Topological substitutions

Nicolas Bédaride

Université Aix Marseille

- $\mathcal{A}$ finite set: alphabet
- A substitution is a morphism of monoid $\mathcal{A}^*$.

$$\begin{cases} a \mapsto abc \\ b \mapsto cc \\ c \mapsto aa \end{cases}$$

- If $a \in \mathcal{A}$ is a prefix of $\sigma(a)$, then there exists a fixed point:

$$\sigma(u) = u$$

Example

$$\sigma : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

$$u = abaabaab\ldots$$

$$u = \lim_n \sigma^n(a)$$

Representation of u: Tiling of $\mathbb{R}$ with two tiles a, b.

Representation of u: Tiling of $\mathbb{R}$ with two tiles a, b.

Dimension two ?

Goal

- Definition of topological substitution.
- Iteration on a tile.
- Tiling of a space.

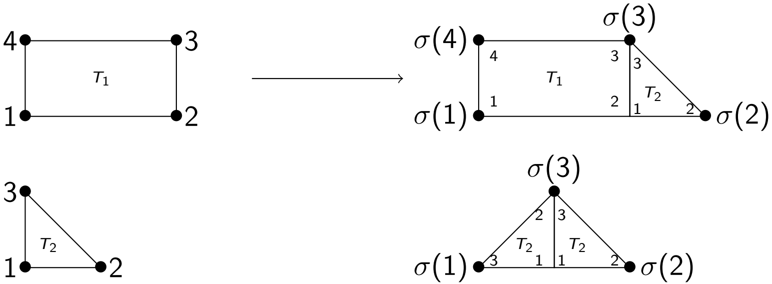
Result

Theorem (B. Hilion)

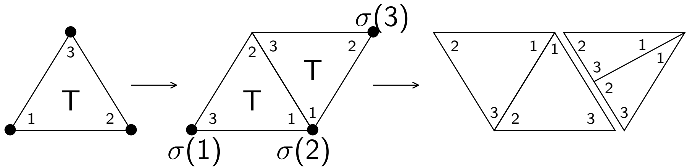
- *There does not exist a primitive substitutive tiling of the hyperbolic plane* $\mathbb{H}^2$.
- *There exists a non primitive substitutive tiling of the hyperbolic plane* $\mathbb{H}^2$.

Pre-substitution

- Cellular complex. Topological polygons $\mathcal{T} = \{T_1, \dots, T_d\}$.
- Patches $\sigma(T_i), i = 1 \dots d$.
- $(\mathcal{T}, \sigma(\mathcal{T}), \sigma)$.

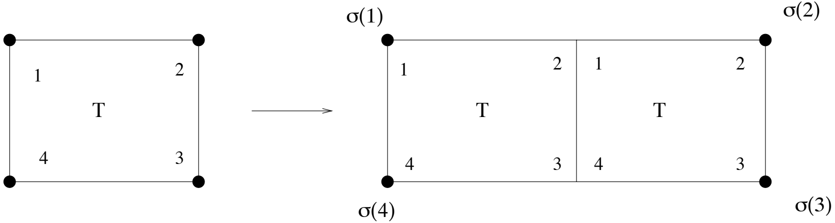


First problem



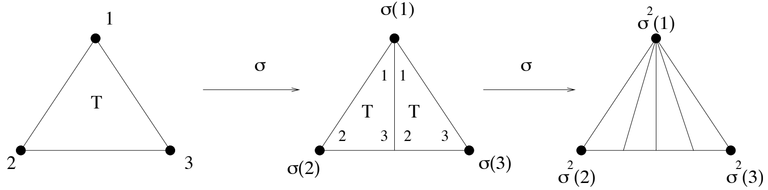
A topological pre-substitution which is not 2-compatible

Second problem



A topological pre-substitution which has no core

Third problem



The valence of $\sigma^k(1)$ in $\sigma^k(T)$ is $2^k + 1$.

The problems can be checked by finite graphs.
A topological substitution is a pre-substitution such that

- can be iterated on a tile T
- $\sigma^n(T)$ has non empty core for each integer n.

Under hypothesis we can define

$$\sigma^{\infty}(T) = \bigcup_{k=0}^{\infty} \sigma^{k}(T).$$

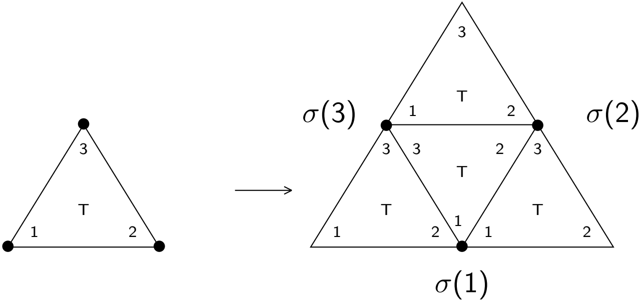
By construction, the complex $\sigma^{\infty}(T)$ is homeomorphic to $\mathbb{R}^2$.

A tiling of $\mathbb{M}$ naturally defines a 2-complex.

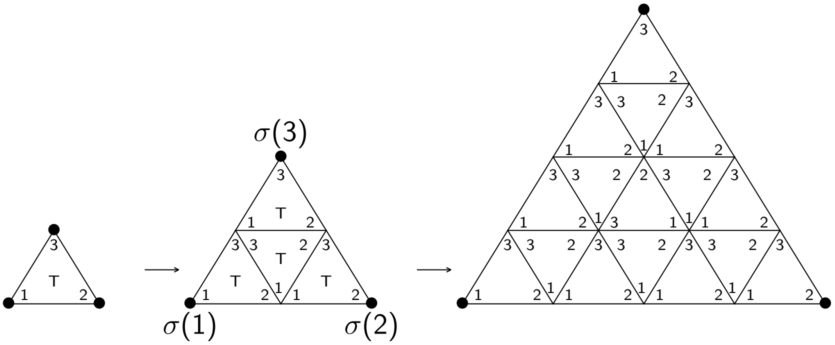
Definition

A tiling of the plane $\mathbb{M}$ is **substitutive** if the labelled complex associated to it can be obtained by inflation from a topological substitution.

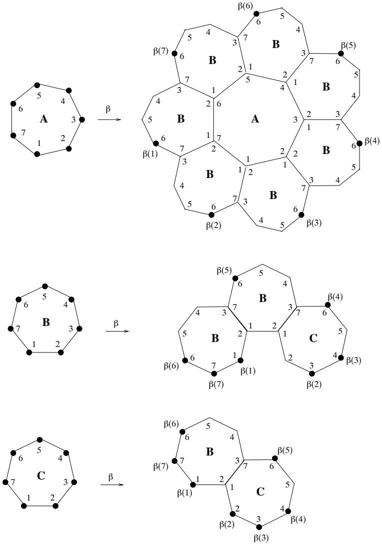
Problem

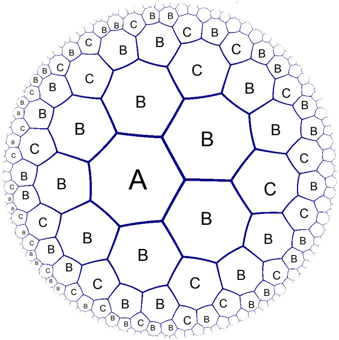


A topological substitution which gives rise to the regular tiling of $\mathbb{E}^2$ by equilateral triangles



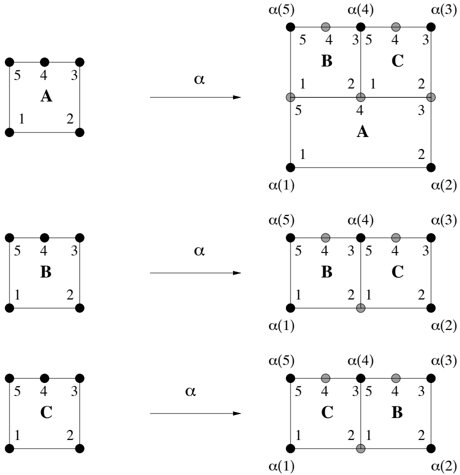
Example of a non-primitive topological substitution β



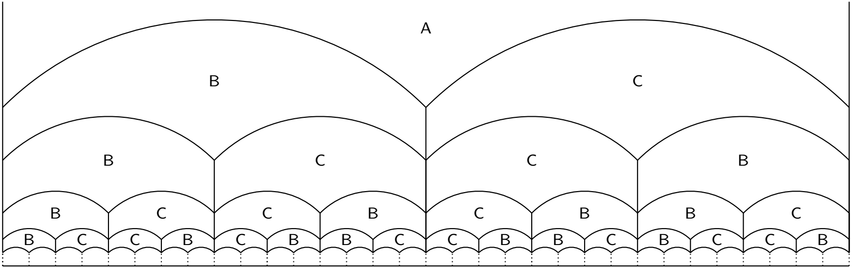


Tiling with the topological substitution β

Other example of a non-primitive topological substitution



Definition of the substitution α



Geometric realization of the complex obtained by iteration of the topological substitution α

Primitive substitution

An integer matrix $M \in \mathcal{M}_d(\mathbb{N})$ is **primitive** if there exists some $k \in \mathbb{N}$ such that all entries of M^k are positive.

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The **transition matrix** $M_\sigma \in \mathcal{M}_d(\mathbb{N})$ associated to the topological substitution σ is the matrix whose entry $m_{i,j}$ is the number of faces of type T_i in the patch $\sigma(T_j)$.

Proof

Lemma

Let σ be a primitive topological substitution. For any tile T and for any integer $n \in \mathbb{N}$, $||\sigma^n(T)|| = \lambda^n ||T||$.

Lemma (Isoperimetric inequality)

Let $\mathcal{C}$ be a piecewise $\mathcal{C}^1$ simple curve in $\mathbb{H}^2$. Then:

$$\mathcal{A}(\mathcal{C}) \leq L(\mathcal{C}).$$

Lemma

Let σ be a primitive topological substitution. There exists $\alpha, 0 < \alpha < 1$, such that for every tile T such that the core of $\sigma(T)$ contains a face of type T:

$$||\partial\sigma^n(T)|| \leq (1-\alpha)^n ||\sigma^n(T)||.$$