ORTHOGONALITY RELATIONS FOR DEEP LEVEL DELIGNE–LUSZTIG SCHEMES OF COXETER TYPE

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Abstract. In this paper we prove some orthogonality relations for representations arising from deep level Deligne–Lusztig schemes of Coxeter type. This generalizes previous results of Lusztig [Lus04], and of Chan and the second author [CI19a]. Potential applications include the study of unipotent representations arising from such deep level Deligne–Lusztig schemes, as well as their geometry, in the spirit of Lusztig's work [Lus76].

1. Introduction

In the last fifteen years various p-adic and deep level analoga of classical Deligne–Lusztig varieties attracted a lot of attention, see in particular [Lus04, Boy12, CS17, CI19b, Iva20]. The interest in them is justified by the fact that they allow to apply methods from classical Deligne–Lusztig theory to study representations of p-adic groups. Furthermore, they are very interesting geometric objects in their own right (like the classical Deligne–Lusztig varieties are). In this article we consider deep level Deligne–Lusztig schemes of Coxeter type, and prove orthogonality relations for the corresponding representations, extending a classical result of [DL76] to the deep level setup.

Let k be a non-archimedean local field with uniformizer ϖ and residue field $\mathbb{F}_q$ with q elements. Let $\breve{k}$ denote the completion of a maximal unramified extension of k with residue field $\overline{\mathbb{F}}_q$. Let $\mathbf{G}$ be an unramified reductive group over k, $\mathbf{T} \subseteq \mathbf{G}$ a k-rational unramified maximal torus, and $\mathbf{U}$ the unipotent radical of a $\breve{k}$-rational Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Let $\mathcal{G} = \mathcal{G}_{\mathbf{x}}$ be a (connected) parahoric $\mathcal{O}_k$-group scheme with generic fiber $\mathbf{G}$, whose corresponding facet $\mathbf{x}$ in the Bruhat–Tits building (over k) of the adjoint group of $\mathbf{G}$ lies in the apartment of $\mathbf{T}$, and let $\mathcal{T} \subseteq \mathcal{G}$ denote the schematic closure of $\mathbf{T}$ in $\mathcal{G}$.

Fix an integer $r \geqslant 1$. Let $G = \mathcal{G}(\mathcal{O}_k/\varpi^r)$ and $T = \mathcal{T}(\mathcal{O}_k/\varpi^r)$. In [Lus04, Sta09, CI19a], a certain (perfect) $\overline{\mathbb{F}}_q$-scheme $S_{\mathbf{T},\mathbf{U}} = S_{\mathbf{x},\mathbf{T},\mathbf{U},r}$ equipped with a natural $G \times T$-action was defined. In a sense, it can be regarded as a deep level analog of a classical Deligne–Lusztig variety[1] As in the classical Deligne–Lusztig theory, the cohomology of $S_{\mathbf{T},\mathbf{U}}$ attaches to any character $\theta\colon T \to \overline{\mathbb{Q}}_\ell^\times$ ($\ell \neq \operatorname{char} \mathbb{F}_q$) the G-representation $R_{\mathbf{T},\mathbf{U}}(\theta) = \sum_{i\in\mathbb{Z}}(-1)^i H^i_c(S_{\mathbf{T},\mathbf{U}}, \overline{\mathbb{Q}}_\ell)_\theta$. One of the central features within the classical Deligne–Lusztig theory is the *Deligne–Lusztig orthogonality relation*, which computes (in the classical case, that is $r = 1$, $\mathbf{x}$ hyperspecial) the inner product of two virtual representations $R_{\mathbf{T},\mathbf{U}}(\theta)$, $R_{\mathbf{T}',\mathbf{U}'}(\theta')$ [DL76, Thm. 6.8].

The goal of the present article is to generalize the abovementioned classical orthogonality relations to deep level schemes $S_{\mathbf{T},\mathbf{U}}$ of Coxeter type. There is a meaningful notion of a *Coxeter pair* $(\mathbf{T}, \mathbf{U})$ (cf. Section 2.6), which essentially means that $S_{\mathbf{T},\mathbf{U}}$ is the deep level analog of a classical Deligne–Lusztig variety of Coxeter type. In that case the intersection of the apartment of $\mathbf{T}$ in the k-rational Bruhat–Tits building of the adjoint group is just one vertex, $\mathbf{x}_{\mathbf{T}}$, and (as we assumed $\mathbf{G}$ to be unramified) this vertex must necessarily be hyperspecial (cf. Section 2.6). The following theorem is our main result.

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[1]For example, if $r = 1$ and $\mathbf{x}$ hyperspecial, then there is a natural map from $S_{\mathbf{T},\mathbf{U}}$ into some classical Deligne–Lusztig variety attached to the special fibers of $\mathcal{G}, \mathcal{T}, \mathcal{U}$; this map induces an isomorphism of ℓ-adic cohomology groups, up to a degree shift.

Theorem 3.2.3. *Let* $(\mathbf{T}, \mathbf{U})$, $(\mathbf{T}', \mathbf{U}')$ *be Coxeter pairs with* $\mathbf{x} = \mathbf{x}_{\mathbf{T}} = \mathbf{x}_{\mathbf{T}'}$ *(then, automatically,* $\mathbf{x}$ *is hyperspecial). Assume that* $q > 5$. *Then for all* $r \geqslant 1$ *and all* $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$, $\theta' \colon T' \to \overline{\mathbb{Q}}_\ell^\times$, *we have*

$$\langle R_{\mathbf{T},\mathbf{U}}(\theta), R_{\mathbf{T}',\mathbf{U}'}(\theta')\rangle_G = \#\left\{w \in W(\mathbf{T},\mathbf{T}')^F : \theta' = {}^w\theta\right\},$$

where $W(\mathbf{T},\mathbf{T}') = \mathbf{T}(\breve{k})\backslash\{g \in \mathbf{G}(\breve{k}) \colon {}^g\mathbf{T}' = \mathbf{T}\}/$.

Note that the assumption on q can be strengthened depending on the root system of $\mathbf{G}$, see Condition (2.7.1).

Our proof follows an idea of [Lus04] (already appearing in [DL76]) which consists in extending the $T \times T'$-action on various subschemes of $\Sigma = G\backslash(S_{\mathbf{T},\mathbf{U}} \times S_{\mathbf{T}',\mathbf{U}'})$ to an action of some torus with finitely many fixed points. This considerably simplifies the computation of the Euler characteristic of Σ. For $\mathbf{G} = \mathbf{GL}_n$ (and, essentially, for any unramified group of type A_n) Theorem 3.2.3 was proven in [CI19b, §3]. However, the general case requires several serious improvements, which are the core of the present work.

Let us explain the importance of Theorem 3.2.3. Under the additional condition that θ (or θ') is *regular*, i.e. "highly non-trivial" on $\ker(\mathcal{T}(\mathcal{O}_k/\varpi^r) \to \mathcal{T}(\mathcal{O}_k/\varpi^{r-1}))$, the result was already shown in [Lus04, Sta09]. However, from the perspective of unipotent representations, the most interesting case is that of $\theta = \theta' = 1$. In the classical situation ($r = 1$, $\mathbf{x}$ hyperspecial), the characters $R_{\mathbf{T},\mathbf{U}}(1)$ as well as the geometry and cohomology of the related Coxeter type Deligne–Lusztig varieties were studied in Lusztig's seminal work [Lus76]; one of the starting points for [Lus76] was the orthogonality relation [DL76, Thm. 6.8] with $\theta = \theta' = 1$. In light of this, Theorem 3.2.3 now opens up the way towards the study of deep level schemes $X_w^{\mathcal{G},r}(1)$ of Coxeter type (which are isomorphic to certain quotients of $S_{\mathbf{T},\mathbf{U}}$) and the corresponding "unipotent" representations (=irreducible constituents of $R_{\mathbf{T},\mathbf{U}}(1)$) of the group $\mathcal{G}(\mathcal{O}_k/\varpi^r)$ and their inflation to $\mathcal{G}(\mathcal{O}_k)$.

The schemes $X_w^{\mathcal{G},r}(1)$ are integral truncated versions of (and closely related to) the "big" p-adic Deligne–Lusztig spaces $X_w(1)$, which carry an $\mathbf{G}(k)$-action, studied in [Iva20]. At least in special cases, the "p-adic Deligne–Lusztig" $\mathbf{G}(k)$-representations in the cohomology of $X_w(1)$ realize interesting supercuspidal representations of $\mathbf{G}(k)$, related to the local Langlands correspondences (cf. [CI19b, Thm. A]), and they are compactly induced from the $\mathcal{G}(\mathcal{O}_k)$-representations $R_{\mathbf{T},\mathbf{U}}(\theta)$ considered here.

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2. Setup and preliminaries

2.1. **Some notation.** Given a group G and $g, x \in G$, we write ${}^gx = gxg^{-1}$ and $x^g = g^{-1}xg$. If θ is an irreducible character of a finite subgroup H of G then ${}^g\theta$ is the character of H^g given by ${}^g\theta(x) := \theta(gxg^{-1})$.

Let p be a prime number. Given a ring R of characteristic p, we denote by Perf_R the category of perfect R-algebras, and by $W(R)$ the (p-typical) Witt vectors of R.

Let k be a non-archimedean local field with residue field $\mathbb{F}_q$, where q is some fixed power of p. The ring of integers of k will be denoted by $\mathcal{O}_k$. Let ϖ be a uniformizer of k. Given $R \in \mathrm{Perf}_{\mathbb{F}_q}$, there is an essentially unique ϖ-adically complete and separated $\mathcal{O}_k$-algebra $\mathbb{W}(R)$, in which ϖ is not a zero-divisor and which satisfies $\mathbb{W}(R)/\varpi\mathbb{W}(R) = R$. Explicitly we have

$$\mathbb{W}(R) = \begin{cases} W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_k & \text{if } \mathrm{char}\, k = 0 \\ R[\![\varpi]\!] & \text{if } \mathrm{char}\, k = p, \end{cases}$$

i.e., in the first case $\mathbb{W}(R)$ are the ramified Witt vectors, details on which can be found for example in [FF18, 1.2]. In particular, $\mathbb{W}(\mathbb{F}_q)[1/\varpi] = k$. Fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ and put $\mathcal{O}_{\breve{k}} = \mathbb{W}(\overline{\mathbb{F}}_q)$ and $\breve{k} = \mathbb{W}(\overline{\mathbb{F}}_q)[1/\varpi]$. The field $\breve{k}$ is the ϖ-adic completion of a maximal unramified extension of k.

2.2. **Loop functors.** Let $\mathcal{X}$ be an $\mathcal{O}_{\breve{k}}$-scheme. We have the functor of positive loops and its truncations (also called Greenberg functors, following [Gre61])

$$L^+\mathcal{X}\colon \mathrm{Perf}_{\overline{\mathbb{F}}_q} \to \mathrm{Sets}, \quad (L^+\mathcal{X})(R) = \mathcal{X}(\mathbb{W}(R))$$
$$L_r^+\mathcal{X}\colon \mathrm{Perf}_{\overline{\mathbb{F}}_q} \to \mathrm{Sets}, \quad (L_r^+\mathcal{X})(R) = \mathcal{X}(\mathbb{W}(R)/\varpi^r\mathbb{W}(R)).$$

If $\mathcal{X}$ is affine of finite type over $\mathcal{O}_{\breve{k}}$, then $L^+\mathcal{X}$ and $L_r^+\mathcal{X}$ are representable by affine perfect schemes, and the latter is of perfectly finite type over $\overline{\mathbb{F}}_q$, as follows from [Gre61].

Moreover, if $\mathcal{X}$ is equipped with an $\mathcal{O}_k$-rational structure, i.e., $\mathcal{X} = \mathcal{X}_0 \otimes_{\mathcal{O}_k} \mathcal{O}_{\breve{k}}$ for an $\mathcal{O}_k$-scheme $\mathcal{X}_0$, then $L^+\mathcal{X}$ and $L_r^+\mathcal{X}$ both come equipped with geometric Frobenius automorphisms (over $\overline{\mathbb{F}}_q$), which we denote by $F\colon L^+\mathcal{X} \to L^+\mathcal{X}$ resp. $F\colon L_r^+\mathcal{X} \to L_r^+\mathcal{X}$.

2.3. **Perfect schemes and ℓ-adic cohomology.** We fix a prime $\ell \neq p$, and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Without further reference we will make use of the formalism of étale cohomology with compact support, as developed in [Del77]. If $f\colon X \to \operatorname{Spec}\overline{\mathbb{F}}_q$ is a (separated) morphism of finite type, then we put $H_c^i(X, \overline{\mathbb{Q}}_\ell) = R^i f_! \overline{\mathbb{Q}}_\ell$, where $\overline{\mathbb{Q}}_\ell$ is the constant local system of rank 1 on X. Then $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space, which is zero for almost all $i \in \mathbb{Z}$, and we may form the ℓ-adic Euler characteristic $H_c^*(X) = \sum_{i\in\mathbb{Z}}(-1)^i H_c^i(X, \overline{\mathbb{Q}}_\ell)$ of X, which is an element of the Grothendieck group of finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces.

If X is a perfect scheme over $\overline{\mathbb{F}}_q$, such that the structure morphism $f\colon X \to \operatorname{Spec}\overline{\mathbb{F}}_q$ is (separated and) of perfectly finite type, we may choose any model $f_0\colon X_0 \to \operatorname{Spec}\overline{\mathbb{F}}_q$ of finite type over $\overline{\mathbb{F}}_q$, such that f is the perfection of f_0. Then we have $H_c^*(X) = H_c^*(X_0)$. Hence, the above cohomological formalism extends to (separated) perfectly finitely presented perfect schemes over $\overline{\mathbb{F}}_q$. In particular, $H_c^*(X)$ makes sense as a virtual (finite) $\overline{\mathbb{Q}}_\ell$-vector space. Moreover, if X is acted on by a finite group G, then by functoriality, $H_c^*(X)$ is a virtual $\overline{\mathbb{Q}}_\ell G$-module.

Below (in Section 7.2) we will often encounter the following situation. Let X and G be as in the preceding paragraph. Suppose that X is affine and that there is a torus $\mathbb{T}$ over $\overline{\mathbb{F}}_q$ which acts on X, and that this action commutes with the G-action. Then

$$H_c^*(X) = H_c^*(X^T) \tag{2.3.1}$$

as $\overline{\mathbb{Q}}_\ell[G]$-modules, as follows from [DM91, 10.15]. This will apply to schemes $\widehat{\Sigma}_w$, $\widetilde{\Sigma}_w$ constructed in Sections 5.1,5.2. We also must apply this to the schemes $Y_{v,w}$ (resp. $Z_{v,w}$) constructed in Section 5.3, which are locally closed subschemes of $\widehat{\Sigma}_{v,w}$ of which we do not know that they are affine. However, the action of the torus $\mathbb{T}$ on $Y_{v,w}$ (resp. $Z_{v,w}$) will be the restriction of an action of the same $\mathbb{T}$ on $\widehat{\Sigma}_w$ (resp. $\widehat{\Sigma}_v$). In this situation the proof of [DM91, 10.15] still applies and hence (2.3.1) still holds for $Y_{v,w}$, $Z_{v,w}$.

In the rest of this article all schemes over $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$ will be separated, perfect and of perfectly finite type (unless specified otherwise). Whenever we consider objects over $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$, we simply write "scheme" for "perfect scheme".

2.4. **Groups, parahoric models and Moy–Prasad quotients.** Let $\mathbf{G}$ be a reductive group over k which splits over $\breve{k}$. For $E \in \{k, \breve{k}\}$, let $\mathcal{B}(\mathbf{G}, E)$ be the Bruhat–Tits building of the adjoint group of $\mathbf{G}$. The Frobenius of $\breve{k}/k$ induces automorphisms of $\mathbf{G}(\breve{k})$ and $\mathcal{B}(\mathbf{G}, \breve{k})$, both denoted by F, and we have $\mathbf{G}(\breve{k})^F = \mathbf{G}(k)$ and $\mathcal{B}(\mathbf{G}, \breve{k})^F = \mathcal{B}(\mathbf{G}, k)$.

Let $\mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$ be the set of k-rational $\breve{k}$-split maximal tori of $\mathbf{G}$. Given $\mathbf{T} \in \mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$, we denote by $X^*(\mathbf{T})$ (resp. $X_*(\mathbf{T})$) the group of characters (resp. cocharacters) of $\mathbf{T}$, and by $\Phi(\mathbf{T}, \mathbf{G}) \subseteq X^*(\mathbf{T})$ the set of roots of $\mathbf{T}$ in $\mathbf{G}$. Given $\alpha \in \Phi(\mathbf{T}, \mathbf{G})$, $\mathbf{U}_\alpha$ denotes the corresponding root subgroup. Furthermore, we denote by F the automorphism of $X^*(\mathbf{T})$ resp. $X_*(\mathbf{T})$ induced by the Frobenius of $\breve{k}/k$. Let $\mathcal{A}(\mathbf{T}, \breve{k})$ denote the apartment of $\mathbf{T}$ in $\mathcal{B}(\mathbf{G}, \breve{k})$.

From the theory of Bruhat–Tits, we can attach to any facet $\mathbf{x} \in \mathcal{B}(\mathbf{G}, k)$ a connected parahoric $\mathcal{O}_k$-model $\mathcal{G}_{\mathbf{x}}$ of $\mathbf{G}$ [BT84, §4.6, 5.2.6]. It is smooth, affine and has generic fiber $\mathbf{G}$. The group $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_{\breve{k}})$ admits a Moy–Prasad filtration by subgroups $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_{\breve{k}})_r$ for $r \in \widetilde{\mathbb{R}}_{\geqslant 0} = \mathbb{R}_{\geqslant 0} \cup \{r+ : r \in \mathbb{R}_{\geqslant 0}\}$ [MP94, §2]. By [Yu15, 8.6 Cor., §9.1], there exists a unique smooth affine $\mathcal{O}_k$-model $\mathcal{G}_{\mathbf{x}}^r$ of $\mathbf{G}$ satisfying $\mathcal{G}_{\mathbf{x}}^r(\mathcal{O}_{\breve{k}}) = \mathcal{G}_{\mathbf{x}}(\mathcal{O}_{\breve{k}})_r$. It is obtained from $\mathcal{G}_{\mathbf{x}}$ by a series of dilatations along the unit section.

For the rest of this article we fix an integer $r \geqslant 1$. We consider the fpqc-quotient

$$\mathbb{G} = \mathbb{G}_r = L^+\mathcal{G}_{\mathbf{x}}/L^+\mathcal{G}_{\mathbf{x}}^{(r-1)+} \tag{2.4.1}$$

of sheaves on $\mathrm{Perf}_{\mathbb{F}_q}$. It is representable by a (perfect) affine $\mathbb{F}_q$-group scheme, perfectly of finite type over $\mathbb{F}_q$ [CI20, Prop. 4.2(ii)]. We denote this group scheme, as well as its base change to $\overline{\mathbb{F}}_q$, again by $\mathbb{G}$. The $\overline{\mathbb{F}}_q$-group $\mathbb{G}$ admits a geometric Frobenius automorphism $F\colon \mathbb{G} \to \mathbb{G}$ attached to its $\mathbb{F}_q$-rational structure. We have

$$\mathbb{G}(\overline{\mathbb{F}}_q) = \mathcal{G}_{\mathbf{x}}(\mathcal{O}_{\breve{k}})/\mathcal{G}_{\mathbf{x}}(\mathcal{O}_{\breve{k}})_{(r-1)+} \quad \text{and} \quad \mathbb{G}(\mathbb{F}_q) = \mathbb{G}^F = \mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)/\mathcal{G}_{\mathbf{x}}^{(r-1)+}(\mathcal{O}_k)$$

(by taking Galois cohomology and using that $\mathcal{G}_{\mathbf{x}}^{r'}$ is pro-unipotent for $r' > 0$, see [MP94, §2.6]). For more details on this setup and for a more explicit description of $\mathbb{G}$ in terms of root subgroups, we refer to [CI19a, §2.4,2.5] (such an explicit description will not be used below).

Remark 2.4.1. Instead of (2.4.1) we could work with the seemingly more natural object $L_r^+\mathcal{G}_{\mathbf{x}}$ (r-truncated positive loops of $\mathcal{G}_{\mathbf{x}}$). However, the advantage of the normalization in (2.4.1) is that $\mathbb{G}_1$ is canonically isomorphic to the reductive quotient of the special fiber $\mathcal{G}_{\mathbf{x}} \otimes_{\mathcal{O}_k} \mathbb{F}_q$ (cf. [MP94, §3.2]), whereas $L_1^+\mathcal{G}_{\mathbf{x}}$ identifies with the special fiber of $\mathcal{G}_{\mathbf{x}}$, which is less useful. On the other side, if $\mathbf{x}$ is hyperspecial (as will be the case in our main result Theorem 3.2.3), then $L_r^+\mathcal{G}_{\mathbf{x}} = \mathbb{G}_r$.

2.5. **Subschemes of $\mathbb{G}$.** Let $\mathbf{H} \subseteq \mathbf{G}$ be a smooth closed $\breve{k}$-subgroup. The schematic closure $\mathcal{H} \subseteq \mathcal{G}_{\mathbf{x}}$ of $\mathbf{H}$ is a flat closed $\mathcal{O}_{\breve{k}}$-subgroup scheme of $\mathcal{G}_{\mathbf{x}}$ by [BT72, 1.2.6,1.2.7]. Applying L_r^+ gives a closed immersion $L_r^+\mathcal{H}_{\mathbf{x}} \subseteq L_r^+\mathcal{G}_{\mathbf{x}}$ by [Gre61, Cor. 2 on p. 639]. We define the closed $\overline{\mathbb{F}}_q$-subgroup $\mathbb{H} \subseteq \mathbb{G}$ as the image of $L_r^+\mathcal{H}$ under $L_r^+\mathcal{G}_{\mathbf{x}} \twoheadrightarrow \mathbb{G}$. If $\mathbf{H}$ is already defined over k, then $\mathcal{H}$ is defined over $\mathcal{O}_k$, and hence $\mathbb{H}$ is defined over $\mathbb{F}_q$. In this case, we usually will write $H := \mathbb{H}(\mathbb{F}_q)$.

Furthermore, for each $0 < r' \leqslant r$, we have a natural homomorphism $\mathbb{H} = \mathbb{H}_r \to \mathbb{H}_{r'}$, and we denote its kernel by $\mathbb{H}_r^{r'}$ (resp. simply $\mathbb{H}^{r'}$).

In particular, this procedure applies to any $\mathbf{T} \in \mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$, any root subgroup $\mathbf{U}_\alpha$ (with $\alpha \in \Phi(\mathbf{T}, \mathbf{G})$) and the unipotent radical $\mathbf{U}$ of any $\breve{k}$-rational Borel subgroup containing $\mathbf{T}$. This gives the subgroups $\mathbb{T}, \mathbb{U}_\alpha, \mathbb{U} \subseteq \mathbb{G}$, etc. and we will use this notation without further reference.

2.6. **Coxeter pairs and Coxeter tori.** Suppose that $\mathbf{G}$ is unramified (that is quasi-split over k and split over $\breve{k}$). Let $\mathbf{T}_0 \subseteq \mathbf{B}_0 \subseteq \mathbf{G}$ be a k-rational Borel subgroup and a k-rational maximal torus of $\mathbf{G}$ contained in it. Let $W_0 = N_{\mathbf{G}}(\mathbf{T}_0)(\breve{k})/\mathbf{T}_0(\breve{k})$ be the Weyl group of $\mathbf{T}_0$. It is a Coxeter group with the set of simple reflections S_0 determined by $\mathbf{B}_0$. The Frobenius of $\breve{k}/k$ induces an automorphism σ of W_0 fixing the set of simple reflections. Changing $(\mathbf{T}_0, \mathbf{B}_0)$ amounts to replacing (W_0, S_0, σ) by a triple canonically isomorphic to it (just as in [DL76, 1.1]). In particular, whenever we have a vertex $\mathbf{x} \in \mathcal{B}(\mathbf{G}, k)$ as in Section 2.4, we may assume that $\mathbf{x} \in \mathcal{A}(\mathbf{T}_0, k)$.

Any pair $(\mathbf{T}, \mathbf{B})$ with $\mathbf{T} \in \mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$ and $\mathbf{B}$ a $\breve{k}$-rational Borel subgroup containing it, determines the triple (W, S, F), where W is the Weyl group of $\mathbf{T}$, S the set of simple reflections determined by $\mathbf{B}$ and $F\colon W \to W$ is induced by the Frobenius. There is a uniquely determined coset $g\mathbf{T}_0(\breve{k}) \subseteq \mathbf{G}(\breve{k})$ with ${}^g(\mathbf{T}_0, \mathbf{B}_0) = (\mathbf{T}, \mathbf{B})$ and we have $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T}_0)(\breve{k})$ mapping to some element $w = w_{\mathbf{T},\mathbf{B}} \in W_0$. In this case, the triples (W, S, F) and $(W_0, S_0, \mathrm{Ad}\, w \circ \sigma)$ are canonically isomorphic and we may (and will) identify them.

Definition 2.6.1. (i) Given $w \in W_0$, we say that $w\sigma$ (or by abuse of language w) is a *twisted Coxeter element* if a (any) reduced expression of $w_{\mathbf{T},\mathbf{B}}$ contains precisely one simple reflection from any σ-orbit on S_0. If W_0 is irreducible, the order h of $w\sigma$ is called the *Coxeter number* of (W_0, σ).

(ii) We say that $(\mathbf{T}, \mathbf{B})$ (resp. $(\mathbf{T}, \mathbf{U})$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$) is a *Coxeter pair*, if $w_{\mathbf{T},\mathbf{B}}\sigma$ is a twisted Coxeter element.

(iii) If $(\mathbf{T}, \mathbf{B})$ is a Coxeter pair we say that $\mathbf{T} \in \mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$ is a *Coxeter torus*.

Recall that a torus $\mathbf{T} \in \mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$ is called *elliptic* (or *k-minisotropic*), if one of the following equivalent conditions holds: (i) $X^*(\mathbf{T})^F = X^*(\mathbf{Z}(\mathbf{G})^\circ)^F$, where $\mathbf{Z}(\mathbf{G})^\circ$ is the connected component of the center of $\mathbf{G}$; (ii) the group $\mathbf{T}(k)$ has a unique fixed point (necessarily a vertex) $\mathbf{x} = \mathbf{x}_\mathbf{T}$ in $\mathcal{B}(\mathbf{G}, k) = \mathcal{B}(\mathbf{G}, \breve{k})^F$. Any Coxeter torus is elliptic. Note that the property of a torus to be Coxeter (resp. elliptic) is stable under the equivalence relation of stable conjugacy.

Lemma 2.6.2. *Suppose* $\mathbf{G}$ *is unramified.*

(i) *If* $\mathbf{T}$ *is a Coxeter torus, then* $\mathbf{x}_\mathbf{T}$ *is a hyperspecial vertex.*

(ii) $\mathbf{T} \mapsto \mathbf{x}_\mathbf{T}$ *induces a natural bijection between* $\mathbf{G}(k)$*-conjugacy classes of Coxeter tori and* $\mathbf{G}(k)$*-orbits on the set of hyperspecial points of* $\mathcal{B}(\mathbf{G}, k)$.

(iii) *If* $(\mathbf{T}, \mathbf{U})$, $(\mathbf{T}', \mathbf{U}')$ *are Coxeter pairs with* $\mathbf{x}_\mathbf{T} = \mathbf{x}_{\mathbf{T}'}$, *then there is some* $g \in \mathcal{G}_\mathbf{x}(\mathcal{O}_{\breve{k}})$ *with* ${}^g(\mathbf{T}, \mathbf{U}) = (\mathbf{T}', \mathbf{U}')$.

Proof. For (i), we may pass to the adjoint group of $\mathbf{G}$. Then $\mathbf{G} \cong \prod_i \mathbf{G}_i$, with $\mathbf{G}_i$ simple and of adjoint type, and $\mathcal{B}(\mathbf{G}, \breve{k}) \cong \prod_i \mathcal{B}(\mathbf{G}_i, \breve{k})$. It thus suffices to prove the result in the case $\mathbf{G}$ is simple and of adjoint type. In this case all Coxeter tori are $\mathbf{G}(k)$-conjugate by [Ree08, Prop. 8.1(i)]. Moreover, by [DeB06, Thm. 3.4.1], there is at least the $\mathbf{G}(k)$-conjugacy class of Coxeter tori attached to a (any) hyperspecial vertex $\mathbf{x}$ of $\mathcal{B}(\mathbf{G}, \breve{k})$ and a (any) Coxeter torus in $\mathcal{G}_\mathbf{x} \otimes_{\mathcal{O}_k} \mathbb{F}_q$. Now (ii) and (iii) follow from (i), [DeB06, Thm. 3.4.1] and the fact that in a finite Weyl group all twisted Coxeter elements are conjugate. □

If $\mathbf{T} \in \mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$ is arbitrary and $\mathbf{x} \in \mathcal{A}(\mathbf{T}, k)$, then we have the torus $\mathcal{T} \subseteq \mathcal{G}_\mathbf{x}$, and the subgroup $\mathbb{T}_r \subseteq \mathbb{G}_r$ for any $r > 0$ (as in Section 2.5). This gives the two Weyl groups

$$W_\mathbf{x}(\mathbf{T}, \mathbf{G}) := W(\mathbb{T}_1, \mathbb{G}_1) \subseteq W(\mathbf{T}, \mathbf{G}).$$

attached to $\mathbf{T}$ (and $\mathbf{x}$). We denote them by $W_\mathbf{x}$ and W if $\mathbf{T}, \mathbf{G}$ are clear from the context. If $\mathbf{x}$ is a hyperspecial vertex – which is by Lemma 2.6.2 necessarily the case whenever $\mathbf{T}$ is Coxeter – then the situation simplifies to $\mathbb{G}_1 = L_1^+\mathcal{G}_\mathbf{x} = \mathcal{G}_\mathbf{x} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and $W_\mathbf{x} = W$.

2.7. A condition on q.

Identifying $X_*(\mathbb{G}_m)$ with $\mathbb{Z}$, we have the perfect pairing of $\mathbb{Z}$-lattices

$$X^*(\mathbb{T}_1) \times X_*(\mathbb{T}_1) \to \mathbb{Z}, \quad \alpha, \nu \mapsto \langle \alpha, \nu \rangle$$

such that $\langle F\alpha, \nu \rangle = \langle \alpha, F\nu \rangle$ for all α, ν. This pairing also induces the analogous pairing for $\mathbb{T}_1^{\mathrm{ad}}$ (where $\mathbf{T}^{\mathrm{ad}}$ is the image of $\mathbf{T}$ in the adjoint quotient of $\mathbf{G}$) and for the $\mathbb{Q}$-vector spaces obtained by extension of scalars.

Recall that the choice of $\mathbf{U}$ is equivalent to the choice of a set of simple roots $\Delta \subseteq \Phi(\mathbf{T}, \mathbf{G})$, and it endows W with a structure of a Coxeter group. The simple roots Δ form a basis of $X_*(\mathbb{T}_1^{\mathrm{ad}})_\mathbb{Q}$. We will denote by $\{\alpha^* : \alpha \in \Delta\} \subseteq X_*(\mathbb{T}_1^{\mathrm{ad}})_\mathbb{Q}$ the set of fundamental coweights, defined as the basis of $X_*(\mathbb{T}_1^{\mathrm{ad}})_\mathbb{Q}$ dual to Δ. We will prove the orthogonality relations of Coxeter-type Deligne–Lusztig characters under the following restriction on q:

$$q > M := \max_{\alpha \in \Delta} \langle \alpha_0, \alpha^* \rangle \tag{2.7.1}$$

Note that this condition depends only on the group $\mathbf{G}_{\breve{k}}$ and on no other choice (like that of Δ). For the irreducible types we can explicitly compute the constant M from Condition (2.7.1): type A_n: $M = 1$; types B_n, C_n: $M = 2$; types G_2, E_6: $M = 3$; types F_4, E_7: $M = 4$; type E_8: $M = 6$. In general, the constant M

for $\mathbf{G}_{\breve{k}}$ is the maximum of the values of M over all connected components of the Dynkin diagram of $\mathbf{G}_{\breve{k}}$. In particular (2.7.1) holds whenever $q > 5$.

3. Deep level Deligne–Lusztig induction

We work in the setup of Section 2.4. In particular, the reductive $\breve{k}$-split group $\mathbf{G}/k$, the point $\mathbf{x} \in \mathcal{B}(\mathbf{G}, k)$, and the integer $r \geqslant 1$ are fixed. We omit $\mathbf{x}$ and r from notation, and write $\mathcal{G}$ for the $\mathcal{O}_k$-group $\mathcal{G}_{\mathbf{x}}$, and $\mathbb{G}$, $\mathbb{T}$, etc. for $\mathbb{G}_r$, $\mathbb{T}_r$, etc.

3.1. **The schemes $S_{\mathbf{T},\mathbf{U}}$.** Let $\mathbf{T} \in \mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$, such that $\mathbf{x} \in \mathcal{A}(\mathbf{T}, k)$. Let $\mathbf{B} = \mathbf{TU}$ be a Borel subgroup, defined over $\breve{k}$, containing $\mathbf{T}$, and with unipotent radical $\mathbf{U}$. As in Section 2.5, we have the corresponding closed subgroups $\mathbb{U} \subseteq \mathbb{G}$, defined over $\overline{\mathbb{F}}_q$. Following [Lus04, CI19a], consider the $\overline{\mathbb{F}}_q$-scheme

$$\begin{array}{ccc} S_{\mathbf{x},\mathbf{T},\mathbf{U},r} & \longrightarrow & F\mathbb{U} \\ \big\downarrow & & \big\downarrow \\ \mathbb{G} & \xrightarrow{\ L_{\mathbb{G}}\ } & \mathbb{G} \end{array}$$

where $L_{\mathbb{G}} \colon \mathbb{G} \to \mathbb{G}$, $g \mapsto g^{-1}F(g)$ is the Lang map. We usually write $S_{\mathbf{T},\mathbf{U}}$ for $S_{\mathbf{x},\mathbf{T},\mathbf{U},r}$, as $\mathbf{x}$, r remain constant throughout the article. The finite group $G \times T = \mathbb{G}(\mathbb{F}_q) \times \mathbb{T}(\mathbb{F}_q)$ acts on $S_{\mathbf{T},\mathbf{U}}$ by $(g,t) \colon x \mapsto gxt$. For a character $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$, we obtain the virtual G-representation

$$R_{\mathbf{T},\mathbf{U}}(\theta) = R_{\mathbf{x},\mathbf{T},\mathbf{U},r}(\theta) := \sum_{i \in \mathbb{Z}} (-1)^i H_c^i(S_{\mathbf{T},\mathbf{U}}, \overline{\mathbb{Q}}_\ell)_\theta,$$

where the subscript θ indicates that we take the θ-isotypic component. By inflation, we may regard $R_{\mathbf{T},\mathbf{U}}(\theta)$ as a virtual smooth $\mathcal{G}(\mathcal{O}_k)$-representation.

Remark 3.1.1. The varieties $S_{\mathbf{T},\mathbf{U}}$ are closely related to classical Deligne–Lusztig varieties. Indeed, the group $\mathbb{U} \cap F\mathbb{U}$ acts by right multiplication on $S_{\mathbf{T},\mathbf{U}}$ and we may form the quotient $X_{\mathbf{T},\mathbf{U}} = S_{\mathbf{T},\mathbf{U}}/\mathbb{U} \cap F\mathbb{U}$. If $r = 1$, then $X_{\mathbf{T},\mathbf{U}}$ is equal to the classical Deligne–Lusztig variety $\widetilde{X}_{\mathbb{T}_1 \subseteq \mathbb{B}_1}$ attached to the reductive $\mathbb{F}_q$-group $\mathbb{G}_1$, cf. [DL76, 1.17(ii), 1.19] and Remark 2.4.1.

Remark 3.1.2. In the light of Remark 3.1.1, $X_{\mathbf{T},\mathbf{U}}$ are deep level analogs of classical Deligne–Lusztig varieties. Moreover, the fibers of the morphism $S_{\mathbf{T},\mathbf{U}} \to X_{\mathbf{T},\mathbf{U}}$ are isomorphic to the perfection of a fixed finite-dimensional affine space over $\mathbb{F}_q$. It follows that $H_c^*(S_{\mathbf{T},\mathbf{U}}) = H_c^*(X_{\mathbf{T},\mathbf{U}})$. In turn, $X_{\mathbf{T},\mathbf{U}}$ is the r-truncated integral version of the p-adic Deligne–Lusztig spaces $X_w(b)$ (or rather their coverings $\dot{X}_{\dot{w}}(b)$) defined in [Iva20]. Cf. Section 4.1 below.

3.2. **Main result.** Let $(\mathbf{T}, \mathbf{U})$, $(\mathbf{T}', \mathbf{U}')$ be two pairs where $\mathbf{T}, \mathbf{T}' \in \mathrm{Tori}_{\breve{k}/k}(\mathbf{G})$ satisfy $\mathbf{x} \in \mathcal{A}(\mathbf{T}, k) \cap \mathcal{A}(\mathbf{T}', k)$, and $\mathbf{U}$ (resp. $\mathbf{U}'$) is the unipotent radical of a $\breve{k}$-rational Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ (resp. $\mathbf{T}'$). We have the groups $\mathcal{T}, \mathbb{T}, T, \mathcal{U}, \mathbb{U}$ attached to $\mathbf{T}, \mathbf{U}$ by Section 2.5, and similarly for $\mathbf{T}', \mathbf{U}'$.

Using Remark 3.1.1, the classical orthogonality relations for Deligne–Lusztig characters [DL76, Thm. 6.8] can be expressed as follows: for $r = 1$ and any characters $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$, $\theta' \colon T' \to \overline{\mathbb{Q}}_\ell^\times$, we have

$$\langle R_{\mathbf{T},\mathbf{U}}(\theta), R_{\mathbf{T}',\mathbf{U}'}(\theta') \rangle_G = \# \left\{ w \in W(\mathbb{T}_1, \mathbb{T}_1')^F : \theta' = {}^w\theta \right\}, \tag{3.2.1}$$

where

$$W_{\mathbf{x}}(\mathbf{T}, \mathbf{T}') = W(\mathbb{T}_1, \mathbb{T}_1') = \mathbb{T}_1 \backslash \{ g \in \mathbb{G}_1 : {}^g\mathbb{T}_1' = \mathbb{T}_1 \} \tag{3.2.2}$$

is the transporter principal homogeneous space under $W_{\mathbf{x}}(\mathbf{T}, \mathbf{G})$. We may ask for a generalization of this to deeper levels.

Question 3.2.1. Does (3.2.1) hold in general, that is for arbitrary $\mathbf{G}$, $\mathbf{x}$, r, $\mathbf{T}$, $\mathbf{T}'$, $\mathbf{U}$, $\mathbf{U}'$, θ, θ'?

Remark 3.2.2. The answer to Question 3.2.1 is affirmative in the following cases:

(i) If $r = 1$ by [DL76, Thm. 6.8].
(ii) If $r \geqslant 2$, and θ or θ' is *regular* in the sense of [Lus04] (roughly, "regular" = "highly non-trivial on $\ker(T_r \to T_{r-1})$") by [Lus04] if $\mathcal{G}$ reductive and $\operatorname{char} k > 0$, resp. [Sta09] if $\mathcal{G}$ reductive and $\operatorname{char} k = 0$, resp. [CI19a] in general.
(iii) If $\mathbf{G}$ = inner form of $\mathbf{GL}_n$, and $(\mathbf{T}, \mathbf{U})$, $(\mathbf{T}', \mathbf{U}')$ are Coxeter pairs, by [CI19b, Thm. 3.1].

One might conjecture an affirmative answer to Question 3.2.1 in general, but there is not enough evidence beyond the known special cases. In this article we concentrate on the Coxeter case, and prove the following generalization of Remark 3.2.2(iii).

Theorem 3.2.3. *Suppose $\mathbf{G}$ is unramified, and $(\mathbf{T}, \mathbf{U})$, $(\mathbf{T}', \mathbf{U}')$ are Coxeter pairs with $\mathbf{x} = \mathbf{x}_{\mathbf{T}} = \mathbf{x}_{\mathbf{T}'}$. Suppose that condition* (2.7.1) *holds for q and the root system of $\mathbf{G}$. Then for all $r \geqslant 1$ and all $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$, $\theta' \colon T' \to \overline{\mathbb{Q}}_\ell^\times$, we have*

$$\langle R_{\mathbf{T},\mathbf{U}}(\theta), R_{\mathbf{T}',\mathbf{U}'}(\theta')\rangle_G = \#\left\{w \in W_{\mathbf{x}}(\mathbf{T}, \mathbf{T}')^F : \theta' = {}^w\theta\right\}, \tag{3.2.3}$$

where $W_{\mathbf{x}}(\mathbf{T}, \mathbf{T}')$ is as in (3.2.2).

We will show Theorem 3.2.3 when $(\mathbf{T}, \mathbf{U}) = (\mathbf{T}', \mathbf{U}')$ is a given Coxeter pair and W is irreducible. The various reductions needed to deduce the theorem from this particular case are studied in the next section.

4. Reductions

The purpose of this section is to show that it is enough to prove Theorem 3.2.3 when $(\mathbf{T}, \mathbf{U}) = (\mathbf{T}', \mathbf{U}')$ is a given Coxeter pair and W is irreducible. There is a small price to pay, and one will actually need to show a stronger statement, namely Theorem 4.2.1, which behaves well with respect to our reductions.

4.1. **Changing Coxeter pairs.** Suppose $\mathbf{G}$ is unramified and $\mathbf{x}$ is hyperspecial. Then $\mathcal{G}$ is a reductive group over $\mathcal{O}_k$ and we have $\mathbb{G} = L_r^+\mathcal{G}$ (cf. Remark 2.4.1). Let $\mathbf{T}_0 \subseteq \mathbf{B}_0 \subseteq \mathbf{G}$ be as in Section 2.6, such that $\mathbf{x} \in \mathcal{A}(\mathbf{T}_0, k)$ and $W_0 = N_{\mathbf{G}}(\mathbf{T}_0)(\breve{k})/\mathbf{T}_0(\breve{k})$. Then $\mathcal{T}_0 \subseteq \mathcal{B}_0 \subseteq \mathcal{G}$ are a maximal torus and a Borel subgroup containing it and defined over $\mathcal{O}_k$. Let $\mathbf{U}_0$ (resp. $\mathcal{U}_0$) be the unipotent radical of $\mathbf{B}$ (resp. $\mathcal{B}_0$).

The $\mathcal{O}_k$-group $\mathcal{G}$ is quasi-split, $\mathcal{B}_0 \subseteq \mathcal{G}$ is a rational Borel subgroup, and the quotient $\mathcal{G}/\mathcal{B}_0$ is projective over $\mathcal{O}_k$ [Con14, Thm. 2.3.6]. Then $\mathcal{G}$ admits a Bruhat decomposition in the following sense: letting $\mathcal{G}$ act diagonally on $(\mathcal{G}/\mathcal{B}_0)^2$, there are $\mathcal{G}$-stable (reduced) subschemes $\mathcal{O}(w) \subseteq (\mathcal{G}/\mathcal{B}_0)^2$ for each $w \in W_0$, flat over $\mathcal{O}_k$, such that for any geometric point $x \in \operatorname{Spec} \mathcal{O}_k$, the fiber $\mathcal{O}(w)_x$ is the $\mathcal{G}_x$-orbit in $(\mathcal{G}/\mathcal{B}_0)^2_x$ like in the usual Bruhat decomposition. We have the following integral analog of [Iva20, Def. 8.3].

Definition 4.1.1. Let $w \in W_0$ and $\dot{w} \in N_{\mathcal{G}}(\mathcal{T}_0)(\mathcal{O}_{\breve{k}})$. Define the *integral p-adic Deligne–Lusztig space* $X_w^{\mathcal{G}}(1)$, and $\dot{X}_{\dot{w}}^{\mathcal{G}}(1)$ by Cartesian diagrams of functors on $\operatorname{Perf}_{\overline{\mathbb{F}}_q}$

$$\begin{array}{ccc} X_w^{\mathcal{G}}(1) & \longrightarrow & L^+\mathcal{O}(w) \\ \downarrow & & \downarrow \\ L^+(\mathcal{G}/\mathcal{B}_0) & \xrightarrow{(\mathrm{id},F)} & L^+(\mathcal{G}/\mathcal{B}_0) \times L^+(\mathcal{G}/\mathcal{B}_0) \end{array} \quad \text{and} \quad \begin{array}{ccc} \dot{X}_{\dot{w}}^{\mathcal{G}}(1) & \longrightarrow & L^+\dot{\mathcal{O}}(\dot{w}) \\ \downarrow & & \downarrow \\ L^+(\mathcal{G}/\mathcal{U}_0) & \xrightarrow{(\mathrm{id},F)} & L^+(\mathcal{G}/\mathcal{U}_0) \times L^+(\mathcal{G}/\mathcal{U}_0) \end{array}$$

Similarly, replacing L^+ by L_r^+ everywhere, define their r-truncations $X_w^{\mathcal{G},r}(1)$, $\dot{X}_{\dot{w}}^{\mathcal{G},r}(1)$.

The functors $X_w^{\mathcal{G}}(1)$, $\dot{X}_{\dot{w}}^{\mathcal{G},r}(1)$ are representable by (perfect) $\overline{\mathbb{F}}_q$-schemes, the latter are of perfectly finite presentation. If $\dot{w}$ maps to w, then there is a natural map $\dot{X}_{\dot{w}}^{\mathcal{G},r}(1) \to X_w^{\mathcal{G},r}(1)$; $G \times T$ acts on $\dot{X}_{\dot{w}}^{\mathcal{G},r}(1)$, G acts on $X_w^{\mathcal{G},r}(1)$, and the above map is G-equivariant finite étale T-torsor. Recall the definition of the space $X_{\mathbf{T},\mathbf{U}}$ from Remark 3.1.1.

Lemma 4.1.2. *Suppose $\mathbf{T} \in \operatorname{Tori}_{\breve{k}/k}(\mathbf{G})$ such that $\mathbf{x} \in \mathcal{A}(\mathbf{T}, k)$. Then $X_{\mathbf{T},\mathbf{U}} \cong \dot{X}_{\dot{w}}^{\mathcal{G},r}(1)$ (equivariant for the $G \times T$-actions), where we identify $W(\mathbf{T}, \mathbf{G})$ with W_0, $w \in W_0$ is the element satisfying $F\mathbf{U} = {}^w\mathbf{U}$, and $\dot{w} \in N_{\mathcal{G}}(\mathcal{T}_0)(\mathcal{O}_{\breve{k}})$ is an arbitrary lift of w.*

Proof. This has the same proof as [DL76, 1.19]. There are no subtleties due to the loop functor, cf. the similar results of [Iva20, Prop. 12.1 and Lem. 12.3]. ☐

To $\dot{X}_{\dot{w}}^{\mathcal{G},r}(1)$ we may apply the technique of Frobenius-cyclic shift. Let ℓ denote the length function on the Coxeter group (W_0, S_0).

Lemma 4.1.3. *Suppose $w = w_1w_2$, $w' = w_2F(w_1) \in W_0$, such that $\ell(w) = \ell(w_1) + \ell(w_2) = \ell(w')$. Then there is a G-equivariant isomorphism $X_w^{\mathcal{G},r}(1) \cong X_{w'}^{\mathcal{G},r}(1)$. If $\dot{w}$, $\dot{w}'$, $\dot{w}_1$, $\dot{w}_2 \in \mathcal{G}(\mathcal{O}_{\breve{k}})$ are lifts of w, w', w_1, w_2, satisfying $\dot{w} = \dot{w}_1\dot{w}_2$, $\dot{w}' = \dot{w}_2F(\dot{w}_1)$, then there is a $G \times T$-equivariant isomorphism $\dot{X}_{\dot{w}}^{\mathcal{G},r}(1) \cong \dot{X}_{\dot{w}'}^{\mathcal{G},r}(1)$.*

Proof. The same proof as in [DL76, 1.6] applies. Again, the use of the (positive, truncated) loop functor causes no problems, cf. [Iva20, Lem. 8.16]. ☐

As a corollary we deduce:

Corollary 4.1.4. *Suppose $\mathbf{G}$ is unramified, and $(\mathbf{T}, \mathbf{U})$, $(\mathbf{T}', \mathbf{U}')$ are Coxeter pairs with $\mathbf{x} = \mathbf{x_T} = \mathbf{x_{T'}}$ (in particular, $\mathbf{x}$ hyperspecial). Then $X_{\mathbf{T},\mathbf{U}} \cong X_{\mathbf{T}',\mathbf{U}'}$ ($G \times T \cong G \times T'$-equivariantly). In particular, $H_c^*(S_{\mathbf{T},\mathbf{U}}) \cong H_c^*(S_{\mathbf{T}',\mathbf{U}'})$. To show Theorem 3.2.3, it suffices to do so under the additional assumption $(\mathbf{T}', \mathbf{U}') = (\mathbf{T}, \mathbf{U})$ is a fixed Coxeter pair.*

Proof. We prove the first statement. By Lemma 4.1.2 it suffices to show that whenever $w, w' \in W_0$ are two twisted Coxeter elements, $\dot{X}_{\dot{w}}^{\mathcal{G},r}(1) \cong \dot{X}_{\dot{w}'}^{\mathcal{G},r}(1)$. First, when $\dot{w}_1, \dot{w}_2 \in N_{\mathcal{G}}(\mathcal{T}_0)(\mathcal{O}_{\breve{k}})$ are two lifts of w, then $\dot{X}_{\dot{w}_1}^{\mathcal{G},r}(1) \cong \dot{X}_{\dot{w}_2}^{\mathcal{G},r}(1)$ equivariantly (same argument as on [DL76, p. 111], along with an application of Lang's theorem to the connected $\overline{\mathbb{F}}_q$-group $L_r^+\mathcal{T}_0$ with Frobenius $\mathrm{Ad}(w) \circ F$). Using this, the first statement of the corollary follows from Lemma 4.1.3 along with the fact that all twisted Coxeter elements are conjugate by a sequence of cyclic shifts in W_0 (cf. the corresponding discussion in [Iva20, §8.4]).

The second claim follows from the first and Remark 3.1.2, and the third claim follows from the second. ☐

4.2. **First step towards the proof of Theorem 3.2.3.** In the proof of Theorem 3.2.3 we follow the general strategy of [DL76, §6] and [Lus04]. Let the setup be as in the beginning of Section 3.2. Attached to $(\mathbf{T}, \mathbf{U})$, $(\mathbf{T}', \mathbf{U}')$ we may consider the $\overline{\mathbb{F}}_q$-scheme

$$\begin{aligned} \Sigma := {}^{\mathbb{U},\mathbb{U}'}\Sigma &:= G\backslash(S_{\mathbf{T},\mathbf{U}} \times S_{\mathbf{T}',\mathbf{U}'}) \\ &\cong \{(x, x', y) \in F\mathbb{U} \times F\mathbb{U}' \times \mathbb{G} \colon xF(y) = yx'\}. \end{aligned} \tag{4.2.1}$$

We will write ${}^{\mathbb{U},\mathbb{U}'}\Sigma$, whenever the choice of $\mathbb{U}, \mathbb{U}'$ is relevant, and simply Σ whenever it is clear from context). In (4.2.1) the group G acts diagonally on $S_{\mathbf{T},\mathbf{U}} \times S_{\mathbf{T}',\mathbf{U}'}$, and the second isomorphism is given by $(g, g') \mapsto (x, x', y)$ with $x = g^{-1}F(g)$, $x' = g'^{-1}F(g')$, $y = g^{-1}g'$, just as in [DL76, 6.6]. Now $T \times T'$ acts on Σ by $(t, t') \colon (x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$ and an application of the Künneth formula shows that

$$\langle R_{\mathbf{T},\mathbf{U}}(\theta), R_{\mathbf{T}',\mathbf{U}'}(\theta')\rangle_G = \dim_{\overline{\mathbb{Q}}_\ell} H_c^*(\Sigma, \overline{\mathbb{Q}}_\ell)_{\theta\otimes\theta'}.$$

Let $\mathrm{pr} \colon \mathbb{G} = \mathbb{G}_r \to \mathbb{G}_1$ denote the natural projection. We have the locally-closed decomposition $\mathbb{G}_1 = \coprod_{v \in W(\mathbb{T}_1, \mathbb{T}_1')} \mathbb{U}_1\dot{v}\mathbb{T}_1'\mathbb{U}_1'$. This induces a $T \times T'$-stable locally closed decomposition $\Sigma = \coprod_{v \in W(\mathbb{T}_1, \mathbb{T}_1')} \Sigma_v$, with

$$\Sigma_v = \{(x, x', y) \in \Sigma \colon y \in \mathrm{pr}^{-1}(\mathbb{U}_1\dot{v}\mathbb{T}_1'\mathbb{U}_1')\}, \tag{4.2.2}$$

where $\dot{v}$ is an arbitrary lift of v to $\mathbb{G}(\overline{\mathbb{F}}_q)$ fixed once and for all. To prove formula (3.2.3) for the given $\mathbf{T}, \mathbf{T}', \mathbf{U}, \mathbf{U}', \theta, \theta'$ it suffices to show

$$\dim_{\overline{\mathbb{Q}}_\ell} H_c^*(\Sigma_v, \overline{\mathbb{Q}}_\ell)_{\theta\otimes\theta'} = \begin{cases} 1 & \text{if } F(v) = v \text{ and } \theta' = {}^v\theta, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.3}$$

We shall show that a stronger statement holds for a specific choice of a Coxeter pair. Let $\mathbb{Z} = Z(\mathbb{G})$ be the center of $\mathbb{G}$ and $Z := \mathbb{Z}^F$ be its rational points. The group Z embeds diagonally in $T \times T'$ and its action on Σ (hence on its cohomology) is trivial. The action of $T \times^Z T'$ on Σ extends to an action of $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T}')^F$ and the cohomology of the cell Σ_v for that action is given by the following theorem.

Theorem 4.2.1. *Suppose* $\mathbf{G}$ *is unramified, and that condition* (2.7.1) *holds for* q *and the root system of* $\mathbf{G}$. *Then there exists a Coxeter pair* $(\mathbb{T}, \mathbb{U})$ *such that for all* $v \in W$

$$H_c^*(\Sigma_v^{\mathbb{U},\mathbb{U}}) = \begin{cases} H_c^0((\dot{v}\mathbb{T})^F, \overline{\mathbb{Q}}_\ell) & \text{if } v \in W^F, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.4}$$

as virtual $\overline{\mathbb{Q}}_\ell(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T}')^F$*-modules.*

Equation (4.2.3) follows easily from this theorem. Indeed, if $\theta_{|Z} \neq \theta'_{|Z}$ then $H_c^*(\Sigma_v)_{\theta \otimes \theta'} = 0$ since Z acts trivially on Σ_v. On the other hand, since $T \times^Z T' \subset (\mathbb{T} \times^{\mathbb{Z}} \mathbb{T}')^F$, Theorem 4.2.1 implies that the cohomology of Σ_v as a virtual $T \times^Z T$-module is the same as the cohomology of $(\dot{v}\mathbb{T})^F$ for which the analog of (4.2.3) clearly holds.

The proof of Theorem 4.2.1 in the case where W is irreducible will be given in Section 7. The reduction to that case is the purpose of the remainder of this section.

4.3. **Reduction to the almost simple case.** Let $\mathbf{G}$ be an arbitrary unramified reductive group over k. Let $\pi \colon \widetilde{\mathbf{G}} \to \mathbf{G}$ be the simply connected covering of the derived group of $\mathbf{G}$. Let $\mathbf{Z}$ denote the center of $\mathbf{G}$. Adjoint buildings of $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ agree, and we have the parahoric $\mathcal{O}_k$-model $\widetilde{\mathcal{G}}$ of $\widetilde{\mathbf{G}}$, corresponding to the same point as $\mathcal{G}$. Moreover, π extends uniquely to a map $\pi \colon \widetilde{\mathcal{G}} \to \mathcal{G}$ [BT84, 1.7.6], which in turn induces the map $\pi \colon \widetilde{\mathbb{G}} \to \mathbb{G}$. Put $\widetilde{\mathbf{T}} = \pi^{-1}(\mathbf{T})$, $\widetilde{\mathcal{T}} = \pi^{-1}(\mathcal{T})$, $\widetilde{\mathbb{T}} = \pi^{-1}(\mathbb{T})$, and similarly for $\mathbf{U}$, $\mathbf{Z}$, etc.

Remark 4.3.1. If $r = 1$, then $\widetilde{\mathbb{G}} \to \mathbb{G}$ is the simply connected cover of the derived group of $\mathbb{G}$, and the situation is precisely as in [DL76, 1.21-1.27]).

The map π induces maps on rational points $\widetilde{G} = \widetilde{\mathbb{G}}(\mathbb{F}_q) \to \mathbb{G}(\mathbb{F}_q) = G$, and similarly $\widetilde{T} \to T$. In particular, any character χ of T pulls back to a character $\widetilde{\chi}$ of $\widetilde{T}$. Now the general case of Theorem 3.2.3 follows from the next proposition.

Proposition 4.3.2. *If Theorem 4.2.1 holds for* $\widetilde{\mathbf{G}}$, $\widetilde{\mathbf{T}}$ *then it holds for* $\mathbf{G}$, $\mathbf{T}$.

Proof. Let $\mathbb{S} = \mathbb{T} \times \mathbb{T}$ and $S = T \times T$ (resp. $\widetilde{\mathbb{S}} = \pi^{-1}(\mathbb{S})$ and $\widetilde{S} = \widetilde{\mathbb{S}}(\mathbb{F}_q)$). We have the space X and $\widetilde{X}$ carrying actions of $G \times T$ and $\widetilde{G} \times \widetilde{T}$ respectively. Moreover, we also have the quotients $\Sigma = (X \times X)/G$ and $\widetilde{\Sigma} = (\widetilde{X} \times \widetilde{X})/\widetilde{G}$ acted on by S and $\widetilde{S}$ respectively. The $G \times T$-action on X factors through the action of the quotient $G \times^Z T = G \times T/\{(z, z^{-1}) \colon z \in Z\}$, which in turn extends to an action of the bigger group $(\mathbb{G} \times^{\mathbb{Z}} \mathbb{T})^F$ given by the same formula. Similarly, the S-action on Σ factors through an action of S/Z (Z embedded diagonally), which extends to an action of $(\mathbb{S}/\mathbb{Z})^F$ given by same formula. These two extensions of actions also hold when we put a $\widetilde{(\cdot)}$ over each of the objects.

Recall the notion of the *induced space* from [DL76, 1.24]: if $\alpha \colon A \to B$ is a homomorphism of finite groups, and Y a space on which A acts, then the induced space $\mathrm{Ind}_A^B Y = \mathrm{Ind}_\alpha Y$ is the (unique up to unique isomorphism) B-space I, provided with an A-equivariant map $Y \to I$, which satisfies $\mathrm{Hom}_B(I, V) = \mathrm{Hom}_A(Y, V)$ for any B-space V.

Lemma 4.3.3. *Let* $\gamma \colon (\widetilde{\mathbb{S}}/\widetilde{\mathbb{Z}})^F \to (\mathbb{S}/\mathbb{Z})^F$ *be the natural map induced by* π. *Then* $\Sigma = \mathrm{Ind}_\gamma \widetilde{\Sigma}$.

Proof. We have the natural map $\alpha \colon (\widetilde{\mathbb{S}} \times^{\widetilde{\mathbb{Z}}} \widetilde{\mathbb{G}})^F \to (\mathbb{S} \times^{\mathbb{Z}} \mathbb{G})^F$. Kernel and cokernel of α are canonically isomorphic to the kernel and cokernel of $\beta \colon \widetilde{\mathbb{S}}^F \to \mathbb{S}^F$ (same argument as [DL76, 1.26] with $\mathbb{S}$ instead of $\mathbb{T}$). One checks that $X = \mathrm{Ind}_{\widetilde{\mathbb{S}}^F}^{\mathbb{S}^F} \widetilde{X}$. Thus, similar as in [DL76, 1.25],

$$X \times X = \mathrm{Ind}_\beta \widetilde{X} \times \widetilde{X} = \mathrm{Ind}_\alpha \widetilde{X} \times \widetilde{X}. \tag{4.3.1}$$

Now, we have the commutative diagram with exact rows:

$$(4.3.2)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{\mathbb{G}}^F & \longrightarrow & (\widetilde{\mathbb{S}} \times^{\widetilde{\mathbb{Z}}} \widetilde{\mathbb{G}})^F & \longrightarrow & (\widetilde{\mathbb{S}}/\widetilde{\mathbb{Z}})^F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & \mathbb{G}^F & \longrightarrow & (\mathbb{S} \times^{\mathbb{Z}} \mathbb{G})^F & \longrightarrow & (\mathbb{S}/\mathbb{Z})^F & \longrightarrow & 1 \end{array}$$

which is obtained from the same diagram for the algebraic groups (with all F's removed) by taking Galois cohomology and using Lang's theorem and connectedness of $\mathbb{G}$, $\widetilde{\mathbb{G}}$. Now the lemma formally follows from (4.3.2) and (4.3.1), using that $\Sigma = (X \times X)/\mathbb{G}^F$ and $\widetilde{\Sigma} = (\widetilde{X} \times \widetilde{X})/\widetilde{\mathbb{G}}^F$. □

On the other hand, if $v \in W$ then we also have $(\dot{v}\mathbb{T})^F = \mathrm{Ind}_\gamma(\dot{v}\widetilde{\mathbb{T}})^F$ as $(\mathbb{S}/\mathbb{Z})^F$-varieties. Therefore Proposition 4.3.2 follows from Lemma 4.3.3. □

Assume now that $\mathbf{G}$ is semisimple and simply connected. In this case, there is some $s \geqslant 1$ such that $\mathbf{G} \cong \prod_{i=1}^s \mathbf{G}_i$, where each $\mathbf{G}_i$ is an almost simple and simply connected unramified reductive k-group. We have then similar product decompositions for the Bruhat–Tits buildings, the parahoric models $\mathcal{G} \cong \prod_i \mathcal{G}_i$, their Moy–Prasad filtrations, the maximal tori, their Weyl groups, the unipotent radicals of the Borels, etc. Upon applying the functor $\mathcal{G} \mapsto L^+\mathcal{G}/L^+\mathcal{G}^{(r-1)+}$, this induces an isomorphism $X_{\mathbf{T},\mathbf{U}} \cong \prod_i X_{\mathbf{T}_i,\mathbf{U}_i}$, and finally an isomorphism $\Sigma \cong \prod_i \Sigma_i$ (with obvious notation), equivariant for the action of $T \times T = \prod_i (T_i \times T_i)$. Applying the Künneth formula shows that Theorem 4.2.1 holds for $\mathbf{G}, \mathbf{T}$ whenever it holds for all $\mathbf{G}_i$, $\mathbf{T}_i$.

Finally if $\mathbf{G}$ is almost simple and simply connected, then there is some $m \geqslant 1$, and an absolutely almost simple group $\widehat{\mathbf{G}}$ over k_m, the degree m subextension of $\breve{k}/k$, such that $\mathbf{G} \cong \mathrm{Res}_{k_m/k} \widehat{\mathbf{G}}$ is the restriction of scalars of $\widehat{\mathbf{G}}$. We thus may assume that $\mathbf{G} = \mathrm{Res}_{k_m/k} \widehat{\mathbf{G}}$. The Bruhat–Tits buildings $\mathscr{B}(\mathbf{G}, k)$ and $\mathscr{B}(\widehat{\mathbf{G}}, k_m)$ are canonically isomorphic. Let $\mathbf{x}$ be a vertex of $\mathscr{B}(\mathbf{G}, k)$ with attached parahoric $\mathcal{O}_k$-model $\mathcal{G}$ of $\mathbf{G}$, and let $\mathbf{x}$ also denote the corresponding vertex of $\mathscr{B}(\widehat{\mathbf{G}}, k_m)$, with attached parahoric $\mathcal{O}_{k_m}$-model $\widehat{\mathcal{G}}$ of $\widehat{\mathbf{G}}$. Then there is a canonical isomorphism $\mathcal{G} = \mathrm{Res}_{\mathcal{O}_{k_m}/\mathcal{O}_k} \widehat{\mathcal{G}}$ inducing the identity on generic fibers [HR20, Prop. 4.7]. Reducing modulo ϖ^r and applying [BGA18, Cor. 13.5] (with $e = 1$), we deduce a canonical identification $\mathbb{G} = \mathrm{Res}_{\mathbb{F}_{q^m}/\mathbb{F}_q} \widehat{\mathbb{G}}$, where $\mathbb{G}, \widehat{\mathbb{G}}$ are attached to $\mathcal{G}, \widehat{\mathcal{G}}$ as in Section 2.5.

We have $\mathbf{T} = \mathrm{Res}_{k_m/k} \widehat{\mathbf{T}}$ for a Coxeter torus of $\widehat{\mathbf{G}}/k_m$, and we may identify $W = \prod_{i=1}^m \widehat{W}$, where $\widehat{W}$ is the Weyl group of $\widehat{\mathbf{T}}$. Furthermore under this identification one can assume that F acts by $F((w_i)_{i=1}^m) = (\widehat{F}(w_m), w_1, \ldots, w_{m-1})$, where $\widehat{F}$ is the Frobenius of $\widehat{W}$. In particular $W^F = \{(w_1, \ldots, w_1) \colon \widehat{F}(w_1) = w_1\} \simeq (\widehat{W})^{\widehat{F}}$. Choose $\mathbf{U}$ such that $F(\mathbf{U}) = {}^c\mathbf{U}$, where $c = (\widehat{c}, 1, \ldots, 1) \in W$ and $\widehat{c} \in \widehat{W}$ is the twisted Coxeter element of $\widehat{W}$ satisfying $\widehat{F}(\widehat{\mathbf{U}}) = {}^{\widehat{c}}\widehat{\mathbf{U}}$. Then $(\mathbf{T}, \mathbf{U})$ is a Coxeter pair. Now, if we consider the decomposition $\mathbb{G}_{\overline{\mathbb{F}}_q} \cong \prod_{i=1}^m \widehat{\mathbb{G}}_{\overline{\mathbb{F}}_q}$ the equation $xF(y) = yx'$ for $(x, x', y) \in \mathbb{U} \times \mathbb{U} \times \mathbb{G}$ can be written

$$(x_1, \ldots, x_m)(\widehat{F}(y_m), y_1, \ldots, y_m) = (y_1, \ldots, y_m)(x'_1, \ldots, x'_m),$$

which in turn is equivalent to

$$\widehat{x}\widehat{F}(y_1) = y_1\widehat{x}' \quad \text{and} \quad \forall i \in \{2, \ldots, m\}\ y_i = x_i y_{i-1}(x'_i)^{-1}$$

where $\widehat{x} := x_1\widehat{F}(x_m x_{m-1} \ldots x_1)$ and $\widehat{x}' := x'_1 F(x'_m x'_{m-1} \ldots x'_1)$. Therefore we can remove all the y_i's for $i \geqslant 2$ to show that

$$\Sigma = \{(x_i), (x'_i), y_1 \in \mathbb{U} \times \mathbb{U} \times \widehat{\mathbb{G}} \colon \widehat{x}\widehat{F}(y_1) = y_1\widehat{x}'\}.$$

This scheme lies over the scheme $\widehat{\Sigma} = \{\widehat{x}, \widehat{x}', y_1 \in \widehat{\mathbb{U}} \times \widehat{\mathbb{U}} \times \widehat{\mathbb{G}} \colon \widehat{x}\widehat{F}(y) = y\widehat{x}'\}$ attached to $\widehat{\mathbb{G}}$, via the natural map $(x_i), (x'_i), y_1 \mapsto (\widehat{x}, \widehat{x}', y_1)$. All fibers of this map are isomorphic to the perfection of a fixed affine space of some dimension, so that $H_c^*(\Sigma) = H_c^*(\widehat{\Sigma})$. This shows that Theorem 4.2.1 holds for $\mathbf{G}$ whenever it holds for $\widehat{\mathbf{G}}$

Summarising the results obtained in this section, we have that Theorem 4.2.1 holds whenever it holds for any absolutely almost simple group. In particular we shall, and we will, only consider the case where W is irreducible in Sections 6 and 7.

5. Extensions of action

Throughout this section we work in the general setup of Section 2.4. We fix two *arbitrary* pairs $(\mathbf{T},\mathbf{U})$, $(\mathbf{T}',\mathbf{U}')$ with $\mathbf{x}\in\mathcal{A}(\mathbf{T},k)\cap\mathcal{A}(\mathbf{T}',k)$. Then we have the corresponding subgroups $\mathbb{U},\mathbb{U}'\subseteq\mathbb{G}$ and for $v\in W(\mathbb{T}_1,\mathbb{T}_1')$, the scheme $\Sigma_v={}^{\mathbb{U},\mathbb{U}'}\Sigma_v$ as in (4.2.2). Pushing further the ideas from [DL76, (6.6.2)] and [CI19b, §3.3 and §3.4], we will extend the action of the finite group $T\times T'$ on Σ_v to the actions of various bigger groups.

5.1. **Lusztig's extension.** First we have the extension of action due to Lusztig (and a minimal variation of it). The geometric points $\mathbb{G}^1(\overline{\mathbb{F}}_q)$ of the group $\mathbb{G}^1=\ker(\mathbb{G}\to\mathbb{G}_1)$ can be written as a product of all "root subgroups" $\mathbb{U}_\alpha(\overline{\mathbb{F}}_q)$, $\alpha\in\Phi(\mathbf{T},\mathbf{G})$ contained in it, and these can be taken in any order [BT72, (6.4.48)], so we have

$$\begin{aligned}\mathrm{pr}^{-1}(\mathbb{U}_1\dot v\mathbb{T}_1'\mathbb{U}_1') &= \mathbb{U}\dot v\mathbb{G}^1\mathbb{T}'\mathbb{U}'\\ &= \mathbb{U}\dot v\left[({}^{v^{-1}}\mathbb{U})^1({}^{v^{-1}}\mathbb{U}^-\cap\mathbb{U}'^-)^1\mathbb{U}'^1\mathbb{T}'^1\right]\mathbb{T}'\mathbb{U}'\\ &= \mathbb{U}\dot v\mathbb{T}'\mathbb{K}^1\mathbb{U}',\end{aligned}$$

where we put

$$\mathbb{K}:=\mathbb{K}_{\mathbb{U},\mathbb{U}',v}:={}^{v^{-1}}\mathbb{U}^-\cap\mathbb{U}'^-$$

Then Σ_v can be rewritten as

$$\Sigma_v=\{(x,x',y)\in\Sigma\colon y\in\mathbb{U}\dot v\mathbb{T}'\mathbb{K}^1\mathbb{U}'\}.$$

Consider

$$\widehat\Sigma_v=\{x,x',y',\tau,z,y''\in F\mathbb{U}\times F\mathbb{U}'\times\mathbb{U}\times\mathbb{T}'\times\mathbb{K}^1\times\mathbb{U}'\colon xF(y'\dot v\tau zy'')=y'\dot v\tau zy''x'\},\tag{5.1.1}$$

with an action of $T\times T'$ given by

$$(t,t')\colon(x,x',y',\tau,z,y'')\mapsto(txt^{-1},t'x't'^{-1},ty't^{-1},\dot v^{-1}t\dot v\tau t',t'zt'^{-1},t'y''t'^{-1}).$$

Then we have an obvious $T\times T'$-equivariant map $\widehat\Sigma_v\to\Sigma_v$, $(x,x',y',\tau,z,y'')\mapsto(x,x',y'\dot v\tau zy'')$, which is a Zariski-locally trivial fibration with fibers isomorphic to the perfection of a fixed affine space. Then the ℓ-adic Euler characteristic does not change, so that we have an equality of virtual $T\times T'$-modules

$$H_c^*(\Sigma_v)=H_c^*(\widehat\Sigma_v).\tag{5.1.2}$$

Now make the change of variables $xF(y')\mapsto x$, $x'F(y'')^{-1}\mapsto x'$, so that

$$\widehat\Sigma_v\cong\{x,x',y',\tau,z,y''\in F\mathbb{U}\times F\mathbb{U}'\times\mathbb{U}\times\mathbb{T}'\times\mathbb{K}^1\times\mathbb{U}'\colon xF(\dot v\tau z)=y'\dot v\tau zy''x'\},\tag{5.1.3}$$

Lemma 5.1.1 ([Lus04], 1.9)**.** *(i) This $T\times T'$-action on $\widehat\Sigma_v$ extends to an action of the closed subgroup*

$$H_v=\{(t,t')\in\mathbb{T}\times\mathbb{T}'\colon\dot v^{-1}t^{-1}F(t)\dot v=t'^{-1}F(t')\text{ centralizes }\mathbb{K}=({}^{v^{-1}}\mathbb{U}\cap\mathbb{U}')^-\}$$

of $\mathbb{T}\times\mathbb{T}'$, given by

$$(t,t')\colon(x,x',y',\tau,z,y'')\mapsto(F(t)xF(t)^{-1},F(t')x'F(t')^{-1},F(t)y'F(t)^{-1},\dot v^{-1}t\dot v\tau t'^{-1},t'zt'^{-1},F(t')y''F(t')^{-1}).$$

(ii) Similarly, the $T\times T'$-action on $\widehat\Sigma_v$ extends to an action of the closed subgroup

$$H_v'=\{(t,t')\in\mathbb{T}\times\mathbb{T}'\colon F(\dot v)^{-1}tF(t)^{-1}F(\dot v)=t'F(t')^{-1}\text{ centralizes }F(\mathbb{K})=F({}^{v^{-1}}\mathbb{U}\cap\mathbb{U}')^-\}$$

of $\mathbb{T}\times\mathbb{T}'$, given by

$$(t,t')\colon(x,x',y',\tau,z,y'')\mapsto(txt^{-1},t'x't'^{-1},ty't^{-1},\dot v^{-1}t\dot v\tau t'^{-1},t'zt'^{-1},t'y''t'^{-1}).$$

Proof. (ii) is proven in [Lus04, 1.9]. The proof of part (i) is completely analogous. □

5.2. **Another extension of action.** To extend the action differently, we replace the resolution $\widehat{\Sigma}_v \to \Sigma_v$ by a different one. For that purpose note that the ($\overline{\mathbb{F}}_q$-points of the) closed subgroup

$$({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}'^-)^1 \cdot ({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}') \tag{5.2.1}$$

of ${}^{v^{-1}}\mathbb{U}^-$ can be described as being cut out by a certain concave function on $\Phi(\mathbf{T}, \mathbf{G})$. More precisely, it is equal to the quotient of U_f (in the sense of Bruhat–Tits [BT72, §6.2]) with $f\colon \Phi \cup \{0\} \to \widetilde{\mathbb{R}}$ being the function

$$f(\alpha) = \begin{cases} \infty & \text{if } \alpha = 0 \text{ or } \alpha \in \Phi(\mathbf{T}, {}^{v^{-1}}\mathbf{U}) \\ 0 & \text{if } \alpha \in \Phi(\mathbf{T}, {}^{v^{-1}}\mathbf{U} \cap \mathbf{U}') \\ 1 & \text{if } \alpha \in \Phi(\mathbf{T}, {}^{v^{-1}}\mathbf{U} \cap \mathbf{U}'^-) \end{cases}$$

by the normal subgroup $\ker(L^+\mathcal{G} \to \mathbb{G}_r)(\overline{\mathbb{F}}_q)$ (which is itself of the form $U_{f'}$ for a further concave function f'). By [BT72, 6.4.48], the order of the roots in the product expression appearing in (5.2.1) can be chosen arbitrary, thus the group (5.2.1) is also equal to

$$\begin{aligned} &({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}'^- \cap F\mathbb{U}'^-)^1 \cdot ({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}' \cap F\mathbb{U}'^-) \cdot ({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}'^- \cap F\mathbb{U}')^1 \cdot ({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}' \cap F\mathbb{U}') \\ &= ({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}'^-)' \cdot ({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}')', \end{aligned} \tag{5.2.2}$$

where $({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}'^-)'$ denotes the closed subgroup of ${}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}'^-$ determined by the appropriate concave function on roots (and similarly for $({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}')'$). Then on $\overline{\mathbb{F}}_q$-points we have

$$\begin{aligned} \mathrm{pr}^{-1}(\mathbb{U}_1\dot{v}\mathbb{T}'_1\mathbb{U}'_1) &= \mathrm{pr}^{-1}(\mathbb{U}_1\dot{v}\mathbb{T}'_1({}^{v^{-1}}\mathbb{U}_1^- \cap \mathbb{U}'_1)) \\ &= \mathbb{U}\dot{v}\mathbb{T}'\mathbb{G}^1({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}'_1) \\ &= \mathbb{U}\dot{v}\mathbb{T}'({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}'^-)^1({}^{v^{-1}}\mathbb{U}^- \cap \mathbb{U}'), \end{aligned}$$

using that $\mathbb{G}^1(\overline{\mathbb{F}}_q)$ decomposes into the product of "root subgroups" $\mathbb{U}_\alpha(\overline{\mathbb{F}}_q)$ contained in it, taken in any order. Using this and the expression (5.2.2) of the group (5.2.1), we can rewrite

$$\Sigma_v = \{x, x', y \in F\mathbb{U} \times F\mathbb{U}' \times \mathbb{U}\dot{v}\mathbb{T}'({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}'^-)'({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}')' \colon xF(y) = yx'\}.$$

Now consider

$$\widetilde{\Sigma}_v := \{x, x', y', \tau, z_1, y''_1 \in F\mathbb{U} \times F\mathbb{U}' \times \mathbb{U} \times \mathbb{T}' \times ({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}'^-)' \times ({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}')' \colon xF(y'\dot{v}\tau z_1 y''_1) = y'\dot{v}\tau z_1 y''_1 x'\},$$

with $T \times T'$-action given by

$$(t, t')\colon (x, x', y', \tau, z_1, y''_1) \mapsto (txt^{-1}, t'x't'^{-1}, ty't^{-1}, \dot{v}^{-1}t\dot{v}\tau t', t'z_1t'^{-1}, t'y''_1t'^{-1})$$

Then the map $\widetilde{\Sigma}_v \to \Sigma_v$, $(x, x', y', \tau, z_1, y''_1) \mapsto (x, x', y'\dot{v}\tau z_1 y''_1)$ is a $T \times T'$-equivariant Zariski-locally trivial fibration, with fibers isomorphic to the perfection of some fixed affine space. In particular, we again have an equality of virtual $T \times T'$-modules

$$H_c^*(\Sigma_v) = H_c^*(\widetilde{\Sigma}_v) \tag{5.2.3}$$

Now we make the change of variables $xF(y') \mapsto x$, $y''_1 x' \mapsto x'$, so that

$$\widetilde{\Sigma}_v := \{x, x', y', \tau, z_1, y''_1 \in F\mathbb{U} \times F\mathbb{U}' \times \mathbb{U} \times \mathbb{T}' \times ({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}'^-)' \times ({}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}')' \colon xF(\dot{v}\tau z_1 y''_1) = y'\dot{v}\tau z_1 x'\},$$

with the action of $T \times T'$ given by the same formula as before.

Lemma 5.2.1. (i) *The action of $T \times T'$ on $\widetilde{\Sigma}_v$ extends to an action of the closed subgroup*

$$H''_v = \{(t, t') \in \mathbb{T} \times \mathbb{T}' \colon \dot{v}^{-1}F^{-1}(t)^{-1}t\dot{v} = t'^{-1}F^{-1}(t') \text{ centralizes } {}^{v^{-1}}\mathbb{U}^- \cap F\mathbb{U}'^-\},$$

of $\mathbb{T} \times \mathbb{T}'$ *given by*

$$(t,t')\colon (x,x',y',\tau,z_1,y_1'') \mapsto (txt^{-1}, t'x't'^{-1}, ty't^{-1}, \dot{v}^{-1}t\dot{v}\tau t'^{-1}, t'z_1t'^{-1}, F(t')y_1''F(t')^{-1})$$

(ii) *The action of* $T \times T'$ *on* $\widetilde{\Sigma}_v$ *extends to an action of the closed subgroup*

$$H_v'''' = \{(t,t') \in \mathbb{T} \times \mathbb{T}' \colon \dot{v}^{-1}F^{-1}(t)^{-1}t\dot{v} = t'^{-1}F^{-1}(t') \text{ centralizes } {}^{v^{-1}}F\mathbb{U}^- \cap \mathbb{U}'^-\},$$

of $\mathbb{T} \times \mathbb{T}'$ *given by*

$$(t,t')\colon (x,x',y',\tau,z_1,y_1'') \mapsto (txt^{-1}, t'x't'^{-1}, ty't^{-1}, \dot{v}^{-1}t\dot{v}\tau t'^{-1}, t'z_1t'^{-1}, F^{-1}(t')y_1''F^{-1}(t')^{-1})$$

Proof. The proof is a computation similar to Lemma 5.1.1. □

5.3. **An isomorphism.** The extensions of actions from Sections 5.1 and 5.2 suffice to prove Theorem 3.2.3 in type A_n, as was done in [CI19b, Thm. 3.1]. The proof was however based on a particular combinatorial property of this type. For the general case, we need the following new idea. One immediately checks that

$$\alpha = {}^{\mathbb{U},\mathbb{U}'}\alpha\colon {}^{\mathbb{U},\mathbb{U}'}\Sigma \to {}^{\mathbb{U},F\mathbb{U}'}\Sigma$$
$$(x,x',y) \mapsto (x, F(x'), yx')$$

is an $T \times T'$-equivariant isomorphism. In general, it does not preserve the locally closed pieces Σ_v. However, we have the following lemma.

Lemma 5.3.1. *For* $v,w \in W(\mathbb{T}_1,\mathbb{T}_1')$*, let* $Y_{v,w} \subseteq {}^{\mathbb{U},F\mathbb{U}'}\widehat{\Sigma}_w$ *be defined by the Cartesian diagram*

$$\begin{array}{ccc} Y_{v,w} & \longrightarrow & {}^{\mathbb{U},F\mathbb{U}'}\widehat{\Sigma}_w \\ \downarrow & & \downarrow \\ \alpha({}^{\mathbb{U},\mathbb{U}'}\Sigma_v) \cap {}^{\mathbb{U},F\mathbb{U}'}\Sigma_w & \longrightarrow & {}^{\mathbb{U},F\mathbb{U}'}\Sigma_w \end{array}$$

where the left lower entry is the scheme-theoretic intersection inside ${}^{\mathbb{U},F\mathbb{U}'}\Sigma_w$*. Then* $Y_{v,w}$ *is stable under the action of* H_w *on* ${}^{\mathbb{U},F\mathbb{U}'}\widehat{\Sigma}_w$ *defined in Lemma 5.1.1(i).*

Proof. In terms of the presentation (5.1.1) of ${}^{\mathbb{U},F\mathbb{U}'}\widehat{\Sigma}_w$ where we denote the coordinates by $x_1, x_1', y_1', \tau_1, z_1, y_1''$, consider the morphism

$$y_1'\dot{w}\tau_1z_1y_1''F^{-1}(x_1')^{-1} \colon \widehat{\Sigma}_w \to \mathbb{G}.$$

The subscheme $Y_{v,w}$ is the preimage under this morphism of $\mathrm{pr}^{-1}(\mathbb{U}_1\dot{v}\mathbb{T}_1'\mathbb{U}_1')$. Now we apply the change of coordinates $(x_1F(y_1) \mapsto x_1,\ x_1'F(y_1'')^{-1} \mapsto x_1')$ from (5.1.1) to (5.1.3). Then the expression $F^{-1}(x_1')^{-1}$ in the old coordinates gets $y_1''^{-1}F^{-1}(x_1')^{-1}$ in the new coordinates. Thus in the new coordinates, $Y_{v,w} \subseteq {}^{\mathbb{U},F\mathbb{U}'}\widehat{\Sigma}_w$ is the preimage under

$$y := y_1'\dot{w}\tau_1z_1F^{-1}(x_1')^{-1} \colon \widehat{\Sigma}_w \to \mathbb{G}$$

of $\mathrm{pr}^{-1}(\mathbb{U}_1\dot{v}\mathbb{T}_1'\mathbb{U}_1') \subseteq \mathbb{G}$. We have to show that for any $\overline{\mathbb{F}}_q$-algebra R and any $(t,t') \in H_w(R)$, the map $(t,t')\colon Y_{v,w,R} \to {}^{\mathbb{U},F\mathbb{U}'}\widehat{\Sigma}_{w,R}$ factors through $Y_{v,w,R} \subseteq {}^{\mathbb{U},F\mathbb{U}'}\widehat{\Sigma}_{w,R}$, that is $y \circ (t,t')\colon Y_{v,w,R} \to \mathbb{G}_R$ factors through the locally closed subset $\mathrm{pr}^{-1}(\mathbb{U}_1\dot{v}\mathbb{T}_1'\mathbb{U}_1')_R \subseteq \mathbb{G}_R$. It suffices to do so on points. Let $(X_1, X_1', Y_1', T_1, Z_1, Y_1'') \in Y_{v,w}(R')$ for some R-algebra R'. Then

$$\begin{aligned} y \circ (t,t')(X_1,X_1',Y_1',T_1,Z_1,Y_1'') &= y(F(t)X_1F(t)^{-1}, F(t')X_1'F(t')^{-1}, F(t)Y_1'F(t)^{-1}, \dot{w}^{-1}t\dot{w}T_1t'^{-1}, t'Z_1t'^{-1}, F(t')Y_1''F(t')^{-1}) \\ &= F(t)Y_1'F(t)^{-1}t\dot{w}T_1Z_1F^{-1}(X_1')^{-1}t'^{-1} \\ &= \underbrace{F(t)Y_1'F(t)^{-1}}_{\in\mathbb{U}(R')} \cdot \underbrace{t}_{\in\mathbb{T}(R')} \cdot \underbrace{Y_1'^{-1}}_{\in\mathbb{U}(R')} \cdot \underbrace{Y_1'\dot{w}T_1Z_1F^{-1}(X_1')^{-1}}_{\substack{\in\mathrm{pr}^{-1}(\mathbb{U}_1\dot{v}\mathbb{T}_1'\mathbb{U}_1')(R') \\ \text{by assumption}}} \cdot \underbrace{t'^{-1}}_{\in\mathbb{T}'(R')}. \end{aligned}$$

The last expression clearly lies in $\mathrm{pr}^{-1}(\mathbb{U}_1\dot{v}\mathbb{T}_1'\mathbb{U}_1')(R')$ and we are done. □

Let us also look at the converse situation. The inverse of α is given by $(x_1, x_1', y_1) \mapsto (x_1, F^{-1}(x), y_1 F^{-1}(x_1')^{-1})$.

Lemma 5.3.2. *For $v, w \in W(\mathbb{T}_1, \mathbb{T}_1')$ let $Z_{v,w} \subseteq {}^{\mathbb{U},\mathbb{U}'}\widehat{\Sigma}_v$ be defined by the Cartesian diagram*

$$\begin{array}{ccc} Z_{v,w} & \longrightarrow & {}^{\mathbb{U},\mathbb{U}'}\widehat{\Sigma}_v \\ \downarrow & & \downarrow \\ {}^{\mathbb{U},\mathbb{U}'}\Sigma_v \cap \alpha^{-1}({}^{\mathbb{U},F\mathbb{U}'}\Sigma_w) & \longrightarrow & {}^{\mathbb{U},\mathbb{U}'}\Sigma_v \end{array}$$

where the left lower entry is the scheme-theoretic intersection inside ${}^{\mathbb{U},F\mathbb{U}'}\Sigma_w$. Then $Z_{v,w}$ is stable under the action of H_v' on ${}^{\mathbb{U},\mathbb{U}'}\widehat{\Sigma}_v$ defined in Lemma 5.1.1(ii).

Proof. In terms of the presentation (5.1.1) of ${}^{\mathbb{U},\mathbb{U}'}\widehat{\Sigma}_v$ where we denote the coordinates by x, x', y', τ, z, y'', consider the morphism

$$y'\dot{v}\tau z y'' x' \colon \widehat{\Sigma}_v \to \mathbb{G}.$$

Then $Z_{v,w}$ is the preimage under this morphism of $\mathrm{pr}^{-1}(\mathbb{U}_1 \dot{w} \mathbb{T}_1' F\mathbb{U}_1')$. Now we make the change of coordinates $(xF(y) \mapsto x,\ x'F(y'')^{-1} \mapsto x')$ from (5.1.1) to (5.1.3). Then the expression x' in the old coordinates becomes $x'F(y'')$ in the new coordinates. Hence in the new coordinates $Z_{v,w}$ is the preimage of $\mathrm{pr}^{-1}(\mathbb{U}_1 \dot{w} \mathbb{T}_1' F\mathbb{U}_1')$ under

$$y_1 := y'\dot{v}\tau z y'' x' F(y'') \colon \widehat{\Sigma}_v \to \mathbb{G}.$$

Let R be an $\overline{\mathbb{F}}_q$-algebra and $(t, t') \in H_v'(R)$. As in the proof of Lemma 5.3.1, we have to show that $y_1 \circ (t, t') \colon Z_{v,w,R} \to \mathbb{G}_R$ factors through $\mathrm{pr}^{-1}(\mathbb{U}_1 \dot{w} \mathbb{T}_1' F\mathbb{U}_1')_R \subseteq \mathbb{G}_R$. Let $(X, X', Y', T, Z, Y'') \in Z_{v,w}(R')$ for some R-algebra R'. Then

$$\begin{aligned} y_1 \circ (t,t')(X, X', Y', T, Z, Y'') &= y_1(tXt^{-1}, t'X't'^{-1}, tY't^{-1}, v^{-1}tvTt'^{-1}, t'Zt'^{-1}, t'Y''t'^{-1}) \\ &= tY'\dot{v}TZY''X't'^{-1}F(t')F(Y'')F(t')^{-1} \\ &= \underbrace{t}_{\in \mathbb{T}(R')} \cdot \underbrace{Y'\dot{v}TZY''X'F(Y'')}_{\substack{\in \mathrm{pr}^{-1}(\mathbb{U}_1\dot{w}\mathbb{T}_1'F\mathbb{U}_1')(R') \\ \text{by assumption}}} \cdot \underbrace{F(Y'')^{-1}}_{\in F\mathbb{U}'(R')} \cdot \underbrace{t'^{-1}}_{\in \mathbb{T}'(R')} \cdot \underbrace{F(t')F(Y'')F(t')^{-1}}_{\in F\mathbb{U}'(R')} \end{aligned}$$

The last expression lies in $\mathrm{pr}^{-1}(\mathbb{U}_1 \dot{w} \mathbb{T}_1' F\mathbb{U}_1')(R')$ and we are done. ☐

Remark 5.3.3. There seem to be no analogs of these lemmas for $\widetilde{\Sigma}_v$ (from Section 5.2) instead of $\widehat{\Sigma}_v$.

6. Regularity of certain subgroups

The purpose of this section is to show that the groups $H_v, H_v', \dots$ produced in Section 5 contain $\overline{\mathbb{F}}_q$-reductive subgroups under which the varieties $\widehat{\Sigma}_v$ and $\widetilde{\Sigma}_v$ have finitely many fixed points. This will be the key for computing their cohomology, as given in Theorem 4.2.1. Note that this strategy was already used in [Lus04, 1.9(e)], but with much bigger versions of H_v.

Throughout this section we work in the setup of Theorem 3.2.3. In particular, $\mathbf{G}$ is unramified, $\mathbf{x}$ is hyperspecial, and $(\mathbf{T}, \mathbf{U})$, $(\mathbf{T}', \mathbf{U}')$ are Coxeter pairs with $\mathbf{x} = \mathbf{x_T} = \mathbf{x_{T'}}$. Thanks to the reduction results in Section 4.3 we will assume in addition that $\mathbf{G}$ is absolutely almost simple over k, i.e., that the Dynkin diagram of the split group $\mathbf{G}_{\breve{k}}$ is connected.

6.1. **Pull-back of a cocharacter under the Lang map.** We may identify the groups of cocharacters $X_*(\mathbb{T}_1)$, $X_*(\mathbf{T})$, and similarly for characters. The Frobenius F acts on $X_*(\mathbb{T}_1)$, $X^*(\mathbb{T}_1)$ and these actions induce $\mathbb{Q}$-linear automorphisms of the $\mathbb{Q}$-vector spaces $X_*(\mathbb{T}_1)_{\mathbb{Q}}$, $X^*(\mathbb{T}_1)_{\mathbb{Q}}$. Let $\mathbf{G}^{\mathrm{ad}}$ be the adjoint quotient of $\mathbf{G}$ and $\mathbf{T}^{\mathrm{ad}}$ the image of $\mathbf{T}$ in $\mathbf{G}^{\mathrm{ad}}$, such that $X_*(\mathbf{T}^{\mathrm{ad}})$ is a quotient of $X_*(\mathbf{T})$, and $X^*(\mathbf{T}^{\mathrm{ad}}) \subseteq X^*(\mathbf{T})$.

For $\chi \in X_*(\mathbb{T}_1)$, we are interested in (the connected component of the) subgroup

$$H_\chi = \{t \in \mathbb{T}_1 \colon t^{-1}F(t) \in \mathrm{im}(\chi \colon \mathbb{G}_m \to \mathbb{T}_1)\} \subseteq \mathbb{T}_1. \tag{6.1.1}$$

Lemma 6.1.1. *Let $\chi \in X_*(\mathbb{T}_1)$. There exists $0 \neq \mu \in X_*(\mathbb{T}_1)$ such that $F\mu - \mu \in \mathbb{Q} \cdot \chi$. Such μ is unique up to a scalar and we have $H_\chi^\circ = \mathrm{im}(\mu)$.*

Proof. By [DM91, Prop. 13.7], the map $F - 1\colon X_*(\mathbb{T}_1) \to X_*(\mathbb{T}_1)$ is injective and has finite cokernel. Therefore there exists $0 \neq \mu \in X_*(\mathbb{T}_1)$, unique up to a scalar, such that $F\mu - \mu \in \mathbb{Q} \cdot \chi$. This implies that $\mathrm{im}(\mu)$ is a one-dimensional subtorus of $\mathbb{T}_1$ contained in H_χ. Since H_χ is one-dimensional, this forces $\mathrm{im}(\mu) = H_\chi^\circ$. □

Recall from §2.7 that $\{\alpha^*\colon \alpha \in \Delta\} \subseteq X_*(\mathbb{T}_1^{\mathrm{ad}})_\mathbb{Q}$ is the set of fundamental coweights, defined as the basis of $X_*(\mathbb{T}_1^{\mathrm{ad}})_\mathbb{Q}$ dual to Δ.

Proposition 6.1.2. *Assume Condition* (2.7.1) *holds for q and $\mathbf{G}$. Then there exists a set of simple roots $\Delta \subset \Phi(\mathbf{T}, \mathbf{G})$ such that*

(i) *F acts on $X^*(\mathbf{T})$ as $qc\sigma$ where σ satisfies $\sigma(\Delta) = \Delta$, $c \in W$ and $c\sigma$ is a twisted Coxeter element of (W, σ) such that*
$$\ell(c\sigma(c)\cdots\sigma^{i-1}(c)) = i\ell(c)$$
for all $0 \leqslant i \leqslant h/2$, where h is the Coxeter number of (W, σ).

(ii) *For all $\alpha \in \Delta$, and all $\gamma \in \Phi(\mathbf{T}, \mathbf{G})$ we have $H_{\alpha^*}^\circ \nsubseteq \ker(\gamma)$.*

Proof. Let $\tau = q^{-1}F$. Then the order of τ is h. Let $g \in \mathbf{G}$ be such that $\mathbf{T}_0 = {}^g\mathbf{T}$. By assumption on $\mathbf{T}$ the endomorphism ${}^g\tau$ of $X^*(\mathbf{T}_0)$ lies in the W_0-conjugacy class of twisted Coxeter elements (this class is unique by [Spr74, Thm. 7.6]). Therefore the same holds for τ in W. Let $\zeta = \exp(2\pi \mathrm{i}/h)$. By [Spr74, Thm. 7.6], the ζ-eigenspace of τ on $X^*(\mathbf{T}^{\mathrm{ad}})_\mathbb{C}$ is one-dimensional and is not contained in any reflection hyperplane. Let $0 \neq v \in X^*(\mathbf{T}^{\mathrm{ad}})_\mathbb{C}$ be an eigenvector of τ for the eigenvalue ζ such that $\mathrm{Re}(\langle v, \alpha^\vee\rangle) \neq 0$ for all $\alpha \in \Phi(\mathbf{T}, \mathbf{G})$. Then as shown in the proof [Spr74, Prop. 4.10] the condition $\mathrm{Re}(\langle v, \alpha^\vee\rangle) > 0$ defines a set of positive roots $\Phi^+ \subset \Phi(\mathbf{T}, \mathbf{G})$, hence a basis Δ. Let $c \in W$ be the unique element in W such that $c(\Delta) = \tau(\Delta)$ and $\sigma = c^{-1}\tau$. Then $\sigma(\Delta) = \Delta$ and (i) follows from [BM97, Prop. 6.5].

Let $\alpha \in \Delta$ be a simple root and $\gamma \in \Phi(\mathbf{T}, \mathbf{G})$ be any root. The orbit of γ under $\tau = q^{-1}F = c\sigma$ has exactly h elements, see [Spr74, Thm. 7.6]. If $V = \langle \tau^i(\gamma)\colon i = 0, \ldots, h-1\rangle$ is the $\mathbb{C}$-vector subspace of $X^*(\mathbf{T}^{\mathrm{ad}})_\mathbb{C}$ spanned by the orbit, then τ restricts to an automorphism of V of order h. In particular it must contain the eigenvector v defined above. Since α^* is a non-negative combination of simple coroots we deduce that $\mathrm{Re}(\langle v, \alpha^*\rangle) > 0$, which forces $\langle \tau^i(\gamma), \alpha^*\rangle \neq 0$ for some i. Let $i_0 \in \{0, \ldots, h-1\}$ be maximal such that $\langle \tau^i(\gamma), \alpha^*\rangle \neq 0$. Then
$$\begin{aligned}\sum_{i=0}^{h-1}\langle F^i(\gamma), \alpha^*\rangle &= \sum_{i=0}^{h-1} q^i\langle \tau^i(\gamma), \alpha^*\rangle \\ &= q^{i_0}\langle \tau^{i_0}(\gamma), \alpha^*\rangle + \sum_{i=0}^{i_0-1} q^i\langle \tau^i(\gamma), \alpha^*\rangle\end{aligned}$$
so that
$$\begin{aligned}\left|\sum_{i=0}^{h-1}\langle F^i(\gamma), \alpha^*\rangle\right| &\geqslant q^{i_0}|\langle \tau^{i_0}(\gamma), \alpha^*\rangle| - \sum_{i=0}^{i_0-1} q^i|\langle \tau^i(\gamma), \alpha^*\rangle| \\ &\geqslant q^{i_0} - \sum_{i=0}^{i_0-1} q^i M = q^{i_0} - M\frac{q^{i_0}-1}{q-1} \\ &\geqslant q^{i_0} - (q-1)\frac{q^{i_0}-1}{q-1} = 1\end{aligned}$$
since by Condition (2.7.1) we have $q - 1 \geqslant M$. This proves that $\sum_{i=0}^{h-1}\langle F^i(\gamma), \alpha^*\rangle \neq 0$. Now recall that $F^h = q^h$ on $X^*(\mathbf{T})$. We have $(F-1)\sum_{i=0}^{h-1} F^i = F^h - 1 = q^h - 1$ therefore $(F-1)$ is invertible on $X^*(\mathbf{T})$ and $(F-1)^{-1} = (q^h-1)^{-1}\sum_{i=0}^{h-1} F^i$. We deduce that
$$\langle (F-1)^{-1}\gamma, \alpha^*\rangle = \langle \gamma, (F-1)^{-1}\alpha^*\rangle \neq 0.$$

Consequently, for any $0 \neq \mu \in (F-1)^{-1}\mathbb{Q}\cdot\alpha^*$ we have $\langle\gamma,\mu\rangle \neq 0$. Using Lemma 6.1.1 we get $H^\circ_{\alpha^*} = \mathrm{im}(\mu)$ and we deduce that $H^\circ_{\alpha^*} \nsubseteq \ker(\gamma)$. □

6.2. **A consequence.** We have the short exact sequence of $\mathbb{F}_q$-groups

$$0 \to \mathbb{T}^1 \to \mathbb{T} \to \mathbb{T}_1 \to 0,$$

which is (canonically) split by the Teichmüller lift. Moreover, we have an isomorphism $\mathbb{T} \cong \mathbb{T}^1 \times \mathbb{T}_1$ which sends the unipotent part $\mathbb{T}_{\mathrm{unip}}$ to $\mathbb{T}^1$ and the reductive part $\mathbb{T}_{\mathrm{red}}$ to $\mathbb{T}_1$. This also applies to $\mathbb{T}'$ instead of $\mathbb{T}$.

Let now $\mathbf{L}$ be a proper Levi subgroup of $\mathbf{G}$ containing $\mathbf{T}$, and let $v \in W(\mathbb{T}_1, \mathbb{T}'_1)$. We will be interested in the closed subgroup

$$H_{\mathbf{L},v,r} = \{(t,t') \in \mathbb{T}\times\mathbb{T}' \colon t^{-1}F(t) = \dot{v}t'^{-1}F(t')\dot{v}^{-1} \text{ centralizes } \mathbb{L}\} \subseteq \mathbb{T}\times\mathbb{T}'.$$

Being affine and commutative, $H_{\mathbf{L},v,r}$ decomposes into the product of its unipotent and reductive parts, $H_{\mathbf{L},v,r} \cong H_{\mathbf{L},v,r,\mathrm{unip}} \times H_{\mathbf{L},v,r,\mathrm{red}}$, and we have $H_{\mathbf{L},v,r,\mathrm{red}} \subseteq \mathbb{T}_{\mathrm{red}} \times \mathbb{T}'_{\mathrm{red}} \cong \mathbb{T}_1 \times \mathbb{T}'_1$.

Proposition 6.2.1. *Assume Condition* (2.7.1) *holds for q and $\mathbf{G}$. Suppose that $(\mathbf{T},\mathbf{U})$, $(\mathbf{T}',\mathbf{U}')$ are such that the corresponding sets of simple roots satisfy the conclusion of Proposition 6.1.2. Let $\mathbf{L}$, v be as above. Consider the connected component $H^\circ_{\mathbf{L},v,r,\mathrm{red}}$ of the reductive part of $H_{\mathbf{L},v,r}$. Let H (resp. H') denote the image of $H^\circ_{\mathbf{L},v,r,\mathrm{red}}$ under*

$$H^\circ_{\mathbf{L},v,r,\mathrm{red}} \hookrightarrow H_{\mathbf{L},v,r} \to \mathbb{T}\times\mathbb{T}' \twoheadrightarrow \mathbb{T}_1\times\mathbb{T}'_1 \stackrel{\mathrm{pr}}{\twoheadrightarrow} \mathbb{T}_1,$$

(resp. the image of the same map with $\mathbb{T}_1$ on the right replaced by $\mathbb{T}'_1$). Then for all $\gamma \in \Phi(\mathbf{T},\mathbf{G})$, H is not contained in the subtorus $\ker(\gamma) \subseteq \mathbb{T}_1$, and similarly for H' and all $\gamma' \in \Phi(\mathbf{T}',\mathbf{G})$.

Proof. Enlarging $\mathbf{L}$ makes its centralizer smaller, hence we may assume that $\mathbf{L}$ is a maximal proper Levi subgroup containing $\mathbf{T}$. We show only the claim for H, the one for H' has a similar proof. Let

$$H_{\mathbf{L},r} = \{t \in \mathbb{T}\colon t^{-1}F(t) \text{ centralizes } \mathbb{L}\} \subseteq \mathbb{T}.$$

The projection to the first factor $H_{\mathbf{L},v,r} \to H_{\mathbf{L},r}$, $(t,t') \mapsto t$ is surjective (by Lang's theorem for the connected group $\mathbb{T}'$), hence induces also a surjection on the reductive parts and hence also on their connected components, so it suffices to show that the connected component of

$$H'_1 := \mathrm{im}\left(H^\circ_{\mathbf{L},r,\mathrm{red}} \hookrightarrow H_{\mathbf{L},r} \to \mathbb{T} \twoheadrightarrow \mathbb{T}_1\right)$$

is not contained in $\ker(\gamma)$ for any $\gamma \in \Phi(\mathbf{T},\mathbf{G})$.

By maximality of $\mathbf{L}$ there exists a system of simple positive roots $\Delta_1 \subseteq \Phi(\mathbf{T},\mathbf{G})$ and some $\alpha \in \Delta_1$ such that $\mathbf{L}$ is generated by $\mathbf{T}$ and all $\mathbf{U}_\beta$, $\mathbf{U}_{-\beta}$ with $\beta \in \Delta_1 \smallsetminus \{\alpha\}$. Alternatively, we can characterize $\mathbf{L}$ as follows: Δ_1 forms a basis of $X^*(\mathbf{T}^{\mathrm{ad}})_\mathbb{Q}$, and we have the fundamental coweights $\{\beta^*\}_{\beta\in\Delta_1}$ which form the dual basis of $X_*(\mathbf{T}^{\mathrm{ad}})_\mathbb{Q}$. Then $\mathbf{L}$ is equal to the centralizer in $\mathbf{G}$ of a(ny) lift of α^* to $X_*(\mathbf{T})_\mathbb{Q}$ (again denoted α^*). By Proposition 6.1.2, the subgroup $H^\circ_{\alpha^*}$ of $\mathbb{T}_1$ studied in Section 6.1 is not contained in $\ker(\gamma)$ for any $\gamma \in \Phi(\mathbf{T},\mathbf{G})$. Thus it suffices to show that $H'_1 \supseteq H^\circ_{\alpha^*}$.

We have the Teichmüller lift $\mathrm{TM}\colon \mathbb{T}_1 \to \mathbb{T}$, inducing an isomorphism $\mathbb{T}_1 \xrightarrow{\sim} \mathbb{T}_{\mathrm{red}}$ onto the reductive part of $\mathbb{T}$. Restricted to $H^\circ_{\alpha^*}$, TM induces an isomorphism $\mathrm{TM}\colon H^\circ_{\alpha^*} \xrightarrow{\sim} \mathrm{TM}(H^\circ_{\alpha^*})$ onto a subgroup of $\mathbb{T}_{\mathrm{red}}$.

Lemma 6.2.2. *For any $\tilde{t} \in \mathrm{TM}(H^\circ_{\alpha^*})$, $\tilde{t}^{-1}F(\tilde{t})$ centralizes $\mathbb{U}_{\beta,r}$ for all $\beta \in \Phi(\mathbf{T},\mathbf{L})$ and consequently, it centralizes $\mathbb{L}_r$. In particular, we have $\mathrm{TM}(H^\circ_{\alpha^*}) \subseteq H_{\mathbf{L},r}$.*

Proof of Lemma 6.2.2. Teichmüller lift commutes with Frobenius F, hence the map $t \mapsto t^{-1}F(t)\colon H^\circ_{\alpha^*} \to \mathrm{im}(\alpha^*)$ induces a map $\tilde{t} \mapsto \tilde{t}^{-1}F(\tilde{t})\colon \mathrm{TM}(H^\circ_{\alpha^*}) \to \mathrm{TM}(\mathrm{im}(\alpha^*))$. Thus we have to show that $\mathrm{TM}(\mathrm{im}(\alpha^*)) \subseteq \mathbb{T}$ centralizes $\mathbb{U}_{\beta,r}$.

We have the homomorphism $\mathbb{T} \to \mathrm{Aut}(\mathbb{U}_\beta)$ given by the action of $\mathbb{T}$ on $\mathbb{U}_\beta$. The group $\mathbb{U}_\beta = \mathbb{U}_{\beta,r}$ comes with a filtration by closed subgroups $\mathbb{U}^i_\beta = \ker(\mathbb{U}_{\beta,r} \to \mathbb{U}_{\beta,i})$ $(0 \leqslant i \leqslant r)$ and the action of $\mathbb{T}_r$ preserves this filtration, i.e., the above homomorphism factors through a homomorphism

$$\mathbb{T} \to \mathrm{Aut}_{\mathrm{fil}}(\mathbb{U}_\beta),$$

where $\mathrm{Aut}_{\mathrm{fil}}(\mathbb{U}_\beta) \subseteq \mathrm{Aut}(\mathbb{U}_\beta)$ is the subgroup of automorphisms preserving the filtration. This subgroup fits into an exact sequence

$$1 \to \mathrm{Aut}_{\mathrm{fil},0}(\mathbb{U}_\beta) \to \mathrm{Aut}_{\mathrm{fil}}(\mathbb{U}_\beta) \to Q \to 1,$$

where $\mathrm{Aut}_{\mathrm{fil},0}(\mathbb{U}_\beta)$ is the subgroup of automorphisms inducing the identity on the graded object $\mathrm{gr}^\bullet \mathbb{U}_\beta = \bigoplus_{i=0}^{r-1} \mathbb{U}^i_{\beta,i+1}$, and Q is defined by exactness of the above sequence. The composition

$$\mathrm{TM}(\mathrm{im}(\alpha^*)) \subseteq \mathbb{T}_r \to \mathrm{Aut}_{\mathrm{fil}}(\mathbb{U}_\beta)$$

factors through $\mathrm{Aut}_{\mathrm{fil},0}(\mathbb{U}_\beta)$: Indeed, the image of $\mathrm{TM}(\mathrm{im}(\alpha^*))$ in $\mathbb{T}_1$ lies in $\mathrm{im}(\alpha^*) \subseteq \ker(\beta)$ (the latter inclusion holds as $\langle \beta, \alpha^* \rangle = 0$), hence it acts trivially on $\mathbb{U}^i_{\beta,i+1}$ for each $0 \leqslant i \leqslant r-1$. But $\mathrm{Aut}_{\mathrm{fil},0}(\mathbb{U}_\beta)$ is unipotent, whereas $TM(\mathrm{im}(\alpha^*)) \cong \mathrm{im}(\alpha^*)$ is a torus, hence the resulting morphism

$$\mathrm{TM}(\mathrm{im}(\alpha^*)) \to \mathrm{Aut}_{\mathrm{fil},0}(\mathbb{U}_\beta)$$

is trivial. This proves the lemma. ☐

By Lemma 6.2.2, $\mathrm{TM}(H^\circ_{\alpha^*}) \subseteq H_{\mathbf{L},r}$. Being reductive and connected, $\mathrm{TM}(H^\circ_{\alpha^*})$ is thus contained in $H^\circ_{\mathbf{L},r,\mathrm{red}}$. This shows that the image of $H^\circ_{\mathbf{L},r,\mathrm{red}}$ in $\mathbb{T}_1$ contains the image of $\mathrm{TM}(H^\circ_{\alpha^*})$, which is just $H^\circ_{\alpha^*}$. ☐

Corollary 6.2.3. *Under the assumptions of Proposition 6.2.1, let $\widetilde{H}$ (resp. $\widetilde{H}'$) denote the image of the map*

$$H^\circ_{\mathbf{L},v,r,\mathrm{red}} \hookrightarrow H_{\mathbf{L},v,r} \to \mathbb{T} \times \mathbb{T}' \overset{\mathrm{pr}}{\twoheadrightarrow} \mathbb{T}$$

(resp. the image of the same map with $\mathbb{T}$ on the right replaced by $\mathbb{T}'$). Let $\mathbb{V}$ be the subgroup of $\mathbb{G}$ corresponding to the unipotent radical $\mathbf{V}$ of an arbitrary Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Then $\mathbb{V}^{\widetilde{H}} = \{1\}$, i.e., the only element of $\mathbb{V}$ fixed by the adjoint action of $\widetilde{H}$ is 1. The analogous statement holds for $\mathbb{T}', \mathbb{V}', \widetilde{H}'$.

Proof. We prove only the first claim. The proof of the second is similar. Any element of $\mathbb{V}(\overline{\mathbb{F}}_q)$ has a unique presentation as a product of elements in the subgroups $\mathbb{U}_\gamma$ corresponding to root subgroups $\mathbf{U}_\gamma \subseteq \mathbf{G}$ for $\gamma \in \Phi(\mathbf{T}, \mathbf{V})$, and this product decomposition is compatible with the adjoint action of $\mathbb{T}$. This reduces the corollary to the claim that $\mathbb{U}_\gamma^{\widetilde{H}} = \{1\}$ for all $\gamma \in \Phi(\mathbf{T}, \mathbf{V})$. For the latter, we can use induction on $1 \leqslant r' \leqslant r$: it suffices to show that if $x \in \mathbb{U}^{\widetilde{H}_i}_{\gamma,r}$ and x projects to 1 under $\mathbb{U}_{\gamma,r} \to \mathbb{U}_{\gamma,r'-1}$, then it projects to 1 under $\mathbb{U}_{\gamma,r} \to \mathbb{U}_{\gamma,r'}$. The adjoint action of $\mathbb{T}_1$ on $\mathbb{U}^{r'-1}_{\gamma,r'}$ can be described as follows: fix an isomorphism $\mathbf{G}_{a,\breve{k}} \xrightarrow{\sim} \mathbf{U}_\gamma$, which is part of an épinglage for $\mathbf{G}$. It induces an isomorphism $u^{r-1}_{\gamma,r'} \colon \mathbb{G}_{a,\overline{\mathbb{F}}_q} \xrightarrow{\sim} \mathbb{U}^{r'-1}_{\gamma,r'}$, and the adjoint action is given by $\mathrm{Ad}(t)(u^{r'-1}_{\gamma,r'}(x)) = u^{r'-1}_{\gamma,r'}(\gamma(t)x)$. Now the result follows, as the image of $\widetilde{H}$ in $\mathbb{T}_1$ is not contained in $\ker(\gamma)$ by Proposition 6.2.1. ☐

7. Cohomology of Σ

As in Section 6 we assume that $\mathbf{G}$ is an unramified absolutely almost simple group. Throughout this section we will assume that $\mathbf{G}$ is an unramified absolutely almost simple group, and that condition (2.7.1) holds for $\mathbf{G}$ and q. We fix a Coxeter pair $(\mathbf{T}, \mathbf{U})$ as in Proposition 6.1.2. In particular the action of F on W is given by $F = qc\sigma$ where σ is an automorphism of W permuting the simple reflections and $c\sigma$ is a twisted Coxeter element of (W, σ). The purpose of this section is to show that (4.2.4) holds for such a Coxeter pair. This will imply Theorem 3.2.3 for general unramified groups.

7.1. **Non-emptyness of cells.** We give here conditions for a cell ${}^{\mathbb{U},\mathbb{U}'}\Sigma_v$ to be empty (see (4.2.2) for the definition of the cell). Unlike Theorem 4.2.1 which is stated in the case where $\mathbb{U} = \mathbb{U}'$, we will work here with more general Coxeter pairs.

Proposition 7.1.1. *Let $a \in \mathbb{Z}$ and set $\mathbf{U}' := F^a(\mathbf{U})$. If $v \in W$ is such that ${}^{\mathbb{U},\mathbb{U}'}\Sigma_v \neq \varnothing$, then at least one of the following holds*

(i) *$v \in W^F$;*

(ii) *${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}'$ is contained in a proper Levi subgroup of $\mathbf{G}$ containing $\mathbf{T}$.*

Proof. Recall from Proposition 6.1.2 that $F = qc\sigma$ with $c\sigma$ a twisted Coxeter element. Given $v \in W$, let us consider the condition

$$v^{-1}\mathbb{B}_1 F(\mathbb{B}_1 v) \cap \mathbb{B}_1' F(\mathbb{B}_1') \neq \varnothing \tag{7.1.1}$$

From the definition of ${}^{\mathbb{U},\mathbb{U}'}\Sigma_v$ we see that if ${}^{\mathbb{U},\mathbb{U}'}\Sigma_v$ is non-empty then there exists $x, x \in F(\mathbb{B}_1)$ and $y \in \mathbb{B}_1 v \mathbb{B}_1'$ such that $xF(y) = yx'$. Writing $y = bvb'$ with $b \in \mathbb{B}_1$ and $b' \in \mathbb{B}_1'$ we deduce that $v^{-1}b^{-1}xF(bv) = b'vxF(b')^{-1}$ so that (7.1.1) holds. Therefore it is enough to show that if (7.1.1) holds for v then (i) or (ii) hold as well.

Since $F(\mathbb{B}) = {}^c\mathbb{B}$ we have $\mathbb{B}_1' = F^a(\mathbb{B}_1) = {}^d\mathbb{B}_1$ with $d = c\sigma(c)\cdots\sigma^{a-1}(c) = (c\sigma)^a\sigma^{-a}$. Using the fact that $d^{-1}F(d) = \sigma^a(c)c^{-1}$ we get

$$\begin{aligned} v^{-1}\mathbb{B}_1 F(\mathbb{B}_1 v) \cap \mathbb{B}_1' F(\mathbb{B}_1') &= \big(v^{-1}\mathbb{B}_1 {}^c\mathbb{B}_1 F(v)\big) \cap \big(d\mathbb{B}_1 d^{-1}F(d){}^c\mathbb{B}_1 F(d^{-1})\big) \\ &= \big(v^{-1}\mathbb{B}_1 c\mathbb{B}_1 \sigma(v)c^{-1}\big) \cap \big(d\mathbb{B}_1\sigma^a(c)\mathbb{B}_1\sigma(d^{-1})c^{-1}\big) \\ &= v^{-1}\Big(\big(\mathbb{B}_1 c\mathbb{B}_1\sigma(vd)\big) \cap \big(vd\mathbb{B}_1\sigma^a(c)\mathbb{B}_1\big)\Big)\sigma(d^{-1})c^{-1}. \end{aligned}$$

Therefore (7.1.1) is equivalent to

$$\big(\mathbb{B}_1 c\mathbb{B}_1\sigma(vd)\big) \cap \big(vd\mathbb{B}_1\sigma^a(c)\mathbb{B}_1\big) \neq \varnothing$$

which in turn is equivalent to

$$(\mathbb{B}_1 c\mathbb{B}_1\sigma(vd)\mathbb{B}_1) \cap (\mathbb{B}_1 vd\mathbb{B}_1\sigma^a(c)\mathbb{B}_1) \neq \varnothing. \tag{7.1.2}$$

Let $\Delta \subseteq \Phi(\mathbf{T},\mathbf{G})$ be the set of simple roots corresponding to $\mathbf{U}$. Since $c\sigma$ is a twisted Coxeter element, there exists representatives of σ-orbits of simple reflections $s_1, \ldots, s_r$ with $r = |\Delta/\sigma|$ such that $c = s_1 s_2 \cdots s_r$. Given $I \subset \{1, \ldots, r\}$ we will denote by W_I the smallest σ-stable parabolic subgroup of W containing s_i for all $i \in I$ and by $c_I = \prod_{i \in I} s_i$ the element of W_I obtained from c by keeping the simple reflections labelled by I. Note that $c_I\sigma$ is a twisted Coxeter element of (W_I, σ).

Assume that (7.1.2) holds. Since c contains each simple reflection at most once, the Bruhat cells $\mathbb{B}_1 u\mathbb{B}_1$ contained in $\mathbb{B}_1 c\mathbb{B}_1\sigma(vd)\mathbb{B}_1$ (resp. in $\mathbb{B}_1 vd\mathbb{B}_1\sigma^a(c)\mathbb{B}_1$) are attached to elements $u \in W$ of the form $u = c_I\sigma(vd)$ for some $I \subset \{1, \ldots, r\}$ (resp. $u = vd\sigma^a(c_J)$ for some $J \subset \{1, \ldots, r\}$). Consequently if (7.1.2) holds then there exists $I, J \subset \{1, \ldots, r\}$ such that $c_I\sigma(vd) = vd\sigma^a(c_J)$. Set $w := w_0 vd$ where w_0 is the longest element of W. Since $\sigma(w_0) = w_0$ we have $({}^{w_0}c_I)\sigma(w) = w\sigma^a(c_J)$. Let $K \subset \{1, \ldots, r\}$ be such that $W_K := {}^{w_0}W_I$. Then ${}^{w_0}c_I\sigma$ is a twisted Coxeter element of W_K (but not necessarily equal to $c_K\sigma$). By [GP00, Prop. 2.1.7], one can write $w = w_1 x w_2$ where $x \in W$ is K-reduced-J (i.e. of minimal length in $W_K x W_I$) and $(w_1, w_2) \in W_K \times W_J$. Since x is K-reduced-J we claim that

$$W_J \cap x^{-1}W_K\sigma(x) = \begin{cases} W_{J\cap K^x} & \text{if } \sigma(x) = x \\ \varnothing & \text{otherwise.} \end{cases}$$

The proof of this claim follows for example from the proof of [GP00, Thm. 2.1.12]. Indeed, if $W_J \cap x^{-1}W_K\sigma(x)$ is non-empty then there exists $y \in W_J$ and $z \in W_K$ such that $xy = z\sigma(x)$. Since x is reduced-J and $\sigma(x)$ is K-reduced we have necessarily $\ell(y) = \ell(z)$. Let $y = y_1 \cdots y_m$ be a reduced expression of y. We define inductively $z_i \in W_K$ and x_i a K-reduced element by the conditions $x_0 = x$ and $x_{i-1}y_i = z_i x_i$ for all $i = 1, \ldots, m$. In particular $z = z_1 \cdots z_m$ and $x_m = \sigma(x)$. By Deodhar's Lemma [GP00, Lem. 2.1.2] we have $\ell(z_i) = 1$ and $x_i = x_{i-1}$ for all i (the case $z_i = 1$ does not happen since $\ell(z) = \ell(y) = m$). In particular $\sigma(x) = x$ and the result of [GP00, Thm. 2.1.12] applies.

The equality $({}^{w_0}c_I)\sigma(w) = w\sigma^a(c_J)$ forces $W_J \cap x^{-1}W_K\sigma(x)$ to be non-empty, therefore $\sigma(x) = x$. Now the element

$$w_2^{-1}\sigma^a(c_J)\sigma(w_2) = x^{-1}w_1^{-1}({}^{w_0}c_I)\sigma(w_1 x) = x^{-1}(w_1^{-1}w_0 c_I w_0^{-1}\sigma(w_1))x$$

lies in $W_J \cap x^{-1}(W_K)x$ and is σ-conjugate to a twisted Coxeter element of W_J. Since Coxeter elements are elliptic, this forces $W_{J\cap K^x} = W_J$, hence $J \subset K^x$. Similarly, we find $K \subset {}^xJ$ hence ${}^xJ = K$. In particular one can write $w = w'x$ with $w' \in W_K$.

Let us now look more precisely at what elements $u \in W$ can appear. If $\mathbb{B}_1 v d\sigma^a(c_J)\mathbb{B}_1 \subset \mathbb{B}_1 v d \mathbb{B}_1 \sigma^a(c)\mathbb{B}_1$ with $J = \{j_1 < j_2 < \cdots < j_m\}$ then for all $i = 0, \ldots, m$ and all $j_i < l < j_{i+1}$ we must have $vd\sigma^a(s_{j_1} \cdots s_{j_i} s_l) < vd\sigma^a(s_{j_1} \cdots s_{j_i})$, with the convention that $j_0 = 0$, $j_{m+1} = r+1$ and $s_{j_0} = 1$. On the other hand, $w = w'x$ with $w' \in W_K$ and x is K-reduced. Since ${}^x J = K$ we can write $K = \{k_1, \ldots, k_m\}$ with $x s_{j_i} x^{-1} = s_{k_i}$. Then the condition

$$w\sigma^a(s_{j_1} \cdots s_{j_i} s_l) > w\sigma^a(s_{j_1} \cdots s_{j_i})$$

can be written

$$w'\sigma^a(s_{k_1} \cdots s_{k_i} x s_l) > w'\sigma^a(s_{k_1} \cdots s_{k_i} x)$$

Now, since $w'\sigma^a(s_{k_1} \cdots s_{k_i}) \in W_K$ and ${}^x s_l \notin W_K$ this forces $x\sigma^a(s_l) > x$ hence $xs_l > x$ (recall that $\sigma(x) = x$). Indeed, if α_l denotes the simple root associated to s_l then $w'\sigma^a(s_{k_1} \cdots s_{k_i} x)(\alpha_l) > 0$ by assumption. Since $x(\alpha_l)$ is not in Φ_K, the root subsystem associated to W_K, the element $w'\sigma^a(s_{k_1} \cdots s_{k_i}) \in W_K$ cannot change the sign of $x(\alpha_l)$, therefore $x(\alpha_l) > 0$. Since l runs over all the elements in $\{1, \ldots, r\} \smallsetminus J$ and x is reduced-J this proves that $xs > x$ for all simple reflections s in I, and therefore $x = 1$ since x is σ-stable. Consequently $vd = w_0 w = w_0 w' \in w_0 W_K = W_J w_0$. If $J \neq \{1, \ldots, r\}$, then ${}^{d^{-1}v^{-1}}\mathbf{U} \cap \mathbf{U} = {}^{d^{-1}}({}^{v^{-1}}\mathbf{U} \cap \mathbf{U}')$ is contained in the Levi subgroup of $\mathbf{G}$ corresponding to W_J, hence (ii) holds. Otherwise $I = J = K = \{1, \ldots, r\}$ and the relation $c_I \sigma(vd) = vdc_J$ is just $c\sigma(vd) = vd\sigma^a(c)$ which, with $d = (c\sigma)^a \sigma^{-a}$ gives $v \in W^{c\sigma} = W^F$, hence (i) holds. ☐

7.2. **Comparison of various cells.** In this section we prove that (4.2.4) holds for the Coxeter pair $(\mathbf{T}, \mathbf{U})$. Note that the Coxeter number h (the order of $c\sigma$) is even unless W is of type A_n with n even. Proposition 6.1.2 implies that when h is even we have $c\sigma(c) \cdots \sigma^{h/2-1}(c) = w_0$, the longest element in W.

Lemma 7.2.1. *Assume that $v \in W \smallsetminus W^F$. Then*

$$H_c^*({}^{\mathbb{U},\mathbb{U}}\Sigma_v) = 0$$

as a virtual $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-module.

Proof. If ${}^{\mathbb{U},\mathbb{U}}\Sigma_v$ is empty then the statement is trivial. Otherwise, Proposition 7.1.1 ensures that ${}^{v^{-1}}\mathbf{U} \cap \mathbf{U}$ is contained in a proper Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ containing $\mathbf{T}$. In particular the torus $\mathbb{H} := H^\circ_{\mathbf{L},v,r,\mathrm{red}}$ defined in §6.2 is contained in H_v, which by 5.1.1 acts on $\widehat{\Sigma}_v$. Using Corollary 6.2.3, we see that $({}^{\mathbb{U},\mathbb{U}}\widehat{\Sigma}_v)^{\mathbb{H}}$ is empty since $F(v) \neq v$. By (2.3.1), this shows that $H_c^*({}^{\mathbb{U},\mathbb{U}}\widehat{\Sigma}_v) = 0$. The same holds for ${}^{\mathbb{U},\mathbb{U}}\Sigma_v$ since it is related to ${}^{\mathbb{U},\mathbb{U}}\widehat{\Sigma}_v$ by a $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-equivariant map $\widehat{\Sigma}_v \to \Sigma_v$ which is a Zariski-locally trivial fibration with fibers isomorphic to the perfection of a fixed affine space. ☐

Given $a \in \mathbb{Z}$ and $v \in W^F$ we define the virtual $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-modules

$$\begin{aligned} h_{a,v} &:= H_c^*({}^{\mathbb{U},F^a(\mathbb{U})}\Sigma_v, \overline{\mathbb{Q}}_\ell) = H_c^*({}^{\mathbb{U},F^a(\mathbb{U})}\widehat{\Sigma}_v, \overline{\mathbb{Q}}_\ell) = H_c^*({}^{\mathbb{U},F^a(\mathbb{U})}\widetilde{\Sigma}_v, \overline{\mathbb{Q}}_\ell), \\ \widetilde{h}_v &:= H_c^0((\dot{v}\mathbb{T})^F, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell(\dot{v}\mathbb{T})^F. \end{aligned}$$

Note that $h_{a,v}$ depends only on the class of a modulo h, the Coxeter number.

Lemma 7.2.2. *Let $c_a = (c\sigma)^a \sigma^{-a}$. Assume that either*

- *h is even and $vc_a \in \{w_0, cw_0, w_0\sigma^a(c^{-1})\}$; or*
- *h is odd and $vc_a = c_{\lfloor h/2 \rfloor \pm 1}$.*

Then $h_{a,v} = \widetilde{h}_v$.

Proof. Assume first that h is even and that $vc_a = w_0$. Then ${}^{v^{-1}}\mathbb{U} \cap F^a(\mathbb{U}) = {}^{v^{-1}}\mathbb{U} \cap {}^{c_a}\mathbb{U} = {}^{c_a}({}^{(vc_a)^{-1}}\mathbb{U} \cap \mathbb{U}) = {}^{c_a}(\mathbb{U}^- \cap \mathbb{U}) = 1$ implies that $H_v = \mathbb{T} \times \mathbb{T}'$, which acts on $\widehat{\Sigma}_v$ by Lemma 5.1.1. By Corollary 6.2.3 applied to the Levi subgroup $\mathbf{L} = \mathbf{T}$, the map

$$\dot{v}\tau \in (\dot{v}\mathbb{T})^F \longmapsto (1, 1, 1, \tau, 1, 1) \in \widehat{\Sigma}_v$$

induces a $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-equivariant isomorphism

$$(\dot{v}\mathbb{T})^F \simeq (\widehat{\Sigma}_v)^{(H_v)_{\mathrm{red}}}$$

and the result follows. Similarly, the other two cases follow by using Lemma 5.2.1(i) resp. 5.2.1(ii) instead of Lemma 5.1.1.

When h is odd then W is of type A_n with n even and $\sigma = 1$. In that case $h = n+1$ and $\ell(c) = n$. By Proposition 6.1.2 we have $\ell(c^{n/2}) = n^2/2$ and $\ell(c^{n/2+1}) = \ell(c^{-n/2}) = n^2/2$. Therefore if $k = \lfloor h/2 \rfloor \pm 1$ we have $\ell(w_0 c_k) = n(n+1)/2 - n^2/2 = n/2 < n$, which forces $w_0 c_k$ to lie in a proper parabolic subgroup of W. Consequently ${}^{v^{-1}}\mathbb{U} \cap F^a(\mathbb{U}) = {}^{v^{-1}}\mathbb{U} \cap {}^{c_a}\mathbb{U} = {}^{c_a}\big({}^{(vc_a)^{-1}}\mathbb{U} \cap \mathbb{U}\big) = {}^{c_a}\big({}^{(w_0 c_k)^{-1}}\mathbb{U}^- \cap \mathbb{U}\big)$ lies in a proper parabolic subgroup of $\mathbf{G}$ containing $\mathbf{T}$. The result follows again by the combination of Lemma 5.1.1 and Corollary 6.2.3. ☐

The key observation is the following proposition.

Proposition 7.2.3. *Let $a \in \mathbb{Z}/h\mathbb{Z}$ and $v \in W^F$. We have $h_{a,v} = h_{a+1,v}$, unless σ is trivial and $v = w_0 c^{-a}$ or $v = w_0 c^{-a-1}$.*

Proof. As in Section 5.3 we have the isomorphism $\alpha\colon {}^{\mathbb{U},F^a(\mathbb{U})}\Sigma \to {}^{\mathbb{U},F^{a+1}(\mathbb{U})}\Sigma$, and the cell ${}^{\mathbb{U},F^a(\mathbb{U})}\Sigma_v$ decomposes into finitely many locally closed $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-stable pieces:

$$(7.2.1)\qquad {}^{\mathbb{U},F^a(\mathbb{U})}\Sigma_v = \bigcup_{w\in W} \alpha^{-1}\left(\alpha\left({}^{\mathbb{U},F^a(\mathbb{U})}\Sigma_v\right) \cap {}^{\mathbb{U},F^{a+1}(\mathbb{U})}\Sigma_w\right),$$

As in Lemma 5.3.1, we have the $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-stable piece ${}^aY_{v,w} := Y_{v,w} \subseteq {}^{\mathbb{U},F^{a+1}(\mathbb{U})}\widehat{\Sigma}_v$, and it satisfies

$$H_c^*({}^aY_{v,w}) = H_c^*\left(\alpha\left({}^{\mathbb{U},F^a(\mathbb{U})}\Sigma_v\right) \cap {}^{\mathbb{U},F^{a+1}(\mathbb{U})}\Sigma_w\right)$$

This and (7.2.1) give

$$(7.2.2)\qquad h_{a,v} = \sum_{w\in W} H_c^*({}^aY_{v,w}, \overline{\mathbb{Q}}_\ell)$$

By Lemma 5.3.1, ${}^aY_{v,w} \subseteq {}^{\mathbb{U},F^{a+1}(\mathbb{U})}\widehat{\Sigma}_w$ is stable under the H_w-action on ${}^{\mathbb{U},F^{a+1}(\mathbb{U})}\widehat{\Sigma}_w$ as in Lemma 5.1.1(i). If ${}^{w^{-1}}\mathbf{U} \cap F^{a+1}(\mathbf{U})$ is contained in a proper Levi subgroup of $\mathbf{G}$ containing $\mathbf{T}$, then using again the argument as in the proof of Lemma 7.2.1 we have $H_c^*({}^aY_{v,w}, \overline{\mathbb{Q}}_\ell) = 0$ whenever $w \notin W^F$ (cf. Section 2.3). Consequently by Proposition 7.1.1 applied to $\mathbb{U}' = F^{a+1}(\mathbb{U})$ we only need to consider the case where $w \in W^F$, so that

$$(7.2.3)\qquad h_{a,v} = \sum_{w\in W^F} H_c^*({}^aY_{v,w}, \overline{\mathbb{Q}}_\ell)$$

Analogously one can decompose the cell ${}^{\mathbb{U},F^{a+1}(\mathbb{U})}\Sigma_v$ into finitely many locally closed $(\mathbb{T} \times^{\mathbb{Z}} \mathbb{T})^F$-stable pieces as follows:

$${}^{\mathbb{U},F^{a+1}(\mathbb{U})}\Sigma_v = \bigcup_{w\in W} \alpha\left({}^{\mathbb{U},F^a(\mathbb{U})}\Sigma_w\right) \cap {}^{\mathbb{U},F^{a+1}(\mathbb{U})}\Sigma_v,$$

and using Lemmas 5.3.2 and 5.1.1(ii) instead of Lemmas 5.3.1 and 5.1.1(i) we show

$$(7.2.4)\qquad h_{a+1,v} = \sum_{w\in W^F} H_c^*({}^aY_{w,v}, \overline{\mathbb{Q}}_\ell).$$

Lemma 7.2.4. *Let $v, w \in W^F$. Assume that ${}^aY_{v,w} \neq \varnothing$ and $v \neq w$. Then σ is trivial, $v = w_0 c^{-a}$ and $v = wc$.*

Proof. The scheme ${}^aY_{v,w}$ can only be non-empty if $\alpha\left({}^{\mathbb{U},F^a(\mathbb{U})}\Sigma_v\right) \cap {}^{\mathbb{U},F^{a+1}(\mathbb{U})}\Sigma_w \neq \varnothing$. If this is the case, there must exist a point $(x, x', y) \in {}^{\mathbb{U},F^a(\mathbb{U})}\Sigma_v$, such that $\alpha(x, x', y) = (x, F(x'), yx') \in {}^{\mathbb{U},F^{a+1}(\mathbb{U})}\Sigma_w$. Let $y_1 = yx'$, and let $\bar{x}', \bar{y}, \bar{y}_1$ denote the images of x', y, y_1 in $\mathbb{G}_1$. Write $\mathbb{B}_1 = \mathbb{T}_1\mathbb{U}_1$. Given $k \in \mathbb{Z}$ we write $c_k = (c\sigma)^k\sigma^{-k}$ so that $F^k(\mathbb{B}) = {}^{c_k}\mathbb{B}$. We then have

$$\bar{y} \in \mathbb{B}_1 v F^a(\mathbb{B}_1) = \mathbb{B}_1 v c_a \mathbb{B}_1 c_a^{-1}, \quad \bar{x}' \in F^{a+1}(\mathbb{U}_1) = {}^{c_{a+1}}\mathbb{U}_1, \quad \text{and}$$

$$\bar{y}_1 \in \mathbb{B}_1 w F^{a+1}(\mathbb{B}_1) = \mathbb{B}_1 w c_{a+1} \mathbb{B}_1 c_{a+1}^{-1} \tag{7.2.5}$$

From the latter two of these three conditions it follows that $\bar{y} = \bar{y}_1 \bar{x}'^{-1} \in \mathbb{B}_1 w c_{a+1} \mathbb{B}_1 c_{a+1}^{-1}$, and we deduce from the first condition in (7.2.5) that $\mathbb{B}_1 w c_{a+1} \mathbb{B}_1 c_{a+1}^{-1} \cap \mathbb{B}_1 v c_a \mathbb{B}_1 c_a^{-1}$ contains $\bar{y}$, hence is non-empty. Multiplying by c^a from the right and using that $c_{a+1} = c_a \sigma^a(c)$ we get

$$\big(\mathbb{B}_1 w c_{a+1} \mathbb{B}_1 \sigma^a(c^{-1})\big) \cap \big(\mathbb{B}_1 v c_a \mathbb{B}_1\big) \neq \varnothing. \tag{7.2.6}$$

By [Spr74, Thm. 7.6(v)], there exists $k, l \in \{0, 1, \ldots, h-1\}$ such that $\sigma^k = \sigma^l = 1$ and $v = c_k$, $w = c_l$. Therefore the previous equation can be written

$$\big(\mathbb{B}_1 c_{l+a+1} \mathbb{B}_1 \sigma^a(c^{-1})\big) \cap \big(\mathbb{B}_1 c_{k+a} \mathbb{B}_1\big) \neq \varnothing.$$

This implies that

$$\big(\mathbb{B}_1 c_{l+a+1} \mathbb{B}_1 \sigma^a(c^{-1}) \mathbb{B}_1\big) \cap \big(\mathbb{B}_1 c_{k+a} \mathbb{B}_1\big) \neq \varnothing \quad \text{and} \quad \big(\mathbb{B}_1 c_{l+a+1} \mathbb{B}_1\big) \cap \mathbb{B}_1 c_{k+a} \mathbb{B}_1 \sigma^a(c) \mathbb{B}_1\big) \neq \varnothing. \tag{7.2.7}$$

As in the proof of Proposition 7.1.1, recall that the elements $u \in W$ such that $\mathbb{B}_1 u \mathbb{B}_1 \subset \mathbb{B}_1 c_{k+a} \mathbb{B}_1 \sigma^a(c) \mathbb{B}_1$ are of the form $c_{k+a}\sigma^a(c_I)$, where c_I is obtained by removing some simple reflections in c. Therefore by (7.2.7) we have $c_{k+a}\sigma^a(c_I) = c_{l+a+1}$ for some $c_I \leqslant c$, yielding $k \in \{l, l+1\}$. In addition when $\ell(c_{k+a+1}) = \ell(c) + \ell(c_{k+a})$ (or when $\ell(c_{l+a}) = \ell(c^{-1}) + \ell(c_{l+a+1})$) only $c_I = c$ can appear, in which case $k = l$ and hence $v = w$, which contradicts the assumptions of the Lemma. Therefore we have $k = l+1$, $\ell(c_{k+a+1}) \neq \ell(c) + \ell(c_{k+a})$ and $\ell(c_{k+a-1}) \neq \ell(c^{-1}) + \ell(c_{k+a})$. Consequently $\ell(c_{k+a}) > \ell(c_{k+a\pm 1})$ therefore $c_{k+a} = w_0$. Note that we also have $\sigma = \sigma^k \sigma^{-l} = 1$ and the lemma follows. □

Now we finish the proof of Proposition 7.2.3. Applying Lemma 7.2.4 we deduce from equation (7.2.3) that $h_{a,v} = H_c^*({}^a Y_{v,v}, \overline{\mathbb{Q}}_\ell)$ unless $\sigma = 1$ and $v = w_0 c^{-a}$. Similarly, equation (7.2.4) yields $h_{a+1,v} = H_c^*({}^a Y_{v,v}, \overline{\mathbb{Q}}_\ell)$ unless $\sigma = 1$ and $vc = w_0 c^{-a}$. Therefore if $\sigma \neq 1$ or if $v \notin \{w_0 c^{-a}, w_0 c^{-a-1}\}$ we have

$$h_{a,v} = H_c^*({}^a Y_{v,v}, \overline{\mathbb{Q}}_\ell) = h_{a+1,v}, \tag{7.2.8}$$

which finishes the proof. □

Proof of Theorem 4.2.1. By Lemma 7.2.1 it suffices to show that $h_{0,v} = \widetilde{h}_v$ for all $v \in W^F$. Recall that by [Spr74, Thm. 7.6(v)] the elements in W^F are of the form $c_k = (c\sigma)^k \sigma^{-k}$ for some $k \in \mathbb{Z}$ with $\sigma^k = 1$.

If σ is non-trivial, then there exists $a \in \mathbb{Z}$ such that $vc_a = w_0$ (for example $a = h/2 - k$). By Lemma 7.2.2 we have $h_{a,v} = \widetilde{h}_v$ and from Proposition 7.2.3 we deduce that $h_{0,v} = h_{a,v} = \widetilde{h}_v$.

If $\sigma = 1$, then $v = c^k$. Without loss of generality we can assume that $v \neq w_0$ (equivalently $k \neq h/2$) since in that case Lemma 7.2.2 applies. Assume first that $0 \leqslant k < h/2$. Then $vc^a = c^{k+a} \neq w_0$ for all $0 \leqslant a < h/2 - k$. Therefore by Proposition 7.2.3 we have $h_{0,v} = h_{a,v}$ in that case. If h is even then $vc^{h/2-k-1} = w_0 c^{-1}$ in which case $h_{h/2-k-1,v}$ equals $\widetilde{h}_v$ by Lemma 7.2.2. If h is odd then $h_{\lfloor h/2 \rfloor - k, v}$ equals $\widetilde{h}_v$ by Lemma 7.2.2 again. When $h/2 < k < h$ we have $h_{0,v} = h_{-a,v}$ for all $0 \leqslant a < h/2 + k$, and a similar argument applies. □

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