\gamma e_i,\gamma e_1\rangle}{2(t-y)(x-y)} + \sum_{k=1}^{2N+M}\frac{\langle\alpha_k,\gamma e_1\rangle}{2(t-y)(z_k-y)}\right)\Psi_{i,1}(x,y)\\ &-I_{tot}\left[\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)\right]\end{aligned}$$

and likewise for the other term that appears there. Recollecting terms and simplifying the expression obtained using that $\beta = -\chi\omega_1$ we then arrive to Equation (4.55). Thanks to this equality we can rewrite Equation (4.54) under the form

$$\begin{aligned}
\mathfrak{I} &= 2\gamma\left(1-\frac{\gamma}{\chi}\right) I_{sing}\times I_{sing}\left[\partial_x\left(\frac{\Psi_{2,1}(x,y)}{(t-y)(x-y)}-\frac{\Psi_{1,2}(x,y)}{(t-x)(x-y)}\right)\right]\\
&-\frac{4}{\chi}I^3_{sing}\left[\partial_x\left(\frac{\gamma^2}{2(t-y)(t-z)}\Psi_{i,1,1}(x,y,z)\right)\right]\\
&-\frac{4}{\chi}I^2_{sing}\times I_{tot}\left[\partial_x\left(\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)-\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)\right)\right]\\
&-\frac{4}{\chi}I_{sing}\times I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-x)(z_k-y)}-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\right)\Psi_{1,1}(x,y)\right)\right]\\
&-\frac{4}{\chi}I_{sing}\times I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_2,\alpha_k\rangle}{2(t-x)(z_k-y)}\Psi_{1,2}(x,y)-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\Psi_{2,1}(x,y)\right)\right)\right].
\end{aligned}$$

The three-fold integrals can be dealt with by using the symmetries in the different variables involved. We see that by doing so the integrals simplify to

$$\begin{aligned}
&\frac{4}{\chi}I^3_{sing}\left[\partial_x\left(\frac{\gamma^2}{2(t-y)(z-y)}\Psi_{i,1,2}(x,y,z)\right)\right]\\
&+\frac{4}{\chi}I^2_{sing}\times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)-\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)\right)\right].
\end{aligned}$$

Since $\mu_{L,2}-\mu_{R,2}=0$, the fact that, in virtue of Lemma 3.2.5, the integral

$$I^2_{sing}\left[\frac{1}{2(t-y)(z-y)}\langle V_{\beta+\gamma e_2}(t)V_{\gamma e_1}(y)V_{\gamma e_2}(z)\mathbf{V}\rangle\right]$$

is absolutely convergent, implies that the term corresponding to $i=2$ in the integral over I^3_{sing} vanishes in the limit. As for the case where $i=1$, along the same lines as in the proof of Theorem 4.2.8 we see that for $\chi=\frac{2}{\gamma}$ this term goes to 0 as well. The case where $i=1$ and $\chi=\gamma$ requires more care: we develop the derivative in x to rewrite the integral over I^3_{sing} as

$$\begin{aligned}
&2\gamma I^3_{sing}\left[\left(\frac{-\gamma^2}{2(t-y)(z-y)(t-x)}+\frac{\gamma^2}{(t-y)(z-y)(y-x)}+\frac{-\gamma^2}{2(t-y)(z-y)(z-x)}\right)\Psi_{1,1,2}(x,y,z)\right]\\
&+2\gamma I^3_{sing}\left[\sum_{k=1}^{2N+M}\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z-y)(z_k-x)}\Psi_{1,1,2}(x,y,z)\right]\\
&-2\gamma I^3_{sing}\times I_{tot}\left[\frac{\langle\gamma e_1,\gamma e_i\rangle}{2(t-y)(z-y)(w-x)}\Psi_{1,1,2,i}(x,y,z,w)\right].
\end{aligned}$$

By symmetry in x,y cancellations occur and the first line is seen to vanish. As for the other terms we can write them as derivatives in y of a function that is well-defined at $y=t$ and as such yield a term that vanishes in the limit. This allows to conclude that the three-fold integrals contribute to the limit only by

$$\frac{4}{\chi} I_{sing}^2 \times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)-\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)\right)\right].$$

And as a consequence we can write that

$$\begin{aligned}
\mathfrak{I} = \ &(P)\text{-class terms } +2\gamma\left(1-\frac{\gamma}{\chi}\right) I_{sing}\times I_{sing}\left[\partial_x\left(\frac{\Psi_{2,1}(x,y)}{(t-y)(x-y)}-\frac{\Psi_{1,2}(x,y)}{(t-x)(x-y)}\right)\right]\\
&-\frac{4}{\chi} I_{sing}\times I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-x)(z_k-y)}-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\right)\Psi_{1,1}(x,y)\right)\right]\\
&-\frac{4}{\chi} I_{sing}\times I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_2,\alpha_k\rangle}{2(t-x)(z_k-y)}\Psi_{1,2}(x,y)-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\Psi_{2,1}(x,y)\right)\right)\right]\\
&+\frac{4}{\chi} I_{sing}^2 \times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)-\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)\right)\right].
\end{aligned}$$

Hence it remains to treat the first line, for which it is readily seen that it vanishes if $\chi=\gamma$ so that we may assume that $\chi=\frac{2}{\gamma}$. Now thanks to Lemma 3.2.5 for $\beta=-\frac{2}{\gamma}\omega_1$ the integral

$$I_{sing}\left[\frac{1}{(t-y)^2}\langle V_{\beta+\gamma e_2}(t)V_{\gamma e_1}(y)\mathbf{V}\rangle\right]$$

is absolutely convergent. Hence and since we have assumed $\mu_{L,2}-\mu_{R,2}=0$ the term in the first line corresponding to $\Psi_{2,1}$ is a $o(1)$. As for the second one it is seen to converge to

$$r_2 := \gamma^2\left(\frac{2}{\gamma}-\gamma\right)(\mu_{L,1}+\mu_{R,1})I_{sing}\left[\frac{\Psi_{1,2}(t,y)}{t-y}\right].$$

To summarize and after all these remarkable simplifications we see that all that is left from Equation (4.54) is, as expected,

$$\begin{aligned}
\mathfrak{I} = \ &(P)\text{-class terms } + r_2\mathbb{1}_{\chi=\frac{2}{\gamma}}\\
&-\frac{4}{\chi} I_{sing}\times I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-x)(z_k-y)}-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\right)\Psi_{1,1}(x,y)\right)\right]\\
&-\frac{4}{\chi} I_{sing}\times I_{sing}\left[\partial_x\left(\sum_{k=1}^{2N+M}\left(\frac{\langle\gamma e_2,\alpha_k\rangle}{2(t-x)(z_k-y)}\Psi_{1,2}(x,y)-\frac{\langle\gamma e_1,\alpha_k\rangle}{2(t-y)(z_k-y)}\Psi_{2,1}(x,y)\right)\right)\right]\\
&+\frac{4}{\chi} I_{sing}^2 \times I_{reg}\left[\partial_x\left(\frac{\langle\gamma e_i,\gamma e_j\rangle}{2(t-x)(z-y)}\Psi_{1,i,j}(x,y,z)-\frac{\langle\gamma e_1,\gamma e_j\rangle}{2(t-y)(z-y)}\Psi_{i,1,j}(x,y,z)\right)\right].
\end{aligned}$$

□

4.4 Proof of Theorem 1.2.11

We explain here how to obtain the differential equations of Theorem 1.2.11. The method is mostly algorithmic, so we will be brief. To start with, the probabilistic representation of the correlation function entails that a correlation function with an insertion at $+\infty$, defined by the limit $\langle V_\beta(+\infty)\mathbf{V}\rangle := \lim_{t\to+\infty} |t|^{2\Delta_\beta}\langle V_\beta(t)\mathbf{V}\rangle$ does make sense and we have

$$\langle V_\beta(+\infty)\mathbf{V}\rangle = C(\boldsymbol{z},\boldsymbol{\alpha})\int_{\mathbb{R}^2} e^{\langle \boldsymbol{s}+\frac{\beta}{2},\boldsymbol{c}\rangle}\mathbb{E}\Big[$$
$$\exp\left(-\sum_{i=1}^2 \mu_{B,i}e^{\langle\gamma e_i,\boldsymbol{c}\rangle}\int_{\mathbb{H}} Z_i\,|x|_+^{\langle\gamma e_i,\beta\rangle} M_{\gamma e_i}(dxd\bar{x}) + e^{\langle\frac{\gamma}{2}e_i,\boldsymbol{c}\rangle}\int_{\mathbb{R}} Z_i^\partial\,|x|_+^{\langle\gamma e_i,\beta\rangle}\mu_i(x)M^\partial_{\gamma e_i}(dx)\right)\Bigg]\,d\boldsymbol{c}.$$

Correlation functions defined in such a way satisfy Ward identities similar to that of Theorem 1.2.6 and obtained from these ones by an appropriate limit as $t\to+\infty$. These represent the starting point for the proof of Theorem 1.2.11: from there this statement is mostly algorithmic and has been conducted using Mathematica. We describe here the method leading to these BPZ equations and refer to [25, Subsection 5.3] for more details.

We start with Equation (1.57) and express the left-hand side, by means of the global Ward identities for the higher-spin current from Theorem 1.2.6, in terms of $W_{-i}V_{\beta^{\circledast}}$, $i=1,2$, and $W_{-1}V_{\beta^*}$. Inverting this system of equations gives for the three-point boundary correlator

$$\begin{aligned}\langle \mathbf{W}_{-3}V_{\beta^{\circledast}}(t)V_{\beta_1}(0)V_{\beta^*}(1)V_{\beta_2}(+\infty)\rangle = \Bigg[&-\left(\frac{2}{t}+\frac{1}{t-1}\right)\mathbf{W}^{(\circledast)}_{-2}-\left(\frac{1}{t^2}+\frac{2}{t(t-1)}\right)\mathbf{W}^{(\circledast)}_{-1}\\ &+\frac{\mathbf{W}^{(*)}_{-1}}{(t(t-1))^2}-\frac{w(\beta^{\circledast})+w(\beta_2)}{t^2(t-1)}-\frac{w(\beta_1)}{t^3(t-1)}+\frac{w(\beta^*)}{t(t-1)^2}\left(\frac{1}{t}+\frac{1}{t-1}\right)\Bigg]\langle\mathbf{V}\rangle.\end{aligned}$$

Now, under the assumption (1.53), the singular vectors $\psi_{-1,\beta^{\circledast}}$ and $\psi_{-2,\beta^{\circledast}}$ are null, and likewise for ψ_{-1,β^*}. This allows to express the right-hand side in the above equation in terms of $\mathbf{L}$-descendants, which act as differential operators on the correlation functions. Moreover

$$\mathscr{L}^{(\circledast)}_{-2}\langle\mathbf{V}\rangle = \left[\frac{2t-1}{t(1-t)}\partial_t+\frac{\Delta_\infty-\Delta}{t(t-1)}-\frac{\Delta_0}{t^2(t-1)}+\frac{\Delta_1}{t(t-1)^2}\right]\langle\mathbf{V}\rangle$$

in virtue of the global Ward identities. In the end, the left-hand side of Equation (1.57) can be expressed under the form $\mathscr{D}_L(t)\langle\mathbf{V}\rangle$ for some differential operator $\mathscr{D}_L(t)$ acting on t.

Besides, we also know that $\psi_{-3,\beta^{\circledast}}$ is null, and therefore Equation (1.57) holds. The right-hand side being expressed in terms of $\mathbf{L}$-descendants, using the global Ward identities for the stress-energy tensor it can thus be written as $\mathscr{D}_R(t)\langle\mathbf{V}\rangle$ for some differential operator $\mathscr{D}_R(t)$ (different from $\mathscr{D}_L(t)$) and acting only on the t variable. As a consequence we obtain the equality $\mathscr{D}_R(t)\langle\mathbf{V}\rangle = \mathscr{D}_L(t)\langle\mathbf{V}\rangle$. Simplifying the latter yields Equation (1.59).

The other equation is obtained in the exact same fashion, that is by combining global Ward identities with explicit expressions for the null vectors to express the two sides of Equation (1.57) in terms of a differential operator in the t variable.

5 Outlook and perspectives

Sommaire

5.1 Introduction

In this last chapter we describe some short- to long-term research prospects regarding Toda theories and related topics. Some projects are already underway and in this case we give more details and state some obtained (unpublished) results.

5.2 Reflection coefficients and computation of the structure constants

The derivation of BPZ equations (Theorem 1.2.11) satisfied by certain correlations of the $\mathfrak{sl}_3\mathbb{C}$ Toda CFT on the half-plane paves the way for the explicit computation of these correlations. The BPZ equations are third-order hypergeometric equations, for which several bases of solutions are known. In order to compute the structure constants, the philosophy is then to exploit explicit changes of basis between these solutions, together with asymptotic analysis of the path integral, in order to derive periodicity equations for the correlation functions, that in turn may determine the latter, submitted to some additional meromorphicity conditions. Starting from the BPZ equations unveiled in this work, it seems possible to obtain explicit formulas for certain families of correlation functions of the form $\langle V_\alpha(i)V_{\beta^*}(0)\rangle$ and $\langle V_{\beta_1}(0)V_{\beta^*}(1)V_{\beta_2}(\infty)\rangle$, which correspond respectively to the two structure constants given by the bulk–boundary correlator and the boundary three-point function. Let us recall that, to date, no explicit formula is known for these quantities.

The next step in this direction is to extend the domain of definition of these correlation functions, which requires having an explicit formula for the so-called *reflection coefficients* of the theory.

5.2.1 The reflection relation in Liouville and Toda CFTs

It is well-known in Liouville and Toda CFTs that the primary fields obey a reflection relation. In the case of Liouville CFT it formally takes the form

$$V_\alpha = R(\alpha)V_{2Q-\alpha} \tag{5.1}$$

where $R(\alpha)$ is a constant called the reflection coefficient of the theory. In Toda CFTs, since the field is vector-valued, these coefficients depend on more complicated reflections, in that they are indexed by the *Weyl group* W associated with the Lie algebra $\mathfrak{g}$ underlying the model (see below). The reflection relations take the form

$$V_\alpha = R_s(\alpha)V_{Q+s(\alpha-Q)} \tag{5.2}$$

for any reflection s in the Weyl group W. At the level of the correlation functions, the reflection coefficients allow one to extend the domain of definition of the correlation functions beyond the Seiberg bounds (Theorem 1.2.1), via meromorphic continuation. Indeed, they arise as coefficients in the asymptotic expansion of the Vertex Operators, as we shall describe below.

On the mathematical side, the Liouville reflection coefficient on the sphere [33, 102] has been computed in [65] along the proof of the DOZZ formula. In the boundary case, the computation of the reflection coefficients [39] is also part of the derivation of the structure constants in [6]. In a work in progress [29] with Cerclé we perform the computation of the reflection coefficients associated with boundary Toda CFTs, building on the machinery developed in [21] —where the bulk reflection coefficients [40] are computed— together with inputs from [6]. We show that these reflection coefficients describe the asymptotics of the Vertex Operators in particular regimes. As a by-product, we obtain the joint tail probabilities of families of correlated bulk/boundary GMC measures, which are described by the (unit volume) reflection coefficients. This fact is well known, and we refer to [86] for a concise treatment of these questions. We introduce now a few notions that are needed in order to state and explain these results.

5.2.2 The setting

5.2.2.1 Reflection groups

Let $(V,\langle\cdot,\cdot\rangle)$ be a Euclidean space, and $\mathscr{R}$ a root system associated with V (recall Definition 2.2.5). To these Euclidean space and root system is naturally attached a *reflection group* W, generated by the reflections with respect to the roots in $\mathscr{R}$. In fact, one can see that W is freely generated by the reflections with respect to the simple roots $(e_i)_{i=1,\dots,r}$ where r is the dimension of V.

Every reflection in W takes the form $s_\alpha : x \mapsto \langle x,\alpha\rangle\alpha^\vee$ where $\alpha \in V^* \simeq V$ and $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$. Moreover, every reflection s can be put under the form $s = s_{i_1}\cdots s_{i_k}$, where s_i is the reflection with respect to the simple root e_i. The minimal number of such simple reflections entering any such decomposition of s is called the *length* of s and is denoted $l(s)$: it is such that $\varepsilon(s) := \det(s) = (-1)^{l(s)}$.

The action of the reflection group naturally divides V into *chambers*: these are the connected components of $V\setminus\bigcup_{i=1}^r H_i$, where $H_i := \{x\in V, \langle x,e_i\rangle = 0\}$ is the hyperplane orthogonal to the simple root e_i. For instance, the *fundamental chamber* is given by

$$\mathscr{C} := \{x\in V, \langle x,e_i\rangle > 0 \quad \text{for all } 1\leq i\leq r\}, \tag{5.3}$$

and W has a free and transitive action on the Weyl chambers. Moreover $\overline{\mathscr{C}}$ is a fundamental domain for the action of W: for any $x\in V$ there is a unique $y\in\overline{\mathscr{C}}$ that is conjugate to x under W. We denote by $\mathscr{C}_- := -\mathscr{C}$ the *negative Weyl chamber* and define the *walls* of $\mathscr{C}$ via:

$$\partial\mathscr{C}_i := \{x\in V, \langle x,e_i\rangle = 0\}\cap\partial\mathscr{C}. \tag{5.4}$$

In the case where the Euclidean space $(V,\langle\cdot,\cdot\rangle)$ arises from a Lie algebra g, which is the setting that motivates this study, the reflection group W is called the *Weyl group* of $\mathfrak{g}$. It does not characterize the Lie algebra $\mathfrak{g}$, and that is why we always have to prescribe a fixed set of simple roots. In the rest of this section we thus fix a Euclidean space $(V,\langle\cdot,\cdot\rangle)$, a reflection group W acting on V, we choose a root system $\mathscr{R}$ for W and set $(e_i)_{i=1,\dots,r=\dim V}$ a basis of simple roots.

5.2.2.2 Radial decomposition of the Gaussian Free Field

Consider the Neumann GFF $\mathbf{X}$[1] defined on the complex upper half-plane by its correlation kernel (2.35). Recall that the bulk and boundary GMC measures associated to $\mathbf{X}$ are defined by

$$A^{\gamma e_i}(d^2x) := \lim_{\varepsilon\to 0} e^{\langle\gamma e_i,\mathbf{X}_\varepsilon(x)\rangle - \frac{1}{2}\mathbb{E}[\langle\gamma e_i,\mathbf{X}_\varepsilon(x)\rangle^2]}d^2x \tag{5.5}$$

1. Since we only deal with Neumann boundary conditions and since the metric is fixed we omit the respective marks of dependency.

for the bulk measure, and

$$L^{\gamma e_i}(\mathrm{d}x) := \lim_{\varepsilon\to 0} e^{\langle \frac{\gamma}{2}e_i,\mathbf{X}_\varepsilon(x)\rangle - \frac{1}{2}\mathbb{E}[\langle \frac{\gamma}{2}e_i,\mathbf{X}_\varepsilon(x)\rangle^2]}\mu_i(\mathrm{d}x) \tag{5.6}$$

for the boundary one. Here $\mu_i(dx)$ is of the form $\mu_i(dx) = \left(\mu_{L,i}\mathbb{1}_{x\le 0} + \mu_{R,i}\mathbb{1}_{x>0}\right)dx$ where $\mu_{L,i}$ and $\mu_{B,i}$ are complex constants. The so-called *radial decomposition* of the GFF $\mathbf{X}$ on the half-disk consists in writing

$$\mathbf{X}(e^{-t+i\theta}) = B_{2t} + \mathbf{Y}(t,\theta), \quad \text{with} \quad B_{2t} := \frac{1}{\pi}\int_0^\pi \mathbf{X}(e^{-t+i\theta})d\theta \tag{5.7}$$

and $\mathbf{Y}(t,\theta) := \mathbf{X}(e^{-t+i\theta}) - B_{2t}$. The Gaussian process $(B_{2t})_{t\ge 0}$ has covariances given by

$$\mathbb{E}[\langle u, B_{2t}\rangle\langle u, B_{2s}\rangle] = 2\langle u, v\rangle(t\wedge s) \tag{5.8}$$

so that $(B_{2t})_{t\ge0}$ is a Brownian motion on $\mathbb{R}^d$ started at 0. Moreover, the processes $(B_{2t})_{t\ge0}$ and $(\mathbf{Y}(t,\theta))_{\substack{t\ge 0\\ \theta\in[0,\pi]}}$ are independent, and the field $\mathbf{Y}$ has the following covariance structure:

$$\mathbb{E}[\langle u,\mathbf{Y}(t,\theta)\rangle\langle u,\mathbf{Y}(s,\theta')\rangle] = \langle u, v\rangle \ln\frac{e^{-2t}\vee e^{-2s}}{|e^{-t+i\theta}-e^{-s+i\theta'}||e^{-t+i\theta}-e^{-s-i\theta'}|}. \tag{5.9}$$

for all $u, v\in V$, $t\ge 0$ and $\theta\in[0,\pi]$.

Using the radial decomposition of the GFF, the bulk and boundary GMC measures associated with the field $\mathbf{X}$ can be expressed in terms of the Brownian motion B_{2t} and the field $\mathbf{Y}(t,\theta)$ as

$$\begin{aligned} A^{\gamma e_i}(dxd\bar{x}) &= e^{\langle \gamma e_i, B_{2t}-(Q+\frac{\gamma e_i}{2})t\rangle} A_{\mathbf{Y}}^{\gamma e_i}(dt,d\theta) \quad \text{and}\\ L^{\gamma e_i}(dx) &= e^{\langle \frac{\gamma}{2} e_i, B_{2t}-(Q+\frac{\gamma e_i}{2})t\rangle} L_{\mathbf{Y}}^{\gamma e_i}(dt) \end{aligned} \tag{5.10}$$

where $A_{\mathbf{Y}}^{\gamma e_i}$ and $L_{\mathbf{Y}}^{\gamma e_i}$ are respectively the bulk and boundary GMC measures associated to the field $\mathbf{Y}$. They are defined by the limits

$$\begin{aligned} A_{\mathbf{Y}}^{\gamma e_i}(dt,d\theta) &= \lim_{\varepsilon\to0} e^{\langle \gamma e_i,\mathbf{Y}_\varepsilon(t,\theta)\rangle-\frac{1}{2}\mathbb{E}[\langle\gamma e_i,\mathbf{Y}_\varepsilon(t,\theta)\rangle^2]}dtd\theta \quad \text{and}\\ L_{\mathbf{Y}}^{\gamma e_i}(dt) &= \lim_{\varepsilon\to0}(\mu_{L,i}e^{\langle\frac{\gamma}{2}e_i,\mathbf{Y}_\varepsilon(t,\pi)\rangle-\frac{1}{2}\mathbb{E}[\langle\frac{\gamma}{2}e_i,\mathbf{Y}_\varepsilon(t,\pi)\rangle^2]} + \mu_{R,i}e^{\langle\frac{\gamma}{2}e_i,\mathbf{Y}_\varepsilon(t,0)\rangle-\frac{1}{2}\mathbb{E}[\langle\frac{\gamma}{2}e_i,\mathbf{Y}_\varepsilon(t,0)\rangle^2]})dt. \end{aligned}$$

We will make clear now the reason for these notations.

5.2.2.3 Vertex Operators

The objects of interest of this section are the boundary Vertex Operators $V_\alpha(0)$, in the setting of the complex upper half-plane. We will be interested in the $\mathbf{c}\to\infty$ asymptotics of the latter, and thus will change slightly our notations. We first fix for $i = 1,\cdots,r$ bulk cosmological constants $\mu_{B,i}\ge 0$ as well as boundary ones $\mu_{L,i}$ and

$\mu_{R,i}$, that we choose to be complex-valued but subject to the following assumptions:

$$\text{Re } \mu_{L,i},\ \text{Re } \mu_{R,i} \geq 0 \quad \text{and} \quad \mu_{B,i} + \text{Re } \mu_{L,i} + \text{Re } \mu_{R,i} > 0 \text{ for all } i = 1,\dots,r. \tag{5.11}$$

We will call such cosmological constants *admissible*. Having fixed these parameters, the Vertex Operators then depend on a weight $\alpha \in V$ and on a *zero-mode* $\boldsymbol{c} \in V$, and are defined by[2]

$$\psi_\alpha(\boldsymbol{c}) := e^{\langle \frac{\alpha-Q}{2}, \boldsymbol{c}\rangle}\mathbb{E}\left[\exp\left(-\sum_{i=1}^r \mu_{B,i} e^{\langle \boldsymbol{c},\gamma e_i\rangle}\int_{\mathbb{D}\cap\mathbb{H}} |x|^{-\langle\alpha,\gamma e_i\rangle}|x-\bar{x}|^{-\frac{\langle\gamma e_i,\gamma e_i\rangle}{2}} A^{\gamma e_i}(dxd\bar{x}) \right.\right.$$
$$\left.\left. -\sum_{i=1}^r e^{\langle \boldsymbol{c},\frac{\gamma}{2}e_i\rangle}\int_{-1}^1 |x|^{-\langle\alpha,\frac{\gamma}{2}e_i\rangle} L^{\gamma e_i}(dx)\right)\right] \tag{5.12}$$

where recall that the boundary cosmological constants are taken into account in the boundary measure $L^{\gamma e_i}(dx)$. The Seiberg bounds 1.2.1 translate as the following for the Vertex Operator $\psi_\alpha(\boldsymbol{c})$.

Lemma 5.2.1. *Assume that $\alpha \in Q + \mathscr{C}_-$ and that Equation (5.11) holds true. Then $\psi_\alpha(\boldsymbol{c})$ is well-defined and non-zero.*

The Vertex Operators can be rewritten using the radial decomposition of the GFF described above. Namely, thanks to Equation (5.10) we see that

$$\psi_\alpha(\boldsymbol{c}) = e^{\langle \nu,\boldsymbol{c}\rangle}\times$$
$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^r \mu_{B,i}e^{\langle \boldsymbol{c},\gamma e_i\rangle}\int_0^{+\infty} e^{\langle B_t^\nu,\gamma e_i\rangle}A_t^i(dt) + e^{\langle \boldsymbol{c},\frac{\gamma}{2}e_i\rangle}\int_0^{+\infty} e^{\langle B_t^\nu,\frac{\gamma e_i}{2}\rangle}L_t^i(dt)\right)\right] \tag{5.13}$$

where we have set $\nu = \frac{\alpha-Q}{2}$ and with $(B_t^\nu)_{t\geq 0}$ a drifted Brownian motion in time $2t$ with drift ν, that is, $B_t^\nu = B_{2t} + (\alpha - Q)t$, while

$$A_t^i(dt) := \int_0^\pi (2\sin(\theta))^{-\frac{\langle\gamma e_i,\gamma e_i\rangle}{2}} A_{\mathbf{Y}}^{\gamma e_i}(dt,d\theta) \quad \text{and} \quad L_t^i(dt) := L_{\mathbf{Y}}^{\gamma e_i}(dt). \tag{5.14}$$

The assumption that $\alpha \in Q + \mathscr{C}_-$ thus translates as $\nu \in \mathscr{C}_-$.

5.2.2.4 Path decomposition of a drifted Brownian motion

One can show that the behavior of $\psi_\alpha(\boldsymbol{c})$ as $\mathbf{c} \to \infty$ in $\mathscr{C}_-$ is governed by the radial part of $\mathbf{X}$, that is itself controlled by the respective maximums of the processes $\langle \gamma e_i, B^\nu\rangle$, up to marginal terms that contribute as constant order terms in the limit. We thus need to understand the law of the process $\mathscr{B}^\nu$ whose law is that of B^ν conditioned on the value of its (vectorial) maximum $\mathbf{M}$. In [21] Cerclé provided a path decomposition of the drifted Brownian motion B^ν —generalizing a celebrated result by Williams [99]— which is a key ingredient in studying the law of the latter.

2. This definition corresponds to the GFF expectation of the Vertex Operator $\mathbf{V}_\alpha(0)$ on the half-disk, without integrating out the zero-mode.

Let us briefly recall this result, adapted to our context. Define a process $\mathscr{B}^\nu$, started from $-\mathbf{M}$ with $\mathbf{M}\in\mathscr{C}$, to be the welding of:

— a diffusion process $\mathbf{X}^1$ started from $-\mathbf{M}$ and with infinitesimal generator

$$\Delta+2\left(\nu+\nabla\log\partial_{1,\cdots,r}h\right)\cdot\nabla.$$

This process will almost surely hit $\partial\mathscr{C}_-$: without loss of generality let us assume that this is done at $z_1\in\partial\mathscr{C}_1$;

— we then run an independent diffusion process $\mathbf{X}^2$ started from z_1 and with infinitesimal generator

$$\Delta+2\left(\nu+\nabla\log\partial_{2,\cdots,r}h\right)\cdot\nabla.$$

Likewise this process will almost surely hit $\partial\mathscr{C}_-\setminus\partial\mathscr{C}_1$, for instance at $z_2\in\partial\mathscr{C}_2$;

— repeat the same procedure until all the components $\partial\mathscr{C}_i$ have been hit, and finally launch an independent diffusion process whose infinitesimal generator is given by

$$\Delta+2\left(\nu+\nabla\log h\right)\cdot\nabla.$$

This last process has the law of B^ν conditioned on staying inside the cone $\mathscr{C}_-$. The factors 2 arise because of the presence of the boundary that entails a time change for the drifted Brownian motion. Then [21, Theorem 2.2] the process $\mathscr{B}^\nu-\mathbf{M}$ has the law of the process B^ν conditioned on its maximum being equal to $\mathbf{M}$. Put differently a drifted Brownian motion B^ν started from the origin can be realized by sampling:

— a random variable $\mathbf{M}$ whose law is defined by

$$\mathbb{P}\left(\langle\mathbf{M},e_i\rangle\leq\boldsymbol{M}_i\quad\forall 1\leq i\leq r\right)=h(\boldsymbol{M})\mathbb{1}_{\boldsymbol{M}\in\mathscr{C}};$$

— the process $B^\nu:=\mathbf{M}+\mathscr{B}^\nu$ where $\mathscr{B}^\nu$ is the process started from $-\mathbf{M}$ described above.

5.2.2.5 Quantum discs and reflection coefficients

The above path decomposition of a drifted Brownian motion allow one to unveil a connection between the reflection coefficients and the *quantum discs* introduced in [36] in the context of the mating-of-trees approach to Liouville Quantum Gravity. Indeed, the one-dimensional version of this decomposition arises in the definition of the quantum disc considered in [6], as we briefly recall now.

Assume that $r=1$ and let B^ν be the process defined by

$$\mathrm{B}^\nu_t:=\mathrm{B}^{1,\nu}_t\mathbb{1}_{t>0}+\mathrm{B}^{2,\nu}_{-t}\mathbb{1}_{t<0}\tag{5.15}$$

with $\mathrm{B}^{1,\nu}$ and $\mathrm{B}^{2,\nu}$ two independent Brownian motions with drift $\nu<0$, in time $2t$ and conditioned to stay negative, that is diffusion processes with infinitesimal generator

$$\Delta+2\left(\nu+\nabla\log h\right)\cdot\nabla=\partial_x^2+2\nu\coth(\nu x)\partial_x.\tag{5.16}$$

Using this process we define (using stationarity of $A_t(dt)$ and $L_t(dt)$)

$$\mathscr{A}_\gamma(\nu) := \int_{\mathbb{R}} e^{\gamma \mathrm{B}_t^\nu} A_t(dt) \quad \text{and} \quad \mathscr{L}_\gamma(\nu) := \int_{\mathbb{R}} e^{\frac{\gamma}{2}\mathrm{B}_t^\nu} L_t(dt). \tag{5.17}$$

Then following [6, Lemma 2.16, Proposition 2.17] we can define for $\gamma \in (0,2)$, $\beta \in (\frac{2}{\gamma}, q)$ and admissible cosmological constants (recall (5.11)):

$$R_{\mu_B,\mu_L,\mu_R}(\beta) := \int_{\mathbb{R}} (-2\nu) e^{2\nu M} \mathbb{E}\left[\exp\left(-\mu_B e^{\gamma M} \mathscr{A}_\gamma(\nu) - e^{\frac{\gamma}{2}M}\mathscr{L}_\gamma(\nu)\right) - 1\right] \mathrm{d}M \tag{5.18}$$

where $\nu = \frac{\beta - q}{2} < 0$ and $q := \frac{\gamma}{2} + \frac{2}{\gamma}$. This quantity has been computed[3] in [6, Theorem 1.3] where it was found to describe the boundary reflection coefficient of Liouville CFT [39] (see one of these articles for the exact formula):

$$R_{\mu_B,\mu_1,\mu_2}(\beta) = \left(\mu_B\right)^{\frac{q-\beta}{2\gamma}} R_{\mathrm{FZZ}}(\beta,\sigma_L,\sigma_R), \quad \text{where} \quad \frac{\mu_L}{\sqrt{\mu_B}} = \frac{\cos\left(\pi\gamma(\sigma_L - \frac{q}{2})\right)}{\sqrt{\sin\left(\frac{\pi\gamma^2}{4}\right)}}. \tag{5.19}$$

As we shall see, these quantities are the building blocks for the boundary Toda reflection coefficients.

5.2.3 Reflection coefficients, asymptotic expansion of Vertex Operators, and tail probabilities of correlated GMC measures

For any $s \in W$ we define the reflection coefficients $R_s(\alpha;\boldsymbol{\mu})$ in a recursive way by setting

$$R_{s_i}(\alpha;\boldsymbol{\mu}) := R_{\mu_{B,i},\mu_{L,i},\mu_{R,i}}(\langle \alpha, e_i^* \rangle) \quad \text{and} \quad R_{s_i s}(\alpha;\boldsymbol{\mu}) := R_{s_i}(\hat{s}\alpha;\boldsymbol{\mu}) R_s(\alpha;\boldsymbol{\mu}) \tag{5.20}$$

for any $1 \le i \le r$, where $e_i^* := \frac{e_i}{|e_i|}$ and where the coupling constant corresponding to R_{s_i} is given by $\gamma_i := \gamma|e_i|$. We state in this subsection that these reflection coefficients describe the asymptotics of the Vertex Operators (5.12), as well as the tail probabilities of correlated bulk and boundary GMC measures (Theorems 5.2.2 and 5.2.4).

Let us denote by A_i (resp. L_i) the total bulk area (resp. boundary length) of the correlated GMC measures that we have defined above. Namely

$$\begin{aligned} A_i &:= \int_{\mathbb{D}\cap\mathbb{H}} |x|^{-\langle \alpha, \gamma e_i \rangle} |x - \bar{x}|^{-\frac{\langle \gamma e_i, \gamma e_i \rangle}{2}} A^{\gamma e_i}(d^2x), \\ L_{L,i} &:= \int_{-1}^{0} |x|^{-\langle \alpha, \frac{\gamma}{2} e_i \rangle} L^{\gamma e_i}(dx) \quad \text{and} \quad L_{R,i} := \int_0^1 |x|^{-\langle \alpha, \frac{\gamma}{2} e_i \rangle} L^{\gamma e_i}(dx) \end{aligned} \tag{5.21}$$

3. Using [6, Proposition 2.17] the evaluation of the reflection coefficient from [6, Theorem 1.3] can be extended to admissible cosmological constants.

with $L_i = L_{L,i} + L_{R,i}$. Using the polar decomposition of the GFF $\mathbf{X}$ together with the path decomposition for drifted Brownian motions described above, we can rewrite for any F continuous and bounded over $\mathbb{R}_+^{2r}$:

$$\begin{aligned}&\mathbb{E}\left[F\Big(A_i,L_i\Big)_{1\le i\le r}\right]=\int_{\mathscr{C}}\mathbb{E}_{\boldsymbol{M}}\left[F\Big(e^{\langle\gamma e_i,\boldsymbol{M}\rangle}\mathscr{A}_i,e^{\langle\frac{\gamma}{2}e_i,\boldsymbol{M}\rangle}\mathscr{L}_i\Big)_{1\le i\le r}\right]\mathrm{d}\mathbb{P}(\boldsymbol{M})\quad\text{with}\\&\mathscr{A}_i:=\int_0^{+\infty}e^{\langle\mathscr{B}_t^\nu,\gamma e_i\rangle}A_t^i(dt)\quad\text{and}\quad\mathscr{L}_i:=\int_0^{+\infty}e^{\langle\mathscr{B}_t^\nu,\frac{\gamma e_i}{2}\rangle}L_t^i(dt)\end{aligned}\tag{5.22}$$

and where under $\mathbb{P}_{\mathbf{M}}$, $\mathscr{B}_t^\nu$ has the law of the process described in Subsection 5.2.2.4 and started from $-\mathbf{M}$, while A^i and L^i are independent of $\mathscr{B}^\nu$ (and $\mathbf{M}$) and defined in (5.14). This new expression of the GMC measures allow one to express the Vertex Operators in terms of the reflection coefficients.

We are now in position to state some of the results that we obtain. We say that $\boldsymbol{x}\to\infty$ inside $\mathscr{C}_-$ when for all $1\le i\le r$, $\langle\boldsymbol{x},e_i\rangle\to-\infty$. We also introduce $W_{1,\cdots,r}$ the subset of W that consists of $s\in W$ that admit a reduced expression that contains all the reflections $s_1,\cdots s_r$. For any $1\le i\le j\le r$ we set $s_{i,j}:=s_is_{i+1}\cdots s_{j-1}s_j$. First we have the following results that describe the asymptotics as $\mathbf{c}\to\infty$ in $\mathscr{C}_-$ of the Vertex Operators $\psi_\alpha(\boldsymbol{c})$.

Theorem 5.2.2. *Assume that $\nu\in\mathscr{C}_-$ satisfies $\langle\nu-s\nu,\omega_i^\vee\rangle<\frac{\gamma}{2}$ for all $1\le i\le r$ and $s\in W_{1,\cdots,r}$. Further assume that $\boldsymbol{c}\to\infty$ in $\mathscr{C}_-$ with $\langle s_{1,r}\nu-s\nu,\boldsymbol{c}\rangle\to+\infty$ for all $s\neq s_{1,r}$ in $W_{1,\dots,r}$. Then we have the asymptotic*

$$\mathbb{E}\left[\prod_{i=1}^r\left(\exp\left(-\mu_{B,i}e^{\langle\boldsymbol{c},\gamma e_i\rangle}A_i-e^{\langle\boldsymbol{c},\frac{\gamma e_i}{2}\rangle}L_i\right)-1\right)\right]\sim e^{\langle s_{1,r}\nu-\nu,\boldsymbol{c}\rangle}R_{s_{1,r}}(\alpha;\boldsymbol{\mu}).\tag{5.23}$$

As a direct consequence of the above result we obtain the asymptotic expansion of the Vertex Operators.

Corollary 5.2.3. *Assume that $\nu=\frac{\alpha-Q}{2}\in\mathscr{C}_-$ satisfies $\langle\nu-s\nu,\omega_i^\vee\rangle<\frac{\gamma}{2}$ for all $1\le i\le r$ and $s\in W$. In the regime where $\boldsymbol{c}\to\infty$ with $\langle s_1s_2\nu-s\nu,\boldsymbol{c}\rangle\to+\infty$ for all $s\neq s_1s_2$ in W with length at least 2, the following asymptotic expansion holds:*

$$\psi_\alpha(\boldsymbol{c})=e^{\langle\nu,\boldsymbol{c}\rangle}+\sum_{i=1}^rR_{s_i}(\alpha)e^{\langle s_i\nu,\boldsymbol{c}\rangle}+R_{s_1s_2}(\alpha)e^{\langle s\nu,\boldsymbol{c}\rangle}+o(e^{\langle s\nu,\boldsymbol{c}\rangle}).$$

The proof of these results relies on several ingredients, that we sketch here. First of all, the law of the maximum $\mathbf{M}$ is explicit, given by

$$d\mathbb{P}(\boldsymbol{M})=\sum_{s\in W_{1,\dots,r}}\varepsilon(s)\left(\prod_{i=1}^r\langle\omega_i^\vee,\nu-s\nu\rangle\right)e^{\langle\boldsymbol{M},\nu-s\nu\rangle}\mathbb{1}_{\boldsymbol{M}\in\mathscr{C}}d\boldsymbol{M}.$$

This formula is obtained using a Doob's h-transform (see [49, 21]). Then using the path decomposition of Subsection 5.2.2.4 one can write

$$\begin{aligned}&\mathbb{E}\left[\prod_{i=1}^{r}\left(\exp\left(-\mu_{B,i}e^{\langle \boldsymbol{c},\gamma e_i\rangle}A_i-e^{\langle \boldsymbol{c},\frac{\gamma e_i}{2}\rangle}L_i\right)-1\right)\right]\\&=\sum_{s\in W_{1,\dots,r}}\varepsilon(s)e^{\langle s\nu-\nu,\boldsymbol{c}\rangle}\int_{\mathscr{C}+\boldsymbol{c}}\prod_{i=1}^{r}\langle\omega_i^\vee,\nu-s\nu\rangle e^{\langle \nu-s\nu,\boldsymbol{M}\rangle}\\&\times\mathbb{E}_{\boldsymbol{M}-\boldsymbol{c}}\left[\prod_{i=1}^{r}\left(\exp\left(-\mu_{B,i}e^{\langle \boldsymbol{M},\gamma e_i\rangle}\mathscr{A}_i-e^{\langle \boldsymbol{M},\frac{\gamma e_i}{2}\rangle}\mathscr{L}_i\right)-1\right)\right]\mathrm{d}\boldsymbol{M}\end{aligned}\tag{5.24}$$

Now, an important fact is that the part of the integral that is near the walls of $\mathscr{C}$ will not contribute in the limit as $\mathbf{c}\to\infty$ in $\mathscr{C}_-$. On the other hand we can show that the expectation term

$$\mathbb{E}_{\boldsymbol{M}-\boldsymbol{c}}\left[\prod_{i=1}^{r}\left(\exp\left(-\mu_{B,i}e^{\langle \boldsymbol{M},\gamma e_i\rangle}\mathscr{A}_i-e^{\langle \boldsymbol{M},\frac{\gamma e_i}{2}\rangle}\mathscr{L}_i\right)-1\right)\right]$$

converges to

$$\prod_{i=1}^{r}\mathbb{E}\left[\exp\left(-\mu_{B,i}e^{\langle \boldsymbol{M},\gamma e_i\rangle}\int_{\mathbb{R}}e^{\sqrt{\langle\gamma e_i,\gamma e_i\rangle}\mathrm{B}_t^{\nu^i}}A_i(dt)+e^{\langle \boldsymbol{M},\frac{\gamma e_i}{2}\rangle}\int_{\mathbb{R}}e^{\sqrt{\langle\frac{\gamma e_i}{2},\frac{\gamma e_i}{2}\rangle}\mathrm{B}_t^{\nu^i}}L_{\mathbf{Y}}^{\gamma e_i}(dt)\right)-1\right],$$

where B_t^ν is defined in Subsection 5.2.2.5 and with $\nu_i:=\langle s_{i+1,r}\nu,e_i^*\rangle$. This is turn proves Theorem 5.2.2.

Following the same lines, we can prove that the (unit volume) reflection coefficients describe the joint tail probabilities of correlated bulk/boundary GMC measures, as it is stated in the following proposition, in the case of length one.

Proposition 5.2.4. *Assume that* $-\frac{\gamma}{2}<\langle\nu,e_i^\vee\rangle<0$ *and take* $\mu_{B,i}$, $\mu_{L,i}$ *and* $\mu_{R,i}$ *any non-negative, admissible cosmological constants. Then as* $u\to+\infty$

$$\mathbb{P}\left(\mu_{B,i}A_i>u^2,\quad \mu_{L,i}L_{L,i}>u,\quad \mu_{R,i}L_{R,i}>u\right)=\frac{\overline{R_{s_i}}(\alpha;\boldsymbol{\mu})}{u^{\frac{1}{\gamma}\langle Q-\alpha,e_i^\vee\rangle}}+o(u^{-\frac{1}{\gamma}\langle Q-\alpha,e_i^\vee\rangle}).\tag{5.25}$$

For $\mu_{B,i}=0$ this statement corresponds to [83, Proposition 1.10], while for $\mu_{B,i}>0$ this generalizes and answers a question raised in [6, Subsection 1.4.2]. It is also possible to give a generalization of this result in the general case where more than one elementary reflections are involved.

Having defined these reflection coefficient and with Theorem 5.2.2 at hand, the next step is to provide analytic continuations of the correlation functions in order to exploit the BPZ equations (Theorem 1.2.11), which we plan to address in future works.

5.3 The semi-classical limit

The semi-classical analysis of Liouville CFT has been performed in the Mathematics literature in [66] for the case of the sphere and recently in [23] for general Riemann surfaces with or without boundary. Let us recall first the classical picture, following [66].

5.3.1 Classical Liouville theory and Picard-Poincaré uniformization

Given a Riemann surface Σ, the uniformization theorem states that its universal cover $\widetilde{\Sigma}$ is biholomorphic to either the Riemann sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$, or the unit disk $\mathbb{D}$, meaning that there is a discrete subgroup of automorphisms Γ of one of the latter such that Σ is biholomorphic to $\widetilde{\Sigma}/\Gamma$. Let suppose for conciseness that Σ is hyperbolic (i.e. $\chi(\Sigma)<0$), in which case its universal cover is the unit disk $\mathbb{D}$. A way to solve the uniformization problem for Σ is then to find 1) a metric g of constant curvature on Σ 2) a conformal, orientation-preserving map f on $\mathbb{D}$ such that g is the pullback of the hyperbolic metric on the unit disk by f.

In the case of the punctured sphere $\Sigma=\hat{\mathbb{C}}\setminus\{z_1,...,z_N\}$, Picard [75, 76] showed that for $\Lambda>0$, the Liouville equation [4]

$$\Delta\Phi=2\pi\Lambda e^{2\Phi} \tag{5.26}$$

admits a unique solution Φ^*, with conical singularities of prescribed order χ_k at the singular points z_k (i.e. $\Phi(z)\sim\chi_k\log\frac{1}{|z-z_k|}$ when $z\to z_k$) and asymptotic behavior at infinity given by $\Phi(z)\sim-4\log|z|$. This entails that $e^{\Phi^*(z)}|dz|^2$ is a compact metric on $\hat{\mathbb{C}}$, with constant negative curvature equal to $-2\pi\Lambda$ on Σ, and conical singularities at the punctures.

In [77] Poincaré related the results of Picard to the problem of uniformization of $\hat{\mathbb{C}}\setminus\{z_1,...,z_N\}$. To this end he introduced the classical stress-energy tensor

$$T^{cla}[\Phi_*](z)=\partial^2\Phi_*-\frac{1}{2}(\partial\Phi_*)^2.$$

The Liouville equation (5.26) together with the prescribed behavior of Φ_* at the punctures and at infinity then constrain T^{cla} to be of the form

$$T^{cla}[\Phi_*](z)=\sum_{k=1}^N\left(\frac{\frac{\chi_k}{2}\left(1-\frac{\chi_k}{4}\right)}{(z-z_k)^2}+\frac{c_k}{z-z_k}\right)\quad\text{and}\quad T^{cla}[\Phi_*](z)=O(z^{-4})\quad\text{as}\quad z\to\infty.$$

with *a priori* unknown parameters c_k called *accessory parameters*. Poincaré showed that the function $f:=u_1/u_2$, where u_1,u_2 are two linearly independent solutions of

4. This is the Liouville equation in the flat metric $|dz|^2$, that is why there is no curvature term.

the Fuschian differential equation

$$u''(z) + \frac{1}{2} T^{cla}[\Phi_*](z) u(z) = 0 \quad ; \quad u \quad \text{holomorphic on} \quad \mathbb{D}$$

is such that the metric $e^{\Phi_*(z)}|dz|^2$ is the pullback of the hyperbolic metric on $\mathbb{D}$ by f, thus solving the uniformization problem for $\hat{\mathbb{C}} \setminus \{z_1, ..., z_N\}$.

The parameters c_k were computed non-rigorously by Polyakov and Zamolodchikov, and found to be equal to

$$c_k = \frac{1}{2} \partial_{z_k} S^{cla}_{L,(\chi_k, z_k)}[\Phi_*] \tag{5.27}$$

the latter being the classical Liouville action (1.9) but with additional conical singularities:

$$S^{cla}_{L,(\chi_k, z_k)}[\Phi] = \frac{1}{4\pi} \int_{\mathbb{C}} \left(|\nabla \Phi|^2 + 4\pi \Lambda e^{\Phi} \right) dz d\bar{z} - \sum_{k=1}^{N} \chi_k \Phi(z_k).$$

This relation has been proven in [93] using a purely geometric approach relying on Teichmüller theory. In [66], the authors study the limit of the Liouville field and correlation functions of Liouville CFT in the semi-classical limit, that is when the coupling constant is taken to 0, and under suitable reparametrization. They prove that the limit is described by the classical Liouville theory, which is not at all straightforward. By taking a suitable limit of the BPZ equation of Liouville CFT on the sphere, they obtain a new "CFT flavored" proof of the Takhtajan-Zograf theorem (5.27). Let us mention here that the more general problem of prescribing the (non-constant) curvature and singularities of a metric on a given Riemann surface, known as the Nirenberg problem, has been extensively studied: see [14, 64] for closed surfaces, [67] for surfaces with boundary and [96, 10] for surfaces with singularities.

5.3.2 Classical Toda theory

As already mentioned in the Introduction, Toda CFTs also have their classical counterpart. Let us describe briefly the geometrical framework underlying the classical Toda equations.

Given a Riemann surface Σ, a *Higgs bundle* on Σ is a pair (E, θ) where E is a holomorphic vector bundle over Σ and θ is a holomorphic section of $\mathrm{End}(E) \otimes K$, whith $K = T^*\Sigma$ the holomorphic cotangent bundle of Σ (the *canonical bundle*). θ is thus a matrix of holomorphic differentials called the *Higgs field*.

Given a Hermitian vector bundle E on Σ, i.e. a complex vector bundle equipped with a Hermitian metric h, and a Higgs field $\theta \in H^0(\Sigma, \mathrm{End}(E) \otimes K)$, the Hitchin's equations are

$$\begin{aligned} F(\nabla_h) + [\theta, \theta^{*_h}] &= 0 \\ \nabla_h^{0,1} \theta &= 0. \end{aligned} \tag{5.28}$$

Here $F(\nabla_h)$ is the curvature of the Chern connection ∇_h, i.e. the only connection compatible with h in the sense that $\nabla_h(h) = 0$ and that defines a holomorphic structure on E (by $\bar\partial_E = \nabla_h^{0,1}$), and $\cdot^{*_h}$ is the adjoint with respect to h. Under a stability assumption on the couple (E,θ) there exist solutions h to the Hitchin's equations, in which case the connection ∇_h is unique. These equations were introduced by Hitchin [56, 55] in the framework of the dimensional reduction of the Yang-Mills self-duality equations. A extensive amount of work followed, regarding the applications of the framework introduced by Hitchin. See for example [90, 91] for applications in relation with local systems and uniformization. Generally, the Hitchin's equation describe harmonic maps from $\widetilde\Sigma$ into symmetric spaces. Namely, to a solution h of Equations (5.28) is naturally attached a flat connection $D_h = \nabla_h + \alpha$ with $\alpha = \theta + \theta^{*_h}$, that in turn provides E with a holomorphic structure. Then it can be seen [2] that the Hitchin's equation is equivalent to a harmonicity condition for a map $f_h : \widetilde\Sigma \to G/K$, where G is the *structure group* and K its maximal compact subgroup, such that $\alpha = -\frac{1}{2} f_h^{-1} d f_h$ in certain coordinates, and such that f_h is equivariant under the action of the fundamental group $\pi_1(\Sigma)$. In the Toda case (see below) one can refine this analysis and obtain a holomorphic embedding $F_h : \Sigma \to \Gamma \setminus G/K_0$ where K_0 is a compact subgroup of the structure group G and Γ is the monodromy group [2].

In a special but nonetheless important case, the Hitchin's equations reduce, and are in fact equivalent, to the Toda equations [2, 9]

$$\Delta\Phi = \sum_{i=1}^{n-1} e^{\langle \Phi, e_i \rangle} e_i \tag{5.29}$$

This connection in turn allows one to link this geometrical picture to the generating currents of the classical W-algebras, as we shall illustrate now. We restrict ourselves to the case of $\mathfrak{sl}_n\mathbb{C}$. A *cyclic* $SL_n(\mathbb{C})$-Higgs bundle is a pair (E,θ) where E is the direct sum $\sum_{j=0}^{n-1} K^{\frac{n-1-2j}{2}}$ [5], and θ is written in matrix form

$$\theta = \begin{pmatrix} 0 & & & \gamma_n \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

with the 1's seen as constant holomorphic functions and $\gamma_n \in H^0(\Sigma, K^n)$. The Hitchin's equation (5.28) for such a Higgs bundle reduces to the Toda equation (5.29) (or its affine version if $\gamma_n \neq 0$). If the Hermitian metric $h = \mathrm{diag}(e^{\langle \Phi, h_i \rangle})_{i=1,\ldots,n}$ solves the Hitchin's equation for the Higgs bundle (E,θ), as explained in [2], there exists an explicit gauge transformation that put D_h in the so-called Drinfeld-Sokolov form:

5. When n is even, one has to make a choice of a *spin structure* on Σ, that is a choice of $K^{\frac{1}{2}}$.

$D^{0,1}_{h,\mathrm{DS}} = \bar\partial$, while

$$D^{1,0}_{h,\mathrm{DS}} = \partial + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -w_n & \dots & -w_2 & 0 \end{pmatrix}$$

with $\bar\partial w_i = 0$ for $i = 2, ..., n$, and where w_i are the generating currents of the W_n algebra. For instance w_2 is the classical stress-energy tensor that, in the $SL_3(\mathbb{C})$ case, writes $w_2 = \langle\partial\Phi, e_1\rangle^2 + \langle\partial\Phi, e_2\rangle^2 - (\langle\partial^2\Phi, \boldsymbol{\rho}\rangle + \langle\partial\Phi, e_1\rangle\langle\partial\Phi, e_2\rangle)$.

In the same fashion as in the Liouville case, the Toda equations should constrain the quantities w_i to take a simple meromorphic form given by the classical analog of the Ward identities. We expect these expressions to exhibit unknown quantities that would be the counterpart of the accessory parameters c_k in this context. By performing the semi-classical analysis of the $\mathfrak{sl}_3\mathbb{C}$ Toda theory we should be able to provide explicit expressions for these quantities in the case of the sphere as well as on the upper half-plane, in terms of the Toda action.

In an other but related direction, it would be interesting to study the existence and unicity of solutions to the Toda equations with the different boundary conditions described in Chapter 2. This would involve deriving new Moser-Trudinger type inequalities adapted to this context, and we expect that these solutions would describe the classical theories associated with the quantum fields described in Chapter 2.

5.4 Other directions of research

5.4.1 Cardy's boundary conditions

In following works, we plan to address the case of Cardy boundary conditions (in the sense of Chapter 2), in the simplest instance where the target Lie algebra is $\mathfrak{sl}_3\mathbb{C}$. Though the conformal Ward identities should remain valid in this setting, in the bulk and on the boundary, this is way much unclear for the higher-spin Ward identities. Indeed, while the bulk case should be tractable, heuristically the higher-spin current $\mathbf{W}$ should vanish on the boundary because of the covariance property Proposition 2.4.14 (see also the discussion in [38]). However, there is a hope that the bulk constraints allow one to recover some predictions for the bulk one-point function [38, 45].

As mentioned previously, we also expect these models to admit a semi-classical limit, the latter would present new features regarding the existing literature on the Toda equations.

5.4.2 Structure of the Verma modules

The setting of Vertex Operator Algebras [15, 47] is the right one to conduct the study of W-algebras [43, 42, 63, 46, 7, 8]. As already mentioned, one consequence of the work carried out on the study of singular vectors in $\mathfrak{sl}_3\mathbb{C}$ Toda theory on the half-plane

(Chapter 4) is a description of the structure of highest-weight representations of the W_3 algebra defined from this model. A natural direction for further research is therefore to complete this probabilistic description and to translate it into the VOA formalism. The two approaches have already proved to be complementary, notably in recent work by Cerclé [22], which in particular establishes a correspondence between W-algebras and probability theory at the level of the Gaussian Free Field.

5.4.3 Construction of the Segal's functor for Toda CFTs

Segal's functorial axiomatization of two-dimensional CFT [89] aims both to capture the algebraic structure of CFTs (VOAs) and to provide a mathematical framework best suited to the BPZ conformal bootstrap method [12]. It is partly inspired by the formulation of Euclidean Quantum Field Theories via path integrals. According to Segal, a CFT is a functor from the category whose objects are disjoint unions of parametrized circles and whose morphisms are Riemann surfaces with boundary (Σ, g), to the category whose objects are Hilbert spaces and whose morphisms are Hilbert–Schmidt operators. A CFT associates to a disjoint union of circles $\cup C_i$ a Hilbert space $H^{\otimes i}$, and to a Riemann surface (Σ, g) with parametrized boundary an amplitude $\mathscr{A}_{\Sigma,g} : H^{\otimes i} \to H^{\otimes j}$, where i is the number of positively oriented circles in $\partial\Sigma$ and j is the number of negatively oriented circles. The operation of gluing surfaces along their boundaries is then required to correspond to the composition of the amplitudes. The probabilistic construction of the amplitudes via the path integral, as well as the verification of Segal's axioms, has been carried out in the context of Liouville CFT [54, 52], and constitutes a fundamental step in the rigorous implementation of the BPZ method. Extending this construction to Toda theories would represent a major advance in the mathematical understanding of these models.

5.4.4 Imaginary theories

Imaginary Liouville theory arises when the coupling constant is chosen to be imaginary (i.e. $\gamma \in \mathrm{i}\mathbb{R}$) These theories are expected in physics to describe the scaling limits of a large class of so-called loop models. Moreover, imaginary Liouville theory is a non-rational and logarithmic CFT. The mathematical construction of the path integral of this model was carried out in [50], and constitutes the first mathematical construction of a logarithmic CFT. The Segal's axioms were also proved in that work. This study was recently generalized to surfaces with boundary in [101]. Let us emphasize that the construction of the path integral in this setting raises non-trivial geometric issues. Furthermore, in order to preserve diffeomorphism invariance of the theory, the field is taken to be valued in a compactified space (the circle). A non-compactified construction was proposed in [97].

A natural direction of research is to generalize these studies to Toda theories, which are themselves expected to describe scaling limits of statistical physics models such as RSOS models [98], as well as loop models. In Physics, a formula for the structure constants on the sphere was proposed in [35], and it was observed that this quantity is not the analytic continuation of its real counterpart (the Fateev–Litvinov formula [40]), in the same way as for imaginary Liouville theory (the DOZZ formula [33, 102] and its imaginary analogue [104]). A mathematical construction of imaginary Toda theories would provide the starting point for a rigorous study of enhanced symmetries (W-algebras) in the context of logarithmic CFTs.

5.4.5 Mating-of-trees for Toda

The different techniques of coupling Liouville theory with Schramm-Loewner evolutions, introduced in [36] under the name *mating-of-trees*, have been used to derive the boundary Liouville structure constants [5, 6]. It would be of great interest to adapt these methods to the case of Toda theories. It is not clear in which way these coupling would be defined, and a first step towards this direction would be to construct a quantum zipper [4] in the vectorial setting. Hopefully, the powerful tools of the mating-of-trees could be imported in the study of Toda CFTs, possibly leading to some new exact formulas.

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