Combinatorics of billiards

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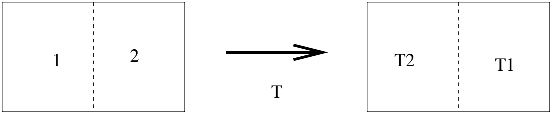
Plan

Symbolic dynamics

Consider a dynamical system (X, T).
Assume there exists a partition $(\mathcal{P}_i)_i$ of X in a finite number of cells.
The **coding** of an orbit $(T^n m)_n$ is a sequence (v_n) defined as

$$v_n = i \iff T^n m \in \mathcal{P}_i.$$

$$\phi : m \mapsto (v_n)_{n \in \mathbb{N}}$$



Example of dynamical system with a partition

Example

Two sequences:

$$\phi(m) = 12121212\ldots$$

$$\phi(p) = 21212121\ldots$$

$1, 2$ are called **letters**. The block $v_i \ldots v_{n-1+i}$ is called a finite word of length n.
For example 212121 is a **finite word** of length 6.

If the partition has some properties then there is a semi-conjugacy:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \phi \downarrow & & \downarrow \phi \\ \phi(X) & \xrightarrow{S} & \phi(X) \end{array}$$

where $\phi(m) = (v_n)_n$ and S is the shift map.

We can study the system (Σ, S), where $\Sigma = \overline{\phi(X)}$.

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- Piecewise isometries.

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- Polyhedral billiard.
- Piecewise isometries.
- Dual billiard.

Complexity

Definition

If v is an infinite word, we define the COMPLEXITY function $p(n, v)$ as the number of different words of length n inside v.

Example

$v = abaaabbabbaaa\dots \quad p(n, v) = 2^n$

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$v = abaaabbbbb\ldots \quad p(n, v) \leq 6$

In fact we can compute two different complexities:

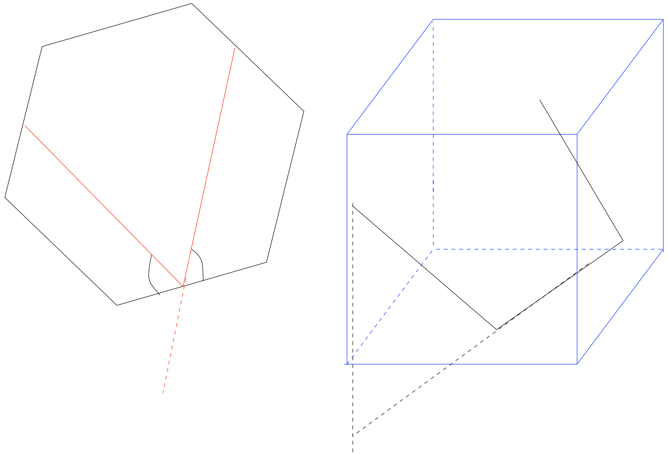
- The complexity of one word: $p(n, v)$.
- The complexity of the union of all the words: $p(n)$.

Let v be a word corresponding to the orbit of m, and $\mathcal{L}_v$ its language. Consider the following language

$$\mathcal{L} = \bigcup_m \mathcal{L}_v, \quad p(n) = card \mathcal{L}(n).$$

Plan

Trajectories



Reflections and billiard.

Definition

Let P be a polyhedron of $\mathbb{R}^d$, $m \in \partial P$ and $\omega \in \mathbb{RP}^{d-1}$.
The point moves along a straight line until it reaches the boundary of P.
On the face: orthogonal reflection of the line over the plane of the face.

$$T: \quad X \longrightarrow \partial P \times \mathbb{RP}^{d-1}.$$

$$T: \quad (m, \omega) \mapsto (m', \omega').$$

If a trajectory hits an edge, it stops.

Coding

Definition

We associate one letter to each face of the polyhedron. In the case of the cube we give the same letters to the parallel faces.

Definition

The complexity of the word v generated by the orbit of (m, ω) is denoted by $p(n, \omega)$.

Definition

Let P be a polyhedron in $\mathbb{R}^d$, and s_i the linear reflections over the faces of P. We denote by $G(P)$ the group generated by all the reflections s_i: $G(P) \subset O(d)$.
The polyhedron is called rational if $G(P)$ is finite.

Consider the orbit of one point (m, ω). Then the differents directions are included inside $G\omega$, where $\omega \in \mathbb{R}^3$.

$$(T^n(m, \omega))_n \subset \partial P * G\omega.$$

Polygons	$p(n,\omega)$	$p(n)$
rational	$an+b$	n^3
Irrational	$n^{1+\varepsilon}$	??

$$\lim \frac{\log p(n)}{n} = 0.$$

Theorem (Boshernitzan-Masur 1986)

In any polygon we have

$$\int_{S^1} \overline{p(n,\omega)}d\omega = Cn.$$

Corollary

For any $\varepsilon > 0$ *and almost every* ω *we have*

$$\overline{p(n,\omega)} = O(n^{1+\varepsilon}).$$

Paper of Gutkin-Rams 2007

Theorem (Arnoux-Mauduit-Shiokawa-Tamura; Baryshnikov; B 94-95-03)

For the cubic billiard, under some hypothesis on ω:

- *In* $\mathbb{R}^3$, $p(n,\omega) = n^2 + n + 1$.
- *In* $\mathbb{R}^{d+1}$, $p(n,\omega) \sim n^d$.

Remark

For the square we obtain

$$p(n,\omega) = n + 1.$$

Theorem (B 07)

Consider a cube of $\mathbb{R}^{d+1}$, then we have:

- *Fix $n, d \in \mathbb{N}$, then the map $\omega \mapsto p(n, d, \omega)$ is constant on the set of B directions.*
- *Moreover if we denote it by $p(n, d)$ we have*

$$p(n+2, d) - 2p(n+1, d) + p(n, d) = d(d-1)p(n, d-2).$$

- $$p(n, d, \omega) = \sum_{i=0}^{min(n,d)} \frac{n!d!}{(n-i)!(d-i)!i!} \quad \forall\ n, d \in \mathbb{N}.$$

Theorem (B 07)

In any convex polyhedron the billiard map T fulfills:

$$h_{top}(T) = 0.$$

Theorem (B-Hubert 07)

Consider the cube of $\mathbb{R}^{d+1}$, then there exists $a, b > 0$ such that

$$a \leq \frac{p(n)}{n^{3d}} \leq b.$$

	Square	Cube
$p(n,\omega)$	$n+1$	$p(n,\omega) \sim n^d$
$p(n)$	$\sim n^3$	$p(n) \approx n^{3d}$

	Polygons	Polyhedrons
$p(n,\omega)$	$an+b$	n^2 ?
$p(n)$	$\approx n^3$	n^6 ?
Entropy	$h_{top}=0$	$h_{top}=0$

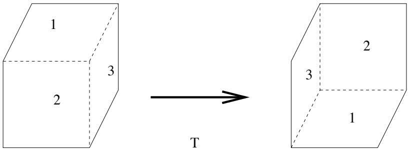
Let P be a rational polyhedron.
The billiard flow acts on $P * \mathbb{RP}^2$.
Consider the first return map of the billiard flow on a transverse set $I * \{\omega\}$ where I is a rectangle:

Lemma

It is a piecewise isometry defined on the compact set I.

Cube

For the cube and a direction ω, the return map T_I is a ROTATION on the torus $\mathbb{T}^2$.



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Isometries

Definition

Consider a finite number of hyperplanes H_i in $\mathbb{R}^d$.

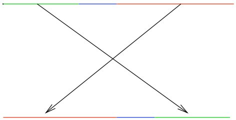
$$X = \mathbb{R}^d \setminus \bigcup H_i.$$

The map T is defined on the connected components of X as an isometry of $\mathbb{R}^d$.

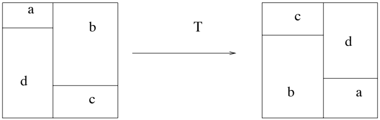
Example

Interval exchanges:

- Piecewise isometry in dimension one, with translations.
- defined on a compact set,
- bijective



Interval exchange



Polygon exchange

Theorem (Buzzi 2001)

Every piecewise isometry has zero topological entropy.

Theorem (Bressaud-Poggiaspalla 06)

Classification of piecewise isometries defined on a triangle.

Interval exchanges

For interval exchanges, there are a lot of results:

- Boshernitzan, Masur: ergodic properties of the subshift.
- Rauzy induction: first return map.
- Marmi-Moussa-Yoccoz: Cohomogical equation.
- Ferenczi-Zamboni: characterization of the language.

Remark

For an interval exchange we have $p(n, v) \leq an$.

- A two interval exchange is called a rotation.
- It corresponds to the map $x \mapsto x + \alpha \quad mod 1$.
- In the coding, the first interval has length $1 - \alpha$.
- $p(n, v) = n + 1$.

Rotations

Theorem (Morse-Hedlund 1940.)

Let v be an infinite word, assume there exists n such that $p(n, v) \leq n$. Then v is an ultimately periodic word.

A word v such that $p(n, v) = n + 1$ for all integer n, is called a Sturmian word.

Theorem (Coven-Hedlund 1973)

Let v be a sturmian word, then there exists m, α in $\mathbb{R}$ such that for the rotation of angle α the orbit of m fulfills:

$$\phi(m) = v.$$

Global complexity

Consider the set of all rotations. **Computation** of $p(n)$.

Proof of:

- Tarannikov, Lipatov: (1982).
- Mignosi: (1991).
- Berstel-Pocchiola.
- Cassaigne-Hubert-Troubetzkoy.

Theorem

For all integer n:

$$p(n)=\sum_{i=0}^{n}\phi(i)(n+1-i)\sim Cn^3,$$

where ϕ is the Euler function.

Dictionnary

Sturmian word.

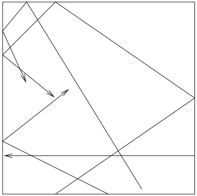
$$\Updownarrow$$

Rotation on the torus $\mathbb{T}^1$.

$$\Updownarrow$$

Coding of a billiard trajectory inside the square.

The complexity $p(n)$ represents the number of different billiard words of length n.

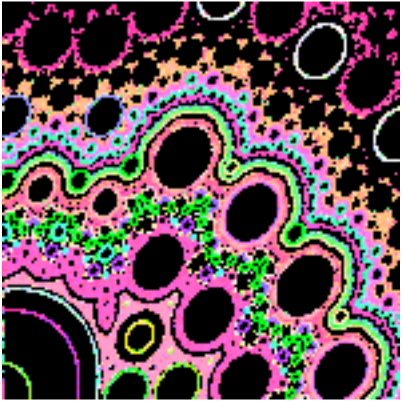


For another polygon there is no equivalence with the interval exchange words.

No general result on piecewise isometry in dimension two:

Problems:

- Classification of polygons exchanges.
- Minimality.
- Size of rational polyhedrons.
- Periodic islands.



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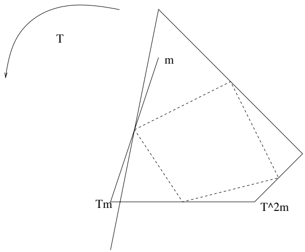
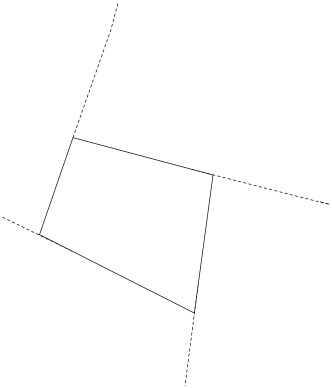
Dual billiard

Consider a convex polygon in $\mathbb{R}^2$. Fix one orientation.



The billiard map is defined in $\mathbb{R}^2 \setminus P$ by reflection through the verteces of P.

There is a natural coding with the following regions:



Question Compute $p(n)$, and describe the symbolic dynamics.

A polygon is rational if there exists a lattice which contains P.

Theorem (Gutkin-Simanyi 92)

For a rational polygon every orbit is periodic.
For a quasi-rational polygon every orbit is bounded.

Every regular polygon is a quasi-rational polygon.

Theorem (Tabachnikov 95)

For the regular pentagon we have:

- *almost all point has a periodic orbit.*
- *There exists some points with non periodic orbit.*
- *The set of non periodic point is a fractal set.*
- *Computation of Hausdorff dimension.*

Theorem (Gutkin-Tabachnikov 06)

If P is rational then $an^2 \leq p(n) \leq bn^2$.
If P is a quasi-rational polygon with k verteces, then

$$an \leq p(n) \leq bn^{k+1}.$$

Theorem (B-Cassaigne 08)

Computation of $p(n)$ for

- *Triangle, square, regular hexagon, regular octogon.*
- *Regular pentagon.*

$$p(n) \sim Cn^2.$$

Remark

If h is an affine map, then $h(P)$ has the same properties than P.

Plan

Example

$$\sigma \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}$$

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$$\sigma^2(a) = aba, \sigma^3(a) = abaab.$$

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$$v = abaababaabaababa \ldots$$

Bispecial words

Let $\mathcal{L}(n)$ the set of words of length n in a language. For $v \in \mathcal{L}(n)$ let

$$
\begin{aligned}
s(n) &= p(n+1) - p(n). \\
m_l(v) &= card\{a \in \Sigma, \quad av \in \mathcal{L}(n+1)\}. \\
m_r(v) &= card\{b \in \Sigma, \quad vb \in \mathcal{L}(n+1)\}. \\
m_b(v) &= card\{(a,b) \in \Sigma^2, \quad avb \in \mathcal{L}(n+2)\}. \\
b(n) &= \sum_{v \in \mathcal{L}(n)} (m_b(v) - m_r(v) - m_l(v) + 1).
\end{aligned}
$$

Definition

A word v is:

- right special if $m_r(v) \geq 2$,
- left special if $m_l(v) \geq 2$,
- bispecial if it is right and left special.

We have

Lemma (Cassaigne 97)

For all integer n we have

$$s(n+1) - s(n) = b(n).$$

Example

The leftt special words are

- a
- ab
- aba
- $abaa$
- Prefix of v.

One left special word for every length:

$$s(n) = 1.$$

$$p(n, v) = n + 1.$$

Example

The right special words are

- a
- ba
- aba
- $aaba$
- Mirror image of prefix of v.

The bispecial words are:

- a
- aba
- $abaaba$
- Palindromic prefixs.

For all bispecial word $i(v) = 3 - 2 - 2 + 1 = 0$.

The fibonacci word corresponds to a billiard trajectory inside the square starting from 0 and with slope $\phi - 1$.