Quotients of Deligne-Lusztig varieties

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Let $\mathbf{G}$ be a connected reductive algebraic group and F be a Frobenius endomorphism defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. The group of fixed points $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$ is a *finite reductive group*. Examples of such groups are the classical groups $\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{Sp}_{2n}(q)$ and the exceptional groups of Lie type, such as $E_8(q)$.

The first approach for studying the representation theory of finite reductive groups is a variant of the classical induction/restriction for representations of abstract groups. Given an F-stable Levi subgroup $\mathbf{L}$ which is the Levi complement of an F-stable parabolic subgroup $\mathbf{P} = \mathbf{L}\mathbf{U}$ of $\mathbf{G}$ one can define a pair of adjoint functors, called *Harish-Chandra induction/restriction functors* as follows.

$$\begin{array}{rcccl} \mathrm{R}_{\mathbf{L}}^{\mathbf{G}} & : & \mathcal{O}\mathbf{L}^F\text{-mod} & \longrightarrow & \mathcal{O}\mathbf{G}^F\text{-mod} \\ & & N & \longmapsto & \mathcal{O}[\mathbf{G}^F/\mathbf{U}^F] \otimes_{\mathcal{O}\mathbf{L}^F} N \end{array}$$

$$\begin{array}{rcccl} {}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} & : & \mathcal{O}\mathbf{G}^F\text{-mod} & \longrightarrow & \mathcal{O}\mathbf{L}^F\text{-mod} \\ & & M & \longmapsto & M^{\mathbf{U}^F}. \end{array}$$

A key tool for working with these functors is the so-called *Mackey formula* which gives the following isomorphism of functors

$$ {}^*\mathrm{R}_{\mathbf{M}}^{\mathbf{G}} \circ \mathrm{R}_{\mathbf{L}}^{\mathbf{G}} \;\simeq\; \sum \mathrm{R}_{\mathbf{L}\cap {}^x\mathbf{M}}^{\mathbf{L}} \circ {}^*\mathrm{R}_{\mathbf{L}\cap {}^x\mathbf{M}}^{{}^x\mathbf{M}} \circ \mathrm{ad}\, x $$

where x runs over a set of representives in $\mathbf{L}^F\backslash\mathbf{G}^F/\mathbf{M}^F$ of elements such that $\mathbf{L}\cap {}^x\mathbf{M}$ contains a maximal torus of $\mathbf{G}$. This can be used for example to prove that $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ do not depend on $\mathbf{P}$ (but only on $\mathbf{L}$).

Deligne and Lusztig have generalised this construction to the case where $\mathbf{P}$ is no longer assumed to be F-stable [6]. The permutation module $\mathcal{O}[\mathbf{G}^F/\mathbf{U}^F]$ is replaced by the cohomology of a quasi-projective variety $\mathrm{Y}_{\mathbf{G}}(\mathbf{U})$ (with coefficients in a finite extension of $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$ or $\mathbb{F}_\ell$). The price to pay is that the new functors $\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and ${}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ are no longer defined on the module category but on its bounded derived category. Furthermore, the naive Mackey formula does not hold for these derived functors, even though it holds for the morphisms induced on the Grothendieck groups [6, 1, 2].

The purpose of this note is to explain how to solve this problem in the specific case where $\mathbf{L}$ is any F-stable Levi subgroup and $\mathbf{M}$ is a Levi complement of an F-stable parabolic subgroup $\mathbf{Q} = \mathbf{M}\mathbf{V}$. In that situation, the composition of induction/restriction is given by the cohomology of a quotient of the Deligne-Lusztig variety

$$ {}^*\mathrm{R}_{\mathbf{M}}^{\mathbf{G}} \circ \mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \;\simeq\; \mathrm{R}\Gamma_c(\mathbf{V}^F\backslash \mathrm{Y}_{\mathbf{G}}(\mathbf{U})) \otimes_{\mathbf{L}^F} - $$

and having a Mackey formula amounts to expressing the cohomology of this quotient in terms of "smaller" Deligne-Lusztig varieties. This provides an inductive method for computing the cohomology of Deligne-Lusztig varieties. We will detail the example of $\mathrm{GL}_n(q)$ for which the representation theory is well-known.

1. Unipotent ℓ-blocks of $\mathrm{GL}_n(q)$

Recall that the unipotent characters of $\mathrm{GL}_n(q)$ are parametrised by partitions of n. The trivial character $1 = \chi_{(n)}$ corresponds to the partition (n) whereas the Steinberg character $\mathrm{St} = \chi_{(1,\ldots,1)}$ corresponds to the conjugate partition $(1,1,\ldots,1)$.

Most of the properties of the unipotent characters (dimensions, restriction/induction...) can be read off from the associated partition. The Nakayama conjectures give the partition of the unipotent characters into ℓ-blocks.

Theorem 1.1 (Brauer-Robinson)**.** *Let ℓ be a prime number. Assume ℓ and q are coprime and let d be the order of q modulo ℓ. Then χ_λ and $\chi_{\lambda'}$ are in the same ℓ-block if and only if λ and λ' have the same d-core.*

Examples.

(i) If λ is a d-core, then χ_λ is the character of a projective module over $\mathbb{Z}_\ell$.

(ii) Assume $n = 3$. Then $\{1, \mathrm{St}, \chi_{(2,1)}\}$ is a single 3-block whereas the partition of unipotent characters into 2-blocks is $\{1, \mathrm{St}\}, \{\chi_{(2,1)}\}$.

(iii) Assume $n = d$. Then the unipotent character in the principal block correspond to n-hooks $(n-i, 1^i)$.

2. Deligne-Lusztig varieties associated with blocks

Let $d \in \{1,\ldots,n\}$. There exists a Deligne-Lusztig variety $\mathrm{X}_{n,d}$ of dimension $2n-d-1$ whose cohomology affords a "minimal" d-induction [3]. More precisely, if $\mu \vdash n-d$, we can from the local system $\mathcal{F}_\mu$ associated to the representation χ_μ of $\mathrm{GL}_{n-d}(q)$ and the consituents of the virtual character $\sum(-1)^i \mathrm{H}_c^i(\mathrm{X}_{n,d}, \mathcal{F}_\mu)$ are exactly the unipotent characters χ_λ where λ is obtained from μ by adding a d-hook. In particular, if μ is a d-core, the cohomology of $\mathrm{X}_{n,d}$ with coefficients in $\mathcal{F}_\mu$ gives the unipotent characters in the ℓ-block associated to μ.

Although there are general methods for computing the alternating sum of the cohomology (such as Lefschetz trace formula), it is a difficult problem to determine each individual cohomology group. When μ is trivial, the cohomology of $\mathrm{X}_{n,d}$ has only been determined when $d = n$ [10], $d = n-1$ [7] and $n = 2$ [8]. Craven has formulated in [5] a conjecture giving the degree in the cohomology where a given unipotent character should appear. Using a good quotient of $\mathrm{X}_{n,d}$ one can prove the following.

Theorem 2.1 (2011 [9])**.** *Craven's formula holds for $\mathrm{X}_{n,d}$ when μ is trivial. Furthermore, Craven's formula holds for any unipotent local system if it holds for $d = 1$.*

The case $d = 1$ corresponds to the Deligne-Lusztig variety $\mathrm{X}(\boldsymbol{\pi})$ associated with the central element $\boldsymbol{\pi} = \mathbf{w_0}^2$ of the Braid group. A precise conjecture for the cohomology of this variety was already formulated in [4].

3. Quotients of $\mathrm{X}_{n,d}$

In this section we assume that $\mathbf{M} = \mathrm{GL}_{n-1}(\overline{\mathbb{F}}_q)$ is the standard Levi subgroup of $\mathbf{G}$. The Harish-Chandra restriction ${}^*\mathrm{R}_{\mathbf{M}}^{\mathbf{G}}$ of a unipotent character χ_λ is given by the usual branching rule for representations of the symmetric group. In particular, if λ is obtained from μ by adding a d-hook, then the restricting χ_λ amounts to

- restricting μ to obtain a character $\chi_{\lambda'}$ which occurs in the cohomology of $\mathrm{X}_{n-1,d}$ with coefficients in the restriction of $\mathcal{F}_\mu$;
- restricting the d-hook (in general in two different ways) to obtain a character $\chi_{\lambda''}$ which occurs in the cohomology of $\mathrm{X}_{n-1,d-1}$ with coefficients in $\mathcal{F}_\mu$.

To understand geometrically why two copies of $\chi_{\lambda''}$ should occur we use Lusztig's result on the case $d = n$. He showed in [10] that the quotient $\mathbf{V}^F\backslash\mathrm{X}_{n,n}$ is isomorphic to $\overline{\mathbb{F}}_q^\times \times \mathrm{X}_{n-1,n-1}$. The cohomology of $\overline{\mathbb{F}}_q^\times$ is given by two copies of the coefficient ring in two consecutive degrees. Note that this part does not contribute to the alterning sum as the two terms cancel out.

Theorem 3.1 (2011 [9])**.** *Assume $d \geq 2$. There is a decomposition of the quotient $\mathbf{V}^F\backslash\mathrm{X}_{n,d} = U \cup Z$ into a disjoint union of $\mathbf{L}^F$-subvarieties such that*

- *Z is a closed subvariety whose cohomology is given by*

$$\mathrm{H}_c^i(Z, \mathcal{F}_{\mu|Z}) \simeq \mathrm{H}_c^{i-2}(\mathrm{X}_{n-1,d}, \mathcal{F}_{\mathrm{Res}\,\mu})(1)$$

- *U is an open subvariety whose cohomology is given by*

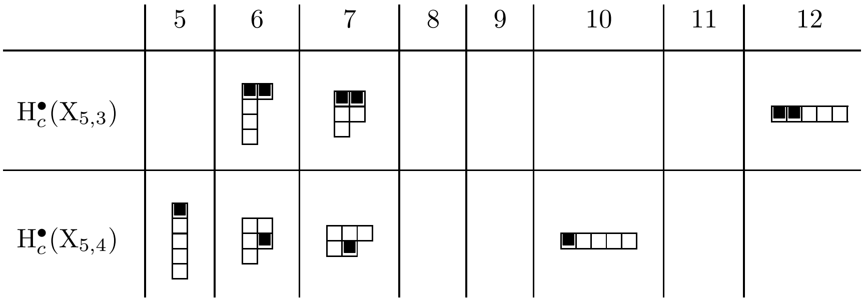
$$\mathrm{H}_c^i(U, \mathcal{F}_{\mu|U}) \simeq \mathrm{H}_c^{i-2}(\mathrm{X}_{n-1,d-1}, \mathcal{F}_\mu)(1) \oplus \mathrm{H}_c^{i-1}(\mathrm{X}_{n-1,d-1}, \mathcal{F}_\mu)$$

From this decomposition we obtain a long exact sequence relating the cohomology of $\mathbf{V}^F\backslash\mathrm{X}_{n,d}$ to the cohomology of $\mathrm{X}_{n-1,d}$ and $\mathrm{X}_{n-1,d-1}$. Together with the action of the Frobenius this determines completely the cohomology of $\mathrm{X}_{n,d}$.

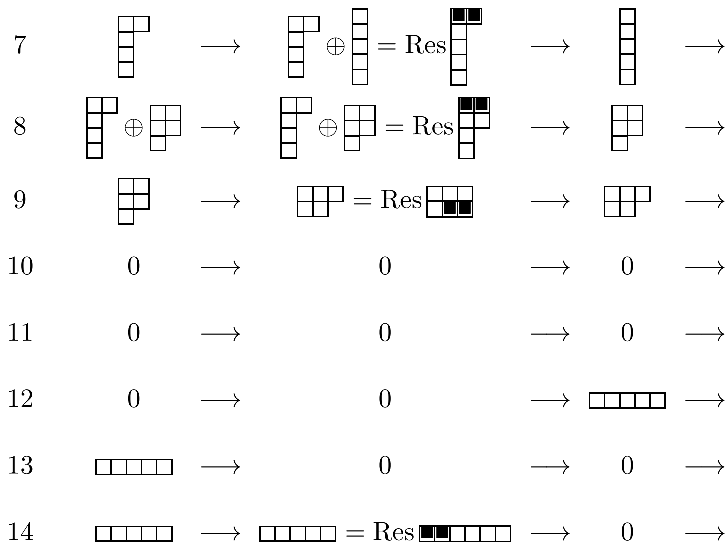
The minimal cases correspond to $d = n$ and $d = 1$. Lusztig solved the first one in [10]. For the second one, we can only prove that Craven's formula hold when μ is the trivial partition. The other cases are work in progress.

4. An example

Assume that Craven's formula hold for the cohomology of $\mathrm{X}_{5,3}$ and $\mathrm{X}_{5,4}$ with coefficients in the trivial local system. We give here the cohomology with compact support; the black boxes in Young diagrams correspond to the partition μ we started with, that is (2) for $\mathrm{X}_{5,3}$ and (1) for $\mathrm{X}_{5,4}$. The white boxes represent the d-hook that we have added.



Using Theorem 3.1 we write the long exact sequence in cohomology and deduce the cohomology of $X_{6,5}$. The groups of degree 10 and 11 are obviously zero, the group of degree 14 is also easily obtained, and the group of degree 7 follows from equivariance of the boundary map $H_c^7(Z) \longmapsto H_c^8(U)$. For the remaining ones, we need to use the action of F together with the fact that the cohomology of $\mathbf{V}^F\backslash X_{6,5}$ should be the restriction of unipotent characters of $GL_6(q)$.



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