Categorical actions on unipotent representations of finite unitary groups

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We are interested in the representation theory of finite classical groups, and in particular finite unitary groups, over a field of positive characteristic ℓ (different from the defining characteristic of the algebraic group). Finite unitary groups are defined by

$$\mathrm{GU}_n(q) = \{M \in \mathrm{GL}_n(q^2) \,|\, M^t\mathrm{Fr}(M) = I_n\}$$

where $\mathrm{Fr} : (m_{i,j}) \longmapsto (m_{i,j}^q)$ is the standard Frobenius endomorphism of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$. Representations of $\mathrm{GU}_n(q)$ over an algebraic closed field $k = \overline{k}$ of characteristic ℓ form the abelian category $k\mathrm{GU}_n(q)$-mod. We will focus on the Serre subcategory formed by the so-called *unipotent representations*, which we will denote by $k\mathrm{GU}_n(q)$-umod. When $\ell \neq q$ (non-defining characteristic case), there is an adjoint pair of exact functors

$$F_{n,n+2} : k\mathrm{GU}_n(q)\text{-umod} \rightleftharpoons k\mathrm{GU}_{n+2}(q)\text{-umod} : E_{n,n+2}$$

given by Harish-Chandra (or parabolic) induction and restriction. These functors yield endofunctors $F = \bigoplus F_{n,n+2}$ and $E = \bigoplus E_{n,n+2}$ of the category

$$\mathcal{C} = \bigoplus_{n\geq 0} k\mathrm{GU}_n(q)\text{-umod}.$$

Motived by the pioneering work of Chuang and Rouquier on the case of finite general linear groups [1], we show that F and E endow $\mathcal{C}$ with a structure of *higher representation* for a well-identified Kac-Moody algebra $\mathfrak{g}$.

A categorical action of $\mathfrak{g}$ should consist of an abelian category acted on by endofunctors inducing an action of the Lie algebra $\mathfrak{g}$ on the Grothendieck group. This notion is actually too weak, and Chuang-Rouquier [1] and Rouquier [4] have axiomatized the good notion of such a higher action. For our category $\mathcal{C}$, it will be given by natural transformations of F and F^2 satisfying the relations of an affine Hecke algebra of type A.

Proposition. There exist natural transformations $X \in \mathrm{End}(F)^\times$ and $T \in \mathrm{End}(F^2)$ such that

(i) $(T^2 - q^2 1_{F^2}) \circ (T^2 + 1_{F^2}) = 0$
(ii) $(T1_F) \circ (1_F T) \circ (T1_F) = (1_F T) \circ (T1_F) \circ (1_F T)$
(iii) $T \circ (1_F X)T = q^2 X1_F$

From this datum we can construct a proper categorical action. First, we define the functors F_a and E_a by the generalized a-eigenspace of X on F and E. Then we consider the Kac-Moody algebra $\mathfrak{g}$ associated with the quiver whose vertices are the eigenvalues of X (here $(-q)^{\mathbb{Z}}$) and with arrows $a \longrightarrow aq^2$. Let e be the order of $-q$ modulo ℓ. If $e = \infty$ (*i.e.* $\mathrm{char}\, k = 0$) then $\mathfrak{g} = \mathfrak{sl}_\infty \oplus \mathfrak{sl}_\infty$; otherwise if e is finite we can distinguish two cases:

- if e is odd then $\mathfrak{g} = \widehat{\mathfrak{sl}}_e$ (*unitary prime* case)
- if e is even then $\mathfrak{g} = \widehat{\mathfrak{sl}}_{e/2} \oplus \widehat{\mathfrak{sl}}_{e/2}$ (*linear prime* case)

Then the Lie algebra $\mathfrak{g}$ acts on the category $\mathcal{C}$ in the following sense:

Theorem. Let $V = \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$ and $f_a = [F_a]$, $e_a = [E_a]$ be the linear endomorphisms of V induced by the functors E_a and F_a. Then

(i) the action of $\langle e_a, f_a \rangle_{a \in (-q)^{\mathbb{Z}}}$ induces an integrable action of $\mathfrak{g}$ on V
(ii) there is a decomposition $\mathcal{C} = \bigoplus \mathcal{C}_\omega$ with $V_\omega = \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}_\omega)$ and such that $F_a\mathcal{C}_\omega \subset \mathcal{C}_{\omega-\alpha_a}$ and $E_a\mathcal{C}_\omega \subset \mathcal{C}_{\omega+\alpha_a}$

It turns out that this action encodes a large part of the representation theory of finite unitary groups, as it does in the case of linear groups. For example, cuspidal representations will correspond to highest weight vectors, and when e is odd (resp. when e is even), blocks correspond to weight spaces (resp. weight spaces in a fixed ordinary Harish-Chandra series).

It is important to note that in our case the structure of $\mathfrak{g}$-module of V can be made explicit. It is isomorphic to direct sum of level 2 Fock space representations and can be related to the level 1 (corresponding to $\mathrm{GL}(q)$) by Uglov's construction. This gives, at least on the Grothendieck group, an explicit relation between Harish-Chandra induction and 2-Harish-Chandra induction (defined by Lusztig induction from 2-split Levi subgroups). It will be interesting to show the same relation between the functors.

To finish, let us give some applications of the existence of categorical actions on $\mathcal{C}$. By [1], the action of the affine Weyl group lifts to derived self-equivalences of $\mathcal{C}$. Using the work of Livesey [3] on the existence of good blocks for linear primes, we deduce the following:

Theorem. Assume the order of $-q$ modulo ℓ is even. Then Broué's abelian defect group conjecture holds for $\mathrm{GU}_n(q)$.

Also, from the explicit structure of $\mathfrak{g}$-module of $K_0(\mathcal{C})$ we can deduce the modular branching rule for Harish-Chandra induction, as well as the various Hecke algebras occuring as endomorphism algebras of induced representations. This gives a partial proof of the recent conjectures of Gerber-Hiss-Jacon [2].

Theorem. The modular branching graph for Harish-Chandra induction is isomorphic to the crystal graph of the level 2 Fock space representation V.

Note that many of these results have already been generalized to other classical groups, but this is still a work in progress.

References

[1] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Annals of Math. 167 (2008), 245-298.
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[3] Livesey, M. *On Rouquier Blocks for Finite Classical Groups at Linear Primes*, J. Algebra 432, 2015.
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