Piecewise Rotations

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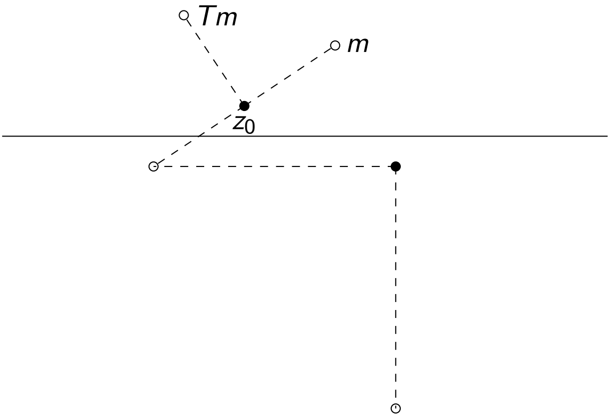
Definition

In $\mathbb{R}^2$ consider two points outside the real line and an angle θ.
Define a map $T : \mathbb{C} \to \mathbb{C}$ by

$$T(z) = \begin{cases} e^{2i\pi\theta}(z - z_0) + z_0 & \Im m z \geq 0 \\ e^{2i\pi\theta}(z - z_1) + z_1 & \Im m z < 0 \end{cases}$$

Picture



Piecewise rotation of angle $\pi/2$.

Among the positions of the centers of rotations the map can be

- Bijective.
- Non injective.
- Non surjective.

The dynamics is different in each case:

Theorem (Boshernitzan-Goetz)

If T is non injective then there exists $M > 0$ such that for any integer n, $|T^n z| \leq M$ for all sufficiently large n. If T is non surjective, then there exists M such that $\lim |T^n(z)| = \infty$ for all z satisfying $|z| \geq M$.

For a bijective map the maps can be reduced to the form, where $\sigma \geq 0$:

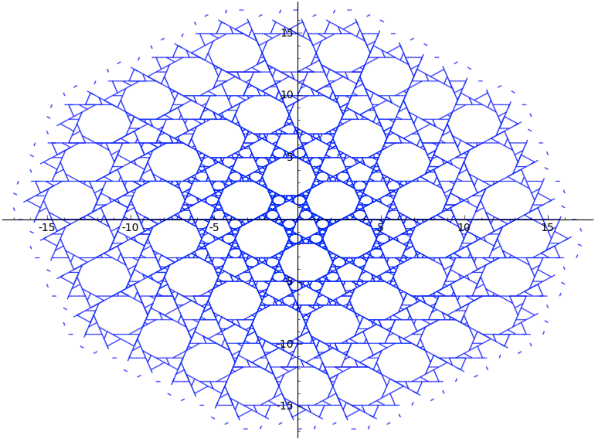
$$T(z) = \begin{cases} e^{2i\pi\theta}(z+\sigma+1) & \Im m z \geq 0 \\ e^{2i\pi\theta}(z+\sigma-1) & \Im m z < 0 \end{cases}$$

If $\sigma = 0$, then the map is called symmetric.

Theorem (Goetz-Quas)

If θ is a rational number, and T is bijective, closed to the symmetric map, then the map has bounded orbits which are included inside invariant rings. Description of the coding of these invariant rings.

Angle $2\pi/5$



The invariant rings made of decagons.

Classification

Bijective case		Non bijective case	
Symmetric	Non symmetric	Non injective	Non surjective

Goal

Description of the symbolic dynamics.

Theorem

For a piecewise rotation: Complete description of the map for the angles $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{4}, \frac{1}{8}, \frac{1}{5}\}$.

The language is given by a substitutive system.

Theorem

For a bijective map

- *If* $\theta \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{6}\}$, *every orbit is periodic.*
- *If* $\theta \in \{\frac{1}{5}, \frac{1}{8}\}$, *every orbit is bounded but some are non periodic.*
- *For non symmetric cases, the dynamics is the same for all value of* $\sigma \in (0, 1)$.

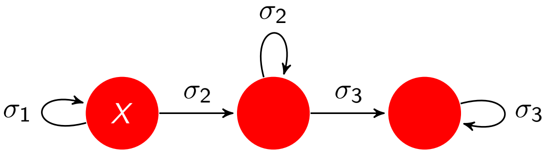
Theorem

For a non bijective map, we have

- *For non surjective map, the attractive set can be disconnected.*
- *For non injective map, in the attractive set there exists some points with non periodic orbit if* $\theta \in \{1/5, 1/8\}$.

Substitutive system

Finite number of substitutions. X is a finite set of words. A word of the language is obtained by a finite path on the following oriented graph.



Example of words:

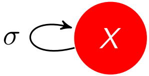
$$\sigma_1^n(x), \sigma_2^m \circ \sigma_1^n(x), \sigma_3^p \circ \sigma_2(x), n, m, p \in \mathbb{N}$$

Example

One substitution:

$$\sigma : \begin{cases} A \mapsto DBC \\ B \mapsto DC \\ C \mapsto DB \\ D \mapsto D \end{cases}$$

$$X = \{A, B, C\}$$



Example

The language is the set of factors of the periodic words of the form z^ω for $z \in Z$, where

$$Z = \bigcup_{n \in \mathbb{N}} \{\sigma^n(A), \sigma^n(B), \sigma^n(C)\}.$$

Bijective piecewise rotation of angle $\pi/2$.

Induction

We find a subset and study the first return map $\hat{T}$ to this set.

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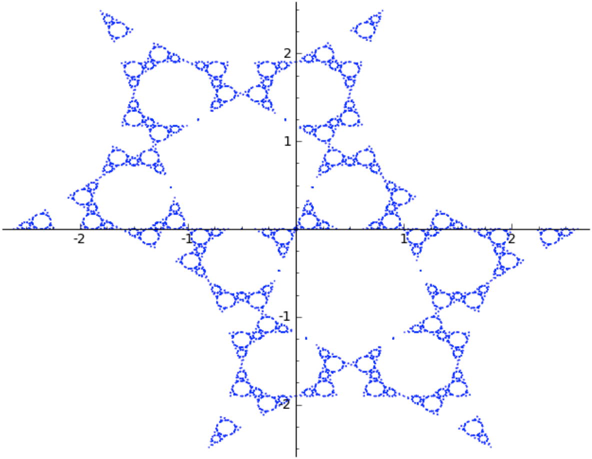
For a good subset, the first return map is conjugated to the initial one.

Compact set

Inside an invariant compact set, the dynamics is closed to some classical piecewise isometries.


2
1
-1
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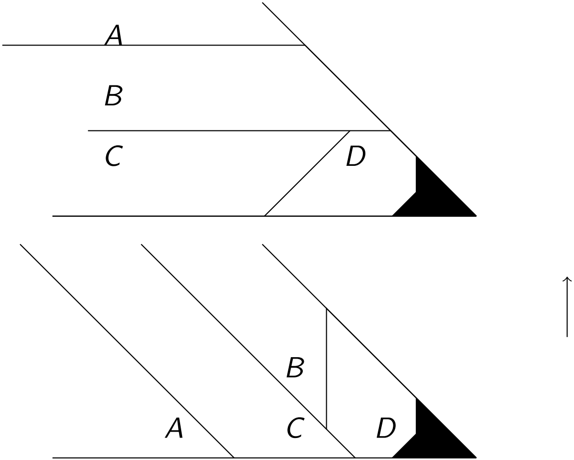
2

Map $\hat{T}$



Three invariant regions:

- Black set
- Other compact set.
- The other part.

Compact sets

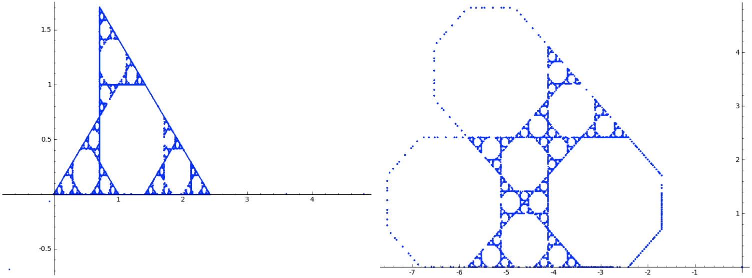


Figure: Decomposition of the dynamics.

- On the compact sets: Similar study to outer billiard outside a regular octagon.
  Work of Schwartz.
  Three substitutions.

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  Renormalisation to the first return map to A.
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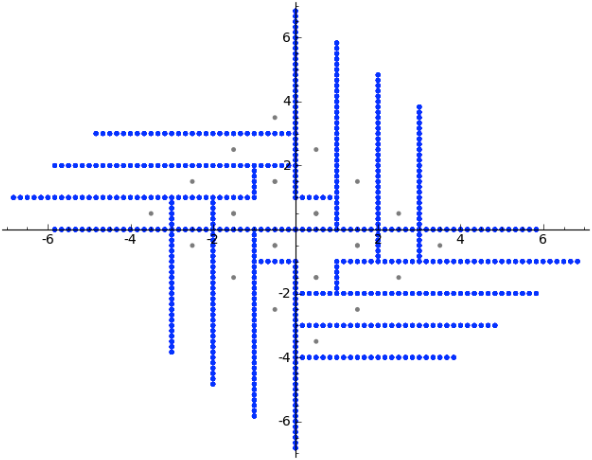
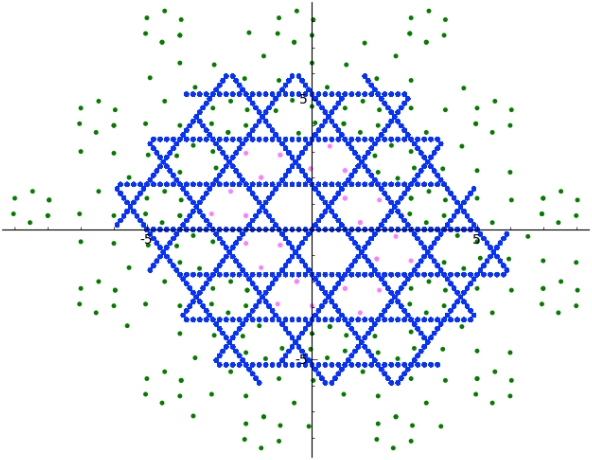
- Non compact set:
  Renormalisation to the first return map to A.
  Other substitution.
- Global description.

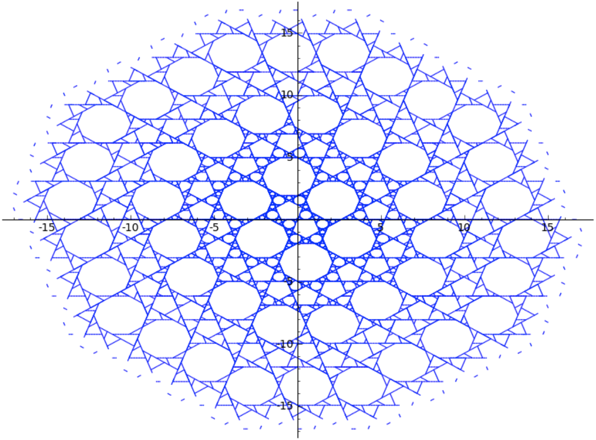
Angle $\pi/2$



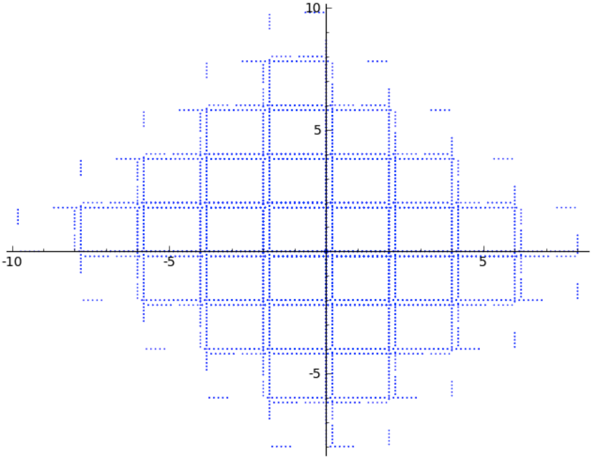
Angle $\pi/3$



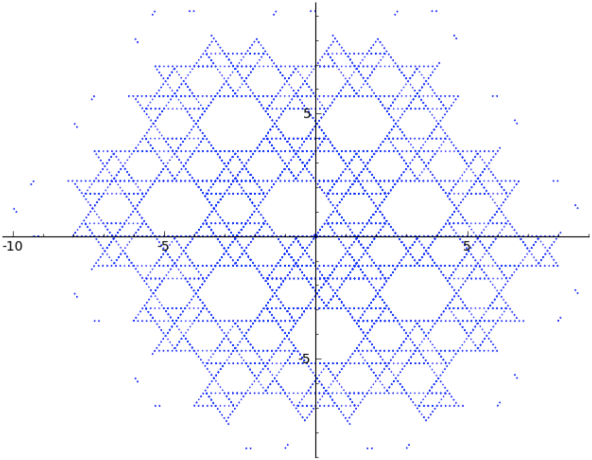
Angle $2\pi/5$



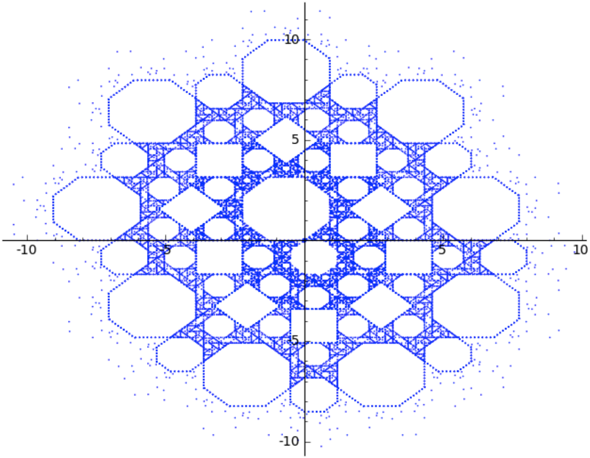
Angle $\pi/2$



Angle $\pi/3$



Angle $\pi/4$



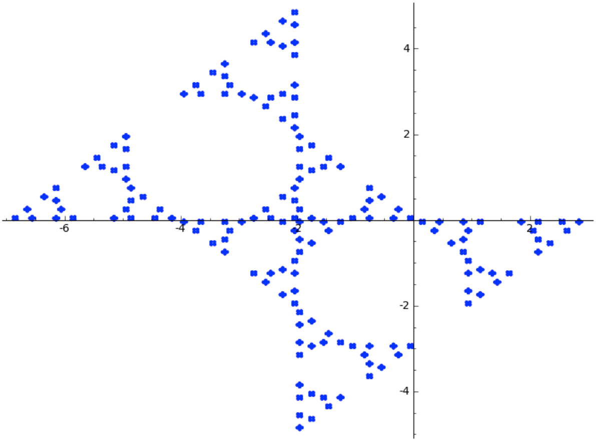
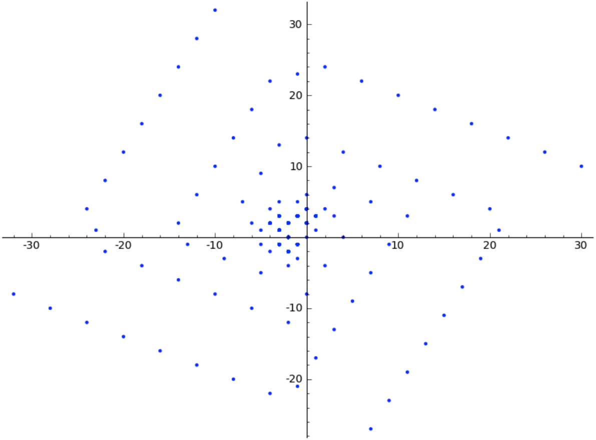
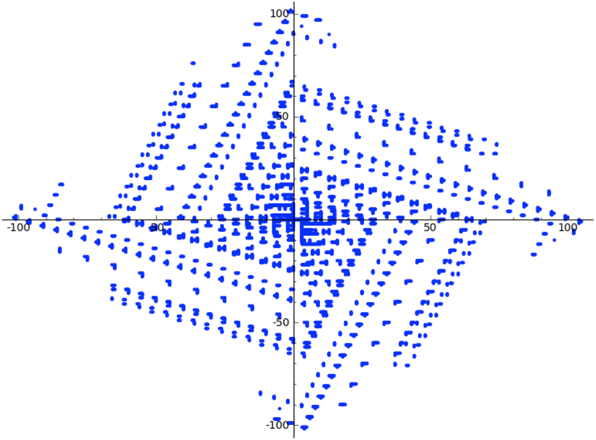


Figure: Angle $\pi/4$ and non injective case

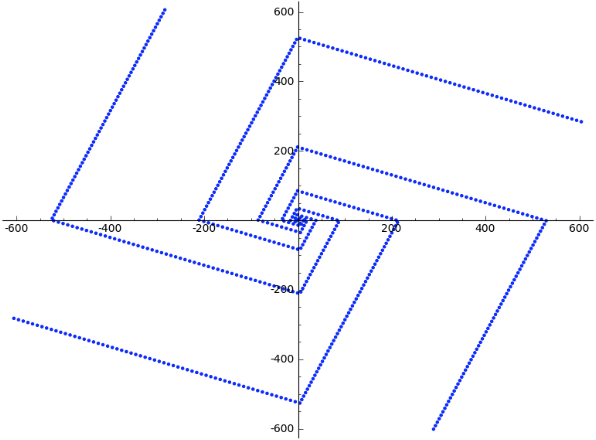
Orbit of a point



Angle $\pi/2$



One orbit with angle $\pi/2$

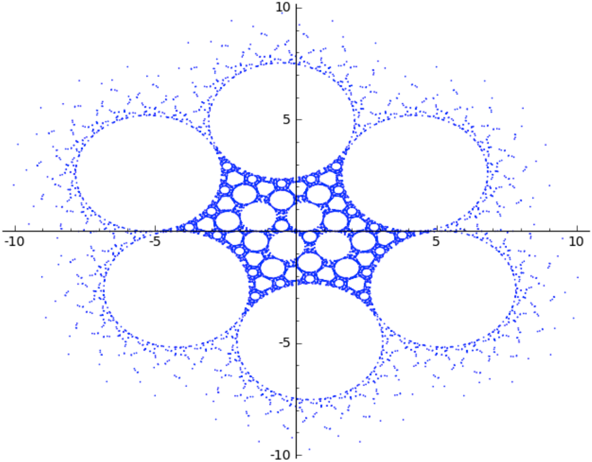


Divergence of an orbit.

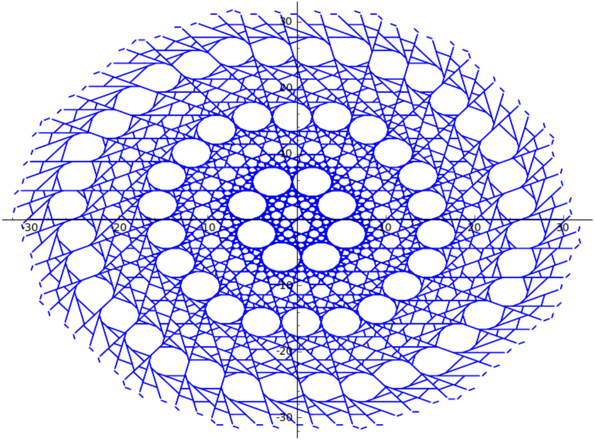
Questions

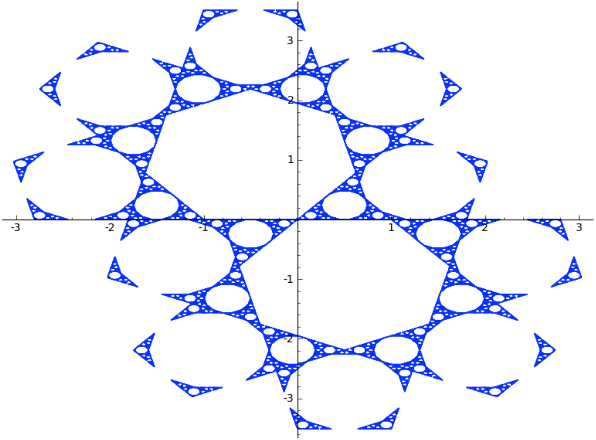
- Other rational angle
- Irrational angle
- Compactification
- Two angles

Irrational angle



Angle $\pi/7$

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Two angles