Dynamical systems

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A **discrete dynamical system** is a map

$$T : X \to X$$

We want to understand the orbit of a point $x \in X$: $\{n \in \mathbb{N}, T^n x\}$

- Is it finite ?
- is it dense in X ?
- Does it depend on x ?
- Can we say something for all x ?

Example

$X = \mathbb{R}/\mathbb{Z}$, and $Tx = 2x \mod 1$.

- Every rational number of the form $\frac{p}{2^n - 1}$ has a periodic orbit.
  For example $\frac{1}{3}$, $\frac{2}{3}$, $T^2(\frac{1}{3}) = \frac{1}{3}$.
- Every irrational number has a non periodic orbit.
- Can you find a point of dense orbit ?

This is a **chaotic map**.

Example

$X = \mathbb{R}/\mathbb{Z}$, and $Tx = x + a \mod 1$.

- If a is rational, then every orbit is periodic.
- If a is irrational, then every orbit is dense.

Translation on the torus, or rotation on the circle.

Work of H. Poincaré $(1854 - 1912)$.

Related example: T homeomorphism of the circle S^1.

Theorem of Denjoy (1932): If f is $\mathcal{C}^2$ with irrational number of rotation, then it is conjugated to a rotation of S^1.

Example

Consider a finite set A, and $X = A^{\mathbb{N}}$. Define a distance on $A^{\mathbb{N}}$ by $d(u, v) = \frac{1}{2^n}$ where $n = \min k \geq 0, u_k \neq v_k$.
Now let T be the shift map, defined by $Tu = (u_{n+1})_{n \in \mathbb{N}}$.

- Find some periodic orbits. For example 010101 . . .
- Find some dense orbits.

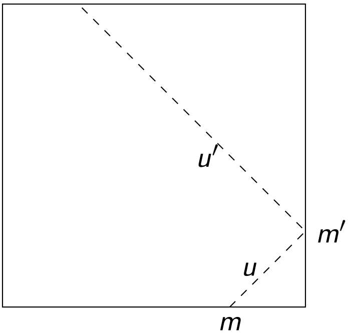
This is a **subshift**. Link with combinatorics.

Find the link with the first example . . .

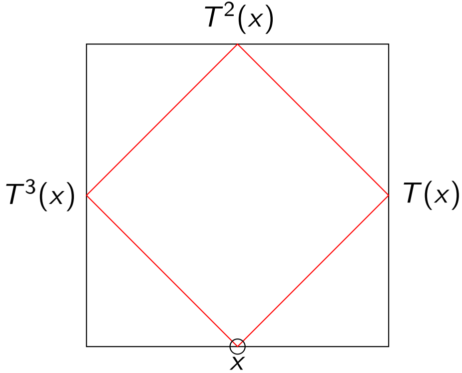
Example

Consider a square P, and let $X = \partial P \times \mathbb{R}^2$. Let T be the billiard map inside P.

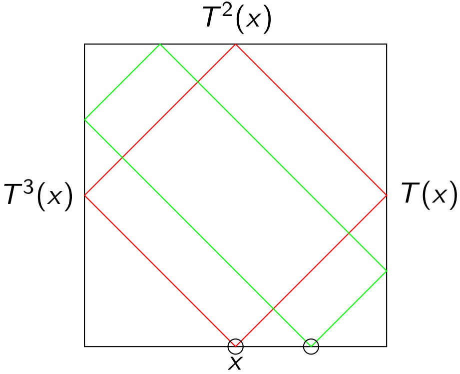
$$T : (m, u) \mapsto (m', u')$$



Example of periodic trajectories in a square



Example of periodic trajectories in a square



In another polygon, it is more difficult.

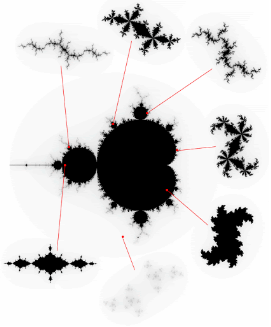
For example in an obtuse triangle, it is unknown if there exists some periodic orbit.

Link with **geometry**.

Example

We can consider $X = \mathbb{C}$ and T a quadratic polynomial as $T(z) = z^2 + c$.

Link with **complex dynamics**.



Previous examples can be seen, if T is invertible, as the action of $\mathbb{Z}$ on X (by $n.x = T^n x$).
But we can use other groups.

We can look at flow with a $\mathbb{R}$ action:

Continuous dynamical systems

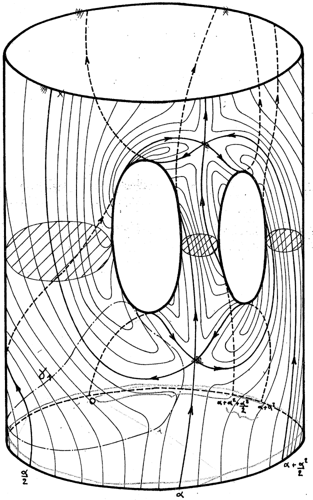
$$\phi : \mathbb{R} * X \to X$$

where $\phi(t+s, x) = \phi(t, \phi(s, x))$

Link with **differential equations.**

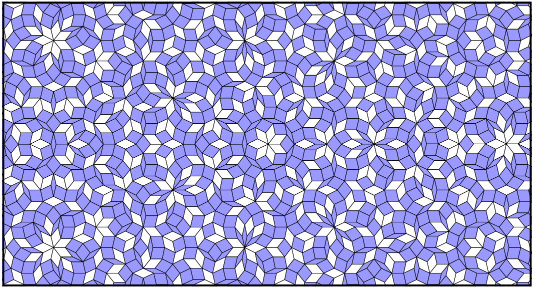
Or flow on surface: geodesic flow on hyperbolic surface,...

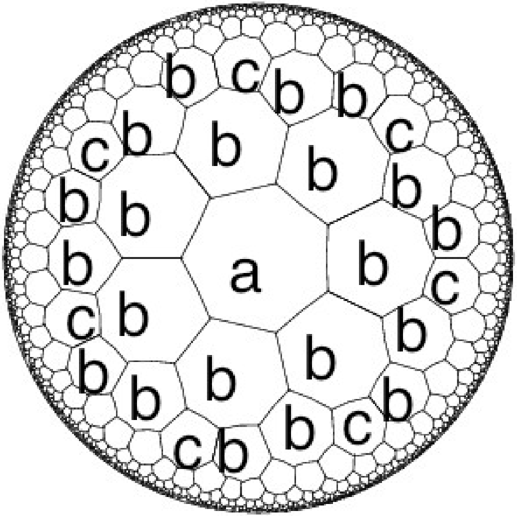
la suspension $(M, \mathcal{F}, \nu, \gamma)$. (Il faut identifier le cercle supérieur et le cercle inférieur par une rotation de $\frac{1}{2}$)



On a représenté en tirets ——— la courbe γ_1

We can also use $\mathbb{R}^2$ as the groups of translations. This means looking at **tilings** (of the plane or other spaces).

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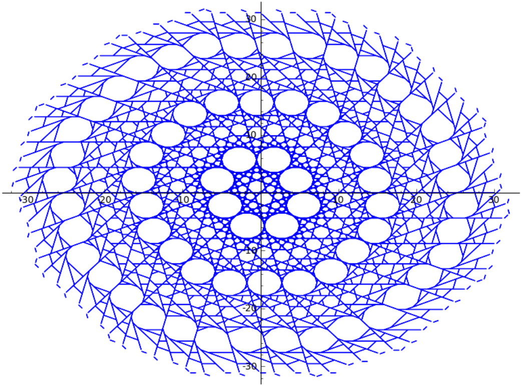
Cantor set

Another aspect is the study of Cantor set: Closed set without isolated point and nowhere dense.

We want to compute the Hausdorff dimension of these sets, and understand the structure.



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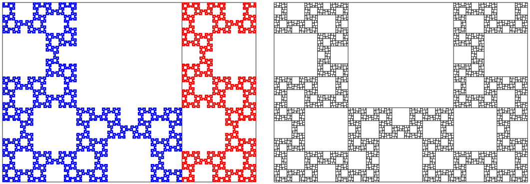
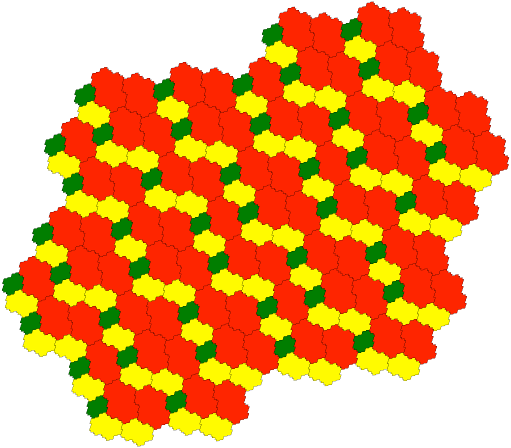


Haudorff dimension ?



Haudorff dimension ?

Tools

Ergodic theory.

Beginning with Birkhoff 1931. To (X, T) we associate some measure μ on X with good properties.

The measure is ergodic if

- $\mu(T^{-1}A) = \mu(A)$ for all measurable sets A.
- $T^{-1}A = A$ implies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Consider some function $f : X \to \mathbb{R}$ in $L^1(X, \mu)$. The **ergodic theorem** says that if μ is an ergodic measure, then for almost all x we have

$$\lim_{+\infty} \frac{1}{N} \sum_{i=0}^{N-1} f \circ T^i(x) = \frac{1}{\mu(X)} \int_X f d\mu$$

Poincaré recurrence theorem is also an important tool.

Examples of results

With this we can prove

- In a polygon with angles in $\pi\mathbb{Q}$: almost every point of the boundary defines a periodic orbit if we start orthogonally to the side.
- If a is an irrational number, then for all x, the sequence $(x + na \mod 1)_{n\in\mathbb{N}}$ is equidistributed in $[0, 1]$.
- The Bernoulii measure $B(\frac{1}{2}, \frac{1}{2})$ is ergodic on $\{0, 1\}^{\mathbb{N}}$.
- Existence of an ergodic measure for a finite Markov chain.

To resume in dynamical system you can do

- Geometry
- Group
- Number theory,
- Analysis
- Combinatorics

References

On the web you can look at (a lot of choices . . .)

- Webpage of Alex Gorodnik: https://www.math.uzh.ch/gorodnik/ds17/index.html
- Webpage of Omri Sarig: http://www.weizmann.ac.il/math/sarigo/ergodic-theory-course
- Vidéo of Stefano Luzzato (ICTP courses)

Books

- Yves Coudène: Ergodic theory in dynamical systems, see his webpage.
- Boris Hasselblatt. Many books.
- Doug Lind: Symbolic dynamics and coding

Master 2 in Marseille 2020-2021

Schedule

- 3 introduction courses for two weeks in September
- 3 courses for 3 months (october-december).
- 2 courses for 3 monthes (january-march 2021).
- English course.
- Student seminar every week: Oral presentation of a research paper.
- Master thesis: ($\sim$ 3 months)
  initiation to research.

Grants

- Grant for the M2 via Institut Archimède.
- Grant for Phd thesis

Web site:

https://maths-sciences.univ-amu.fr/master-maap/

Do not hesitate to contact me.