When Periodicity Enforces Aperiodicity

Nicolas Bédaride & Thomas Fernique

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Consider n pairwise non-colinear vectors of $\mathbb{R}^2 \rightsquigarrow$ tilings of $\mathbb{R}^2$ by $\binom{n}{2}$ rhombi.

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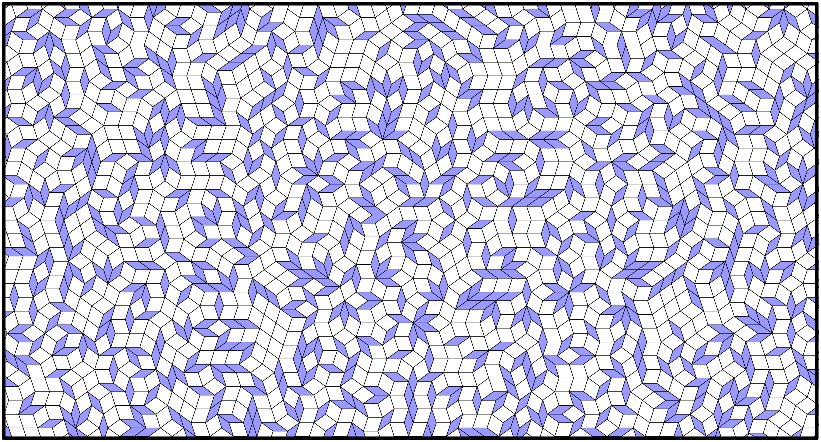
Shadow: projection of $\mathcal{S}$ on a space generated by three basis vectors of $\mathbb{R}^n$.

A *subperiod* is an integer vector $(p, q, r) \in \mathbb{Z}^3$ such that a shadow is periodic by translation by this vector.

Planar rhombus tilings



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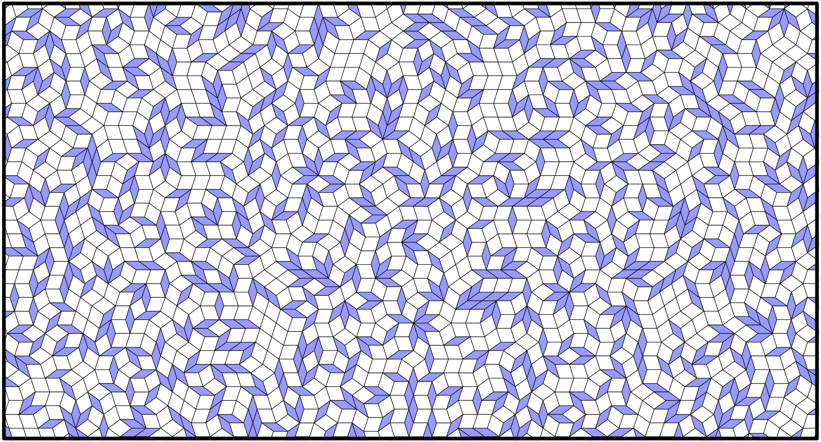

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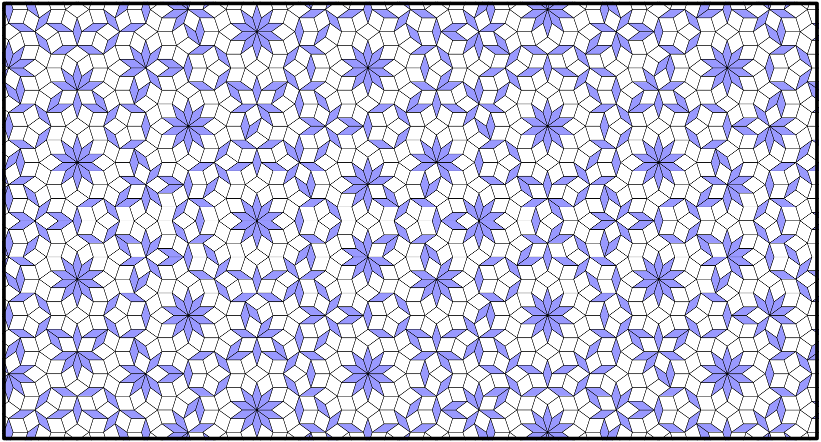
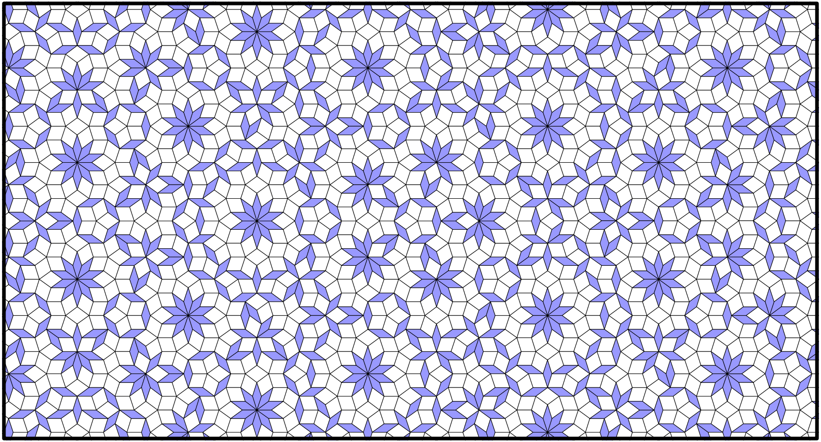


Shadows and subperiods



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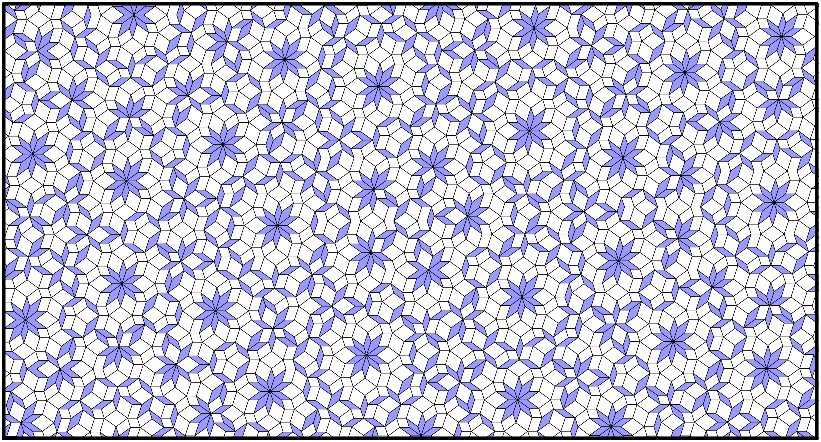

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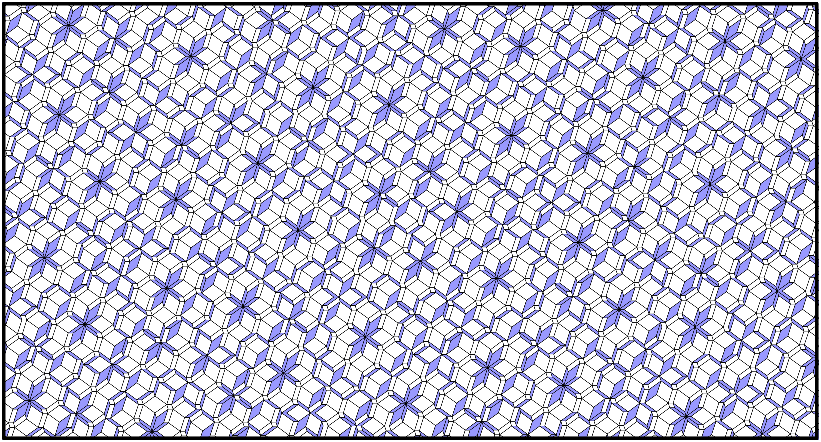

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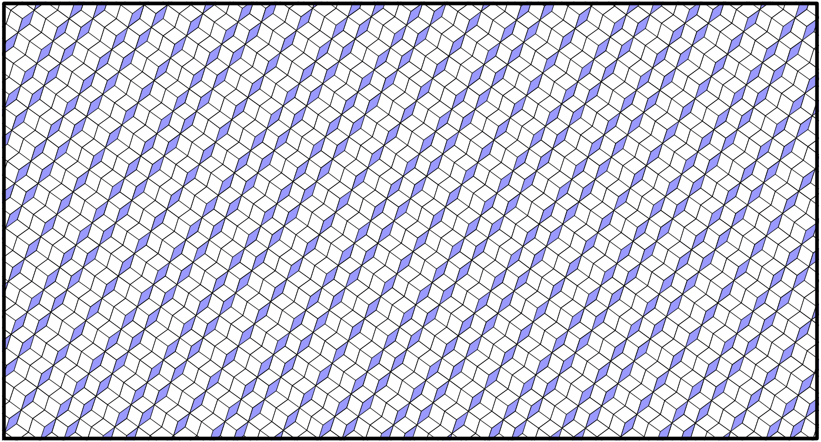

Lemma

From a 2-plane E in $\mathbb{R}^n$ we can construct a tiling by rhombi: Orthogonal projection of the lattice inside $E + [0,1]^n$ onto E.

Definitions

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An irrational plane is a 2-plane in $\mathbb{R}^n$ which does not contain a vector of $\mathbb{Z}^n$.

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An irrational plane is a 2-plane in $\mathbb{R}^n$ which does not contain a vector of $\mathbb{Z}^n$.

Definition
A slope satisfies the *P-condition* if it contains three non-collinear vectors which project onto subperiods in three irrational shadows.

Local rules

Definition

A slope E has *local rules* (LR) if there is a finite set of *patches* s. t. any rhombus tiling without any such patch is planar with slope E.

LR are said to be

- *strong* if the tilings satisfying them have thickness 1;
- *natural* if the thickness 1 tilings satisfy them;
- *weak* otherwise (the thickness is just bounded).

We here focus on natural LR.

Subperiod and SFT

Lemma

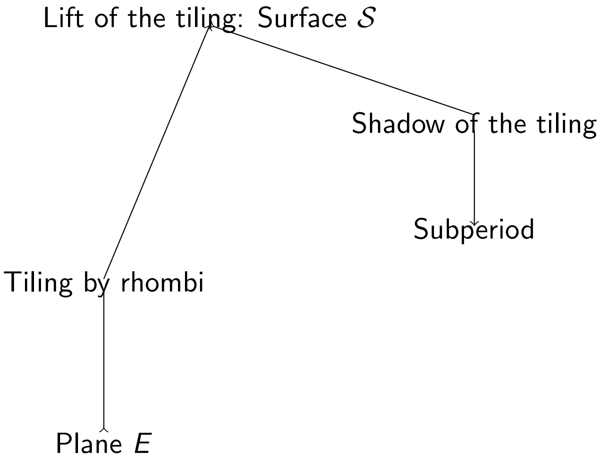
Subperiods can be enforced by forbidding finitely many patches.

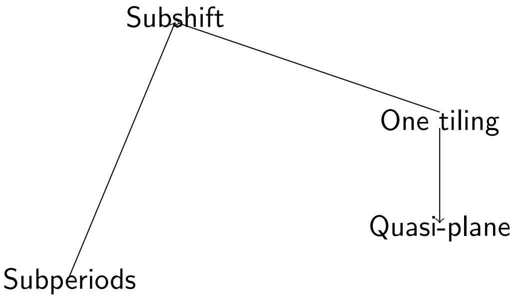
Corollary

There exists a subshift of finite type in which tilings all have the same subperiods.

Definition

Subperiods force planarity if every tiling in the subshift is a quasi-plane.


Lift of the tiling: Surface $\mathcal{S}$
Shadow of the tiling
Subperiod
Tiling by rhombi
Plane E



The thickness of the quasi-planes are not uniformly bounded !

A Characterization ?

Conjecture

A slope has natural LR iff finitely many slopes have its subperiods.

A result

Theorem

Consider a slope in $\mathbb{R}^4$. Then we have:
P-condition $\Leftrightarrow$ planarity of the tilings with the same subperiods.

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Subperiods characterize finitely many slopes $\Rightarrow$ P-condition holds.

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Theorem

Consider a slope in $\mathbb{R}^4$. Then we have:
P-condition $\Leftrightarrow$ planarity of the tilings with the same subperiods.

Lemma

Subperiods characterize finitely many slopes $\Rightarrow$ P-condition holds.

Corollary

If subperiods characterize finitely many slopes in $\mathbb{R}^4$, then there are natural local rules.

Our method allows us to obtain one proof for (almost) all known examples. In particular:

Theorem

The slope of a n-fold tiling has natural LR if n is an odd multiple of $5, 7$.

n-fold plane

Consider the following 2-planes:

- The 2-plane in $\mathbb{R}^{2p+1}$ generated by the vectors

$$(\cos \frac{2k\pi}{2p+1})_{0 \leq k \leq 2p}, (\sin \frac{2k\pi}{2p+1})_{0 \leq k \leq 2p}$$

- The 2-plane in $\mathbb{R}^{p}$ generated by the vectors

$$(\cos \frac{k\pi}{p})_{0 \leq k \leq p-1}, (\sin \frac{k\pi}{p})_{0 \leq k \leq p-1}$$

These planes are called n-fold planes where $n = 2p+1$ or $2p$.

Local rules 2

Definition

A slope E is *sofic* if there is a finitely many *colored* patterns s. t.

- slope E and thickness 1 colored tilings can avoid such patterns;
- colored tilings with no such patterns are planar with slope E.

Theorem (Levitov 89)

If a slope has strong rules, then its shadows are periodic.

Theorem (Fernique-Sablik 12)

A slope is sofic iff it is computable.

History

Tiling	undecorated rules	
5,10-fold	strong	
8-fold	none	Burkov 88
(4k+i)-fold, $i \neq 0$	weak	Socolar 90
quadratic slope in $\mathbb{R}^4$	a.e weak	Levitov 88
non algebraic slope	none	Le

Grassmann-Plücker coordinates

Definition

The plane $\mathbb{R}\vec{u}+\mathbb{R}\vec{v}$ has GP-coordinates $(G_{ij})_{i<j}=(u_iv_j-u_jv_i)_{i<j}$.

Proposition (Grassmann-Plücker)

GP-coordinates satisfy all the relations $G_{ij}G_{kl}=G_{ik}G_{jl}-G_{il}G_{jk}$.

Proposition

Whenever a planar tiling admits $p\vec{e}_i+q\vec{e}_j+r\vec{e}_k$ *as a subperiod, the GP-coordinates of its slope satisfy* $pG_{jk}-qG_{ik}+rG_{ij}=0$.

Lemma

The P-condition can be checked by a polynomial system on GP-coordinates.

Corollary

We can find an algorithm to find the solutions by use of Grobner Basis.

Assume we have P- condition.

Part 1

Lemma

We can reduce the problem to the case where:

- *E has GP-coordinates in $\mathbb{Q}[\sqrt{D}]$.*
- *There exists another plane E' with the same subperiods and $E \cap E' = \emptyset$.*

Let p_i be one subperiod and q_i, r_i its lifts in E, E'.

Lemma

The lifted surface is equal to

$$\mathcal{S} = \{x + z_1(x)r_1 + z_2(x)r_2, x \in E\}$$

It remains to prove that z_1, z_2 are closed to linear maps.

Consider $\pi_i(x+E') \cap \pi_i(\mathcal{S})$, this curve is parametrized by

$$\{\pi_i(x) + z_1(x+\lambda q_i)\pi_i(r_1) + z_2(x+\lambda q_i)\pi_i(r_2), \lambda \in \mathbb{R}\}$$

Lemma

We deduce of periods:

- $\lambda \mapsto z_1(x+\lambda q_1)$ *is uniformely bounded.*
- $\lambda \mapsto z_2(x+\lambda q_2)$ *is uniformely bounded.*
- *There exists* $\alpha \in \mathbb{R}^*$ *such that* $\lambda \mapsto (z_2 - \alpha z_1)(x+\lambda q_3)$ *is uniformely bounded. Remark that* $q_3 = q_1 + \alpha q_2$*.*

Lemma

The functions z_1, z_2 *are closed to linear maps.*

Corollary

Thus we have:
P-condition enforces the planarity.

Part 2.

Lemma

The equations on GP-coordinates associated with the subperiods of a n-fold tiling form a system of dimension either one if 4 divides n or zero otherwise.

Lemma

If $5, 7, 8$ or 12 divides n then the subperiods of the n-fold tilings enforce planarity of the four-dimensional shadows.

Lemma

Planarity of three four-dimensional shadows implies planarity of the tiling.

Corollary

Natural rules for odd multiples of $5, 7$.

n-fold tiling

Lemma

For such a tiling we have $G_{i,j} = \sin 2(j-i)\pi/n$.
We deduce

$$G_{i,j} = G_{j,k}, \quad k = 2j - i \mod n.$$

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Three examples in the following:

- $n = 8$
- $n = 7$
- $n = 5$

Example $n = 8$. Amman-Beenker tilings

$$\begin{cases} G_{12} = G_{23} = G_{34} = G_{14} \\ 2G_{12}^2 = G_{13}G_{24} \end{cases}$$

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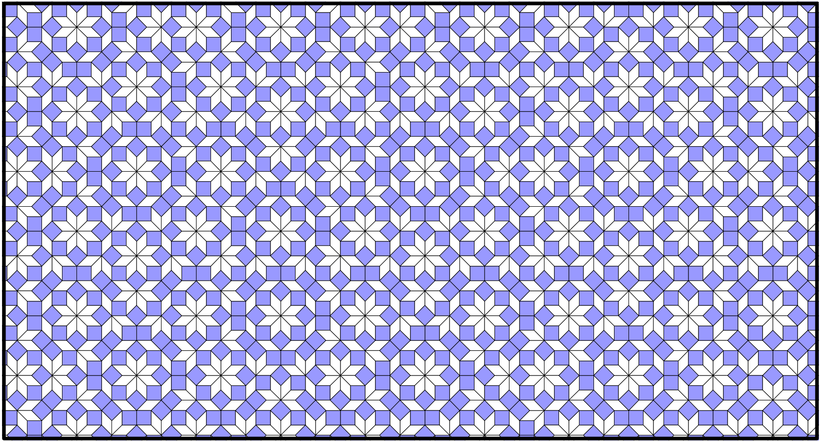
$$\begin{cases} G_{12} = G_{23} = G_{34} = G_{14} = 1 \\ 2 = G_{13}G_{24} \end{cases}$$

Infinity of planes solutions.
Subperiods thus characterize all the slopes $(1, t, 1, 1, 2/t, 1)$, $t \in \mathbb{R}$.
First theorem implies the planarity of these planes.
The AB tilings are those maximizing the rhombus frequencies.

Ammann-Beenker tilings



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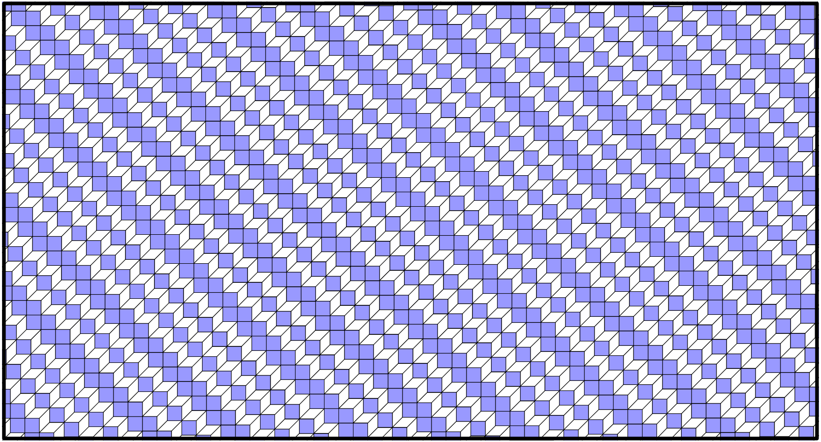

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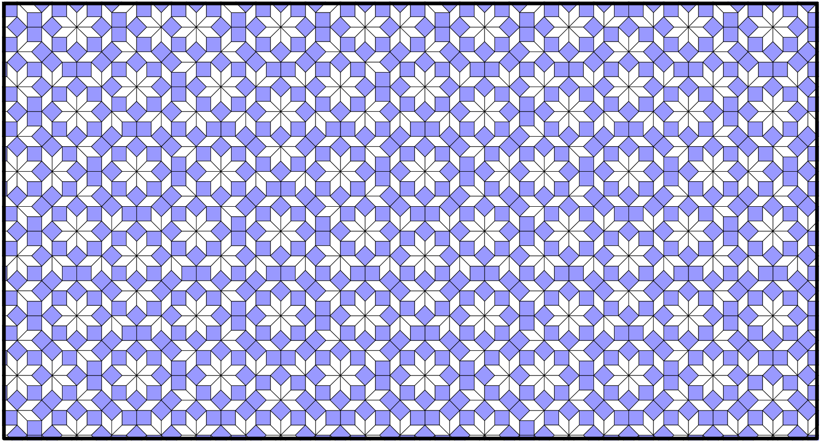

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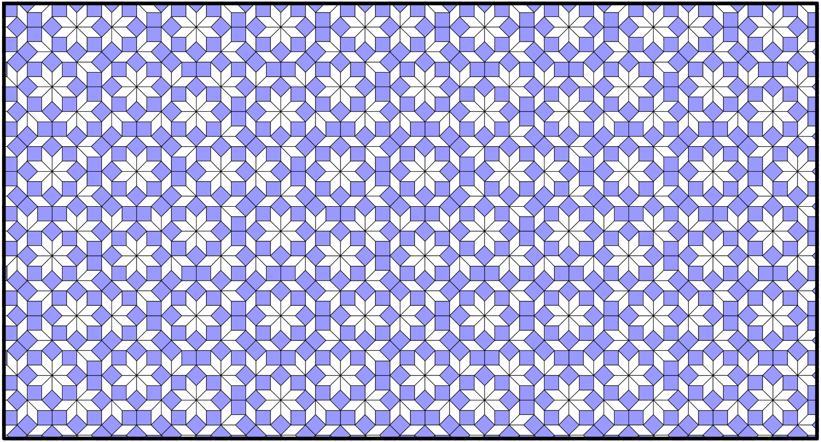

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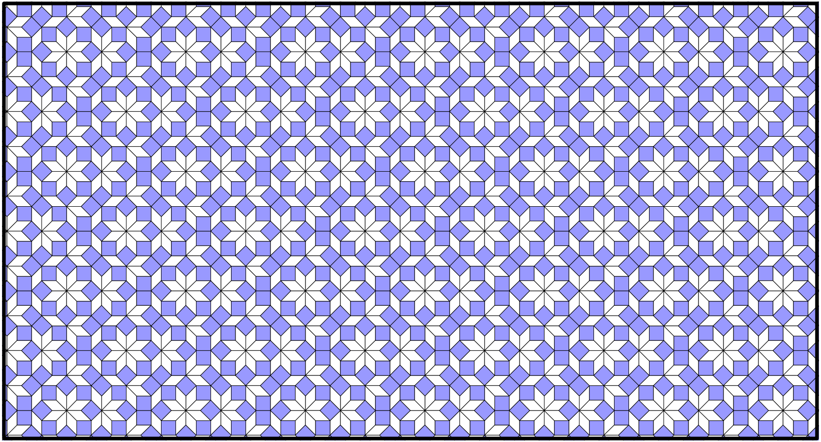

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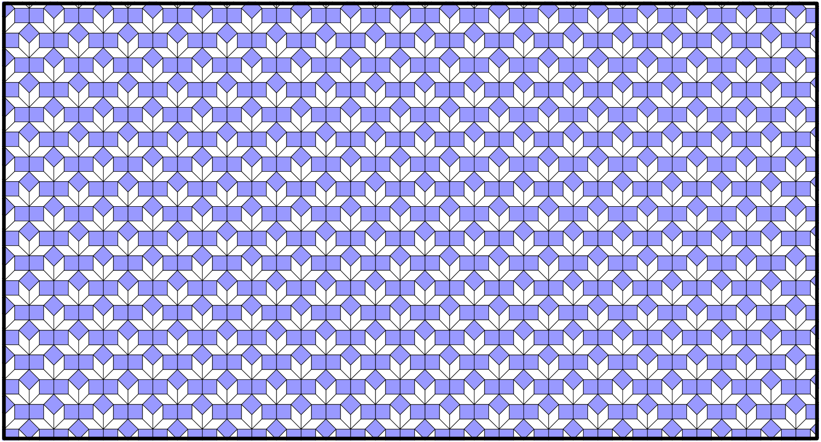

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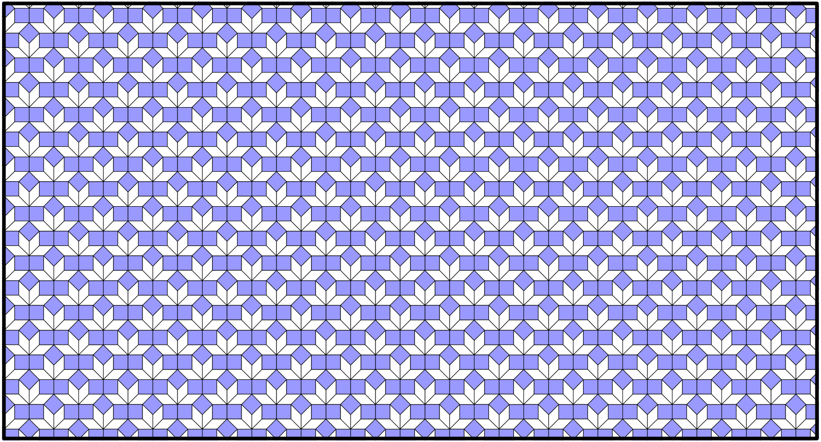

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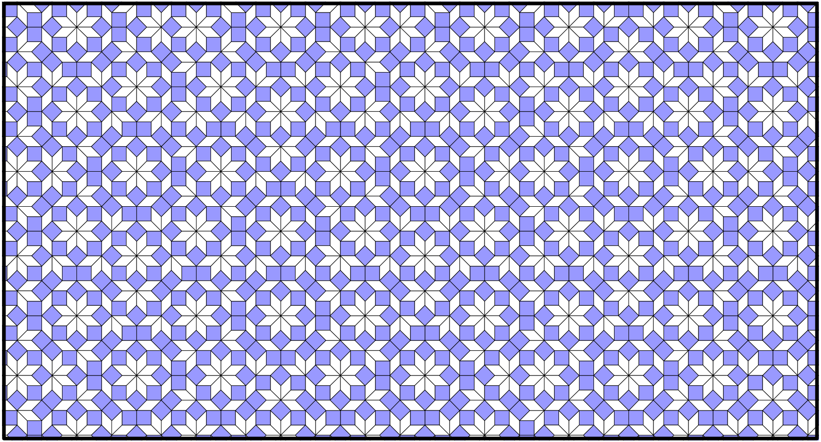


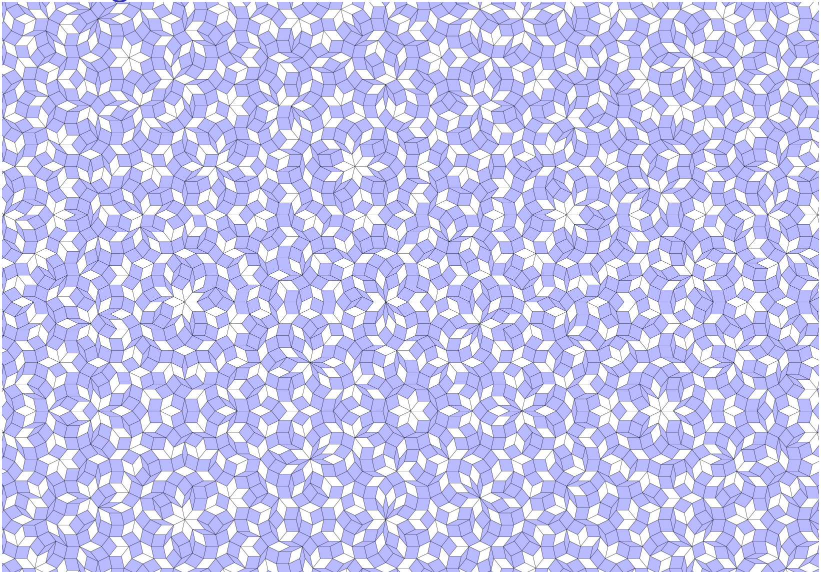
Example $n = 7$.

$$\begin{cases} G_{12} = G_{23} = G_{34} = G_{45} = G_{56} = G_{67} = G_{71} \\ G_{13} = G_{35} = G_{57} = G_{72} = G_{24} = G_{46} = G_{61} \\ G_{14} = G_{47} = G_{73} = G_{36} = G_{62} = G_{25} = G_{51} \end{cases}$$

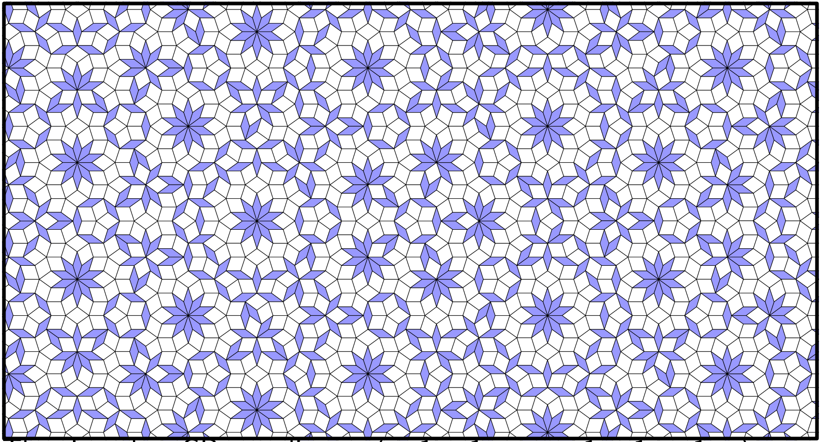
$$\begin{cases} G_{12}^2 = G_{13}^2 - G_{14}G_{12} \\ G_{12}G_{13} = G_{12}G_{14} + G_{13}G_{14} \\ G_{12}^2 = G_{14}^2 - G_{13}G_{14} \\ G_{14}^2 = G_{13}^2 - G_{13}G_{12} \end{cases}$$

Solution of $X^3 - X^2 - 2X + 1 = 0$.

7-fold tiling



Generalized Penrose tilings



The slope has GP-coordinates $(\varphi, 1, -1, -\varphi, \varphi, 1, -1, \varphi, 1, \varphi)$.

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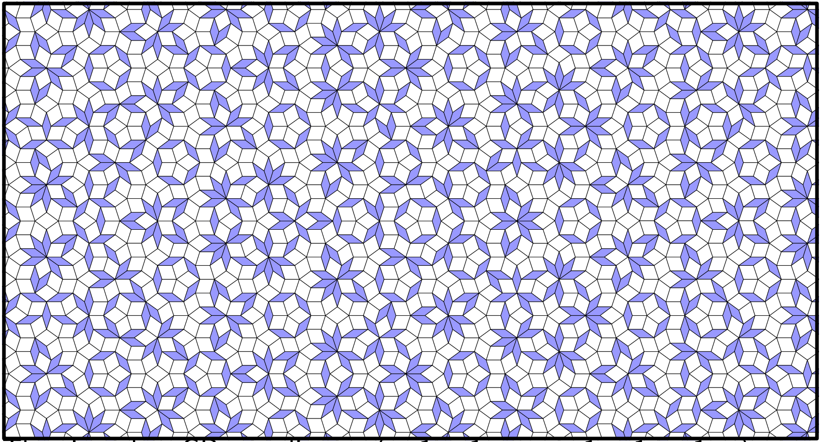

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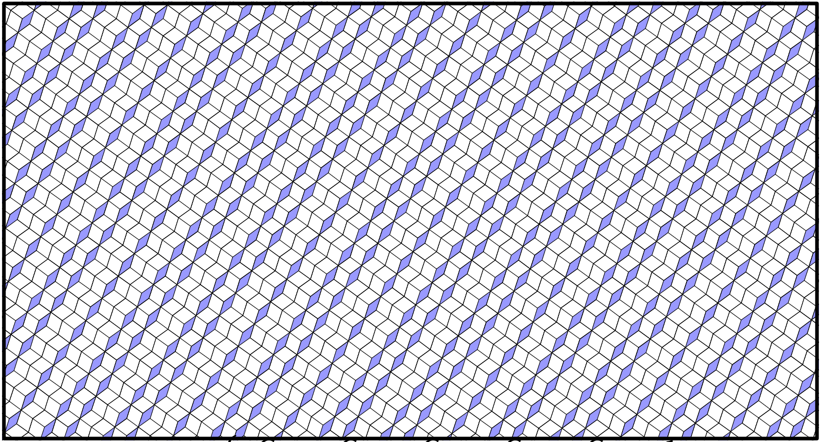
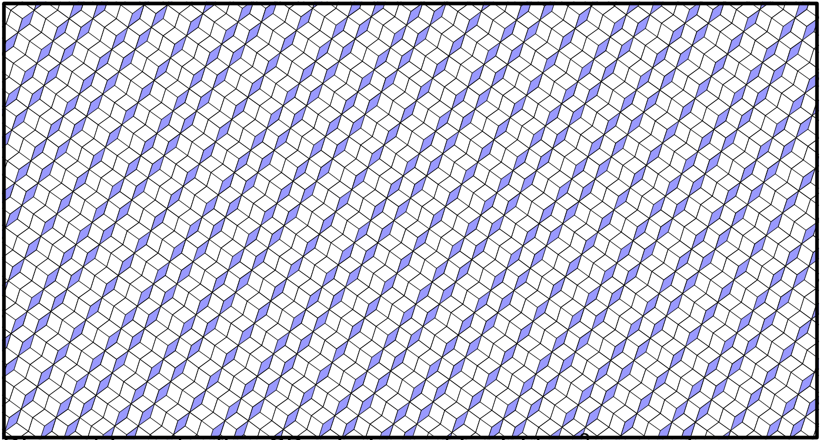


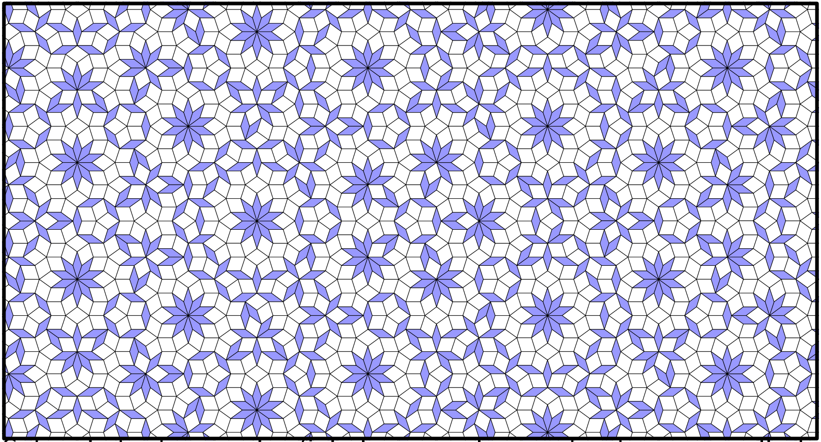
Subperiods yield $\begin{cases} G_{13} = G_{41} = G_{24} = G_{52} = G_{35} = 1 \\ G_{12} = G_{51} = G_{45} = G_{34} = G_{23} =: x \end{cases}$

Generalized Penrose tilings



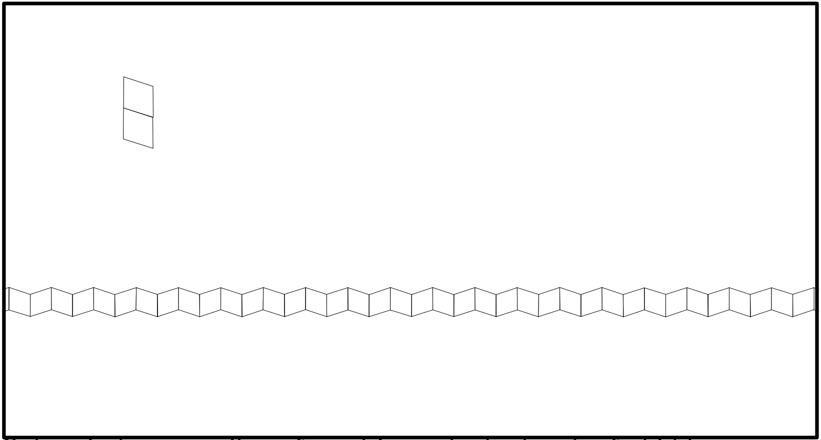
Plugged into the five GP-relations, this yields $x^2 = x + 1$.

Generalized Penrose tilings



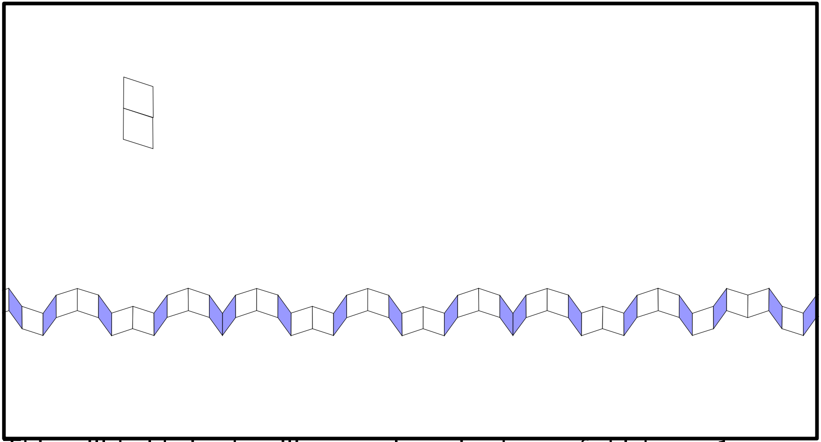
Subperiods characterize finitely many slopes: the theorem applies!

Generalized Penrose tilings



Subperiods are easily enforced in each shadow by forbidden patches.

Generalized Penrose tilings



This still holds in the tilings, at least in those of thickness 1.

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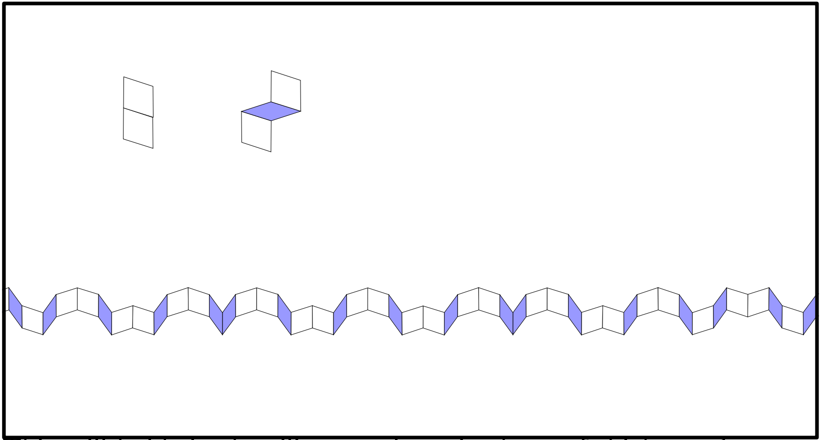

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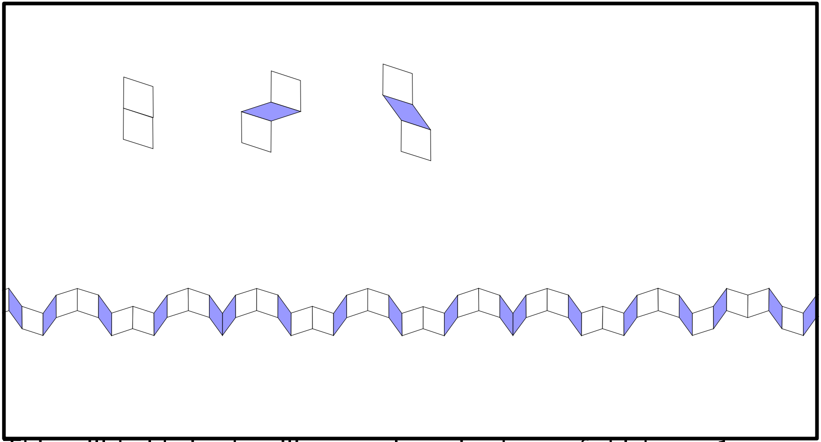

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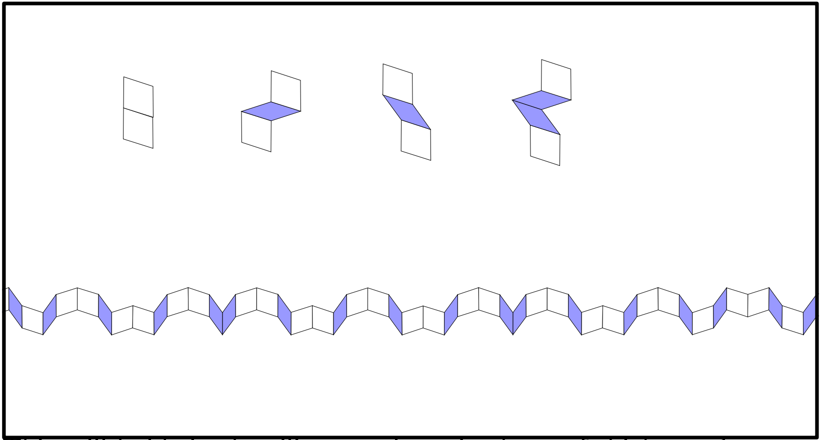

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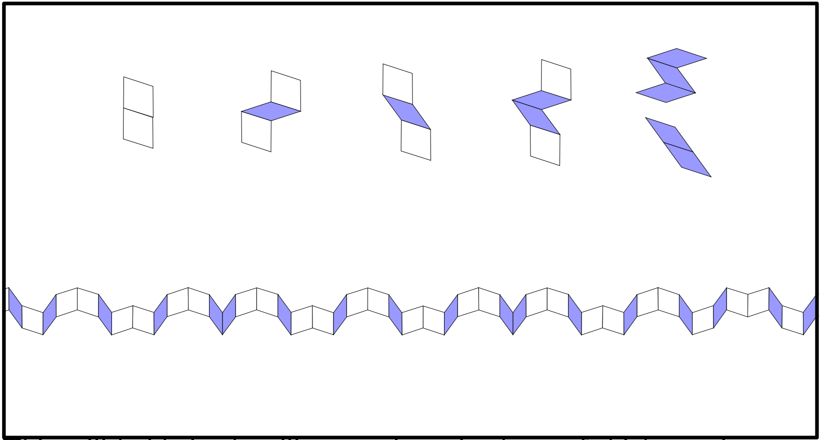

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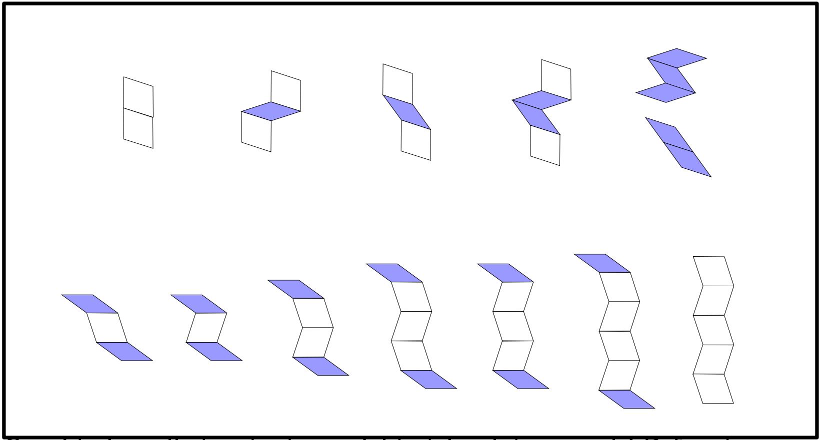

Considering all the shadows yields (simple) natural LR for the tilings.

To be followed

- A general 2-plane in $\mathbb{R}^n$.
- Dynamical properties of these tilings.
- Tilings $n - d$.