Symbolic coding of linear complexity for generic translations on the torus, using continued fractions

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Abstract

In this paper, we prove that almost every translation on $\mathbb{T}^2$ admits a symbolic coding which has linear complexity $2n+1$. The partitions are constructed with Rauzy fractals associated with sequences of substitutions, which are produced by a particular extended continued fraction algorithm in projective dimension 2. More generally, in dimension $d \geq 1$, we study extended measured continued fraction algorithms, that associate to each direction a subshift generated by substitutions, called S-adic subshift. We give some conditions which imply the existence, for almost every direction, of a translation on the torus $\mathbb{T}^d$ and a nice generating partition, such that the associated coding is a measurable conjugacy with the subshift that it defines.

Keywords: symbolic dynamics, continued fraction, renormalization, Rauzy fractal, bounded remainder set, S-adic system, S-adic subshift, Lyapunov exponent, torus translation, Pisot substitution conjecture

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1 Introduction

The first motivation of this paper is to find symbolic codings with low complexity of translations on the d-dimensional torus $\mathbb{T}^d$. In dimension 1, every irrational translation on $\mathbb{T}^1$ admits a generating partition made of two intervals giving a symbolic coding of complexity $n+1$, generating the famous *Sturmian words* [43][46]. However, the endpoints of the intervals must be chosen carefully, since most partitions into two intervals lead to a symbolic coding of complexity $2n$ [24].

In higher dimension $d \geq 2$, a result of Chevallier [23] ensures that, for any minimal translation on $\mathbb{T}^d$, and for any generating partition of $\mathbb{T}^d$ with polygonal atoms, the corresponding symbolic coding has a complexity in $\Omega(n^d)$. Hence, if we want to go below this bound, we will have to abandon the smooth shape of the atoms, while keeping their topological and measure-theoretic regularity to avoid trivial constructions: the partitions must still be generating, the atoms should be the closure of their interior, and their boundaries should have zero Lebesgue measure.

In the seminal paper [48], for the special case of the translation on $\mathbb{T}^2$ with vector (ρ, ρ^2), where $\rho = 1.839286755214161\ldots$ is the real root of $X^3 - X^2 - X - 1$, Rauzy constructs such a generating partition whose associated subshift is the Tribonacci subshift with complexity $2n + 1$ (see also [22]). This construction was generalized to (countably many) other parameters [33] [17] [2], and highly relies on the algebraic nature of the translation vector, which is witnessed in the self-similarity of the fractal generating partition.

Actually, Rauzy does not start from a translation on $\mathbb{T}^2$, but first constructs a piecewise translation on a fundamental domain of the plane (for the action of $\mathbb{Z}^2$ by translation), and the projection modulo $\mathbb{Z}^2$ of each piece forms an atom of a partition in $\mathbb{T}^2$: the translation on $\mathbb{T}^2$ is deduced from the domain exchange. If a minimal translation on $\mathbb{T}^2$ is coded with such a liftable generating partition, the resulting complexity is at least $2n+1$ [11] (this result is generalized in [10]: in dimension d, the bound becomes $dn+1$). Hence, looking for generating partitions with complexity $2n + 1$ for translations on $\mathbb{T}^2$ seems to be a reasonable target.

Some known families of subshifts with complexity $2n + 1$ can be tried out. They are generated by continued fraction algorithms. The first candidate is the Arnoux-Rauzy algorithm. Unfortunately, the set of points where this algorithm can be iterated is too narrow; this set is known as the Rauzy gasket, see [9] for references. Another candidate is the continued fraction algorithm associated with the set of 3-interval exchange transformations. It is defined for almost every direction and produces subshifts with complexity $2n + 1$, but we know since [34] that almost all of them are weakly mixing, see also [7]. Thus, they cannot be conjugate to a translation on a torus. Recently, Cassaigne introduced a continued fraction algorithm which has nice combinatorial properties and which is defined on the full space of parameters [3, 19].

The first objective of this paper is to use Cassaigne algorithm to construct, for almost every translation on $\mathbb{T}^2$, a regular generating partition giving a symbolic coding of complexity $2n+1$ (Theorem A).

To this end, we first develop a general framework for constructing Rauzy fractals out of infinite sequences of substitutions, and use their pieces as the atoms of the generating partitions (Theorem C). Our approach is direct and provides an alternative to the "dual" construction of [13]. For this, we use particular topologies on $\mathbb{Z}^{d+1}$, introduced in [42], that we extend to the S-adic context.

We then prove that, when the sequences of substitutions are generated by an ergodic extended continued fraction algorithm whose second Lyapunov exponent is negative, the existence of a single direction that fulfills certain requirements, called a seed point, gives a set of full measure of good directions that produce nice Rauzy fractals (Theorem B). Theorem A follows by applying Theorem B to the Cassaigne algorithm.

As byproducts of those constructions, the atoms of the partitions provide bounded remainder sets (see [49] for some motivation); also, we get a renormalization scheme that relates the continued fraction algorithm to the first return map on some of the atoms.

Regarding further applications, the symbolic codings that are constructed in this paper with the Cassaigne algorithm are shown, in [21], to satisfy the Boshernitzan criterion for unique ergodicity [16] (under a mild combinatorial condition involving the existence of a word builder). In particular, the authors deduce that for almost every translation of $\mathbb{T}^2$, every continuous function $\mathbb{T}^2 \to \mathbb{R}$ can be uniformly approximated by measurable functions whose associated discrete Schrödinger operator has a Cantor spectrum of zero Lebesgue measure.

During the revision of the present paper, Berthé, Steiner and Thuswaldner proved independently similar results on the same subject [14]. We discuss differences between the two approaches in Section 10.1.

2 Statement of the results and outline of the proofs

The three theorems of this paper, that were informally presented in the introduction, involve a number of objects that will be precisely defined in following sections. In this section, we give formal statements for the three theorems, with pointers to the definitions.

Our main theorem is the following (see Definition 9 for a definition of a nice generating partition).

Theorem A. *Lebesgue-almost every translation on $\mathbb{T}^2$ admits a nice generating partition giving a symbolic coding with complexity $2n+1$.*

In order to prove it, we use the Cassaigne algorithm [3, 19] and prove that it fulfills the hypotheses of Theorem B below.

Theorem B involves an extended measured continued fraction algorithm (X, s_0, μ), see Definition 76. The Pisot condition for such an algorithm is defined in Definition 86. The set of seed points $G_0 \subseteq X$ in defined in Definition 90.

We fix a euclidean hyperplane $P \subseteq \mathbb{R}^{d+1}$ with a cocompact lattice Λ of P, a quotient map $q\colon P \to P/\Lambda$, a map $v\colon G_0 \to \mathbb{R}^{d+1}$ and a vector $e_0 \in \mathbb{R}^{d+1}$ such that the map $e_0 - v$ takes values in P (see Section 3.1 and 3.2) and prove the following theorem.

Theorem B. *Let (X, s_0, μ) be an extended measured continued fraction algorithm satisfying the Pisot condition. Assume $G_0 \neq \emptyset$. Then, for μ-almost every point $x \in X$, there exist a translation $z \mapsto z + t_x$ on the torus $\mathbb{T}^d$ and a nice generating partition such that the associated symbolic coding is a measurable conjugacy with the uniquely ergodic subshift associated with x.*

Moreover, we can take $t_x = \psi(q(e_0 - v(x)))$ for a given isomorphism $\psi\colon P/\Lambda \to \mathbb{T}^d$.

We prove Theorem B by defining, for μ-almost every point $x \in X$, a Rauzy fractal $R \subseteq P$. We show that its pieces define a nice generating partition of $\mathbb{T}^d$ (identified with P/Λ). This is done with Theorem C below, see Definition 50 and Definition 15 for the definitions of a good directive sequence and its direction and Section 3.4 for the alphabet A.

Theorem C. *Let s be a good directive sequence. Then the Rauzy fractal $R(s)$ is a measurable fundamental domain of P for the lattice Λ. It can be decomposed as a union $R(s) = \bigcup_{a \in A} R_a(s)$ which is disjoint up to sets of Lebesgue measure 0, and each piece $R_a(s)$ is the closure of its interior.*

Moreover, the pieces $R_a(s)$, $a \in A$, of the Rauzy fractal induce a nice generating partition of the translation by $q(e_0 - v)$ on the torus P/Λ, where v is the unit vector of the direction of s.

This partition defines a symbolic coding of the translation by $q(e_0 - v)$, and this coding is a measurable conjugacy with the uniquely ergodic subshift associated with s.

Theorem C does not depend on a continued fraction algorithm. It is proven in Section 4. We introduce some topologies in Section 3.7, which play a central role in the proof of Theorem C. We prove that every good directive sequence gives a nice Rauzy fractal with all the wanted properties.

Theorem B is proven in Section 6. We first establish in Proposition 92 that the existence of a seed point implies that μ-almost all points of X are good. Then we use Theorem C.

Theorem A is proven in Section 8. We first recall some facts about the Cassaigne algorithm, one of its invariant measures, and the associated Lyapunov exponents. Then we consider a particular periodic point for this algorithm, and we prove that it is a seed point. Being a seed point is a decidable property for such periodic points, see Proposition 108. This allows to apply Theorem B.

3 Tools

3.1 Geometrical setting

Let $d \geq 1$ be an integer. In Section 5 we will work with continued fraction algorithms. To define them in dimension d it is convenient to work in the $d+1$-dimensional space $\mathbb{R}^{d+1}$, or rather its positive cone $\mathbb{R}_+^{d+1} \setminus \{0\}$. This is why we introduce some notations here.

Let $(e_i)_{0 \leq i \leq d}$ be the canonical basis of $\mathbb{R}^{d+1}$ (note the unusual numbering of dimensions). The space $\mathbb{R}^{d+1}$ is equipped with the classic norm $\|.\|_1$ defined by $\|(y_0, \ldots, y_d)\|_1 =$

$\sum_{i=0}^{d} |y_i|$. The space we are really interested in is $\mathbb{PR}_+^d = \left(\mathbb{R}_+^{d+1} \setminus \{0\}\right)/\mathbb{R}_+^*$, the set of positive directions.

For a vector $y \in \mathbb{R}_+^{d+1} \setminus \{0\}$, we denote by $[y] = \mathbb{R}_+^* y \in \mathbb{PR}_+^d$ the corresponding direction. Conversely, for every $x \in \mathbb{PR}_+^d$, we denote by $v(x) \in \mathbb{R}_+^{d+1}$ the unique representative of x such that $\|v(x)\|_1 = 1$. When $y \in \mathbb{R}_+^{d+1} \setminus \{0\}$, we write $v(y) = v([y]) = \frac{y}{\|y\|_1}$.

For every matrix $M \in \mathcal{M}_{d+1}(\mathbb{R})$ and $x \in \mathbb{PR}_+^d$, we write $Mx = [Mv(x)]$ if $Mv(x) \in \mathbb{R}_+^{d+1} \setminus \{0\}$.

We define a distance on $\mathbb{PR}_+^d$ by $d(x, y) = \|v(y) - v(x)\|_1$, making it a metric space. Note that $\mathbb{PR}_+^d$ is thus isometric to the simplex $\Delta = \{y \in \mathbb{R}_+^{d+1} \mid \|y\|_1 = 1\}$. Open balls in $\mathbb{PR}_+^d$ are denoted $B(x, r)$.

Let h denote the linear form on $\mathbb{R}^{d+1}$ defined by $h(y_0, \ldots, y_d) = \sum_{i=0}^{d} y_i$. Note that, when $y \in \mathbb{R}_+^{d+1}$, $h(y) = \|y\|_1$. Let P be the hyperplane $\{y \in \mathbb{R}^{d+1} \mid h(y) = 0\}$. Open balls in P are also denoted $B(p, r)$.

In the following we consider the lattice $\Lambda = P \cap \mathbb{Z}^{d+1} = \langle e_1 - e_0, \ldots, e_d - e_0 \rangle$. Let us denote by λ the Lebesgue measure on P.

For $y \in \mathbb{R}_+^{d+1} \setminus \{0\}$, let π_y denote the projection along y onto P (note that y does not belong to P as $h(y) > 0$). This map sends a vector $z \in \mathbb{R}^{d+1}$ to $z - h(z)v(y)$. For $x \in \mathbb{PR}_+^d$, we also denote $\pi_x = \pi_{v(x)}$. Remark that $\Lambda \subseteq P$, so Λ is preserved by every projection π_y. It will be convenient (for instance in Lemma 1 below) to also denote by $\pi_x \in \mathcal{M}_{d+1}(\mathbb{R})$ the matrix of the projection π_x in the canonical basis.

Lemma 1. *For every matrix $M \in GL_{d+1}(\mathbb{R})$ and every $x \in \mathbb{PR}_+^d \cap M\mathbb{PR}_+^d$, we have the identity $\pi_x M \pi_{M^{-1}x} = \pi_x M$.*

Proof. Since $x \in M\mathbb{PR}_+^d$, the direction $M^{-1}x$ is well-defined. The identity comes from the fact that $M\pi_{M^{-1}x}(y) = My - h(y)Mv(M^{-1}x)$, and $\pi_x(Mv(M^{-1}x)) = 0$. □

We say that $y = (y_0, \ldots, y_d) \in \mathbb{R}_+^{d+1}$ has a *totally irrational direction*, or that $[y] \in \mathbb{PR}_+^d$ is a totally irrational direction, if $y_0, \ldots, y_d$ are linearly independent over $\mathbb{Q}$.

3.2 Translations on the torus

We define the d-dimensional torus as P/Λ, and let $q\colon P \to P/\Lambda$ denote the quotient map. We still denote by λ the Lebesgue measure transported on P/Λ. Note that our definition of a torus differs from the usual one $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, but they are isomorphic, in a non-canonical way that depends on a choice of a basis of Λ. To fix an isomorphism, let $L\colon P \to \mathbb{R}^d$ be the restriction to P of the linear map which sends $(x_i)_{0 \le i \le d}$ to $(x_i)_{1 \le i \le d}$. Since $L(\Lambda) = \mathbb{Z}^d$, L induces a torus isomorphism $\psi\colon P/\Lambda \to \mathbb{T}^d$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{L} & \mathbb{R}^d \\ \downarrow & & \downarrow \\ P/\Lambda & \xrightarrow{\psi} & \mathbb{T}^d \end{array}$$

Let $t \in P/\Lambda$, and $\widehat{t} = (t_0, \ldots, t_d) \in P$ be a representative of t. Then t is said to be a *totally irrational vector* if $\widehat{t} + e_0$ has a totally irrational direction, *i.e.*, if $1, t_1, \ldots, t_d$ are linearly independent over $\mathbb{Q}$ (this is independent of the choice of $\widehat{t}$). Note that totally irrational vectors should not be confused with totally irrational directions.

Remark that any isomorphism $\psi\colon P/\Lambda \to \mathbb{T}^d$ preserves the Lebesgue measure (up to a multiplicative constant) and totally irrational vectors. This will be used in the proof of Theorem B.

For $t \in P/\Lambda$, we consider the associated translation

$$T_t = \left(\begin{array}{ccc} P/\Lambda & \longrightarrow & P/\Lambda \\ z & \longmapsto & z+t \end{array} \right).$$

We recall that a translation T_t is *minimal* if, and only if, t is a totally irrational vector [35].

If $x \in \mathbb{PR}^d_+$ is a direction, we denote $T_x = T_{q(\pi_x(e_0))}$. Note that $\pi_x(e_0) = e_0 - v(x)$. As a consequence, $q(\pi_x(e_i)) = q(\pi_x(e_0))$ for all $i \in \{0, \dots, d\}$, so e_0 could be replaced with any other vector of the canonical basis in the definition of T_x. Also, it implies that $\|\pi_x(e_0) - \pi_{x'}(e_0)\|_1 = d(x, x')$, which will be used later (see Lemma 25).

3.3 Matrices

Continued fraction algorithms are often expressed with matrices. Here we recall some facts about matrices, and establish some properties of matrix sequences.

As the space $\mathbb{R}^{d+1}$ is equipped with the norm $\|.\|_1$, the operator norm of a matrix $M \in \mathcal{M}_{d+1}(\mathbb{R})$ is defined by

$$|||M|||_1 = \sup_{v \in \mathbb{R}^{d+1} \setminus \{0\}} \frac{\|Mv\|_1}{\|v\|_1}.$$

Moreover, we also define a semi-norm for a non-trivial subspace V:

$$|||M_{|V}|||_1 = \sup_{v \in V \setminus \{0\}} \frac{\|Mv\|_1}{\|v\|_1}.$$

Finally we write $M > 0$ if every coefficient of M is positive.

A matrix $M \in \mathcal{M}_{d+1}(\mathbb{R})$ is said to be *Pisot* if it has non-negative integer entries, its dominant eigenvalue is simple and all other eigenvalues have moduli less than one.

Lemma 2. *Let M be an invertible matrix with non-negative integer entries, and $x \in \mathbb{PR}^d_+$. Then $M\mathbb{PR}^d_+ \subseteq B(x, |||\pi_x M|||_1)$.*

Proof. Let $y \in \mathbb{PR}^d_+$. Then

$$\begin{aligned} d(x, My) &= \|v(Mv(y)) - v(x)\|_1 \\ &= \frac{\|Mv(y) - h(Mv(y))v(x)\|_1}{h(Mv(y))}. \end{aligned}$$

Observe that $\|Mv(y) - h(Mv(y))v(x)\|_1 = \|\pi_x Mv(y)\|_1 \le |||\pi_x M|||_1$. Moreover, since M is invertible and has non-negative integer entries, $h(Me_i) \ge 1$ for any vector e_i of the canonical basis, hence also $h(Mv(y)) \ge 1$. It follows that $d(x, My) \le |||\pi_x M|||_1$. □

We now consider sequences of matrices $(M_k) = (M_k)_{k\geq 0} \in (\mathcal{M}_{d+1}(\mathbb{R}))^{\mathbb{N}}$, where $\mathbb{N}$ denotes the set of non-negative integers. For the product of consecutive terms in such a sequence, we use the shorthand notation

$$M_{[k,n)} = M_k \dots M_{n-1}.$$

The following definition generalizes the classical notion of a primitive matrix (a matrix M is primitive when $M^n > 0$ for some $n \in \mathbb{N}$).

Definition 3. *Let (M_k) be a sequence of matrices with non-negative entries. We say that (M_k) is* primitive *if*

$$\forall k \in \mathbb{N},\ \exists n \geq k,\ M_{[k,n)} > 0.$$

We shall use several notions of convergence for a sequence of matrices.

Definition 4. *Let (M_k) be a sequence of invertible matrices with non-negative integer entries. We say that (M_k) is*

- cone convergent*, if there exists $x \in \mathbb{PR}^d_+$ such that $\bigcap_{n\in\mathbb{N}} M_{[0,n)}\mathbb{PR}^d_+ = \{x\}$;*
- sum convergent*, if there exists $x \in \mathbb{PR}^d_+$ such that $\sum_n \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1$ is a convergent series;*
- exponentially convergent*, if there exists $x \in \mathbb{PR}^d_+$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1 < 0.$$

When (M_k) is cone convergent, the unique direction x such that $\bigcap_{n\in\mathbb{N}} M_{[0,n)}\mathbb{PR}^d_+ = \{x\}$ is called the direction *of (M_k).*

Lemma 5. *Every exponentially convergent sequence of matrices is sum convergent. Every sum convergent sequence of matrices is cone convergent. Moreover, the convergences are for the same direction $x \in \mathbb{PR}^d_+$.*

Proof. The first assertion is clear. Assume that $\sum_n \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1$ converges, for some $x \in \mathbb{PR}^d_+$. Then $\lim_{n\to\infty} \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1 = 0$. By Lemma 2, $M_{[0,n)}\mathbb{PR}^d_+ \subseteq B(x, \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1)$. As their diameters tend to 0, the intersection of these balls is $\{x\}$. Then $\bigcap_{n\in\mathbb{N}} M_{[0,n)}\mathbb{PR}^d_+$, which is non-empty as it is a decreasing intersection of compact sets, must be $\{x\}$ too. □

Lemma 6. *Let (M_k) be cone convergent (respectively, sum convergent, or exponentially convergent) with direction x. For each $n \in \mathbb{N}$, the direction $M_{[0,n)}^{-1}x$ is well-defined and the shifted sequence $(M_{n+k})_{k\in\mathbb{N}}$ is cone convergent (respectively, sum convergent, or exponentially convergent) with direction $M_{[0,n)}^{-1}x$.*

Conversely, if for some $n \in \mathbb{N}$, $(M_{n+k})_{k\in\mathbb{N}}$ is cone convergent (respectively, sum convergent, or exponentially convergent) with direction x, then (M_k) is cone convergent (respectively, sum convergent, or exponentially convergent) with direction $M_{[0,n)}x$.

Proof. Assume that (M_k) is cone convergent with direction x. Let $n \in \mathbb{N}$. As $x \in M_{[0,n)}\mathbb{P}\mathbb{R}^d_+$, there exists $x' \in \mathbb{P}\mathbb{R}^d_+$ such that $M_{[0,n)}x' = x$, so $M_{[0,n)}^{-1}x = x'$ is well-defined. We then have

$$\bigcap_{k\in\mathbb{N}} M_{[n,n+k)}\mathbb{P}\mathbb{R}^d_+ = M_{[0,n)}^{-1} \bigcap_{k\in\mathbb{N}} M_{[0,n+k)}\mathbb{P}\mathbb{R}^d_+ = \{x'\}$$

so $(M_{n+k})_{k\in\mathbb{N}}$ is cone convergent.

Applying Lemma 1 to the matrix $M = M_{[0,n)}^{-1}$ and the projection $\pi_{x'} = \pi_{Mx}$, we get $\pi_{x'}M = \pi_{x'}M\pi_x$, so that

$$\pi_{x'}M_{[n,n+k)} = \pi_{x'}M\pi_x M_{[0,n+k)},$$

hence

$$\left|\left|\left|\pi_{x'}M_{[n,n+k)}\right|\right|\right|_1 \leq \left|\left|\left|\pi_{x'}M\right|\right|\right|_1 \left|\left|\left|\pi_x M_{[0,n+k)}\right|\right|\right|_1.$$

Therefore, if (M_k) is sum convergent (respectively, exponentially convergent), then so is $(M_{n+k})_{k\in\mathbb{N}}$.

The proof of the converse is similar. □

Lemma 7. *Let (M_k) be a sequence of invertible matrices with non-negative integer entries. If (M_k) is cone convergent with a totally irrational direction x, then it is primitive.*

Proof. By Lemma 6, the hypotheses are stable under multiplication by $M_{[0,k)}^{-1}$, so it is enough to prove that $M_{[0,n)} > 0$ for some n. Assume that (M_k) is cone convergent with direction x. Let $\epsilon = \min_i(v(x)_i)$. As x is totally irrational, $\epsilon > 0$. As (M_k) is cone convergent, there exists $n \in \mathbb{N}$ such that $M_{[0,n)}\mathbb{P}\mathbb{R}^d_+ \subseteq B(x, \epsilon/2)$. For every i, every coordinate of $v(M_{[0,n)}e_i)$ is therefore greater than $\epsilon/2$. Thus, we have $M_{[0,n)} > 0$. □

3.4 Words

We recall usual definitions and notations on words, see [41]. For a fixed $d \geq 1$, we define the *alphabet* A as the finite set $A = \{0, \dots, d\}$ (note that A has $d+1$ elements). Its elements are called *letters*. A *finite word* is an element of the monoid $A^* = \bigcup_{n\in\mathbb{N}} A^n$. The *length* of a finite word $w \in A^n$ is denoted by $|w| = n$. The set of non-empty words is the semigroup $A^+ = \bigcup_{n\geq 1} A^n$. An *infinite word* is an element of $A^\mathbb{N}$. A *word* can be finite or infinite.

The set of words $A^* \cup A^\mathbb{N}$ is endowed with the topology of coordinatewise convergence.

When a word w can be written as a product of three words pfs, p is called a *prefix* of w, f is called a *factor* of w, s is called a *suffix* of w, and the length of p is called an *occurrence* of f in w. The number of occurrences of a finite word f in a word w is denoted by $|w|_f$.

The *complexity* of an infinite word w is the map $p\colon \mathbb{N} \to \mathbb{N}$ which gives, for any non-negative integer n, the number of factors of w of length n.

A *substitution* is an element σ of $\hom(A^*, A^*)$: for all finite words $w_1, w_2 \in A^*$, we have $\sigma(w_1w_2) = \sigma(w_1)\sigma(w_2)$. A substitution is characterized by the images of letters. A *non-erasing substitution* is an element of $\hom(A^+, A^+)$: it is a substitution that maps every letter to a non-empty word.

The *abelianization map* is the monoid morphism $\mathrm{ab}\colon A^* \to \mathbb{Z}^{d+1}$ such that $\mathrm{ab}(a) = e_a$ for every letter a in A (recall that $(e_a)_{a\in A}$ is the canonical basis of $\mathbb{R}^{d+1}$). We use the same notation for the map from $\hom(A^*, A^*)$ to $\mathcal{M}_{d+1}(\mathbb{Z})$ such that $\mathrm{ab}(\sigma)\,\mathrm{ab}(w) = \mathrm{ab}(\sigma(w))$

for every substitution σ and every finite word w. A substitution σ is said to be *unimodular* if $|\det(\mathrm{ab}(\sigma))| = 1$. A substitution σ is said to be *Pisot* if the matrix $\mathrm{ab}(\sigma)$ is Pisot.

The action of a non-erasing substitution σ can be extended to infinite words by the limit procedure:

$$\sigma(w) = \lim_{\substack{w=ps \\ |p|\to\infty}} \sigma(p)$$

An infinite word w is a *fixed point* of σ if $\sigma(w) = w$. An infinite word w is a *periodic point* of σ if there exists an integer $m \geq 1$ such that $\sigma^m(w) = w$.

For an integer $k \geq 1$, an infinite word w is *k-balanced* if for any two factors f_1, f_2 of w of the same length, and any $a \in A$, we have $||f_1|_a - |f_2|_a| \leq k$. An infinite word is *balanced* if it is *k-balanced* for some integer k.

For an infinite word w, the (possibly undefined) *frequency vector* of w is

$$\mathrm{freq}(w) = \lim_{\substack{w=ps \\ |p|\to\infty}} \frac{\mathrm{ab}(p)}{|p|} \in \Delta \subseteq \mathbb{R}_+^{d+1}.$$

When this limit exists, we say that w admits a frequency vector. This is in particular the case if w is balanced (see Proposition 28) or if it is an element of a uniquely ergodic subshift.

For a non-empty finite word $w \in A^+$, we denote by w^ω the infinite word $\lim_{n\to\infty} w^n$.

Finally, we define the *shift map* T on $A^{\mathbb{N}}$ that maps an infinite word w to its suffix Tw such that $w = aTw$ with $a \in A$. Remark that with the coordinatewise topology, the shift map T is continuous. A non-empty subset $X \subseteq A^{\mathbb{N}}$ is called a *subshift* if X is closed and shift-invariant.

The *orbit* of $w \in A^{\mathbb{N}}$ is the set $\mathcal{O}(w) = \{T^n w \mid n \in \mathbb{N}\}$ and the *subshift generated* by w is its orbit closure $\Omega_w = \overline{\mathcal{O}(w)}$. Given a finite factor f of w, we define the *cylinder* $[f] = \{x \in \Omega_w \mid \exists y \in \Omega_w, x = fy\}$.

3.5 Symbolic coding

A *measured topological dynamical system* is a triple (X, T, μ) such that X is a compact topological space, μ is a finite Borel measure, and $T\colon X \to X$ is a μ-almost everywhere continuous map such that $\mu(T^{-1}(B)) = \mu(B)$ for any Borel set B of X.

Given a measured topological dynamical system (X, T, μ) and a measurable partition $(P_i)_{i\in I}$ of X, we associate the map $\mathrm{cod}\colon X \to I^{\mathbb{N}}$ defined by $\mathrm{cod}(y) = (i_n)_{n\in\mathbb{N}}$ when $\forall n \in \mathbb{N}$, $T^n y \in P_{i_n}$. The map cod is called a *symbolic coding* of the system (X, T, μ) and the closure of $\mathrm{cod}(X)$ defines a subshift over the alphabet I. A *generating partition* of the map T is a measurable partition such that the associated coding is injective μ-almost everywhere.

The atoms of the partitions we will construct will not be smooth, but they will keep some topological and measure-theoretic regularity: a generating partition $(P_i)_{i\in I}$ of X is *regular* if every set $\overline{P}_i$ is the closure of its interior and if the boundary of each P_i has zero measure.

A measurable subset A of X is said to be a *bounded remainder set* for the map T if there exists a constant K such that, for μ-almost every x in X and every positive integer N,

$$\left|\sum_{n=0}^{N-1} \mathbb{1}_A(T^n(x)) - N\frac{\mu(A)}{\mu(X)}\right| \leq K,$$

where $\mathbb{1}_A$ is the indicator function of the subset A. As we shall see, the atoms of the generating partition we will construct are bounded remainder sets.

Now, let T_t be a translation by t on the torus P/Λ. The triple $(P/\Lambda, T_t, \lambda)$ is a measured topological dynamical system, where λ denotes the Lebesgue measure inherited from P. The generating partitions we will construct on P/Λ actually comes from a piecewise translation on a measurable fundamental domain of P for the action of Λ:

Definition 8. *A finite measurable partition $(P_i)_{i\in I}$ of P/Λ is said to be* liftable *with respect to the translation $T_t\colon z \mapsto z+t$ on P/Λ if there exist a measurable fundamental domain $D \subseteq P$ for the action of Λ, a measurable partition $(D_i)_{i\in I}$ of D, and some vectors $(t_i)_{i\in I}$ in P^I, such that, for every i in I, $D_i + t_i \subseteq D$, $q(D_i) = P_i$, and $q(t_i) = t$.*

The measurable map $E = \begin{pmatrix} D & \longrightarrow & D \\ y & \longmapsto & y + t_i \text{ if } y \in D_i \end{pmatrix}$ is called a *piecewise translation* or a *domain exchange*, and is measurably conjugate to the translation T_t via the quotient map $q\colon P \to P/\Lambda$.

Definition 9. *A finite measurable partition $(P_i)_{i\in I}$ of P/Λ is said to be a* nice generating partition *with respect to the translation T_t on P/Λ if it is generating, regular, liftable, and if every P_i is a bounded remainder set.*

3.6 S-adic systems and S-adic subshifts

Let $S \subseteq \hom(A^+, A^+)$ be a finite set of non-erasing substitutions on the alphabet A.

An *S-adic system* is a shift-invariant subset of $S^{\mathbb{N}}$. Note that we do not impose that S-adic systems are topologically closed. For instance, in Section 7.1 we will consider $S = \{\tau_0, \tau_1\}$ and the S-adic system $\{s \in S^{\mathbb{N}} \mid \text{both } \tau_0 \text{ and } \tau_1 \text{ occur infinitely often in } s\}$.

An element $s = (s_k)$ of an S-adic system is called a *directive sequence*.

Given a directive sequence s, we define $M_k(s) = \mathrm{ab}(s_k)$, denoted simply by M_k when there is no ambiguity on what the directive sequence is.

Definition 10 (S-adic subshift)**.** *Let s be a directive sequence. The* S-adic subshift *associated with s is the subshift Ω_s defined as follows. Let first $L \subseteq A^*$ be the language of all factors of finite words of the form $s_{[0,n)}(a)$ for all $n \in \mathbb{N}$ and $a \in A$, where $s_{[k,n)} = s_k \circ \cdots \circ s_{n-1}$. Then Ω_s is the set of infinite words $w \in A^{\mathbb{N}}$ such that all factors of w are in L.*

S-adic subshifts were introduced by Ferenczi [28], where he proves that every word of linear complexity is an element of some S-adic subshift in an S-adic system with some additional conditions. This notion has been used in many places thereafter. We refer to [27] and [12].

Remark 11. *There are alternative ways to define subshifts from a directive sequence. One is to consider the set*

$$\Omega'_s = \bigcap_{n\in\mathbb{N}} \{T^k s_{[0,n)}(w) \mid w \in A^{\mathbb{N}}, k \in \mathbb{N}\}.$$

The set Ω'_s is the set of words that are infinitely desubstitutable by s, and it always holds that $\Omega_s \subseteq \Omega'_s$.

Another way is to first define an infinite word w by starting from a fixed letter $a \in A$ and taking a limit point of the sequence of finite words

$$s_0(a), s_0(s_1(a)), s_0(s_1(s_2(a))), \dots$$

*then consider the subshift Ω_w generated by w (*i.e., *the smallest closed subset of $A^{\mathbb{N}}$ invariant by the shift and containing w), which is a subset of Ω_s.*

Here, we will let the directive sequence act on sequences of infinite words, each word representing a scale on which the corresponding substitution acts.

A *word sequence* is an element $u = (u_k)$ of $(A^{\mathbb{N}})^{\mathbb{N}}$. Directive sequences act naturally on word sequences as follows:

$$\left(\begin{array}{ccc} S^{\mathbb{N}} \times (A^{\mathbb{N}})^{\mathbb{N}} & \longrightarrow & (A^{\mathbb{N}})^{\mathbb{N}} \\ (s, (u_k)_{k\in\mathbb{N}}) & \longmapsto & (s_k(u_{k+1}))_{k\in\mathbb{N}} \end{array} \right).$$

Definition 12. *A* fixed point *of a directive sequence s is a fixed point for the above action, that is, a word sequence u satisfying:*

$$\forall k \in \mathbb{N},\ s_k(u_{k+1}) = u_k.$$

We denote by $\mathrm{Fix}(s)$ *the set of fixed points of s.*

Example 18 gives an example of a fixed point of a directive sequence.

Directive sequences always admit fixed points. Indeed, choose a letter $a \in A$ and for each $n \in \mathbb{N}$, consider the word sequence $u^{(n)} = (u_k^{(n)})_{k\in\mathbb{N}}$ defined by $u_k^{(n)} = a^\omega$ when $k \geq n$ and $u_k^{(n)} = s_{[k,n)}(a^\omega)$ when $k < n$, where $s_{[k,n)} = s_k \circ \dots \circ s_{n-1}$. Then let u be a limit point of this sequence of word sequences when n tends to infinity, in the compact space $(A^{\mathbb{N}})^{\mathbb{N}}$ (with the coordinatewise topology). This u is a fixed point of s.

Fixed points of a directive sequence are not unique in general.

This generalizes both the notion of fixed point and the notion of periodic point for a single substitution σ. Let σ^ω denote the constant directive sequence with all terms equal to σ. Similarly, for $v \in A^{\mathbb{N}}$, let v^ω denote the word sequence with all terms equal to v.

Lemma 13. *Let $\sigma \in S$ be a substitution. We have*

- *$v \in A^{\mathbb{N}}$ is a fixed point of σ if, and only if, $v^\omega \in (A^{\mathbb{N}})^{\mathbb{N}}$ is a fixed point of σ^ω;*
- *if $u \in (A^{\mathbb{N}})^{\mathbb{N}}$ is a fixed point of σ^ω, then u_0 is a periodic point of σ, whose period divides* $\mathrm{lcm}(\{1, \dots, d+1\})$.

More generally, assume that u is a fixed point of a directive sequence s, and that there is a positive integer m such that s is periodic with period m, i.e., *$s_n = s_{n+m}$ for all $n \in \mathbb{N}$. Then u_0 is a periodic point of the substitution $s_{[0,m)}$.*

Proof. The first point is clear. Let u be a fixed point of σ^ω, and for all $n \in \mathbb{N}$, let $a_n \in A$ be the first letter of u_n. Then, the sequence (a_n) is periodic, with a period $k \leq d+1$ since a_n is completely determined by a_{n+1}. Now, if $\lim_{n\to\infty} |\sigma^n(a_0)| = \infty$, then $\sigma^{nk}(a_0)$ converges as n tends to infinity to the infinite word $u_0 = u_k$, so u_0 is a periodic point of σ. Otherwise, for all $n \in \mathbb{N}$, $\sigma^n(a_n) = a_0$, and the word sequence $(Tu_n)_{n\in\mathbb{N}}$ is also a fixed point of σ^ω, where $T\colon A^\mathbb{N} \to A^\mathbb{N}$ is the shift map. If we iterate the argument and take the least common multiple of the periods obtained (each being bounded by $d+1$), it gives a period m for which u_0 is a fixed point of σ^m.

If $u = (u_n)$ is a fixed point of $s = (s_n)$, then $(u_{mn})_{n\in\mathbb{N}}$ is a fixed point of the directive sequence $(s_{[mn,mn+m)})_{n\in\mathbb{N}}$. When s has period m, then $(s_{[mn,mn+m)})_{n\in\mathbb{N}} = \sigma^\omega$, where $\sigma = s_{[0,m)}$, and by the previous result it follows that u_0 is a periodic point of σ. □

Definition 14. *For a fixed point $u \in (A^\mathbb{N})^\mathbb{N}$ of a directive sequence s, we define the subshift Ω_u as the subshift Ω_{u_0}, that is the smallest closed subset of $A^\mathbb{N}$ invariant by the shift and containing u_0.*

We extend Definitions 3 and 4 to directive sequences:

Definition 15. *We say that a directive sequence $s \in S^\mathbb{N}$ is* primitive*,* cone convergent*,* sum convergent*, or* exponentially convergent *if the sequence of matrices* $(\mathrm{ab}(s_n))$ *has the said property (see Definition 4). When s is cone convergent, the direction of* $(\mathrm{ab}(s_n))$ *is simply called the* direction of s.

Definition 16. *We say that a directive sequence $s \in S^\mathbb{N}$ is* everywhere growing *if for every $a \in A$, we have*

$$\lim_{n\to\infty} \left|s_{[0,n)}(a)\right| = \infty.$$

It is equivalent to say that the 1-norm of each column of the matrix $\mathrm{ab}(s_{[0,n)})$ *tends to infinity.*

Remark that if a directive sequence s is primitive, then for all $k \in \mathbb{N}$, and all $a \in A$, we have $\left|s_{[k,n)}(a)\right| \xrightarrow[n\to\infty]{} \infty$. In particular, s is everywhere growing.

Proposition 17. *Let $s \in S^\mathbb{N}$ be a primitive directive sequence. Then the subshift Ω_s is minimal. In particular, for every fixed point $u \in (A^\mathbb{N})^\mathbb{N}$ of s, we have $\Omega_u = \Omega_s$. Thus, Ω_u does not depend on the choice of the fixed point u.*

Proof. Let w and $w' \in \Omega_s$ be two words of the subshift. Let p be a prefix of w. Then, there exist $n \in \mathbb{N}$ and $a \in A$ such that p is a factor of $s_{[0,n)}(a)$. Using primitivity, let $N \geq n$ such that $\mathrm{ab}(s_{[n,N)}) > 0$. Now, take a factor f of w' of length at least $2\max_{c\in A}\left|s_{[0,N)}(c)\right|$. There exist $k \in \mathbb{N}$ and $b \in A$ such that f is a factor of $s_{[0,k)}(b)$. Necessarily, we have $k \geq N$, and $s_{[0,k)}(b)$ is a concatenation of words $s_{[0,N)}(c)$, for each letter c of $s_{[N,k)}(b)$. Hence, there exists $c \in A$ such that $s_{[0,N)}(c)$ is a factor of f. Then, the letter a appears in the word $s_{[n,N)}(c)$. So p is a factor of $s_{[0,n)}(a)$ which is a factor of $s_{[0,N)}(c)$ which is a factor of f which is a factor of w'. We conclude that for every $w, w' \in \Omega_s$, every prefix of w is a factor of w', thus the subshift Ω_s is minimal.

To end the proof, remark that for any fixed point u of s, we have $u_0 \in \Omega_s$ since $\lim_{n\to\infty} s_{[0,n)}(a_n) = u_0$, where a_n is the first letter of u_n. Hence, by minimality we get $\Omega_u = \Omega_s$. □

Let us give an example of a fixed point of a directive sequence. The reader will recognize that each u_k is a Sturmian word, see [46].

Example 18. *Let $S = \{\tau_0, \tau_1\}$, with $\tau_0 = \begin{pmatrix} 0 & \mapsto & 0 \\ 1 & \mapsto & 01 \end{pmatrix}$, $\tau_1 = \begin{pmatrix} 0 & \mapsto & 10 \\ 1 & \mapsto & 1 \end{pmatrix}$, and let us consider the directive sequence $s = \tau_0\tau_1\tau_1\tau_0\tau_0\tau_1\tau_0\tau_0\tau_1\ldots$. Then, there exists a fixed point $u \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ of s beginning with*

$$\begin{aligned}
u_0 &= 0101001010010101001010010100101010010100101001010 1\ldots \\
u_1 &= 110110111011011011101101101110110111011011011101 10\ldots \\
u_2 &= 1010110101011010101101011010101101010110101101010 1\ldots \\
u_3 &= 0010001000100100010001001000100010001001000100010 0\ldots \\
u_4 &= 0100100101001001010010010010100100101001001001010 0\ldots \\
u_5 &= 1010110101101010110101101010110101101010110101101 0\ldots \\
u_6 &= 0010010001001000100100010010010001001000100100010 0\ldots
\end{aligned}$$

Fixed points of directive sequences encompass both the time and scale dynamics in a single object. In the context of this paper, the time dynamics corresponds to the action of the translation on the torus and the scale dynamics corresponds to the action of the continued fraction algorithm on the space of translations. Symbolically, the shift map on Ω_{u_0} encodes the time dynamics, while shifting the fixed point $(u_k) \mapsto (u_{k+1})$ corresponds to accelerating the time dynamics.

In Example 18 above, we can visualize how fixed points grasp the multi-scale structure of the dynamical system with the following alignment:

$u_0 =$ 01010010100101010010100101001010100101001010010101...
$u_1 =$ 1 1 01 1 01 1 1 01 1 01 1 01 1 1 01 1 01 1 01 1 1 ...
$u_2 =$ 1 0 1 0 1 1 0 1 0 1 0 1 1 0 1 0 1 0 1 1 0 ...
$u_3 =$ 0 0 1 0 0 0 1 0 0 0 1 0 ...
$u_4 =$ 0 1 0 0 1 0 0 1 0 ...
$u_5 =$ 1 0 1 0 1 1 ...
$u_6 =$ 0 0 1 0 ...

As we will see in Section 9, when the substitutions enjoy some recognizability properties, the scale dynamics corresponds to inducing on some atoms of the partition.

Rokhlin towers and ordered Bratteli diagrams are other combinatorial objects that account for the multi-scale structure of dynamical systems [26]. An ordered Bratteli diagram $\mathcal{B}$ can be associated with a directive sequence s. When s is everywhere growing, the minimal infinite paths of $\mathcal{B}$ are in bijective correspondence with the fixed points u of s: the kth edge of the infinite path is encoded by the first letter of the word u_k.

3.7 Topologies on the integer half-space and worms

Let us define the *integer half-space*: $\mathbb{H} = \{z \in \mathbb{Z}^{d+1} \mid h(z) \geq 0\}$. For $i \in \mathbb{N}$, let $\mathbb{H}_i = \{z \in \mathbb{Z}^{d+1} \mid h(z) = i\} = \Lambda + ie_0$.

Remark 19. *When x is a totally irrational direction, then $x \cap \mathbb{Z}^{d+1} = \emptyset$ (recall that a direction $x \in \mathbb{PR}^d_+$ is a half-line $x \subseteq \mathbb{R}^{d+1}_+ \setminus \{0\}$), the projection π_x is injective on $\mathbb{Z}^{d+1}$, and $\pi_x(\mathbb{H})$ is dense in P (this follows from the minimality of T_x, see Section 3.2).*

The following two definitions are crucial in the rest of the paper.

Definition 20. *For any fixed $x \in \mathbb{PR}^d_+$, we define the topology $\mathcal{T}(x)$ on $\mathbb{H}$: a subset $V \subseteq \mathbb{H}$ is* open *if there exists an open set $U \subseteq P$ such that $V = \pi_x^{-1}(U) \cap \mathbb{H}$.*

Remark that $\mathcal{T}(x)$ is the finest topology on $\mathbb{H}$ that makes $\pi_x \colon \mathbb{H} \to P$ continuous. It is metrizable if, and only if, $x \cap \mathbb{H} = \emptyset$, which is the case if x is a totally irrational direction.

We introduce the notion of *worm*:

Definition 21. *Given an infinite word $w \in A^{\mathbb{N}}$, its* worm *is the set*

$$W(w) = \{\mathrm{ab}(p) \mid p \text{ prefix of } w\} \subseteq \mathbb{N}^{d+1} \subseteq \mathbb{H}.$$

For a worm and a letter a we also define the subset

$$W_a(w) = \{\mathrm{ab}(p) \mid pa \text{ prefix of } w\} = W(w) \cap (W(w) - e_a).$$

Remark 22. *The subsets $W_a(w)$, $a \in A$, form a partition of $W(w)$: $W(w) = \sqcup_{a\in A} W_a(w)$.*

An example of worm is depicted in Figure 1.

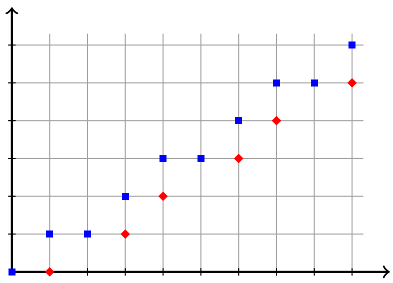

Figure 1: Worm of the word $w = (01001)^\omega$. $W_0(w)$ in blue squares and $W_1(w)$ in red diamonds.

Lemma 23 (tiling)**.** *A worm W tiles the integer half-space by translations: $\mathbb{H} = W \oplus \Lambda$.*

Figure 2 shows an example of such a tiling by the worm $(01001)^\omega$, for $d = 1$.

Proof. The lattice Λ acts on $\mathbb{H}$ by translation. The orbits of this action are the cosets $\mathbb{H}_i$. A worm intersects each coset exactly once. □

Lemma 24 (automatic balance)**.** *If a worm $W(w)$ has non-empty interior for some topology $\mathcal{T}(x)$ with $x \in \mathbb{PR}^d_+$, then $\pi_x(W(w))$ is bounded.*

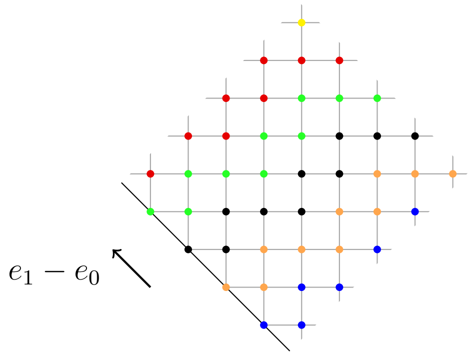

Figure 2: Tiling of $\mathbb{H}$ by a worm, by translation by the group $\Lambda = \langle e_1 - e_0 \rangle$.

Proof. Let U be an open subset of P such that $\emptyset \neq \pi_x^{-1}(U) \cap \mathbb{H} \subseteq W(w)$. Up to restricting it, we assume that U is included in an open ball $B(0, r)$ for some $r > 0$.

Let us consider the translation by $q(\pi_x(e_0))$ on the d-dimensional torus P/Λ:

$$T_x = T_{q(\pi_x(e_0))} = \begin{pmatrix} P/\Lambda & \longrightarrow & P/\Lambda \\ z & \longmapsto & z + q(\pi_x(e_0)) \end{pmatrix}$$

For any integer i, $\pi_x^{-1}(U)$ intersects $\mathbb{H}_i$ if, and only if, $T_x^i(0)$ belongs to the open subset $q(U)$ of the torus. The translation T_x acts minimally on every orbit closure. By hypothesis, the open set $q(U)$ intersects the orbit of 0, hence $T_x^i(0)$ belongs to $q(U)$ for i in a syndetic subset of $\mathbb{N}$: $\exists K \geq 1, \forall i \in \mathbb{N}, \exists 0 < k \leq K, T_x^{i+k}(0) \in q(U)$ [32]. If, for each integer i, we denote by m_i the single element of $W(w) \cap \mathbb{H}_i$, we have $\|m_{i+1} - m_i\|_1 = 1$. Consider a point m_i of $W(w)$. Then there exists $0 < k \leq K$ such that $T_x^{i+k}(0) \in q(U)$. Therefore, $\pi_x^{-1}(U)$ intersects $\mathbb{H}_{i+k}$, and as $\pi_x^{-1}(U) \subseteq W(w)$ this intersection must be $\{m_{i+k}\}$. As $\|m_{i+k} - m_i\|_1 \leq k \leq K$, it follows that m_i is at distance at most K of a point of $\pi_x^{-1}(U)$ (see Figure 3).

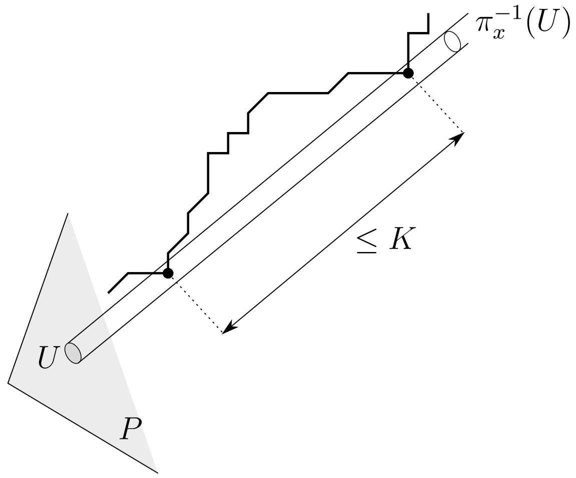

Figure 3: A worm cannot escape too far between consecutive interior points.

Since the direction x is in $\mathbb{PR}_+^d$, the projection π_x is 2-Lipschitz. Hence, $\pi_x(W(w)) \subseteq B(0, r + 2K)$, which concludes the proof. $\square$

Lemma 25 (uniform automatic balance)**.** *If there exist a radius $r > 0$, a sequence of directions $x^{(n)} \in \mathbb{PR}^d_+$ such that $x^{(n)} \xrightarrow[n\to\infty]{} x^\infty$ with x^∞ a totally irrational direction, and a sequence of worms $W(u_n)$ such that $\forall n \in \mathbb{N}$, $\exists c_n \in P$, $\pi^{-1}_{x^{(n)}}(B(c_n, r)) \cap \mathbb{H} \subseteq W(u_n)$, then $\pi_{x^{(n)}}(W(u_n))$ is uniformly bounded for $n \in \mathbb{N}$.*

Proof. Following the proof of the previous lemma, we consider the translation T_{x^∞} by $q(\pi_{x^\infty}(e_0))$ on the torus P/Λ. For every $n \in \mathbb{N}$, let $B_n = q(B(c_n, r))$ and $B'_n = q(B(c_n, r/2))$. Since x^∞ is a totally irrational direction, the translation T_{x^∞} acts minimally on the whole torus P/Λ. Moreover, every point of P/Λ comes back in uniform finite time in every B'_n. More precisely, there exists a constant K such that for all $n \in \mathbb{N}$ and for all $y \in P/\Lambda$, there exists $0 < k \leq K$ such that $T^k_{x^\infty}(y) \in B'_n$. If we take n_0 such that for all $n \geq n_0$, $d(\pi_{x^{(n)}}(e_0), \pi_{x^\infty}(e_0)) = d(x^{(n)}, x^\infty) \leq \frac{r}{2K}$, then for all $y \in P/\Lambda$ and all $n \geq n_0$, there exists $0 < k \leq K$ such that $T^k_{x^{(n)}}(y) \in B_n$. Hence, we deduce as in the proof of the previous lemma that for all $n \geq n_0$, $\pi_{x^{(n)}}(W(u_n)) \subseteq B(c_n, r + 2K)$. In particular, $0 \in B(c_n, r + 2K)$. Thus, $\pi_{x^{(n)}}(W(u_n)) \subseteq B(0, 2r + 4K)$. The result follows since $\pi_{x^{(n)}} W(u_n)$ is bounded for $n < n_0$ by Lemma 24. □

The following lemma is useful to propagate non-emptiness of the interior of a subset of $\mathbb{H}$ when an affine map is applied.

Lemma 26. *Let $W \subseteq \mathbb{N}^{d+1} \subseteq \mathbb{H}$, let $x \in \mathbb{PR}^d_+$ be a totally irrational direction, let $t \in \mathbb{H}$, and let $M \in GL_{d+1}(\mathbb{Z}) \cap \mathcal{M}_{d+1}(\mathbb{N})$. If W has non-empty interior for the topology $\mathcal{T}(x)$, then $MW + t$ has non-empty interior for the topology $\mathcal{T}(Mx)$.*

Proof. Let $U \subseteq P$ be a bounded non-empty open subset such that $\pi^{-1}_x(U) \cap \mathbb{H} \subseteq W$. We have $M \in GL_{d+1}(\mathbb{Z})$, so $M^{-1}\mathbb{Z}^{d+1} = \mathbb{Z}^{d+1}$, and $M^{-1}(\mathbb{H} - t)$ is a half-space

$$M^{-1}(\mathbb{H} - t) = \{z \in \mathbb{Z}^{d+1} \mid h(Mz + t) \geq 0\} = \{z \in \mathbb{Z}^{d+1} \mid \sum_{i=0}^{d} \alpha_i z_i \geq \beta\},$$

for some integer coefficients α_i and β. Since the matrix M is non-negative and invertible, we have $\alpha_i = h(Me_i) > 0$ for all i, so $v(x)$ is in the half-space $\{z \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} \alpha_i z_i > 0\}$. For every $t' \in \mathbb{R}^{d+1}$, the intersection of the line $t' + \mathbb{R}v(x)$ with the set

$$\{z \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} \alpha_i z_i \geq \beta\} \setminus \{z \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} z_i \geq 0\}$$

is included in an interval with one extremity in P and with a finite length that depends continuously on t'. Using that moreover U is bounded we deduce that the set

$$L = \pi^{-1}_x(U) \cap M^{-1}(\mathbb{H} - t) \setminus \mathbb{H} \subseteq \mathbb{Z}^{d+1}$$

is finite. Moreover, we have $\pi^{-1}_x(\pi_x(L)) \cap M^{-1}\mathbb{H} = L$ since π_x is injective on $\mathbb{Z}^{d+1}$, and we have $M\pi^{-1}_x(U \setminus \pi_x(L)) = \pi^{-1}_{Mx}(\pi_{Mx}(M(U \setminus \pi_x(L))))$, so

$$\begin{aligned} MW + t &\supseteq M(\pi^{-1}_x(U) \cap \mathbb{H}) + t \\ &\supseteq M(\pi^{-1}_x(U) \cap (M^{-1}(\mathbb{H} - t) \setminus L) + t \\ &= (M(\pi^{-1}_x(U \setminus \pi_x(L))) + t) \cap \mathbb{H} \\ &= \pi^{-1}_{Mx}(\pi_{Mx}(M(U \setminus \pi_x(L))) + \pi_{Mx}(t)) \cap \mathbb{H}. \end{aligned}$$

Finally $\pi_{Mx}(M(U \setminus \pi_x(L))) \neq \emptyset$ is open, so $MW + t$ has non-empty interior for $\mathcal{T}(Mx)$. □

Remark 27. *The statement and the proof of Lemma 26 would have been immediate for the topology defined by the open sets $\pi_x^{-1}(U) \cap \mathbb{Z}^{d+1}$ rather than $\pi_x^{-1}(U) \cap \mathbb{H}$.*

We finish this subsection with a result that shows how to relate properties of the worm to some combinatorial properties of the infinite word:

Proposition 28. *An infinite word $w \in A^{\mathbb{N}}$ is balanced if, and only if, there exists a direction $x \in \mathbb{PR}_+^d$ such that $\pi_x(W(w))$ is bounded.*

Proof. First of all, remark that a word u is balanced if, and only if, there exists a constant K such that, for any two factors f_1, f_2 of w, $\|\mathrm{ab}(f_1)\|_1 = \|\mathrm{ab}(f_2)\|_1$ implies $\|\mathrm{ab}(f_1) - \mathrm{ab}(f_2)\|_1 \leq K$.

Assume that $\pi_x(W(w)) \subseteq B(0, L)$. First of all remark that for a finite word f its length fulfills $|f| = h(\mathrm{ab}(f)) = \|\mathrm{ab}(f)\|_1$. Moreover, if p is a prefix of w, then we have $\mathrm{ab}(p) - |p|\, v(x) = \pi_x(\mathrm{ab}(p)) \in \pi_x(W(w))$, so we get $\|\mathrm{ab}(p) - |p|\, v(x)\|_1 \leq L$.

Let f be a factor of w and let p be a prefix of w such that pf is a prefix of w. Since $\mathrm{ab}(f) = \mathrm{ab}(pf) - \mathrm{ab}(p)$ and $|f| = |pf| - |p|$ we obtain:

$$\|\mathrm{ab}(f) - |f|\, v(x)\|_1 \leq \|\mathrm{ab}(pf) - |pf|\, v(x)\|_1 + \|\mathrm{ab}(p) - |p|\, v(x)\|_1 \leq 2L.$$

Thus, if we consider two factors f_1 and f_2 of w of the same length, we deduce:

$$\|\mathrm{ab}(f_1) - \mathrm{ab}(f_2)\|_1 \leq \|\mathrm{ab}(f_1) - |f_1|\, v(x)\|_1 + \|\mathrm{ab}(f_2) - |f_2|\, v(x)\|_1 \leq 4L.$$

Thus, the word w is balanced.

Now, assume that w is balanced, and let K be such that for any two factors f_1, f_2 of w, $\|\mathrm{ab}(f_1)\|_1 = \|\mathrm{ab}(f_2)\|_1$ implies $\|\mathrm{ab}(f_1) - \mathrm{ab}(f_2)\|_1 \leq K$. Let p_n be the prefix of w of length n. For every $k \geq 1$, by cutting p_n into $\lfloor \frac{n}{k} \rfloor$ parts of length k and a remaining factor of length less than k, we get

$$\left\| \mathrm{ab}(p_n) - \left\lfloor \frac{n}{k} \right\rfloor \mathrm{ab}(p_k) \right\|_1 \leq K \left\lfloor \frac{n}{k} \right\rfloor + k,$$

so that

$$\left\| \frac{1}{n} \mathrm{ab}(p_n) - \frac{1}{k} \mathrm{ab}(p_k) \right\|_1 \leq \frac{K}{k} + \frac{2k}{n}.$$

Hence, for every $N \geq n \geq 1$ and every $k \geq 1$, we have

$$\left\| \frac{1}{n} \mathrm{ab}(p_n) - \frac{1}{N} \mathrm{ab}(p_N) \right\|_1 \leq \frac{2K}{k} + \frac{4k}{n}.$$

Thus, by taking $k = \lfloor \sqrt{n} \rfloor$, we see that $(\frac{1}{n} \mathrm{ab}(p_n))_{n \geq 1}$ is a Cauchy sequence, so it converges to some vector $v \in \Delta$. Then, for every $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \big(2\, \mathrm{ab}(p_{n2^k}) - \mathrm{ab}(p_{n2^{k+1}})\big) = \mathrm{ab}(p_n) - nv,$$

since $\lim_{k\to\infty} \frac{1}{2^k} \operatorname{ab}(p_{n2^k}) = nv$. Moreover,

$$\|2\operatorname{ab}(p_n) - \operatorname{ab}(p_{2n})\|_1 = \|\operatorname{ab}(p_n) - \operatorname{ab}(q_n)\|_1 \leq K,$$

where q_n is such that $p_{2n} = p_n q_n$. It follows that, for all $n \in \mathbb{N}$,

$$\|\pi_v(\operatorname{ab}(p_n))\|_1 = \|\operatorname{ab}(p_n) - nv\|_1 \leq \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \|2\operatorname{ab}(p_{n2^k}) - \operatorname{ab}(p_{n2^{k+1}})\|_1 \leq \sum_{k=0}^{\infty} \frac{K}{2^{k+1}} = K,$$

Hence, $\pi_v(W(w)) \subseteq B(0, K+1)$. □

3.8 Worms and Dumont-Thomas numeration

In all the following we consider $S \subseteq \hom(A^+, A^+)$ a finite set of unimodular substitutions on the alphabet A. We give a definition of the Dumont-Thomas numeration, which is a generalization, for a finite set S of substitutions, of the one given for a single substitution in [25].

Definition 29. *The* Dumont-Thomas alphabet *of S is defined as*

$$\Sigma = \{\operatorname{ab}(p) \mid \exists \sigma \in S,\ \exists a, b \in A,\ pb \text{ prefix of } \sigma(a)\} \subseteq \mathbb{N}^{d+1}.$$

Remark that the Dumont-Thomas alphabet is a finite set, since S and A are finite. We also introduce an automaton:

Definition 30. *We call* abelianized prefix automaton *of the set of substitutions S, the automaton $\mathcal{A}$ defined as follows:*

- *the alphabet is $\Sigma \times S$,*
- *the set of states is A,*
- *there is a transition $a \xrightarrow{t,\sigma} b$, with $(a, t, \sigma, b) \in A \times \Sigma \times S \times A$ if, and only if, there exist $p, v \in A^*$ such that $\sigma(a) = pbv$, with* $\operatorname{ab}(p) = t$.

We write $a \xrightarrow{t_n, s_n} \ldots \xrightarrow{t_0, s_0} b$ if we have a path of length $n+1$ in the automaton $\mathcal{A}$: there exist states $a = a_{n+1}, a_n, \ldots, a_1, a_0 = b$ such that for every $0 \leq k \leq n$, there is a transition $a_{k+1} \xrightarrow{t_k, s_k} a_k$. Similarly, we write $\ldots \xrightarrow{t_n, s_n} \ldots \xrightarrow{t_0, s_0} b$ if there is a left-infinite path in $\mathcal{A}$ ending in b.

The automaton $\mathcal{A}$ is depicted in Figure 10 for the set of Arnoux-Rauzy substitutions, and in Figure 11 for the set of Cassaigne substitutions.

The following lemma, whose proof is left to the reader, follows from the construction of the abelianized prefix automaton.

Lemma 31. *For every $w \in A^{\mathbb{N}}$, $\sigma \in S$, and $a \in A$, the following relation holds*

$$W_a(\sigma(w)) = \bigcup_{b \xrightarrow{t,\sigma} a} (\operatorname{ab}(\sigma) W_b(w) + t)\,.$$

Example 32. *Let* $S = \{\tau_0, \tau_1\}$*, with* $\tau_0 = \begin{pmatrix} 0 & \mapsto & 0 \\ 1 & \mapsto & 01 \end{pmatrix}$ *and* $\tau_1 = \begin{pmatrix} 0 & \mapsto & 10 \\ 1 & \mapsto & 1 \end{pmatrix}$ *as in Example 18. Then, the Dumont-Thomas alphabet is* $\Sigma = \{0, e_0, e_1\}$*, where* (e_0, e_1) *is the canonical basis of* $\mathbb{R}^2$*, and the abelianized prefix automaton* $\mathcal{A}$ *is depicted in Figure 4. For every word* $w \in \{0,1\}^{\mathbb{N}}$ *we have the relations*

$$\begin{aligned} W_0(\tau_0(w)) &= \mathrm{ab}(\tau_0)W_0(w) \sqcup \mathrm{ab}(\tau_0)W_1(w) \\ W_1(\tau_0(w)) &= \mathrm{ab}(\tau_0)W_1(w) + e_0 \\ W_0(\tau_1(w)) &= \mathrm{ab}(\tau_1)W_0(w) + e_1 \\ W_1(\tau_1(w)) &= \mathrm{ab}(\tau_1)W_0(w) \sqcup \mathrm{ab}(\tau_1)W_1(w). \end{aligned}$$

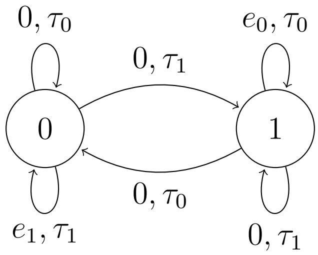

Figure 4: Abelianized prefix automaton $\mathcal{A}$ for the set of substitutions S of Example 32.

By iterating Lemma 31, we obtain expressions for the worms associated with a fixed point of a directive sequence.

Lemma 33. *Let* $s \in S^{\mathbb{N}}$ *be a directive sequence and* (M_k) *the corresponding matrix sequence. Consider a fixed point* $u \in \mathrm{Fix}(s)$*. For all integers* k*,* l *such that* $0 \leq k \leq l$*, and every* $a \in A$*, we have*

$$W_a(u_k) = \bigcup_{b \xrightarrow{t_{l-1}, s_{l-1}} \dots \xrightarrow{t_k, s_k} a} M_{[k,l)} W_b(u_l) + \sum_{i=k}^{l-1} M_{[k,i)} t_i.$$

Lemma 34 (Dumont-Thomas numeration)**.** *Let* $s \in S^{\mathbb{N}}$ *be a directive sequence and* $u \in (A^{\mathbb{N}})^{\mathbb{N}}$ *be a fixed point of* s*. Let* b_n *be the first letter of the word* u_n*. We assume that* $\left|s_{[0,n)}(b_n)\right| \xrightarrow[n\to\infty]{} \infty$*. Then, for every* $a \in A$ *we have*

$$W_a(u_0) = \bigcup_{n \in \mathbb{N}} \{\sum_{i=0}^{n-1} M_{[0,i)} t_i \mid b_n \xrightarrow{t_{n-1}, s_{n-1}} \dots \xrightarrow{t_0, s_0} a\}.$$

Proof. Applying Lemma 33 with $k = 0$ and $l = n$, we get

$$W_a(u_0) = \bigcup_{b \xrightarrow{t_{n-1}, s_{n-1}} \dots \xrightarrow{t_0, s_0} a} M_{[0,n)} W_b(u_n) + \sum_{i=0}^{n-1} M_{[0,i)} t_i.$$

As b_n is the first letter of u_n, we have $0 \in W_{b_n}(u_n)$, so that

$$\sum_{i=0}^{n-1} M_{[0,i)} t_i \in W_a(u_0)$$

when $b_n \xrightarrow{t_{n-1},s_{n-1}} \ldots \xrightarrow{t_0,s_0} a$.

Conversely, recall that $W_a(u_0) = \{\mathrm{ab}(p) \mid pa \text{ prefix of } u_0\}$, and consider such a prefix pa. The hypothesis that $\left|s_{[0,n)}(b_n)\right|$ is unbounded ensures that there exists n such that pa is a prefix of $s_{[0,n)}(b_n)$. Then there is a path $b_n \xrightarrow{t_{n-1},s_{n-1}} \ldots \xrightarrow{t_0,s_0} a$ in $\mathcal{A}$ such that $p = s_{[0,n-1)}(p_{n-1})\ldots s_0(p_1)p_0$, where $t_i = \mathrm{ab}(p_i)$, and thus $\mathrm{ab}(p) = \sum_{i=0}^{n-1} M_{[0,i)}t_i$. □

Finally, the following lemma follows from the definition of transitions in $\mathcal{A}$.

Lemma 35. *Let $s \in S^{\mathbb{N}}$ be a directive sequence. For every $a, b \in A$, the number of paths $b \xrightarrow{t_{l-1},s_{l-1}} \ldots \xrightarrow{t_k,s_k} a$ in the automaton $\mathcal{A}$ is equal to $(M_{[k,l)})_{a,b} = \left|s_{[k,l)}(b)\right|_a$.*

3.9 Rauzy fractals

In this section, we generalize the classical notion of Rauzy fractals, and we give some useful properties.

Definition 36. *Let $w \in A^{\mathbb{N}}$ be an infinite word admitting a frequency vector $v = \mathrm{freq}(w)$ (see Section 3.4). We define $R(w)$ as the closure of $\pi_v W(w) \subseteq P$. For a letter $a \in A$, we also define $R_a(w)$ as the closure of $\pi_v W_a(w)$. The set $R(w)$ is called* Rauzy fractal *and the subsets $R_a(w)$ are called its* pieces.

Our definition of a Rauzy fractal is a generalization of the classical notion for a fixed point of a substitution, see [48, 46] for references. Note however that in those classical definitions, the projection hyperplane is often chosen to be orthogonal to the frequency vector v. In our setting, we want to be able to compare Rauzy fractals associated with words with different frequency vectors, so it is more convenient to project onto a fixed hyperplane P.

Example 37. *For $w = (01001)^\omega$, we have $\mathrm{freq}(w) = (3/5, 2/5)$. So we can define the Rauzy fractal by projecting on the hyperplane (*i.e., *line) $x + y = 0$, and we get a Rauzy fractal with only 5 points. See Figure 5.*

Using Rauzy fractals, we can give a characterization of the interior of $W_a(w)$ for the topology $\mathcal{T}(x)$, with the following lemma.

Lemma 38. *For every open subset B of the hyperplane P, for every totally irrational direction $x \in \mathbb{PR}_+^d$, for every infinite word $w \in A^{\mathbb{N}}$ admitting a frequency vector $\mathrm{freq}(w) \in x$ and for every letter $a \in A$, we have the equivalence between*

1. *$\mathbb{H} \cap \pi_x^{-1}(B) \subseteq W_a(w)$,*
2. *$\forall b \in A \setminus \{a\},\ \forall t \in \Lambda \setminus \{0\},\ B \cap R_b(w) = \emptyset = B \cap (R(w) + t)$.*

In particular, if $R(w)$ is bounded, then $p \in \mathbb{H}$ is in the interior of $W_a(w)$ for the topology $\mathcal{T}(x)$ if, and only if,

$$\pi_x(p) \notin \bigcup_{b \in A\setminus\{a\}} R_b(w) \cup \bigcup_{t \in \Lambda\setminus\{0\}} R(w) + t.$$

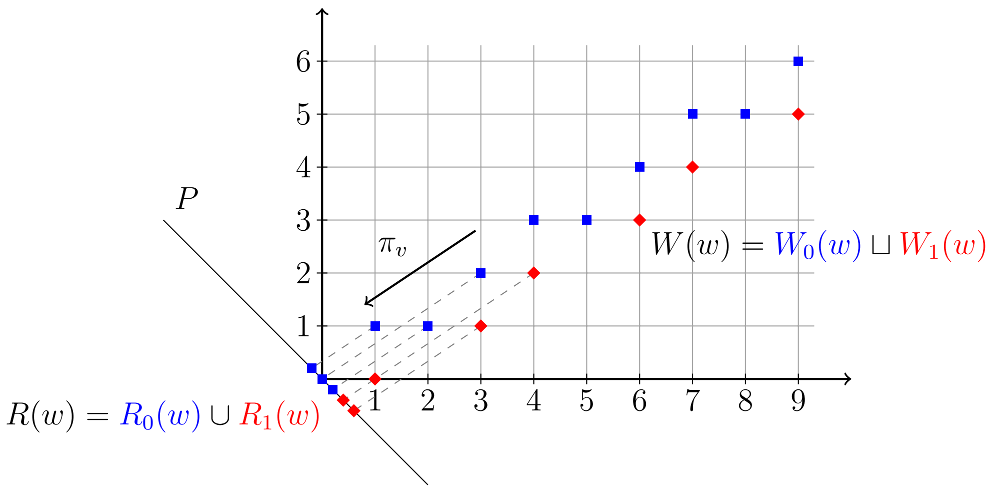

Figure 5: The Rauzy fractal $R(w)$ as the closure of the projection of the worm $W(w)$ on the hyperplane P. Example for $w = (01001)^\omega$, so $v = (3/5, 2/5)$.

Proof. By Remark 22, we have

$$W_a(w) = \mathbb{H} \setminus \left(\bigcup_{b \in A \setminus \{a\}} W_b(w) \cup \bigcup_{t \in \Lambda \setminus \{0\}} W(w) + t \right),$$

As π_x is injective on $\mathbb{H}$ by Remark 19, and B is open, we have the equivalences

$$\begin{aligned} \mathbb{H} \cap \pi_x^{-1}(B) \subseteq W_a(w) &\iff B \cap \pi_x(\mathbb{H}) \subseteq \pi_x(W_a(w)) \\ &\iff B \cap \pi_x \left(\bigcup_{b \in A \setminus \{a\}} W_b(w) \cup \bigcup_{t \in \Lambda \setminus \{0\}} W(w) + t \right) = \emptyset \\ &\iff B \cap \left(\bigcup_{b \in A \setminus \{a\}} R_b(w) \cup \bigcup_{t \in \Lambda \setminus \{0\}} R(w) + t \right) = \emptyset. \end{aligned}$$

When $R(w)$ is bounded, the set $\bigcup_{b \in A \setminus \{a\}} R_b(w) \cup \bigcup_{t \in \Lambda \setminus \{0\}} R(w) + t$ is closed since it is a locally finite union. Let B be its complement: then the interior of $W_a(w)$ for the topology $\mathcal{T}(x)$ is $\mathbb{H} \cap \pi_x^{-1}(B)$. □

Remark 39. *We emphasize the fact that the interior of $R(w)$ need not correspond to the interior of $W(w)$ for the topology $\mathcal{T}(x)$. As x is a totally irrational direction, if an open set O of P is such that $\pi_x^{-1}(O) \cap \mathbb{H} \subseteq W_a(w)$, then O is included in the interior of $R_a(w)$, but the converse may be false in general.*

Rauzy fractals associated with fixed points of directive sequences play a particular role. We recall the following result, which alreay appears in different form in [31] or [52].

Proposition 40. *[12, Theorem 5.7] Let $s \in S^{\mathbb{N}}$ be a directive sequence, which is everywhere growing and cone convergent with direction x (see Definitions 15 and 16). Then the subshift Ω_s is uniquely ergodic, and for every word $w \in \Omega_s$, we have $\mathrm{freq}(w) = v(x)$. In particular, if $u \in \mathrm{Fix}(s)$, then $\mathrm{freq}(u_0) = v(x)$.*

Corollary 41. *Let $s \in S^{\mathbb{N}}$ be a directive sequence, which is cone convergent with a totally irrational direction x. Then the subshift Ω_s is uniquely ergodic, and for every word $w \in \Omega_s$, we have* $\mathrm{freq}(w) = v(x)$.

Proof. By Lemma 7, total irrationality of x implies primitivity of s. In particular s is everywhere growing and the hypotheses of Proposition 40 are satisfied. □

Remark 42. *If $s \in S^{\mathbb{N}}$ is a cone convergent directive sequence with totally irrational direction x, then, by Corollary 41, for every fixed point $u \in \mathrm{Fix}(s)$, the infinite word u_0 admits $v(x)$ as frequency vector, hence we can define the Rauzy fractal $R(u_0)$.*

The following proposition allows to show that the Rauzy fractal does not depend on the choice of a fixed point u of a directive sequence s, and it gives a useful characterization with left-infinite paths in the abelianized prefix automaton.

Proposition 43. *Let $s \in S^{\mathbb{N}}$. We assume that s is primitive and sum convergent, with direction x. Then, for every letter $a \in A$ and every fixed point $u \in \mathrm{Fix}(s)$, we have*

$$R_a(u_0) = \{\sum_{n=0}^{\infty} \pi_x(M_{[0,n)}(s)t_n) \mid ... \xrightarrow{t_n,s_n} ... \xrightarrow{t_0,s_0} a\}.$$

In particular, the Rauzy fractal does not depend on the choice of the fixed point u and is compact.

Proof. Let u be a fixed point of s. We denote by b_n the first letter of the word u_n. Note that, since s is primitive, $\lim_{n\to\infty} \left|s_{[0,n)}(b_n)\right| = \infty$. By Lemma 34, for every letter $a \in A$, we have the equality

$$R_a(u_0) = \overline{\bigcup_{n\in\mathbb{N}} \{\sum_{k=0}^{n-1} \pi_x(M_{[0,k)}t_k) \mid b_n \xrightarrow{t_{n-1},s_{n-1}} ... \xrightarrow{t_0,s_0} a\}}.$$

Let $Q_a = \{\sum_{k=0}^{\infty} \pi_x(M_{[0,k)}t_k) \mid ... \xrightarrow{t_k,s_k} ... \xrightarrow{t_0,s_0} a\}$. We first show the inclusion $Q_a \subseteq R_a(u_0)$. Let $... \xrightarrow{t_k,s_k} ... \xrightarrow{t_0,s_0} a$ be a left-infinite path in the automaton $\mathcal{A}$. Let $\epsilon > 0$. By the sum convergence hypothesis, and using the fact that the Dumont-Thomas alphabet Σ is finite, there exists $n \in \mathbb{N}$ such that

$$\max_{t\in\Sigma-\Sigma} \|t\|_1 \sum_{k=n}^{\infty} \left|\left|\left|\pi_x M_{[0,k)}\right|\right|\right|_1 \le \epsilon$$

where $\Sigma - \Sigma = \{t - t' \mid t, t' \in \Sigma\}$. There is a path $b \xrightarrow{t_{n-1},s_{n-1}} ... \xrightarrow{t_0,s_0} a$, where b may be different from b_n. Thanks to primitivity and Lemma 35, we extend it to a path $b_N \xrightarrow{t'_{N-1},s_{N-1}} ... \xrightarrow{t'_0,s_0} a$, for some $N > n$ such that $M_{[n,N)} > 0$, with $t'_k = t_k$ when $k < n$. We have

$$\left\|\sum_{k=0}^{\infty} \pi_x(M_{[0,k)}t_k) - \sum_{k=0}^{N-1} \pi_x(M_{[0,k)}t'_k)\right\|_1 \le \max_{t\in\Sigma-\Sigma} \|t\|_1 \sum_{k=n}^{\infty} \left|\left|\left|\pi_x M_{[0,k)}\right|\right|\right|_1 \le \epsilon.$$

Since $\sum_{k=0}^{N-1} \pi_x(M_{[0,k)} t'_k) \in R_a(u_0)$ and since $R_a(u_0)$ is closed, we deduce that

$$\sum_{k=0}^{\infty} \pi_x(M_{[0,k)} t_k) \in R_a(u_0).$$

Therefore, $Q_a \subseteq R_a(u_0)$.

Let us show the other inclusion. We have

$$\bigcup_{n \in \mathbb{N}} \{\sum_{k=0}^{n-1} \pi_x(M_{[0,k)} t_k) \mid b_n \xrightarrow{t_{n-1}, s_{n-1}} \dots \xrightarrow{t_0, s_0} a\} \subseteq Q_a$$

because for every $n \in \mathbb{N}$, there exists a left-infinite path $\dots \xrightarrow{0, s_k} \dots \xrightarrow{0, s_n} b_n$ as b_k is the first letter of $s_k(b_{k+1})$. Therefore, $R_a(u_0) \subseteq \overline{Q}_a$.

To end the proof, it remains to show that the set Q_a is compact. We define a natural distance on the set of left-infinite paths in the automaton $\mathcal{A}$ by taking a distance 2^{-n} between two paths that coincide for the last n transitions. This makes the set of left-infinite paths compact, and the map sending a left-infinite path $\dots \xrightarrow{t_k, s_k} \dots \xrightarrow{t_0, s_0} a$ to the corresponding sum $\sum_{k=0}^{\infty} \pi_x(M_{[0,k)} t_k)$ is continuous by the sum convergence hypothesis. So we get the compactness of Q_a, and consequently $R_a(u_0) = Q_a$. □

Definition 44. *Let s be a directive sequence. When the Rauzy fractal $R(u_0)$ does not depend on the fixed point u of s, we denote it by $R(s)$, or just R when there is no ambiguity on s, and its pieces by $R_a(s)$ or R_a.*

Examples of non-substitutive Rauzy fractals associated with a directive sequence are drawn in Figures 6 and 12.

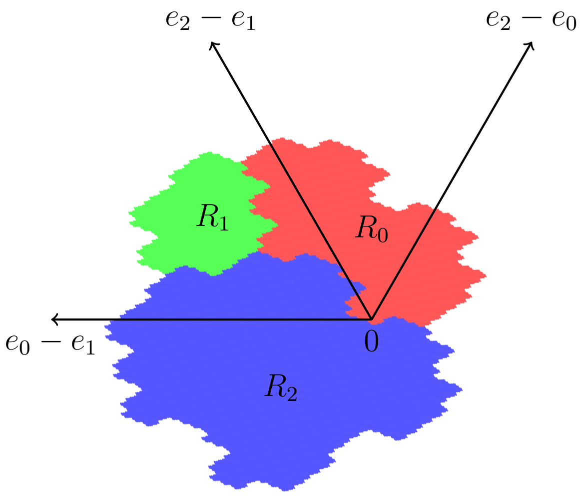

Figure 6: Approximation of the Rauzy fractal of a directive sequence beginning with $c_1c_1c_0c_1c_0c_0c_0c_1c_1c_1c_0c_1c_0c_0c_1c_0c_1c_0c_0c_0c_0c_1c_0c_1c_1c_0c_0c_1c_1c_1c_0c_1c_1c_1c_0c_0c_0c_1c_0c_0c_0c_1c_1c_0c_0c_0$, where c_0 and c_1 are defined in Section 8.
Here, $x = [(0.279291082100669\dots, 0.1294709739854265\dots, 0.5912379439139045\dots)]$.

The primitivity hypothesis of Proposition 43 may be replaced with the hypothesis that x is a totally irrational direction.

Corollary 45. *Let $s \in S^{\mathbb{N}}$ be a sum convergent directive sequence with a totally irrational direction $x \in \mathbb{PR}_+^d$.*

Then, for every letter $a \in A$ and every fixed point $u \in \mathrm{Fix}(s)$, we have

$$R_a(u_0) = \{\sum_{n=0}^{\infty} \pi_x(M_{[0,n)}(s)t_n) \mid ... \xrightarrow{t_n, s_n} ... \xrightarrow{t_0, s_0} a\}.$$

In particular, we have the properties:

- *the Rauzy fractal $R(s)$ and its pieces do not depend on the choice of a fixed point,*
- *$R(s)$ is compact,*
- *$R(s)$ and its translates by Λ cover the hyperplane: $\bigcup_{t\in\Lambda} R(s) + t = P$,*
- *$R(s)$ has non-empty interior.*

Proof. Thanks to Lemma 5 and Lemma 7, s is primitive, so we can apply Proposition 43, hence we deduce the formula, the fact that the Rauzy fractal and its pieces do not depend on the choice of a fixed point, and their compactness. Now, for any fixed point $u \in \mathrm{Fix}(s)$, we have $W(u_0) \oplus \Lambda = \mathbb{H}$, and $\pi_x(\mathbb{H})$ is dense in P since x is a totally irrational direction. Hence, we deduce that the union $\bigcup_{t\in\Lambda} R(s)+t$ is dense in P. Since $R(s)$ is bounded, this union is locally finite, thus locally closed. Hence, we get the wanted covering. The last point is a consequence of the Baire category theorem: if the interior of $R(s)$ were empty, then the interior of the countable union $\bigcup_{t\in\Lambda} R(s) + t = P$ would be empty, which is absurd. □

3.10 Technical lemmas

The following lemma is a quantitative improvement of Lemma 6. It says that the rate of exponential convergence is invariant under a shift of the directive sequence.

Lemma 46. *Let s be a directive sequence and let $x \in \mathbb{PR}_+^d$ be a direction. Then, for every $k \in \mathbb{N}$, we have the equality*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| \pi_x M_{[0,n)} \right|\right|\right|_1 = \limsup_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| \pi_{x^{(k)}} M_{[k,k+n)} \right|\right|\right|_1,$$

where $x^{(k)} = M_{[0,k)}^{-1} x$. In particular, if s is exponentially convergent with direction x, then

$$\exists C > 0,\ \forall k \in \mathbb{N},\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ \left|\left|\left| \pi_{x^{(k)}} M_{[k,k+n)} \right|\right|\right|_1 \leq e^{-nC}.$$

Proof. Let η be the linear endomorphism of P such that $\pi_x M_{[0,k)} = \eta \pi_{x^{(k)}}$. Remark that η is invertible. We have the inequalities

$$\left|\left|\left| \pi_x M_{[0,k+n)} \right|\right|\right|_1 \leq \left|\left|\left| \eta \pi_{x^{(k)}} \right|\right|\right|_1 \left|\left|\left| \pi_{x^{(k)}} M_{[k,k+n)} \right|\right|\right|_1,$$

and

$$\left|\left|\left| \pi_{x^{(k)}} M_{[k,k+n)} \right|\right|\right|_1 \leq \left|\left|\left| \eta^{-1} \pi_x \right|\right|\right|_1 \left|\left|\left| \pi_x M_{[0,k+n)} \right|\right|\right|_1.$$

So we get the wanted equality

$$\limsup_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| \pi_x M_{[0,n)} \right|\right|\right|_1 = \limsup_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| \pi_x M_{[0,k+n)} \right|\right|\right|_1$$
$$= \limsup_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| \pi_{x^{(k)}} M_{[k,k+n)} \right|\right|\right|_1.$$

We deduce the second part of the lemma by taking

$$C = -\frac{1}{2} \limsup_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| \pi_x M_{[0,n)} \right|\right|\right|_1. \qquad \square$$

The remaining lemmas in this subsection are topology exercises and are not specific to the subject of this paper.

Lemma 47. *Let B, C, D be open subsets of P. If $D \subseteq \overline{B} \cap \overline{C}$, then $D \subseteq \overline{B \cap C}$.*

Proof. Let $x \in D$. Let $r_0 > 0$ be small enough to have $B(x, r_0) \subseteq D$ (balls are assumed open in this proof). Let $r > 0$ be such that $r \leq r_0$.

As $D \subseteq \overline{B}$, we get that $B(x, r) \subseteq D \subseteq \overline{B}$. If $B(x, r) \cap B = \varnothing$, then $x \notin \overline{B}$ which is absurd. So there exists $y \in B(x, r) \cap B$, and since these are open sets, there exists $r' > 0$ such that $B(y, r') \subseteq B(x, r) \cap B$.

Also, $B(y, r') \subseteq B(x, r) \subseteq \overline{C}$ thus there exists z such that $z \in B(y, r') \cap C$.

Finally, for every $r > 0$ we have found $z \in B \cap C$ such that $d(x, z) < r$ (since $z \in B(x, r)$). Therefore, $x \in \overline{B \cap C}$. $\square$

The next technical lemmas are useful in the proof of Proposition 92.

Lemma 48. *Let H be a closed subset of a metric space X, and let μ be a finite measure on X such that $\mu(H) = 0$. Then for every $\epsilon > 0$ there exists an open subset O such that $\mu(O) \leq \epsilon$ and $H \subseteq O$.*

Proof. For $n \geq 1$, let $H_n = \{x \in X \mid d(x, H) < \frac{1}{n}\}$. We have $\bigcap_{n\geq 1} H_n = H$, because H is closed. The sequence $(H_n)_n$ is decreasing for inclusion, thus we deduce $\lim_{n\to\infty} \mu(H_n) = \mu\big(\bigcap_{n\geq 1} H_n\big) = \mu(H) = 0$. Let $\epsilon > 0$. There exists $n \geq 1$ such that $\mu(H_n) \leq \epsilon$. Then, the open set $O = H_n$ suits. $\square$

Lemma 49. *Let $X \subseteq \mathbb{PR}_+^d$ and let μ be a probability measure on X. Let $N \subseteq X$ be the set of non totally irrational directions of X. We assume that $\mu(N) = 0$. Then, for every $\epsilon > 0$ there exists an open set O of X such that O contains all the non totally irrational directions and such that $\mu(O) \leq \epsilon$.*

Proof. The set N is the union of kernels of linear forms with rational coefficients. Thus, it is a countable union of closed subsets. Let $(N_n)_{n\in\mathbb{N}}$ be closed subsets such that $N = \bigcup_{n\in\mathbb{N}} N_n$. Let $\epsilon > 0$. For every $n \in \mathbb{N}$, let O_n be an open set given by Lemma 48 such that $\mu(O_n) \leq \frac{\epsilon}{2^{n+1}}$ and $N_n \subseteq O_n$. Then, the open set $O = \bigcup_{n\in\mathbb{N}} O_n$ satisfies what we want: we have $N \subseteq O$ and

$$\mu(O) \leq \sum_{n\in\mathbb{N}} \mu(O_n) \leq \sum_{n\in\mathbb{N}} \frac{\epsilon}{2^{n+1}} = \epsilon. \qquad \square$$

4 General conditions for the existence of nice Rauzy fractals

4.1 Statement

Definition 50. *Let S be a finite set of unimodular substitutions on A. We say that a directive sequence $s \in S^{\mathbb{N}}$ is* good *if the following four conditions are satisfied:*

1. *the sequence s is exponentially convergent,*
2. *the direction x of s is totally irrational,*
3. *there exist a fixed point $u \in \mathrm{Fix}(s)$, an increasing sequence of integers $(k_n)_{n\in\mathbb{N}}$, and a positive radius $r > 0$ such that*

   $$\forall n \in \mathbb{N},\ \forall a \in A,\ \exists p \in P,\ \mathbb{H} \cap \pi^{-1}_{x^{(k_n)}}(B(p,r)) \subseteq W_a(u_{k_n}),$$

   where $x^{(k_n)} = M^{-1}_{[0,k_n)}x$,
4. *the sequence $(x^{(k_n)})$ defined above has a limit which is a totally irrational direction.*

Remark 51. *The set of good directive sequences is shift-invariant: if a directive sequence $(s_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ is good, then the shifted directive sequence $(s_{n+1})_{n\in\mathbb{N}}$ is also good, and vice versa.*

Remark that for a good directive sequence, the Rauzy fractal does not depend on the choice of a fixed point, is compact and has non-empty interior by Corollary 45. We recall Theorem C that will be proven in the rest of this section:

Theorem C. *Let s be a good directive sequence. Then the Rauzy fractal $R(s)$ is a measurable fundamental domain of P for the lattice Λ. It can be decomposed as a union $R(s) = \bigcup_{a\in A} R_a(s)$ which is disjoint up to sets of Lebesgue measure 0, and each piece $R_a(s)$ is the closure of its interior.*

Moreover, the pieces $R_a(s)$, $a \in A$, of the Rauzy fractal induce a nice generating partition of the translation by $q(e_0 - v)$ on the torus P/Λ, where v is the unit vector of the direction of s.

This partition defines a symbolic coding of the translation by $q(e_0 - v)$, and this coding is a measurable conjugacy with the uniquely ergodic subshift associated with s.

4.2 Proof of Theorem C

In all this subsection we assume that s is a good directive sequence with direction x, and we let $x^{(k)} = M^{-1}_{[0,k)}x$, so that

$$\bigcap_{n\geq k} M_{[k,n)}\mathbb{P}\mathbb{R}^d_+ = \{x^{(k)}\}.$$

We let u, (k_n) and r be as in Definition 50. We denote by $R^{(k)} = R(u_k)$ the Rauzy fractal uniquely defined by the directive sequence $(s_n)_{n\geq k}$ (which is good by Remark 51), and by $R^{(k)}_a = R_a(u_k)$ its pieces, for $a \in A$, see Definition 36.

4.2.1 Step 1: proof that we have a topological tiling

Lemma 52. *The Rauzy fractals $R^{(k_n)}$ are uniformly bounded.*

Proof. Conditions 3 and 4 of Definition 50 ensure that we can apply Lemma 25 to the directions $x^{(k_n)}$ and the worms $W(u_{k_n})$. We get that the sets $\pi_{x^{(k_n)}} W(u_{k_n})$ are uniformly bounded for $n \in \mathbb{N}$, and the result follows by taking the closure. □

Lemma 53. *For every $k \in \mathbb{N}$ and every $a \in A$, the set $W_a(u_k)$ has non-empty interior for $\mathcal{T}(x^{(k)})$.*

Proof. Consider $k \in \mathbb{N}$ and $a \in A$. By Condition 3 in Definition 50, we can find $l \geq k$ such that the interior of $W_b(u_l)$ is non-empty for $\mathcal{T}(x^{(l)})$ for all $b \in A$. Since $M_{[k,l)}$ is invertible, by Lemma 35 there exists at least one $b \in A$ such that there is a path $b \xrightarrow{t_{l-1},s_{l-1}} \dots \xrightarrow{t_k,s_k} a$ in $\mathcal{A}$. Then, by Lemma 33, $M_{[k,l)} W_b(u_l) + t \subseteq W_a(u_k)$ for some $t \in \mathbb{N}^{d+1}$. By Lemma 26, the interior of $M_{[k,l)} W_b(u_l) + t$ is non-empty for $\mathcal{T}(x^{(k)})$, and we deduce the result. □

Lemma 54. *For every $k \in \mathbb{N}$ and every $a \in A$, the interior of $W_a(u_k)$ is dense in $W_a(u_k)$ for $\mathcal{T}(x^{(k)})$. If $U \subseteq P$ is an open set such that $\pi^{-1}_{x^{(k)}}(U) \cap \mathbb{H}$ is the interior of $W_a(u_k)$, then the set U is dense in $R^{(k)}_a$.*

Proof. Consider $m \in W_a(u_k)$ and V open set containing m. We want to find an element of V in the interior of $W_a(u_k)$. By Lemma 33, m belongs to a set of the form $M_{[k,l)} W_b(u_l) + t_l$ for each $l \geq k$. By Lemma 52, the sets $\pi_{x^{(k_n)}} W(u_{k_n})$ are uniformly bounded, thus we deduce with Lemma 46 that the diameter of

$$\pi_{x^{(k)}} M_{[k,k_n)} W_b(u_{k_n})$$

is arbitrarily small for $n \in \mathbb{N}$ large enough, so that $m \in M_{[k,k_n)} W_b(u_{k_n}) + t_{k_n} \subseteq W_a(u_k) \cap V$. As this set has non-empty interior by Lemmas 53 and 26, it follows that V intersects the interior of $W_a(u_k)$. This proves that the interior of $W_a(u_k)$ is dense in $W_a(u_k)$.

Now, if U is an open subset of P such that $\pi^{-1}_{x^{(k)}}(U) \cap \mathbb{H}$ is the interior of $W_a(u_k)$, then the projection $U \cap \pi_{x^{(k)}}(\mathbb{H})$ is dense in $\pi_{x^{(k)}}(W_a(u_k))$ which is dense in $R^{(k)}_a$. Thus, U is dense in $R^{(k)}_a$. □

Lemma 55.

- *For every $t \in \Lambda \setminus \{0\}$, $R \cap (R + t)$ has empty interior.*
- *For $a \neq b \in A$, $R_a \cap R_b$ has empty interior.*

Proof. We denote $\pi = \pi_{x^{(0)}}$. By Lemma 23, we have $W(u_0) \cap (W(u_0) + t) = \emptyset$. Now consider $U \subseteq P$ an open set such that $\pi^{-1}(U) \cap \mathbb{H}$ is the interior of $W(u_0)$. Then, we have

$$\begin{aligned} \pi^{-1}(U) \cap (\pi^{-1}(U) + t) \cap \mathbb{H} = \emptyset &\Longrightarrow U \cap (U + t) \cap \pi(\mathbb{H}) = \emptyset \\ &\Longrightarrow U \cap (U + t) = \emptyset, \end{aligned}$$

because $\pi(\mathbb{H})$ is dense in P since the direction $x^{(0)}$ is totally irrational, and $\pi(t) = t$ since $t \in \Lambda$. Moreover, by Lemma 54, the set U is dense in R. Then, by Lemma 47, the empty set $U \cap (U+t)$ is dense in the interior of $R \cap (R+t)$. We deduce that the interior of $R \cap (R+t)$ is empty.

For $a \neq b$, we have $W_a(u_0) \cap W_b(u_0) = \emptyset$. Let U_a and U_b be open subsets of P such that $\mathring{W}_a(u_0) = \pi^{-1}(U_a) \cap \mathbb{H}$ and $\mathring{W}_b(u_0) = \pi^{-1}(U_b) \cap \mathbb{H}$. By Lemma 54, the set U_a is dense in R_a and U_b is dense in R_b. Then, by Lemma 47, the empty set $U_a \cap U_b$ is dense in the interior of $R_a \cap R_b$. We deduce that the interior of $R_a \cap R_b$ is empty. $\square$

4.2.2 Step 2: proof that the boundary has zero Lebesgue measure

For every $k \in \mathbb{N}$, we let $v^{(k)} = v(x^{(k)})$. Remark that the direction $x^{(k)}$ being totally irrational, the numbers $v_a^{(k)}$ cannot be equal to zero. Let us then define $g_k = \max_{a \in A} \frac{\lambda(R_a^{(k)})}{v_a^{(k)}}$ and $f_k = \max_{a \in A} \frac{\lambda(\partial R_a^{(k)})}{v_a^{(k)}}$. Let η_k be the linear endomorphism of P such that $\eta_k \circ \pi_{x^{(k+1)}} = \pi_{x^{(k)}} \circ M_k$ (where the matrix M_k is treated like an endomorphism of $\mathbb{R}^{d+1}$). This map is well-defined since $M_k x^{(k+1)} = x^{(k)}$. Observe that η_k is an invertible map. We write $\eta_{[k,l)} = \eta_k \circ \cdots \circ \eta_{l-1}$, so that $\eta_{[k,l)} \circ \pi_{x^{(l)}} = \pi_{x^{(k)}} \circ M_{[k,l)}$.

Lemma 56. *For all $l > k$, we have*

$$\left|\det \eta_{[k,l)}\right| = \frac{1}{\left\|M_{[k,l)} v^{(l)}\right\|_1}.$$

Proof. Let γ be any fixed basis of P, and let us consider the following two bases of $\mathbb{R}^{d+1}$: $\gamma_l = (\gamma, v^{(l)})$ and $\gamma_k = (\gamma, v^{(k)})$. Since $h(v^{(l)}) = h(v^{(k)}) = 1$, we have $v^{(l)} - v^{(k)} \in P$, hence the transition matrix between the two bases is of the form $\begin{pmatrix} \mathrm{Id} & * \\ 0 & 1 \end{pmatrix}$, its determinant is 1. The linear map $M_{[k,l)}$ sends $v^{(l)}$ to $\left\|M_{[k,l)} v^{(l)}\right\|_1 v^{(k)}$ so that, if $\left[\eta_{[k,l)}\right]$ denotes the matrix of $\eta_{[k,l)}$ in the basis γ, the matrix of $M_{[k,l)}$ in the pair of bases (γ_k, γ_l) is

$$\begin{pmatrix} \left[\eta_{[k,l)}\right] & 0 \\ * & \left\|M_{[k,l)} v^{(l)}\right\|_1 \end{pmatrix}.$$

Hence, $\det M_{[k,l)} = \det \eta_{[k,l)} \left\|M_{[k,l)} v^{(l)}\right\|_1$. As the matrices M_k are unimodular, we have $\left|\det M_{[k,l)}\right| = 1$, and the result follows. $\square$

The endomorphisms η_k allow to translate Lemma 33 into a relation between Rauzy fractals.

Lemma 57. *For all integers k, l such that $0 \leq k \leq l$, and every $a \in A$,*

$$R_a^{(k)} = \bigcup_{b \xrightarrow{t_{l-1}, s_{l-1}} \cdots \xrightarrow{t_k, s_k} a} \eta_{[k,l)} R_b^{(l)} + \sum_{i=k}^{l-1} \pi_{x^{(k)}} M_{[k,i)} t_i.$$

In particular,

$$R_a^{(k)} = \bigcup_{b \xrightarrow{t, s_k} a} \eta_k R_b^{(k+1)} + \pi_{x^{(k)}}(t).$$

Proof. From Lemma 33, projecting both sides by $\pi_{x^{(k)}}$, we get

$$\pi_{x^{(k)}} W_a(u_k) = \bigcup_{b \xrightarrow{t_{l-1},s_{l-1}} \dots \xrightarrow{t_k,s_k} a} \pi_{x^{(k)}} M_{[k,l)} W_b(u_l) + \sum_{i=k}^{l-1} \pi_{x^{(k)}} M_{[k,i)} t_i.$$

We then obtain the first equality by using $\pi_{x^{(k)}} M_{[k,l)} = \eta_{[k,l)} \pi_{x^{(l)}}$ and taking the closure. The second equality follows when $l = k+1$. □

Lemma 58. *For every $k \in \mathbb{N}$, we have $g_k \leq g_{k+1}$ and $f_k \leq f_{k+1}$. The limits $g = \lim_{k\to\infty} g_k$ and $f = \lim_{k\to\infty} f_k$ exist and are finite.*

Proof. As $\lambda\big(\eta_k R_b^{(k+1)} + \pi_{x^{(k)}}(t)\big) = |\det \eta_k|\, \lambda\big(R_b^{(k+1)}\big)$, we deduce from Lemma 57 that, for every $k \in \mathbb{N}$,

$$\begin{aligned}
\big(\lambda(R_a^{(k)})\big)_{a\in A} &\leq \left(\sum_{b \xrightarrow{t,s_k} a} |\det \eta_k|\, \lambda(R_b^{(k+1)}) \right)_{a \in A} \\
&= |\det \eta_k|\, M_k \big(\lambda(R_b^{(k+1)})\big)_{b\in A} \\
&\leq |\det \eta_k|\, M_k g_{k+1} v^{(k+1)}.
\end{aligned}$$

By Lemma 56, we have

$$(\lambda(R_a^{(k)}))_{a\in A} \leq \frac{1}{\|M_k v^{(k+1)}\|_1} g_{k+1} M_k v^{(k+1)} = g_{k+1} v^{(k)},$$

thus the sequence (g_k) is non-decreasing. The proof is similar for the sequence (f_k).

By Lemma 52, $\lambda(R^{(k_n)})$ is bounded by some constant $D > 0$. Moreover, as $v^{(k_n)}$ is totally irrational and converges to a totally irrational vector, the numbers $v_a^{(k_n)}$ are bounded from below by a constant $\epsilon > 0$. Therefore, the subsequence (g_{k_n}) is bounded by D/ϵ. As (f_k) and (g_k) are non-decreasing and $f_k \leq g_k$, it follows that both sequences have a finite limit. □

Let $K = \{k_n \mid n \in \mathbb{N}\}$, and fix $a \in A$. For every $b \in A$, $k \in K$, and $l \geq k$, let

$$\begin{aligned}
L_b^{k,l} &= \{\pi_{x^{(k)}} \left(\sum_{i=k}^{l-1} M_{[k,i)} t_i \right) \in P \mid b \xrightarrow{t_{l-1},s_{l-1}} \dots \xrightarrow{t_k,s_k} a\}, \\
I_b^{k,l} &= \{t \in L_b^{k,l} \mid \eta_{[k,l)} R_b^{(l)} + t \subseteq B(p_k, r)\},
\end{aligned}$$

where the $p_k \in P$ are such that $\mathbb{H} \cap \pi_{x^{(k)}}^{-1} B(p_k, r) \subseteq W_a(u_k)$. Such p_k exist by Condition 3 in Definition 50. Lemma 57 then yields

$$R_a^{(k)} = \bigcup_{b\in A} \bigcup_{t \in L_b^{k,l}} \eta_{[k,l)} R_b^{(l)} + t.$$

Lemma 59. *There exists a uniform constant $C > 0$ such that for every $k \in K$, there exists $l_0 \geq k$ such that for every $l \geq l_0$, we have*

$$\sum_{b\in A} v_b^{(l)} \# I_b^{k,l} \geq C \sum_{b \in A} v_b^{(l)} \# L_b^{k,l}.$$

Proof. Let $k \in K$. We first choose $l_0 \in K$, $l_0 \geq k$, such that the diameter of $\eta_{[k,l_0)} R^{(l_0)}$ is less than $r/2$. It is possible, using Lemma 46, as the sets $R^{(l_0)}$ are uniformly bounded when $l_0 \in K$ by Lemma 52. Then, for every $l \geq l_0$ and every $b \in A$, some translate of the set $\eta_{[k,l)} R_b^{(l)}$ is included in $\eta_{[k,l_0)} R^{(l_0)}$ by Lemma 57, so the diameter of $\eta_{[k,l)} R_b^{(l)}$ is also less than $r/2$. For every $t \in L_b^{k,l}$, if $\eta_{[k,l)} R_b^{(l)} + t$ meets $B(p_k, r/2)$, then it is included in $B(p_k, r)$. Thus,

$$\bigcup_{b \in A} \bigcup_{t \in I_b^{k,l}} \eta_{[k,l)} R_b^{(l)} + t \supseteq B(p_k, r/2),$$

so that

$$g_l \sum_{b \in A} v_b^{(l)} \# I_b^{k,l} \geq \sum_{b \in A} \lambda(R_b^{(l)}) \# I_b^{k,l} \geq \frac{\lambda(B(p_k, r/2))}{\left|\det \eta_{[k,l)}\right|}.$$

Moreover, by Lemma 35, we have $\sum_{b \in A} v_b^{(l)} \# L_b^{k,l} = (M_{[k,l)} v^{(l)})_a = \left\| M_{[k,l)} v^{(l)} \right\|_1 v_a^{(k)}$. We deduce by Lemma 56, and as $v_a^{(k)} \leq \left\| v^{(k)} \right\|_1 = 1$:

$$\frac{\sum_{b \in A} v_b^{(l)} \# I_b^{k,l}}{\sum_{b \in A} v_b^{(l)} \# L_b^{k,l}} \geq \frac{1}{g_l} \lambda(B(p_k, r/2)) \frac{1}{v_a^{(k)}} \geq \frac{1}{g} \lambda(B(0, r/2))$$

where $g = \lim_{l \to \infty} g_l$ as in Lemma 58 (note that $g \geq g_l > 0$). Taking

$$C = \frac{1}{g} \lambda(B(0, r/2)),$$

we deduce the result. □

Lemma 60. *There exists $c < 1$ such that, for every $k \in K$, there exists $l > k$ such that $f_k \leq c f_l$.*

Proof. Let $k \in K$. For every $l \geq k$, by Lemma 57, we have

$$\partial R_a^{(k)} \subseteq \bigcup_{b \in A} \bigcup_{t \in L_b^{k,l}} \eta_{[k,l)} \partial R_b^{(l)} + t.$$

Now if $t \in I_b^{k,l}$, then $\eta_{[k,l)} \partial R_b^{(l)} + t$ is included in the interior of $R_a^{(k)}$, thus we deduce

$$\partial R_a^{(k)} \subseteq \bigcup_{b \in A} \bigcup_{t \in L_b^{k,l} \setminus I_b^{k,l}} \eta_{[k,l)} \partial R_b^{(l)} + t,$$

With Lemma 59, we get

$$\begin{aligned}
\lambda(\partial R_a^{(k)}) &\leq \left|\det \eta_{[k,l)}\right| \sum_{b \in A} \lambda(\partial R_b^{(l)}) \#(L_b^{k,l} \setminus I_b^{k,l}) \\
&\leq \left|\det \eta_{[k,l)}\right| f_l \sum_{b \in A} v_b^{(l)} \#(L_b^{k,l} \setminus I_b^{k,l}) \\
&\leq \left|\det \eta_{[k,l)}\right| f_l (1 - C) \sum_{b \in A} v_b^{(l)} \# L_b^{k,l} \\
&= (1 - C) f_l \left|\det \eta_{[k,l)}\right| \left\| M_{[k,l)} v^{(l)} \right\|_1 v_a^{(k)}.
\end{aligned}$$

We conclude with Lemma 56 that $\lambda(\partial R_a^{(k)}) \leq (1 - C) f_l v_a^{(k)}$, thus $f_k \leq (1 - C) f_l$. □

Proposition 61. *Let s be a good directive sequence. Then, for every $a \in A$, we have $\lambda(\partial R_a(s)) = 0$.*

Proof. We deduce from Lemmas 58 and 60 that $f_k = 0$ for all $k \in \mathbb{N}$. Then $\lambda(\partial R_a^{(k)}) = 0$ for every $a \in A$. $\square$

4.2.3 Step 3: proof that the translation is conjugate to the subshift

We refer to [42]. In the theorem that we recall below, the authors give conditions to prove that the translation by $q(\pi_x(e_0))$ on the torus $P/\Lambda \simeq \mathbb{T}^d$ is measurably conjugate to the subshift Ω_w generated by a word $w \in A^{\mathbb{N}}$.

Theorem 62. *[42, Theorem 2.3] Let $w \in A^{\mathbb{N}}$ be an infinite word, and let $x \in \mathbb{PR}_+^d$ be a totally irrational direction. Let $R = \overline{\pi_x(W(w))}$ and for all $a \in A$, $R_a = \overline{\pi_x(W_a(w))}$. We assume the following:*

- *the set $\pi_x(W(w))$ is bounded,*
- *the subshift (Ω_w, T) generated by w is minimal,*
- *the boundaries of R_a, $a \in A$, have zero Lebesgue measure,*
- *the union $\bigcup_{t\in\Lambda} R + t = P$ is disjoint up to sets of Lebesgue measure 0.*

Then there exists a Borel T-invariant measure μ such that the subshift (Ω_w, T, μ) is measurably conjugate to the translation on the torus $(P/\Lambda, T_x, \lambda)$.

We check that each condition is satisfied for the word $w = u_0$:

- boundedness is given by Corollary 45,
- minimality of the subshift $\Omega_{u_0} = \Omega_s$ is given by Proposition 17 since s is primitive by Lemma 7,
- boundaries of R_a, $a \in A$, have zero Lebesgue measure thanks to Proposition 61,
- thanks to Lemma 55, we have a topological tiling, and thanks to Proposition 61 the boundaries have zero Lebesgue measure; thus the union $\bigcup_{t\in\Lambda} R + t = P$ is disjoint up to sets of Lebesgue measure 0.

Therefore, we can use Theorem 62 for $w = u_0$.

Remark 63. *Under the hypotheses of Theorem 62, the domain exchange*

$$E_x = \begin{pmatrix} R & \longrightarrow & R \\ p & \longmapsto & p + \pi_x(e_a) \text{ if } p \in R_a \end{pmatrix}$$

is almost everywhere well-defined, and the quotient map q is a measurable conjugacy between the domain exchange (R, E_x) and the torus translation $(P/\Lambda, T_x)$.

Hence, we have:

Remark 64. *Consider a good sequence s, then by Corollary 41 (Ω_s, T) is uniquely ergodic. Moreover, for an irrational vector the torus translation $(P/\Lambda, T_x)$ is also uniquely ergodic. These two systems are measurably conjugate: The symbolic coding coming from the partition of the Rauzy fractal into pieces $R_a(s)$, $a \in A$ is a measurable conjugacy between the torus translation $(P/\Lambda, T_x)$ and the subshift (Ω_s, T). In particular, it gives a generating partition.*

Figure 7 shows the tiling of P by the Rauzy fractal of Figure 6 for the lattice Λ. The vector $\pi_x(e_0)$ giving the translation in the quotient P/Λ is also depicted.

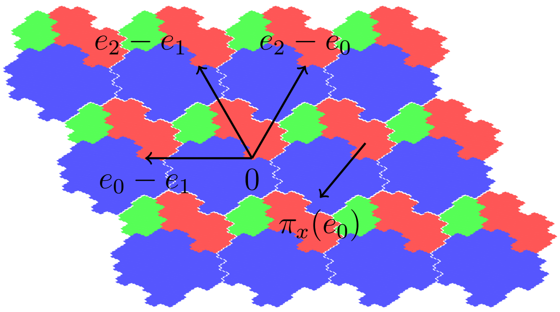

Figure 7: Tiling of P by a Rauzy fractal, and the translation vector $\pi_x(e_0)$.

4.2.4 Nice partition

Lemma 65. *For every $a \in A$, $q(R_a)$ is a bounded remainder set for the translation T_x by $q(\pi_x(e_0))$ on the torus P/Λ. For every $a \in A$, we have $v(x)_a = \frac{\lambda(R_a)}{\lambda(R)}$.*

Proof. The Rauzy fractal being bounded, there exists a constant K such that $R - R \subseteq B(0, K)$, where $R - R = \{p - p' \mid p, p' \in R\}$ is the set of differences. Let u be a fixed point of s. By the above conjugacy, for λ-almost every $z \in P/\Lambda$, there exists an infinite word $w \in \Omega_u$ such that w is the coding of the orbit of z under the translation for the measurable partition $(q(R_a))_{a \in A}$ of the torus P/Λ. Then, for every $a \in A$ and every $N \in \mathbb{N}$ we have the equality

$$\sum_{n=0}^{N-1} \mathbb{1}_{q(R_a)}(T_x^n(z)) = |p_N|_a \,,$$

where p_N is the prefix of length N of the word w. Since $w \in \Omega_u$, for every $N \in \mathbb{N}$ the word p_N is a factor of u_0. Thus

$$\mathrm{ab}(p_N) - Nv(x) = \pi_x(\mathrm{ab}(p_N)) \in \pi_x(W(u_0) - W(u_0)) \subseteq R - R \subseteq B(0, K).$$

Hence, for λ-almost every $z \in P/\Lambda$ and for every $a \in A$ we get the inequality

$$\left| \sum_{n=0}^{N-1} \mathbb{1}_{q(R_a)}(T_x^n(z)) - Nv(x)_a \right| \leq K.$$

Now, since T_x is ergodic for λ, by the Birkhoff ergodic theorem, we have for every $a \in A$, $v(x)_a = \frac{\lambda(R_a)}{\lambda(R)}$, so $q(R_a)$ is a bounded remainder set. □

Lemma 66. *The partition $(q(R_a))_{a\in A}$ is a liftable, generating and regular partition of the torus.*

Proof. The partition is liftable since the domain exchange E_x is a translation on each $q(R_a)$, $a \in A$, see Definition 8. By Remark 64, it is a generating partition. Finally it is also regular by Theorem 62. Altogether, we get that $(q(R_a))_{a\in A}$ is a nice generating partition. □

4.2.5 Conclusion

Now, we prove Theorem C. Starting from a good directive sequence, there exists a Rauzy fractal R by Remark 42, and it is independent from the choice of a fixed point by Corollary 45. By Lemma 55 (step 1), we know that R and $R+t$, $t \in \Lambda\setminus\{0\}$, have intersection of empty interior. Since $\bigcup_{t\in\Lambda} R+t = P$, by Corollary 45, we deduce that R is a topological fundamental domain of P for the lattice Λ. By Proposition 61 (step 2), we know that the boundaries of R and of the pieces R_a have zero Lebesgue measure. Thus, we deduce that up to a set of zero Lebesgue measure, R is a measurable fundamental domain of P for the action of Λ. By Lemma 55, we know that the interior of the intersection of two pieces is empty. So such intersection $R_a \cap R_b$ is included in $\partial R_a \cup \partial R_b$ and it has zero Lebesgue measure. Thus, we get that the union $R = \bigcup_{a\in A} R_a$ is disjoint up to sets of Lebesgue measure 0. By Lemma 54, for every $a \in A$, the piece R_a is the closure of an open set, thus it is the closure of its interior. Theorem 62 and Remark 64 (step 3) give the expected conjugacy.

By Lemma 65, $q(R_a)$ is a bounded remainder set. Finally, by Lemma 66, $(q(R_a))_{a\in A}$ is a nice partition of the torus.

4.3 Rauzy fractal for a periodic point of a single substitution

If we consider a directive sequence of the form σ^ω, where σ is a unimodular substitution, then we are back in the classical setting of the Rauzy fractal associated with a single substitution. In this sense, Theorem C gives a generalization of [42, Theorem 1.3.3].

We say that a substitution is *irreducible* if the characteristic polynomial of its abelianized matrix is irreducible over $\mathbb{Q}$. See Section 3.3 for the definition of a Pisot substitution.

Lemma 67. *Let σ be an irreducible Pisot unimodular substitution, and $x \in \mathbb{PR}^d_+$ be the class of a Perron eigenvector of $\mathrm{ab}(\sigma)$. Then x is a totally irrational direction.*

Proof. The characteristic polynomial is irreducible over $\mathbb{Q}$ and splits with simple roots over the splitting field. Thus, the Galois group acts transitively on the eigenvectors. Hence, if x were not a totally irrational direction, it would give a rational non-zero vector in the left kernel of the matrix of eigenvectors. And this is absurd because this matrix is invertible. We deduce that x is a totally irrational direction. □

Lemma 68. *Let σ be an irreducible Pisot unimodular substitution, $x \in \mathbb{PR}^d_+$ the class of a Perron eigenvector of $\mathrm{ab}(\sigma)$, $u_0 \in A^{\mathbb{N}}$ a periodic point of σ, and $a \in A$ a letter such that the interior of $W_a(u_0)$ is not empty for the topology $\mathcal{T}(x)$. Then the directive sequence σ^ω is good.*

Proof. We check that $s = \sigma^\omega$ satisfies the four conditions of Definition 50.

1. We have

$$\lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1 = \lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\pi_x M^n\right|\right|\right|_1 = \ln |\beta| ,$$

where β is the second largest eigenvalue of $M = \mathrm{ab}(\sigma)$ in modulus. We have $\ln |\beta| < 0$ since σ is Pisot, so s is exponentially convergent.

2. By Lemma 67, x is a totally irrational direction.

3. Assume that u_0 is a periodic point of σ of period m. Let $u \in (\mathcal{A}^{\mathbb{N}})^{\mathbb{N}}$ be the word sequence defined by

$$u_n = \sigma^{m-(n \bmod m)}(u_0),$$

where $n \bmod m$ is the remainder in the division of n by m. We easily check that u is a fixed point of the directive sequence σ^ω. By Lemma 7, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M_{[0,n)} = M^n > 0$. We choose $n_1 \geq n_0$ divisible by m. Hence, we have $u_{n_1} = u_0$. Now, for every $b \in A$, we have by Lemma 33

$$W_b(u_0) = \bigcup_{c \xrightarrow{t_{n_1-1},\sigma} \dots \xrightarrow{t_0,\sigma} b} M^{n_1} W_c(u_0) + \sum_{k=0}^{n_1-1} M^k t_k.$$

As $M^{n_1} > 0$, by Lemma 35 there exists a path $a \xrightarrow{t_{n_1-1},\sigma} \dots \xrightarrow{t_0,\sigma} b$, so $M^{n_1} W_a(u_0) + \sum_{k=0}^{n_1-1} M^k t_k \subseteq W_b(u_0)$. It follows from Lemma 26 that $W_b(u_0)$ has non-empty interior for all $b \in A$. Hence, we get Condition 3, with the sequence $(k_n)_{n\in\mathbb{N}} = (mn)_{n\in\mathbb{N}}$, and with $r > 0$ small enough so that

$$\forall b \in A, \exists p \in P, \mathbb{H} \cap \pi_x^{-1}(B(p,r)) \subseteq W_b(u_0).$$

4. For all $n \in \mathbb{N}$, $x^{(n)} = x$, so $\lim_{n\to\infty} x^{(k_n)} = x$ exists and is a totally irrational direction. □

The *Pisot substitution conjecture* [2] states (or is equivalent to the fact) that the conclusion of our Theorem C is true for every directive sequence of the form σ^ω, with σ an irreducible Pisot unimodular substitution: the subshift is measurably conjugate to a translation on a torus. In [42, Theorem 3.3], the following is proved:

Theorem 69. *Let σ be an irreducible Pisot unimodular substitution and $x \in \mathbb{PR}_+^d$ be the class of a Perron eigenvector of* $\mathrm{ab}(\sigma)$*. The following are equivalent:*

- *σ satisfies the Pisot substitution conjecture,*
- *there exist a periodic point $u_0 \in A^{\mathbb{N}}$ of σ and a letter $a \in A$ such that $W_a(u_0)$ has non-empty interior for $\mathcal{T}(x)$.*

From Lemma 68 and Theorem 69 we deduce the following

Corollary 70. *The converse of Theorem C is true for directive sequences of the form σ^ω, where σ is an irreducible Pisot unimodular substitution. In other words, if Ω_{σ^ω} is measurably conjugate to a translation on a torus, then the directive sequence σ^ω is good.*

The Pisot substitution conjecture is further discussed in Section 10.6.

5 Dynamics of continued fractions

5.1 Extended continued fraction algorithms

Definition 71. *An* extended continued fraction algorithm*, denoted* (X, s_0)*, is given by*

- *a subset* $X \subseteq \mathbb{PR}_+^d$,
- *a finite alphabet* $A = \{0, \dots, d\}$*, with* $d \geq 1$,
- *a finite set* $S \subseteq \hom(A^+, A^+)$ *of unimodular substitutions on the alphabet* A,
- *a map* $s_0 \colon X \to S$ *such that for every* $x \in X$, $\mathrm{ab}(s_0(x))^{-1}x \in X$.
- *a map defined by*

$$F = \begin{pmatrix} X & \longrightarrow & X \\ x & \longmapsto & \mathrm{ab}(s_0(x))^{-1}x \end{pmatrix}.$$

We use the word *extended* to indicate that the algorithm uses substitutions. If we do not use the substitutions, we can retain their matrices $\mathrm{ab}(s_0(x))$ only, or even just the map F.

Definition 72. *A* continued fraction algorithm*, denoted* (X, F)*, is given by*

- *a subset* $X \subseteq \mathbb{PR}_+^d$,
- *a map* $M_0 \colon X \to GL_{d+1}(\mathbb{Z}) \cap \mathcal{M}_{d+1}(\mathbb{N})$*, taking a finite number of values, such that for every* $x \in X$, $M_0(x)^{-1}x \in X$.
- *a map defined by*

$$F = \begin{pmatrix} X & \longrightarrow & X \\ x & \longmapsto & M_0(x)^{-1}x \end{pmatrix}.$$

Given a continued fraction algorithm (X, F), there are several possible choices for S and s_0 to turn it into an extended continued fraction algorithm (X, s_0). These choices do not yield the same subshifts, and not the same complexity function.

Definition 73. *When* (X, s_0) *is an extended continued fraction algorithm, we define* X_0 *as the set of* $x \in X$ *such that*

- x *is a totally irrational direction,*
- $s_0 \circ F^n$ *is continuous at* x *for all* $n \in \mathbb{N}$.

When (X, F) *is a continued fraction algorithm, we define* X_0 *as the set of* $x \in X$ *such that*

- x *is a totally irrational direction,*
- $M_0 \circ F^n$ *is continuous at* x *for all* $n \in \mathbb{N}$.

Remark 74. *For every* $n \in \mathbb{N}$, F^n *is continuous on* X_0.

Remark 75. *If s_0 is continuous at every totally irrational direction, then so is $s_0 \circ F^n$ for all n, and X_0 is the set of totally irrational directions. This is in particular the case if the regions on which s_0 is constant are delimited by rational hyperplanes.*

For $x \in X$ and $i \in \mathbb{N}$, we define $s_i = s_0 \circ F^i$. The matrices associated with the substitutions are denoted by $M_i = M_i(x) = \mathrm{ab}(s_i(x))$, and we recall that $M_{[i,j)}$ stands for the product of matrices $M_i M_{i+1} \dots M_{j-1}$. With the map s_0 we can do some higher-level symbolic dynamics: the map

$$s = \left(\begin{array}{ccc} X & \longrightarrow & S^{\mathbb{N}} \\ x & \longmapsto & s(x) = (s_n(x))_{n\in\mathbb{N}} \end{array} \right).$$

is a symbolic coding of the continued fraction algorithm. In particular, we have $s \circ F = T \circ s$, where T is the shift map on $S^{\mathbb{N}}$.

Definition 76. *Let (X, s_0) be an extended continued fraction algorithm, and μ a measure on X. We say that (X, s_0, μ) is an* extended measured continued fraction algorithm *if*

1. *μ is an ergodic F-invariant Borel probability measure,*
2. *the map s_0 is measurable with respect to μ,*
3. *$\mu(X_0) = 1$,*
4. *for every measurable $Y \subseteq X$, we have $\mu(Y) = 0 \Longrightarrow \mu(F(Y)) = 0$,*
5. *$\exists \epsilon > 0, \forall x \in X_0, \forall n \geq 1, \mu(F^n(\{y \in X_0 \mid M_{[0,n)}(y) = M_{[0,n)}(x)\})) > \epsilon$.*

As above, if we are not interested in the particular choice of substitutions, we will consider instead the *measured continued fraction algorithm* (X, F, μ).

Definition 77. *Let (X, F) be a continued fraction algorithm, and μ a measure on X. We say that (X, F, μ) is a* measured continued fraction algorithm *if*

1. *μ is an ergodic F-invariant Borel probability measure,*
2. *the map M_0 is measurable with respect to μ,*
3. *$\mu(X_0) = 1$,*
4. *for every measurable $Y \subseteq X$, we have $\mu(Y) = 0 \Longrightarrow \mu(F(Y)) = 0$,*
5. *$\exists \epsilon > 0, \forall x \in X_0, \forall n \geq 1, \mu(F^n(\{y \in X_0 \mid M_{[0,n)}(y) = M_{[0,n)}(x)\})) > \epsilon$.*

By ergodicity the hypothesis $\mu(X_0) = 1$ is equivalent to $\mu(X_0) > 0$.

Now we give a criterion to prove that the last condition of Definition 76 is satisfied.

Proposition 78. *Assume that we have an extended continued fraction algorithm (X, s_0) and a finite union H of rational hyperplanes of $\mathbb{PR}_+^d$ that partition $X \setminus H$ into a finite number of pieces $(X_i)_{i\in I}$ such that for every $i \in I$,*

- *s_0 is constant on X_i,*

- *the set $F(X_i) \setminus H$ is a union of pieces: $F(X_i) \setminus H = \bigcup_{j \in J} X_j$ for some $J \subseteq I$.*

If μ is an ergodic F-invariant Borel probability measure on X such that

- *for all $i \in I$, $\mu(X_i) > 0$,*
- *for every measurable $Y \subseteq X$, we have $\mu(Y) = 0 \Longrightarrow \mu(F(Y)) = 0$,*
- *the measure of the set of non totally irrational directions is zero,*

then (X, s_0, μ) is an extended measured continued fraction algorithm as defined in Definition 76.

Such a family $(X_i)_{i \in I}$ is sometimes called a Markov partition.

Proof. Conditions 1 and 4 are satisfied by assumption. Condition 2 holds since s_0 is constant on the pieces. By Remark 75, with such hypotheses X_0 is the set of totally irrational directions of X, and we have by assumption $\mu(X \setminus X_0) = 0$, so Condition 3 holds. It remains to show Condition 5:

$$\exists \epsilon > 0,\ \forall x \in X_0,\ \forall n \geq 1,\ \mu(F^n(\{y \in X_0 \mid M_{[0,n)}(y) = M_{[0,n)}(x)\})) > \epsilon.$$

Observe that we can replace $y \in X_0$ with $y \in X$ since $F(X_0) \subseteq X_0$ and $\mu(X_0) = 1$. Let us show that for $x \in X_0$ and for all $n \geq 1$, we have

$$F^n(\{y \in X \mid M_{[0,n)}(y) = M_{[0,n)}(x)\}) \supseteq M_{n-1}^{-1}(x) X_{i(F^{n-1}x)} = F(X_{i(F^{n-1}x)}),$$

where $i(y) \in I$ is such that $y \in X_{i(y)}$. It will end the proof since the sets $F(X_{i(F^{n-1}x)})$ have positive measure

$$\mu(F X_{i(F^{n-1}x)}) = \mu(F^{-1}(F X_{i(F^{n-1}x)})) \geq \mu(X_{i(F^{n-1}x)}) > 0,$$

and there are finitely many of them.

The inclusion is equivalent to

$$\{y \in X \mid M_{[0,n)}(y) = M_{[0,n)}(x)\} \supseteq M_{[0,n-1)}(x) X_{i(F^{n-1}x)}.$$

We show the above inclusion for every $x \in X_0$ by induction on n.

Let $y \in M_{[0,n-1)}(x) X_{i(F^{n-1}x)}$. If $n = 1$, we have $y \in X_{i(x)}$, so $M_0(y) = M_0(x)$. Otherwise, by assumption, we have $X_{i(Fx)} \subseteq F(X_{i(x)})$, so

$$M_0(x) X_{i(Fx)} \subseteq X_{i(x)}.$$

If we iterate this, we see that we have $M_{[0,n-1)}(x) X_{i(F^{n-1}x)} \subseteq X_{i(x)}$. So, $M_0(y) = M_0(x)$ and we have $F(y) \in M_{[0,n-2)}(F(x)) X_{i(F^{n-1}x)}$. By induction hypothesis with x replaced with $F(x)$, we get that $M_{[0,n)}(y) = M_{[0,n)}(x)$. □

The hypotheses of Proposition 78 are satisfied by the usual continued fraction algorithms. See Sections 7 and 8.

Lemma 79. *Let (X, s_0) be an extended continued fraction algorithm. We have*

$$\forall x \in X_0, \forall k \in \mathbb{N}, \exists r > 0, d(x, y) \leq r \Longrightarrow \forall i < k, s_i(x) = s_i(y).$$

Proof. This is an obvious consequence of the definition of the set X_0, where s_n is continuous for every $n \in \mathbb{N}$. $\square$

Remark 80. *We have* $x = M_{[0,k)}(x)F^k(x)$*, so that* $x \in \bigcap_{k\geq 0} M_{[0,k)}(x)\mathbb{PR}_+^d$*. If* $(M_k(x))$ *is cone convergent, then its direction must therefore be* x*.*

Remark 81. *If* $x \in X_0$ *is such that* $(M_k(x))$ *is sum convergent, then its direction is* x *by Remark 80, which is totally irrational, so by Corollary 45 there is a unique Rauzy fractal associated with the directive sequence* $s(x)$*, which we will denote* $R(x) = R(s(x))$*, and its pieces* $R_a(x) = R_a(s(x))$*.*

5.2 Lyapunov exponents

Consider a dynamical system (X, T) with a T-invariant Borel probability measure μ on X. A *cocycle* of the dynamical system (X, T) is a map $\mathbf{M}\colon X \times \mathbb{N} \to GL_{d+1}(\mathbb{R})$ such that

- $\mathbf{M}(x, 0) = \mathrm{Id}$ for all $x \in X$,
- $\mathbf{M}(x, n + m) = \mathbf{M}(T^n(x), m)\mathbf{M}(x, n)$ for all $x \in X$ and $n, m \in \mathbb{N}$.

We denote $\mathbf{M}(x, -n) = \mathbf{M}(x, n)^{-1}$ for $n > 0$. Let $\|.\|$ be any norm on $\mathbb{R}^{d+1}$.

Theorem 82 (Oseledets)**.** *Let* (X, T) *be a dynamical system and* μ *be an invariant probability measure for this system. Let* $\mathbf{M}$ *be a cocycle of* (X, T) *in* $GL_{d+1}(\mathbb{R})$ *such that the maps* $x \mapsto \ln |||\mathbf{M}(x, 1)|||, x \mapsto \ln |||\mathbf{M}(x, -1)|||$ *are* L^1*-integrable with respect to* μ*.*

Then there exist a measurable set $Z \subseteq X$ *with* $\mu(Z) = 1$ *and measurable functions* r *and* θ_i *from* Z *to* $\mathbb{R}$*, such that for all* $x \in Z$ *there exist*

- *an integer* $r(x)$ *with* $0 < r(x) \leq d + 1$*,*
- $r(x)$ *distinct numbers* $\theta_1(x) > \cdots > \theta_{r(x)}(x)$*,*
- *a sequence of linear subspaces*

$$\mathbb{R}^{d+1} = E_1(x) \supsetneq \cdots \supsetneq E_{r(x)}(x) \supsetneq E_{r(x)+1}(x) = \{0\}$$

such that

$$y \in E_i(x) \setminus E_{i+1}(x) \iff \lim_{n\to\infty} \frac{1}{n} \ln \|\mathbf{M}(x, n)y\| = \theta_i(x).$$

If in addition μ *is an ergodic measure, then* Z *can be chosen so that the functions that map* x *to* $r(x), \theta_1(x), \ldots, \theta_{r(x)}(x)$*,* $\dim E_1(x) \ldots, \dim E_r(x)$ *are constant on* Z*. Then we denote* $\theta_i(x)$ *by* $\theta_i(\mu) = \theta_i(\mathbf{M}, \mu)$*.*

The numbers $\theta_i(\mu), i = 1 \ldots m$ are called *Lyapunov exponents* of the cocycle [45, 31]. We also use the following formulas, see [12, Theorem 6.3].

Corollary 83. *In the ergodic case we have, for every* $x \in Z$*,*

$$\theta_1(\mu) = \lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\mathbf{M}(x, n)\right|\right|\right|, \quad \theta_2(\mu) = \lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\mathbf{M}(x, n)_{|E_2(x)}\right|\right|\right|.$$

5.3 Lyapunov exponents for a continued fraction algorithm

In the following, since we use matrix transposition, we consider the dual space $\left(\mathbb{R}^{d+1}\right)^*$ of $\mathbb{R}^{d+1}$, with the norm $\|.\|_\infty$ defined by $\|\varphi\|_\infty = \max_{i\in A} |\varphi(e_i)|$ for $\varphi \in \left(\mathbb{R}^{d+1}\right)^*$. For a vector $v \in \mathbb{R}^{d+1}$ we denote by v° the orthogonal in the dual space, *i.e.*, the set of linear forms which vanish on v.

Lemma 84. *Let (X, F) be a continued fraction algorithm.*

- $\forall y \in X$, $\forall n \in \mathbb{N}$, $\forall N \geq n$, $M_{[0,N)}(y) = M_{[0,n)}(y)M_{[0,N-n)}(F^n y)$.
- $\forall y \in X$, $\forall n \in \mathbb{N}$, $\forall N \geq n$, $\pi_y M_{[0,N)}(y) = \pi_y M_{[0,n)}(y)\pi_{F^n y} M_{[0,N-n)}(F^n y)$.
- *The map*
$$\begin{pmatrix} X \times \mathbb{N} & \longrightarrow & GL_{d+1}(\mathbb{R}) \\ (x, n) & \longmapsto & \mathbf{M}(x, n) = M^t_{[0,n)}(x) \end{pmatrix}$$
defines a cocycle.

Proof. For the first point, consider $i \geq n$, then $F^i(y) = F^{i-n}(F^n(y))$, thus $M_i(y) = M_{i-n}(F^n(y))$. Then the second point follows from Lemma 1. Finally, the third point is a consequence of the definition of a cocycle. □

Let now (X, F, μ) be a measured continued fraction algorithm. We use Theorem 82 and Corollary 83 for the cocyle defined in Lemma 84. Remark that the hypothesis of integrability is automatically satisfied since M_0 is measurable and takes only a finite number of values. In the following we consider the set Z given by Theorem 82 for this cocycle $(x, n) \mapsto M^t_{[0,n)}(x)$. In particular we have the following corollary.

Corollary 85. *For every $x \in Z$ and $\varphi \in (\mathbb{R}^{d+1})^*$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln \left\|\varphi M_{[0,n)}(x)\right\|_\infty = \theta_i(\mu) \iff \varphi \in E_i(x) \setminus E_{i+1}(x),$$

and

$$\theta_1(x) = \theta_1(\mu) = \lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| M^t_{[0,n)}(x)\right|\right|\right|_\infty = \lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| M_{[0,n)}(x)\right|\right|\right|_1,$$
$$\theta_2(x) = \theta_2(\mu) = \lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left| M^t_{[0,n)}(x)_{|E_2(x)}\right|\right|\right|_\infty.$$

Definition 86. *A measured continued fraction algorithm (X, F, μ) is said to satisfy the* Pisot condition *if $\theta_1(\mu) > 0 > \theta_2(\mu)$ and, for μ-almost every point x,* $\operatorname{codim}(E_2(x)) = 1$.

Lemma 87. *Let $x \in Z$. Assume that $(M_n(x))$ is cone convergent and* $\operatorname{codim}(E_2(x)) = 1$. *Then*
$$E_2(x) = v(x)^\circ = \{\varphi \circ \pi_x \mid \varphi \in \left(\mathbb{R}^{d+1}\right)^*\}.$$

Proof. Let $\varphi \in E_2(x)$ (a linear form, or a line vector), and let $w = e_0 + \cdots + e_d \in \mathbb{R}^{d+1}_+$.

By the Hölder inequality,

$$\left|\varphi M_{[0,n)} w\right| \leq \left\|\varphi M_{[0,n)}\right\|_\infty \|w\|_1 .$$

Let us denote $w_n = v(M_{[0,n)}w)$. We obtain

$$\frac{1}{n}\ln|\varphi w_n| \leq \frac{1}{n}\ln\left\|\varphi M_{[0,n)}\right\|_\infty + \frac{1}{n}\ln\|w\|_1 - \frac{1}{n}\ln\left\|M_{[0,n)}w\right\|_1.$$

On the one hand,

$$\left\|M_{[0,n)}w\right\|_1 = \sum_{i=0}^{d}\left|e_i^* M_{[0,n)}w\right|.$$

Choosing i such that $e_i^* \notin E_2(x)$, which is possible since $E_2(x) \neq \left(\mathbb{R}^{d+1}\right)^*$, we have $\lim_{n\to\infty}\frac{1}{n}\ln\left\|e_i^* M_{[0,n)}\right\|_\infty = \theta_1(\mu)$ by Corollary 85. Moreover, $\left\|e_i^* M_{[0,n)}\right\|_\infty \leq e_i^* M_{[0,n)}w$ since $M_{[0,n)}$ has non-negative entries, so

$$\liminf_{n\to\infty}\frac{1}{n}\ln\left\|M_{[0,n)}w\right\|_1 \geq \lim_{n\to\infty}\frac{1}{n}\ln\left\|e_i^* M_{[0,n)}\right\|_\infty = \theta_1(\mu).$$

On the other hand, $\lim_{n\to\infty}\frac{1}{n}\ln\left\|\varphi M_{[0,n)}\right\|_\infty \leq \theta_2(\mu)$ since $\varphi \in E_2(x)$, so we get

$$\limsup_{n\to\infty}\frac{1}{n}\ln|\varphi w_n| \leq \theta_2(\mu) - \theta_1(\mu) < 0.$$

We also have $w_n \xrightarrow[n\to\infty]{} v(x)$ by hypothesis of cone convergence and by Remark 80. We deduce

$$\varphi v(x) = \lim_{n\to\infty}\varphi w_n = 0,$$

so $\varphi \in v(x)^\circ$. Since $\operatorname{codim}(E_2(x)) = 1 = \operatorname{codim}(v(x)^\circ)$ we obtain that $E_2(x) = v(x)^\circ$. The last equality follows from the fact that $\pi_x(v(x)) = 0$ and $\operatorname{Im}(\pi_x) = P$ is a hyperplane. $\square$

Lemma 88. *Let $x \in Z$. Assume that $(M_n(x))$ is cone convergent and* $\operatorname{codim}(E_2(x)) = 1$. *Then*

$$\left\|\left\|\pi_x M_{[0,n)}(x)\right\|\right\|_1 \leq (d+1)\left\|\left\|M^t_{[0,n)}(x)_{|E_2(x)}\right\|\right\|_\infty.$$

Proof. Recall that π_x is the projection on P with respect to the direction x.

$$\begin{aligned}
\left\|\left\|\pi_x M_{[0,n)}\right\|\right\|_1 &= \sup_{w\in\mathbb{R}^{d+1},\|w\|_1\leq 1}\left\|\pi_x M_{[0,n)}w\right\|_1 \\
&= \sup_{w\in\mathbb{R}^{d+1},\|w\|_1\leq 1}\sum_{i=0}^{d}\left|e_i^*\pi_x M_{[0,n)}w\right| \\
&\leq \sum_{i=0}^{d}\sup_{w\in\mathbb{R}^{d+1},\|w\|_1\leq 1}\left|e_i^*\pi_x M_{[0,n)}w\right| \leq \sum_{i=0}^{d}\left\|e_i^*\pi_x M_{[0,n)}\right\|_\infty.
\end{aligned}$$

By Lemma 87,

$$e_i^* \circ \pi_x \in E_2(x) = \{\varphi\circ\pi_x \mid \varphi \in (\mathbb{R}^{d+1})^*\}.$$

It follows that

$$\left\|e_i^*\pi_x M_{[0,n)}\right\|_\infty \leq \left\|\left\|M^t_{[0,n)|E_2(x)}\right\|\right\|_\infty \left\|e_i^*\circ\pi_x\right\|_\infty.$$

For $i \in A$, let $v_i = e_i^*(v(x))$. As $\pi_x(e_j) = e_j - v(x)$, we have $e_i^* \circ \pi_x(e_j) = \delta_{i,j} - v_i$, so that $\|e_i^* \circ \pi_x\|_\infty = \max(v_i, 1 - v_i) \leq 1$. We conclude

$$\left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1 \leq \left|\left|\left|M^t_{[0,n)|E_2(x)}\right|\right|\right|_\infty \sum_{i=0}^{d} \|e_i^* \circ \pi_x\|_\infty$$
$$\leq (d+1)\left|\left|\left|M^t_{[0,n)|E_2(x)}\right|\right|\right|_\infty. \qquad \square$$

From Lemma 88 and Corollary 85, we deduce the following

Corollary 89. *Let $x \in Z$. Assume that the sequence of matrices $(M_n(x))$ is cone convergent and that* $\operatorname{codim}(E_2(x)) = 1$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\pi_x M_{[0,n)}(x)\right|\right|\right|_1 = \theta_2(\mu).$$

Proof. By Lemma 87, $E_2(x) = \{\varphi \circ \pi_x \mid \varphi \in (\mathbb{R}^{d+1})^*\}$. Let $\varphi \in (\mathbb{R}^{d+1})^* \setminus \{0\}$ be a linear form such that $\varphi \circ \pi_x \in E_2(x) \backslash E_3(x)$. Then, using Lemma 88, we have the inequalities

$$\left\|\varphi \pi_x M_{[0,n)}\right\|_\infty \leq \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1 \|\varphi\|_\infty \leq (d+1)\left|\left|\left|M^t_{[0,n)|E_2(x)}\right|\right|\right|_\infty \|\varphi\|_\infty.$$

By Corollary 85, as $\varphi \circ \pi_x \in E_2(x) \setminus E_3(x)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \ln \left\|\varphi \pi_x M_{[0,n)}\right\|_\infty = \lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|M^t_{[0,n)|E_2(x)}\right|\right|\right|_\infty = \theta_2(\mu).$$

Thus, by the squeeze theorem we get that the limit

$$\lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1$$

exists and is equal to $\theta_2(\mu)$. $\square$

6 A lot of good points

The aim of this section is to prove that existence of one seed point ensures a set of full μ-measure with good directive sequences (see Proposition 92). With this result, and with Theorem C, the proof of Theorem B will be easy.

In all this section, (X, s_0, μ) is an extended measured continued fraction algorithm satisfying the Pisot condition (see Definition 86).

6.1 Preliminaries

Definition 90. *We define the set of* seed points G_0 *as the set of points $x \in X_0$ such that*

- $(M_n(x))$ *is exponentially convergent,*
- *there exist a letter $a \in A$ and a fixed point $u \in \operatorname{Fix}(s(x))$ such that $W_a(u_0)$ has non-empty interior for the topology $\mathcal{T}(x)$ (see Definitions 12, 20, and 21).*

Recall that X_0 is defined in Definition 73.

Definition 91. *We define the set of* good points

$$G = \{x \in X \mid s(x) \text{ is a good directive sequence}\},$$

where a good directive sequence is defined in Definition 50.

Note that, if $x \in G$, then by Remark 80 the directive sequence $s(x)$ has direction x.
The goal of this section is to prove:

Proposition 92. *Let (X, s_0, μ) be an extended measured continued fraction satisfying the Pisot condition. If $G_0 \neq \emptyset$, then $\mu(G) = 1$.*

In the proof of Proposition 92, we need a variant of the notion of seed point:

Definition 93. *We define G_1 as the set of $x \in X_0$ such that*

- *$(M_n(x))$ is exponentially convergent,*
- *there exists a fixed point $u \in \mathrm{Fix}(s(x))$ such that, for all $a \in A$, $W_a(u_0)$ has non-empty interior for the topology $\mathcal{T}(x)$.*

The definitions of G_1 and G_0 differ only by their last properties where we ask that the interior is not empty for every $a \in A$ rather than for one letter.
In the following, we introduce some more notations.

Definition 94. *For $C > 0$, let*

$$Z_C = \{x \in X_0 \mid \forall n \in \mathbb{N}, \left|\left|\left|\pi_x M_{[0,n)}(x)\right|\right|\right|_1 \leq Ce^{-\frac{n}{C}}\}.$$

Let B be a ball of positive radius in P and let $C > 0$. For every $a \in A$, we define

$$G^a_{B,C} = \{x \in Z_C \mid \exists u \in \mathrm{Fix}(s(x)), \pi_x^{-1}(B) \cap \mathbb{H} \subseteq W_a(u_0)\}.$$

Remark 95. *By Lemma 38 the inclusion $\pi_x^{-1}(B) \cap \mathbb{H} \subseteq W_a(u_0)$ is equivalent to a property about the Rauzy fractal and its pieces, but by Proposition 43 the Rauzy fractal doesn't depend on the choice of a fixed point, thus we have*

$$G^a_{B,C} = \{x \in Z_C \mid \forall u \in \mathrm{Fix}(s(x)), \pi_x^{-1}(B) \cap \mathbb{H} \subseteq W_a(u_0)\}$$

Remark 96. *These sets are linked to G_0 and G_1 by*

$$G_0 = \bigcup_{C>0} \bigcup_{\substack{(B_a)_{a\in A} \\ \text{balls of positive radius}}} \bigcup_{a \in A} G^a_{B_a,C},$$

$$G_1 = \bigcup_{C>0} \bigcup_{\substack{(B_a)_{a\in A} \\ \text{balls of positive radius}}} \bigcap_{a \in A} G^a_{B_a,C}.$$

Lemma 97. *Let $x \in X_0$. Then $(M_n(x))$ is exponentially convergent if, and only if, there exists $C > 0$ such that $x \in Z_C$.*

Proof. Let $l = \limsup_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1$. Clearly, if $x \in Z_C$ for some $C > 0$, then $l \leq -\frac{1}{C}$, so $(M_n(x))$ is exponentially convergent.

Conversely, if $(M_n(x))$ is exponentially convergent, then $l < 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1 \leq e^{nl/2}.$$

If we take $C = \max(\max_{n<n_0} \left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1 e^{-nl/2},\ 1,\ -\frac{2}{l})$, we have $x \in Z_C$. □

Lemma 98. *We have*

$$\lim_{C\to\infty} \mu(Z_C) = 1.$$

Proof. Let Y be the set of directions $x \in X_0$ such that $(M_n(x))$ is exponentially convergent. We have $\mu(Y) = 1$ by Corollary 89, as $\theta_2(\mu) < 0$. And by Lemma 97, we have $Y = \bigcup_{C>0} Z_C$. Since Z_C is increasing with C, we get $\lim_{C\to\infty} \mu(Z_C) = \mu(Y) = 1$. □

6.2 Proof of Proposition 92

The strategy is to prove

$$G_0 \neq \emptyset \Longrightarrow \mu(G_0) > 0 \Longrightarrow \mu(G_1) > 0 \Longrightarrow \mu(G) = 1.$$

In the following each step corresponds to one of these implications.

6.2.1 Step 1: $G_0 \neq \emptyset \Longrightarrow \mu(G_0) > 0$

Lemma 99. *We have*

$$\forall x \in G_0, \exists C > 0, \forall r > 0, \mu(B(x,r) \cap Z_C) > 0.$$

Proof. Let $x \in G_0$. Let $\epsilon > 0$ such that

$$\forall n \geq 1,\ \mu(F^n(\{y \in X_0 \mid M_{[0,n)}(y) = M_{[0,n)}(x)\})) \geq 2\epsilon.$$

This is given by our hypotheses on the measured continued fraction algorithm (Condition 5 in Definition 77).

Now, for every $K > 1$, let

$$O_K = \{y \in X \mid 1 < K \min_i(v(y)_i)\}.$$

We have $\mu\big(\bigcup_{K>1} O_K\big) = 1$, since $X_0 \subseteq \bigcup_{K>1} O_K$. So there exists $K > 1$ such that $\mu(O_K) > 1 - \epsilon$.

By Lemma 98, there exists $C' \geq 1$ such that $\mu(Z_{C'}) > 1 - \epsilon$. We choose the constant $C = (K+1)C_x C'$, where $C_x \geq 1$ is such that $x \in Z_{C_x}$ (see Lemma 97). Let $r > 0$. By hypothesis of exponential convergence, we can take $n \in \mathbb{N}$ large enough so that $\left|\left|\left|\pi_x M_{[0,n)}\right|\right|\right|_1 \leq r$. By Lemma 2, $M_{[0,n)}(x)X$ is included in $B(x,r)$.

Then, we take

$$Y = M_{[0,n)}(x)\,(Z_{C'} \cap O_K) \cap \{y \in X_0 \mid M_{[0,n)}(y) = M_{[0,n)}(x)\}.$$

By the previous inequalities, we have $\mu(F^n(Y)) = \mu(M_{[0,n)}^{-1}(x)Y) > 0$. By the definition of a measured continued fraction algorithm (Condition 4 in Definition 77), it follows that $\mu(Y) > 0$.

We have $Y \subseteq X_0 \cap B(x,r)$ by construction. Let us show the inclusion $Y \subseteq Z_C$. Let $y \in Y$. For all $N \geq n$, we have by Lemma 84

$$\begin{aligned}\left|\!\left|\!\left|\pi_y M_{[0,N)}(y)\right|\!\right|\!\right|_1 &\leq \left|\!\left|\!\left|\pi_y M_{[0,n)}(y)\right|\!\right|\!\right|_1 \left|\!\left|\!\left|\pi_{F^n y} M_{[0,N-n)}(F^n y)\right|\!\right|\!\right|_1 \\ &\leq \left|\!\left|\!\left|\pi_y M_{[0,n)}(x)\right|\!\right|\!\right|_1 \left|\!\left|\!\left|\pi_{F^n y} M_{[0,N-n)}(F^n y)\right|\!\right|\!\right|_1\end{aligned}$$

because $M_{[0,n)}(y) = M_{[0,n)}(x)$ by construction of Y. Now, let us show that we have $\left|\!\left|\!\left|\pi_y M_{[0,n)}(x)\right|\!\right|\!\right|_1 \leq (K+1)C_x e^{-n/C}$.

Recall that $h\colon \mathbb{R}^{d+1} \to \mathbb{R}$ is the linear form such that for every $w \in \mathbb{R}_+^{d+1}$, $h(w) = \|w\|_1$. Now, for every $z \in X$ and $w \in \mathbb{R}^{d+1}$, we have $\pi_z w = w - h(w)v(z)$. Let $M = M_{[0,n)}(x)$. We have $x \in Z_{C_x}$ and $C \geq C_x$, so for every $w \in \mathbb{R}^{d+1}$,

$$\|Mw - h(Mw)v(x)\|_1 = \|\pi_x Mw\|_1 \leq C_x e^{-n/C} \|w\|_1 .$$

Let $y' = F^n(y)$, so that $My' = y$. The previous inequality applied with $w = v(y')$ gives

$$\|Mv(y') - h(Mv(y'))v(x)\|_1 \leq C_x e^{-n/C}.$$

By the triangle inequality, and using that $v(y) = \frac{Mv(y')}{h(Mv(y'))}$, we have

$$\begin{aligned}\|\pi_y Mw\|_1 &= \|Mw - h(Mw)v(y)\|_1 \\ &\leq \|Mw - h(Mw)v(x)\|_1 + \left|\frac{h(Mw)}{h(Mv(y'))}\right| \|Mv(y') - h(Mv(y'))v(x)\|_1 ,\end{aligned}$$

so we get

$$\|\pi_y Mw\|_1 \leq C_x e^{-n/C}\left(\|w\|_1 + \left|\frac{h(Mw)}{h(Mv(y'))}\right|\right).$$

Since $y' \in O_K$, we have $h(Mv(y')) \geq \frac{1}{K}|\!|\!|M|\!|\!|_1$, and $|h(Mw)| \leq |\!|\!|M|\!|\!|_1 \|w\|_1$. Thus,

$$\|w\|_1 + \left|\frac{h(Mw)}{h(Mv(y'))}\right| \leq (K+1)\|w\|_1 .$$

We deduce that $\left|\!\left|\!\left|\pi_y M_{[0,n)}(x)\right|\!\right|\!\right|_1 \leq (K+1)C_x e^{-n/C}$. And by construction of Y we have $F^n(y) \in Z_{C'}$, and $C \geq C'$, so

$$\left|\!\left|\!\left|\pi_{F^n y} M_{[0,N-n)}(F^n y)\right|\!\right|\!\right|_1 \leq C' e^{-(N-n)/C}.$$

We deduce that

$$\left|\!\left|\!\left|\pi_y M_{[0,N)}\right|\!\right|\!\right|_1 \leq C e^{-N/C}.$$

Hence, we get that $Y \subseteq B(x,r) \cap Z_C$ with $\mu(Y) > 0$, so $\mu(B(x,r) \cap Z_C) > 0$. □

The next lemma says that if x is in G_0, then there exists a set of positive measure of points close to x where the Rauzy fractals are close to each other for the Hausdorff distance δ in P defined by

$$\delta(A_1, A_2) = \max\left(\sup_{p\in A_1} d(p, A_2),\ \sup_{p\in A_2} d(p, A_1)\right),$$

for any subsets A_1, A_2 of P.

Lemma 100. *For every $x \in G_0$, and every $\epsilon > 0$, there exist $C > 0$ and $V \subseteq B(x,\epsilon) \cap Z_C$ such that $\mu(V) > 0$ and $\forall y \in V,\ \forall a \in A,\ \delta(R_a(x), R_a(y)) \leq \epsilon$.*

Proof. Let $x \in G_0$. Let $C > 0$ be given by Lemma 99. Let $k \in \mathbb{N}$ be big enough to have

$$\frac{Ce^{-k/C}}{1 - e^{-1/C}} \max_{t \in \Sigma} \|t\|_1 \leq \frac{\epsilon}{3},$$

where $\Sigma \subseteq \mathbb{Z}^{d+1}$ is the finite Dumont-Thomas alphabet for our S-adic system, see Definition 29. Then, we choose $r^{(k)} = \frac{\epsilon}{3 \max_{(t_i) \in \Sigma^k} \left| h\left(\sum_{i=0}^{k-1} M_{[0,i)}(x) t_i\right)\right|}$, so that for all $y \in B(x, r^{(k)})$,

$$\forall (t_i)_{0 \leq i < k} \in \Sigma^k, \left\| \pi_x \left(\sum_{i=0}^{k-1} M_{[0,i)}(x) t_i \right) - \pi_y \left(\sum_{i=0}^{k-1} M_{[0,i)}(x) t_i \right) \right\|_1 \leq \frac{\epsilon}{3}.$$

Then, we take $r > 0$ given by Lemma 79 such that $d(x,y) \leq 2r \implies \forall i < k, s_i(x) = s_i(y)$. We can assume that $r \leq r^{(k)}$ and $r \leq \epsilon$ by taking the minimum of the three values, and we let $V = B(x,r) \cap Z_C$. Let us show that the set V has the desired property. We have $\mu(V) > 0$ by Lemma 99.

Let $y \in V$. Then $(M_n(y))$ is sum convergent with a totally irrational direction. Hence, we can use Corollary 45, and we get that

$$R_a(y) = \{ \sum_{n=0}^{\infty} \pi_y(M_{[0,n)}(y) t_n) \mid \dots \xrightarrow{t_n, s_n(y)} \dots \xrightarrow{t_0, s_0(y)} a \},$$

and we get a similar description for $R_a(x)$.

Let $p \in R_a(x)$, and let $\dots \xrightarrow{t_n, s_n(x)} \dots \xrightarrow{t_0, s_0(x)} a$ be a left-infinite path in the abelianized prefix automaton $\mathcal{A}$ such that

$$p = \sum_{n=0}^{\infty} \pi_x(M_{[0,n)}(x) t_n).$$

We have $s_i(x) = s_i(y)$ for all $i < k$, and the matrices of the substitutions in S are invertible, so we can take a left-infinite path $\dots \xrightarrow{t'_n, s_n(y)} \dots \xrightarrow{t'_0, s_0(y)} a$ in the automaton $\mathcal{A}$ such that $t_i = t'_i$ for all $i < k$. This defines a point $p' \in R_a(y)$ by

$$p' = \sum_{n=0}^{\infty} \pi_y(M_{[0,n)}(y) t'_n).$$

We have the inequalities

$$\begin{aligned} \|p - p'\|_1 \leq & \left\| \pi_x \left(\sum_{i=0}^{k-1} M_{[0,i)}(x) t_i \right) - \pi_y \left(\sum_{i=0}^{k-1} M_{[0,i)}(y) t_i \right) \right\|_1 \\ & + \left\| \sum_{n=k}^{\infty} \pi_x \left(M_{[0,n)}(x) t_n \right) \right\|_1 + \left\| \sum_{n=k}^{\infty} \pi_y \left(M_{[0,n)}(y) t'_n \right) \right\|_1 . \end{aligned}$$

Then using that $x \in Z_C$ and $y \in Z_C$, we have

$$\begin{aligned}\|p - p'\|_1 &\leq \frac{\epsilon}{3} + \sum_{n=k}^{\infty} Ce^{-n/C} \|t_n\|_1 + \sum_{n=k}^{\infty} Ce^{-n/C} \|t'_n\|_1 \\ &\leq \frac{\epsilon}{3} + 2\frac{Ce^{-k/C}}{1-e^{-1/C}} \max_{t\in\Sigma} \|t\|_1 \\ &\leq \epsilon.\end{aligned}$$

By reverting the role of x and y, we also show that for any point $p \in R_a(y)$, there exists a point $p' \in R_a(x)$ such that $\|p - p'\|_1 \leq \epsilon$, so we get the wanted inequality

$$\delta(R_a(x), R_a(y)) \leq \epsilon.$$

□

Lemma 101. *If $G_0 \neq \emptyset$ then $\mu(G_0) > 0$.*

Proof. Let $x \in G_0$. Let $a \in A$, and let u be a fixed point of $s(x)$. Assume that there exists an open ball $B_a = B(c_a, r_a)$ of positive radius $r_a > 0$ in P such that $\mathbb{H} \cap \pi_x^{-1}(B_a) \subseteq W_a(u_0)$. Then, by Lemma 38, for all $b \in A \setminus \{a\}$ and $t \in \Lambda \setminus \{0\}$,

$$B_a \cap R_b(x) = \emptyset = B_a \cap (R(x) + t).$$

We take the $C > 0$ and $V \subseteq B(x, \epsilon) \cap Z_C$ given by Lemma 100 for $\epsilon = r_a/2$. Let $B'_a = B(c_a, r_a/2)$ be the open ball with half the radius of B_a and same center. Let us show that for all $b \in A \setminus \{a\}$ and $t \in \Lambda \setminus \{0\}$ we have

$$\forall y \in V,\ B'_a \cap R_b(y) = \emptyset = B'_a \cap (R(y) + t).$$

If $b \in A \setminus \{a\}$, then for all $y \in V$ and for all $p \in R_b(y)$,

$$d(c_a, p) \geq d(c_a, R_b(x)) - d(p, R_b(x)) \geq r_a - \delta(R_b(x), R_b(y)) \geq r_a - r_a/2 = r_a/2,$$

so $B'_a \cap R_b(y) = \emptyset$. If $t \in \Lambda \setminus \{0\}$, then for all $y \in V$ and all $p \in R(y) + t$,

$$d(c_a, p) \geq d(c_a, R(x) + t) - d(p, R(x) + t) \geq r_a - \delta(R(x) + t, R(y) + t) \geq r_a/2,$$

so $B'_a \cap (R(y) + t) = \emptyset$.

By Lemma 38, we deduce that for every $y \in V$, we have the inclusion $\pi_y^{-1}(B'_a) \cap \mathbb{H} \subseteq W_a(u'_0)$, for any choice of a fixed point u' of $s(y)$. We get that $V \subseteq G^a_{B'_a,C}$, so the set G_0 has positive measure by Remark 96. □

6.2.2 Step 2: $\mu(G_0) > 0 \Longrightarrow \mu(G_1) > 0$

Lemma 102. *We have $\mu(G_0) > 0 \Longrightarrow \mu(G_1) > 0$.*

Proof. If $\mu(G_0) > 0$, then by the Poincaré recurrence theorem

$$\mu(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} F^{-k} G_0) > 0.$$

Let $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} F^{-k}G_0$. As $(M_n(x))$ is cone convergent by Lemma 5, and as x is a totally irrational direction, we deduce from Lemma 7 that there exists $n_0 \in \mathbb{N}$ such that $M_{[0,n_0)}(x) > 0$. Choose $n \geq n_0$ such that $F^n(x) \in G_0$. Let u be a fixed point of $s(x)$, and $a \in A$ a letter, such that $W_a(u_n)$ has non-empty interior for the topology $\mathcal{T}(F^n x)$. For every $b \in A$, we have by Lemma 33

$$W_b(u_0) = \bigcup_{c \xrightarrow{t_{n-1}, s_{n-1}(x)} \cdots \xrightarrow{t_0, s_0(x)} b} M_{[0,n)}(x) W_c(u_n) + \sum_{k=0}^{n-1} M_{[0,k)}(x) t_k.$$

Since $M_{[0,n)}(x) > 0$, we know that for every b, the letter $c = a$ appears in this union. The interior of $W_a(u_n)$ is non-empty for the topology $\mathcal{T}(F^n x)$, so by Lemma 26 the interior of $W_b(u_0)$ is non-empty for the topology $\mathcal{T}(x)$, for every $b \in A$. By Lemma 6, $(M_k(x))$ is exponentially convergent as $(M_k(F^n x))$ is. Thus, $x \in G_1$. So G_1 contains the set $\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} F^{-k}G_0$ which has positive measure. □

6.2.3 Step 3: $\mu(G_1) > 0 \Longrightarrow \mu(G) = 1$

Lemma 103. *The set G is measurable and F-invariant.*

Proof. Since we assume that μ is a Borel measure, the fact that G is a measurable set is an exercise left to the reader. By Remark 51, G is F-invariant. □

Lemma 104. *If $\mu(G_1) > 0$, then $\mu(G) = 1$.*

Proof. By Remark 96,

$$G_1 = \bigcup_{C>0} \bigcup_{\substack{(B_a)_{a\in A} \\ \text{balls of positive radius}}} \bigcap_{a\in A} G^a_{B_a,C}.$$

If $\mu(G_1) > 0$, then, as the union can be reduced to a countable union, there exists a family $(B_a)_{a\in A}$ of balls of positive radius and a real number $C > 0$ such that

$$\mu\Big(\bigcap_{a\in A} G^a_{B_a,C}\Big) > 0.$$

By Lemma 49, there exists on open set $O \subseteq X$ containing all the non totally irrational directions such that $\mu(O) < \mu\big(\bigcap_{a\in A} G^a_{B_a,C}\big)$. Let $Y = \bigcap_{a\in A} G^a_{B_a,C} \setminus O$.

First of all we claim that

$$\bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} F^{-n}(Y) \subseteq G.$$

Indeed, the first two conditions of Definition 50 are clearly satisfied since $Y \subseteq Z_C$ ensures exponential convergence and total irrationality. If m is inside $\bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} F^{-n}(Y)$, then there exist infinitely many n such that $F^n(m)$ belongs to Y. Using Remark 95, we obtain the third condition of Definition 50. The last condition follows from the fact that the set O is open.

Now we apply the Poincaré recurrence theorem: We have $\mu(Y) > 0$, thus

$$\mu\Big(\bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} F^{-n}(Y)\Big) > 0.$$

We deduce $\mu(G) > 0$. By Lemma 103, G is an F-invariant set, thus by ergodicity we have $\mu(G) = 1$. □

Combining Lemmas 101, 102, and 104 yields a proof of Proposition 92.

6.3 Proof of Theorem B

By hypothesis, we can apply Proposition 92. We deduce that $\mu(G) = 1$. Now we apply Theorem C for each point $x \in G$: the pieces of the Rauzy fractal induce a nice generating partition of the translation by $q(e_0 - v(x))$ on the torus P/Λ. The associated symbolic coding is a measurable conjugacy with the uniquely ergodic subshift $\Omega_{s(x)}$ associated with x. If $\psi\colon P/\Lambda \to \mathbb{T}^d$ is an isomorphism, then the subshift is measurably conjugate to the translation by $t_x = \psi(q(e_0 - v(x)))$ on the torus $\mathbb{T}^d$.

6.4 Finding seed points

The conditions to be a seed point are not easily checked in general. The goal of this section is to see how seed points with a periodic directive sequence can effectively be found.

In this section, for a finite sequence $(\sigma_0, \dots, \sigma_{k-1})$ in S^+, we denote by τ the substitution $\tau = \sigma_{[0,k)} = \sigma_0 \circ \cdots \circ \sigma_{k-1}$, and by s the periodic directive sequence $s = (\sigma_0 \dots \sigma_{k-1})^\omega$. The following subroutines will be used in the proof of Propositions 108 and 110.

Subroutine 105.
Input: *A finite sequence of substitutions* $(\sigma_0, \dots, \sigma_{k-1}) \in S^+$.
Algorithm: *If* τ *is Pisot and if the characteristic polynomial of the matrix* $ab(\tau)$ *is irreducible, then return the direction* $x \in \mathbb{PR}^d_+$ *which is the class of a Perron eigenvector of* $\mathrm{ab}(\tau)$. *Otherwise, reject.*

If Subroutine 105 rejects, then s cannot be exponentially convergent or there does not exist a direction x with directive sequence s that is totally irrational. If the subroutine returns a direction x, then x is a totally irrational direction (Lemma 67), s is exponentially convergent with direction x, and τ is a primitive substitution since the characteristic polynomial of $\mathrm{ab}(\tau)$ is irreducible. Moreover, the direction x is computable, and we can decide whether x satisfies a given rational linear inequality.

Subroutine 106.
Input: *A finite sequence of substitutions* $(\sigma_0, \dots, \sigma_{k-1}) \in S^+$ *such that Subroutine 105 returns a direction* x.
Algorithm: *Iterate over all letters* $a \in A$ *and all the finitely many fixed points* u_0 *of* τ^m, *where* $m = \mathrm{lcm}(\{1, \dots, d+1\})$. *For each pair* (a, u_0), *use [42, Theorem 5.12] to decide whether the interior of* $W_a(u_0)$ *is empty for the topology* $\mathcal{T}(x)$, *by checking whether the regular language describing it (denoted by* $\mathring{L}$ *in that paper) is empty, which is decidable. If it is the case for some pair* (a, u_0), *accept. Otherwise, reject.*

By the second point of Lemma 13, for every fixed point $u \in \mathrm{Fix}(s)$, the infinite word u_0 is a fixed point of τ^m. The primitivity of τ implies that each fixed point of τ^m is determined by its first letter, so there are finitely many of them. Hence, Subroutine 106 accepts if, and only if, there exist a letter $a \in A$ and a fixed point $u \in \mathrm{Fix}(s)$ such that $W_a(u_0)$ has non-empty interior for the topology $\mathcal{T}(x)$, which corresponds to the last condition of Definition 90.

Remark that the computation in [42] is done for the bi-infinite topology, but it is possible to adapt it to our setting.

Definition 107. *A* rational polytope *is a subset Q of $\mathbb{PR}_+^d$ such that there exists an integer $k \geq 1$ and a matrix $M \in \mathcal{M}_{k,d+1}(\mathbb{Q})$ with no zero row such that*

$$\{x \in \mathbb{PR}_+^d \mid Mx < 0\} \subseteq Q \subseteq \{x \in \mathbb{PR}_+^d \mid Mx \leq 0\}.$$

We say that an extended continued fraction algorithm $(\mathbb{PR}_+^d, s_0)$ is rational *if, for every σ in S, $s_0^{-1}\{\sigma\}$ is a rational polytope.*

Proposition 108. *Let $(\mathbb{PR}_+^d, s_0)$ be a rational extended continued fraction algorithm. Determining, given a finite sequence of substitutions $(\sigma_0, \ldots, \sigma_{k-1}) \in S^+$, whether there exists a seed point $x \in G_0$ for $(\mathbb{PR}_+^d, s_0)$ such that $s(x) = (\sigma_0 \ldots \sigma_{k-1})^\omega$, is decidable.*

Proof. Let us provide a decision algorithm, together with a proof of correctness.

Let us consider an input $(\sigma_0, \ldots, \sigma_{k-1}) \in S^+$.

We run Subroutine 105, and reject if the subroutine rejects.

Otherwise, we check whether the direction x returned by Subroutine 105 belongs to the open rational polytope

$$\bigcap_{l=0}^{k-1} \mathrm{ab}(\sigma_{[0,l)}) \overset{\circ}{\widehat{s_0^{-1}\{\sigma_l\}}}$$

where $\overset{\circ}{\widehat{s_0^{-1}\{\sigma_l\}}}$ denotes the interior of $s_0^{-1}\{\sigma_l\}$.

If x does not belong to this polytope, we reject since s cannot be a directive sequence produced by $(\mathbb{PR}_+^d, s_0)$. If it does, we know that:

- the directive sequence of x is s,
- x belongs to X_0, by Remark 75,
- $(M_n(x))$ is exponentially convergent.

Then, we run Subroutine 106 to decide whether the last condition of Definition 90 is satisfied. We accept if, and only if, this subroutine accepts.

□

Proposition 108 allows to decide whether a given substitution τ produces a seed point. But the substitution has to be guessed first, and the extended continued fraction algorithm has to be rational. Now, we will see how we can find a seed point with periodic directive sequence as long at it exists (semi-decision), and weaken the rationality hypothesis on the continued fraction algorithm. If we remove the rationality hypothesis on the continued fraction algorithm, we should still assume some computability to query it. Since computable functions are continuous and since $s_0 \colon X \to S$ is in general not continuous (S is discrete), the classical notion of computability is too restrictive. However, we are not interested in the behaviour of s_0 on the boundary of the sets $s_0^{-1}\{\sigma\}$ (for $\sigma \in S$) since being a seed point requires being a continuity point of s_0, so we can define a weaker notion of computability.

Definition 109. *We say that an extended continued fraction (X, s_0) is* interior-computable *if*

- *it is possible to semi-decide whether a computable element $x \in \mathbb{PR}_+^d$ belongs to X,*
- *for every σ in S, the interior of $s_0^{-1}\{\sigma\}$ is a recursively enumerable open set (see e.g. [53]): there exists a Turing machine that outputs a sequence of pairs $(c_i, r_i) \in \mathbb{PQ}_+^d \times \mathbb{Q}_+^*$ such that*

$$\overset{\circ}{\overbrace{s_0^{-1}\{\sigma\}}} = \bigcup_{i \in \mathbb{N}} B(c_i, r_i) \cap X.$$

Proposition 110. *Let (X, s_0) be an interior-computable extended continued fraction algorithm. Determining whether there exists a seed point $x \in X$ with a periodic directive sequence $s(x)$ is semi-decidable.*

Proof. Let us provide a semi-decision algorithm, together with a proof of correctness.

Using a scheduler, we start a parallel computation for every finite sequence of substitutions $(\sigma_0, \ldots, \sigma_{k-1}) \in S^+$.

For each sequence $(\sigma_0, \ldots, \sigma_{k-1})$, we run Subroutine 105. We ignore the sequence if the subroutine rejects. If the subroutine returns an element $x \in \mathbb{PR}_+^d$, we start a computation to semi-decide whether $x \in X$. If this computation finishes by accepting x as an element of X, from the interior-computability of (X, s_0), we compute a recursive enumeration (c_i, r_i) of the open set

$$\bigcap_{l=0}^{k-1} \mathrm{ab}(\sigma_{[0,l)}) \overset{\circ}{\overbrace{s_0^{-1}\{\sigma_l\}}}.$$

In parallel, we check whether x belongs to some $B(c_i, r_i)$: this might loop forever, but if x belongs to some $B(c_i, r_i)$, it will eventually be found. When it is the case, we deduce that:

- the directive sequence of x is s,
- x is totally irrational (by Subroutine 105),
- $s_0 \circ F^n$ is continuous at x for all $n \in \mathbb{N}$ since it is continuous for all $n \in \{0, \ldots, k-1\}$ and $F^k(x) = x$,
- $(M_n(x))$ is exponentially convergent.

Then, we run Subroutine 106 to decide whether the last condition of Definition 90 is satisfied. If this subroutine accepts, we return the point x as a seed point with periodic directive sequence s. Otherwise, we ignore (and let the algorithm continue with other finite sequences of substitutions).

Hence, if there exists a periodic seed point, it will eventually be found.

□

Remark 111. *The last condition of Definition 90 can also be checked for directions with periodic directive sequence using the balanced pair algorithm, see [40] and [51].*

7 Examples of continued fraction algorithms

Here we list some classical examples of continued fraction algorithms and we check if the hypotheses of Theorem B are fulfilled.

7.1 Classical one-dimensional continued fraction algorithm

Let $d = 1$. The algorithm is defined on the whole $X = \mathbb{PR}^1_+$. Let $S = \{\tau_0, \tau_1\}$, where

$$\tau_0 = \left(\begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 01 \end{array} \right), \quad \tau_1 = \left(\begin{array}{ccc} 0 & \mapsto & 10 \\ 1 & \mapsto & 1 \end{array} \right).$$

Remark that this example is constructed on the same set S as in Example 18, and that the abelianization of the substitutions are

$$\mathrm{ab}(\tau_0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{ab}(\tau_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We define the extended continued fraction algorithm as:

$$s_0 = \left(\begin{array}{ccc} X & \longrightarrow & S \\ [(x_0, x_1)] & \longmapsto & \begin{cases} \tau_0 & \text{if } x_0 \geq x_1 \\ \tau_1 & \text{if } x_0 < x_1 \end{cases} \end{array} \right).$$

The associated continued fraction algorithm is:

$$F = \left(\begin{array}{ccc} X & \longrightarrow & X \\ [(x_0, x_1)] & \longmapsto & \begin{cases} [(x_0 - x_1, x_1)] & \text{if } x_0 \geq x_1 \\ [(x_0, x_1 - x_0)] & \text{if } x_0 < x_1 \end{cases} \end{array} \right).$$

This algorithm is known as the *additive continued fraction algorithm* in dimension one, see [4].

Remark that with the change of coordinates $x = \frac{x_0}{x_1}$, we obtain the map

$$\left(\begin{array}{ccc} [0, +\infty] & \longrightarrow & [0, +\infty] \\ x & \longmapsto & \begin{cases} x - 1 & \text{if } x \geq 1 \\ \frac{x}{1-x} & \text{if } x < 1 \end{cases} \end{array} \right).$$

There exists an ergodic invariant measure for this algorithm which is absolutely continuous with respect to the Lebesgue measure, its density can be expressed as $\frac{1}{x}$ in the above coordinate system, but this measure has infinite volume. So we cannot apply our Theorem B.

The usual acceleration of this algorithm restricted to $(0,1)$ is given by the map $\left(\begin{array}{ccc} (0,1) & \longrightarrow & (0,1) \\ x & \longmapsto & \{\frac{1}{x}\} \end{array}\right)$, defined almost everywhere. This map has an invariant ergodic probability measure which is absolutely continuous with respect to the Lebesgue measure, with density $\frac{1}{\log 2}\frac{1}{1+x}$, see [4]. But it cannot be described with a finite number of matrices, so we cannot apply our Theorem B with this acceleration either.

However, this additive algorithm is well-known, and for every totally irrational direction, fixed points of the directive sequence $s(x)$ are constituted of Sturmian words. See [46] for more details. It could be shown that for every totally irrational direction $x \in \mathbb{PR}^1_+$, the directive sequence $s(x)$ is good. Hence, we deduce by Theorem C that for such a direction x there exists a generating partition of the translation by $q(e_0 - v(x))$ on the torus $P/\Lambda \simeq \mathbb{T}^1$ such that the associated symbolic coding is a measurable conjugacy with the subshift $\Omega_{s(x)}$. We easily check that, when x is a totally irrational direction, $T_x = T_{q(e_0 - v(x))}$ spans the set of irrational translations of P/Λ.

On the other hand, the complexity of the subshift $\Omega_{s(x)}$ is $p(n) = n+1$. Thus, we get, for every irrational translation on $\mathbb{T}^1$, a generating partition giving a symbolic coding of complexity $n+1$. This partition is made of two intervals, and we recognize the classical conjugacy between irrational translations of $\mathbb{T}^1$ and Sturmian subshifts, see [46] for more details.

7.2 Brun algorithm

Now we give an example of a continued fraction algorithm which does not have associated substitutions (*i.e.*, not an extended continued fraction algorithm). Let $d = 2$ (but the definition easily generalizes to any $d \geq 1$). Let X be $\mathbb{PR}^2_+$. For $\zeta \in S_3$ (the permutation group on the set $\{0,1,2\}$) we define $X_\zeta = \{[(x_0,x_1,x_2)] \in X \mid x_{\zeta(0)} < x_{\zeta(1)} < x_{\zeta(2)}\}$. Then we define the six matrices

$$B_{012} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B_{021} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{120} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_{102} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B_{201} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{210} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we define $M_0 \colon X \to GL_3(\mathbb{Z})$ by $M_0(x) = B_\zeta$ if $x \in X_\zeta$. If x is in X and not in some X_ζ, then we extend the definition arbitrarily. The following will not depend on these choices.

The Brun algorithm is then defined, as any algorithm of continued fraction, by

$$F = \left(\begin{array}{ccc} X & \longrightarrow & X \\ x & \longmapsto & (M_0(x))^{-1}x \end{array}\right).$$

In other words, the algorithm subtracts from the largest coordinate the largest of the remaining ones.

Lemma 112. *[3] The following function is a density function of an invariant probability measure for F:*

$$\frac{(x_0+x_1+x_2)^3}{2x_{\zeta(1)}(x_{\zeta(0)}+x_{\zeta(2)})x_{\zeta(2)}}$$

where (x_0, x_1, x_2) is any representative of $x \in X_\zeta$.

Lemma 113. *Let μ be the invariant probability measure given by Lemma 112. Then (X, F, μ) is a measured continued fraction algorithm.*

Proof. We have to check the hypotheses of Definition 77. Property 1 is proved in [38, 50]. Properties 2, 3 and 4 follow from the fact that μ is absolutely continuous with respect to the Lebesgue measure. Note that here X_0 is just the set of totally irrational directions by Remark 75. Property 5 is obtained by applying Proposition 78 on the partition (X_ζ). □

General conditions that permit to check the Pisot condition for the Brun algorithm with the measure μ are given in [6].

As said at the beginning of this section, the Brun algorithm is not an extended continued fraction algorithm, but we can extend it. In [36], some choices have been made to associate a finite set of substitutions with this algorithm. Denoting b_ζ the substitution with matrix B_ζ such that $b_\zeta(a)$ starts with a for every letter $a \in \{0, 1, 2\}$, we find that

$$b_{210}b_{021}b_{102} = \begin{pmatrix} 0 & \mapsto & 0210 \\ 1 & \mapsto & 10 \\ 2 & \mapsto & 210 \end{pmatrix} = \begin{pmatrix} 0 & \mapsto & 10 \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 0 \end{pmatrix}^3.$$

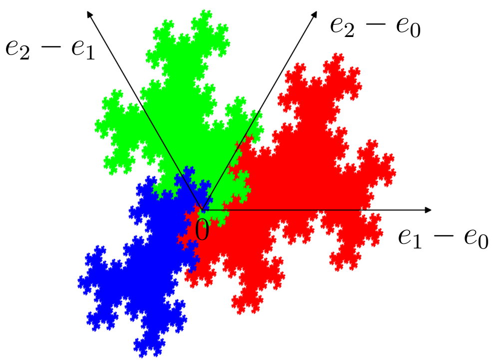

Figure 8: Rauzy fractal of the directive sequence $(b_{210}b_{021}b_{102})^\omega$.

The Brun extended continued fraction algorithm is rational (see Definition 107), so we can use Proposition 108 and check that the Perron eigenvector of the substitution $b_{210}b_{021}b_{102}$ is a seed point. Therefore, we can apply Theorem B, and we get:

Proposition 114. *For μ-almost every point x of X, the S-adic subshift associated with x is measurably conjugate to a translation on the torus P/Λ.*

7.3 Arnoux-Rauzy algorithm

Let again $d = 2$. The Arnoux-Rauzy extended continued fraction algorithm is defined by

$$s_0 = \left(\begin{array}{ccc} X & \longrightarrow & S \\ [(x_0, x_1, x_2)] & \longmapsto & \begin{cases} \mathrm{ar}_0 & \text{if } x_0 > x_1 + x_2 \\ \mathrm{ar}_1 & \text{if } x_1 > x_0 + x_2 \\ \mathrm{ar}_2 & \text{if } x_2 > x_0 + x_1 \end{cases} \end{array} \right)$$

where $S = \{\mathrm{ar}_0, \mathrm{ar}_1, \mathrm{ar}_2\}$ with

$$\mathrm{ar}_0 = \left(\begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 10 \\ 2 & \mapsto & 20 \end{array} \right), \quad \mathrm{ar}_1 = \left(\begin{array}{ccc} 0 & \mapsto & 01 \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \end{array} \right), \quad \mathrm{ar}_2 = \left(\begin{array}{ccc} 0 & \mapsto & 02 \\ 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \end{array} \right).$$

The associated continued fraction algorithm is

$$F = \left(\begin{array}{ccc} X & \longrightarrow & X \\ [(x_0, x_1, x_2)] & \longmapsto & \begin{cases} [(x_0 - x_1 - x_2, x_1, x_2)] & \text{if } x_0 > x_1 + x_2 \\ [(x_0, x_1 - x_0 - x_2, x_2)] & \text{if } x_1 > x_0 + x_2 \\ [(x_0, x_1, x_2 - x_1 - x_0)] & \text{if } x_2 > x_0 + x_1 \end{cases} \end{array} \right).$$

In other words, the algorithm subtracts from the largest coordinate the sum of the other ones. Here again we extend this definition to the boundaries of the sets with an arbitrary choice. In this case, the set X is defined a posteriori as the subset of points of $\mathbb{PR}_+^2$ for which F^n is defined for all $n \in \mathbb{N}$. This set is known as *the Rauzy gasket*, and is depicted in Figure 9.

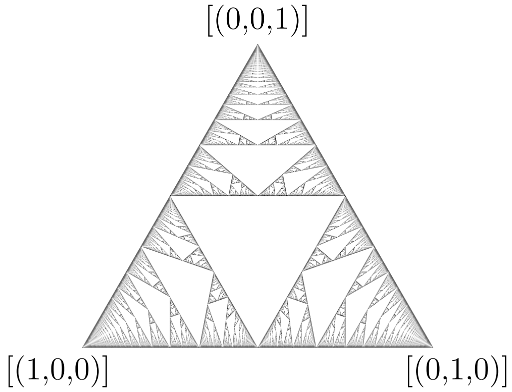

Figure 9: The Rauzy gasket $X \subseteq \mathbb{PR}_+^2$.

For this set S of substitutions, the Dumont-Thomas alphabet is $\Sigma = \{0, e_0, e_1, e_2\}$, and the abelianized prefix automaton $\mathcal{A}$ is depicted in Figure 10.

The Arnoux-Rauzy algorithm has been well studied, see [5, 9]. In [6], some sufficient conditions for a measured continued fraction algorithm to satisfy the Pisot condition are given. One of these conditions is independent of the ergodic measure. It is called the Pisot property. The Arnoux-Rauzy algorithm satisfies the Pisot property [6].

It appears that this algorithm has a lot of ergodic measures. One of them has been introduced in [8]. This measure is a good candidate, but we have not checked whether it fulfils all the hypotheses needed in Definition 77.

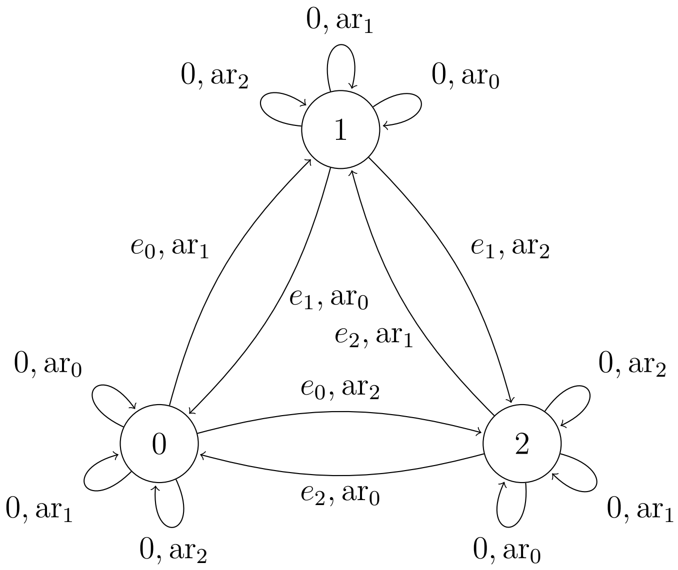

Figure 10: Abelianized prefix automaton for the Arnoux-Rauzy substitutions.

8 Application: Cassaigne algorithm and two-dimensional translations

First we define the Cassaigne extended measured continued fraction algorithm, and show that it fulfills the hypotheses of Theorem B. Then we will prove Theorem A. In all this section $d = 2$.

8.1 Description of the algorithm

The algorithm is defined on the whole $X = \mathbb{PR}_+^2$. Let μ be the measure on Δ with density $\frac{a}{(1-x_0)(1-x_2)}$ with respect to the Lebesgue measure on Δ, where a is chosen so that $\mu(\Delta) = 1$. It can be viewed as a probability measure on X thanks to the bijection v between X and Δ.

Let $S = \{c_0, c_1\}$, where

$$c_0 = \begin{pmatrix} 0 & \mapsto & 0 \\ 1 & \mapsto & 02 \\ 2 & \mapsto & 1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 0 & \mapsto & 1 \\ 1 & \mapsto & 02 \\ 2 & \mapsto & 2 \end{pmatrix}.$$

We define the extended continued fraction algorithm as:

$$s_0 = \begin{pmatrix} X & \longrightarrow & S \\ [(x_0, x_1, x_2)] & \longmapsto & \begin{cases} c_0 & \text{if } x_0 \geq x_2 \\ c_1 & \text{if } x_0 < x_2 \end{cases} \end{pmatrix}.$$

The associated continued fraction algorithm is:

$$F = \begin{pmatrix} X & \longrightarrow & X \\ [(x_0, x_1, x_2)] & \longmapsto & \begin{cases} [(x_0 - x_2, x_2, x_1)] & \text{if } x_0 \geq x_2 \\ [(x_1, x_0, x_2 - x_0)] & \text{if } x_0 < x_2 \end{cases} \end{pmatrix}.$$

The matrices associated with the substitutions c_0 and c_1 are:

$$\mathrm{ab}(c_0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{ab}(c_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

For this set S of substitutions, the Dumont-Thomas alphabet is $\Sigma = \{0, e_0\}$, and the abelianized prefix automaton $\mathcal{A}$ is depicted in Figure 11.

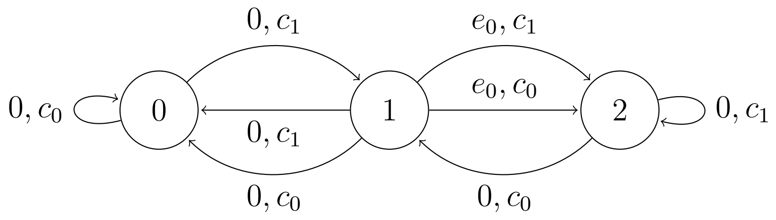

Figure 11: Abelianized prefix automaton for $S = \{c_0, c_1\}$.

Lemma 115. *(X, s_0, μ) is an extended measured continued fraction algorithm and satisfies the Pisot condition.*

Proof. We refer to [3] for a proof of the F-invariance of the measure μ.

By [50], we know that the Selmer algorithm is ergodic. Moreover, we know that the Cassaigne algorithm is conjugate to the semi-sorted form of Selmer algorithm [20]. Ergodicity of the semi-sorted form of Selmer algorithm cannot be simply deduced from ergodicity of the usual (sorted) form, but arguments of [50] can be adapted to prove it, thus we deduce the ergodicity of the measure μ. See also [29] for a direct proof.

It is well-known that for the Selmer algorithm the second Lyapunov exponent is strictly negative, with Lebesgue-almost surely $\mathrm{codim}(E_2(x)) = 1$, see [44]. As before we cannot directly deduce the Pisot condition, but the same method can be applied with the semi-sorted Selmer algorithm. Thus, by conjugacy we deduce $\theta_2(\mu) < 0$ and $\mathrm{codim}(E_2(x)) = 1$ for μ-almost every $x \in X$, so the Pisot condition is satisfied. The Cassaigne algorithm fulfills the conditions of Proposition 78 with the hyperplane $\{x \in \mathbb{PR}_+^2 \mid x_0 = x_2\}$, since μ is absolutely continuous with respect to the Lebesgue measure. Hence, (X, s_0, μ) is an extended measured continued fraction algorithm. □

Figure 12 illustrates approximations of Rauzy fractals $R(x)$ obtained by choosing points $x \in \mathbb{PR}_+^2$ randomly for the Lebesgue measure and applying the Cassaigne algorithm to compute the directive sequence up to a certain integer n that we choose in order to have enough points, and enough precision. We then plot the set of points

$$Q_a = \{\sum_{k=0}^{n} \pi_x(M_{[0,k)} t_k) \mid b \xrightarrow{t_n, s_n} \ldots \xrightarrow{t_0, s_0} a\}$$

with a color depending on the letter a.

If the Rauzy fractal associated with the point $F^{n+1}(x)$ is bounded and not too large (which occurs with high probability), then the Hausdorff distance between the

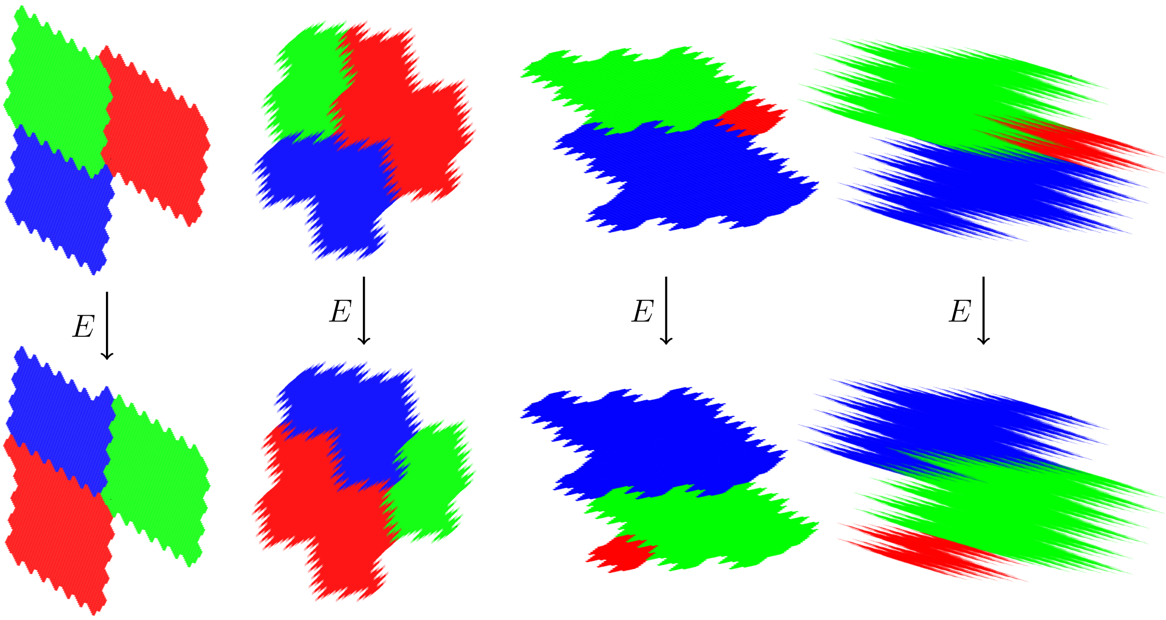

Figure 12: Approximations of random Rauzy fractals, and the associated domain exchanges.

approximation Q_a and the Rauzy fractal $R(x)$ is at most a few pixels. Rauzy fractals of this article have been drawn using the Sage mathematical software and the badic package. These are available here: `https://www.sagemath.org/` and `https://gitlab.com/mercatp/badic`.

8.2 There exists a seed point

In this section, we show that a particular substitution gives a seed point for the Cassaigne continued fraction algorithm. Since the Cassaigne algorithm is rational, it would be sufficient to apply the algorithm of Proposition 108 to (c_0, c_1). However, we would like to present a more geometric approach, that relies on an identification of the projection plane P with the complex field $\mathbb{C}$.

We consider the substitution $c_0 c_1 = \begin{pmatrix} 0 & \mapsto & 02 \\ 1 & \mapsto & 01 \\ 2 & \mapsto & 1 \end{pmatrix}$. Let $w = (c_0 c_1)^\omega(0) \in A^{\mathbb{N}}$ be its unique fixed point. Its abelianization is the matrix $M = \mathrm{ab}\,(c_0 c_1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

The characteristic polynomial of M is $X^3 - 2X^2 + X - 1$. It is an irreducible polynomial over $\mathbb{Q}$, with one real root greater than 1 and two complex roots of modulus less than 1. Thus, the substitution $c_0 c_1$ is Pisot unimodular.

Let β be the eigenvalue of M with negative imaginary part. Let $x_0 \in \mathbb{PR}^2_+$ be the class of a Perron eigenvector of M. The goal of this section is to prove that x_0 is a seed point:

Proposition 116. *The point x_0 is in G_0 (see Definition 90).*

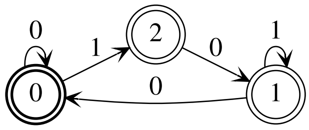

Figure 13: Automaton describing the projection by ϕ of the worm $W(w)$.

We need to prove several lemmas first.

Lemma 117. *The point x_0 is in X_0, see Definition 73.*

Proof. The point x_0 is a totally irrational direction by Lemma 67. We deduce the continuity of $s_0 \circ F^n$ at x_0 for every $n \in \mathbb{N}$ by Remark 75. □

The projection plane P can be identified with $\mathbb{C}$, so that $\pi_{x_0}M$ acts by multiplication by β:

Lemma 118. *Consider the linear map ϕ from $\mathbb{R}^3$ to $\mathbb{C}$ given by $\phi(v) = ev$, for the line vector $e = (1, \beta^2 - \beta, \beta - 1)$. This map induces a bijection between P and $\mathbb{C}$. For every $v \in \mathbb{R}^3$, we have $\phi(Mv) = \beta\phi(v)$ and $\phi(\pi_{x_0}v) = \phi(v)$.*

Proof. Remark that the line vector e is a left-eigenvector of M for the eigenvalue β. Let $v \in \mathbb{R}^3$. We have $\phi(v) = ev$, so $\phi(Mv) = eMv = \beta ev = \beta\phi(v)$. As x_0 is the class of a right eigenvector of M for an eigenvalue different from β, we have $ex_0 = 0$, thus $\phi(\pi_{x_0}v) = \phi(v - h(v)v(x_0)) = \phi(v)$.

Now we check that the rank of ϕ is 2, so its kernel is the vector space spanned by x_0, which intersects P only at 0. Thus, ϕ induces a bijection between P and $\mathbb{C}$. □

With Lemmas 34 and 118, we can project the worm $W(w)$ on the complex plane

$$\phi(W_a(w)) = \{\sum_{k=0}^{n-1} t_k\beta^k \mid n \in \mathbb{N},\ 0 \xrightarrow{t_{n-1}} \ldots \xrightarrow{t_0} a\} \subseteq \mathbb{C},$$

where $0 \xrightarrow{t_{n-1}} \ldots \xrightarrow{t_0} a$ denotes a path in the automaton $\phi(\mathcal{A})$ of Figure 13 (we do not label the edges by the substitution since there is only one in this case). This automaton $\phi(\mathcal{A})$ is obtained from the abelianized prefix automaton $\mathcal{A}$ for the substitution c_0c_1 by replacing edge labels (t, c_0c_1) with $\phi(t)$.

Figure 14 shows the Rauzy fractal R of the directive sequence $(c_0c_1)^\omega$ and its image $\phi(R)$ by ϕ.

The following lemma permits to find good bounding boxes (for example good disks) that contain the pieces of the Rauzy fractal.

Lemma 119. *If there exist $(O_a)_{a\in A}$ bounded open subsets of $\mathbb{C}$ and an integer $n \in \mathbb{N}$ such that for every $a \in A$ we have*

$$\bigcup_{b \xrightarrow{t_{n-1}} \ldots \xrightarrow{t_0} a} \left(\beta^n\overline{O_b} + \sum_{k=0}^{n-1} t_k\beta^k\right) \subseteq O_a,$$

then for all $a \in A$, we have $\phi(R_a) \subseteq O_a$.

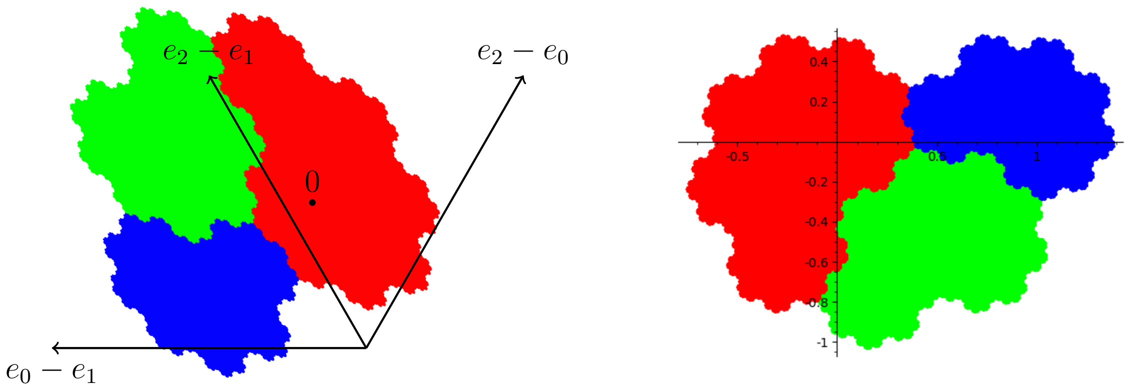

Figure 14: Rauzy fractal of $(c_0c_1)^\omega$ in P (left) and its projection by ϕ on $\mathbb{C}$ (right).

Proof. Let $y \in \phi(R_a)$. By Corollary 45, there exists an infinite path $\ldots \xrightarrow{t_n} \ldots \xrightarrow{t_0} a$ in the automaton $\phi(\mathcal{A})$ of Figure 13 such that

$$y = \sum_{k=0}^{\infty} t_k \beta^k.$$

Let $b \in A$ such that $b \xrightarrow{t_{n-1}} \ldots \xrightarrow{t_0} a$ is a path in the automaton. Let l be the distance between $\beta^n \overline{O_b} + \sum_{k=0}^{n-1} t_k \beta^k$ and the complement of O_a. As O_b is bounded, the first set is compact so that $l > 0$. We denote by D the usual distance on $\mathbb{C}$. Let $k \in \mathbb{N}$ be large enough such that $\left|\beta^{kn}\right| \max_{c \in A} \sup_{z \in \phi(R)} D(z, \overline{O_c}) < l$, and let $c \in A$ such that $c \xrightarrow{t_{kn-1}} \ldots \xrightarrow{t_n} b \xrightarrow{t_{n-1}} \ldots \xrightarrow{t_0} a$ is a path in the automaton. We have $\sum_{j=kn}^{\infty} t_j \beta^{j-kn} \in \phi(R)$, again by Corollary 45. So $D(\sum_{j=kn}^{\infty} t_j \beta^j, \beta^{kn} \overline{O_c}) < l$.

Thus, $D(y, \beta^{kn} \overline{O_c} + \sum_{j=0}^{kn-1} t_j \beta^j) < l$. Moreover, we have the inclusion

$$\beta^{kn} \overline{O_c} + \sum_{j=0}^{kn-1} t_j \beta^j \subseteq \beta^n \overline{O_b} + \sum_{k=0}^{n-1} t_k \beta^k$$

by iterating $k-1$ times the inclusion of the hypothesis. So, we get that y is in O_a. □

Corollary 120. *For the Rauzy fractal associated with $(c_0c_1)^\omega$ we have the inclusions*

$$\begin{aligned} \phi(R_0) &\subseteq O_0 = B(-0.19 - 0.15i, 0.75), \\ \phi(R_1) &\subseteq O_1 = B(0.5 - 0.6i, 0.655), \\ \phi(R_2) &\subseteq O_2 = B(0.865 + 0.123i, 0.566). \end{aligned}$$

Proof. We use Lemma 119 for $n = 8$, and check the result by computer, see Figure 15. □

In the following we denote by z_a and r_a the center and the radius of the ball O_a for $a = 0, 1, 2$.

Now we can prove Proposition 116:

Proof of Proposition 116. We check all the conditions in Definition 90 to show that x_0 is a seed point:

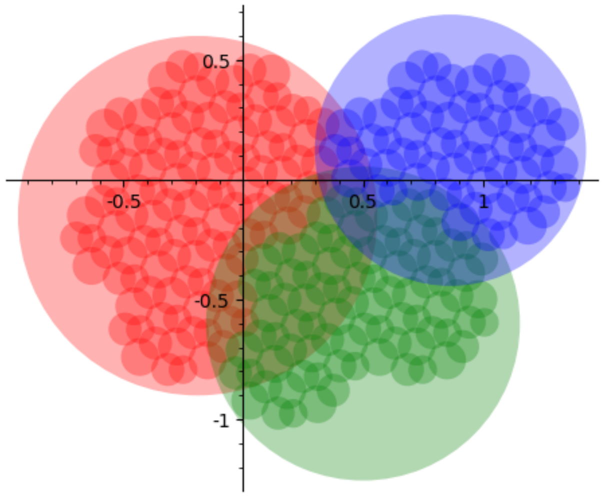

Figure 15: Bounding balls found from Lemma 119 for $n = 8$.

- By Lemma 117, $x_0 \in X_0$.
- We have

$$\lim_{n\to\infty} \frac{1}{n} \ln \left|\left|\left|\pi_{x_0} M_{[0,n)}(x_0)\right|\right|\right|_1 = \lim_{n\to\infty} \frac{1}{2n} \ln \left|\left|\left|\pi_{x_0} M^n\right|\right|\right|_1 = \frac{1}{2} \ln |\beta| < 0,$$

so $(M_n(x_0))$ is exponentially convergent.

- We have $0 \notin R_1 \cup R_2$ by Corollary 120 since $0 \notin B(0.5 - 0.6i, 0.655) \cup B(0.865 + 0.123i, 0.566)$.
- For $t \in \Lambda \setminus \{0, e_1 - e_2, e_2 - e_1\}$, we check that $|\phi(t)| > 1.5 > \max_{a\in A} r_a + |z_a|$, so by Corollary 120, we get that $0 \notin R + t$.

  For $t \in \{e_1 - e_2, e_2 - e_1\}$, we check that for all $a \in A$, $|z_a + \phi(t)| > r_a$, thus $0 \notin R + t$.
- We have $0 \notin R_1 \cup R_2 \cup \bigcup_{t\in\Lambda\setminus\{0\}} R + t$, so by Lemma 38 we have that 0 is in the interior of $W_0(w)$. In particular, the interior of $W_0(w)$ is non-empty. Furthermore, there exists a fixed point $u \in (A^{\mathbb{N}})^{\mathbb{N}}$ of the directive sequence $s(x_0) = (c_0c_1)^\omega$ such that $u_0 = w$ is the fixed point of the substitution c_0c_1. □

8.3 Proof of Theorem A

We refer to [19, Proposition 6] for the proof of the following result:

Lemma 121. *Consider a directive sequence s in $S^{\mathbb{N}}$, where $S = \{c_0, c_1\}$. Assume that s cannot be written as a finite sequence followed by an infinite concatenation of c_0^2 and c_1^2 (which is the case if the direction of s is totally irrational). Then Ω_s is minimal and has complexity $2n + 1$.*

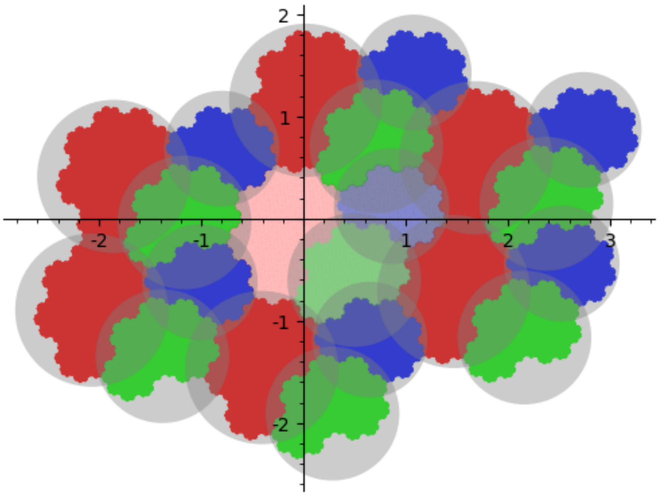

Figure 16: Proof that $0 \notin R_1 \cup R_2 \cup \bigcup_{t \in \Lambda \setminus \{0\}} R + t$ thanks to covering with balls.

Now we proceed with the proof of Theorem A. Let (X, s_0, μ) be the Cassaigne extended measured continued fraction algorithm. By Lemma 115 and Proposition 116, it satisfies the hypotheses of Theorem B. Since μ is absolutely continuous with respect to the Lebesgue measure, the conclusion holds for Lebsegue-almost every $x \in \mathbb{PR}_+^2$.

The map $x \mapsto t_x$ of Theorem B does not map $\mathbb{PR}_+^2$ to $\mathbb{T}^2$, though: as $t_x = \psi(q(e_0 - v(x)))$, its image is $\psi(q(e_0 - \Delta))$. The translation vectors $t_x = (a_1, a_2) \in [0,1]^2$ satisfy $a_1 + a_2 \geq 1$. However, remark that a symbolic coding of the translation by $-t_x$ can be easily constructed from the coding of the translation by t_x, and that $(e_0 - \Delta) \cup (\Delta - e_0)$ is a measurable fundamental domain of P for the action of Λ. Thus, the set $\{\psi(q(e_0 - v(x))) \mid x \in G\} \cup \{\psi(q(v(x) - e_0)) \mid x \in G\}$ is of full measure in $\mathbb{T}^2$.

Hence, we get for Lebesgue-almost every translation on $\mathbb{T}^2$ a nice generating partition such that the associated symbolic coding is a conjugacy with the subshift. With Lemma 121 we deduce the result.

9 Renormalization schemes

The Poincaré recurrence theorem ensures that, given a measurable map $T\colon Z \to Z$ which preserves a finite measure on Z, and for a measurable set $U \subseteq Z$ of positive measure, we can define the *induced map* (or *first return map*) on the set U as:

$$T_U = \left(\begin{array}{ccc} U & \longrightarrow & U \\ z & \longmapsto & T^{r_U(z)}(z) \end{array} \right),$$

where $r_U(z) = \inf\{n > 0 \mid T^n(z) \in U\}$ is finite for almost every point z of U, so T_U is defined almost everywhere on U.

Finding consistent induction subsets for a family of dynamical systems is an important step in the construction of a renormalization scheme.

9.1 Dimension 1

In dimension 1, the following is well-known [46]. There is, up to translation, only one fundamental interval for the action of $\mathbb{Z}$ on $\mathbb{R}$, $I = [0,1]$. If we lift a translation T_t on the 1-dimensional torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ to the fundamental interval I, we get a piecewise translation on I with a discontinuity at $1-t$ (assuming $t \in [0,1]$). Hence, it is natural to partition the unit interval into two pieces $([0,1-t],[1-t,1])$. If we induce on the larger piece, and rescale linearly to recover an interval of length 1, we get a new translation on the torus $\mathbb{T}^1$. The resulting translation vector is obtained from the original one by applying one step of the classical additive continued fraction algorithm (see Section 7.1). The process can be iterated indefinitely unless t is rational.

This construction translates to the following renormalization scheme. Let $\mathcal{P}$ denote the family of irrational translations on the torus endowed with a partition of a fundamental interval into two pieces as above:

$$\mathcal{P} = \{(T_t, ([0,1-t],[1-t,1])) \mid t \in [0,1] \setminus \mathbb{Q}\}$$

The renormalization scheme, which is the composition of inducing on the larger piece, rescaling, and recreating the new pieces, can be written as:

$$\mathcal{R} = \left(\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{P} \\ (T_t, ([0,1-t],[1-t,1])) & \longmapsto & \begin{cases} (T_{\frac{t}{1-t}}, ([0,\frac{1-2t}{1-t}],[\frac{1-2t}{1-t},1])) & \text{if } t < 1/2 \\ (T_{\frac{2t-1}{t}}, ([0,\frac{1-t}{t}],[\frac{1-t}{t},1])) & \text{if } t > 1/2 \end{cases} \end{array} \right).$$

This renormalization scheme does not depend on the fundamental interval chosen when lifting, the induction set is prescribed by the discontinuity and the (positive) scaling factor is prescribed by the length of the induction set. Hence, the renormalization scheme is somehow canonical.

9.2 Dimension 2 with the Cassaigne algorithm

In higher dimension, there is no obvious induction set. There is not even an obvious fundamental domain to lift the translation. The goal of this section is to describe, in an informal discussion, a renormalization scheme in dimension 2 that relies on the Rauzy fractals that we constructed for the Cassaigne algorithm.

Let us first look at the symbolic level. Let us consider a good directive sequence $s = (s_k)$ for the Cassaigne algorithm, and let u be one of its fixed points. Assume first that s starts with c_0. The word $u_0 = c_0(u_1)$ is a concatenation of the three finite words 0, 02 and 1. Those three words are *return words* on the pair $\{0,1\}$, *i.e.*, any word in Ω_{u_0} starting with 0 or 1 can be written in a unique way as a concatenation of 0, 02 and 1, and 0 and 1 appear only at the first positions of those three words. Hence, inducing the subshift Ω_{u_0} on the clopen set $[0] \cup [1] = \Omega_{u_0} \setminus [2]$ leads to a subshift conjugate to Ω_{u_1}, whose directive sequence is (s_{k+1}).

Now, assume that the directive sequence s starts with c_1. In this case, the images of the letters by c_1 are not return words, and we have to look backwards: the reverse of the images of the letters by c_1, that is 1, 20 and 2, are return words on the pair $\{1,2\}$ in the left-infinite word obtained by reversing u_0. A workaround could be to reverse

$c_1(1)$ in the definition of c_1 to be 20 as this would not change the continued fraction algorithm; however, this would increase the complexity of the associated subshift, which we cannot afford. Instead, we remark that inducing on $T([1]) \cup T([2]) = T(\Omega_{u_0} \setminus [0])$, where T denotes the shift map, leads again to a subshift conjugate to Ω_{u_1}, whose directive sequence is again (s_{k+1}).

All those remarks translate to the geometrical level. As in Section 8.3, we work on half the translations, namely those of the form T_x. Let T_x be a translation on the torus P/Λ for some good point $x \in G$, let E_x be the associated lifted domain exchange on the Rauzy fractal $R(x) \subseteq P$, which is partitioned into three pieces $(R_0(x), R_1(x), R_2(x))$. To define the induction set, we distinguish two cases:

- (*bottom type*) If $\lambda(R_0(x)) \geq \lambda(R_2(x))$, let $U = R_0(x) \cup R_1(x)$.
- (*top type*) If $\lambda(R_0(x)) < \lambda(R_2(x))$, let $U = E_x(R_1(x)) \cup E_x(R_2(x))$.

Let us notice that the vector $(\lambda(R_0(x)), \lambda(R_1(x)), \lambda(R_2(x)))$ belongs to the direction x by Lemma 65 and that the conditions for *bottom* and *top* are the same as the conditions defining the sets $s_0^{-1}\{c_0\}$ and $s_0^{-1}\{c_1\}$.

Then, the induced map $(T_x)_{q(U)}$ is conjugate to the translation $T_{F(x)}$. Indeed, U is a measurable fundamental domain of P for the action of the lattice $\Lambda' = \eta\Lambda$, where η is the linear endomorphism of P such that $\eta \circ \pi_{Fx} = \pi_x \circ \mathrm{ab}(s_0(x))$ that was introduced in Section 4.2.2. The induced map $(T_x)_{q(U)}$ can be *rescaled* to the translation $T_{F(x)}$ on the reference torus P/Λ: the linear map that sends U to $R(F(x))$ is η^{-1}.

Given a translation equipped with the partition of the torus P/Λ into three pieces, we described a way to define an induction set and to rescale the induced map into a new translation on the original torus P/Λ. This induced-rescaled map inherits the partition $(R_0(F(x)), R_1(F(x)), R_2(F(x)))$. This partition can be described without relying on the knowledge of $F(x)$, so that the renormalization scheme sustains itself. To this end, let us consider the *bottom* type (the *top* type is similar): the induction set U is the union of the two pieces $R_0(x)$ and $R_1(x)$. But while the return time in U of any element of $R_1(x)$ is 1, the return time of elements of $R_0(x)$ is either 1 or 2. Hence, the piece $R_0(x)$ can be subdivided into two pieces $R_0(x) \cap r_U^{-1}\{1\}$ and $R_0(x) \cap r_U^{-1}\{2\} = r_U^{-1}\{2\}$, and we get the partition into three pieces $(R_0(x) \cap r_U^{-1}\{1\}, r_U^{-1}\{2\}, R_1(x))$ which refines the partition $(R_0(x), R_1(x))$ (note that the order of the new pieces matters for the next induction step).

To sum up, if $G \subseteq \mathbb{PR}_+^2$ denotes the set of good points and $\mathcal{G} \subseteq S^{\mathbb{N}}$ denotes the set of good sequences, let us denote $\mathcal{P} = \{(T_x, (R_0(x), R_1(x), R_2(x))) \mid x \in G\}$ the set of translations on the torus P/Λ endowed with the partition of a fundamental domain of P/Λ into three pieces that can be obtained by applying Theorem B for the Cassaigne algorithm. Let $\mathcal{R}\colon \mathcal{P} \to \mathcal{P}$ denote the renormalization scheme which consists in applying the operations of inducing, rescaling and partitioning as described above. Let $\vec{\lambda}\colon \mathcal{P} \to \mathbb{PR}_+^2$ denote the map $(T, (P_0, P_1, P_2)) \mapsto [(\lambda(P_0), \lambda(P_1), \lambda(P_2))]$. Let $\iota\colon \mathcal{G} \to \mathcal{P}$ denote the map $s \mapsto (T_x, (R_0(x), R_1(x), R_2(x)))$, where x is the direction of the directive sequence s. Then, the following diagram is commutative:

$$\begin{array}{ccccccc}
\mathcal{P} & \xrightarrow{\vec{\lambda}} & G \subseteq \mathbb{PR}_+^2 & \xrightarrow{s} & \mathcal{G} \subseteq S^{\mathbb{N}} & \xrightarrow{\iota} & \mathcal{P} \\
\mathcal{R}\big\downarrow & & \big\downarrow F & & \big\downarrow \text{shift} & & \big\downarrow \mathcal{R} \\
\mathcal{P} & \xrightarrow{\vec{\lambda}} & G \subseteq \mathbb{PR}_+^2 & \xrightarrow{s} & \mathcal{G} \subseteq S^{\mathbb{N}} & \xrightarrow{\iota} & \mathcal{P}
\end{array}$$

It is remarkable that this scheme is pretty similar to the famous Rauzy-Veech induction for interval exchange maps [47] (we named the *top* and *bottom* types after the naming scheme of [54]).

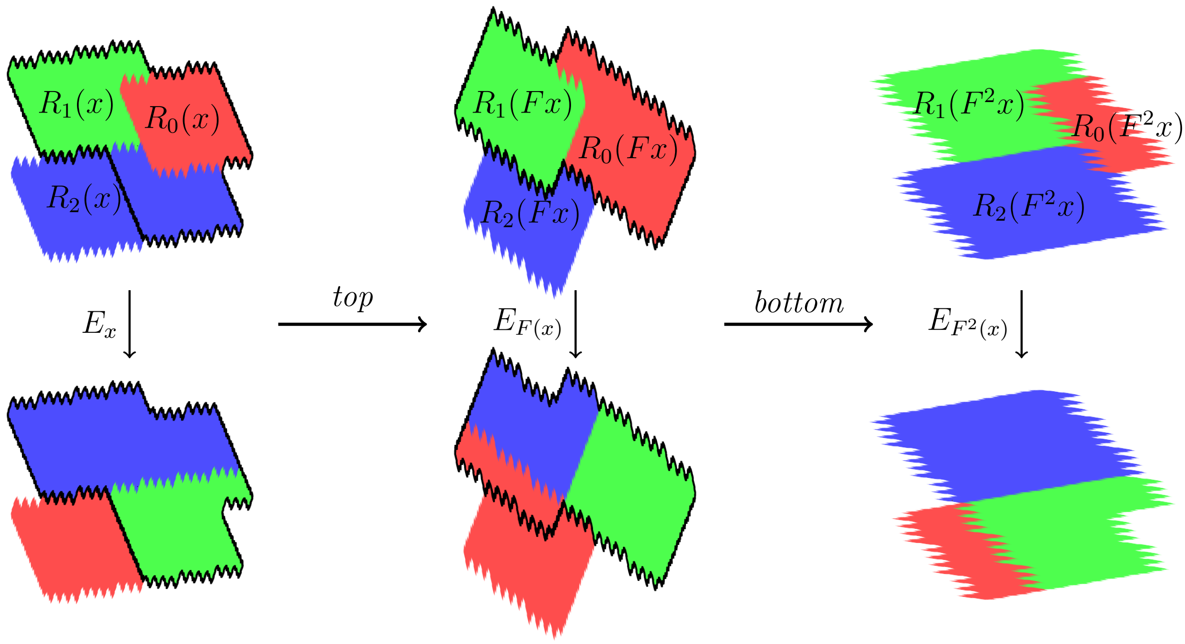

Figure 17: Two steps of induction, rescaling, partitioning. The sets of induction are outlined in black. Note that the pictures look flipped (and stretched) from one step to the next one, this is due to the fact that the rescaling matrices have negative determinant.

Figure 17 shows two steps of induction, rescaling and partitioning starting from the direction $x = [(0.256005715380561..., 0.286881483823029..., 0.457112800796410...)]$. The associated directive sequence is $s(x) = c_1c_0c_1c_0c_1c_0c_0c_0c_1c_0c_0c_0c_1c_1c_0c_0c_0c_0c_0c_0c_1c_1c_0c_1c_0...$ It corresponds to the translation by

$$t = (0.743994284619438..., -0.286881483823029..., -0.457112800796410...)$$

on the torus P/Λ. The figure shows the Rauzy fractals $R(x)$, $R(Fx)$ and $R(F^2x)$, with R_0 in red, R_1 in green and R_2 in blue. The upper row of the figure shows the decomposition $R(y) = R_0(y) \cup R_1(y) \cup R_2(y)$, while the lower row shows the decomposition $R(y) = (R_0 + \pi_y(e_0)) \cup (R_1(y) + \pi_y(e_1)) \cup (R_2(y) + \pi_y(e_2))$ obtained after applying the domain exchange E_y, for $y = x, F(x), F^2(x)$ respectively.

9.3 Other algorithms

This construction of a renormalization scheme can be generalized to some other extended continued fraction algorithms. For the Brun algorithm, which was extended in Section 7.2 with the substitutions as in [36], the complement of the image of the second largest piece of the Rauzy fractal is a suitable induction set U (in this case, all types are *top*). Another choice of substitutions will lead to a different renormalization scheme.

For the Arnoux-Rauzy algorithm (see Section 7.3), we cannot conclude so easily since, due to the lack of a probability measure satisfying the hypotheses of Definition 77, we could not identify a large subset of the Rauzy gasket for which we could construct Rauzy fractals. However, from the symbolic description of the algorithm, we could bet that once there will be such a construction, the image of the largest piece of the Rauzy fractal will be a convenient induction set.

10 Remarks and open problems

10.1 Comments on the results of other papers

In [14] the authors prove some theorems on the same subject. Their Theorem 3.1 is in the same spirit as our Theorem B. Their proof follows the same lines as ours: start from a single "seed" directive sequence, use the control provided by the Pisot condition to extend that property to a set of positive measure, then propagate it almost everywhere by ergodicity. Regarding the construction of Rauzy fractals, they rely on the paper [13] which is in the same spirit as our Theorem C but needs some hypotheses such as irreducibility and balancedness for the directive sequences or coincidence conditions on the subshift. In our theorem these conditions are not assumed, and are replaced with our notion of good directive sequence. Also, their Theorem 3.8 is in the same spirit as our Lemma 65. Finally, their Corollary 6.3 is in the same spirit as our Theorem A.

10.2 Translation vectors *vs* directions

The link between a continued fraction algorithm and torus translations was done by mapping every direction $x \in \mathbb{PR}^d_+$ to a translation vector of $\mathbb{T}^d$ via the composition:

$$\chi\colon\ \mathbb{PR}^d_+ \xrightarrow{v} \Delta \xrightarrow{f} P \xrightarrow{q} P/\Lambda \xrightarrow{\psi} \mathbb{T}^d\,.$$

The map $f = \begin{pmatrix} \Delta & \longrightarrow & P \\ y & \longmapsto & e_0 - y \end{pmatrix}$ is affine (and injective), which is why we could transport results holding for almost every direction to results holding for almost every torus translation. Note however that the map χ is not surjective, so that we had to project the simplex twice to cover all possible torus translations in dimension 2 in the proof of Theorem A (Section 8.3). If t is an element of $\mathbb{T}^2$, either t or $-t$ is the image of some direction $x \in \mathbb{PR}^2_+$. Since the translation T_t is conjugate to the translation T_{-t}, we get the result for almost every translation on $\mathbb{T}^2$.

In higher dimensions, if we identify $\mathbb{T}^d$ with the unit hypercube $[0,1]^d$, the image of $\mathbb{PR}^d_+$ is the convex hull S_d of $\{0, e_1, \ldots, e_d\}$, whose Lebesgue measure is only $1/d!$.

If $\alpha \in GL_d(\mathbb{Z})$ is an automorphism of $\mathbb{T}^d$, T_t is conjugate to $T_{\alpha(t)}$. As shown in [30], there exist an explicit finite family $(\alpha_i)_{0\leq i<d!}$ of elements of $GL_d(\mathbb{Z})$ and a family $(n_i)_{0\leq i<d!}$ of elements of $\mathbb{Z}^d$ such that $[0,1]^d = \bigcup_{0\leq i<d!} \alpha_i(S_d) + n_i$, that is,

$$\mathbb{T}^d = \bigcup_{0\leq i<d!} \alpha_i(\chi(\mathbb{PR}^d_+)).$$

Such a tiling is also known as a *Kuhn triangulation* [39].

Therefore, if we want to go from a particular translation T_t on $\mathbb{T}^d$ to a projective direction and study its dynamics through continued fractions, it suffices to find to which atom $\alpha_i(S_d)$ of the triangulation it belongs, and to associate the direction $x = \chi^{-1}(\alpha_i^{-1}(t))$ (note that χ is injective, except on the finite set $\{[e_0], \ldots, [e_d]\}$, which is mapped to $\{0\}$).

10.3 Exceptional directions in the Cassaigne algorithm

To finish with the Cassaigne algorithm, we list some sets of non-generic directions. Let $x = [(x_0, x_1, x_2)] \in \mathbb{PR}^2_+$ be a direction and $s(x)$ be its associated directive sequence. We have equivalence between $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}(x_0, x_1, x_2) = 1$ and the fact that the sequence $s(x)$ is eventually constant. Moreover, $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}(x_0, x_1, x_2) = 1$ if, and only if, $s(x)$ is not everywhere growing. Also, the property $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}(x_0, x_1, x_2) \leq 2$ is equivalent to the fact that $s(x)$ can be written as the concatenation of a finite sequence followed by an infinite concatenation of c_0^2 and c_1^2 (even runs) [19, Lemma 1].

Those directions are somehow exceptional for trivial reasons. A natural question is to understand the set of translation vectors in $\mathbb{T}^2$ for which the conclusion of Theorem A holds. Our proof shows that it is a subset of measure one in the set of totally irrational translations. Can we extend the result of Theorem A to all totally irrational translations? It is not possible with our proof, but maybe we can use some other continued fraction algorithm, or some unrelated method. Indeed, there are subshifts defined by the Cassaigne algorithm which are not balanced [1], so there are directions where we cannot use this algorithm to construct symbolic codings of translations on $\mathbb{T}^2$. More generally, one may ask whether some subshifts defined by the Cassaigne algorithm are weakly mixing, see [18].

10.4 Optimality of the complexity

Recall that for a minimal translation on $\mathbb{T}^2$, the complexity $2n + 1$ obtained by the Cassaigne algorithm is the lowest that can be reached using nice generating partitions, since they are liftable [11].

However, without the liftability condition, it is possible to artificially lower the complexity by a simple recoding. For example, we can replace the partition $(q(R_a))_{a\in A}$ with the partition $(q(R_0), q(R_1 \cup R_2))$, and still get a regular generating partition (however not liftable) that yields a subshift with complexity at most $2n$ for $n \geq 1$. See [20] for more on computing the complexity of the subshifts produced by the Cassaigne algorithm.

10.5 Higher dimensions

Another natural question is to generalize Theorem A for $d \geq 3$. A good candidate could be the Brun algorithm which can be defined in any dimension d. Numerical experiments tend to show that that the second Lyapunov exponent should be negative if $d < 10$ [15] [37]. It is not an extended continued fraction algorithm, but we can extend it. It seems that for any reasonable choice of substitutions, the complexity is linear. It would remain to find a seed point in order to generalize Theorem A for $d < 10$, with some higher but linear complexity.

The example of Section 7.2 (the Rauzy fractal of which is shown on Figure 8) is a seed point for $d = 2$. Its natural generalization gives also a seed point for $d = 3$, but fails for $d \geq 4$ as the substitution is no longer a Pisot substitution. However, if we instead consider

$$b_{210}^2 b_{021}^2 b_{102}^2 = \begin{pmatrix} 0 & \mapsto & 100 \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 0 \end{pmatrix}^3$$

then its generalization to any dimension seems to give a seed point for the Brun algorithm:

$$\begin{pmatrix} 0 & \mapsto & 100 & \\ i & \mapsto & (i+1) & \text{if } 1 \leq i \leq d-1 \\ d & \mapsto & 0 & \end{pmatrix}^{d+1}.$$

10.6 Pisot substitution conjecture

As explained in Corollary 70, we can relate Theorem C to the Pisot substitution conjecture. We can restate the Pisot substitution conjecture [2] as:

Conjecture 122 (Reformulation of the Pisot substitution conjecture)**.** *For every irreducible Pisot unimodular substitution σ, the directive sequence σ^ω is good.*

A generalization of the Pisot substitution conjecture could be:

Conjecture 123 (Generalization of the Pisot substitution conjecture)**.** *Let S be a set of unimodular substitutions. Let $s \in S^{\mathbb{N}}$ be a directive sequence such that there exist a totally irrational direction $x \in \mathbb{PR}_+^d$ and a constant $C > 0$ such that for every k and $n \in \mathbb{N}$, $\left\vert\left\vert\left\vert \pi_{M_{[0,k)}^{-1} x} M_{[k,k+n)} \right\vert\right\vert\right\vert_1 \leq Ce^{-n/C}$. Then, s is good.*

The conjecture could be made even more general:

Conjecture 124 (Further generalization of the Pisot substitution conjecture)**.** *Let S be a set of unimodular substitutions. Let $s \in S^{\mathbb{N}}$ be a directive sequence such that there exists a totally irrational direction $x \in \mathbb{PR}_+^d$ such that $\sum_n \left\vert\left\vert\left\vert \pi_{M_{[0,k)}^{-1} x} M_{[k,k+n)} \right\vert\right\vert\right\vert_1$ converges uniformly in k. Then the subshift associated with s is measurably conjugate to a translation on the torus $\mathbb{T}^d$.*

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Nomenclature

Greek alphabet

β	complex eigenvalue, 58
Δ	simplex, 6
δ	Hausdorff distance, 45
η_k	endomorphism of P, 29
$\eta_{[k,n)}$	product $\eta_k \dots \eta_{n-1}$, 29
$\theta_i(x)$	Lyapunov exponent, 39
ι	map from $\mathcal{G}$ to $\mathcal{P}$, 64
Λ	integer lattice in P, 6
Λ'	integer lattice after induction, 64
λ	Lebesgue measure, 6
$\vec{\lambda}$	map from $\mathcal{P}$ to $\mathbb{PR}^d_+$, 64
μ	Borel measure, 10
π_y, π_x	projection on P along y or $v(x)$, 6
ρ	Tribonacci number, 3
Σ	Dumont-Thomas alphabet, 19
σ	generic substitution, 9
τ_0, τ_1	Sturmian substitutions, 52
φ	linear form on $\mathbb{R}^{d+1}$, 40
ϕ	linear map from $\mathbb{R}^3$ to $\mathbb{C}$, 59
χ	map from $\mathbb{PR}^d_+$ to $\mathbb{T}^d$, 66
ψ	torus isomorphism from P/Λ to $\mathbb{T}^d$, 6
$\Omega(n^d)$	growth rate, 3
Ω_s	S-adic subshift, 11
Ω_w	subshift generated by w, 10
w^ω	periodic word, 10
σ^ω	constant directive sequence, 12

Latin alphabet

A	alphabet, 9
A^*	finite words, 9
$A^{\mathbb{N}}$	infinite words, 9
$\mathcal{A}$	abelianized prefix automaton, 19
ab	abelianization map, 9
$\mathrm{ar}_0, \mathrm{ar}_1, \mathrm{ar}_2$	Arnoux-Rauzy substitutions, 55
$B(p, r)$	ball in the hyperplane P, 6
$B(x, r)$	ball in the projective space, 6
$\mathcal{B}$	Bratteli diagram, 14
b_ζ	substitution for Brun algorithm, 54
c_0, c_1	Cassaigne substitutions, 56
cod	symbolic coding, 10
D	distance on $\mathbb{C}$, 60
d	dimension, 5
$d(x, y)$	distance, 6
E	domain exchange, 11

$E_i(x)$	Lyapunov space, 39
E_x	domain exchange, 32
$(e_i)_{0\le i\le d}$	basis of $\mathbb{R}^{d+1}$, 5
F	extended continued fraction algorithm, 36
f	limit of (f_k), 30
f_k	measure of the boundary of the Rauzy fractal, 29
$\mathrm{Fix}(s)$	set of fixed points, 12
$\mathrm{freq}(w)$	frequency vector, 10
G	good points, 43
G_0	seed points, 42
G_1	auxiliary set related to good points, 43
$G^a_{B,C}$	seed set with explicit bound, 43
$\mathcal{G}$	good sequences, 64
g	limit of (g_k), 30
g_k	measure of Rauzy fractal, 29
$\mathbb{H}$	integer half-space, 14
h	sum of coordinates, 6
$\mathrm{hom}(A^*, A^*)$	substitutions, 9
$\mathrm{hom}(A^+, A^+)$	non-erasing substitutions, 9
(k_n)	integer sequence, 27
L	linear map from P to $\mathbb{R}^d$, 6
$M_k(s)$	kth matrix of s, 11
$M_{[k,n)}$	product $M_k \dots M_{n-1}$, 8
$\mathcal{M}_{d+1}(\mathbb{R})$	square matrices, 6
$\mathcal{M}_{k,d+1}(\mathbb{Q})$	rectangular matrices, 50
n	integer, 3
$\mathcal{O}(w)$	orbit, 10
P	hyperplane where h cancels, 6
$\mathcal{P}$	family of translations, 63
$p(n)$	complexity function, 9
$\mathbb{PR}^d_+$	set of positive directions, 6
q	quotient map $P \to P/\Lambda$, 6
$R(s)$	Rauzy fractal of a directive sequence, 24
$R(w)$	Rauzy fractal of a word, 21
$R(x)$	Rauzy fractal associated with a direction, 39
$R_a(w)$	piece of the Rauzy fractal, 21
$\mathcal{R}$	renormalization scheme, 63
r_U	return time, 62
S	finite set of substitutions, 11
$s = (s_k)$	directive sequence, 11
$s = s(x)$	directive sequence associated with x, 37
$s_{[k,n)}$	product of substitutions, 11
s_0	map defining an extended continued fraction algorithm, 36
T	shift map, 10
T_t	translation on the torus by vector t, 7
T_x	translation on the torus associated with direction x, 7

T_U	induced map, 62
$\mathcal{T}(x)$	topology on $\mathbb{H}$, 15
$\mathbb{T}^d$	torus, 6
t_x	translation vector associated with direction x, 5
U	open subset of P, 15
$u = (u_k)$	word sequence, usually fixed point, 12
V	open subset of $\mathbb{H}$ for some topology $\mathcal{T}(x)$, 15
$v(x)$	representative of x of norm 1, 6
$v^{(k)}$	vector of norm 1, 29
w	word, 9
$W(w)$	worm, 15
$W_a(w)$	subset of a worm, 15
X	base set of a dynamical system, 10
$x^{(k)}$	kth element of a sequence of directions, 25
Z	subset of measure one in Oseledets theorem, 39
Z_C	set of points with explicit exponential convergence, 43

Other symbols

$[f]$	cylinder, 10
$[y]$	direction of the vector y, 6
$\|w\|$	length of a word, 9
$\|w\|_f$	number of occurrences of f in w, 9
$\mathbb{1}_A$	indicator function, 11
$\\|\cdot\\|_1$	norm, 5
$\\|\\|\\|M\\|\\|\\|_1$	operator norm, 7
$\\|\\|\\|M_{\\|V}\\|\\|\\|_1$	operator semi-norm, 7
v°	orthogonal in the dual space, 40
$\overline{\square}$	topological closure, 10
$\overset{\circ}{\widehat{\square}}$	topological interior, 50

References

[1] M. Andrieu. *Exceptional trajectories in the symbolic dynamics of multidimensional continued fraction algorithms*. PhD thesis, Aix-Marseille Université, 2021.

[2] P. Arnoux and S. Ito. Pisot substitutions and Rauzy fractals. *Bull. Belg. Math. Soc. Simon Stevin*, 8(2):181–207, 2001.

[3] P. Arnoux and S. Labbé. On some symmetric multidimensional continued fraction algorithms. *Ergodic Theory Dynam. Systems*, 38(5):1601–1626, 2018.

[4] P. Arnoux and A. Nogueira. Mesures de Gauss pour des algorithmes de fractions continues multidimensionnelles. *Ann. Sci. École Norm. Sup. (4)*, 26(6):645–664, 1993.

[5] P. Arnoux and G. Rauzy. Représentation géométrique de suites de complexité $2n+1$. *Bull. Soc. Math. France*, 119(2):199–215, 1991.

[6] A. Avila and V. Delecroix. Some monoids of Pisot matrices. In *New trends in one-dimensional dynamics*, volume 285 of *Springer Proc. Math. Stat.*, pages 21–30. Springer, Cham, 2019.

[7] A. Avila and G. Forni. Weak mixing for interval exchange transformations and translation flows. *Ann. Math. (2)*, 165(2):637–664, 2007.

[8] A. Avila, P. Hubert, and A. Skripchenko. Diffusion for chaotic plane sections of 3-periodic surfaces. *Invent. Math.*, 206(1):109–146, 2016.

[9] A. Avila, P. Hubert, and A. Skripchenko. On the Hausdorff dimension of the Rauzy gasket. *Bull. Soc. Math. France*, 144(3):539–568, 2016.

[10] N. Bédaride and J.-F. Bertazzon. Minoration of the complexity function associated to a translation on the torus. *Monatsh. Math.*, 171(3-4):291–304, 2013.

[11] J.-F. Bertazzon. Fonction complexité associée à une application ergodique du tore. *Bull. London Math. Soc.*, 44(6):1155–1168, 2012.

[12] V. Berthé and V. Delecroix. Beyond substitutive dynamical systems: S-adic expansions. In *Numeration and substitution 2012*, RIMS Kôkyûroku Bessatsu, B46, pages 81–123. Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.

[13] V. Berthé, W. Steiner, and J. M. Thuswaldner. Geometry, dynamics, and arithmetic of S-adic shifts. *Ann. Inst. Fourier (Grenoble)*, 69(3):1347–1409, 2019.

[14] V. Berthé, W. Steiner, and J. M. Thuswaldner. Multidimensional continued fractions and symbolic codings of toral translations, 2020. Electronic preprint arXiv:2005.13038. To appear in *J. Eur. Math. Soc.*

[15] V. Berthé, W. Steiner, and J. M. Thuswaldner. On the second Lyapunov exponent of some multidimensional continued fraction algorithms. *Math. Comput.*, 90(328):883–905, 2021.

[16] M. D. Boshernitzan. A condition for unique ergodicity of minimal symbolic flows. *Ergodic Theory Dyn. Syst.*, 12(3):425–428, 1992.

[17] V. Canterini and A. Siegel. Geometric representation of substitutions of Pisot type. *Trans. Am. Math. Soc.*, 353(12):5121–5144, 2001.

[18] J. Cassaigne, S. Ferenczi, and A. Messaoudi. Weak mixing and eigenvalues for Arnoux-Rauzy sequences. *Ann. Inst. Fourier (Grenoble)*, 58(6):1983–2005, 2008.

[19] J. Cassaigne, S. Labbé, and J. Leroy. A set of sequences of complexity $2n + 1$. In *Combinatorics on words*, volume 10432 of *Lecture Notes in Comput. Sci.*, pages 144–156. Springer, Cham, 2017.

[20] J. Cassaigne, S. Labbé, and J. Leroy. Almost everywhere balanced sequences of complexity $2n + 1$. *Mosc. J. Comb. Number Theory*, 11(4):287–333, 2022.

[21] J. Chaika, D. Damanik, J. Fillman, and P. Gohlke. Zero measure spectrum for multi-frequency Schrödinger operators. *J. Spectr. Theory*, 12(2):573–590, 2022.

[22] N. Chekhova, P. Hubert, and A. Messaoudi. Propriétés combinatoires, ergodiques et arithmétiques de la substitution de Tribonacci. *J. Théor. Nombres Bordeaux*, 13(2):371–394, 2001.

[23] N. Chevallier. Coding of a translation of the two-dimensional torus. *Monatsh. Math.*, 157(2):101–130, 2009.

[24] G. Didier. Combinatoire des codages de rotations. *Acta Arith.*, 85(2):157–177, 1998.

[25] J.-M. Dumont and A. Thomas. Systèmes de numération et fonctions fractales relatifs aux substitutions. *Theoret. Comput. Sci.*, 65:153–169, 1989.

[26] F. Durand. Combinatorics on Bratteli diagrams and dynamical systems. In *Combinatorics, automata and number theory*, volume 135 of *Encyclopedia Math. Appl.*, pages 324–372. Cambridge Univ. Press, Cambridge, 2010.

[27] F. Durand, J. Leroy, and G. Richomme. Do the properties of an S-adic representation determine factor complexity? *J. Integer Seq.*, 16(2):Article 13.2.6, 30, 2013.

[28] S. Ferenczi. Rank and symbolic complexity. *Ergodic Theory Dynam. Systems*, 16(4):663–682, 1996.

[29] C. Fougeron. Dynamical properties of simplicial systems and continued fraction algorithms, 2020. Electronic preprint arXiv:2001.01367.

[30] H. Freudenthal. Simplizialzerlegungen von beschränkter Flachheit. *Annals of Mathematics*, 43(3):580–582, 1942.

[31] H. Furstenberg and H. Kesten. Products of random matrices. *Ann. Math. Statist.*, 31:457–469, 1960.

[32] W. H. Gottschalk and G. A. Hedlund. *Topological dynamics*. Number 36 in Colloquium Publications. American Mathematical Society, Providence, RI, 1955.

[33] B. Host. Représentation géométrique des substitutions sur 2 lettres. Unfortunately unpublished manuscript, 1992.

[34] A. B. Katok and A. M. Stepin. Approximations in ergodic theory. *Uspehi Math. Nauk*, 22(5):81–106, 1967.

[35] L. Kronecker. Näherungsweise ganzzahlige Auflösung linearer Gleichungen. *Berl. Ber.*, 1884:1179–1193, 1884.

[36] S. Labbé. 3-dimensional continued fraction algorithms cheat sheets, 2015. Electronic preprint arXiv:1511.08399.

[37] S. Labbé. Computer experiments for the Lyapunov exponent for MCF algorithms when dimension is larger than 3. `http://www.slabbe.org/blogue/2020/03/computer-experiments-for-the-lyapunov-exponent-for-mcf-algorithms-when-dimension-is-larger-than-3/`, 2020.

[38] J. C. Lagarias. The quality of the Diophantine approximations found by the Jacobi-Perron algorithm and related algorithms. *Monatsh. Math.*, 115(4):299–328, 1993.

[39] C. W. Lee and F. Santos. Subdivisions and triangulations of polytopes. In C. D. Tóth, J. O'Rourke, and J. E. Goodman, editors, *Handbook of Discrete and Computational Geometry*, pages 415–447. CRC Press, 3rd edition, 2017.

[40] A. N. Livshits. On spectra of adic transformations of Markovian compacta. *Usp. Mat. Nauk*, 42(3(255)):189–190, 1987.

[41] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia Math. Appl.* Cambridge University Press, 2002.

[42] P. Mercat and S. Akiyama. Yet another characterization of the Pisot substitution conjecture. In *Substitution and tiling dynamics: introduction to self-inducing structures. Lecture notes from the research school on tiling dynamical systems, CIRM Jean-Morlet Chair, Marseille, France, Fall 2017*, pages 397–448. Springer, Cham, 2020.

[43] M. Morse and G. A. Hedlund. Symbolic dynamics II. Sturmian trajectories. *American Journal of Mathematics*, 62(1):1–42, 1940.

[44] K. Nakaishi. Strong convergence of additive multidimensional continued fraction algorithms. *Acta Arith.*, 121(1):1–19, 2006.

[45] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

[46] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[47] G. Rauzy. Échanges d'intervalles et transformations induites. *Acta Arith.*, 34(4):315–328, 1979.

[48] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[49] G. Rauzy. Ensembles à restes bornés. *Seminaire de Théorie des Nombres de Bordeaux*, 13:1–12, 1984.

[50] F. Schweiger. *Multidimensional continued fractions.* Oxford Science Publications. Oxford University Press, Oxford, 2000.

[51] V. F. Sirvent and B. Solomyak. Pure discrete spectrum for one-dimensional substitution systems of Pisot type. *Can. Math. Bull.*, 45(4):697–710, 2002.

[52] W. A. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. Math. (2)*, 115:201–242, 1982.

[53] K. Weihrauch. *Computable analysis. An introduction.* Springer, Berlin, 2000.

[54] J.-C. Yoccoz. Interval exchange maps and translation surfaces. In *Homogeneous flows, moduli spaces and arithmetic. Proceedings of the Clay Mathematics Institute summer school, Centro di Recerca Mathematica Ennio De Giorgi, Pisa, Italy, June 11–July 6, 2007*, pages 1–69. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2010.