# Dual billiards

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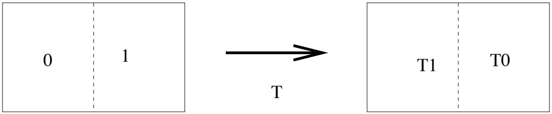
# Coding

Consider a dynamical system $(X, T)$.
Assume there exists a partition $(\mathcal{P}_i)_i$ of $X$ in a finite number of cells.
The **coding** of an orbit $(T^n m)_n$ is a sequence $(v_n)$ defined as

$$v_n = i \iff T^n m \in \mathcal{P}_i.$$

$$\phi : m \mapsto (v_n)_{n\in\mathbb{N}}$$




Example of dynamical system with a partition

# Example

Two sequences:

$$\phi(m) = 0101010101\ldots$$

$$\phi(p) = 10101010\ldots$$

The $0, 1$ are called **letters**. The block $v_i \ldots v_{n-1+i}$ is called a finite word of length $n$.
For example 101010 is a **finite word** of length 6.
The sequence $\phi(m)$ is an **infinite word**.

If the partition has some properties then there is a semi-conjugacy:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \phi\downarrow & & \downarrow\phi \\ \phi(X) & \xrightarrow{S} & \phi(X) \end{array}$$

where $\phi(m) = (v_n)_n$ and $S$ is the shift map.

We can study the system $(\Sigma, S)$, where $\Sigma = \overline{\phi(X)}$.

# Complexity

**Definition**

If $v$ is an infinite word, we define the COMPLEXITY function $p(n, v)$ as the number of different words of length $n$ inside $v$.

**Example**

$v = 0100011011000\ldots \quad p(n, v) = 2^n$

# Complexity

## Definition

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## Example

$v = 0100011111\ldots \;\; p(n, v) \leq 6$

In fact we can compute two different complexities:

- The complexity of one word: $p(n, v)$.
- The complexity of the union of all the words: $p(n)$.

Let $v$ be a word corresponding to the orbit of $m$, and $\mathcal{L}_v$ its language. Consider the following language

$$\mathcal{L} = \bigcup_m \mathcal{L}_v, \quad p(n) = card \mathcal{L}(n).$$

# Fibonacci

A substitution is a morphism of free monoid. For example for $\{0;1\}^*$ we have:

$$\phi \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$$

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$$v = 0100101001001010\dots$$

For a fixed point $v$ of a substitution, the dynamical system is $(\Sigma, S)$ where

$$\Sigma = \overline{\bigcup_{n\in\mathbb{N}} S^n v}.$$

**Theorem (Buzzi 2001)**

*Every piecewise isometry has zero topological entropy.*

**Theorem (Bressaud-Poggiaspalla 06)**

*Classification of piecewise isometries defined on a triangle.*

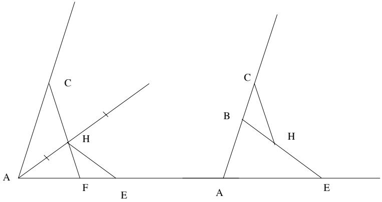
# Example

We define a piecewise isometry $(Y, R)$ on the union of two triangles

$$Y = AFC \bigcup HFE,$$

where $R : Y \mapsto Y$ is defined as follows:

- The triangles $AFC, HFE$ are isoscele triangles, and $\hat{A} = 2\pi/5$.
- a rotation of center $O_1$ and angle $-3\pi/5$ which sends $C$ to $E$, if $m$ belongs to $AFC$.
- a rotation of center $O_2$ and angle $-\pi/5$ which sends $H$ to $C$ otherwise.




$$AFC \mapsto BAE$$

$$HFE \mapsto CBH$$

Tabachnikov 1995

# Substitution and first return map

We denote the first rotation by $a$ and the second by $b$. For some affine map $D$ we remark

$$\begin{cases} Da(x) = aababaaD(x) & x \in AFC, \\ Db(y) = aaaD(y) & y \in HFE \end{cases}$$

Thus we introduce the substitution $\sigma$:

$$\begin{cases} a \mapsto aababaa \\ b \mapsto aaa \end{cases}$$

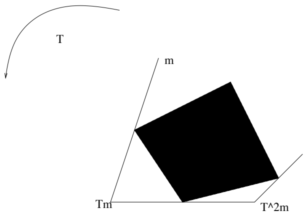
The language of this piecewise isometry is given by the fixed point of the susbtitution $\sigma$ for the aperiodic points:

$$aababaaaababaaaaaaababaaaaa\ldots$$

There is a complete description of the dynamics.

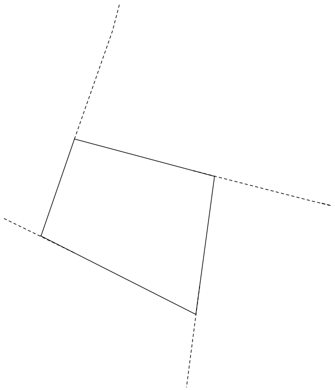
# Dual billiard

Consider a convex polygon in $\mathbb{R}^2$. Fix one orientation.




The billiard map is defined in $\mathbb{R}^2 \setminus P$ by reflection through the verteces of $P$.

There is a natural coding with the following regions:



Question Compute $p(n)$, and describe the symbolic dynamics.

A polygon is said to be rational if there exists a lattice which contains $P$.

**Theorem**

*For a rational polygon every orbit is periodic.*
*For a quasi-rational polygon every orbit is bounded.*

- Vivaldi-Shaidenko 87.
- Kolodziej 89.
- Gutkin-Simanyi 92

Every regular polygon is a quasi-rational polygon.

**Theorem (Tabachnikov 95)**

*For the regular pentagon we have:*

- *Almost every point has a periodic orbit.*
- *There exists some points with non periodic orbit.*
- *The set of non periodic point is a fractal set.*
- *Computation of Hausdorff dimension.*

# Background

## Theorem (Gutkin-Tabachnikov 06)

*If $P$ is rational then $an^2 \leq p(n) \leq bn^2$.*
*If $P$ is a polygon with $k$ verteces, then*

$$an \leq p(n) \leq bn^{r+2}.$$

*The integer $r$ is the rank of the abelian group generated by translations in the sides of $P$.*

We have $r \leq k-1$ and if $P$ is rational then $r = 2$. For a regular pentagon we find $r = 3$.

## Remark

*If $h$ is an affine map, then $h(P)$ has the same properties than $P$.*

## Theorem (B-Cassaigne 08)

*With the notation $p(n) = kp_{L'}(n-1)$ where $k$ is the number of vertices of the polygon we have:*

| *Polygons* | $p_{L'}(n)$ |
|---|---|
| *Triangle* | $\frac{5n^2+14n+f(r)}{24}$ |
| *Square* | $\frac{1}{2}\lfloor\frac{(n+2)^2}{2}\rfloor$ |
| *Hexagon* | $\lfloor\frac{5n^2+16n+15}{12}\rfloor$ |

$$n = 12q + r.$$

| r | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| f(r) | 24 | 29 | 24 | 9 | 8 | 21 | 24 | 17 | 0 | -3 | 8 | 9 |

**Theorem (B-Cassaigne 08)**

*For the regular pentagon we obtain*

$$p_{L'}(n) \sim Cn^2$$

*where*

$$C = \frac{1}{5} + 2\sum_{n\geq 0}\left(\frac{7}{52.6^n + 28 - 10(-1)^n} + \frac{7}{192.6^n + 28 - 10(-1)^n}\right).$$

# Method

The method consists in

- Define a new map $\hat{T}$ on one cone.
- Find a set where the first return map of $\hat{T}$ is "simple".
- Describe the language of $\hat{T}$.
- Compute the complexity function.

We will explain the method for the square and the regular pentagon.

# New map

We consider one regular polygon $P$ and one cone $V$. We denote by $R$ the rotation centered in the center of the polygon with angle $2\pi/k$. Then we denote by $\hat{T}$ the map defined on $V$:

$$\hat{T}(x) = R^{n_{Tx}}(Tx), \quad n_y = min\{n, R^n y \in V\}.$$

**Lemma**

*The map $\hat{T}$ is a piecewise isometry defined on $\lfloor \frac{k+1}{2} \rfloor$ sets.*

This map has a natural coding due to preceding Lemma.

# Link between the codings

If $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ are two sequences obtained as coding for a point $m$, then

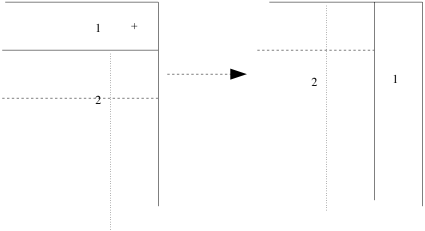
$$v_n = u_{n+1} - u_n \qquad \mod k.$$

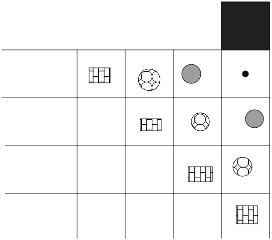
For the square the codings of $m$ are

$$u = 012301230123\ldots$$

$$v = 111111\ldots$$

For the square the map $\hat{T}$ has the following form:



# Language

We deduce a description of the language of $\hat{T}$

## Lemma

*The language of $\hat{T}$ is the union of finite words included in*

$$\bigcup_{n\in\mathbb{N}} (12^n)^\omega.$$

There are only periodic words of different periods.

$$121212\ldots$$

$$122122122\ldots$$

$$122212221222\ldots$$

It remains to compute the complexity of this language.

# Pentagon

Consider the three substitutions:

$$\sigma : \begin{cases} 1 \to 1121211 \\ 2 \to 111 \end{cases} \qquad \psi : \begin{cases} 1 \to 2223223 \\ 2 \to 223 \\ 3 \to 2^{-1} \end{cases} \qquad \xi : \begin{cases} 1 \to 31111 \\ 2 \to 2 \end{cases}$$

## Theorem

*The language of the dual billiard map for the regular pentagon is given by*

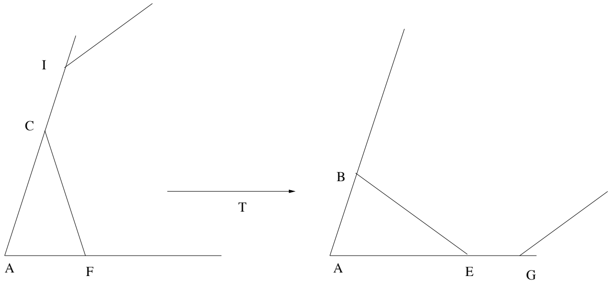
$$\bigcup_{n,m\in\mathbb{N}} \sigma^n(1) \cup \psi^m \circ \sigma^n(1) \cup \psi^m \circ \xi \circ \sigma^n(1).$$

The first return map $\hat{T}$ for the regular pentagon is given by

**Lemma**

*The map $\hat{T}$ is defined on three subsets: the triangle $ACF$, the triangle $HFE$, the infinite quadrilateral $CHE$.*

There is an invariant subset where the map coincides with the piecewise isometry defined by Tabachnikov.


I
C
A
F
T
B
A
E
G

# Bispecial words

Let $\mathcal{L}(n)$ the set of words of length $n$ in a language. For $v \in \mathcal{L}(n)$ let

$$\begin{aligned}
s(n) &= p(n+1) - p(n). \\
m_l(v) &= card\{a \in \Sigma, \quad av \in \mathcal{L}(n+1)\}. \\
m_r(v) &= card\{b \in \Sigma, \quad vb \in \mathcal{L}(n+1)\}. \\
m_b(v) &= card\{(a,b) \in \Sigma^2, \quad avb \in \mathcal{L}(n+2)\}. \\
b(n) &= \sum_{v \in \mathcal{L}(n)} (m_b(v) - m_r(v) - m_l(v) + 1).
\end{aligned}$$

## Definition

A word $v$ is:

- right special if $m_r(v) \geq 2$,
- left special if $m_l(v) \geq 2$,
- bispecial if it is right and left special.

We have

## Lemma (Cassaigne 97)

*For all integer $n$ we have*

$$s(n+1) - s(n) = b(n).$$

# Fibonacci word

$$v = 0100101001001010\ldots$$

The left special words are

# Fibonacci word

$$v = 0100101001001010\ldots$$

The left special words are

- 0

# Fibonacci word

$$v = 0100101001001010\ldots$$

The left special words are

- 0
- 01

# Fibonacci word

$$v = 0100101001001010\ldots$$

The left special words are

- 0
- 01
- 010

# Fibonacci word

$$v = 0100101001001010\dots$$

The left special words are

- 0
- 01
- 010
- 0100

# Fibonacci word

$$v = 0100101001001010\ldots$$

The left special words are

- 0
- 01
- 010
- 0100
- Prefix of $v$.

One left special word for every length $n$.

# Example

$$v = 0100101001001010\ldots$$

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The right special words are

- 0

# Example

$$v = 0100101001001010\ldots$$

The right special words are

- 0
- 10

# Example

$$v = 0100101001001010\dots$$

The right special words are

- 0
- 10
- 010

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$$v = 0100101001001010\ldots$$

The right special words are

- 0
- 10
- 010
- 0010

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$$v = 0100101001001010\dots$$

The right special words are

- 0
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- Mirror image of prefix of $v$.

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The right special words are

- 0
- 10
- 010
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- Mirror image of prefix of $v$.

The bispecial words are:

- 0
- 010
- 010010
- Palindromic prefixs.

For example 010 can be extended in

- 00100
- 00101
- 10100

Thus we have $i(010) = 3 - 2 - 2 + 1 = 0$.

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For all bispecial word $i(v) = 3 - 2 - 2 + 1 = 0$.

## Questions

- Complexity for a quasi-rational polygon.
- Geometry of $\lim \frac{p(n)}{n^2}$.
- Complexity for a non quasi-rational polygon.