RATIONALITY OF EXTENDED UNIPOTENT CHARACTERS

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Abstract. We determine the rationality properties of unipotent characters of finite reductive groups arising as fixed points of disconnected reductive groups under a Frobenius map.

1. Introduction

The work of George Lusztig has shown the singular importance of unipotent characters in the representation theory of finite reductive groups and hence of finite simple groups of Lie type. Recent research has focussed on rationality properties of characters of almost simple groups, and this naturally leads to the problem of understanding fields of values, and fields of realisation, of extensions of unipotent characters to groups of Lie type extended by graph or graph-field automorphisms. The latter can be viewed as groups of rational points of suitable disconnected algebraic groups. While the rationality properties of unipotent characters themselves have long been known, due to the work of Lusztig [13] and Geck [5] (see e.g. [8, Cor. 4.5.6]), their extensions to disconnected groups have so far not been studied systematically; Digne–Michel [4, Thm II.3.3] considered characters in the principal series and [14, Prop. 2] deals with $\mathrm{SU}_3(q)$.

The field of values of a unipotent character ρ is generated by its Frobenius eigenvalue (see [8, Prop. 4.5.5]). Here, we show that this Frobenius eigenvalue also governs the field of values of extensions of ρ. Our first result concerns cuspidal characters:

Theorem 1. *Let $\mathbf{G}$ be a simple algebraic group with a Frobenius map F and a commuting non-trivial graph automorphism σ. Then any cuspidal unipotent character ρ of $G = \mathbf{G}^F$ has an extension $\widehat{\rho}$ to $G\langle\sigma\rangle$ with $\mathbb{Q}(\widehat{\rho}) = \mathbb{Q}(\rho)$, unless $G = {}^2A_{n-1}(q)$ with $n = \binom{t}{2} \equiv 2, 3 \pmod 4$ for some $t \geq 2$, in which case $\mathbb{Q}(\widehat{\rho}) = \mathbb{Q}(\sqrt{-q})$.*

In particular, $\mathbb{Q}(\widehat{\rho})$ is generated by a δth root of the Frobenius eigenvalue of ρ, where $\delta \geq 1$ is minimal such that F^δ acts trivially on the Weyl group of $\mathbf{G}$.

Using earlier results on Frobenius–Schur indicators exhibits the following connection:

Corollary 2. *In the situation of Theorem 1, ρ has a rational extension to $G\langle\sigma\rangle$ if and only if ρ has Frobenius–Schur indicator $+1$.*

For arbitrary unipotent characters, we obtain:

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Theorem 3. *Let* $\mathbf{G}$ *be a simple algebraic group with a Frobenius map* F *and a commuting non-trivial graph automorphism* σ*. Then any* σ*-invariant rational unipotent character* ρ *of* $G = \mathbf{G}^F$ *has a rational extension to* $G\langle\sigma\rangle$*, unless one of:*

(1) $G = A_{n-1}(q)$*,* q *is not a square and* ρ *is labelled by a partition* $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ *of* n *with*

$$\sum_i \binom{\lambda_i}{2} - \sum_i \binom{\lambda'_i}{2} + \binom{n}{2} \equiv 1 \pmod 2,$$

where $\lambda' = (\lambda'_1 \geq \ldots \geq \lambda'_s)$ *is the partition conjugate to* λ*;*

(2) $G = E_6(q)$*,* q *is not a square and* ρ *is one of* $\phi_{64,4}$ *or* $\phi_{64,13}$*; or*

(3) ρ *lies in the Harish-Chandra series of a cuspidal unipotent character of a group of type* ${}^2A_{n-1}(q)$ *labelled by a 2-core of size* $n \equiv 2, 3 \pmod 4$*.*

In cases (1) and (2), the extensions have character field $\mathbb{Q}(\sqrt{q})$*, in the third* $\mathbb{Q}(\sqrt{-q})$*.*

The case of cuspidal characters is settled in Section 2, in Section 3 we reduce the general case to the former, thus proving Theorem 3. In Section 4 we discuss extensions by an exceptional graph automorphism for types B_2, G_2 and F_4.

2. Cuspidal unipotent characters

We consider the following setup. Let $\mathbf{G}$ be a connected reductive linear algebraic group with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ defining an $\mathbb{F}_q$-structure, and set $G := \mathbf{G}^F$, the finite group of F-fixed points. We further assume that $\mathbf{G}$ has a graph automorphism σ commuting with F. We set $\widehat{\mathbf{G}} = \mathbf{G}\langle\sigma\rangle$ the semidirect product of $\mathbf{G}$ with σ, and $\widehat{G} := \widehat{\mathbf{G}}^F$ the corresponding extension of G with σ.

2.1. **Deligne–Lusztig varieties and unipotent characters.** Let $\mathcal{B}$ be the flag variety of $\mathbf{G}$, that is, the variety of Borel subgroups of $\mathbf{G}$. The actions of F and σ on $\mathbf{G}$ induce commuting endomorphisms of $\mathcal{B}$. The group $\mathbf{G}$ acts by simultaneous conjugation on $\mathcal{B}\times\mathcal{B}$ and the orbits are parametrized by the elements of the Weyl group W of $\mathbf{G}$. Given $w \in W$, we denote by $\mathcal{O}(w)$ the corresponding orbit and we define the Deligne–Lusztig variety as in [12, 3.3] by

$$X_w = \{\mathbf{B} \in \mathcal{B} \mid (\mathbf{B}, F(\mathbf{B})) \in \mathcal{O}(w)\}.$$

The action of $\mathbf{G}$ on $\mathcal{B}\times\mathcal{B}$ restricts to an action of the finite group G on X_w. Furthermore, F (resp. σ) induces a finite morphism (resp. an isomorphism) between X_w and $X_{F(w)}$ (resp. $X_{\sigma(w)}$). Consequently, if δ is the smallest integer such that F^δ acts trivially on W, then any Deligne–Lusztig variety has an action of F^δ commuting with G.

For ℓ a prime not dividing q we denote by R_w the corresponding Deligne–Lusztig character of G, given by

$$R_w(g) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}\big(g \mid H^i_c(X_w, \mathbb{Q}_\ell)\big) \qquad \text{for } g \in G.$$

This generalised character of G does not depend on ℓ (see e.g. [12, 1.2]). The complex-valued characters which appear as constituents of the various R_w are called the *unipotent characters* of G; we denote them $\operatorname{Uch}(G)$. By a result of Lusztig [12, 3.9], for any unipotent character ρ of G, F^δ acts by the same *eigenvalue of Frobenius* ω_ρ on any ρ-isotypic component $H^i_c(X_w, \overline{\mathbb{Q}}_\ell)_\rho$ of any ℓ-adic cohomology group of any X_w, up to multiplication by integral powers of q^δ.

2.2. **Reduction to simple groups.** Here we follow the arguments in [8, Rem. 4.2.1] building upon [8, 1.5.9–1.5.13]. The centre $Z(\mathbf{G})$ is characteristic in $\mathbf{G}$, so σ induces a graph automorphism on $(\mathbf{G}/Z(\mathbf{G}))^F$. As unipotent characters have $Z(\mathbf{G})^F$ in their kernel, for the purpose of studying rationality of extensions of unipotent characters we may therefore assume $\mathbf{G}$ is semisimple. Let $\mathbf{G}_{\rm sc}$ and $\mathbf{G}_{\rm ad}$ be simply connected respectively adjoint groups of the same type as $\mathbf{G}$. Then these possess corresponding graph automorphisms again denoted σ. Furthermore, there are natural F- and σ-equivariant epimorphisms $\mathbf{G}_{\rm sc} \to \mathbf{G} \to \mathbf{G}_{\rm ad}$ such that the images of the respective F-fixed points contain the derived subgroup of the F-fixed points of the target. Now unipotent characters have the centre in their kernel and restrict irreducibly to the derived subgroup. Since σ stabilises the centre and the derived subgroup, the rationality properties of extensions of unipotent characters of $\mathbf{G}^F$ agree with those of any group isogenous to it. By passing to a group of adjoint type we see that we may hence assume for our purposes that $\mathbf{G}$ is simple (of a chosen isogeny type), which we do from now on.

In particular, we can always assume that $\delta \in \{1, 2, 3\}$ and $F = \sigma^r$ in its action on W, for some $r \in \{1, 2, 3\}$ with one of r or δ being equal to 1.

2.3. **Eigenvalues of F and character extensions.** We keep the above setting. We first look at the extensions over local fields given by the ℓ-adic cohomology of the Deligne–Lusztig varities.

Proposition 2.1. *Let d be the order of σ (recall that $d \in \{2, 3\}$). Let $\rho \in \mathrm{Uch}(G)$ be rational valued and σ-invariant. Assume that there is $w \in W$ such that*
(1) *the $\langle F\rangle$-orbit of w has length δ and is σ-stable; and*
(2) *the multiplicity of ρ in R_w is not divisible by d.*
Then for every extension $\widehat{\rho}$ of ρ to $G\langle\sigma\rangle$, the field of values $\mathbb{Q}_\ell(\widehat{\rho})$ is contained in the splitting field of $x^d - \omega_\rho^{d/\delta}$ over $\mathbb{Q}_\ell$. Furthermore, there is at least one extension $\widehat{\rho}$ which is $\mathbb{Q}_\ell$-valued if and only if there is a δth root of ω_ρ in $\mathbb{Q}_\ell$.

Remark 2.2. Since the Frobenius eigenvalue ω_ρ is uniquely determined up to integral powers of q^δ, the conclusion of Proposition 2.1 is well-defined.

Proof. We consider the subvariety

$$X = X_w \sqcup X_{F(w)} \sqcup \cdots \sqcup X_{F^{\delta-1}(w)}$$

of $\mathcal{B}$. By (1) it has an action of both F and σ and for all i we have

$$H^i_c(X)_\rho \cong H^i_c(X_w)_\rho \oplus H^i_c(X_{F(w)})_\rho \oplus \cdots \oplus H^i_c(X_{F^{\delta-1}(w)})_\rho$$

as $\mathbb{Q}_\ell G$-modules with F permuting cyclically the δ summands. By (2) there is some i for which the multiplicity of ρ in $H^i_c(X_w)_\rho$ is not divisible by d. Thus there also is a generalised eigenspace $H^i_c(X_w)_{\rho,\mu}$ of F^δ on $H^i_c(X_w)_\rho$ with the same property. Here, as cited above, the eigenvalue μ differs from ω_ρ by an integral power of q^δ. First assume $\delta = 1$. Then $H := H^i_c(X_w)_{\rho,\mu}$ is a $\mathbb{Q}_\ell\widehat{G}$-module in which not all extensions of ρ can occur with the same multiplicity. Since $d \in \{2, 3\}$, at least one of the extensions must then be distinguished by its multiplicity and thus have values in $\mathbb{Q}_\ell$. The others are obtained by tensoring with linear characters of $\widehat{G}/G$, which have values in the splitting field of $x^d - 1$, so of $x^d - \omega_\rho^d$.

Now assume $\delta = d$. Then F has characteristic polynomial $(x^d - \mu)^m$ in its action on $\oplus_{j=0}^{\delta-1} H_c^i(X_{F^j(w)})_{\rho,\mu}$, with $m = \dim H_c^i(X_w)_{\rho,\mu}$ and $m/\rho(1)$ not divisible by d. Let K be the splitting field of $x^d - \mu$ over $\mathbb{Q}_\ell$. Then $\mathrm{Gal}(K/\mathbb{Q}_\ell)$ permutes the generalised F-eigenspaces as it permutes the eigenvalues, that is, as it acts on the roots of $x^d - \mu$. If K contains a zero of $x^d - \mu$, there is a $\mathbb{Q}_\ell$-rational eigenspace and we can argue as before.

If $\mathbb{Q}_\ell$ does not contain a zero of $x^\delta - \omega_\rho$, then $K/\mathbb{Q}_\ell$ is an extension of degree δ (recall that $\delta \leq 3$) and the δ different generalised eigenspaces of F are Galois conjugate over $\mathbb{Q}_\ell$. Thus, the same holds for the δ different extensions of ρ. □

We now lift the rationality properties to $\mathbb{Q}$.

Lemma 2.3. *Let ρ be a rational valued unipotent character of G. Assume that for every $\ell \neq p$, ρ has an extension $\widehat{\rho}_\ell$ to $\widehat{G}$ which takes values in $\mathbb{Q}_\ell$. Then ρ has an extension to $\widehat{G}$ which takes values in $\mathbb{Q}$.*

Proof. We argue by contradiction. Assume all extensions of ρ are defined over a proper extension K of $\mathbb{Q}$, generated by a root of $f \in \mathbb{Q}[x]$, say. Since the sum of the extensions has values in $\mathbb{Q}$ and $\delta \in \{2, 3\}$ this means that K (and hence f) has degree δ over $\mathbb{Q}$. Note that $K/\mathbb{Q}$ is abelian. Thus by Dirichlet there are infinitely many primes ℓ such that f is also irreducible over $\mathbb{Q}_\ell$, that is, its roots generate an extension of degree δ. Hence ρ does not have any $\mathbb{Q}_\ell$-rational extension, contradicting our assumption. □

2.4. **Extensions of cuspidal unipotent characters.** We keep the above setting. For the classification and properties of unipotent characters we refer the reader to [8, §§4.3–4.5]. Recall that cuspidal unipotent characters have values in $\mathbb{Q}(\omega_\rho)$.

Proposition 2.4. *Let $\mathbf{G}$ be simple of type D_n or E_6, σ a graph automorphism of $\mathbf{G}$ and F a commuting Frobenius map with $\delta = 1$. Then any cuspidal unipotent character ρ of $G = \mathbf{G}^F$ has an extension to $\widehat{G} = G\langle\sigma\rangle$ defined over $\mathbb{Q}(\rho)$.*

Proof. We consider the various cases according to Lusztig's classification of cuspidal unipotent characters. By [8, Thm 4.5.11] they are all σ-invariant. If G is of type D_n with $n = (2t)^2$ for some $t \geq 1$ and $o(\sigma) = 2$, then the class of W labelled by the bi-partition $(-; 4t-1, 4t-3, \ldots, 1)$ contains σ-stable elements (since the centraliser in W of σ, of type B_{n-1}, contains the class labelled $(-; 4t-1, 4t-3, \ldots, 3)$), and by [13, Prop. 2.14], the unique cuspidal unipotent character ρ occurs exactly once in the corresponding Deligne–Lusztig character. Thus Proposition 2.1 applies to show that ρ has rational extensions. If G is of type D_4 with $o(\sigma) = 3$, then there exists a σ-stable element $w \in W$ in the class labelled by the bi-partition $(-; 31)$, and using Chevie [15] the unique cuspidal unipotent character ρ of G appears with multiplicity 1 in R_w. Thus Proposition 2.1 applies again.

Finally, for $\mathbf{G}$ of type E_6, there are two cuspidal characters $E_6[\theta], E_6[\theta^2]$ with Frobenius eigenvalue a primitive third root of unity θ, respectively θ^2. Let $w \in W$ be in class E_6. Then w can be chosen σ-stable. Again using [15], both $E_6[\theta]$ and $E_6[\theta^2]$ occur in R_w with multiplicity 1. Let H be a cohomology group of X_w containing ρ with odd multiplicity, for ℓ a prime with $\ell \equiv 1 \pmod 3$, so $\sqrt{-3} \in \mathbb{Q}_\ell$. Since σ fixes w, it acts on H and so

H_ρ contains the two extensions ρ_1, ρ_2 of ρ to $\widehat{G}$ with different multiplicities. Thus, they must be $\mathbb{Q}_\ell$-rational. Since this is true for all $\ell \equiv 1 \pmod 3$, the character field of ρ_i is contained in $\mathbb{Q}(\theta) = \mathbb{Q}(\rho)$, hence equal to $\mathbb{Q}(\theta)$. □

Proposition 2.5. *Let $G = A_{n-1}(-q) = {}^2A_{n-1}(q)$ where $n = \binom{t+1}{2}$ with $t \geq 1$, and let ρ be the cuspidal unipotent character of G. Let σ be the graph-field automorphism of G of order 2. Then the two extensions of ρ to $\widehat{G} = G\langle\sigma\rangle$ are rational-valued if $\binom{n}{2}$ is even, and are algebraically conjugate over $\mathbb{Q}(\sqrt{-q})$ if $\binom{n}{2}$ is odd.*

Proof. The cuspidal unipotent character of ${}^2A_{n-1}(q)$ is labelled by the staircase partition $\lambda = (t, t-1, \ldots, 1)$ (see e.g. [8, Prop. 4.3.6]). The Frobenius eigenvalue of ρ is $\omega_\rho = (-q)^{\binom{n}{2}}$ (up to multiplication by powers of q^2) by [12, Rem. (a) after Thm 3.34] and [8, Prop. 4.3.7]). An application of the Murnaghan–Nakayama rule shows that the irreducible character of $\mathfrak{S}_n$ labelled by λ takes value ± 1 on elements $w \in W = \mathfrak{S}_n$ of cycle type $(2t-1, 2t-5, \ldots)$. Since the multiplicities of unipotent characters in Deligne–Lusztig characters in type A_{n-1} are given by the character table of W (see [8, Cor. 2.4.19]), this means that ρ has multiplicity ± 1 in the Deligne–Lusztig character R_w. Also, no conjugate of w is centralised by σ, hence neither by F. Thus the assumptions of Proposition 2.1 are satisfied and the conclusion follows. □

Remark 2.6. Alternatively, by the arguments given for the principal series case in the proof of Theorem 3 below, the conclusion of Proposition 2.5 would follow if the full endomorphism algebra of the cohomology of X_{w_0}, where $w_0 \in W$ is the longest element, is indeed the Iwahori–Hecke algebra of type A_{n-1} at parameter $-q$ as speculated in [12, 3.10(b)] (see also the general conjectures in [1, 1B]).

Proposition 2.7. *Let $\mathbf{G}$ be simple of type D_n or E_6, σ a graph automorphism of $\mathbf{G}$ and F a commuting Frobenius map with $\delta = o(\sigma)$. Then any cuspidal unipotent character ρ of $G = \mathbf{G}^F$ has an extension to $\widehat{G} = G\langle\sigma\rangle$ with field of values $\mathbb{Q}(\rho)$.*

Proof. First assume that $G = {}^2D_n(q)$ where $n = (2t+1)^2$ with $t \geq 1$, and let ρ be the cuspidal unipotent character of G. Let $w \in W$ be in the class labelled by $(-; 4t+1, 4t-1, \ldots, 1)$. An application of Asai's formula [8, Thm 4.6.9] shows that ρ appears with multiplicity ± 1 in R_w (see also [13, 2.19]). By [7, Thm 4.11] the Frobenius eigenvalue of ρ is $\omega_\rho = 1$, up to multiplication by integral powers of q^2, so an application of Proposition 2.1 allows us to conclude.

Now assume $G = {}^3D_4(q)$, with σ the graph-field automorphism of G of order 3. If ρ is the cuspidal unipotent character ${}^3D_4[-1]$ of G then (using Chevie[15]), ρ occurs with multiplicity 1 in X_w for w of type F_4. The class of w is not F-stable, so Proposition 2.1 applies. In this case the eigenvalue of F^3 for ρ equals $\omega_\rho = -1$ (up to integral powers of q^3) by [11, (7.3)]. Next, let ρ be the cuspidal unipotent character ${}^3D_4[1]$, with Frobenius eigenvalue $\omega_\rho = 1$, by [7, Rem. 4.9]. It occurs with multiplicity 1 in X_w for w of type $F_4(a_1)$, which can be chosen not σ-invariant. Again Proposition 2.1 applies.

Finally, let $G = {}^2E_6(q)$. The cuspidal unipotent character $\rho = {}^2E_6[1]$ with $\omega_\rho = 1$ (by [7, Rem. 4.9]) appears with multiplicity 1 in the Deligne–Lusztig character R_w for w in class $3A_2$. Choosing w not σ-stable, we conclude as in the previous case. Let now ρ be one of the cuspidal unipotent characters ${}^2E_6[\theta]$, ${}^2E_6[\theta^2]$ of G with $\omega_\rho = \theta$, θ^2 respectively [11, (7.4)(e)]. For these the claim follows precisely as for the non-rational cuspidal characters of $E_6(q)$ in the proof of Proposition 2.4. □

We are now ready to show our first main result:

Proof of Theorem 1. This follows from Propositions 2.4, 2.5 and 2.7 as those cover all relevant cases. □

Proof of Corollary 2. If ρ is not real-valued the assertion holds trivially. We now discuss the real-valued cuspidal unipotent characters. All of them are rational by [8, Cor. 4.5.6]. The Frobenius–Schur indicators of all these characters of untwisted groups are $+1$ by Lusztig [13, Thm 0.2], and by Proposition 2.4 they possess rational extensions.

By Ohmori [16] the cuspidal unipotent character ρ of the unitary group $G = \mathrm{SU}_n(q)$ with $n = t(t+1)/2$ has Frobenius–Schur indicator $(-1)^{\lfloor n/2\rfloor}$. Now $\lfloor n/2\rfloor$ is the $\mathbb{F}_q$-rank of G, which in turn is congruent modulo 2 to the exponent i in the Frobenius eigenvalue $\omega_\rho = (-q)^i$ of ρ by [12, Rem. (a) after Thm 3.34]. The claim in this case thus follows from Proposition 2.5, observing that $\lfloor n/2\rfloor$ and $\binom{n}{2}$ have the same parity.

For the orthogonal group $G = \mathrm{SO}_{2n}^-(q)$ where $n = (2t+1)^2$, the Frobenius–Schur indicator of the cuspidal unipotent character equals $+1$ by [13, 1.13], the Frobenius–Schur indicator of the cuspidal unipotent character ${}^2E_6[1]$ of ${}^2E_6(q)$ is $+1$ by [5, 6.2], and similarly the indicators of the two cuspidal unipotent characters of ${}^3D_4(q)$ are also $+1$ by [11, (7.6)] and [5, 6.2]. So for the latter groups we may conclude by Proposition 2.7. □

3. Harish-Chandra theory

We now consider arbitrary unipotent characters.

Proof of Theorem 3. Let $\rho \in \mathrm{Uch}(G)$ be σ-invariant, so it has an extension $\widehat{\rho}$ to $\widehat{G} = G\langle\sigma\rangle$. First assume ρ lies in the principal series, so it occurs as constituent in the permutation module $\mathbb{Q}_\ell[G/B]$, whence $\widehat{\rho}$ occurs in $M := \mathbb{Q}_\ell[\widehat{G}/B]$. Now as a $\mathbb{Q}_\ell\widehat{G} \times \mathrm{End}_{\widehat{G}}(M)$-bimodule, M decomposes as the direct sum of irreducible submodules M_ϕ indexed by $\phi \in \mathrm{Irr}(\mathrm{End}_{\widehat{G}}(M))$ affording $\chi_\phi \otimes \phi$ for some $\chi_\phi \in \mathrm{Irr}(\widehat{G})$ in the principal series. Clearly, any Galois automorphism of $\mathbb{Q}_\ell$ permutes the M_ϕ as it permutes the characters ϕ, and hence it permutes the χ_ϕ in the same way. So the rationality statement for $\widehat{\rho}$ follows from the corresponding one for the extended Hecke algebra in [4, Thm II.3.3]. (See also the proof of [5, Prop. 5.5].)

Now assume we are not in that case. Let $L \leq G$ be a (split) Levi subgroup and λ a cuspidal unipotent character of L such that ρ lies in the Harish-Chandra series of λ, so $\langle\rho, R_L^G(\lambda)\rangle \neq 0$. Thus, ρ corresponds to a character ϕ of the relative Weyl group $W' := W_G(L,\lambda)$ (see [8, Thm 3.2.5]). It is known that this relative Weyl group is of type A_2, B_n, G_2 or F_4 in the cases we consider and that L can also be chosen to be σ-stable (see e.g. [8, Tab. 4.8]). Moreover, as λ is the unique cuspidal unipotent character of L, it is also σ-stable. Furthermore, σ acts trivially on W' except possibly if W' has type A_2. By [9, Thm 3.3 and (3.6)] any such ϕ is of parabolic type, that is, there is a parabolic subgroup W'_J of W' such that $\langle 1_{W'_J}^{W'}, \phi\rangle = 1$. Note that we may assume W'_J is proper in W' if $W' \neq 1$. This is clear if $\phi \neq 1_{W'}$, and $\phi = 1_{W'}$ occurs with multiplicity 1 in the permutation character on any parabolic subgroup. Let $M \geq L$ be the Levi subgroup of G

corresponding to W'_J and χ be the unipotent character of M in the Harish-Chandra series (L,λ) corresponding to $1_{W'_J}$. Then by the comparison theorem [8, Thm 3.2.7] this means that $\langle R_M^G(\chi),\rho\rangle = 1$.

First assume σ has order 2 and let ρ_1,ρ_2 be the two extensions of ρ to $\widehat{G}$. We claim that ρ_1 has the same rationality property as some extension λ_1 of λ to $\widehat{L} = L\langle\sigma\rangle$. If $W' = 1$ then $L = G$ and the claim is obvious. If $W' \neq 1$ then $W'_J < W'$ as argued above. Exclude for the moment the case that W' has type A_2. Then σ stabilises (W'_J,χ), so χ has two extensions χ_1,χ_2 to $\widehat{M} = M\langle\sigma\rangle$. Then

$$\langle R_{\widehat{M}}^{\widehat{G}}(\chi_1+\chi_2),\rho_1\rangle = \langle R_{\widehat{M}}^{\widehat{G}}(\mathrm{Ind}_M^{\widehat{M}}\chi),\rho_1\rangle = \langle \mathrm{Ind}_G^{\widehat{G}} R_M^G(\chi),\rho_1\rangle = \langle R_M^G(\chi),\rho\rangle = 1$$

and thus (after possibly interchanging ρ_1 and ρ_2) $\langle R_{\widehat{M}}^{\widehat{G}}(\chi_1),\rho_1\rangle = 1$ and $\langle R_{\widehat{M}}^{\widehat{G}}(\chi_1),\rho_2\rangle = 0$. Consequently ρ_1 has the same rationality properties as χ_1, which by induction has the same rationality properties as λ_1, showing our claim. Now, the rationality properties of extensions of the cuspidal unipotent character λ were discussed in Section 2, which gives the desired conclusion.

Now assume that W' has type A_2. Then by [8, Tab. 4.8] we have $G = E_6(q)$ and λ is the cuspidal unipotent character of L of type D_4. Of the three characters ρ of G in this Harish-Chandra series, two have multiplicity one in $R_L^G(\lambda)$ and thus by our previous argument, possess rational extensions to $\widehat{G}$. The Deligne–Lusztig character R_w for w in the class E_6 (with characteristic polynomial $\Phi_3\Phi_{12}$) contains the third character D_4,r with multiplicity -1. Since the class E_6 contains elements centralised by σ, Proposition 2.1 shows that D_4,r has a rational extension to $\widehat{G}$.

If σ has order 3, then necessarily G has type D_4 or ${}^3D_4$. Here, only the cuspidal characters do not lie in the principal series, and for those, the claim was shown above. □

Example 3.1. Let ρ be any of the two unipotent characters of $G = {}^2E_6(q)$ in the Harish-Chandra series of type ${}^2A_5$. Then the extensions of ρ to $\widehat{G}$ have character field $\mathbb{Q}(\sqrt{-q})$, by Proposition 2.5 in conjunction with Theorem 3. These are Ennola-dual to the principal series characters $\phi_{64,4}$ and $\phi_{64,13}$ of $E_6(q)$, thus we see that the occurring irrationalities in Theorem 3 do obey the Ennola principle.

We can also understand completely the groups of type D_4 extended by its full group of grah-automorphisms:

Corollary 3.2. *Let $G = D_4(q)$ and $\Gamma \cong \mathfrak{S}_3$ its full group of graph automorphisms. Then all Γ-invariant unipotent characters of G have a rational extension to $\widehat{G} = G.\Gamma$.*

Proof. Let $\rho \in \mathrm{Uch}(G)$ be Γ-invariant. Let $\sigma \in \Gamma$ have order 3. By Proposition 2.4, ρ has one rational extension ρ_1 to $G\langle\sigma\rangle$ and two algebraically conjugate ones. Thus ρ_1 must be Γ-invariant as well and further extends to two characters $\widehat{\rho}_1,\widehat{\rho}_2$ of $\widehat{G}$. The restrictions of these to $G\langle\tau\rangle$, where $\tau \in \Gamma$ has order 2, are the two extensions of ρ to $G\langle\tau\rangle$, so rational again by Proposition 2.4. But then $\widehat{\rho}_1,\widehat{\rho}_2$ must also be rational valued. □

4. Exceptional graph automorphisms

The groups $B_2(2^{2f+1})$, $G_2(3^{2f+1})$, $F_4(2^{2f+1})$ with $f \geq 0$ possess exceptional outer graph automorphisms of order 2 not induced by an automorphism of the ambient algebraic group (in particular one can not use the results in Section 2). In the smaller two cases, the rationality properties of extended unipotent characters were determined by Brunat [2, 3]. More precisely we have the following:

Proposition 4.1 (Brunat)**.** (a) *For $G = B_2(2^{2f+1})$ all four invariant unipotent characters have rational extensions to the extension of G by the exceptional graph automorphisms.*

(b) *For $G = G_2(3^{2f+1})$, eight unipotent characters are invariant under the exceptional graph automorphism. Of these, the characters labelled $\phi_{1,0}, \phi_{2,1}, \phi_{1,6}$ and $G_2[-1]$ have rational extensions, $\phi_{2,2}$ has an extension with character field $\mathbb{Q}(\sqrt{3})$, while $G_2[1], G_2[\theta]$ and $G_2[\theta^2]$ have extensions with character field $\mathbb{Q}(\sqrt{-3})$.*

Proof. The first statement follows by inspection of [2, Tab. 6], the second from [3, Tab. 11]. For the characters in the principal series this also already follows from [6, Tab. IV], using the arguments in the proof of Theorem 3. (Note that the entries in the first rows of both tables in loc. cit. should correctly read $u^{l(w)}$, as confirmed by the authors.) □

Proposition 4.2. *For $G = F_4(2^{2f+1})$ with the exceptional graph automorphism σ of order 2, 21 unipotent characters are σ-invariant. Of these, the principal series characters*

$$\phi_{1,0},\ \phi_{4,1},\ \phi_{9,2},\ \phi_{12,4},\ \phi_{6,6'},\ \phi_{6,6''},\ \phi_{4,8},\ \phi_{9,10},\ \phi_{4,13},\ \phi_{1,24}$$

and the four characters $B_2{,}1$, $B_2{,}\epsilon$, $F_4^I[1], F_4^{II}[1]$ possess rational extensions to $\widehat{G} = G\langle\sigma\rangle$. The character $\phi_{16,5}$ has extensions with character field $\mathbb{Q}(\sqrt{2})$, the cuspidal characters $F_4[\pm i]$ have extensions with character field $\mathbb{Q}(i)$, and $F_4[-1]$ has extensions with character field $\mathbb{Q}(\sqrt{-2})$.

Proof. Let $\mathbf{G}$ be of type F_4 with a Steinberg endomorphism F_0 such that $G = \mathbf{G}^F$ for $F = F_0^2$. We may assume that F_0 induces σ on G. For the characters in the principal series the claim follows from the character table of the extended Hecke algebra in [6, Tab. VI]. For the characters $B_2{,}1$, $B_2{,}\epsilon$ we can argue as in the proof of Theorem 3 since they are parametrised by linear characters of the relative Weyl group, of type B_2 (see [8, Tab. 4.8]), which thus have multiplicity 1 in its regular character.

Now let $w \in W$ be a σ-invariant element, and μ be an eigenvalue of F on $H = H_c^i(X_w)$ for some i. Let $\lambda = \sqrt{\mu}$ be a root of μ. Then the action of G on $\tilde{H} := (H \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(\lambda))_\mu$ extends to an action of $\widehat{G}$ where σ acts as $\lambda^{-1}F_0$ and the traces of all $g\sigma$, for $g \in G$, lie in $\lambda\mathbb{Q}_\ell$. Thus, any Galois automorphism sending λ to $-\lambda$ will interchange the multiplicities in $\tilde{H}$ of the two extensions of any irreducible character ρ of G. In particular, if ρ has odd multiplicity in H, then its two extensions are only defined over $\mathbb{Q}_\ell(\lambda)$.

Now the eigenvalues of F can be computed by evaluating [4, III, Prop. 1.2 with Thm 1.3] with Chevie [15]. Specifically, if $w \in W$ is a σ-invariant regular element of order 8, then $F_4[-1]$ occurs with odd multiplicity in some $H_c^i(X_w)$ and with an eigenvalue of F equal to $-q^3$ .Since q is an odd power of 2 we have $\mathbb{Q}_\ell(\sqrt{-q^3}) = \mathbb{Q}_\ell(\sqrt{-2})$ and the previous argument shows that $F_4[-1]$ has an extension in $\tilde{H}$ with character field $\mathbb{Q}(\sqrt{-2})$.

For the two characters $\rho = F_4[\pm i]$, we use the same element w. In that case the eigenvalues of F are $\pm iq^3$. As before, this means that both characters possess extensions to $\widehat{G}$ with values in $\mathbb{Q}(\sqrt{2i})$. Now note that $\sqrt{2i} = \pm(1+i)$ lies in $\mathbb{Q}(i)$, the character field of ρ.

Finally, the two cuspidal characters $F_4^I[1]$, $F_4^{II}[1]$ appear with odd mulitplicity in the q^6-eigenspace of F on $H_c^i(X_w)$ for w a σ-stable element in class $D_4(a_1)$. We can now argue as before to see that these characters possess rational extensions. □

Remark 4.3. We expect (from Ennola duality and the known character table of $F_4(2).2$), but have not been able to prove, that the unipotent character $B_2{,}r$ possesses rational extensions, and the cuspidal characters $\rho = F_4[\theta], F_4[\theta^2]$ have extensions with character field $\mathbb{Q}(\rho)$.

Remark 4.4. A way to see that the cuspidal unipotent character $F_4[-1]$ of $G = F_4(q)$ has non-real extensions to $\widehat{G}$ is as follows: Let $\ell > 2$ be a prime dividing $q^4 + 1$ and P a Sylow ℓ-subgroup of G. Then P has larger automiser in $\widehat{G}$ than in G. Thus the Brauer tree of the principal ℓ-block of $\widehat{G}$ is obtained by unfolding the Brauer tree of the principal ℓ-block of G around the exceptional vertex. Since $F_4[-1]$ and the trivial character lie on opposite sides of the exceptional vertex by [10, Thm 2.1(3)] and the trivial character has real extensions to $\widehat{G}$, the extensions of $F_4[-1]$ cannot lie on the real stem of the Brauer tree of the principal ℓ-block of $\widehat{G}$ and thus can't be real-valued. This does, however, not exhibit the precise character field.

The argument also applies to the cuspidal unipotent characters of ${}^2A_2(q)$ and ${}^2A_5(q)$, and to $G_2[1]$ of $G_2(q)$ (with suitable ℓ).

References

[1] M. Broué, G. Malle, Zyklotomische Heckealgebren. *Astérisque* No. **212** (1993), 119–189. 5

[2] O. Brunat, The Shintani descents of Suzuki groups and their consequences. *J. Algebra* **303** (2006), 869–890. 8

[3] O. Brunat, On the extension of $G_2(3^{2n+1})$ by the exceptional graph automorphism. *Osaka J. Math.* **44** (2007), 973–1023. 8

[4] F. Digne, J. Michel, Fonctions L des variétés de Deligne–Lusztig et descente de Shintani. *Mém. Soc. Math. France (N.S.)* No. **20** (1985). 1, 6, 8

[5] M. Geck, Character values, Schur indices and character sheaves. *Represent. Theory* **7** (2003), 19–55. 1, 6

[6] M. Geck, S. Kim, G. Pfeiffer, Minimal length elements in twisted conjugacy classes of finite Coxeter groups. *J. Algebra* **229** (2000), 570–600. 8

[7] M. Geck, G. Malle, Fourier transforms and Frobenius eigenvalues for finite Coxeter groups. *J. Algebra* **260** (2003), 162–193. 5

[8] M. Geck, G. Malle, *The Character Theory of Finite Groups of Lie Type. A Guided Tour.* Cambridge University Press, Cambridge, 2020. 1, 3, 4, 5, 6, 7, 8

[9] M. Geck, G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras.* The Clarendon Press, Oxford University Press, New York, 2000. 6

[10] G. Hiss, F. Lübeck, The Brauer trees of the exceptional Chevalley groups of types F_4 and ${}^2E_6$. *Arch. Math. (Basel)* **70** (1998), 16–21. 9

[11] G. Lusztig, Coxeter orbits and eigenspaces of Frobenius. *Invent. Math.* **38** (1976/77), 101–159. 5, 6

[12] G. Lusztig, *Representations of Finite Chevalley Groups.* CBMS Regional Conference Series in Mathematics, 39. American Mathematical Society, Providence, R.I., 1978. 2, 5, 6

[13] G. Lusztig, Rationality properties of unipotent representations. *J. Algebra* **258** (2002), 1–22. 1, 4, 5, 6

[14] G. Malle, Some unitary groups as Galois groups over $\mathbb{Q}$. *J. Algebra* **131** (1990), 476–482. 1

[15] J. Michel, The development version of the CHEVIE package of GAP3. *J. Algebra* **435** (2015), 308–336. 4, 5, 8

[16] Z. Ohmori, The Schur indices of the cuspidal unipotent characters of the finite unitary groups. *Proc. Japan Acad. Ser. A Math. Sci.* **72** (1996), 111–113. 6

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